doi:10.1103/PhysRevLett.125.103601, HAL:hal-02467011, arXiv:2002.02407

Supplemental Material to the article: Morphing Supermodes: A Full Characterization for Enabling Multimode Quantum Optics

Élie Gouzien,[1] Sébastien Tanzilli,[1] Virginia D'Auria,[1] and Giuseppe Patera[2,*]
[1] *Université Côte d'Azur, CNRS, Institut de Physique de Nice, Parc Valrose, 06108 Nice Cedex 2, France*
[2] *Université de Lille, CNRS, UMR 8523 — PhLAM — Physique des Lasers Atomes et Molécules, F-59000 Lille, France*

PROOF THAT $S(\omega)$ IS ω-SYMPLECTIC

For any ω we have

$$S(\omega) = \sqrt{2\Gamma}(\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma} - \mathbb{I}. \tag{1}$$

We note that, for general mode-dependent dumping coefficients, the matrices $\mathcal{M}$ and Γ do not commute. In order to prove that $S(\omega)$ is ω-symplectic (see expression (7) in the main text) we have to evaluate

$$S(\omega)\ \Omega\ S^{\mathrm{T}}(-\omega) = \left[\sqrt{2\Gamma}(\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma} - \mathbb{I}\right]\Omega\left[\sqrt{2\Gamma}(-\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma} - \mathbb{I}\right]^{\mathrm{T}} \tag{2}$$

and prove that it is equal to Ω, with Ω the symplectic form introduced in the main text. To this purpose we use the property of $\mathcal{M}$ and Γ of being a Hamiltonian matrix $(\Omega\mathcal{M})^{\mathrm{T}} = \Omega\mathcal{M}$ and skew-Hamiltonian $(\Omega\,\Gamma)^{\mathrm{T}} = -\Omega\,\Gamma$, respectively. This allows to write $\mathcal{M}^{\mathrm{T}} = \Omega^{-1}\mathcal{M}\Omega^{-1}$ and $\Gamma^{\mathrm{T}} = -\Omega^{-1}\Gamma\Omega^{-1}$ in the third term on the right-hand side. Then by replacing $\mathbb{I} = -\Omega^{-1}\mathbb{I}\Omega^{-1}$, this term can be simplified as following

$$\left[\sqrt{2\Gamma}(-\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma} - \mathbb{I}\right]^{\mathrm{T}} = \sqrt{2\Gamma}\Omega(\mathrm{i}\omega\,\mathbb{I} - \Gamma - \mathcal{M})^{-1}\Omega\sqrt{2\Gamma} - \mathbb{I}. \tag{3}$$

We use this result in (2) together with the fact that Γ and Ω commute and that $\Omega\Omega = -\mathbb{I}$:

$$\begin{aligned}
&S(\omega)\ \Omega\ S^{\mathrm{T}}(-\omega)\\
&\quad= \left[\sqrt{2\Gamma}(\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma} - \mathbb{I}\right]\Omega\left[\Omega\sqrt{2\Gamma}(\mathrm{i}\omega\,\mathbb{I} - \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma}\Omega - \mathbb{I}\right]\\
&\quad= -\sqrt{2\Gamma}(\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})^{-1}\,(2\Gamma)\,(\mathrm{i}\omega\,\mathbb{I} - \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma}\Omega\\
&\qquad - \sqrt{2\Gamma}(\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma}\Omega + \sqrt{2\Gamma}(\mathrm{i}\omega\,\mathbb{I} - \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma}\Omega + \Omega\\
&\quad= \sqrt{2\Gamma}(\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})^{-1}\underbrace{\Big[-(2\Gamma) - (\mathrm{i}\omega\,\mathbb{I} - \Gamma - \mathcal{M}) + (\mathrm{i}\omega\,\mathbb{I} + \Gamma - \mathcal{M})\Big]}_{=0}(\mathrm{i}\omega\,\mathbb{I} - \Gamma - \mathcal{M})^{-1}\sqrt{2\Gamma}\Omega + \Omega\\
&\quad= \Omega
\end{aligned} \tag{4}$$

which proves that $S(\omega)$ is a conjugate-symplectic matrix for every ω.

