TECHNISCHE UNIVERSITÄT WIEN

Two-neighbour bootstrap percolation and the Fredrickson–Andersen model

Ivailo Hartarsky

Probability seminar ICJ/UMPA
Lyon, 7 December 2023

European Research Council
Established by the European Commission

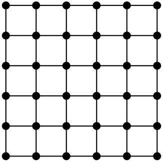
Der Wissenschaftsfonds.

This project has received funding from the ERC under the EU's Horizon 2020 research and innovation programme (Grant agreement No. 680275) and Austrian Science Fund (FWF): P35428-N

Two neighbour bootstrap percolation

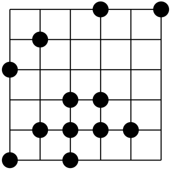
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.



Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).



Two neighbour bootstrap percolation

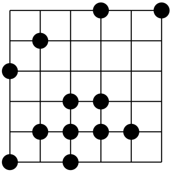
- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.

Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.

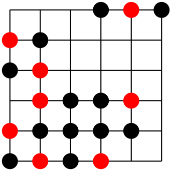
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



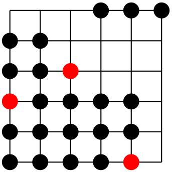
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



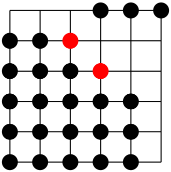
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



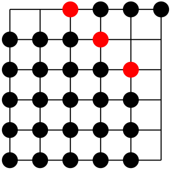
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



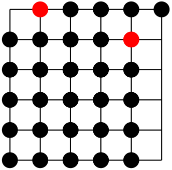
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



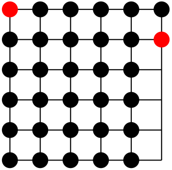
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



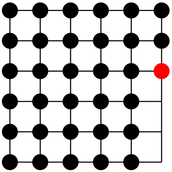
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t + 1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



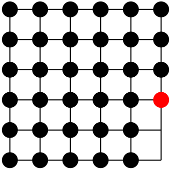
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



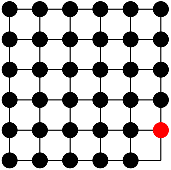
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



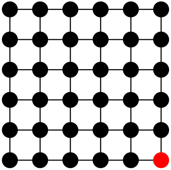
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



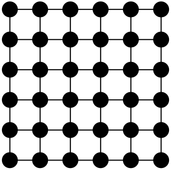
Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.



Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.
- Infection time: $\tau^{\mathrm{BP}} = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.

Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.
- Infection time: $\tau^{\mathrm{BP}} = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $q \in [0,1]$.

Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.
- Infection time: $\tau^{\mathrm{BP}} = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $q \in [0, 1]$.
- Initial distribution: $Ber(q)^{\otimes \mathbb{Z}^2}$.

Two neighbour bootstrap percolation

- Geometry: $\mathbb{Z}^2$.
- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$ ($\circ/\bullet$ =healthy/infected).
- If at time $t \in \mathbb{N}$ site $x \in \mathbb{Z}^2$ has at least 2 neighbours in state $\bullet$, then at time $t+1$ its state also becomes $\bullet$.
- $\bullet$ never becomes $\circ$.
- Infection time: $\tau^{\mathrm{BP}} = \inf\{t \in \mathbb{N} : 0 \text{ is } \bullet\} \in \mathbb{N} \cup \{\infty\}$.
- Density of $\bullet$: $q \in [0,1]$.
- Initial distribution: $Ber(q)^{\otimes \mathbb{Z}^2}$.
- Low temperature regime: all bounds will hold a.a.s. as $q \to 0$.

Previous results

- [Van Enter'87] For all $q > 0$ we have $\tau^{\mathrm{BP}} < \infty$ a.s.

