COALESCING AND BRANCHING SIMPLE SYMMETRIC EXCLUSION PROCESS

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Motivated by kinetically constrained interacting particle systems (KCM), we consider a reversible coalescing and branching simple exclusion process on a general finite graph $G = (V, E)$ dual to the biased voter model on G. Our main goal are tight bounds on its logarithmic Sobolev constant and relaxation time, with particular focus on the delicate slightly supercritical regime in which the equilibrium density of particles tends to zero as $|V| \to \infty$. Our results allow us to recover very directly and improve to ℓ^p-mixing, $p \geqslant 2$, and to more general graphs, the mixing time results of Pillai and Smith for the Fredrickson-Andersen one spin facilitated (FA-1f) KCM on the discrete d-dimensional torus. In view of applications to the more complex FA-jf KCM, $j > 1$, we also extend part of the analysis to an analogous process with a more general product state space.

1. Introduction. In this work we study a coalescing and branching simmetric exclusion process (CBSEP) on a general finite graph $G = (V, E)$. The model was first introduced by Schwartz [35] in 1977 (also see Harris [19]) as follows. Consider a system of particles performing independent continuous time random walks on the vertex set of a (finite or infinite) graph G by jumping along each edge with rate 1, which coalesce when they meet (a particle jumping on top of another one is destroyed) and which branch with rate $\beta > 0$ by creating an additional particle at a empty neighbouring vertex. The process is readily seen to be reversible w.r.t. the Bernoulli(p)-product measure with $p = \frac{\beta}{(1+\beta)}$. Initially the model was introduced in order to study the biased voter model [35] (also known as Williams-Bjerknes tumour growth model [42]), which turns out to be its dual additive interacting particle system [18].[1] A further duality in between the two processes in the Sudbury–Lloyd sense [37] has been established since then, which shows that the law of CBSEP at a fixed time can be obtained as a p-thinning of the biased voter model (see [39, Exercise 3.6]). When $\beta = 0$ this model reduces to coalescing random walks, additive dual to the standard voter model, which have both been extensively studied (see e.g. [25, 26]).

When the graph is the d-dimensional hypercubic lattice, the first results were obtained by Bramson and Griffeath [6, 7]. In particular, they showed that the law of CBSEP converges weakly to its unique invariant measure starting from any non-empty set of particles and for any dimension d. Moreover, building on their work, Durrett and Griffeath [13] proved a shape theorem for this process, which easily implies that CBSEP on the discrete torus of side length $L \to \infty$ exhibits mixing time cutoff (but

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[1]In fact, biased voter and Williams-Bjerknes models slightly differ on non-regular graphs. For such graphs CBSEP is the additive dual of the former.

without any control on the critical window). In the case of the regular tree a complete convergence result is due to Louidor, Tessler, and Vandenberg-Rodes [27]. In the particular setting of $\mathbb{Z}$ a key observation is that the rightmost (or leftmost) particle performs a biased random walk with explicit constant drift (see Griffeath [18]). For more advanced results see e.g. the work by Sun and Swart [38].

While the main focus of the above-mentioned works was the long-time behavior of the process on *infinite* graphs, our interest will concentrate instead on the mixing time for *finite* graphs. We determine the logarithmic Sobolev constant and relaxation time of the model quite precisely on a wide spectrum of relatively sparse finite graphs and for values of the branching rate β which are o(1) as $|V| \to \infty$ (see Theorem 1 and Corollary 3.1). For instance, our results imply that for transitive bounded degree graphs the inverse of the logarithmic Sobolev constant and relaxation time when $\beta = 1/|V|$ are, up to a logarithmic correction, equal to the cover time of the graph. We will then use these results to strengthen and extend the findings of Pillai and Smith [31, 32] on the mixing time for the FA-1f kinetically constrained model in the same regime (see Corollary 3.2). Motivated by a different application to the kinetically constrained models FA-jf with $j > 1$ (see [20]), we then investigate a version of the model in which the single vertex state space $\{0, 1\}$ is replaced by an arbitrary finite set and we bound its mixing time (see Theorem 2).

1.1. *The CBSEP and g-CBSEP models.* Let $G = (V, E)$ be a finite connected graph with n vertices. The degree of $x \in V$ is denoted by d_x. Minimum, maximum, and average degrees in G are denoted by $d_{\min}, d_{\max}$ and d_{avg}, respectively. For any $\omega \in \Omega = \{0, 1\}^V$ and any vertex $x \in V$ we say that x is *filled/empty*, or that there is a *particle/hole* at x, if $\omega_x = 1/0$. We define $\Omega_+ = \Omega \setminus \{0\}$ to be the event that there exists at least one particle. Similarly, for any edge $e = \{x, y\} \in E$ we refer to $(\omega_x, \omega_y) \in \{0, 1\}^{\{x,y\}}$ as the state of e in ω and write $E_e = \{\omega \in \Omega \mid \omega_x + \omega_y \neq 0\}$ for the event that e is not empty.

Given $p \in (0, 1)$, let $\pi = \bigotimes_{x \in V} \pi_x$ be the product Bernoulli measure, in which each vertex is filled with probability p, and let $\mu(\cdot) := \pi(\cdot \mid \Omega_+)$. Given an edge $e = \{x, y\}$ we write $\pi_e := \pi_x \otimes \pi_y$ and $\lambda(p) := \pi(E_e) = p(2 - p)$. In the sequel we will always assume for simplicity that p is bounded away from 1 (e.g. $p \leqslant 1/2$).

The CBSEP, the main object of this work, is a continuous-time Markov chain on Ω_+ for which the state of any edge $e \in E$ such that E_e occurs is resampled with rate one w.r.t. $\pi_e(\cdot \mid E_e)$. Thus, any edge containing exactly one particle with rate $(1-p)/(2-p)$ moves the particle to the opposite endpoint (the *SEP move*) and with rate $p/(2 - p)$ creates an extra particle at its empty endpoint (the *branching move*). Moreover, any edge containing two particles with rate $2(1 - p)/(2 - p)$ kills one of the two particles chosen uniformly (the *coalescing move*). The chain is readily seen to be reversible w.r.t. μ and ergodic on Ω_+, because it can reach the configuration with a particle at each vertex. If $c(\omega, \omega')$ denotes the jump rate from ω to ω', the Dirichlet form $\mathcal{D}(f)$ of the chain has the expression

$$(1) \qquad \mathcal{D}(f) = \frac{1}{2} \sum_{\omega, \omega'} \mu(\omega) c(\omega, \omega') \left(f(\omega') - f(\omega) \right)^2 = \sum_{e \in E} \mu(\mathbb{1}_{E_e} \operatorname{Var}_e(f \mid E_e)).$$

It is also not hard to check that CBSEP is equivalent to the coalescing and branching random walks described in the introduction up to a global time-rescaling.

We will also consider a *generalised* version of CBSEP, in the sequel g-CBSEP, defined as follows. We are given a graph G as above together with a probability space (S, ρ), where S is a finite set and ρ a probability measure on S. We still write

$\rho = \bigotimes_{x \in V} \rho_x$ for the product probability on $\Omega^{(g)} := S^V$. In the *state space* S, we are given a bipartition $S_1 \sqcup S_0 = S$, and we write $p := \rho(S_1) \in (0, 1)$. We define the *projection* $\varphi : \Omega^{(g)} \to \Omega = \{0, 1\}^V$ by $\varphi(\omega) = (\mathbb{1}_{\{\omega_x \in S_1\}})_{x \in V}$ and we let $\Omega_+^{(g)} = \{\omega \in \Omega : \sum_x (\varphi(\omega))_x \geqslant 1\} = \varphi^{-1}(\Omega_+)$. For any edge $e = \{x, y\} \in E$ we also let $E_e^{(g)}$ be the event that there exists a particle at x or at y for $\varphi(\omega)$. In g-CBSEP every edge $e = \{x, y\}$ such that $E_e^{(g)}$ is satisfied is resampled with rate 1 w.r.t. $\rho_x \otimes \rho_y(\cdot \mid E_e^{(g)})$. A key property is that its projection chain onto the variables $\varphi(\omega)$ coincides with CBSEP on G with parameter p. As with CBSEP, the g-CBSEP is reversible w.r.t. $\rho_+ = \rho(\cdot \mid \Omega_+^{(g)})$ and ergodic on $\Omega_+^{(g)}$. For the main motivation behind g-CBSEP we refer the reader to Section 1.2.2.

1.2. *The FA-jf KCM.* We next define another class of models of interest—the j-neighbour or the Fredrickson-Andersen j-facilitated kinetically constrained spin model (FA-jf KCM for short). In the setting of Section 1.1 for CBSEP, these chains evolve as follows. With rate one and independently w.r.t. the other vertices, the state of each vertex $x \in V$ with at least j neighbouring particles is resampled w.r.t. π_x. In this paper we will focus on the simplest case $j = 1$, the case $j = 0$ being trivial. As for CBSEP it is immediate to check that, on Ω_+, the chain is ergodic with μ as the unique reversible measure, and that its Dirichlet form is

$$\mathcal{D}^{\mathrm{FA}}(f) = \sum_x \mu \left(\mathbb{1}_{\left\{ \sum_{\{x,y\} \in E} \omega_y \geqslant 1 \right\}} p(1-p)(f(\omega^x) - f(\omega))^2 \right),$$

where ω^x denotes the configuration ω flipped at x.

