Catalan percolation

joint with E. Archer, B. Kolesnik, S. Olesker-Taylor, B. Schapira, D. Valesin]

1. Model [Gravner - Kolesnik '23]

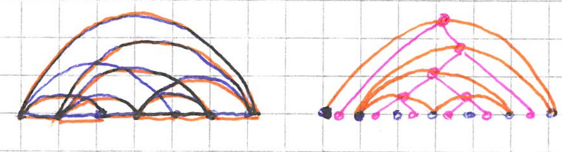
triadic closure (Δ-completion), polluted graph bootstrap percolation.

vx set $\mathbb{Z}$. • $\forall j > i+1$: $\{i,j\}$ is open indep. with probability $p \in (0,1)$.

• $\forall i$: $\{i, i+1\}$ is occupied.

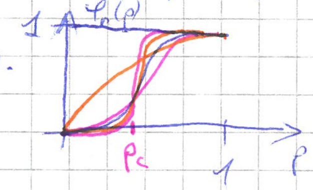
• $\forall j > i+1$: $\{i,j\}$ is occupied if $\exists k \in (i,j)$ s.t. $\{i,k\}$ and $\{k,j\}$ are occup.

$\varphi_n(p) = \mathbb{P}_p(\{0,n\} \text{ is occup.} \mid \{0,n\} \text{ is open})$ $\quad p_c = \inf\{p > 0 : \liminf_{n\to\infty} \varphi_n(p) > 0\}$.



⚠ φ_n (and $p \cdot \varphi_n$) is not ~~monotone in n~~.

$\cancel{\varphi_1(p) = \frac{1}{p}}$, $\varphi_2(p) = 1$, $\varphi_3(p) = 2p - p^2$



2. Lower bounds on p_c.

Th. (G, K) $p_c \geq \frac{1}{4}$.

Obs. $\{0,n\}$ is occupied iff all internal nodes of some dual binary tree are open.

Proof of Th. $C_n = \frac{1}{n+1}\binom{2n}{n} \leq 4^n$ is # bin trees with n leaves.

$p \cdot \varphi_n(p) = \mathbb{P}_p(\{0,n\} \text{ is occup.}) \leq C_n \cdot p^{n-1} \leq 4 \cdot (4p)^{n-1} \xrightarrow[n\to\infty]{} 0$ if $p < \frac{1}{4}$ $\square$

Th. [AHKOTSV 24+] $p_c > 1/4$ $\quad (0{,}254 ; \ldots ; 0{,}29)$ $\quad p_c \approx 0{,}40$

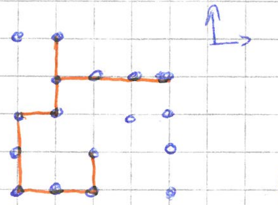
Proof. Generating functions, plugging $\varphi_3 = 2p - p^2 < 2p$ in Catalan recurrence

$$C_n = \sum_{k=0}^{n-1} C_k C_{n-k-1}. \quad \square$$

3. Upper bounds on p_c

Th. [GK] $p_c \leq p_c^o$

Oriented site percolation.

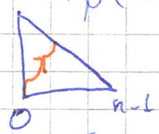


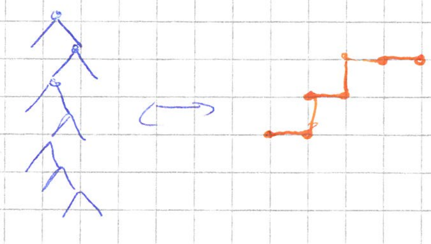
each site is open indep. with prob. $p \in (0,1)$

$p_c^o = \inf\{p > 0 : \mathbb{P}_p(0 \to \infty) > 0\}$

Proof of Th.: Trees s.t. each int. node has at least 1 leaf child

$p \cdot \varphi_n(p) \geq \mathbb{P}_p(\exists \text{ open leaf-child tree rooted at } \{0,n\})$

$= \mathbb{P}_p($  $) \geq \mathbb{P}_p(0 \to \infty) \underset{p > p_c^o}{>} 0$ $\square$



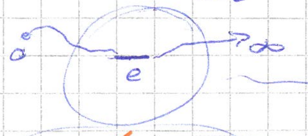
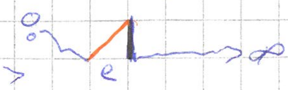
Th. [AHKOTSV] $p_c < p_c^o$.

4. Essential enhancements [Aizenman–Grimmett'91]

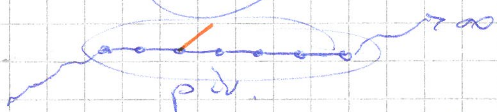
Typical example $\mathbb{Z}^2$ unorient. bond perco with param p with ⟋ added indep with proba $q\in[0,1)$. $p_c(q)<p_c(0)\ (=1/2)$

"Russo fla" $\frac{\partial}{\partial p}\mathbb{P}_{p,q}(0\to\infty)=\sum_{e\in E^2}\mathbb{P}_{p,q}\left(0\xleftrightarrow{\text{with } e}\infty,\ 0\overset{\text{without } e}{\not\leftrightarrow}\infty\right)$

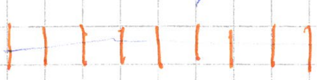
$\frac{\partial}{\partial p}\ldots\le c\,\frac{\partial}{\partial q}\ldots$


EE for OP?



