

THE MAXIMAL RUNNING TIME OF HYPERGRAPH BOOTSTRAP PERCOLATION

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Abstract

We show that for every $r \geq 3$, the maximal running time of the K_{r+1}^r -bootstrap percolation in the complete r -uniform hypergraph on n vertices K_n^r is $\Theta(n^r)$. This answers a recent question of Noel and Ranganathan in the affirmative, and disproves a conjecture of theirs. Moreover, we show that the prefactor is of the form $r^{-r}e^{O(r)}$ as $r \rightarrow \infty$.

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1 Introduction

1.1 Background

Bootstrap percolation is a widely studied model for spreading infection or information on a graph. Given an initial set of infections, this process iteratively adds infections if a certain local pattern of infections is already present. While the original statistical physics perspective suggests selecting initial infections at random, interesting computational and extremal combinatorial questions arise when considering deterministic instances. The most classical extremal problems are determining the minimal number of infections necessary for infecting the entire graph and the maximal time until stationarity or complete infection.

These maximal times have been studied in several settings, including grids [2] and hypercubes [5, 8]. Bootstrap percolation in which one infects the edges of a graph when they complete a copy of a fixed graph is referred to graph bootstrap percolation or weak saturation. In that setting, the maximal times were studied in [1, 3, 6]. Finally, our setting of interest is the more general hypergraph bootstrap percolation, where maximal times were recently investigated by Noel and Ranganathan [7].

1.2 Model and result

For a positive integer $r \geq 2$ and an r -uniform hypergraph H (or r -graph for short), we identify H with its edge set and denote by $V(H)$ the set of vertices of H . In this paper, we pay particular attention to the complete r -graph on n vertices denoted K_n^r , especially the r -graph K_{r+1}^r , which we denote F_r for short.

For a positive integer $r \geq 2$ and an r -graph F , the F -bootstrap process in K_n^r is a monotone cellular automaton described as follows. At the beginning, a subgraph G_0 of K_n^r is infected. At every step $i \geq 1$, we define

$$G_i = G_{i-1} \cup \{e \in K_n^r : \text{there is a copy } F' \subset K_n^r \text{ of } F \text{ such that } e = F' \setminus G_{i-1}\}. \quad (1)$$

The *running time* until stationarity (that is, the first step i at which $G_i = G_{i+1}$) of the F -bootstrap percolation with initially infected set G_0 is denoted by $T(F, G_0, n)$. In [7] it was proved that

$$\max_{G_0 \subset K_n^r} T(K_m^r, G_0, n) = \Theta(n^r)$$

for all fixed $m \geq r + 2 \geq 5$. In this paper, we are interested in the *maximum running time* of the F_r -bootstrap percolation $T_r(n)$ defined as

$$T_r(n) = \max_{G_0 \subset K_n^r} T(F_r, G_0, n).$$

In this case, [7] only established that for all $r \geq 3$, $T_r(n) \geq c_r n^{r-1}$ for some $c_r > 0$. Our work closes the gap between this lower bound and the trivial upper bound $T_r(n) \leq \binom{n}{r}$. More precisely, we answer Question 5.2 from [7] in a strong sense, and disprove Conjecture 5.1 from the same paper suggesting that $T_3(n) = O(n^2)$.

Question 1 (Question 5.2 from [7]). *Does there exist an integer r_0 such that, if $r \geq r_0$, then $T_r(n) = \Theta(n^r)$?*

Our main result is the following lower bound.

Theorem 2. *For every $r \geq 3$ and $n \geq 2r^2$, we have*

$$\frac{n^r}{2^{r+3} r^r} \leq T_r(n) \leq \frac{n^r e^r}{r^r}.$$

Let us note that, while our result shows that the constant prefactor is $r^{-r} e^{O(r)}$ as $r \rightarrow \infty$, our construction has not been optimised for this purpose, but rather for the clarity of the exposition. We also observe that the condition $n \geq 2r^2$ is imposed mostly for technical reasons and is only used in (2).

1.3 Outline of the proof

Theorem 2 is proved as follows. We start with a simple construction showing that $T_3(2(4k-3)+1) \geq 8k^2 - 12k + 4$ for any $k \geq 1$ (see Section 3), which reproves the corresponding result of [7]. The construction (see Fig. 2) consists of two vertex-disjoint paths, each containing $4k-3$ vertices, and an additional vertex v_1^3 . The edges in the initial graph are: v_1^3 together with any 2-edge in some of the paths, v_1^3 together with the initial vertices of the two paths, and some 3-edges that do not contain v_1^3 . This is done so that, firstly, only one edge is infected per time step, secondly, all edges which become infected contain v_1^3 and one vertex in each path, and moreover, the process is reversible in the sense that, if we replace the edge joining the beginnings of the two paths by the last edge infected by the process, the same edges become infected but in reverse order.