The FA-1f KCM has been extensively studied (see e.g. [5, 9, 10, 14, 15]). Of particular relevance for us are the beautiful works of Pillai and Smith [31, 32] that we present next. For any positive integers d and L, set $n = L^d$, and let $\mathbb{Z}_L = \{0, 1, \ldots, L-1\}$ be the set of remainders modulo L. The d-dimensional discrete torus with n vertices, $\mathbb{T}_n^d$ in the sequel, is the set $\mathbb{Z}_L^d$ endowed with the graph structure inherited from $\mathbb{Z}^d$. For the *discrete time* version of FA-1f on $\mathbb{T}_n^d$ with $p = c/n$ [31, 32] provide a rather precise bound for the (total variation) mixing time $T_{\mathrm{mix}}^{\mathrm{FA}}$. Translated into the continuous time setting described above, their results read

(2)
$$C^{-1} n^2 \leqslant T_{\mathrm{mix}}^{\mathrm{FA}} \leqslant C n^2 \log^{14}(n) \quad d = 2$$
$$C^{-1} n^2 \leqslant T_{\mathrm{mix}}^{\mathrm{FA}} \leqslant C n^2 \log(n) \quad\; d \geqslant 3,$$

where $C > 0$ may depend on d but not on n.

REMARK 1.1. In [31, Section 2] it was argued that $T_{\mathrm{mix}}^{\mathrm{FA}}$ should be lower bounded by the time necessary to get two well-separated particles starting from one. By reversibility, and since to move an isolated particle by one step, we should first create a particle at a neighbouring site at rate p, this time should correspond $p^{-1} T_{\mathrm{meet}}^{\mathrm{rw}}$, where $T_{\mathrm{meet}}^{\mathrm{rw}}$ is the meeting time of two independent continuous time random walks on $\mathbb{T}_n^2$ with independent uniformly distributed starting points. In particular, since in two dimensions $T_{\mathrm{meet}}^{\mathrm{rw}} = \Theta(n \log(n))$, in [32, Remark 1.1] it was conjectured that $T_{\mathrm{mix}}^{\mathrm{FA}} = \Omega(p^{-1} n \log(n)) = \Omega(n^2 \log(n))$ in the regime $p = \Theta(1/n)$ and this was recently confirmed by Shapira [36, Theorem 1.2]. As it will be apparent in the proof of Theorem 1(d), this heuristics together with the attractiveness of CBSEP will allow us to prove a lower bound on the logarithmic Sobolev constant and relaxation time of CBSEP on a general graph.

1.2.1. *Relationship between CBSEP and FA-*1*f.* Notice that the branching and coalescing moves of CBSEP are exactly the moves allowed in FA-1f. Moreover, the SEP move for the edge $\{x,y\}$ from $(1,0)$ to $(0,1)$ can be reconstructed using two consecutive FA-1f moves, the first one filling the hole at y and the second one emptying x. If we also take into account the rate for each move, we easily get the following comparison between the respective Dirichlet forms (see e.g. [24, Ch. 13.4]): there exists an absolute constant $c > 0$ such that for all $f : \Omega_+ \to \mathbb{R}$ it holds that

$$(3) \qquad c^{-1}\mathcal{D}^{\mathrm{FA}}(f) \leqslant \mathcal{D}(f) \leqslant cd_{\max}p^{-1}\mathcal{D}^{\mathrm{FA}}(f).$$

In our application of the main results for CBSEP to the estimate of the mixing time of the FA-1f chain (see Corollary 3.2) for $p \ll 1$ only the upper bound, which we believe to be sharper, will count.

Although the two models are clearly closely related, we would like to emphasise that CBSEP has many advantages over FA-1f, making its study simpler. Most notably, CBSEP is *attractive* in the sense that there exists a grand-coupling (see e.g [24]) which preserves the partial order on Ω given by $\omega \prec \omega'$ iff $\omega_x \leqslant \omega'_x$ for all $x \in V$ (as it can be readily verified via the construction of Section 5.1). Furthermore, it is also natural to embed in CBSEP a continuous time random walk $(W_t)_{t \geqslant 0}$ on G such that CBSEP has a particle at W_t for all $t \geqslant 0$. The latter is a particularly fruitful feature, which we will use in Section 5, and which is challenging to reproduce for FA-1f [5].

1.2.2. *Motivations for g-CBSEP: the FA-2f KCM.* The generalisation of CBSEP covering the case of general (finite) single vertex state space S was introduced in [20, Sections 1.5, 5.1] in order to model the effective random evolution of the *mobile droplets* of the FA-2f KCM (see Section 1.2 for $j = 2$). Modelling the dynamics of mobile droplets as a suitable g-CBSEP combined with precise bounds on the relaxation time and probability of occurrence of the latter, proved to be a key tool to determine with very high precision the *infection time* τ_0 of the FA-2f KCM, i.e. the first time the state of the origin is 0, as the density p of the infection vanishes. In particular our Theorems 1 and 2 play a key role in [20] for proving that, for the FA-2f model in $\mathbb{Z}^2$ at density p, w.h.p.

$$\tau_0 = \exp\left(\frac{\pi^2 + o(1)}{9p}\right), \quad \text{as } p \to 0.$$

2. Preliminaries. In order to state our results we need first to recall some classical material on *mixing times* for finite Markov chains (see e.g. [21, 34]) and on the *resistance distance* on finite graphs (see [11, 40], [24, Ch. 9] and [28, Ch.2]).

2.1. *Mixing times and logarithmic Sobolev constant.* Given a finite state space Ω and a uniformly positive probability measure μ on Ω, let $(\omega(t))_{t \geqslant 0}$ be a continuous time ergodic Markov chain on Ω reversible w.r.t. μ, and write $P_\omega^t(\omega') = \mathbb{P}(\omega(t) = \omega' \,|\, \omega(0) = \omega)$. Let also $h_\omega^t(\cdot) = P_\omega^t(\cdot)/\mu(\cdot)$ be the relative density of the law $P_\omega^t(\cdot)$ w.r.t. μ. The total variation mixing time of the chain, T_{mix}, is defined as

$$T_{\mathrm{mix}} = \inf\left\{t > 0 : \max_{\omega \in \Omega_+} \|P_\omega^t(\cdot) - \mu(\cdot)\|_{\mathrm{TV}} \leqslant 1/(2e)\right\},$$

where $\|\cdot\|_{\mathrm{TV}}$ denotes the *total variation distance* defined as

$$\|P_\omega^t(\cdot) - \mu(\cdot)\|_{\mathrm{TV}} = \frac{1}{2}\|h_\omega^t(\cdot) - 1\|_1,$$

where $\|g\|_\alpha^\alpha = \mu(|g|^\alpha), \alpha \geqslant 1$. The ℓ^2-mixing time T_2 or, more generally, the ℓ^q-mixing times T_q, $q \geqslant 1$, are defined by

$$T_q = \inf \left\{ t > 0 : \max_{\omega \in \Omega_+} \|h_\omega^t(\cdot) - 1\|_q \leqslant 1/e \right\}.$$

Clearly $T_{\mathrm{mix}} \leqslant T_q$ for $q > 1$ and it is known that for all $1 < q \leqslant \infty$ the ℓ^q-convergence profile is determined entirely by that for $q = 2$ (see e.g. [34, Lemma 2.4.6]). Moreover, (see e.g. [34, Corollary 2.2.7],

$$(4) \qquad \frac{1}{2}\alpha^{-1} \leqslant T_2 \leqslant \alpha^{-1}(1 + \frac{1}{4}\log\log(1/\mu_*)),$$

where $\mu_* = \min_{\omega \in \Omega_+} \mu(\omega)$ and α is the *logarithmic Sobolev constant* defined as the inverse of the best constant C in the logarithmic Sobolev inequality valid for any $f : \Omega_+ \to \mathbb{R}$

$$(5) \qquad \mathrm{Ent}(f^2) := \mu(f^2 \log(f^2/\mu(f^2))) \leqslant C\mathcal{D}(f).$$

Finally, the *relaxation time* T_{rel} is then defined as the best constant C in the Poincaré inequality

$$(6) \qquad \mathrm{Var}(f) \leqslant C\mathcal{D}(f).$$

It is not difficult to prove that $T_{\mathrm{rel}} \leqslant T_{\mathrm{mix}}$ and that (see e.g. [12, Corollary 2.11])

$$(7) \qquad 2T_{\mathrm{rel}} \leqslant \alpha^{-1} \leqslant (2 + \log(1/\mu_*)) \times T_{\mathrm{rel}}.$$

Notation warning. In the sequel, unless otherwise indicated, all the quantities introduced above will not carry any additional label when referring to CBSEP. On the contrary, the same quantities referring to other chains, e.g. the FA-1f KCM or g-CBSEP, will always carry an appropriate superscript.

2.2. *Resistance distance.* Given a finite connected simple graph $G = (V, E)$, let $\vec{E}$ denote the set of *ordered pairs* of vertices forming an edge of E. For $\vec{e} = (u, v) \in \vec{E}$ we set $-\vec{e} = (v, u)$. Given an anti-symmetric function θ on $\vec{E}$ (that is $\theta(\vec{e}) = -\theta(-\vec{e})$) and two vertices x, y we say that θ is a *unit flow from x to y* iff $\sum_{v:(u,v)\in\vec{E}} \theta((u,v)) = 0$ for all $u \notin \{x, y\}$ and $\sum_{v:(x,v)\in\vec{E}} \theta((x,v)) = 1$. The *energy of the flow θ* is the quantity $\mathcal{E}(\theta) = \frac{1}{2}\sum_{\vec{e}\in\vec{E}} \theta(\vec{e})^2$ and we set

$$(8) \qquad \mathcal{R}_{x,y} = \inf\{\mathcal{E}(\theta) : \theta \text{ is a unit flow from } x \text{ to } y\}.$$

The Thomson principle [41] states that the infimum in (8) is actually attained at a unique unit flow.

