On the entanglement of quasiparticles in a Bose-Einstein Condensate

From Faraday Waves to the Dynamical Casimir Effect

Seminar @LPL

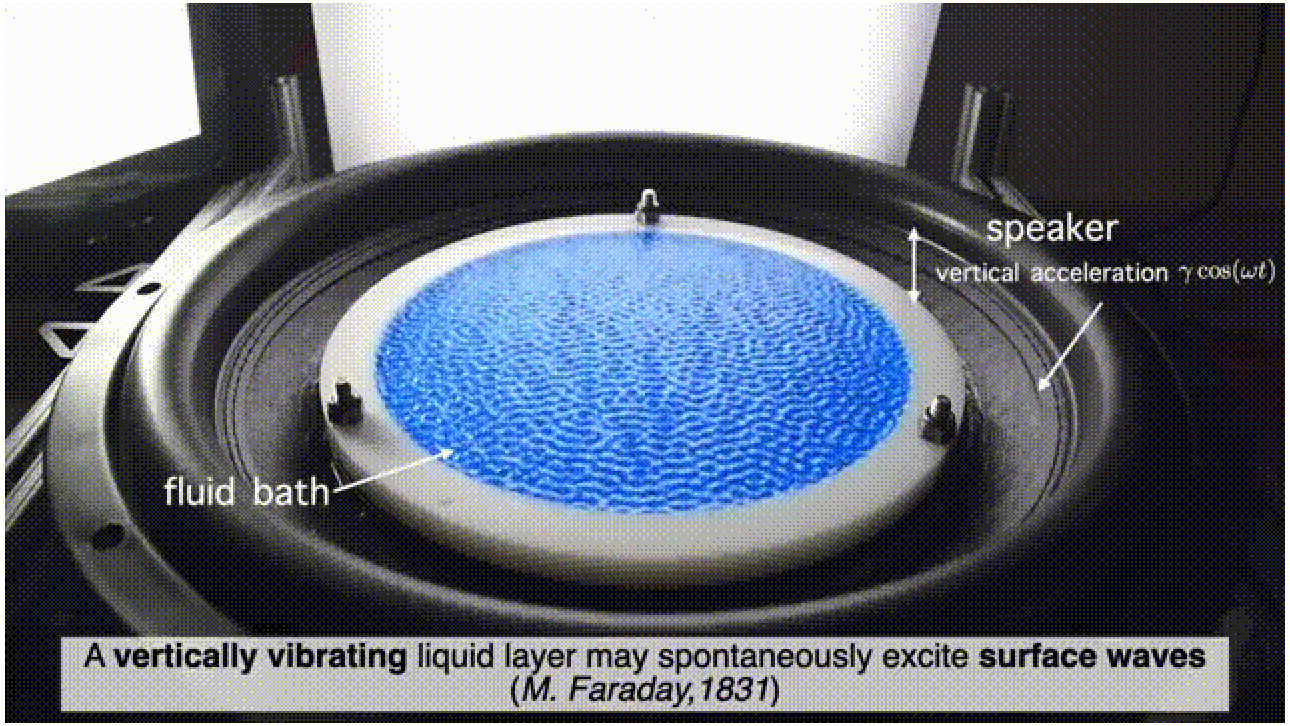
Slides available at
www.normalesup.org/~gondret/talk.pdf

Victor Gondret

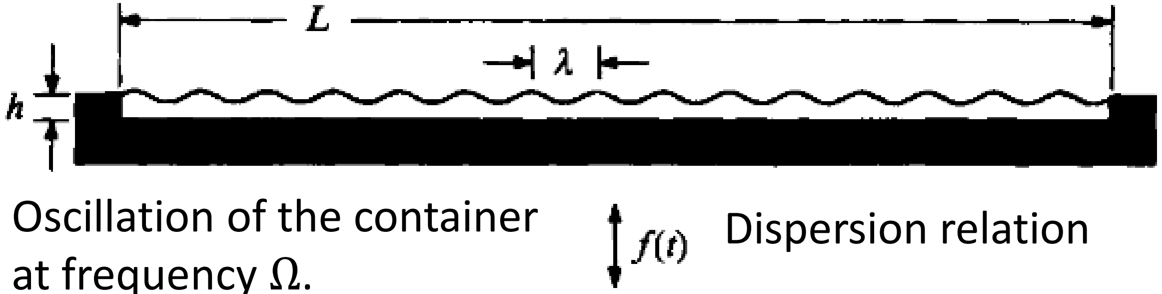
Clothilde Lamirault, Rui Dias, Léa Camier, Charlie Leprince, Quentin Marolleau, Denis Boiron & Chris Westbrook