BOGOLIUBOV-DE GENNES HAMILTONIANS AND THEIR CANONICAL TRANSFORMATIONS

In this section we would like to explicit some of the connections that can be established between our approach and the Bogoliubov canonical transformation adopted in quantum field theory. In the mean field approximation, the system dynamics is driven by a Hamiltonian that is quadratic in the field operators and looks like (1) in the main text. This expression is usually recast into the following

$$H = \frac{\hbar}{2}(\boldsymbol{\xi}^{\dagger})^{\mathrm{T}}\mathcal{H}\boldsymbol{\xi}, \tag{5}$$

where $\boldsymbol{\xi} = (a_1, \ldots, a_N | a_1^{\dagger}, \ldots, a_N^{\dagger})^{\mathrm{T}}$, $\boldsymbol{\xi}^{\dagger} = (a_1^{\dagger}, \ldots, a_N^{\dagger} | a_1, \ldots, a_N)^{\mathrm{T}}$ and

$$\mathcal{H} = \left(\begin{array}{c|c} G & F \\ \hline F^{*} & G^{*} \end{array}\right) \tag{6}$$

is an Hermitian matrix best known as bosonic Bogoliubov-de Gennes (BdG) Hamiltonian. The BdG Hamiltonian $\mathcal{H}$ has, then, a simple connection with the Hamiltonian matrix $\mathcal{M}$ we introduced in the main text

$$\mathcal{M} = L\,(-\mathrm{i}K\mathcal{H})\,L^{-1}. \tag{7}$$

In this expression L is the unitary matrix, performing a change of basis from complex field amplitudes $\boldsymbol{\xi}$ to quadratures representation $\boldsymbol{R} = L\boldsymbol{\xi}$, given by

$$L = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} I & I \\ \hline -\mathrm{i}I & \mathrm{i}I \end{array} \right) \tag{8}$$

and

$$K = \left(\begin{array}{c|c} I & 0 \\ \hline 0 & -I \end{array} \right). \tag{9}$$

We notice that the BdG Hamiltonian is not a Hamiltonian matrix by itself, but this is the case for the term $-\mathrm{i}K\mathcal{H}$ (this amounts to the verification of the following property $(\varOmega(-\mathrm{i}K\mathcal{H}))^{\mathrm{T}} = \varOmega(-\mathrm{i}K\mathcal{H})$). This property guarantees that $-\mathrm{i}K\mathcal{H}$ generates a symplectic transformation via the exponential map or, in other terms, that the time evolution driven by $\exp(-\mathrm{i}K\mathcal{H}t)$ preserves the structure of the commutators algebra.

The Hamiltonian (6) can be diagonalized by a canonical Bogoliubov transformation B [1] when $\mathcal{H} \geq 0$ (in this case the system is said to be thermodynamically stable) such that [2]

$$K\mathcal{H} = B\varLambda B^{-1} \tag{10}$$

where $\varLambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_N | -\lambda_1, \ldots, -\lambda_N\}$ and $B \in \mathbb{C}^{2N\times 2N}$ is a symplectic matrix that has following the block-form

$$B = \left(\begin{array}{c|c} \alpha & \beta \\ \hline \beta^* & \alpha^* \end{array} \right). \tag{11}$$

As an element of the real symplectic group in the complex representation (see ref [38] in the main text) this matrix is symplectic in the sense $BKB^\dagger = K$ or, equivalently, when its blocks satisfy the following conditions

$$\alpha\alpha^\dagger - \beta\beta^\dagger = I, \tag{12}$$
$$\alpha\beta^{\mathrm{T}} - \beta\alpha^{\mathrm{T}} = 0. \tag{13}$$

We note that, since $\mathcal{H} \geq 0$ is equivalent to $-\varOmega\mathcal{M} \geq 0$, this result is nothing else that the application of the Williamson theorem to the real positive definite matrix $-\varOmega\mathcal{M}$. The Bogoliubov transformation allows to define new modes $\boldsymbol{\zeta} = B^{-1}\boldsymbol{\xi} = (b_1, \ldots, b_N | b_1^\dagger, \ldots, b_N^\dagger)^{\mathrm{T}}$ for which the Hamiltonian (6) results to be $H = \sum_m \hbar\lambda_m b_m^\dagger b_m + \text{const}$.