The quantity $\mathcal{R}_{x,y}$ can be interpreted as the effective resistance in the electrical network obtained by replacing the vertices of G with nodes and the edges with unit resistances. In graph theory it is sometimes referred to as the *resistance distance*. It is also connected to the behaviour of the simple random walk on G via the formula $2|E|\mathcal{R}_{x,y} = C_{x,y}$, where $C_{x,y}$ is the expected commute time between x and y. Furthermore, if we let $T_{\mathrm{rel}}^{\mathrm{rw}}$ be the relaxation time of the random walk, the bound $\max_{x,y} \mathcal{R}_{x,y} \leqslant c\sqrt{T_{\mathrm{rel}}^{\mathrm{rw}}}/d_{\min}$ holds [30, Corollary 1.1] where $c > 0$ is a universal constant (see also [1, Corollary 6.21] for regular graphs). Finally, by taking the shortest path between x, y and the flow θ which assigns unit flow to each edge of the path, $\mathcal{R}_{x,y} \leqslant d(x, y)$, where $d(\cdot, \cdot)$ is the graph distance, with equality iff x, y are linked by a unique path. In the sequel and for notation convenience we shall write $\bar{\mathcal{R}}_y$ for the spatial average $\frac{1}{n}\sum_x \mathcal{R}_{x,y}$.

REMARK 2.1. For later use we present bounds on $\max_y \bar{\mathcal{R}}_y$ for certain special graphs. If G is the d-hypercube ($n := |V| = 2^d$) it follows from [33] that $\bar{\mathcal{R}}_y = \Theta(1/\log n)$ for all $y \in V$. If instead G is the regular b-ary tree with $b \geqslant 2$ then $\max_y \bar{\mathcal{R}}_y = \Theta(\log n)$. If G is a uniform random d-regular graph with $n \to \infty$, and d independent of n, then w.h.p. $T^{\mathrm{rw}}_{\mathrm{rel}} = O(1)$ [8, 16], and therefore w.h.p. $\bar{\mathcal{R}}_y = \Theta(1)$ for all $y \in V$. Finally, if G is the discrete d-dimensional torus $\mathbb{T}^d_n \subset \mathbb{Z}^d$ with n vertices, then, as $n \to \infty$ and d is fixed, it follows from [28, Proposition 2.15] that

$$\max_y \bar{\mathcal{R}}_y = \Theta(1) \times \begin{cases} n & \text{if } d = 1, \\ \log(n) & \text{if } d = 2, \\ 1 & \text{if } d \geqslant 3. \end{cases}$$

3. Main results. Our first theorem establishes upper and lower bounds for the inverse of the logarithmic Sobolev constant, α^{-1}, and relaxation time, T_{rel}, of CBSEP in the general setting described in the introduction.

Let $T^{\mathrm{rw}}_{\mathrm{meet}}$ denote the expected meeting time for two *continuous time* random walks jumping along each edge at rate 1 and started from two uniformly chosen vertices of G. We refer the reader to [1, 22] for the close connections between $T^{\mathrm{rw}}_{\mathrm{meet}}$ and $\mathcal{R}_{x,y}$. Let also $T^{\mathrm{rw}}_{\mathrm{mix}}$ denote the mixing time of the *discrete time* lazy simple random walk on G (i.e. staying at its position with probability $1/2$).

THEOREM 1. *Let $p_n \in (0, 1)$ and consider CBSEP with parameter p_n on a sequence of graphs $G = G_n = (V_n, E_n)$ with $|V_n| = n$, maximum degree $d_{\max} = d_{\max}(n)$, minimum degree $d_{\min} = d_{\min}(n)$, and average degree $d_{\mathrm{avg}} = d_{\mathrm{avg}}(n)$.*

(a) *If $p_n = \Omega(1)$, then*

(9) $$\alpha^{-1} \leqslant O(n)$$

(10) $$T_{\mathrm{rel}} \leqslant O(1).$$

(b) *If $p_n \to 0$, then for some absolute constant $c > 0$*

(11) $$\alpha^{-1} \leqslant c \max\left(\frac{d_{\mathrm{avg}} d^2_{\max}}{d^2_{\min}} T^{\mathrm{rw}}_{\mathrm{mix}} \log(n), \left(\max_y \bar{\mathcal{R}}_y \right) n |\log(p_n)| \right)$$

(12) $$T_{\mathrm{rel}} \leqslant c n \max_y \bar{\mathcal{R}}_y.$$

(c) *There exists an absolute constant $c > 0$ such that for all $p_n \in (0, 1)$*

(13) $$\alpha^{-1} \geqslant \frac{cn}{d_{\mathrm{avg}}}$$

(14) $$T_{\mathrm{rel}} \geqslant \frac{1 - \mu(\sum_x \omega_x = 1)}{p d_{\mathrm{avg}}}.$$

(d) *If $p_n = O(1/n)$, then we have the stronger bound*

(15) $$\alpha^{-1} \geqslant T^{\mathrm{rw}}_{\mathrm{meet}} \Omega(1 + |\log(np_n)|)$$

(16) $$T_{\mathrm{rel}} \geqslant T^{\mathrm{rw}}_{\mathrm{meet}} \Omega(1).$$

For the reader's convenience, and in view of our application to the FA-1f KCM, we detail the above bounds for the graphs discussed in Remark 2.1 when $p_n = \Theta(1/n)$.

COROLLARY 3.1. *In the setting of Theorem 1 assume that $p_n = \Theta(1/n)$. Then:*

(1) *hypercube:*

$$\Theta\left(\frac{n}{\log n}\right) = 2T_{\mathrm{rel}} \leqslant \alpha^{-1} \leqslant O(n),$$

(2) *regular b-ary tree, $b \geqslant 2$ independent of n:*

$$\Theta(n\log(n)) = 2T_{\mathrm{rel}} \leqslant \alpha^{-1} \leqslant O(n\log^2(n)),$$

(3) *uniform random d-regular graph, d independent of n:* w.h.p.

$$\Theta(n) = 2T_{\mathrm{rel}} \leqslant \alpha^{-1} \leqslant O(n\log(n)),$$

(4) *discrete torus $\mathbb{T}_n^d$ with d independent of n:*

$$\alpha^{-1} \leqslant O(1) \times \begin{cases} n^2\log(n) & d=1, \\ n\log^2(n) & d=2, \\ n\log(n) & d\geqslant 3, \end{cases}$$

and

$$\alpha^{-1} \geqslant 2T_{\mathrm{rel}} = \Theta(1) \times \begin{cases} n^2 & d=1, \\ n\log(n) & d=2, \\ n & d\geqslant 3. \end{cases}$$

The corollary follows immediately from Theorem 1(b) and (d) together with Remark 2.1, the well-known results on $T_{\mathrm{mix}}^{\mathrm{rw}}$ for each graph and the fact that (see [2, 11, 22]) for the graphs in Remark 2.1 it holds that

$$T_{\mathrm{meet}}^{\mathrm{rw}} = \Theta(n)\max_y \bar{\mathcal{R}}_y.$$

Indeed, the upper bounds on $T_{\mathrm{mix}}^{\mathrm{rw}}$ are only needed to see that for these graphs the maximum in the r.h.s. of (11) is achieved by the second term. Using Corollary 3.1 together with (4) and (3), we immediately get the following consequences for the FA-1f KCM to be compared with the r.h.s. of (2).

COROLLARY 3.2. *Consider the FA-1f KCM on $G = \mathbb{T}_n^d$ with parameter $p_n = \Theta(1/n)$ and let $T_{\mathrm{mix}}^{\mathrm{FA}}$ and T_2^{FA} be its mixing time and ℓ^2-mixing time respectively. Then*

$$(17) \quad T_{\mathrm{mix}}^{\mathrm{FA}} \leqslant T_2^{\mathrm{FA}} \leqslant \left(\alpha^{\mathrm{FA}}\right)^{-1}\log(n) \leqslant cn\log(n)\alpha^{-1} \leqslant O(1) \times \begin{cases} n^3\log^2(n) & d=1 \\ n^2\log^3(n) & d=2 \\ n^2\log^2(n) & d\geqslant 3. \end{cases}$$

REMARK 3.3. Our results in $d \geqslant 2$, besides being more directly proved than in [31, 32], hold in the stronger logarithmic Sobolev sense, and extend to other graphs, e.g. all the graphs discussed in Corollary 3.1. Furthermore, contrary to the approach followed in [31, 32], our methods can be easily adapted to cover other regimes of p_n. For $d=1$ the above upper bound on $T_{\mathrm{mix}}^{\mathrm{FA}}$ can be proved to also be sharp up to logarithmic corrections, using the technique discussed in [9, Section 6.2].

Our second theorem concerns the total variation mixing time of the generalised model, g-CBSEP.

