Coalescing and branching simple exclusion and Fredrickson-Andersen models[1]

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Coalescing Random Walks with Neighbour Births

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CRWNB representation

Random walk jumping along each edge at rate 1.

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CRWNB representation

Coalescing independent random walks jumping along each edge at rate 1 **and giving birth to a particle at each neighbour independently at rate** β.

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- On $\mathbb{Z}^d$ for $\beta > 0$ – limit shape, cutoff [Bramson,Griffeath'80,81; Durrett,Griffeath'82]
- On $\mathbb{Z}$ for $\beta \to 0$ Brownian net [Sun,Swart'08]

Coalescing and Branching Simple Exclusion Process

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- (B) a particle fills the adjacent hole with rate $p/(2-p)$;
- (C) two particles coalesce at uniformly chosen of the two positions at rate $2(1-p)/(2-p)$.

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- CBSEP is the same as CRWNB with $\beta = p/(1-p)$ slowed down by a factor $(1-p)/(2-p)$.
- Nice dual model (in two distinct ways).
- Lots of embedded random walks (even more than those in the CRWNB representation).

Mixing times

Let $h^t_\omega(\cdot) = P^t_\omega(\cdot)/\mu(\cdot)$ be the density of the law of CBSEP started at ω w.r.t. the reversible measure μ.

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$h^t_\omega(\cdot) = P^t_\omega(\cdot)/\mu(\cdot)$

Let $\|f\|_q = \left(\int f^q \,\mathrm{d}\mu\right)^{1/q} = (\mu(f^q))^{1/q}$ for $q \in [1, \infty]$.

Mixing times

$h_\omega^t(\cdot) = P_\omega^t(\cdot)/\mu(\cdot)$
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$$\forall q \in [1, \infty], \quad T_q \leqslant O\left(\log\log\frac{1}{\mu_*}\right) T_{\mathrm{Sob}},$$

$\mu_* = \min_\omega \mu(\omega)$; T_{Sob} is 'the inverse rate of decay of entropy'

Commuting and meeting

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- It's also $2|V|\mathcal{R}_{x,y}$, where $\mathcal{R}_{x,y}$ is the resistance between x, y.
- $T_{\mathrm{meet}}^{x,y}$ is the expected meeting time of x and y.
- In all examples we will encounter (and many others) we have

$$T_{\mathrm{meet}} := \frac{1}{|V|^2}\sum_{x,y} T_{\mathrm{meet}}^{x,y} \asymp \frac{1}{|V|^2}\sum_{x,y} T_{\mathrm{com}}^{x,y} \asymp \max_{x,y} T_{\mathrm{meet}}^{x,y} \asymp \max_{x,y} T_{\mathrm{com}}^{x,y} =: T_{\mathrm{com}}$$

and these are known up to a constant factor (or better).

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Theorem (Martinelli, Toninelli, H.'20)

Let $p_n = \Theta(1/n)$ and $G_n = (V_n, E_n)$ be a sequence of 'nice'[a] graphs with $|V_n| = n$. Then

$$\Omega(T_{\text{meet}}) \leqslant T_{\text{Sob}}^{\text{CBSEP}} \leqslant O(T_{\text{com}} \log n).$$

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Corollary

If G_n is the d-dimensional torus, then

$$\begin{aligned} \Omega(n^2) \leqslant T_{\mathrm{Sob}}^{\mathrm{CBSEP}} &\leqslant O(n^2 \log n) & d&=1\\ \Omega(n \log n) \leqslant T_{\mathrm{Sob}}^{\mathrm{CBSEP}} &\leqslant O(n \log^2 n) & d&=2\\ \Omega(n) \leqslant T_{\mathrm{Sob}}^{\mathrm{CBSEP}} &\leqslant O(n \log n) & d&\geqslant 3 \end{aligned}$$

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– No other (known) nice representations.
– No (known) embedded random walks.
– Not well understood even for $p = 1/10$ on $\mathbb{Z}$.

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T_{Sob} is the smallest constant such that

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Corollary

$$T_{\mathrm{Sob}}^{\mathrm{FA1f}} \leqslant O(d_{\mathsf{max}}/p)\, T_{\mathrm{Sob}}^{\mathrm{CBSEP}}$$

Corollary

With $p = \Theta(1/n)$ on the torus of dimension d, for all $q \geqslant 1$

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Theorem (Pillai,Smith'17; Pillai,Smith'19)

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- Simpler proof.
- Stronger mixing notion.
- General graphs and choices of p.

Definition

$G = (V, E)$, $\Omega = S^V$, $S = S_1 \sqcup S_0$ is finite, ρ is a product probability measure on Ω.

We say there is a particle at $v \in V$ if $\omega_v \in S_1$.