Applying the Bogoliubov transformation to our case means writing (7) as $\mathcal{M} = (LB)(-\mathrm{i}\varLambda)(LB)^{-1}$. This expression can, then, be injected into (1) in order to give

$$S(\omega) = L\,B\,\frac{(\gamma - \mathrm{i}\omega)\mathbb{I} - \mathrm{i}\varLambda}{(\gamma + \mathrm{i}\omega)\mathbb{I} + \mathrm{i}\varLambda}\,B^{-1}\,L^{-1} \tag{14}$$

under the assumption of equal damping coefficients $\varGamma = \gamma\mathbb{I}$. In the representation of complex field amplitudes $\boldsymbol{\xi}$ we have that $\boldsymbol{\xi}_{\mathrm{out}}(\omega) = (L^{-1}S(\omega)L)\,\boldsymbol{\xi}_{\mathrm{in}}(\omega)$, where $L^{-1}S(\omega)L$ is ω-symplectic, the output field amplitudes of Bogoliubov modes read

$$\boldsymbol{\zeta}_{\mathrm{out}}(\omega) = \frac{(\gamma - \mathrm{i}\omega)\mathbb{I} - \mathrm{i}\varLambda}{(\gamma + \mathrm{i}\omega)\mathbb{I} + \mathrm{i}\varLambda}\,\boldsymbol{\zeta}_{\mathrm{in}}(\omega) \tag{15}$$

and present a squeezed (antisqueezed) noise spectrum. In particular, since $\varLambda$ is real (this comes from the assumption that $\mathcal{H} \geq 0$) the transformation from $\boldsymbol{\zeta}_{\mathrm{in}}(\omega)$ to $\boldsymbol{\zeta}_{\mathrm{out}}(\omega)$ is unitary and does not produce squeezing (antisqueezing). As a consequence the input and output spectral covariance matrices are unitarily similar. Since a Bogoliubov transformation is not necessarily unitary, B corresponds to an active transformation so that the input Bogoliubov modes $\boldsymbol{\zeta}_{\mathrm{in}} = B^{-1}\boldsymbol{\xi}_{\mathrm{in}}$ are squeezed (antisqueezed) with respect to $\boldsymbol{\xi}_{\mathrm{in}}$ that are in the vacuum state. This amount of squeezing (antisqueezing) can be, then, quantified by applying a Bloch-Messiah decomposition to $B = L^{-1}(O\varLambda_B O'^{\mathrm{T}})L$. Here the ω-independent squeezing (antisqueezing) part is in the diagonal matrix Λ_B, and O and O' are real symplectic orthogonal matrices. At the output, the Bogoliubov modes present the same amount of squeezing (antisqueezing) as the input because of the unitary transformation. However, since this transformation is ω-dependent, the output Bogoliubov modes are squeezed (antisqueezed) on different quadratures for different frequencies. We conclude then that it is not possible to find physical observables that are both statistically independent and ω-independent.

Two more important points have to be considered when comparing the approach of morphing supermodes to Bogoliubov transformation: first, the detection, characterization and manipulation of the Bogoliubov modes would require the use of active interferometers, while in the case of the morphing supermodes the passive transformations generated by symplectic unitary matrices can be easily implemented by means of passive interferometers. Additionally, we note that requiring B to be unitary would need $\beta = 0$ in (13). In this case the diagonalization (10) would be possible only for $F = 0$ in (6). This corresponds to the paradoxal situation for which our system does not present parametric down-conversion-like processes responsible for the generation of squeezing (in other terms, for creating squeezing we need $F \neq 0$). The second point is that in the Bogoliubov approach, the diagonal form (10) exists only when $\mathcal{H} \geq 0$ a condition that is often not met in a driven quantum system such as those we are interested in. On the contrary, in the case of morphing supermodes, the existence of the analytical Bloch-Messiah decomposition of an ω-symplectic transformation is always guaranteed.