Previous results

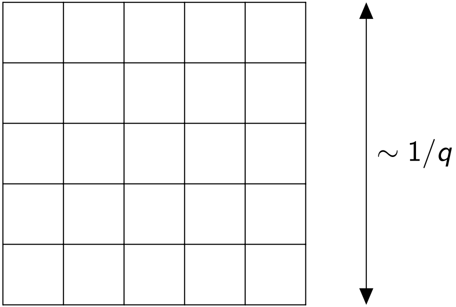
- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

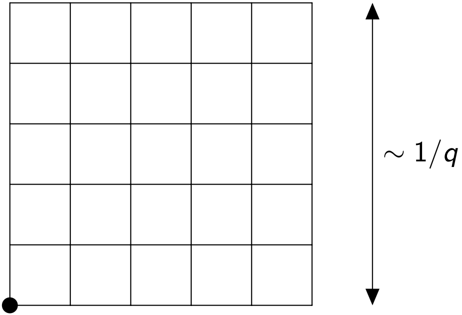
$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

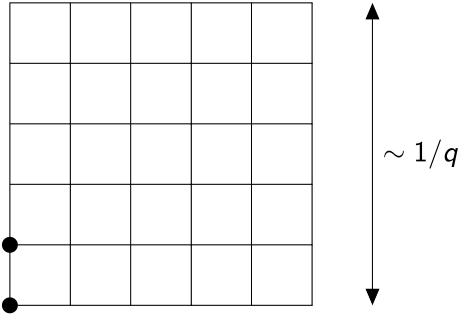
$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

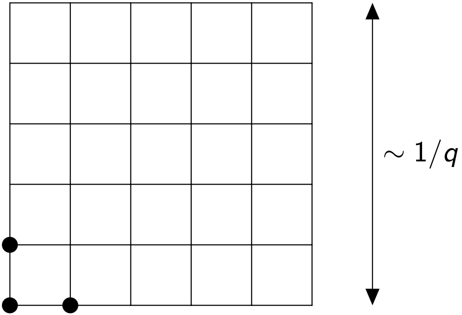
$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

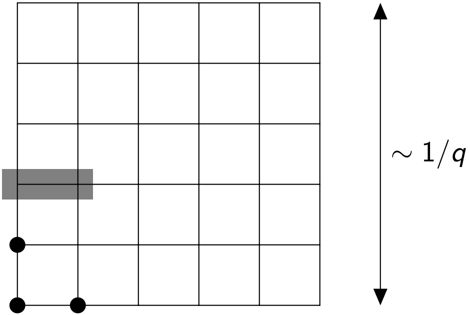
$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

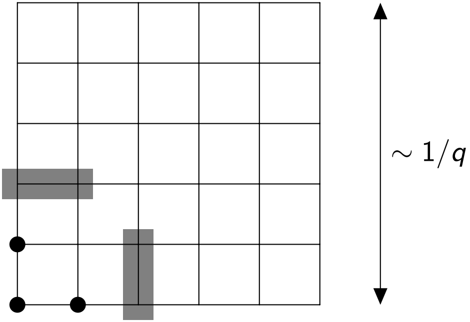
$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

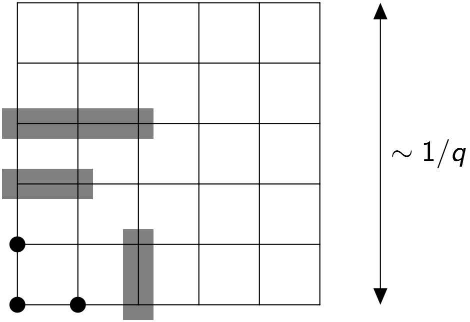
$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

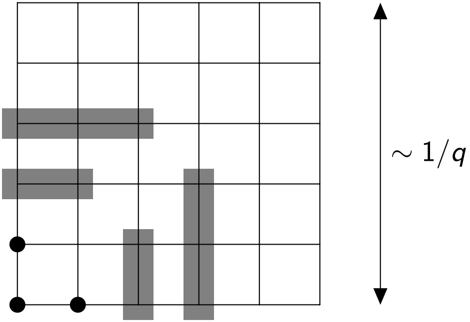
$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