Theory: Amaury Micheli & Scott Robertson

Faraday waves



Guan *et al.* PR Fluids (2023), Edwards & Fauve J. Fluid Mech. (1994)



$\vec{g}$

Oscillation of the container at frequency Ω.

Dispersion relation

$$\omega_k = \sqrt{\tanh(hk)\,[gk + \gamma k^3]} = \Omega/2$$

Broughton Suspension Bridge collapsed in 1831

Parametric oscillation ≠ forced oscillation

$\Omega/2$	Ω
Variation of an internal parameter	External force

Parametric or forced excitation of a swing

Forced excitation

Parametric excitation

Parametric oscillation $\neq$ forced oscillation

Parametric oscillation	Forced oscillation
$\Omega/2$	Ω
Variation of an internal parameter	External force

Parametric or forced excitation of a swing

Forced excitation

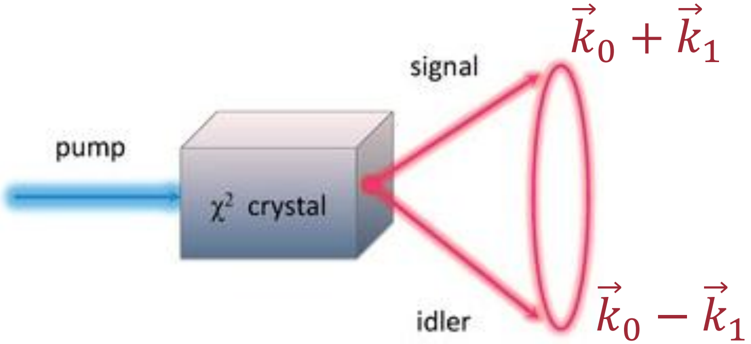
Parametric excitation

Parametric oscillation ≠ forced oscillation

$\Omega/2$	Ω
Variation of an internal parameter	External force
Growth triggered by fluctuations	Growth initialized by the force

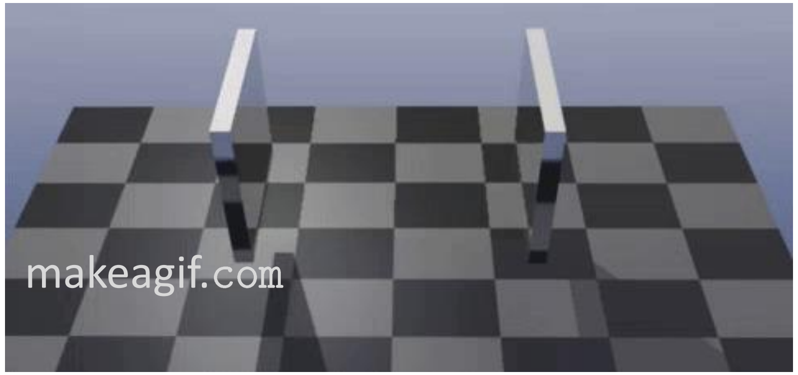
- experimental imperfections,
- thermal fluctuations,
- quantum fluctuations.