Let τ_{cov} denote the cover time of the simple random walk on G (see e.g. [24, Chap. 11] and also [11] for a close connection between the average cover time and the resistance distance), and let

$$T_{\mathrm{cov}}^{\mathrm{rw}} = \inf\left\{t > 0 : \max_{x\in V}\mathbb{P}_x(\tau_{\mathrm{cov}} > t) \leqslant 1/e\right\}.$$

THEOREM 2. *Consider g-CBSEP on a finite connected graph G of minimum degree $d_{\min}$ with parameter $p = \rho(S_1)$ and let T_{mix} be the mixing time of CBSEP on G with parameter p. Then there exists a universal constant $c > 0$ such that*

$$T_{\mathrm{mix}} \leqslant T_{\mathrm{mix}}^{g\text{-}CBSEP} \leqslant c(T_{\mathrm{mix}} + T_{\mathrm{cov}}^{\mathrm{rw}}/d_{\min}).$$

The main reason to bound the total variation mixing time of g-CBSEP, instead of the ℓ^q-mixing times as for CBSEP, is that the scaling of the logarithmic Sobolev constant for g-CBSEP is very different from that of the CBSEP, as the following example shows.

EXAMPLE 3.4. Let $G = \mathbb{T}_n^2$, $p_n = 1/n$, $S = \{0, 1, 2\}$, and $\rho(1) = p, \rho(0) = \rho(2) = (1-p)/2$. Then,

(18)
$$\left(\alpha^{g\text{-}CBSEP}\right)^{-1} = n^{3/2 + o(1)}.$$

In the same setting Corollary 3.1 gives $\alpha^{-1} = O(n \log^2(n))$. To prove (18) it is enough to take as test function in the logarithmic Sobolev inequality for g-CBSEP the indicator that a vertical strip of width $\lfloor \sqrt{n}/2 \rfloor$ of the torus is in state 0.

4. CBSEP—Proof of Theorem 1. For this section we work with CBSEP in the setting of Theorem 1 and abbreviate $p = p_n$. In the sequel c shall denote an absolute constant whose value may change from line to line.

4.1. *Upper bounds—Proof of Theorem 1(a) and 1(b)* . Let us first prove the easy upper bound Theorem 1(a), assuming that $p = \Omega(1)$. We know from [10, Theorem 6.4] that $T_{\mathrm{rel}}^{\mathrm{FA}} = O(1)$. Recalling (3) and the definition of the relaxation time, we get that for CBSEP $T_{\mathrm{rel}} = O(1)$, yielding (10). By (7) this gives $\alpha^{-1} = O(n)$ and concludes the proof of Theorem 1(a).

The rest of this section is dedicated to the proof of the main upper bound—Theorem 1(b). The starting point is the following decomposition of the entropy of any $f : \Omega_+ \to \mathbb{R}$

(19)
$$\mathrm{Ent}(f^2) = \mu\left(\mathrm{Ent}(f^2 \,|\, N)\right) + \mathrm{Ent}\left(\mu(f^2 \,|\, N)\right),$$

where $N(\omega) = \sum_{x \in V} \omega_x$ is the number of particles and $\mathrm{Ent}(f^2 \,|\, N)$ is the entropy of f^2 w.r.t. the conditional measure $\mu(\cdot \,|\, N)$ (see (5)). The first term in the r.h.s. above is bounded from above using the logarithmic Sobolev constant of the SEP on G with a fixed number of particles.

PROPOSITION 4.1. *There exists an absolute constant $c > 0$ such that*

$$\mu\left(\mathrm{Ent}(f^2 \,|\, N)\right) \leqslant c \log(n) \frac{d_{\mathrm{avg}} d_{\max}^2}{d_{\min}^2} T_{\mathrm{mix}}^{\mathrm{rw}} \mathcal{D}(f),$$

PROOF. Let

$$\mathcal{D}_G^{\mathrm{SEP}}(f) = \frac{1}{2} \sum_{e \in E} \mu\left((f(\omega^e) - f(\omega))^2\right),$$

where ω^e is the configuration obtained from ω by swapping the states at the endpoints of the edge e, denote the Dirichlet form of the symmetric simple exclusion process on G. Similarly let

$$\mathcal{D}_{K_n}^{\mathrm{BL}}(f) = \frac{1}{2n} \sum_{e \in E(K_n)} \mu\left((f(\omega^e) - f(\omega))^2\right)$$

be the Dirichlet form of the Bernoulli-Laplace process on the complete graph K_n. The main result of [3, Theorem 1] implies that[2]

$$\mathcal{D}_{K_n}^{\mathrm{BL}}(f) \leqslant c\frac{2|E|}{n}\frac{d_{\max}^2}{d_{\min}^2}T_{\mathrm{mix}}^{\mathrm{rw}}\mathcal{D}_G^{\mathrm{SEP}}(f).$$

Using $2|E| = \sum_x d_x$ we get, in particular, that

$$\mathcal{D}_{K_n}^{\mathrm{BL}}(f) \leqslant c\frac{d_{\mathrm{avg}}d_{\max}^2}{d_{\min}^2}T_{\mathrm{mix}}^{\mathrm{rw}}\mathcal{D}_G^{\mathrm{SEP}}(f).$$

On other hand, the logarithmic Sobolev constant of the Bernoulli-Laplace process on K_n with $k \in \{1, \ldots, n-1\}$ particles is bounded by $c \log n$ uniformly in k [23, Theorem 5]. Hence,

$$\mu\left(\mathrm{Ent}(f^2\,|\,N)\right) \leqslant c\log(n)\mathcal{D}_{K_n}^{\mathrm{BL}}(f) \leqslant c\log(n)\frac{d_{\mathrm{avg}}d_{\max}^2}{d_{\min}^2}T_{\mathrm{mix}}^{\mathrm{rw}}\mathcal{D}_G^{\mathrm{SEP}}(f).$$

The proposition then follows using $p \leqslant 1/2$ and

$$\mathcal{D}_G^{\mathrm{SEP}}(f) \leqslant \frac{(2-p)}{(1-p)}\mathcal{D}(f). \qquad \square$$

We now examine the second term $\mathrm{Ent}(\mu(f^2\,|\,N))$ in the r.h.s. of (19). Let

$$g(k) := \mu\left(f^2\,|\,N=k\right)^{1/2}$$

for $k \geqslant 1$, so that $\mathrm{Ent}(\mu(f^2\,|\,N)) = \mathrm{Ent}_\gamma(g^2)$, where γ is the probability law of N on $\{1, \ldots, n\}$. Clearly, γ is $\mathrm{Bin}(n,p)$ conditioned to be positive, so that for any $2 \leqslant k \leqslant n$ we have

(20) $$\gamma(k)(1-p)k = \gamma(k-1)p(n-k+1).$$

PROPOSITION 4.2. *There exists an absolute constant $c > 0$ such that*

$$\mathrm{Ent}_\gamma(g^2) \leqslant c\log(1/p) \times p\sum_{y \in V}\mu\left([f(\omega^y)-f(\omega)]^2\,(1-\omega_y)\right).$$

where we recall that ω^y denotes the configuration ω flipped at y.

PROOF. The proof starts with a logarithmic Sobolev inequality for γ w.r.t. a suitably chosen reversible death and birth process on $\{1, \ldots, n\}$.

LEMMA 4.3. *There exists an absolute constant $c > 0$ such that for any non-negative function $g : \{1, \ldots, n\} \to \mathbb{R}$*

$$\mathrm{Ent}_\gamma(g^2) \leqslant c\log(1/p) \times \sum_{k=2}^{n}\gamma(k)k[g(k)-g(k-1)]^2.$$

Leaving the tedious proof to the Appendix, we move on to bounding the r.h.s. above for the special choice $g = \mu(f^2\,|\,N)^{1/2}$.

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[2]Actually the comparison result proved in [3] is much stronger, since it concerns weighted exchange processes on G and on K_n.

CLAIM 4.4. *For any $2 \leqslant k \leqslant n$ we have*

$$(g(k) - g(k-1))^2 \leqslant \frac{A_k^2}{g^2(k-1) + g^2(k)},$$

where

$$A_k = \frac{1}{n-k+1} \sum_{y \in V} \mu\left((1-\omega_y)\left[f^2(\omega) - f^2(\omega^y)\right] \mid N = k-1\right).$$

PROOF. We first observe that

(21) $\quad [g(k) - g(k-1)]^2 = \dfrac{[g^2(k) - g^2(k-1)]^2}{[g(k) + g(k-1)]^2} \leqslant \dfrac{[g^2(k) - g^2(k-1)]^2}{g^2(k) + g^2(k-1)}.$

Next we write

$$g^2(k-1) = \sum_{\omega:\, N(\omega)=k-1} \frac{\mu(\omega)}{\gamma(k-1)} f^2(\omega)$$

$$= \frac{1}{n-k+1} \frac{1}{\gamma(k-1)} \sum_{y \in V} \sum_{\omega:\, N(\omega)=k-1} \mu(\omega)(1-\omega_y)f^2(\omega).$$

With the change of variable $\eta = \omega^y$ we get that the r.h.s. above is equal to

$$\frac{1}{n-k+1} \frac{1}{\gamma(k-1)} \sum_{y \in V} \sum_{\omega:\, N(\omega)=k-1} \mu(\omega)(1-\omega_y)\left[f^2(\omega) - f^2(\omega^y)\right]$$

$$+ \frac{\gamma(k)(1-p)}{p(n-k+1)\gamma(k-1)} \sum_{y \in V} \sum_{\eta:\, N(\eta)=k} \frac{\mu(\eta)}{\gamma(k)} \eta_y f^2(\eta),$$

the second line being equal to $g^2(k)$ by (20). In conclusion $g^2(k-1) = g^2(k) + A_k$ and the claim follows from (21). $\qquad\square$

CLAIM 4.5. *For any $2 \leqslant k \leqslant n$ we have*

$$A_k^2 \leqslant \frac{2\left(g^2(k-1) + g^2(k)\right)}{n-k+1} \sum_{y \in V} \mu\left([f(\omega) - f(\omega^y)]^2(1-\omega_y) \mid N = k-1\right).$$