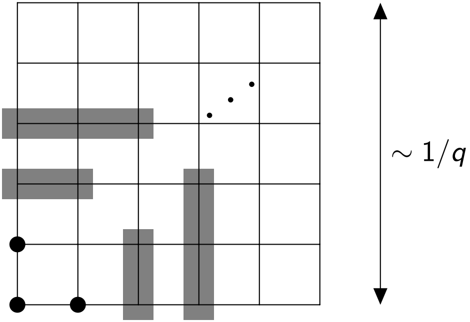
$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$

$$q\prod_{k=1}^{\infty}(1-(1-q)^k)^2 \approx \exp\left(2\int_0^{\infty}\log\left(1-e^{-qx}\right)\mathrm{d}x\right) = \exp\left(\frac{-\pi^2}{3q}\right)$$

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] For some $c, C > 0$

$$\exp\left(\frac{c}{q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{C}{q}\right).$$

Lower bound: Rectangles process—if $1/q < \tau^{\mathrm{BP}} < \infty$, there exists an *internally filled* rectangle of size $1/q$.

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$

Previous results

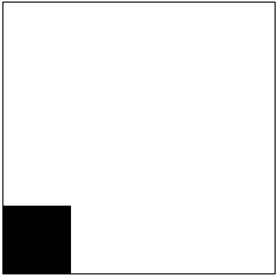
- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

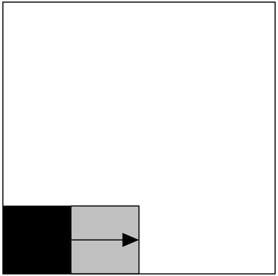
$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

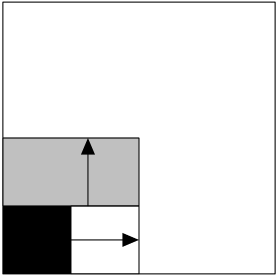
$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

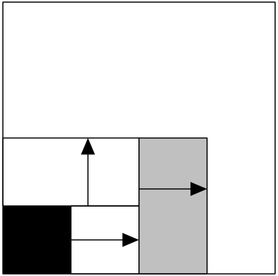
$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

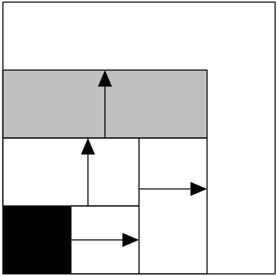
$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

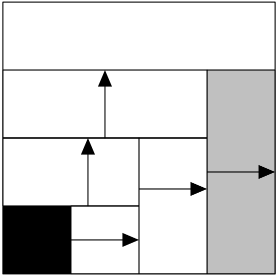
$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

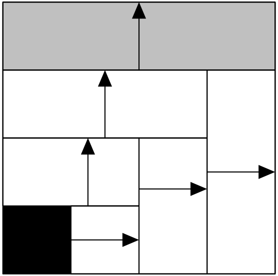
$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

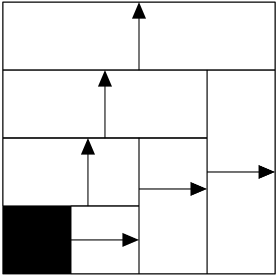
$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

$$\exp\left(\frac{\pi^2-\varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2+\varepsilon}{18q}\right).$$



Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] For every $\varepsilon > 0$

$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 + \varepsilon}{18q}\right).$$

- [Lenormand–Zarcone'84] 0.33 for $q = 0.086$, size 2.10^3.
- [Nakanishi–Takano'86] 0.27 for $q = 0.03$, size 1.10^3.
- [Adler–Stauffer–Aharony'89] 0.245 ± 0.015 for $q = 0.03$, size 2.10^4.
- [Teomy–Shokef'14] 0.274 for 8 cpu years, q=0.016, size 3.10^7.