Based on this construction, we prove Theorem 2 for $r = 3$ by gluing $2k-1$ instances of it, as shown in Fig. 1 (see Proposition 4). Namely, using an auxiliary vertex v_2^3 , we propagate the infection from the end of the process for the vertex v_1^3 to the end of the same process for a new vertex v_3^3 instead of v_1^3 , which will then run the other way around. Repeating this procedure $2k-2$ times, we obtain a cubic running time. The key feature of this strategy is that no edges that do not contain v_1^3 are created in the first construction, so we do not alter the configuration induced by the first $2(4k-3)$ vertices. Moreover, the same gluing technique allows us to extend the construction from r to $r+1$ by adding a vertex to all edges, adding all edges that do not contain this vertex, and then gluing copies of this construction that differ only in this special vertex while keeping the remaining ones unchanged.

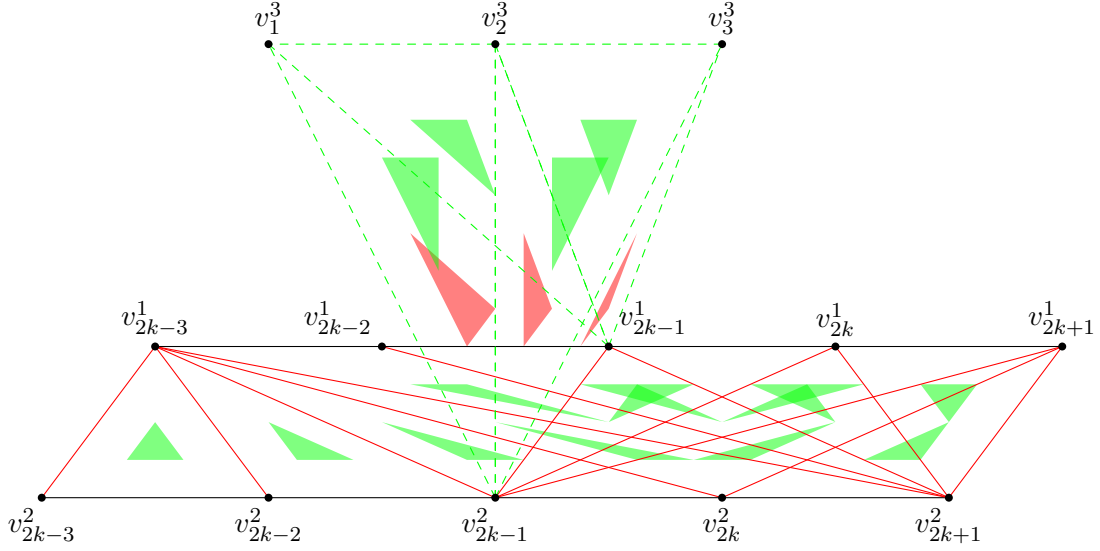


Figure 1: Illustration of the transition between the last step of stage $k - 1$ for the vertex v_1^3 and the first step of the first stage for the vertex v_3^3 . On this intermediate step, only the edge $v_{2k-1}^1 v_{2k-1}^2 v_2^3$ is infected. Note that 3-edges are represented by smaller homothetic copies of the corresponding triangles.

2 Proof of Theorem 2

To start with, we fix a positive integer $k = k(n)$ to be chosen appropriately later. Then, for all integers $i \geq 1$, we define the vertex sets

$$A_i = \{v_1^i, v_2^i, \dots, v_{4k-3}^i\}, \quad A_i^* = \bigcup_{j=1}^i A_j.$$

In particular, the sets $(A_i)_{i \geq 1}$ are disjoint while the sets $(A_i^*)_{i \geq 1}$ form an increasing sequence with respect to inclusion. Next, we consider a particular case of F_r -bootstrap percolation where a sequence of edges can be infected both in the given and in a reverse order under a suitable initial condition.

Definition 3. Let $r \geq 3$ be an integer and H be an r -graph with vertex set $V(H) \subset A_r^*$ and edge set $E(H)$. Given a sequence $(e_i)_{i=0}^T$ of edges in $E(H)$, we say that H is $(e_i)_{i=0}^T$ -sequential if the F_r -bootstrap percolation process in the complete r -graph on $V(H)$ with initially infected r -graph:

- (i) H runs for T steps, infecting only the r -edge e_i at step $i \in [1, T]$;
- (ii) $H \setminus \{e_0\}$ is stationary (does not infect any r -edge);
- (iii) $(H_1 \cup \{e_T\}) \setminus \{e_0\}$ infects only the r -edge e_{T-i} at step $i \in [1, T]$.

Definition 3 is central in the key construction presented in the next proposition.

Proposition 4. Fix integers $r \geq 3$ and $k \geq 2$. Let $T_1 \geq 2$ be an integer and H_1 be an r -graph with vertices $A_{r-1}^* \cup \{v_1^r\}$. Suppose that H_1 is $(e_i)_{i=0}^{T_1}$ -sequential with $v_1^r \in \bigcap_{i=0}^{T_1} e_i$. For all $j \in [1, 2k - 1]$, let H_{2j-1} be a copy of $H_1 \setminus \{e_0\}$ obtained by replacing v_1^r by v_{2j-1}^r . Moreover, define $e_i^- = e_i \setminus \{v_1^r\}$, and for all $j \in [1, k - 1]$, let

$$H_{4j-2} = \{e \subset \{v_{4j-3}^r, v_{4j-2}^r, v_{4j-1}^r\} \cup e_{T_1}^- : |e| = r, \{v_{4j-3}^r, v_{4j-1}^r\} \not\subset e, e_{T_1}^- \not\subset e\}$$

$$H_{4j} = \{e \subset \{v_{4j-1}^r, v_{4j}^r, v_{4j+1}^r\} \cup e_0^- : |e| = r, \{v_{4j-1}^r, v_{4j+1}^r\} \not\subset e, e_0^- \not\subset e\}$$