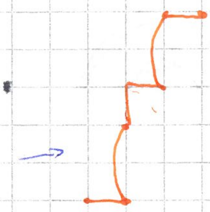
Brochette percolation (Duminil-Copin, Hilário, Kozma, Sidoravicius '18)

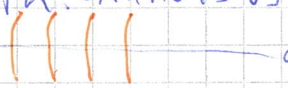
Th. Unoriented bond percolation on $\mathbb{Z}^2$ with  added with proba q

$\forall q>0\quad p_c(q)<p_c(0)$.

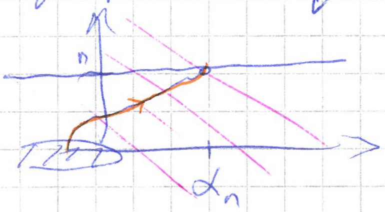
5. Proof idea.

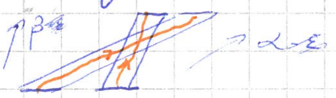
Trees with each int. node has:
- left leaf child
- OR right leaf child
- OR right cherry child.

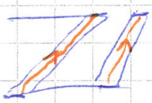



Th. [AHKOTSV] For oriented site perco with  added with proba q, we have $p_c(q)<p_c$ for all $q>0$.

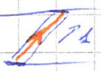
- Edge speed, looking up $\alpha=\lim\frac{\alpha_n}{n}$



- α ↗↗ in q.
- $\mathbb{P}_{p,q}($  $)>1-\varepsilon$
- View length 2 edges as random environment



- At bad times use RSW by Duminil-Copin-Tassion-Teixeira'17 to cross a k-interval of bad times with proba $\geq\varepsilon^k$



- Renormalise to OP with geometric defects [Hilário, Sá, Sanchis, Teixeira '23]

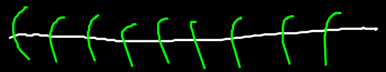
bonds open with proba $p^{1+\xi_i}$ $\quad\xi_i\overset{\text{iid}}{\sim}$ Geo (small expectation), p close to 1.

+ multi-scale ⇝ Hoffman stretched lattices, Kesten-Sidoravicius, Vares)

⚠ LSS needs care, because of dependencies.

Catalan percolation 2

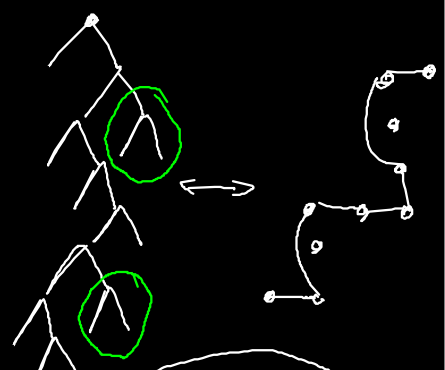
Th 1 (AHKOTSV'24+) $p_c < p_c^\circ$.

Th2 (Duminil-Copin, Hilário, Kozma, Sidoravicius '18)
For Bernoulli bond perco on $\mathbb{Z}^2$ with parameter p and  added indep. with proba. q, we have $\forall q>0$ $p_c(q) < p_c(0)$ $\left(=\frac{1}{2}\right)$.

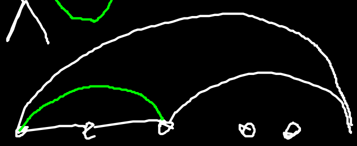
Ideas of the proof of Th 1

Consider trees with each int. node having
- a leaf child

OR
- a right cherry child

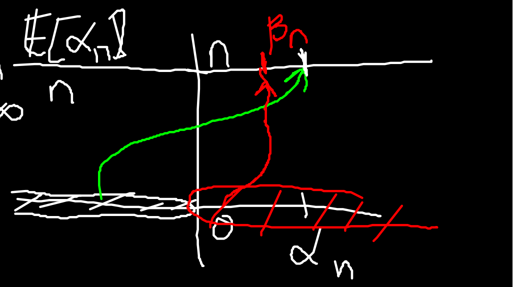


Th 1 follows from this and:

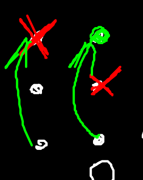


Th3 (us) For Bernoulli oriented site perco on $\mathbb{Z}^2$ with parameter p and  indep. with proba $q\in[0,1]$, for any $q>0$, we have $p_c(q) < p_c(0) = p_c^\circ$.