PROOF. Using $f^2(\omega) - f^2(\omega^y) = (f(\omega) - f(\omega^y))(f(\omega) + f(\omega^y))$ and the Cauchy-Schwarz inequality w.r.t. $\mu(\cdot \mid N = k-1, \omega_y = 0)$, we get

$$A_k \leqslant \mathrm{Av}\Big(\mu([f(\omega) - f(\omega^y)]^2 \mid N = k-1, \omega_y = 0)^{1/2} \times$$

$$\mu([f(\omega) + f(\omega^y)]^2 \mid N = k-1, \omega_y = 0)^{1/2}\Big),$$

where for any $h : V \to \mathbb{R}$

$$\mathrm{Av}(h) := \frac{1}{n-k+1} \sum_{y \in V} \mu\left((1-\omega_y) \mid N = k-1\right) h(y).$$

Another application of the Cauchy-Schwarz inequality, this time w.r.t. $\mathrm{Av}(\cdot)$, gives

$$A_k^2 \leqslant \frac{1}{n-k+1} \sum_{y \in V} \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \,|\, N = k-1 \right) \times$$

$$\frac{2}{n-k+1} \sum_{z \in V} \mu \left([f^2(\omega) + f^2(\omega^z)] (1 - \omega_z) \,|\, N = k-1 \right).$$

Inside the second factor in the above r.h.s. the term containing $f^2(\omega)$ is equal to $2\mu(f^2 \,|\, N = k-1) = 2g^2(k-1)$. Similarly, the term containing $f^2(\omega^y)$, after the change of variable $\eta = \omega^y$ and recalling (20), equals

$$\frac{2}{n-k+1} \frac{\gamma(k)k(1-p)}{p\gamma(k-1)} \mu \left(f^2(\eta) \,|\, N = k \right) = 2g^2(k). \qquad \square$$

Combining Claims 4.4 and 4.5, we get that

(22)
$$\begin{aligned}
(g(k) - g(k-1))^2 &\leqslant \frac{A_k^2}{g^2(k-1) + g^2(k)} \\
&\leqslant \frac{2}{n-k+1} \sum_{y \in V} \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \,|\, N = k-1 \right).
\end{aligned}$$

Using (22) together with (20), we get

$$\sum_{k=2}^{n} \gamma(k)k[g(k) - g(k-1)]^2$$

$$\leqslant \sum_{k=2}^{n} \frac{2k\gamma(k)}{n-k+1} \sum_{y \in V} \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \,|\, N = k-1 \right)$$

$$= \frac{2}{1-p} p \sum_{y \in V} \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \right).$$

Using the above bound together with Lemma 4.3 we get the statement of Proposition 4.2. $\qquad \square$

The final step in the proof of (11) is the following comparison between the quantity $p \sum_{y \in V} \mu \left([f(\omega) - f(\omega^y)]^2 (1 - \omega_y) \right)$ and the Dirichlet form $\mathcal{D}(f)$ using electrical networks. Recall the definition of the resistance distance and of $\max_y \bar{\mathcal{R}}_y$ given in Section 2.2.

PROPOSITION 4.6.

$$p \sum_{x \in V} \mu((f(\omega^x) - f(\omega))^2 (1 - \omega_x)) \leqslant 4n \max_{y \in V} \bar{\mathcal{R}}_y \times \mathcal{D}(f).$$

PROOF. We will identify $\omega \in \{0,1\}^V$ with its set of particles $\{x \in V : \omega_x = 1\}$ and we set $F_\omega(u) := f(\omega \cup \{u\}), u \in V$. For each $\vec{e} = (u,v) \in \vec{E}$ we also write $\nabla_{\vec{e}} F_\omega := F_\omega(v) - F_\omega(u)$. Given $x \in V$ and $\omega \in \Omega_+$, let $y_\omega \in V$ be an arbitrarily chosen vertex such that $\omega_{y_\omega} = 1$, and let θ_* be the optimal (i.e. with the smallest energy) unit flow

from x to y_ω. By applying [28, Lemma 2.9] to the function F_ω and using the Cauchy-Schwarz inequality, we get that for any $\omega \in \Omega_+$ and $x \in V$ such that $\omega_x = 0$

$$(f(\omega^x) - f(\omega))^2 = (F_\omega(x) - F_\omega(y_\omega))^2$$

$$= \left(\frac{1}{2} \sum_{\vec{e} \in \vec{E}} \theta_*(\vec{e}) \nabla_{\vec{e}} F_\omega \right)^2 \leqslant \mathcal{E}_{x, y_\omega} \times \frac{1}{2} \sum_{\vec{e} \in \vec{E}} (\nabla_{\vec{e}} F_\omega)^2.$$

Hence,

$$\sum_{x \in V} (f(\omega^x) - f(\omega))^2 (1 - \omega_x) \leqslant n \left(\max_{y \in V} \bar{\mathcal{R}}_y \right) \times \frac{1}{2} \sum_{\vec{e}} (\nabla_{\vec{e}} F_\omega)^2.$$

We next transform the generic term in the sum above into a Dirichlet form term for CBSEP. For any $\vec{e} = (u, v) \in \vec{E}$ we have

$$p \mu(\omega)(\nabla_{\vec{e}} F_\omega)^2$$

$$= \mu(\omega \cup \{u\}) \times \begin{cases} 0 & \{u, v\} \subset \omega \\ p(f(\omega \cup \{v\}) - f(\omega))^2 & u \in \omega \not\ni v \\ (1 - p)(f(\omega \cup \{v\}) - f(\omega \cup \{u\}))^2 & \{u, v\} \cap \omega = \varnothing. \end{cases}$$

Comparing with the expression of $\mathcal{D}(f)$, (1), we get immediately that

$$\frac{1}{2} \sum_{\omega \in \Omega_+} p \mu(\omega) \sum_{\vec{e} \in \vec{E}} (\nabla_{\vec{e}} F_\omega)^2 \leqslant 4\mathcal{D}(f). \qquad \square$$

We are now ready to prove (11). Using Proposition 4.1 the first term in the r.h.s. of (19) is bounded from above by

$$c \log(n) \frac{d_{\mathrm{avg}} d_{\max}^2}{d_{\min}^2} T_{\mathrm{mix}}^{\mathrm{rw}} \mathcal{D}(f).$$

In turn, Proposition 4.2 combined with Proposition 4.6 gives that the second term in the r.h.s. of (19) is bounded from above by

$$c \log(1/p) \times 4n \max_{y \in V} \bar{\mathcal{R}}_y \times \mathcal{D}(f).$$

In conclusion,

$$\mathrm{Ent}(f^2) \leqslant c \max \left(\log(n) \frac{d_{\mathrm{avg}} d_{\max}^2}{d_{\min}^2} T_{\mathrm{mix}}^{\mathrm{rw}}, \log(1/p) \times 4n \max_{y \in V} \bar{\mathcal{R}}_y \right) \times \mathcal{D}(f),$$

so that the best constant in the logarithmic Sobolev inequality (5) satisfies (11).

Turning to (12), Proposition 4.6 alone is enough to conclude. Indeed, using the two-block argument of [5, Lemma 6.6] (see also Lemma 6.5 and Proposition 6.2 therein) and the well-known fact that the variance w.r.t. a product measure is at most the average of the sum of variances over single spins (see e.g. [4, Chapter 1]), we get

$$\mathrm{Var}(f) \leqslant cp \sum_{x \in V} \mu((f(\omega^x) - f(\omega))^2 (1 - \omega_x)).$$

The desired bound (12) then follows from (3), (6) and Proposition 4.6.

4.2. *Lower bounds—Proof of Theorem 1(c) and 1(d).* Inject $f = \mathbb{1}_{\{N=1\}}$, the indicator of having exactly one particle, in the logarithmic Sobolev inequality (5). For $c > 0$ small enough we have

$$\frac{\mathrm{Ent}(f^2)}{\mathcal{D}(f)} = \frac{\mu(N=1)|\log(\mu(N=1))|}{\frac{2|E|\mu(N=1)}{n} \cdot \frac{p}{2-p}} \geqslant \frac{|\log \mu(N=1)|}{p d_{\mathrm{avg}}} \geqslant c \frac{n}{d_{\mathrm{avg}}},$$

since $\mu(N=1) = np(1-p)^{n-1}/(1-(1-p)^n)$. To check the last inequality, one may distinguish the cases np sufficiently large/of order 1/sufficiently small. This proves (13). Using the same function, so that $\mathrm{Var}(f) = \mu(N=1)(1-\mu(N=1))$, we obtain (14) in the same way. This concludes the proof of Theorem 1(c).