Parametric amplification across various scales



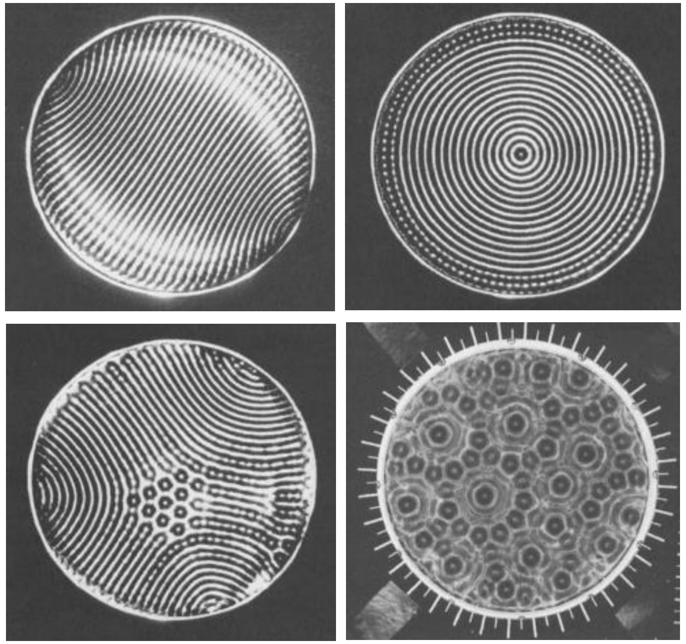
Quantum vacuum fluctuations trigger amplification which leads to entanglement.

Dynamical Casimir effect
Moore (1970)



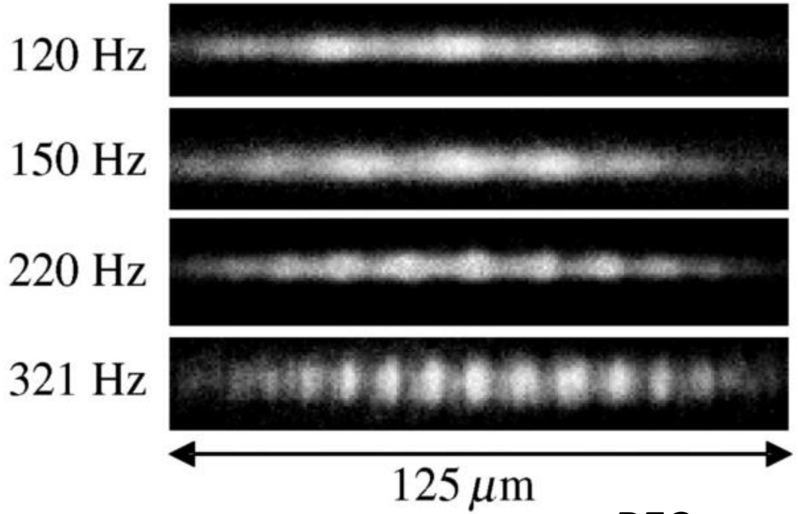
Photons

Fluid



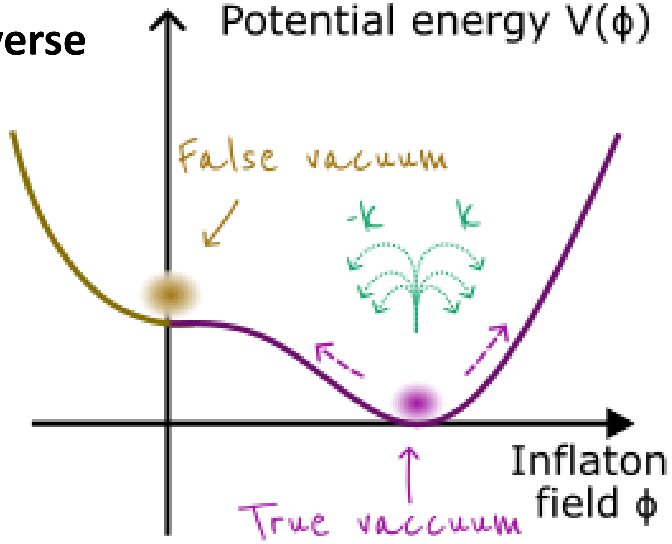
Edwards & Fauve J. Fluid Mech. (1994)

Engels *et al* PRL (2007)



BEC

Preheating in the early universe



Reheating after inflation

The **inflaton** *slowly rolls* from its initial false vacuum state. Its almost constant potential energy **drives the inflation.**

A. Linde, Phys. Lett. 129B, 177 (1983).

It starts to oscillate around its minimum and, coupled to matter fields, it creates particles through broad **parametric resonance**.