(see Fig. 1 for an illustration of H_2 with $r = 3$, one choice of H_1 and $e_{T_1}^- = \{v_{2k-1}^1, v_{2k-1}^2\}$, so that $H_2 = \{v_{2k-1}^1 v_1^3 v_2^3, v_{2k-1}^2 v_1^3 v_2^3, v_{2k-1}^1 v_2^3 v_3^3, v_{2k-1}^2 v_2^3 v_3^3\}$). Let $H = \bigcup_{j=1}^{4k-3} H_j$. Then, for $T = (2k-1)T_1 + 4(k-1)$, there exists a sequence of r -edges $(e_i)_{i=0}^T$ extending $(e_i)_{i=0}^{T_1}$ such that H is $(e_i)_{i=0}^T$ -sequential.

We prove Theorem 2 by induction on r . The induction step is based on Proposition 4, leading to the following statement.

Proposition 5. *Let $k \geq 2$. There exist r -edges $(e_i)_{i=0}^{T_1}$ all containing v_1^r and a $(e_i)_{i=0}^{T_1}$ -sequential r -graph H_1 with $V(H_1) = A_{r-1}^* \cup \{v_1^r\}$ and $T_1 = (2k-1)^{r-3}(8k^2 - 12k + 6) - 2$.*

The base case $r = 3$ of Proposition 5 is left to Section 3.

Proof of Proposition 5 assuming the case $r = 3$. We argue by induction on r , the base being established in Section 3. Assume the induction hypothesis to be true for some $r \geq 3$. Then, Proposition 4 provides us with an $(e_i)_{i=0}^T$ -sequential r -graph denoted H^r with $V(H^r) = A_r^*$ and

$$T = (2k-1)T_1 + 4(k-1) = (2k-1)^{r-2}(8k^2 - 12k + 6) - 2.$$

Let K be the complete $(r+1)$ -graph with $V(K) = A_r^*$. Define the $(r+1)$ -graph

$$H_1^{r+1} = \{e \cup \{v_1^{r+1}\} : e \in H^r\} \cup K.$$

We claim that H_1^{r+1} is $(e_i \cup \{v_1^{r+1}\})_{i=0}^T$ -sequential. To see this, observe that $H_1^{r+1} \setminus \{e_0 \cup \{v_1^{r+1}\}\} \supset K$ and proceed as follows.

Fix any $(r+1)$ -graph $H' \supset K$ with $V(H') = A_r^* \cup \{v_1^{r+1}\}$. Let H'_i be the set of $(r+1)$ -edges infected in the F_{r+1} -bootstrap percolation process on the complete graph on $V(H')$ and initially infected $(r+1)$ -graph H' at step i . Similarly, let H''_i be the set of r -edges infected in the F_r -bootstrap percolation process on the complete r -graph with vertex set A_r^* with initially infected r -graph $H'' = \{e \setminus \{v_1^{r+1}\} : v_1^{r+1} \in e \in H'\}$ at step i . Then, by induction on i , it follows from the definition of these processes that

$$H''_i = \{e \setminus \{v_1^{r+1}\} : e \in H'_i \setminus K\}.$$

Applying this to $H' \in \{H_1^{r+1}, H_1^{r+1} \setminus \{e_0 \cup \{v_1^{r+1}\}\}, (H_1^{r+1} \cup \{e_T \cup \{v_1^{r+1}\}\}) \setminus \{e_0 \cup \{v_1^{r+1}\}\}\}$, it follows from Definition 3 that H_1^{r+1} is $(e_i \cup \{v_1^{r+1}\})_{i=0}^T$ -sequential, as desired. \square

We are now ready to conclude the proof of Theorem 2.

Proof of Theorem 2. Fix $r \geq 3$ and set $k = \lfloor (n/r + 3)/4 \rfloor \geq n/(4r) - 1/4$. For the lower bound, we apply Proposition 5 and then Proposition 4 to obtain an r -graph H^r with $|A_r^*| = (r-1)(4k-3) + (4k-3) \leq n$ vertices such that the F_r -bootstrap percolation process on the complete graph on A_r^* with initially infected r -graph H^r runs for

$$\begin{aligned} T &= (2k-1)^{r-2}(8k^2 - 12k + 6) - 2 \geq (2k-1)^r \geq \left(\frac{n}{2r} - \frac{3}{2}\right)^r \geq \left(\frac{n}{2r}\right)^r \left(1 - \frac{3r}{n}\right)^r \\ &\geq \left(\frac{n}{2r}\right)^r \left(1 - \frac{3}{2r}\right)^r \geq \left(\frac{n}{2r}\right)^r \left(1 - \frac{3}{2 \cdot 3}\right)^3 = \frac{1}{8} \left(\frac{n}{2r}\right)^r. \end{aligned} \quad (2)$$

steps, since $n \geq 2r^2$ and $r \geq 3$. Thus, the lower bound of Theorem 2 is witnessed by the r -graph obtained by adding $n - |A_r^*| \geq 0$ isolated vertices to H^r . The upper bound simply states that the process cannot run more than $\binom{n}{r} \leq n^r/r! \leq (ne/r)^r$ steps. \square

In the sequel, we call a vertex *odd* (resp. *even*) if its subscript is odd (resp. even).

