Weakly constrained-degree percolation

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joint with Bernardo de Lima

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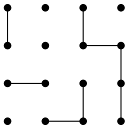

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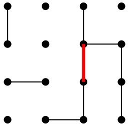

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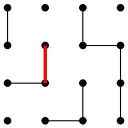

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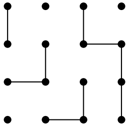

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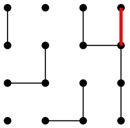

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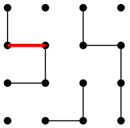

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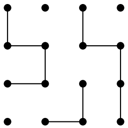
$d = \kappa = 2$ 

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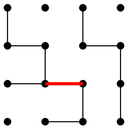

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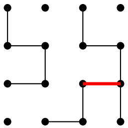

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$$t_c^\kappa(d) = \inf\{t \in [0, 1] : \mathbb{P}(0 \leftrightarrow \infty \text{ at time } t) > 0\} \in [0, 1] \cup \{\infty\}.$$

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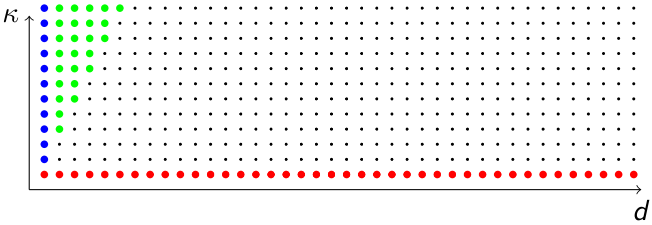
- No FKG inequality (having some edges may influence other negatively).
- No finite energy property (adding an edge may change the probability by more than a bounded factor).
- The range of dependency is unbounded.

What do we know about it?



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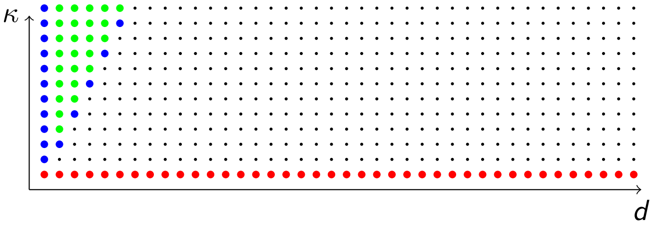
- For $\kappa = 2d - 1$, $d \geqslant 2$ we have $t_c^\kappa(d) \neq \infty$. [Teodoro'14]



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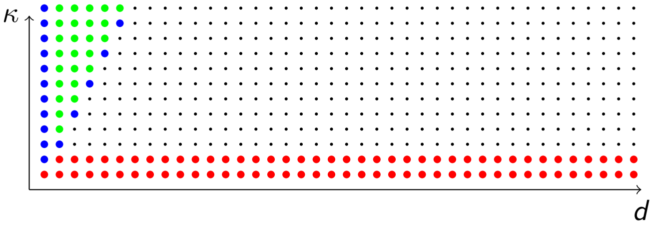

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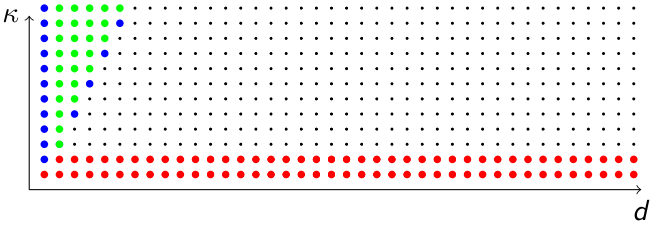
- For $d = 2, \kappa = 3$ we have $t_c^4(2) = 1/2 < t_c^3(2) < 1$. [dLSdSST]



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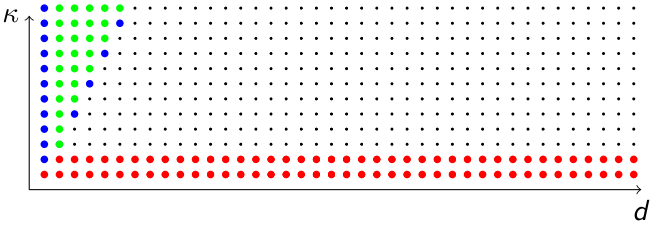
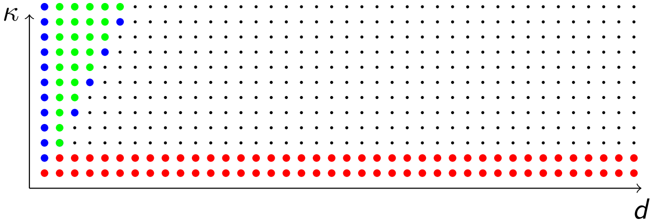
- For $d = 2$ and random constraints concentrated on $\kappa = 3$ and t close to 1 there is percolation. [Sanchis, dos Santos, Silva'21+]

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Theorem

We have $t_c^\kappa(d) < 1.9/d$ for:

- $\kappa \geqslant 10$ *and* $d > \kappa/2$*;*
- $\kappa \geqslant 9$ *and* $d \in \{7, 8, 9, 10, 11, 12, 14, 16\}$*;*
- $\kappa \geqslant 8$ *and* $d \in \{5, 6\}$*;*
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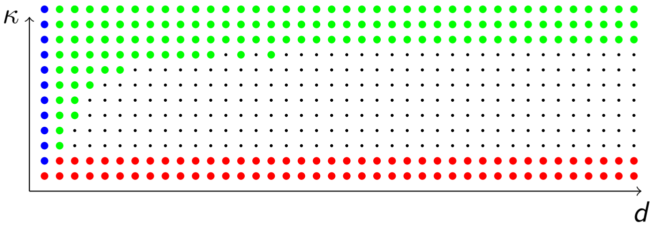

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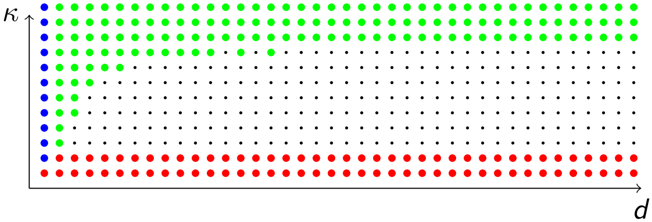
Theorem

For any κ_n, d_n *such that* $\kappa_n \to \infty$ *and* $d_n \to \infty$ *as* $n \to \infty$ *we have*

$$\lim_{n\to\infty} d_n \cdot t_c^{\kappa_n}(d_n) = \frac{1}{2}.$$

Conclusion



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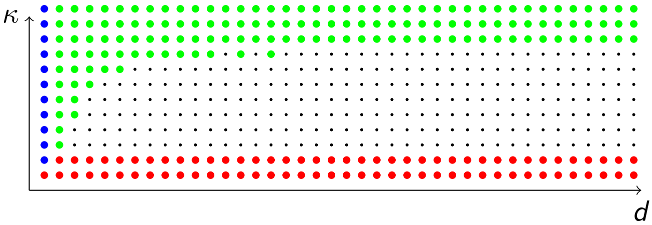

When stuck, make events non-monotone.

Thank you.

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Based on IH, Bernardo de Lima, *Weakly constrained-degree percolation on the hypercubic lattice.* ArXiv:2010.08955.