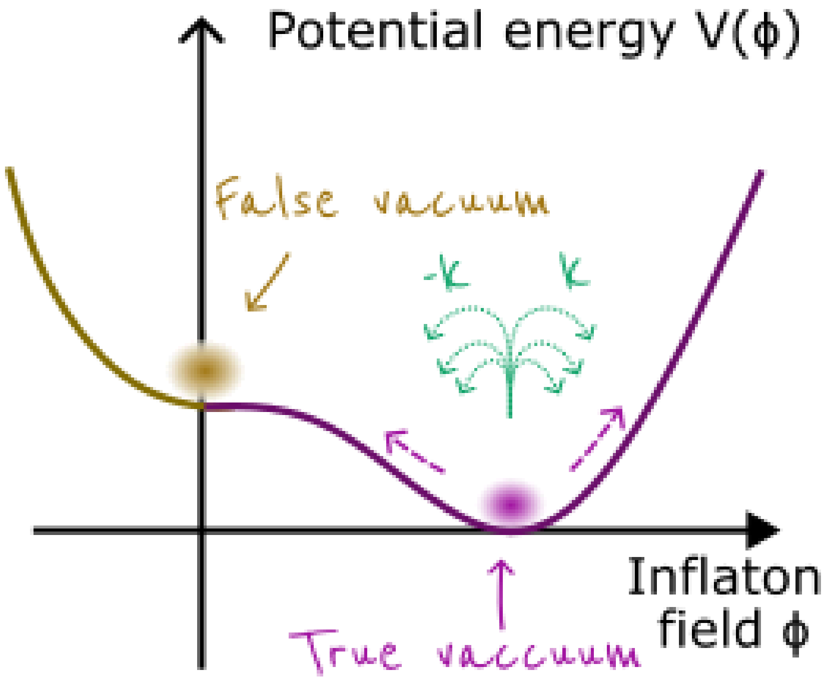
L. Kofman, A. Linde & A. Starobinsky, Phys. Rev. D 56, (1997).

Particles are created in **pairs** with **opposite momenta from vacuum** in a two modes squeezed state.

Interactions lead to decoherence and thermalization

D. Campo & R. Parentani, Phys. Rev. D **74**, 025001 (2006).

Analog gravity & cosmology: reproduce and study effects of quantum field theory in curved space time with collective excitations on strong background. Unruh, Phys. Rev. Lett. **46**, 1351 (1981)

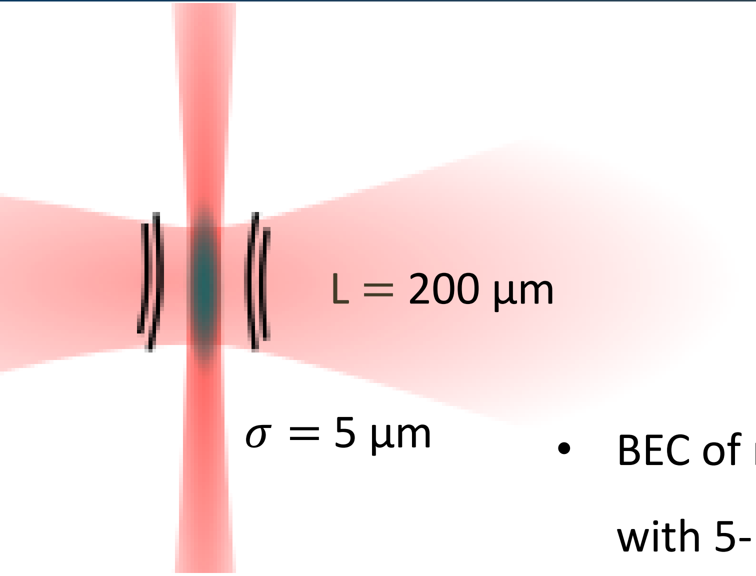


- Does parametric amplification of quasiparticles in a BEC lead to an entangled state?
- How these quasiparticles thermalize?