Proof of Proposition 4. First of all, let us show that the process in $H \setminus \{e_0\}$ is stationary. Suppose for contradiction that this is not the case, and suppose that some $(r+1)$ -tuple of vertices f spans exactly r edges. To begin with, f cannot contain two non-consecutive vertices in A_r since they are not contained in a common r -edge of H . Moreover, f cannot be entirely contained in A_{r-1}^* since otherwise $H_1 \setminus \{e_0\}$ would not be stationary. Two cases remain:

- Suppose that $|f \cap A_r| = 1$. Then the vertex in $f \cap A_r$ must be even (otherwise, we obtain a contradiction with the stationarity of $H_1 \setminus \{e_0\}$). However, even vertices in A_r are not contained in edges with $r-1$ other vertices in A_{r-1}^* , so this case is impossible.
- Suppose that $f \cap A_r = \{v_i^r, v_{i+1}^r\}$ for some $i \in [1, 4k-4]$. We further assume that $i = 4j$ is divisible by 4, the other cases being treated identically. Since the only edges containing v_{4j}^r are those in H_{4j} , necessarily $f = \{v_{4j}^r, v_{4j+1}^r\} \cup e_0^-$. However, in this case the two edges $\{v_{4j}^r\} \cup e_0^-$ and $\{v_{4j+1}^r\} \cup e_0^-$ are both missing, so we obtain the desired contradiction.

Now, for every $j \in [1, 2k-1]$ and every $i \in [0, T_1]$, denote $e_{i,j} = e_i^- \cup \{v_{2j-1}^r\}$. In particular, $e_{i,1} = e_i$. Further set

$$H_\infty = H \cup \{e_{i,j} : i \in [0, T_1], j \in [1, 2k-1]\} \cup \{\{v_{4j-2}^r\} \cup e_{T_1}^- : j \in [1, k-1]\} \cup \{\{v_{4j}^r\} \cup e_0^- : j \in [1, k-1]\}.$$

We claim that H_∞ is also stationary. As above, the $r+1$ tuple f spanning exactly r edges of H_∞ falls in one of the following two cases.

- Suppose that $|f \cap A_r| = 1$. Then, the vertex in $f \cap A_r$ must be even as otherwise we obtain a contradiction with the fact that $H_1 \cup \{e_{i,1} : i \in [0, T_1]\}$ is stationary (which is true by the definition of T_1). However, even vertices in A_r are contained in only one edge with $r-1$ other vertices in A_{r-1}^* , so this case is impossible.
- Suppose that $f \cap A_r = \{v_i^r, v_{i+1}^r\}$ for some $i \in [1, 4k-4]$. Again, we assume that $i = 4j$. Since the only edges containing v_{4j}^r are those in $H_{4j} \cup \{\{v_{4j}^r\} \cup e_0^-\}$, necessarily $f = \{v_{4j}^r, v_{4j+1}^r\} \cup e_0^-$. However, f already spans a copy of F_r in H_∞ , which leads to a contradiction.

Moreover, since $H \subset H_\infty$, all edges infected by H are contained in H_∞ .

From Definition 3 one may conclude that for every $i \in [0, T_1]$, the only copies of F_r in $H_1 \cup \{e_j\}_{j=1}^{T_1}$ that e_i participates in are spanned by $e_{i-1} \cup e_i$ (if $i > 0$) and by $e_i \cup e_{i+1}$ (if $i < T_1$). Moreover, note that:

- for every $j \in [1, 2k-1]$, v_{2j-1}^r is contained in a common edge of H_∞ with v_{2j-2}^r (if $j \geq 2$) and with v_{2j}^r (if $j \leq 2k-2$), but with no other vertex in A_r ,
- no edge among $(e_{i,j})_{i=1}^{T_1-1}$ participates in a copy of F_r in H_∞ with any of v_{2j-2}^r and v_{2j}^r ,
- for every $j \geq 2$, $e_{0,j}$ spans a copy of F_r in H_∞ together with v_{2j-2}^r and $e_1 \setminus e_0$ (if j is odd) or $e_{T_1-1} \setminus e_{T_1}$ (if j is even) only; for every $j \leq 2k-2$, $e_{T_1,j}$ spans a copy of F_r in H_∞ together with v_{2j}^r and $e_{T_1-1} \setminus e_{T_1}$ (if j is odd) or $e_1 \setminus e_0$ (if j is even) only.