But $\pi^2/18 \approx 0.548$.

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp threshold
- [Gravner–Holroyd'08] For every $\varepsilon > 0$ and some $c > 0$

$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 - c\sqrt{q}}{18q}\right).$$

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp threshold
- [Gravner–Holroyd'08] Upper bound for the second term
- [Gravner–Holroyd–Morris'12] For some $c, C > 0$

$$\exp\left(\frac{\pi^2 - C(\log(1/q))^3\sqrt{q}}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 - c\sqrt{q}}{18q}\right).$$

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp threshold
- [Gravner–Holroyd'08] Upper bound for the second term
- [Gravner–Holroyd–Morris'12] Almost matching lower bound
- [Bringmann–Mahlburg'12] 'Morally,' for some $c, C > 0$

$$\exp\left(\frac{\pi^2 - C(\log(1/q))^{5/2}\sqrt{q}}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 - c\sqrt{q}}{18q}\right).$$

Previous results

- [Van Enter'87] Trivial transition
- [Aizenman-Lebowitz'88] Scaling
- [Holroyd'03] Sharp threshold
- [Gravner–Holroyd'08] Upper bound for the second term
- [Gravner–Holroyd–Morris'12] Almost matching lower bound
- [Bringmann–Mahlburg'12] Slightly better lower bound

Theorem (H–Morris'19)

For some $c, C > 0$

$$\exp\left(\frac{\pi^2 - C\sqrt{q}}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 - c\sqrt{q}}{18q}\right).$$

Theorem (H–Morris'19)

For some $c, C > 0$

$$\exp\left(\frac{\pi^2 - C\sqrt{q}}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 - c\sqrt{q}}{18q}\right).$$

Theorem (H'22, but morally Gravner–Holroyd'08)

For some $C > 0$ *and all* $\varepsilon > 0$

$$\frac{F^{-1}_{\tau^{\mathrm{BP}}}(1-\varepsilon)}{F^{-1}_{\tau^{\mathrm{BP}}}(\varepsilon)} \leqslant \frac{1}{q^C}.$$

Theorem (H–Morris'19)

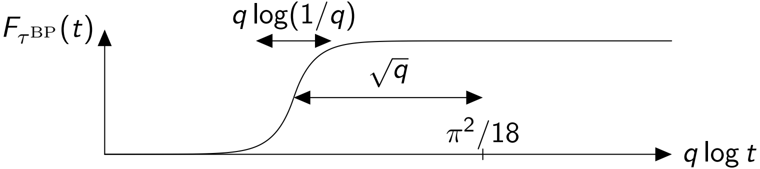
For some $c, C > 0$

$$\exp\left(\frac{\pi^2 - C\sqrt{q}}{18q}\right) \leqslant \tau^{\mathrm{BP}} \leqslant \exp\left(\frac{\pi^2 - c\sqrt{q}}{18q}\right).$$

Theorem (H'22, but morally Gravner–Holroyd'08)

For some $C > 0$ *and all* $\varepsilon > 0$

$$\frac{F^{-1}_{\tau^{\mathrm{BP}}}(1-\varepsilon)}{F^{-1}_{\tau^{\mathrm{BP}}}(\varepsilon)} \leqslant \frac{1}{q^C}.$$



Fredrickson–Andersen 2-spin facilitated model

Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.

Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Glauber dynamcis

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$.

Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Glauber dynamcis

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is the reversible measure.

Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.