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Remark

The projection on $\{0,1\}^V$ of g-CBSEP is CBSEP with $p=\rho(S_1)$.

Theorem (Martinelli,Toninelli,H.'20)

$$T_{\text{mix}}^{\text{CBSEP}} \leqslant T_{\text{mix}}^{\text{g}-\text{CBSEP}} \leqslant O(T_{\text{mix}}^{\text{CBSEP}} + T_{\text{cov}}^{\text{rw}}).$$

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Proof idea.

Once CBSEP couples, wait for one of the random walks to cover G. □

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Remark

Not true for $T_{\mathrm{Sob}}^{\mathrm{g-CBSEP}}$.

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Remark

It is known that $T_{\mathrm{com}} \leqslant T_{\mathrm{cov}}^{\mathrm{rw}} \leqslant O(T_{\mathrm{com}} \log |V|)$, so on 'nice' graphs with $p_n = \Theta(1/n)$ we get

$$\Omega(T_{\mathrm{meet}}) \leqslant T_{\mathrm{mix}}^{\mathrm{g-CBSEP}} \leqslant O(T_{\mathrm{com}} \log^2 |V|).$$

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Remark

It is known that $T_{\mathrm{com}} \leqslant T_{\mathrm{cov}}^{\mathrm{rw}} \leqslant O(T_{\mathrm{com}} \log |V|)$, so on 'nice' graphs with $p_n = \Theta(1/n)$ we get

$$\Omega(T_{\mathrm{meet}}) \leqslant T_{\mathrm{mix}}^{\mathrm{g-CBSEP}} \leqslant O(T_{\mathrm{com}} \log^2 |V|).$$

Corollary

On $\{1, \ldots, L\}^d$, $d \geqslant 2$ with $p = \Theta(1/L^d)$ we have

$$T_{\mathrm{mix}}^{\mathrm{g-CBSEP}} = L^d (\log L)^{O(1)}.$$

FAjf

$d \geqslant j \geqslant 2$, $\Omega = \{0,1\}^{\mathbb{Z}^d}$, $0 < p < 1$, $\pi = Ber(p)^{\otimes \mathbb{Z}^d}$

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Definition (j-neighbour bootstrap percolation)

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Bootstrap percolation

Theorem (Gravner,Holroyd'08+Morris,H.'19; first term: Holroyd'03)

For $d = j = 2$ w.h.p. the origin becomes filled at time

$$\exp\left(\frac{\pi^2}{18p} - \frac{\Theta(1)}{\sqrt{p}}\right).$$

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Morally: a particle reaches the origin starting from a small anomalously occupied region called droplet, which occurs with probability

$$q = \exp\left(-\frac{\pi^2}{9p} + \frac{\Theta(1)}{\sqrt{p}}\right)$$

and invades space at linear speed.

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Theorem (Balogh,Bollobás,Duminil-Copin,Morris'12+Uzzell'19)

For $d \geqslant j \geqslant 2$ there exists an explicit constant[a] *$\lambda(d,j) > 0$ such that w.h.p. the filling time τ of the origin satisfies*

$$\exp^{j-1}\left(\frac{\lambda(d,j) - o(1)}{p^{1/(d-j+1)}}\right) \leqslant \tau \leqslant \exp^{j-1}\left(\frac{\lambda(d,j)}{p^{1/(d-j+1)}} - \frac{\Omega(1)}{p^{1/(2(d-j+1))}}\right)$$

[a] This notation is not the standard one in bootstrap percolation.

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- Sharp thresholds for most other models available in bootstrap percolation transfer to KCM.
- The proof extends to all models for which similar results hold (called isotropic). [H.'20+]
- The techniques allow proving tight upper bounds completing universality for critical KCM. [Marêché,H.20+;H.'20+]

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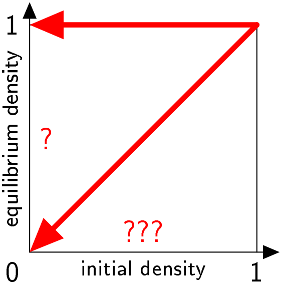

Thank you.

?

Theorem

There exists $c > 0$ s.t. for any $p_n \to 0$

$$T_{\mathrm{Sob}} \leqslant c \max\left(\frac{d_{\mathrm{avg}} d_{\mathrm{max}}^2}{d_{\mathrm{min}}^2} T_{\mathrm{mix}}^{\mathrm{rw}} \log(n), \left(\max_y \bar{\mathcal{R}}_y\right) n |\log(p_n)|\right),$$

where $T_{\mathrm{mix}}^{\mathrm{rw}}$ is the mixing time of the lazy simple random walk on G.

[Alon-Kozma'18+Lee-Yau'98]