The rest of this subsection is dedicated to the proof of the main lower bound—Theorem 1(d), so we assume that $p_n = O(1/n)$. Let $\lambda_0 > 0$ be the smallest eigenvalue, restricted to the event $\{N \geqslant 2\}$ that there are at least two particles, of $-\mathcal{L}$, where $\mathcal{L}$ is the generator of CBSEP. By [17, Lemma 4.2, Equation (1.4)] we have that

$$\alpha^{-1} \geqslant \lambda_0^{-1} |\log(\mu(N \geqslant 2))|,$$

$$T_{\mathrm{rel}} \geqslant \lambda_0^{-1}(1 - \mu(N \geqslant 2)),$$

the second inequality being easy to check from the definition. It is well known (see e.g. [21, Section 3.4]) that

$$\lambda_0^{-1} \geqslant \mathbb{E}_{\mu(\cdot \,|\, N \geqslant 2)}(\tau),$$

where τ is the first time when $N = 1$. Putting these together and recalling that $p_n = O(1/n)$, we obtain

$$\alpha^{-1} \geqslant \mathbb{E}_{\mu(\cdot \,|\, N \geqslant 2)}(\tau)|\log(\mu(N \geqslant 2))| \geqslant \mathbb{E}_{\mu(\cdot \,|\, N \geqslant 2)}(\tau)\Omega(1 + |\log(np_n)|),$$

$$T_{\mathrm{rel}} \geqslant \mathbb{E}_{\mu(\cdot \,|\, N \geqslant 2)}(\tau)\mu(N=1) \qquad \geqslant \mathbb{E}_{\mu(\cdot \,|\, N \geqslant 2)}(\tau)\Omega(1).$$

In turn, again using that $p_n = O(1/n)$, we get

$$\mathbb{E}_{\mu(\cdot \,|\, N \geqslant 2)}(\tau) \geqslant \mu(N=2|N \geqslant 2)\mathbb{E}_{\mu(\cdot \,|\, N=2)}(\tau) \geqslant \Omega(1)\mathbb{E}_{\mu(\cdot \,|\, N=2)}(\tau).$$

It is not hard to see (e.g. via a graphical construction—see Section 5.1) that CBSEP stochastically dominates a process of coalescing random walks with birth rate 0, which we will call CSEP. Therefore, $\mathbb{E}_\omega(\tau) \geqslant \mathbb{E}_\omega^{\mathrm{CSEP}}(\tau)$ for any $\omega \in \Omega_+$. Furthermore, CSEP started with two particles has the law of two independent continuous time random walks which jump along each edge with rate $(1-p)/(2-p)$ and coalesce when they meet. Hence, we obtain (15) and (16), concluding the proof of Theorem 1(d).

5. g-CBSEP—Proof of Theorem 2.

5.1. *Graphical construction.* We start by introducing a graphical construction of g-CBSEP for all initial conditions. The graphical construction of CBSEP can then be immediately deduced by considering the special case $S_1 := \{1\}$ and $S_0 := \{0\}$.

To each edge $e \in E$ we associate a Poisson process of rate $p/(2-p)$ of arrival times $(t_n^e)_{n=1}^\infty$. Similarly, to each oriented edge $\vec{e} \in \vec{E}$ we associate a Poisson process of rate $(1-p)/(2-p)$ of arrival times $(t_n^{\vec{e}})_{n=1}^\infty$. All the above processes are independent as $e, \vec{e}$ vary in $E, \vec{E}$ respectively. Furthermore, for $e \in E$, $\vec{e} \in \vec{E}$ and $n \geqslant 1$, we define X_n^e and $X_n^{\vec{e}}$ to be mutually independent random variables taking values in S^2. We assume that for all n and $(u,v) \in \vec{E}$, the law of $X_n^{(u,v)}$ is $\rho_u(\cdot \,|\, S_1) \otimes \rho_v(\cdot \,|\, S_0)$. Similarly, for $\{u,v\} \in E$, the law of $X_n^{\{u,v\}}$ is $\rho_u(\cdot \,|\, S_1) \otimes \rho_v(\cdot \,|\, S_1)$. Given an initial configuration

$\omega(0) \in \Omega^{(g)}$ and a realization of the above variables, we define the realization of g-CBSEP $\omega(t)$ as follows.

Fix $t \geqslant 0$, let t^* be the first arrival time after t, and let $\{x, y\}$ be the endpoints of the edge where it occurs. We set $\omega_z(t^*) = \omega_z(t)$ for all $z \in V \setminus \{x, y\}$. If $E^{(g)}_{\{x,y\}}$ does not occur, that is $\omega_x(t) \in S_0$ and $\omega_y(t) \in S_0$, we set $\omega(t^*) = \omega(t)$. Otherwise, we set

$$(\omega_x(t^*), \omega_y(t^*)) = \begin{cases} X_n^{\{x,y\}} & \text{if } t^* = t_n^{\{x,y\}}, \\ X_n^{(x,y)} & \text{if } t^* = t_n^{(x,y)}. \end{cases}$$

OBSERVATION 5.1. Let $\omega(t)$ and $\omega'(t)$ be two g-CBSEP processes constructed using the same Poisson processes $(t_n^e)_{n=1}^\infty$, $(t_n^{\vec{e}})_{n=1}^\infty$ and variables X_n^e, $X_n^{\vec{e}}$ above, but with different initial conditions $\omega, \omega' \in \Omega^{(g)}$ satisfying $\varphi(\omega) = \varphi(\omega') = \eta \in \Omega$. Fix $t \geqslant 0$ and let $\mathcal{F}_t$ be the sigma-algebra generated by the arrival times smaller than or equal to t (but not the X_n^e and $X_n^{\vec{e}}$ variables). Then $\varphi(\omega(t)) = \varphi(\omega'(t)) =: \eta(t)$ is $\mathcal{F}_t$-measurable and only depends on ω through its projection η.

We say that a vertex $v \in V$ is *updated* if $v \in e \in E$ so that there exists $0 \leqslant t^* \leqslant t$ and n such that $t^* \in \{t_n^e, t_n^{\vec{e}}, t_n^{-\vec{e}}\}$ and the event $E_e^{(g)}$ occurs for $\omega(t^*)$, i.e. a successful update occurs at v. Denoting the set of updated vertices by Ξ_t, we have

- Ξ_t is $\mathcal{F}_t$-measurable and only depends on ω through its projection η,
- if $x \in \Xi_t$, then $\omega_x(t) = \omega'_x(t)$ and, conditionally on $\mathcal{F}_t$, the law of $\omega_x(t)$ is $\rho(\cdot \mid S_{\eta_x(t)})$,
- if $x \in V \setminus \Xi_t$, then $\omega_x(t) = \omega_x(0)$ and, in particular, $\eta_x(t) = \eta_x$.

In particular, for all $x \in V$ such that there exists $t_x \leqslant t$ satisfying $(\varphi(\omega(t_x)))_x \neq (\varphi(\omega(0)))_x$, we have $\omega_x(t) = \omega'_x(t)$ (since $x \in \Xi_t$).

5.2. *Proof of Theorem 2.*

We are now ready to prove Theorem 2. The lower bound is an immediate consequence of the fact that the projection chain on the variables $\varphi(\omega)$ coincides with CBSEP.

For the upper bound, let μ_t^η be the law of the CBSEP η_t at time t with parameter $p = \rho(S_1)$ and starting point $\eta \in \Omega_+$. Further denote $\nu^\eta = \rho(\cdot \mid \varphi(\omega) = \eta)$, the measure ρ conditioned on whether or not a particle is present at each site. Since ρ is itself product, we have

$$(23) \qquad \nu^\eta = \bigotimes_{x \in V} \rho_x(\cdot \mid S_{\eta_x}) = \bigotimes_{x : \eta_x = 1} \rho_x(\cdot \mid S_1) \otimes \bigotimes_{x : \eta_x = 0} \rho_x(\cdot \mid S_0).$$

CLAIM 5.2. *The law $\rho_t^{\nu^\eta}$ of g-CBSEP with initial law ν^η at time t takes the form*

$$(24) \qquad \rho_t^{\nu^\eta}(\cdot) = \mu_t^\eta\big(\nu^{\eta_t}(\cdot)\big),$$

i.e. it is the average of $\nu^{\eta'}$ over η' distributed as the CBSEP configuration η_t started from η at time t.

PROOF. Fix ω satisfying $\varphi(\omega) = \eta$. Denote by $\mathbb{P}^\omega$ the probability w.r.t. the graphical construction of g-CBSEP of Section 5.1 with initial condition ω, by $\mathcal{F}_t$ the sigma-algebra generated by the arrival times up to time t, as in Observation 5.1, and by $\mathbb{E}_{\mathcal{F}_t}$ the corresponding expectation. Then for any $\omega' \in \Omega^{(g)}$

$$(25) \quad \rho_t^\omega(\omega') = \mathbb{E}_{\mathcal{F}_t}\big[\mathbb{P}^\omega(\omega(t) = \omega' \mid \mathcal{F}_t)\big] = \mathbb{E}_{\mathcal{F}_t}\left[\prod_{x \notin \Xi_t} \mathbb{1}_{\omega'_x = \omega_x} \prod_{x \in \Xi_t} \rho_x(\omega'_x \mid S_{(\varphi(\omega(t)))_x})\right],$$

the last equality reflecting that by Observation 5.1 Ξ_t and $\varphi(\omega(t))$ are $\mathcal{F}_t$-measurable. Again by Observation 5.1, Ξ_t and $\varphi(\omega(t))$ are the same for all ω in the support of ν^η, so we denote the latter by $\eta(t)$. If we now average (25) over the initial condition ω w.r.t. ν^η and use (23), we obtain $\rho^{\nu^\eta}(\omega') = \mathbb{E}_{\mathcal{F}_t}[\nu^{\eta(t)}(\omega')]$, which is exactly (24), since $\eta(t)$ has the law μ_t^η of CBSEP with initial state η, as it is the projection of g-CBSEP with initial condition ω such that $\varphi(\omega) = \eta$ □

Next we write

$$(26) \qquad \max_{\omega \in \Omega_+^{(g)}} \|\rho_t^\omega - \rho_+\|_{\mathrm{TV}} \leqslant \max_{\eta \in \Omega_+} \left(\max_{\omega:\varphi(\omega)=\eta} \|\rho_t^\omega - \rho_t^{\nu^\eta}\|_{\mathrm{TV}} + \|\rho_t^{\nu^\eta} - \rho_+\|_{\mathrm{TV}} \right),$$

where ρ_+ was defined in Section 1.1. Using Claim 5.2, it follows that

$$(27) \qquad \max_{\eta \in \Omega_+} \|\rho_t^{\nu^\eta} - \rho_+\|_{\mathrm{TV}} = \max_{\eta \in \Omega_+} \|\mu_t^\eta - \mu\|_{\mathrm{TV}},$$

where μ is the reversible measure of CBSEP with parameter p.