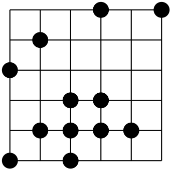
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



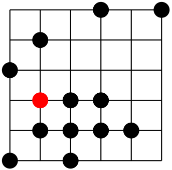
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



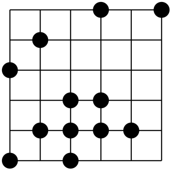
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



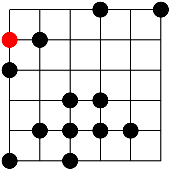
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0,1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



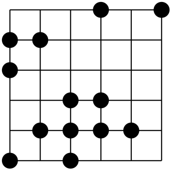
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



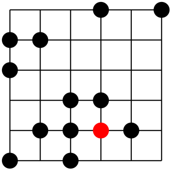
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



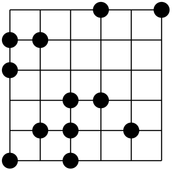
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



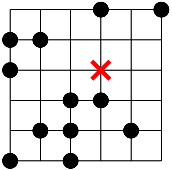
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



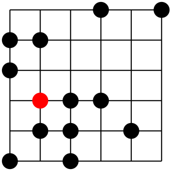
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



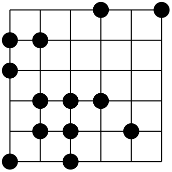
Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.



Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.
- $\tau = \inf\{t > 0 : 0 \text{ is } \bullet\} \in [0, \infty]$.

Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0, 1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.
- $\tau = \inf\{t > 0 : 0 \text{ is } \bullet\} \in [0, \infty]$.
- Initial state $Ber(q)^{\otimes \mathbb{Z}^2}$: equilibrium.

Fredrickson–Andersen 2-spin facilitated model

- State space: $\{\circ, \bullet\}^{\mathbb{Z}^2}$.
- Equilibrium density of $\bullet$: $q \in [0,1]$.

Fredrickson–Andersen 2-spin facilitated model (FA-2f) [FA'84]

The state of each site $x \in \mathbb{Z}^2$ is resampled independently at rate 1 from $Ber(q)$. However, the update is rejected unless x has at least 2 neighbours in state $\bullet$.

- $Ber(q)^{\otimes \mathbb{Z}^2}$ is a reversible measure.
- $\tau = \inf\{t > 0 : 0 \text{ is } \bullet\} \in [0, \infty]$.
- Initial state $Ber(q)^{\otimes \mathbb{Z}^2}$: equilibrium.
- Low temperature: $q \to 0$.

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] For all $q > 0$ we have $\tau < \infty$ a.s.

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] For some $C > 0$ and every $\varepsilon > 0$

$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau \leqslant \exp\left(\frac{C}{q^5}\right).$$

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] For some $C > 0$ and every $\varepsilon > 0$

$$\exp\left(\frac{\pi^2 - \varepsilon}{18q}\right) \leqslant \tau \leqslant \exp\left(\frac{C}{q^5}\right).$$

Idea: FA-2f cannot be faster than 2-neighbour bootstrap percolation.

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] Very rough scaling ($\log\log\tau$ up to constant)
- [Martinelli–Toninelli'19] For some $C > 0$

$$\exp\left(\frac{\pi^2 - o(1)}{18q}\right) \leqslant \tau \leqslant \exp\left(\frac{C(\log(1/q))^2}{q}\right).$$

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] Very rough scaling ($\log\log\tau$ up to constant)
- [Martinelli–Toninelli'19] For some $C > 0$

$$\exp\left(\frac{\pi^2 - o(1)}{18q}\right) \leqslant \tau \leqslant \exp\left(\frac{C(\log(1/q))^2}{q}\right).$$

Idea: take an infected frame of slightly supercritical size and move it in a FA-1f fashion.

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] Very rough scaling ($\log\log\tau$ up to constant)
- [Martinelli–Toninelli'19] Rough scaling ($\log\tau$ up to log corrections)

Non-rigorous predictions

- [Nakanishi–Takano'86] For some $C(q)\to\infty$ as $q\to 0$

$$\tau \approx \exp\left(\frac{C(q)}{q}\right).$$

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] Very rough scaling ($\log\log\tau$ up to constant)
- [Martinelli–Toninelli'19] Rough scaling ($\log\tau$ up to log corrections)