PROOF OF THE EXISTENCE OF ANALYTICAL BLOCH-MESSIAH DECOMPOSITION

In [3–5], constructive methods for finding the analytic singular value decomposition of a matrix smoothly depending on a real parameter are given, while in [6] the analytic singular value decomposition on the real symplectic group has been considered. In the case with no degenerate singular values, at each ω the Bloch-Messiah decomposition is unique up to the order of the singular values and vectors, and up to a phase for each singular vector. Those vectors form a conjugate-symplectic base. Let $x(\omega)$ and $y(\omega)$ be two normed eigenvectors given by the application of smooth decomposition without taking into account symplecticity. By quasi-unicity of the singular value decomposition, up to a phase they are part of a conjugate-symplectic base, $\left|x^\dagger \Omega y\right|$ can only take one of the values 0 or 1. As $\omega \mapsto x(\omega)$ and $\omega \mapsto y(\omega)$ are continuous, $f : \omega \mapsto \left|x^\dagger(\omega) \Omega y(\omega)\right|$ is also continuous. Assuming a connected domain for ω implies that f is constant. Phase can be continuously corrected when needed by replacing y with $\dfrac{y}{x^\dagger \Omega y}$. This being true for all possible pairs of x and y, we conclude that if for a given ω the change-of-basis matrix is conjugate-symplectic, it keeps this property for all ω. We call it an analytical Bloch-Messiah decomposition. In this section we show how to constructively express the decomposition for a given $S(\omega) \in \mathbb{S}_\omega$ of the form

$$S(\omega) = U(\omega) D(\omega) V^\dagger(\omega) \tag{16}$$

where U and V are $2N \times 2N$ unitary and conjugate-symplectic matrix-valued functions and D is a $2N \times 2N$ diagonal matrix-valued function

$$D(\omega) = \begin{pmatrix} D_1(\omega) & 0 \\ 0 & D_2(\omega) \end{pmatrix} \tag{17}$$

were $D_1(\omega) = \operatorname{diag}\{d_1(\omega), \ldots, d_N(\omega)\}$ and $D_2(\omega) = \operatorname{diag}\{d_1^{-1}(\omega), \ldots, d_N^{-1}(\omega)\}$ with $d_k(\omega) > 0$ and $1 \leq d_k(0) \leq d_{k+1}(0)$ (we note that for $\omega > 0$ the order of the singular values can change with respect to the initial one for an analytical decomposition [3]). It is easy to prove that as elements of the intersection between the conjugate-complex symplectic and the unitary groups the matrices U and V have the following block-form

$$U(\omega) = \begin{pmatrix} U_1(\omega) & U_2(\omega) \\ -U_2(\omega) & U_1(\omega) \end{pmatrix} \quad \text{and} \quad V(\omega) = \begin{pmatrix} V_1(\omega) & V_2(\omega) \\ -V_2(\omega) & V_1(\omega) \end{pmatrix} \tag{18}$$

By differentiating (16) with respect to ω (we designate the symbol $'$ for derivation with respect to ω and temporary drop the dependence on ω for space-saving),

$$S' = U' D V^\dagger + U D' V^\dagger + U D V'^\dagger. \tag{19}$$

After multiplying (19) by $U^\dagger$ from the left and by V from the right

$$D' = U^\dagger S' V - U^\dagger U' D - D V'^\dagger V \tag{20}$$

Now we define $H = U^\dagger U'$ and $K = V^\dagger V'$ and, then, we multiply from the left these definitions by U and V and we get

$$U' = UH, \tag{21}$$
$$V' = VK. \tag{22}$$

Since U and V are unitary, then

$$U^\dagger U = I, \tag{23}$$
$$V^\dagger V = I. \tag{24}$$

After differentiating (23) and (24), we find that $H^\dagger = -H$ and $K^\dagger = -K$ are skew Hermitian. Moreover, by construction, H and K have the block-structure

$$H = \begin{pmatrix} H_1 & H_2 \\ -H_2 & H_1 \end{pmatrix} \quad \text{and} \quad K = \begin{pmatrix} K_1 & K_2 \\ -K_2 & K_1 \end{pmatrix}. \tag{25}$$

These properties guarantee that the matrices H and K are Hamiltonian matrices in the sense $\Omega H = (\Omega H)^\dagger$ and $\Omega K = (\Omega K)^\dagger$ and, as a consequence, that the solutions U and V of (21) and (22) are conjugate-symplectic and unitary matrices.