To bound the first term in the r.h.s. of (26) the key ingredient is to use the graphical construction to embed into g-CBSEP of a suitable continuous time simple random walk $(W_t)_{t \geqslant 0}$ on G with the property that g-CBSEP at time t has a "particle" at the location of W_t.

Given $\omega \in \Omega_+^{(g)}$, let $v \in V$ be such that $\varphi(\omega_v) = 1$, and let $t^* = \min\{t_n^{(u,v)} > 0\}$ be the first time an edge of the form (u, v) is resampled to produce a configuration ω' with $\omega'_u \in S_1$ and $\omega'_v \in S_0$. We then set $W_s = v$ for $s < t^*$ and $W_{t^*} = u$. By iterating the construction we construct $(W_t)_{t \geqslant 0}$ with $W_0 = v$. It is clear that $\varphi(\omega_{W_t(\omega)}(t)) = 1$ for all t and that the law $\mathbb{P}_v(\cdot)$ of $(W_t)_{t \geqslant 0}$ is that of a continuous-time random walk started at v and jumping to a uniformly chosen neighbour at rate $d_{W_t}(1-p)/(2-p)$. We denote by σ_{cov} the cover time of $(W_t)_{t \geqslant 0}$.[3]

Observation 5.1 then implies

$$(28) \qquad \max_{\eta \in \Omega_+} \max_{\omega:\varphi(\omega)=\eta} \|\rho_t^\omega - \rho_t^{\nu^\eta}\|_{\mathrm{TV}} \leqslant \max_{\substack{\omega,\omega' \in \Omega_+^{(g)} \\ \varphi(\omega)=\varphi(\omega')}} \|\rho_t^\omega - \rho_t^{\omega'}\|_{\mathrm{TV}} \leqslant \max_{v \in V} \mathbb{P}_v(\sigma_{\mathrm{cov}} > t).$$

The upper bound given in the theorem now follows immediately from (26), (27), and (28) together with a standard comparison between σ_{cov} and the cover time of the *discrete time* simple random walk on G.

APPENDIX: PROOF OF LEMMA 4.3

Recall that

$$\gamma(k) = \binom{n}{k} \frac{p^k}{(1-p)^k} \frac{(1-p)^n}{1-(1-p)^n}$$

and consider the birth and death process on $\{1, \ldots, n\}$ reversible w.r.t. γ with Dirichlet form

$$\mathcal{D}_\gamma(g) = \sum_{k=2}^n \gamma(k)k[g(k) - g(k-1)]^2,$$

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[3]We use σ_{cov} to distinguish it from the cover time τ_{cov} of the *discrete time* simple random walk on G.

corresponding to the jump rates $c(1,0) = 0$ and

$$c(k, k-1) = k \qquad\qquad\qquad k = 2, \ldots, n$$

$$c(k, k+1) = (n-k)\frac{p}{1-p} \qquad\qquad k = 1, \ldots, n-1,$$

Let $m = \lceil pn \rceil$ and $i = \max(2, m)$. Using [29, Proposition 4] (see also [43]) the logarithmic Sobolev constant of the above chain is bounded from above, up to an absolute multiplicative constant, by the number $C_* = C_- \vee C_+$, where

$$C_+ = \max_{j \geqslant i+1}\left(\sum_{k=i+1}^{j}\frac{1}{\gamma(k)c(k, k-1)}\right)\gamma(N \geqslant j)|\log\left(\gamma(N \geqslant j)\right)|,$$

(29) $$C_- = \max_{j \leqslant i-1}\left(\sum_{k=j}^{i-1}\frac{1}{\gamma(k)c(k, k+1)}\right)\gamma(N \leqslant j)|\log\left(\gamma(N \leqslant j)\right)|.$$

Assume first that $i = m$ and let us start with C_+. For $\ell \geqslant 1$ write $a_\ell = \frac{1}{(m+\ell)\gamma(m+\ell)}$ and $S_k = \sum_{\ell=1}^{k}a_\ell$. We have

$$\frac{a_{\ell+1}}{a_\ell} = \frac{1-p}{p}\frac{m+\ell}{n-m-\ell} \geqslant 1,$$

from which it follows that for $0 < \delta < 1$ we have

$$\frac{a_{\ell+1}}{a_\ell} = 1 + \Theta(\ell/m) = e^{\Theta(\ell/m)} \qquad\qquad \ell \leqslant m,$$

(30) $$\frac{a_{\ell+1}}{a_\ell} \geqslant \frac{(1-p)(m+\delta m)}{p(n-m)} \geqslant 1+\delta \qquad\qquad \ell \geqslant \delta m.$$

In particular, for any two integers $s \leqslant t \leqslant m$ such that $t - s \geqslant \min(\sqrt{m}, m/s)$, it holds that for some absolute constant $\beta > 1$

(31) $$\frac{a_t}{a_s} = \prod_{\ell=s}^{t-1}\frac{a_{\ell+1}}{a_\ell} = e^{\Theta((t-s)t/m)} \geqslant \beta.$$

We first analyse the behaviour of $S_k\gamma(N \geqslant m+k)|\log(\gamma(N \geqslant m+k))|$ for $k \leqslant \delta m$ where $\delta > 0$ is a sufficiently small constant depending on β.

LEMMA A.3. *There exists a constant $c > 0$ such that for $\delta > 0$ small enough and $k \leqslant \delta m$ we have*

$$S_k\gamma(N \geqslant m+k)|\log(\gamma(N \geqslant m+k))| \leqslant c$$

PROOF. Let $0 < \delta < 1$. Define recursively

$$k_0 = 1, \quad k_1 = \lceil\sqrt{m}\rceil, \quad k_{t+1} = k_t + \lceil m/k_t \rceil,$$

and let T be the first index such that $k_T \geqslant \delta m$. Using (31) together with $a_{\ell+1} \geqslant a_\ell$, $k_{t+1} - k_t \leqslant k_t - k_{t-1}$, and $k_t/m \leqslant \delta$, we claim that for any $2 \leqslant t \leqslant T-1$

(32) $$\frac{(S_{k_{t+1}} - S_{k_t})}{(S_{k_t} - S_{k_{t-1}})} = \frac{\sum_{\ell=k_t+1}^{k_{t+1}}a_\ell}{\sum_{\ell=k_{t-1}+1}^{k_t}a_\ell} \geqslant \beta\frac{k_{t+1} - k_t}{k_t - k_{t-1}}$$

$$\geqslant \beta\left(\frac{k_t}{k_{t-1}} + \frac{k_t}{m}\right)^{-1} \geqslant \beta\left(\frac{k_t}{k_{t-1}} + \delta\right)^{-1}.$$

To prove the first inequality in (32), observe that for any positive non-decreasing sequence $(a_j)_{j=1}^\infty$ and positive integers $m \leqslant n$,

$$\frac{a_{n+1} + \cdots + a_{n+m}}{a_1 + \cdots + a_n} \geqslant \min_j \left(\frac{a_{j+m}}{a_j}\right) \left(\frac{a_{n-m+1} + \cdots + a_n}{a_1 + \cdots + a_n}\right)$$

$$\geqslant \min_j \left(\frac{a_{j+m}}{a_j}\right) \left(\frac{\sum_{j=n-m+1}^n a_j}{(n-m)a_{n-m} + \sum_{j=n-m+1}^n a_j}\right)$$

$$\geqslant \min_j \left(\frac{a_{j+m}}{a_j}\right) \frac{m}{n},$$

because $\sum_{j=n-m+1}^n a_j \geqslant a_{n-m} m$.

If now δ, t are chosen small enough and large enough, respectively, depending on the constant β above, the r.h.s. of (32) is greater than e.g. $\beta^{1/2} > 1$. In other words, fixing δ small enough and t_0 large enough, the sequence $\left((S_{k_{t+1}} - S_{k_t})\right)_{t=t_0}^T, t_0 \gg 1$, is exponentially increasing.