Non-rigorous predictions

- [Nakanishi–Takano'86] Log corrections
- [Reiter'91] For some $C > 0$

$$\tau \approx \exp\left(\frac{\pi^2 + C}{9q}\right).$$

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] Very rough scaling ($\log\log\tau$ up to constant)
- [Martinelli–Toninelli'19] Rough scaling ($\log\tau$ up to log corrections)

Non-rigorous predictions

- [Nakanishi–Takano'86] Log corrections
- [Reiter'91] No log corrections; different constant; exponentially slow movement of droplets
- [Toninelli–Biroli–Fisher'05] For some $C \in \mathbb{R}$

$$\tau \approx \exp\left(\frac{\pi^2 + C\sqrt{q}}{9q}\right).$$

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] Very rough scaling ($\log\log\tau$ up to constant)
- [Martinelli–Toninelli'19] Rough scaling ($\log\tau$ up to log corrections)

Non-rigorous predictions

- [Nakanishi–Takano'86] Log corrections
- [Reiter'91] No log corrections; different constant; exponentially slow movement of droplets
- [Toninelli–Biroli–Fisher'05] No log corrections; same constant; stretched exponential movement of droplets
- [Teomy–Shokef'15]

$$\tau \approx \exp\left(\frac{2\pi^2}{9q}\right).$$

Previous results

- [Cancrini–Martinelli–Roberto–Toninelli'08] Trivial transition
- [CMRT08] Very rough scaling ($\log\log\tau$ up to constant)
- [Martinelli–Toninelli'19] Rough scaling ($\log\tau$ up to log corrections)

Non-rigorous predictions

- [Nakanishi–Takano'86] Log corrections
- [Reiter'91] No log corrections; different constant; exponentially slow movement of droplets
- [Toninelli–Biroli–Fisher'05] No log corrections; same constant; stretched exponential movement of droplets
- [Teomy–Shokef'15] No log corrections; doubled constant

Theorem (H–Martinelli–Toninelli'23)

For some $C > 0$

$$\exp\left(\frac{\pi^2 - C\sqrt{q}}{9q}\right) \leqslant \tau \leqslant \exp\left(\frac{\pi^2 + C\sqrt{q}(\log(1/q))^3}{9q}\right).$$

Theorem (H–Martinelli–Toninelli'23)

For some $C > 0$

$$\exp\left(\frac{\pi^2 - C\sqrt{q}}{9q}\right) \leqslant \tau \leqslant \exp\left(\frac{\pi^2 + C\sqrt{q}(\log(1/q))^3}{9q}\right).$$

In particular,

$$\tau = \left(\tau^{\mathrm{BP}}\right)^{2+o(1)}.$$

Lower bound

Lower bound

Let τ_0 be the first time when the origin can become infected only using infections at distance at most $\sim \log(1/q)/q$.

Lower bound

Let τ_0 be the first time when the origin can become infected only using infections at distance at most $\sim \log(1/q)/q$.
We have $\tau_0 > 0$ w.h.p. since τ^{BP} is exponentially large w.h.p.

Lower bound

Let τ_0 be the first time when the origin can become infected only using infections at distance at most $\sim \log(1/q)/q$.
We have $\tau_0 > 0$ w.h.p. since τ^{BP} is exponentially large w.h.p.
If $\tau_0 > 0$, then at τ_0 there is an internally filled rectangle of size $\sim \log(1/q)/q$ at the origin.

Lower bound

Let τ_0 be the first time when the origin can become infected only using infections at distance at most $\sim \log(1/q)/q$.

We have $\tau_0 > 0$ w.h.p. since τ^{BP} is exponentially large w.h.p.

If $\tau_0 > 0$, then at τ_0 there is an internally filled rectangle of size $\sim \log(1/q)/q$ at the origin.