Furthermore, for every odd (resp. even) $j \in [1, 2k-2]$, $v_{2j}^r \cup e_{T_1}^-$ (resp. $v_{2j}^r \cup e_0^-$) spans a copy of F_r in H_∞ together with v_{2j-1}^r and v_{2j+1}^r only. We conclude that in the sequence of edges

$$(e_{i,1})_{i=0}^{T_1}, v_2^r \cup e_{T_1}^-, (e_{T_1-i,2})_{i=0}^{T_1}, v_4^r \cup e_0^-, (e_{i,3})_{i=0}^{T_1}, \dots, (e_{T_1-i,2k-2})_{i=0}^{T_1}, v_{4k-4}^r \cup e_0^-, (e_{i,2k-1})_{i=0}^{T_1}, \quad (3)$$

every edge except the first one and the last one participate in exactly two copies of F_r in H_∞ , one spanned together with the edge preceding it and another one with the edge succeeding it. Moreover, the first and the last edges participate in exactly one copy of F_r in H_∞ , spanned together with $e_{1,1}$ and with $e_{T_1-1,2k-1}$, respectively. Recalling Definition 3, this completes the proof of Proposition 4 since (3) is an enumeration of all $T+1$ edges in $H_\infty \setminus (H \setminus \{e_0\})$. \square

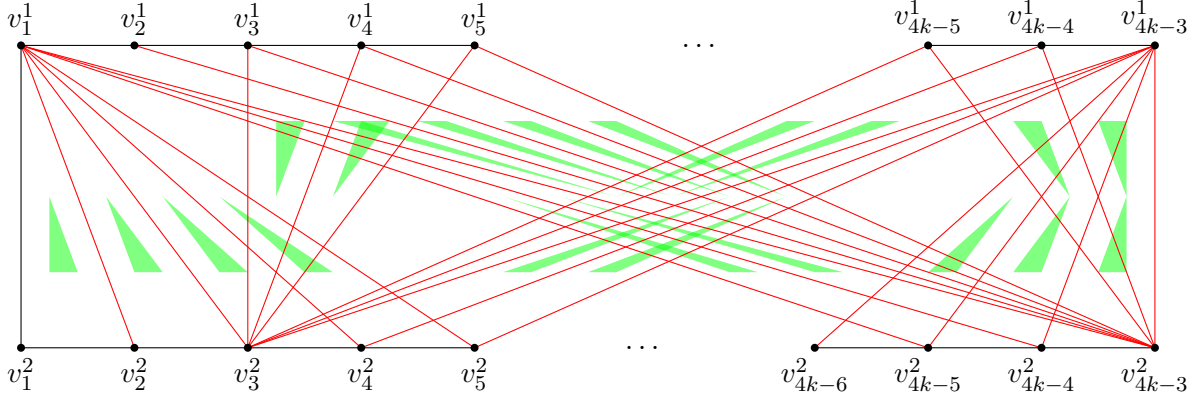


Figure 2: The first stage of the bootstrap percolation process in H_1 , where all 2-edges depicted are parts of 3-edges including v_1^3 . The edges in \mathcal{F} (defined in (5)) and the ignition edge e_0 are represented by black 2-edges. They are infected initially, while red edges are created one by one by the process. Green triangles represent the 3-edges in \mathcal{E}_1 defined in (4); they are all infected initially as well. As in Fig. 1, we draw a 3-edge as a smaller homothetic copy of the corresponding triangle.

3 The base case—proof of Proposition 5 for $r = 3$

In order to prove Proposition 5 for $r = 3$, our goal is to construct a sequential 3-graph H_1 with vertices $A_2^* \cup \{v_1^3\}$ and running time $8k^2 - 12k + 4$. For all $i \in [1, k - 1]$, define the disjoint edge sets

$$\begin{aligned}
\mathcal{E}_{1,i} &= \{v_{2i-1}^1 v_j^2 v_{j+1}^2 : j \in [2i - 1, 4k - 2 - 2i]\}, \\
\mathcal{E}_{2,i} &= \{v_j^1 v_{j+1}^1 v_{4k-1-2i}^2 : j \in [2i - 1, 4k - 2 - 2i]\}, \\
\mathcal{E}_{3,i} &= \{v_{4k-1-2i}^1 v_j^2 v_{j+1}^2 : j \in [2i + 1, 4k - 2 - 2i]\}, \\
\mathcal{E}_{4,i} &= \{v_j^1 v_{j+1}^1 v_{2i+1}^2 : j \in [2i + 1, 4k - 2 - 2i]\}, \\
\mathcal{E}_i &= \mathcal{E}_{1,i} \cup \mathcal{E}_{2,i} \cup \mathcal{E}_{3,i} \cup \mathcal{E}_{4,i}
\end{aligned} \tag{4}$$

(see Fig. 2 for an illustration of \mathcal{E}_1). Also, denote $\mathcal{E} = \bigcup_{i=1}^{k-1} \mathcal{E}_i$. This is the set of edges with vertices in A_2^* that we infect at the beginning.

In addition to \mathcal{E} , some edges containing v_1^3 are also infected initially (see Fig. 2). Denote

$$\mathcal{F} = \bigcup_{j=1}^{4k-4} \bigcup_{l=1}^2 \{v_j^l v_{j+1}^l v_1^3\}. \tag{5}$$

In other words, the set \mathcal{F} consists of all triplets containing the vertex v_1^3 and a consecutive pair of vertices in one of the paths $v_1^1 v_2^1 \dots v_{4k-3}^1$ and $v_1^2 v_2^2 \dots v_{4k-3}^2$. Finally, we need the *ignition edge* $e_0 = v_1^1 v_1^2 v_1^3$ to start the growth process.