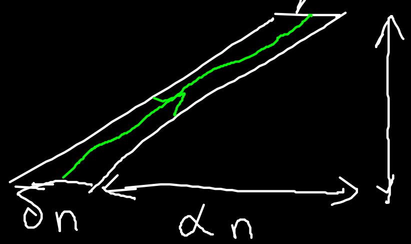
Step 1. Edge speeds $\alpha = \lim_{n\to\infty} \frac{\alpha_n}{n} = \lim_{n\to\infty} \frac{\mathbb{E}[\alpha_n]}{n}$ exists and a.s. const by a subadd. ergo theorem.



Step 2. $\alpha(p_c^\circ,0) = 1 = \beta(p_c^\circ,0)$ and α is strictly ↑ increasing in q (and p)

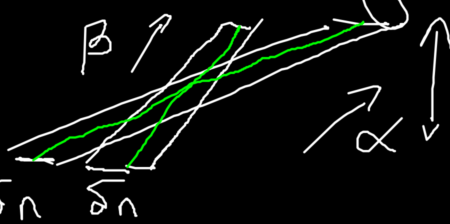
$\mathbb{E}_{p,q}[\alpha_2] - \mathbb{E}_{p,0}[\alpha_2] \geq \mathbb{P}_{p,q}\Big($  $\Big) = q\,p(1-p)^2 > 0$.

Step 3. If $\alpha(p,q) > -\infty$, then for n large dep. on ε

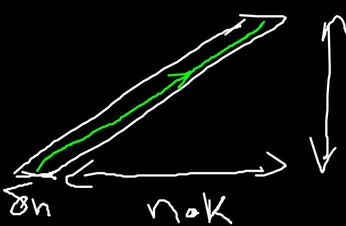
$\mathbb{P}_{p,q}\Big($  $\Big) > 1-\varepsilon$

Step 1-3 – see perco today 13.4.2021 R. Szabó.

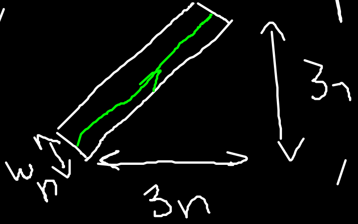
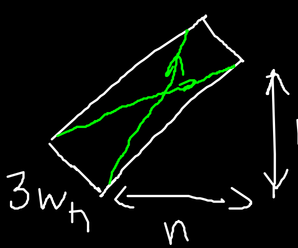
Step 4. View length 2 edges as a random environment. There exists a good event $\mathcal{E}$ s.t. $\mathbb{P}_q(\mathcal{E}) > 1-\delta'$ and

$\mathbb{P}_{p_c^0,q}\Big($  $\Big|$ length 2 edges are good $\Big) > 1-\delta'$

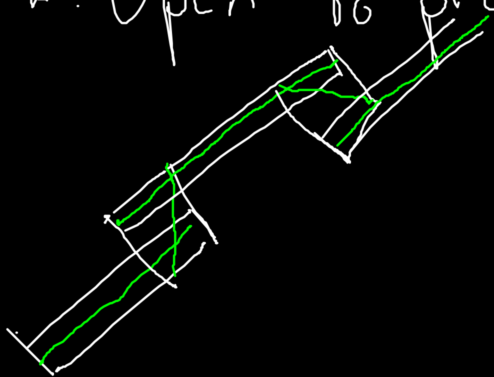
Step 5. Bad times. Use only length 1 edges and show

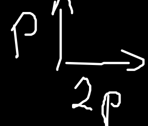
$\mathbb{P}_{p_c^0,0}\Big($  $\Big) \geq \varepsilon^K$, using RSW.

Th 4 (Duminil-Copin, Tassion, Teixeira '17) $\exists (w_n) \; \exists \varepsilon > 0$ $w_n = o(n)$

$\forall n \; \mathbb{P}_{p_c^0,0}\Big($ ,  $\Big) \geq \varepsilon$

Remark. Open to prove Th 4 without symmetry e.g





Step 6 Renormalisation to oriented percolation with geometric defects.

Th 5 (Hilário, Sá, Sanchis, Teixeira '23+) Let ξ_i iid geometric with expectation δ. Do oriented bond perco on $\mathbb{Z}^2$ with edges from $(i+\xi, i-\xi)$ open with proba p^{ξ_i+1}. If p is close to 1, $\mathbb{P}(0 \to \infty) > 0$.

If renormalisation works at (p_c^0, q), $q>0$, then it also works at $(p_c^0 - \eta, q)$ $\square$

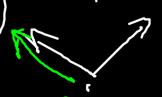
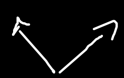
Remark. Liggett-Schonmann-Stacey '97 requires some care, because some marginals are small.

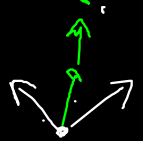
Corollary. We can do essential enhancements for OP in $1+1$ d.

Proof. Steps 1-3 + conclusion.

This was also (not) noticed by Andjel-Rolla '23+

Corollary. $p_c(1+1 \text{ dim}) < p_c(2+1 \text{ dim})$

Proof  is an enhancement of 

 is also. $\square$

This was also proved by exploration coupling by de Lima, Ungaretti, Vares '24+.

Problem. Essential enhancements for OP on $d \geq 2$?