Theorem (HM19)

The probability of this under the stationary measure is at most

$$\exp\left(\frac{-\pi^2 + C\sqrt{q}}{9q}\right).$$

Upper bound

Upper bound

Problems

Find a good droplet structure such that:

Upper bound

Problems

Find a good droplet structure such that:

- it has probability at least

$$\exp\left(\frac{-\pi^2}{9q}\right),$$

Upper bound

Problems

Find a good droplet structure such that:

- it has probability at least

$$\exp\left(\frac{-\pi^2}{9q}\right),$$

- it can move fast locally—at rate at least

$$\exp\left(\frac{-C(\log(1/q))^3}{\sqrt{q}}\right),$$

Upper bound

Problems

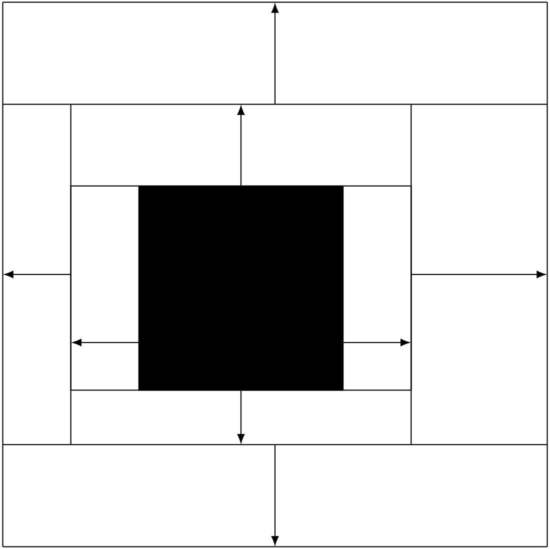
Find a good droplet structure such that:

- it has probability at least
$$\exp\left(\frac{-\pi^2}{9q}\right),$$
- it can move fast locally—at rate at least
$$\exp\left(\frac{-C(\log(1/q))^3}{\sqrt{q}}\right),$$
- it can move globally without recreating itself.

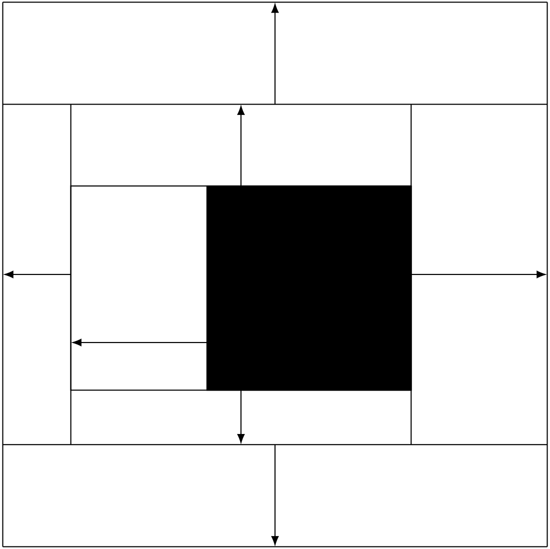
https://www.youtube.com/watch?v=7pR7TNzJ_pA