Lemma 6. *The set $\mathcal{E} \cup \mathcal{F}$ contains no 3 edges with a total of 4 vertices.*

Proof. Let us first consider 4-tuples of vertices in A_2^* . Note that all edges in \mathcal{E} contain two odd vertices and one even vertex. Hence, a set of 4 odd vertices in A_2^* spans no edges, and a set of 4 vertices in A_2^* with at most 2 of them odd spans at most 2 edges. Finally, a set of 4 vertices in A_2^* with 3 of them odd cannot span 3 edges since at least two of the odd vertices are either both in A_1 , or both in A_2 , and hence cannot participate in the same edge as they are not consecutive.

Now, consider 4-tuples containing the vertex v_1^3 . On the one hand, A_1 and A_2 are independent sets in the 3-graph \mathcal{E} , and moreover the graph with edges $\{e \setminus v_1^3 : v_1^3 \in e \in \mathcal{F}\}$ contains no triangles. Thus,

every 4-tuple containing v_1^3 and three vertices in A_1 (respectively in A_2) spans no edges of \mathcal{E} and at most 2 edges of \mathcal{F} . It remains to observe that every 4-tuple containing v_1^3 and at least one vertex from both A_1 and A_2 cannot span more than 1 edge of \mathcal{F} (so 2 edges in total) since \mathcal{F} contains no edges intersecting both A_1 and A_2 . \square

We are now ready to conclude the proof of Proposition 5 for $r = 3$ by showing the following statement, illustrated in Fig. 2 for $s = 1$ and in Fig. 1 for $s = k - 1$. The proof is a straightforward but tedious verification.

Lemma 7. *The 3-graph $H_1 = \mathcal{E} \cup \mathcal{F} \cup \{e_0\}$ is $(e_i)_{i=0}^{T_1}$ -sequential for $T_1 = 8k^2 - 12k + 4$, $e_0 = v_1^1 v_1^2 v_1^3$ and e_i defined as follows for every $i \in [1, T_1]$. Let $s = s(i) \in [1, k - 1]$ be the unique integer such that*

$$\sum_{j=1}^{s-1} (16(k-j) - 4) < i \leq \sum_{j=1}^s (16(k-j) - 4)$$

(which clearly exists since $T_1 = \sum_{j=1}^{k-1} (16(k-j) - 4)$). Moreover, define

$$\alpha = \alpha(i) = i - \sum_{j=1}^{s-1} (16(k-j) - 4) \in [1, 16(k-s) - 4].$$

Then,

- if $\alpha \in [1, 4(k-s)]$, then $e_i = v_{2s-1}^1 v_{2s-1+\alpha}^2 v_1^3$;
- if $\alpha \in [4(k-s) + 1, 8(k-s)]$, then $e_i = v_{6s-4k-1+\alpha}^1 v_{4k-2s-1}^2 v_1^3$;
- if $\alpha \in [8(k-s) + 1, 12(k-s) - 2]$, then $e_i = v_{4k-2s-1}^1 v_{12k-10s-1-\alpha}^2 v_1^3$;
- if $\alpha \in [12(k-s) - 1, 16(k-s) - 4]$, then $e_i = v_{16k-14s-3-\alpha}^1 v_{2s+1}^2 v_1^3$.

Note that the sequence of 3-edges indicated in Lemma 7 may be alternatively described in the following more intuitive way (see Fig. 2). At the beginning, the process first infects the edges $v_1^1 v_j^2 v_1^3$ for $j \in [2, 4k - 3]$, then $v_j^1 v_{4k-3}^2 v_1^3$ for $j \in [2, 4k - 3]$, then $v_{4k-3}^1 v_j^2 v_1^3$ for $j \in [3, 4k - 4]$ in reverse order, then $v_j^1 v_3^2 v_1^3$ for $j \in [3, 4k - 4]$ in reverse order. At this point, s increases from 1 to 2. The process continues to propagate in a similar way except that, every time s increases, the ‘‘ends’’ shift two vertices towards the middle (so v_3^1 plays the role of v_1^1 when s is incremented for the first time).

Proof. To begin with, the F_3 -bootstrap percolation process on the complete 3-graph with vertices $A_2^* \cup \{v_1^3\}$ and initially infected 3-graph $H_1 \setminus \{e_0\} = \mathcal{E} \cup \mathcal{F}$ is stationary by Lemma 6. Now, set $G_{-1} = \mathcal{E} \cup \mathcal{F}$, $G_0 = H_1$ and let the sequence $(G_i)_{i \geq 1}$ be defined as in (1). We will show by induction on $i \geq 0$ that $G_i \setminus G_{i-1} = \{e_i\}$.

For $i = 0$ the statement is clear by construction. Suppose that for some $i \geq 0$, the induction hypothesis is satisfied for i . Then, all edges in $G_{i+1} \setminus G_i$ must span 4 vertices with the edge e_i , and in particular must intersect e_i in 2 vertices. We consider 4 cases according to the interval containing $\alpha = \alpha(i)$. For convenience, we extend the notation by setting $s(0) = 1$ and $\alpha(0) = 0$ (so that $\alpha(i) = 0$ only when $s(i) = 1$), which is consistent with $e_0 = v_{2s-1}^1 v_{2s-1+\alpha}^2 v_1^3$.