On the other side, we define $Q = U^\dagger S' V$, then re-write (20) as

$$D' = Q - HD + DK \tag{26}$$

Eqs. (21), (22) and (26) define a system of differential equations for the elements of $U(\omega)$, $V(\omega)$ and $D(\omega)$ that we endow with the initial conditions $U(0)$, $V(0)$ and $D(0)$ obtained from the Bloch-Messiah decomposition at $\omega = 0$

$$S(0) = U(0)D(0)V^\dagger(0). \tag{27}$$

We now re-write Eq. (26) in the block structure

$$\begin{pmatrix} D_1' & 0 \\ 0 & D_2' \end{pmatrix} = \begin{pmatrix} Q_1 & Q_2 \\ Q_3 & Q_4 \end{pmatrix} - \begin{pmatrix} H_1 & H_2 \\ -H_2 & H_1 \end{pmatrix} \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} + \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \begin{pmatrix} K_1 & K_2 \\ -K_2 & K_1 \end{pmatrix} \tag{28}$$

This expression give rise to two sets of differential equations for the singular values

$$D_1' = Q_1 - H_1 D_1 + D_1 K_1, \tag{29}$$
$$D_2' = Q_4 - H_1 D_2 + D_2 K_1 \tag{30}$$

and two sets of algebraic equations

$$H_2 D_2 - D_1 K_2 = Q_2, \tag{31}$$
$$H_2 D_1 - D_2 K_2 = -Q_3. \tag{32}$$

First we solve Eqs. (31) and (32) in H_2 and K_2. For $i, j = 1, \ldots, N$

$$(H_2)_{ij} d_j^{-1} - (K_2)_{ij} d_i = (Q_2)_{ij}, \tag{33}$$
$$(H_2)_{ij} d_j - (K_2)_{ij} d_i^{-1} = -(Q_3)_{ij}. \tag{34}$$

For $i = j$ and $d_i \neq 1$ the solutions are

$$(H_2)_{ii} = \frac{(Q_3)_{ii} d_i^3 + (Q_2)_{ij} d_i}{1 - d_i^4}, \tag{35}$$
$$(K_2)_{ii} = \frac{(Q_2)_{ii} d_i^3 + (Q_3)_{ii} d_i}{1 - d_i^4}. \tag{36}$$

In the case where $d_i = 1$ it must be $(Q_3)_{ii} = -(Q_2)_{ii}$. As a consequence the system is underdetermined, and we have the freedom to choose $(K_2)_{ii} = 0$. Hence $(H_2)_{ii} = (Q_2)_{ii}$. For $i \neq j$, $d_i \neq 1$ and $d_j \neq 1$ (remember $1 \leq d_i$), the solutions are

$$(H_2)_{ij} = \frac{(Q_3)_{ij} d_i^2 d_j + (Q_2)_{ij} d_j}{1 - d_i^2 d_j^2}, \tag{37}$$

$$(K_2)_{ij} = \frac{(Q_2)_{ij} d_i^2 d_j + (Q_3)_{ij} d_j}{1 - d_i^2 d_j^2}. \tag{38}$$

Otherwise, if $d_i = d_j = 1$, it must be $(Q_3)_{ij} = -(Q_2)_{ij}$ and the underdetermined system allows us to choose $(K_2)_{ij} = 0$ and $(H_2)_{ij} = (Q_2)_{ij}$.

From Eqs. (29) and its Hermitian conjugate we consider first the case $i = j$, for $i = 1, \ldots, N$:

$$d_i' = (Q_1)_{ii} - (H_1 - K_1)_{ii} d_i, \tag{39}$$

$$d_i' = (Q_1)_{ii}^* + (H_1 - K_1)_{ii} d_i. \tag{40}$$

By summing Eqs. (39) and (40) we get a set of differential equation for the singular values

$$d_i' = \frac{(Q_1)_{ii} + (Q_1)_{ii}^*}{2}. \tag{41}$$

By subtracting Eqs. (39) and (40) we get an algebraic equation that allows to obtain the diagonal elements of H_1 and K_1,

