Bisection for Kinetically Constrained Models[1]

Ivailo Hartarsky
CEREMADE, Université Paris Dauphine, PSL University

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- $\{\circ, \bullet\}^{\mathbb{Z}}$ state space.
- $q \in [0, 1]$ equilibrium density of $\bullet$. (Think of q small.)
- $Ber(q)^{\otimes \mathbb{Z}}$ is **a** reversible measure.

East model (Jäckle–Eisinger'91)

The state of each site $x \in \mathbb{Z}$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless $x - 1$ is in state $\bullet$.

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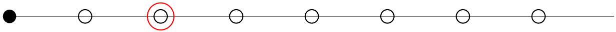

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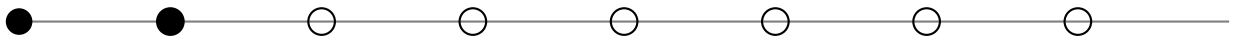

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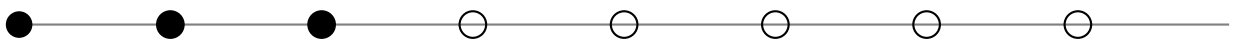

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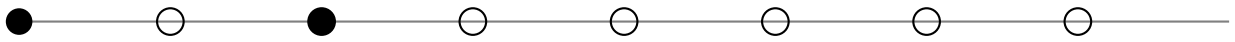

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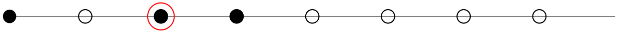

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Theorem (Aldous–Diaconis'02,Cancrini–Martinelli–Roberto–Toninelli'08)

$$T_{\mathrm{rel}} \leqslant \exp\left(C \log^2(1/q)\right)$$

Lemma (Two-block Poincaré inequality of CMRT)

Let X_1 *and* X_2 *be two independent RV valued in the finite sets* $\mathbb{X}_1, \mathbb{X}_2$*. Let* $\mathcal{H} \subset \mathbb{X}_1$ *with* $p := \mathbb{P}(X_1 \in \mathcal{H}) > 0$*. Then for any* $f : \mathbb{X}_1, \mathbb{X}_2 \to \mathbb{R}$

$$\mathrm{Var}(f) \leqslant \frac{1}{1-\sqrt{1-p}} \mathbb{E}[\mathrm{Var}(f|X_2) + \mathbb{1}_{X_1 \in \mathcal{H}} \mathrm{Var}(f|X_1)]$$

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Idea: $e^{-1/T_{\mathrm{rel}}} = \lim_{t\to\infty}(d_{\mathrm{TV}}(\mu_t, \pi))^{1/t}$.

Probabilistic proof

Two chains couple as soon as we update X_1 so that $\mathcal{H}$ occurs and then X_2. There are $\lfloor N/2 \rfloor$ attempted updates at X_2 preceded by an update at X_1, where $N \sim \mathcal{P}(t)$. Each succeeds with probability p, so

$$d_{\mathrm{TV}}(\mu_t, \pi) \leqslant \mathbb{P}(\text{not coupled at time } t) \leqslant \mathbb{E}\left[(1-p)^{\lfloor N/2 \rfloor}\right]$$

$$\approx \mathbb{E}\left[\left(\sqrt{1-p}\right)^N\right] = \exp\left(-t\left(1-\sqrt{1-p}\right)\right).$$

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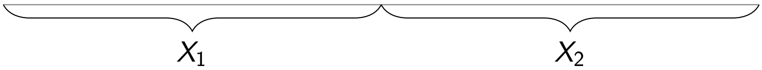

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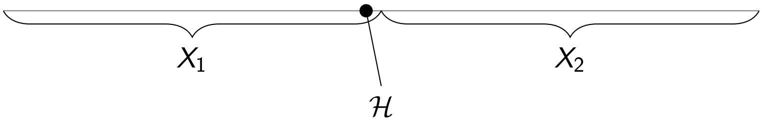

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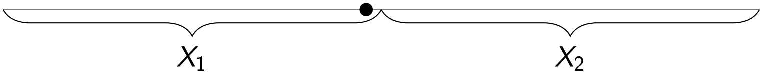

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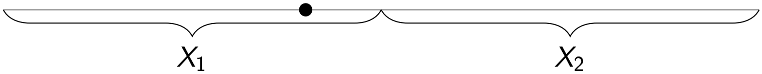

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- an irreducible component $\mathcal{C} \subset \prod_{x \in L} \mathcal{S}_x$ of the dynamics.

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There exists $C_R > 0$ *such that*

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We focus on proving

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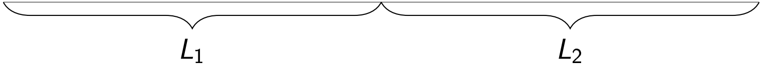
Update L_1 at rate 1 conditionally on the irreducible component induced by $\eta_{L_2 \setminus L_1}$ and similarly for L_2. The relaxation time of this chain is at most $(2/q)^{C_R}$.



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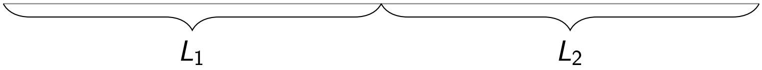

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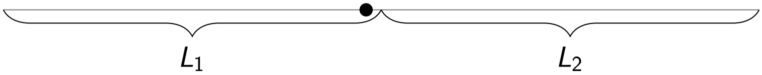

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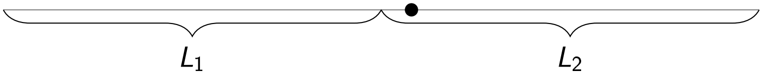

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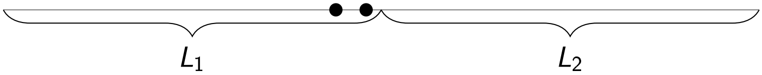

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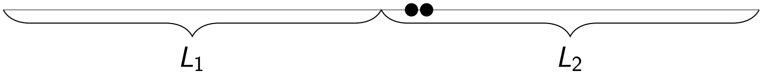

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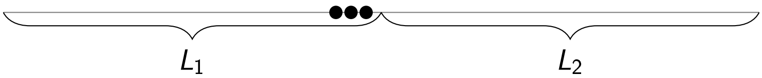

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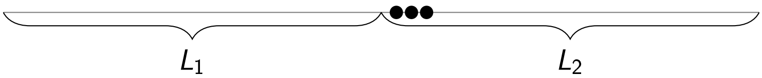

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