If $\alpha \in [0, 4(k-s)]$, then $e_i = v_{2s-1}^1 v_{2s-1+\alpha}^2 v_1^3$. One may easily check that the vertices in A_2^* that complete the pair $v_{2s-1}^1 v_{2s-1+\alpha}^2$ to an edge in G_{i-1} are

$$\begin{cases} v_{2s+\alpha}^2 & \alpha = 0, \\ v_{2s-2+\alpha}^2, v_{2s+\alpha}^2 & \alpha \in [1, 4(k-s) - 1], \\ v_{2s-2+\alpha}^2, v_{2s}^1 & \alpha = 4(k-s). \end{cases}$$

The vertices that complete the pair $v_{2s-1}^1 v_1^3$ to an edge in G_{i-1} are

$$\begin{cases} (v_{2s-1+j}^2)_{j \in [0, \alpha-1]}, v_{2s}^1 & s = 1, \\ (v_{2s-1+j}^2)_{j \in [0, \alpha-1]}, v_{2s}^1, v_{2s-2}^1, (v_{2j-1}^2)_{j \in [2, s-1] \cup [2k-s+1, 2k-1]} & s \geq 2. \end{cases}$$

Finally, the vertices that complete the pair $v_{2s-1+\alpha}^2 v_1^3$ to an edge in G_{i-1} are

$$\begin{cases} v_{2s+\alpha}^2 & \alpha = 0, \\ (v_{2j-1}^1)_{j \in [1, s-1] \cup [2k-s+1, 2k-1]}, v_{2s-2+\alpha}^2, v_{2s+\alpha}^2 & \alpha \in [1, 4k-5], \\ v_{2s-2+\alpha}^2 & \alpha = 4k-4. \end{cases}$$

Then, the only vertices that complete at least two pairs are

$$\begin{cases} v_2^2 & \alpha = 0, \\ v_{2s+\alpha}^2, v_{2s+\alpha}^2 & \alpha \in [1, 4(k-s)-1], \\ v_{2s+\alpha}^2, v_{2s}^1 & \alpha = 4(k-s). \end{cases}$$

However, for $\alpha \neq 0$, $\{v_{2s-2+\alpha}^2\} \cup e_i$ already spans a copy of F_3 in G_i , so the only edge that is generated from G_i is e_{i+1} .

If $\alpha \in [4(k-s)+1, 8(k-s)]$, then $e_i = v_{6s-4k-1+\alpha}^1 v_{4k-2s-1}^2 v_1^3$. One may easily check that the vertices in A_2^* that complete the pair $v_{6s-4k-1+\alpha}^1 v_{4k-2s-1}^2$ to an edge in G_{i-1} are

$$\begin{cases} v_{6s-4k-2+\alpha}^1, v_{6s-4k+\alpha}^1 & \alpha \neq 8(k-s), \\ v_{4k-2s-2}^1, v_{4k-2s-2}^2 & \alpha = 8(k-s). \end{cases}$$

The vertices that complete the pair $v_{6s-4k-1+\alpha}^1 v_1^3$ to an edge in G_{i-1} are

$$\begin{cases} v_{4k-4}^1 & \alpha = 8k-8, \\ (v_{2j-1}^2)_{j \in [2, s-1] \cup [2k-s+1, 2k-1]}, v_{6s-4k-2+\alpha}^1, v_{6s-4k+\alpha}^1 & \text{otherwise.} \end{cases}$$

Finally, the vertices that complete the pair $v_{4k-2s-1}^2 v_1^3$ to an edge in G_{i-1} are

$$\begin{cases} (v_{5-4k+j}^1)_{j \in [4(k-1), \alpha-1]}, v_{4k-4}^2 & s = 1, \\ (v_{6s-4k-1+j}^1)_{j \in [4(k-s), \alpha-1]}, (v_{2j-1}^1)_{j \in [1, s-1] \cup [2k-s+1, 2k-1]}, v_{4k-2s-2}^2, v_{4k-2s}^2 & s \neq 1. \end{cases}$$

Then, the only vertices that complete at least two pairs are

$$\begin{cases} v_{4k-2s-2}^1, v_{4k-2s-2}^2 & \alpha = 8(k-s), \\ v_{6s-4k-2+\alpha}^1, v_{6s-4k+\alpha}^1 & \alpha \neq 8(k-s). \end{cases}$$

However, $\{v_{6s-4k-2+\alpha}^1\} \cup e_i$ already spans a copy of F_3 in G_i , so the only edge that is generated from G_i is e_{i+1} .