Amoeba motion

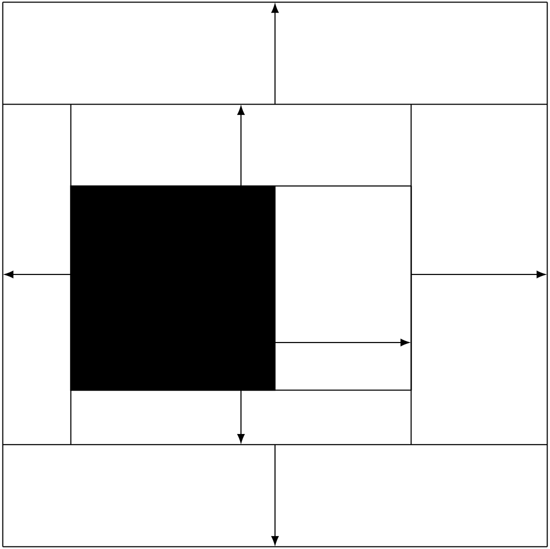
Amoeba motion



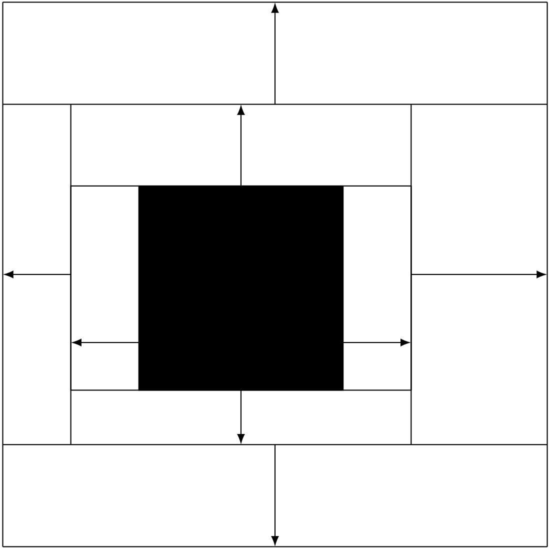
Amoeba motion



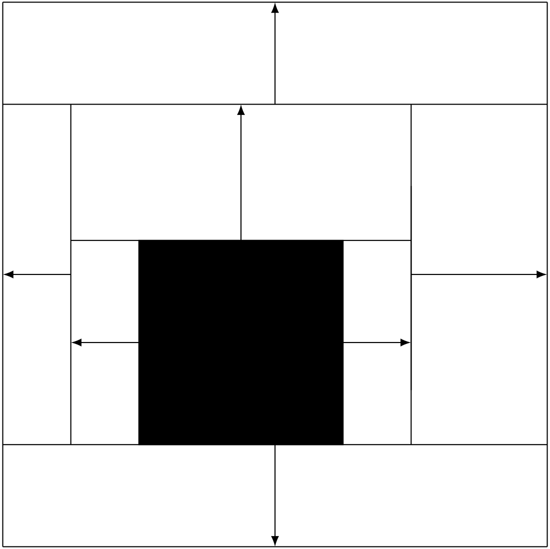
Amoeba motion



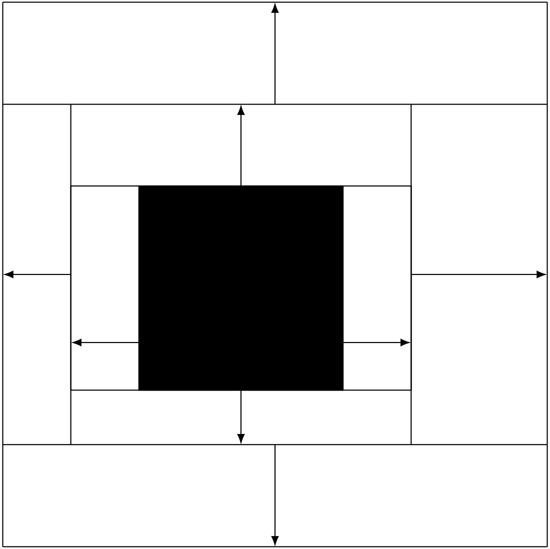
Amoeba motion



Amoeba motion



Amoeba motion



Global movement: coalescing and branching simple symmetric exclusion process.

Global movement: coalescing and branching simple symmetric exclusion process.

- The amoeba moves by 1 step.

Global movement: coalescing and branching simple symmetric exclusion process.

- The amoeba moves by 1 step.
- The amoeba splits.

Global movement: coalescing and branching simple symmetric exclusion process.

- The amoeba moves by 1 step.
- The amoeba splits.
- The amoeba meets another one and eats it.

Global movement: coalescing and branching simple symmetric exclusion process.

- The amoeba moves by 1 step.
- The amoeba splits.
- The amoeba meets another one and eats it.

Theorem (H–Martinelli–Toninelli'22)

The relaxation time on a box of volume V such that there is on average one amoeba is $|V|$ (up to logarithms).

?