Now fix $k \leqslant \delta m$ and t such that $k_t \leqslant k < k_{t+1}$. Assume first that $t_0 \leqslant t < T$. Then, for some positive constant c allowed to depend on β and t_0 and to change from line to line, we have

$$S_k \leqslant \sum_{s=t_0}^{t+1} \left(S_{k_s} - S_{k_{s-1}}\right) + S_{t_0} \leqslant c\left(S_{k_{t+1}} - S_{k_t}\right) + S_{t_0}$$

$$\leqslant c\frac{k_{t+1} - k_t}{m\gamma(m + k_{t+1})} \leqslant c\frac{k_{t+1} - k_t}{m\gamma(m + k)}.$$

If instead $0 \leqslant t < t_0$, we directly have that

$$S_k \leqslant ka_k \leqslant c\frac{k_{t+1} - k_t}{m\gamma(m + k)}.$$

Using the bounds

$$\gamma(N \geqslant m + k) \leqslant c\frac{m + k}{k}\gamma(m + k), \qquad |\log(\gamma(N \geqslant m + k))| \leqslant ck^2/m,$$

we finally get that $k_t \leqslant k < k_{t+1}, t < T$,

$$S_k\gamma(N \geqslant m + k)|\log(\gamma(N \geqslant m + k))| \leqslant c\frac{(k_{t+1} - k_t)k_{t+1}}{m} \leqslant c. \qquad \square$$

Let δ be as in Lemma A.3. We next consider the easier case, $k \geqslant \delta m$. By (30), for c large enough depending on δ and allowed to change from line to line, we have that $S_k \leqslant S_{k_T} + ca_k \leqslant ca_k$ and $\gamma(N \geqslant m + k) \leqslant c\gamma(m + k)$. Thus, for $k \geqslant \delta m$, we have that

$$S_k\gamma(N \geqslant m + k)|\log(\gamma(N \geqslant m + k))| \leqslant \frac{c}{m + k}|\log(\gamma(N \geqslant m + k))| \leqslant c\log(1/p),$$

since for all k we trivially have $\gamma(m + k) \geqslant p^{m+k}$. In conclusion, we have proved that $C_+ \leqslant O(\log(1/p))$ if $m \geqslant 2$. If instead $m = 1$, then the very same computations still give $C_+ \leqslant O(\log(1/p))$, Lemma A.3 being void.

The bound of C_- follows the same pattern. If $m = O(1)$, the reader may readily check that $C_- = O(1)$ because all terms in (29) are $O(1)$. If instead $m \gg 1$, we still obtain $C_- = O(1)$, concluding the proof of Lemma 4.3.

References.

[1] D. Aldous and J. A. Fill, *Reversible markov chains and random walks on graphs* (2002). Unfinished monograph, recompiled 2014, available at http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[2] D. J. Aldous, *Meeting times for independent Markov chains*, Stochastic Process. Appl. **38** (1991), no. 2, 185–193.MR1119980

[3] G. Alon and G. Kozma, *Comparing with octopi*, Ann. Inst. Henri Poincaré Probab. Stat. **56** (2020), no. 4, 2672–2685.MR4164852

[4] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, and G. Scheffer, *Sur les inégalités de Sobolev logarithmiques*, Panoramas et synthèses, vol. 10, Société Mathématique de France, Paris, 2000.MR1845806

[5] O. Blondel, N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Fredrickson-Andersen one spin facilitated model out of equilibrium*, Markov Process. Related Fields **19** (2013), no. 3, 383–406.MR3156958

[6] M. Bramson and D. Griffeath, *On the Williams-Bjerknes tumour growth model: II*, Math. Proc. Camb. Phil. Soc. **88** (1980), no. 2, 339–357.MR578279

[7] M. Bramson and D. Griffeath, *On the Williams-Bjerknes tumour growth model. I*, Ann. Probab. **9** (1981), no. 2, 173–185.MR606980

[8] A. Broder and E. Shamir, *On the second eigenvalue of random regular graphs*, 28th Annual Symposium on Foundations of Computer Science, 1987, pp. 286–294.

[9] N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Kinetically constrained spin models*, Probab. Theory Related Fields **140** (2008), no. 3-4, 459–504.MR2365481

[10] N. Cancrini, F. Martinelli, C. Roberto, and C. Toninelli, *Facilitated spin models: recent and new results*, Methods of contemporary mathematical statistical physics, 2009, pp. 307–340.MR2581609

[11] A. K. Chandra, P. Raghavan, W. L. Ruzzo, and R. Smolensky, *The electrical resistance of a graph captures its commute and cover times*, Proceedings of the twenty-first annual ACM symposium on theory of computing, 1989, pp. 574–586.

[12] P. Diaconis and L. Saloff-Coste, *Logarithmic Sobolev inequalities for finite Markov chains*, Ann. Appl. Probab. **6** (1996), no. 3, 695–750.MR1410112

[13] R. Durrett and D. Griffeath, *Contact processes in several dimensions*, Z Wahrsch Verw Geb. **59** (1982), no. 4, 535–552.MR656515

[14] G. H. Fredrickson and H. C. Andersen, *Kinetic Ising model of the glass transition*, Phys. Rev. Lett. **53** (1984), no. 13, 1244–1247.

[15] G. H. Fredrickson and H. C. Andersen, *Facilitated kinetic Ising models and the glass transition*, J. Chem. Phys. **83** (1985), no. 11, 5822–5831.

[16] J. Friedman, J. Kahn, and E. Szemerédi, *On the second eigenvalue of random regular graphs*, Proceedings of the twenty-first annual ACM symposium on Theory of computing - STOC '89, 1989, pp. 587–598.

[17] S. Goel, R. Montenegro, and P. Tetali, *Mixing time bounds via the spectral profile*, Electron. J. Probab. **11** (2006), no. 0, 1–26.MR2480546

[18] D. Griffeath, *Additive and cancellative interacting particle systems*, Lecture Notes in Mathematics, vol. 724, Springer Berlin, Berlin, Heidelberg, 1979.MR538077

[19] T. E. Harris, *On a class of set-valued Markov processes*, Ann. Probab. **4** (1976), no. 2, 175–194.MR400468

[20] I. Hartarsky, F. Martinelli, and C. Toninelli, *Sharp threshold for the FA-2f kinetically constrained model*, ArXiv E-Prints (2020).

[21] J. Hermon and Y. Peres, *A characterization of L_2 mixing and hypercontractivity via hitting times and maximal inequalities*, Probab. Theory Related Fields **170** (2018), no. 3-4, 769–800.MR3773799

[22] V. Kanade, F. Mallmann-Trenn, and T. Sauerwald, *On coalescence time in graphs: when is coalescing as fast as meeting?*, Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms, 2019, pp. 956–965.MR3909528

[23] T.-Y. Lee and H.-T. Yau, *Logarithmic Sobolev inequality for some models of random walks*, Ann. Probab. **26** (1998), no. 4, 1855–1873.MR1675008

[24] D. A. Levin, Y. Peres, and E. L. Wilmer, *Markov chains and mixing times*, American Mathematical Society, Providence, RI, 2009. With a chapter by J. G. Propp and D. B. Wilson.MR2466937

[25] T. M. Liggett, *Stochastic interacting systems: contact, voter and exclusion processes*, Grundlehren der mathematischen wissenschaften [Fundamental principles of mathematical sciences], vol. 324, Springer Berlin, Berlin, Heidelberg, 1999.MR1717346

[26] T. M. Liggett, *Interacting particle systems*, Classics in mathematics, Springer Berlin, Berlin, Heidelberg, 2005. Reprint of the 1985 original.MR2108619

[27] O. Louidor, R. Tessler, and A. Vandenberg-Rodes, *The Williams–Bjerknes model on regular trees*, Ann. Appl. Probab. **24** (2014), no. 5, 1889–1917.MR3226167

[28] R. Lyons and Y. Peres, *Probability on trees and networks*, Cambridge series in statistical and probabilistic mathematics, vol. 42, Cambridge University Press, Cambridge, 2016.MR3616205

[29] L. Miclo, *An example of application of discrete Hardy's inequalities*, Markov Process. Related Fields **5** (1999), no. 3, 319–330.MR1710983

[30] R. I. Oliveira and Y. Peres, *Random walks on graphs: new bounds on hitting, meeting, coalescing and returning*, 2019 Proceedings of the Sixteenth Workshop on Analytic Algorithmics and Combinatorics (ANALCO), 2019, pp. 119–126.MR3909448

[31] N. S. Pillai and A. Smith, *Mixing times for a constrained Ising process on the torus at low density*, Ann. Probab. **45** (2017), no. 2, 1003–1070.MR3630292

[32] N. S. Pillai and A. Smith, *Mixing times for a constrained Ising process on the two-dimensional torus at low density*, Ann. Inst. Henri Poincaré Probab. Stat. **55** (2019), no. 3, 1649–1678.MR4010947

[33] N. Pippenger, *The hypercube of resistors, asymptotic expansions, and preferential arrangements*, Math. Mag. **83** (2010), no. 5, 331.MR2762645

[34] L. Saloff-Coste, *Lectures on finite Markov chains*, Lectures on probability theory and statistics, 1997, pp. 301–413.

[35] D. L. Schwartz, *Applications of duality to a class of Markov processes*, Ann. Probab. **5** (1977), no. 4, 522–532.MR448631

[36] A. Shapira, *A note on the spectral gap of the Fredrickson–Andersen one spin facilitated model*, J. Stat. Phys. (2020).MR4179809

[37] A. Sudbury and P. Lloyd, *Quantum operators in classical probability theory. IV. Quasi-duality and thinnings of interacting particle systems*, Ann. Probab. **25** (1997), no. 1, 96–114.MR1428501

[38] R. Sun and J. M. Swart, *The Brownian net*, Ann. Probab. **36** (2008), no. 3, 1153–1208.MR2408586

[39] J. M. Swart, *Duality and intertwining of Markov chains*, 2013. Lecture.

[40] P. Tetali, *Random walks and the effective resistance of networks*, J. Theoret. Probab. **4** (1991), no. 1, 101–109.MR1088395

[41] W. Thomson and P. Tait, *Treatise on natural philosophy*, Oxford University Press, 1867.

[42] T. Williams and R. Bjerknes, *Stochastic model for abnormal clone spread through epithelial basal layer*, Nature **236** (1972), no. 5340, 19–21.

[43] Q. Yang, H. Liu, and F. Gao, *Logarithmic Sobolev inequalities for two-sided birth-death processes*, Wuhan Univ. J. Nat. Sci. **13** (2008), no. 2, 133–136.MR2491203