If $\alpha \in [8(k-s)+1, 12(k-s)-2]$, then $e_i = v_{4k-2s-1}^1 v_{12k-10s-1-\alpha}^2 v_1^3$. One may easily check that the vertices in A_2^* that complete the pair $v_{4k-2s-1}^1 v_{12k-10s-1-\alpha}^2$ to an edge in G_{i-1} are

$$\begin{cases} v_{4k-2s-2}^1, v_{2s+2}^2 & \alpha = 12(k-s)-2, \\ v_{12k-10s-2-\alpha}^2, v_{12k-10s-\alpha}^2 & \alpha \neq 12(k-s)-2. \end{cases}$$

The vertices that complete the pair $v_{4k-2s-1}^1 v_1^3$ to an edge in G_{i-1} are

$$\begin{cases} (v_{12k-11-j}^2)_{j \in [8k-8, \alpha-1]}, v_{4k-3}^2, v_{4k-4}^1 & s = 1, \\ (v_{12k-10s-1-j}^2)_{j \in [8(k-s), \alpha-1]}, (v_{2j-1}^2)_{j \in [2, s] \cup [2k-s, 2k-1]}, v_{4k-2s-2}^1, v_{4k-2s}^1 & s \neq 1. \end{cases}$$

The vertices that complete the pair $v_{12k-10s-1-\alpha}^2 v_1^3$ to an edge in G_{i-1} are

$$(v_{2j-1}^1)_{j \in [1,s] \cup [2k-s+1, 2k-1]}, v_{12k-10s-2-\alpha}^2, v_{12k-10s-\alpha}^2.$$

Then, the only vertices that complete at least two pairs are

$$\begin{cases} v_{4k-2s-2}^1, v_{2s+2}^2 & \alpha = 12(k-s) - 2 \\ v_{12k-10s-2-\alpha}^2, v_{12k-10s-\alpha}^2 & \alpha \neq 12(k-s) - 2. \end{cases}$$

However, $\{v_{12k-10s-\alpha}^2\} \cup e_i$ already spans a copy of F_3 in G_i , so the only edge that is generated from G_i is e_{i+1} .

If $\alpha \in [12(k-s) - 1, 16(k-s) - 4]$, then $e_i = v_{16k-14s-3-\alpha}^1 v_{2s+1}^2 v_1^3$. One may easily check that the vertices in A_2^* that complete the pair $v_{16k-14s-3-\alpha}^1 v_{2s+1}^2$ to an edge in G_{i-1} are

$$\begin{cases} v_{2k}^1 & \alpha = 16(k-s) - 4, s = k - 1, \\ v_{2s+2}^1, v_{2s+2}^2 & \alpha = 16(k-s) - 4, s \neq k - 1 \\ v_{16k-14s-4-\alpha}^1, v_{16k-14s-2-\alpha}^1 & \alpha \neq 16(k-s) - 4. \end{cases}$$

The vertices that complete the pair $v_{16k-14s-3-\alpha}^1 v_1^3$ to an edge in G_{i-1} are

$$(v_{2j-1}^2)_{j \in [2,s] \cup [2k-s, 2k-1]}, v_{16k-14s-4-\alpha}^1, v_{16k-14s-2-\alpha}^1.$$

Finally, the vertices that complete the pair $v_{2s+1}^2 v_1^3$ to an edge in G_{i-1} are

$$(v_{16k-14s-3-j}^1)_{j \in [12(k-s)-2, \alpha-1]}, (v_{2j-1}^1)_{j \in [1,s] \cup [2k-s+1, 2k-1]}, v_{2s}^2, v_{2s+2}^2.$$

Then, the only vertices that complete at least two pairs are

$$\begin{cases} v_{2k}^1 & \alpha = 16(k-s) - 4, s = k - 1, \\ v_{2s+2}^1, v_{2s+2}^2 & \alpha = 16(k-s) - 4, s \neq k - 1, \\ v_{16k-14s-4-\alpha}^1, v_{16k-14s-2-\alpha}^1 & \alpha \neq 16(k-s) - 4. \end{cases}$$

However, $\{v_{16k-14s-2-\alpha}^1\} \cup e_i$ already spans a copy of F_3 in G_i , so the only edge that is generated from G_i is e_{i+1} , and the process stops when it infects e_{T_1} .

It remains to show that the F_3 -bootstrap percolation process on the complete 3-graph with vertices $A_2^* \cup \{v_1^3\}$ and initially infected 3-graph $\mathcal{E} \cup \mathcal{F} \cup \{e_{T_1}\}$ infects the edges $(e_i)_{i=0}^{T_1-1}$ in reverse order. Notice that in the process started from G_0 , every copy of F_3 that becomes completely infected contains two edges in $\mathcal{E} \cup \mathcal{F}$ and the edges e_i and e_{i+1} for some $i \geq 0$. Therefore, starting from $\mathcal{E} \cup \mathcal{F} \cup \{e_{T_1}\}$, the edges that do become infected are exactly $(e_i)_{i=0}^{T_1}$. Moreover, all copies of F_3 in G_{T_1} are spanned by some of the vertex sets $(e_{i-1} \cup e_i)_{i=1}^{T_1}$, which implies that the bootstrap percolation with initial condition $\mathcal{E} \cup \mathcal{F} \cup \{e_{T_1}\}$ infects the edges $(e_i)_{i=0}^{T_1-1}$ in the reverse order. \square

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Note. A few days after the current article appeared on arxiv, an independent proof of a version of Theorem 2 was found by Espuny Díaz, Janzer, Kronenberg and Lada [4].

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