$$(H_1 - K_1)_{ii} = \frac{(Q_1)_{ii} - (Q_1)_{ii}^*}{2d_i}. \tag{42}$$

We can choose, then, $(K_1)_{ii} = 0$ and determine $(H_1)_{ii}$. Notice that this result is consistent with the fact that H and K are skew Hermitian so that their diagonal must be purely imaginary. Now we consider Eq. (29) and its Hermitian conjugate for $i \neq j$, for $i, j = 1, \ldots, N$. In this case, after using the fact that H and K are skew Hermitian, we get

$$(Q_1)_{ij} - (H_1)_{ij} d_j + (K_1)_{ij} d_i = 0, \tag{43}$$

$$(Q_1)_{ji}^* + (H_1)_{ij} d_i - (K_1)_{ij} d_j = 0. \tag{44}$$

We can solve Eqs. (43) and (44) with respect to $(H_1)_{ij}$ and $(K_1)_{ij}$. This system of algebraic equation is solvable if $d_i \neq d_j$, which means that the spectrum of singular values is not degenerate. In this case we obtain

$$(H_1)_{ij} = \frac{(Q_1)_{ij} d_j - (Q_1)_{ij}^* d_i}{d_j^2 - d_i^2}, \tag{45}$$

$$(K_1)_{ij} = \frac{(Q_1)_{ij} d_i - (Q_1)_{ij}^* d_j}{d_j^2 - d_i^2}. \tag{46}$$

The case where the path of two or more singular values collide thus giving rise to degeneracies can also be treated by adapting to our case the strategy developed in [4, 5] for the case of the analytic singular value decomposition.

We notice also that in the case of transformations like (1), some of the degeneracies in the spectrum of $S(\omega)$ can derive from degeneracies in the spectrum of the eigenvalues of $\mathcal{M}$. In this case if a degeneracy is present at $\omega = 0$ it will persist at any other $\omega \neq 0$.

Finally, the algorithm that allows to find the analytical Bloch-Messiah decomposition of $S(\omega)$ is the following. We start at $\omega_0 = 0$ and we find the standard Bloch-Messiah decomposition $S(0) = U(0)D(0)V^\dagger(0)$. From $U(0)$, $V(0)$ and $D(0)$, we evaluate $Q(0) = U^\dagger(0)S'(0)V(0)$ as well as $H_1(0)$ and $K_1(0)$ from Eqs. (42), (45) and (46), and $H_2(0)$ and $K_2(0)$ from the solutions of the system (31) and (32). Then we can find, from Euler approximation of Eq. (41) the matrix $D(\omega_1)$. On the other side, for solving Eqs. (21) and (22), we use the Magnus perturbative approach that has the advantage of preserving the symplectic structure at any order of approximation. The solutions $U(\omega_1)$ and $V(\omega_1)$, with $\omega_1 = \omega_0 + d\omega$ (with $d\omega \ll 1$), are thus evaluated at the first Magnus order as

$$U(\omega_1) \approx U(\omega_0) \exp\Big(H(\omega_0)\,\mathrm{d}\omega\Big). \tag{47}$$

These results are used for obtaining the values of $H(\omega_1)$ and $K(\omega_1)$, then the procedure can be iterated for $\omega_m = \omega_0 + m\,d\omega$ with $m > 1$.

* giuseppe.patera@univ-lille.fr

[1] N. N. Bogoliubov, On the theory of superfluidity, J. Phys. **11**, 23 (1947).
[2] J.-P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems* (MIT Press, Cambridge, MA, 1986).
[3] A. Bunse-Gerstner, R. Byers, V. Mehrmann, and N. K. Nichols, Numerical computation of an analytic singular value decomposition of a matrix valued function, Numer. Math. **60**, 1 (1991).
[4] K. Wright, Differential equations for the analytic singular value decomposition of a matrix, Numer. Math. **63**, 283 (1992).
[5] L. Dieci and T. Eirola, On smooth decompositions of matrices, SIAM J. Matrix Anal. Appl. **20**, 800 (1999).
[6] L. Dieci and L. Lopez, Smooth singular value decomposition on the symplectic group and lyapunov exponents approximation*, Calcolo **43**, 1 (2006).