Can we witness the role played by vacuum fluctuations i.e. observe momentum space entanglement between quasiparticles in a BEC?

1. Parametric amplification of quasiparticles in an elongated BEC
2. Experimental setup and protocol
3. Observation of the growth and decay of quasiparticles
4. Quantifying entanglement from number correlation functions
5. Observation of quasiparticle entanglement

Parametric excitation of an elongated BEC



- BEC of metastable helium He^4 in 10 s with 5-15 000 atoms at 50(10) nK
- 1 kHz & 50 Hz: effective 1D dynamics

Description: Bose gas with contact interaction

- $\hat{\Psi} \sim \frac{\sqrt{n_1}}{\sigma} e^{-r^2/2\sigma^2}[1 + \hat{\phi}(z)]$
- $n_1 = N/L$
- $g_1 \sim 1/\sigma^2$ 1D effective interaction

Theoretical approach: Bogoliubov 1D (linearize)

We study collective excitations:

- $\hat{b}_k$ annihilates a longitudinal quasiparticle at k
- $\hat{\phi}_k = u_k \hat{b}_k + v_k \hat{b}^\dagger_{-k}$

$\hat{b}_k$ diagonalizes the Hamiltonian

$$i\hbar\partial_t \hat{b}_k = \omega_k \hat{b}_k \quad + i\frac{\dot{\omega}_k}{2\omega_k}\hat{b}^\dagger_{-k}$$

with Bogoliubov dispersion relation

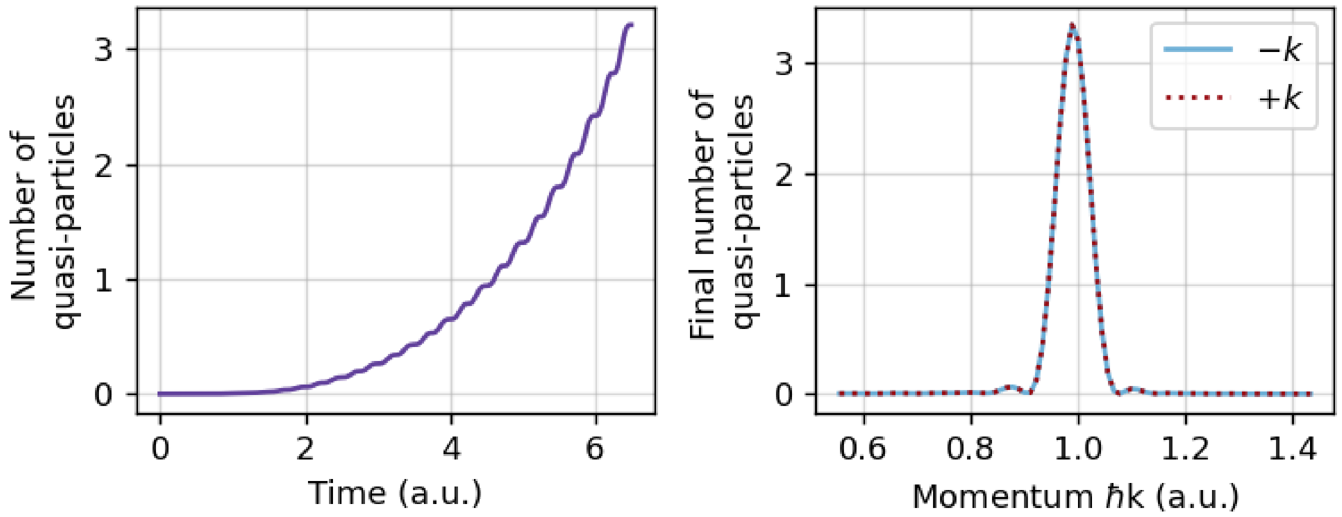
$$\omega_k = \sqrt{2g_1 n_1 \frac{\hbar^2 k^2}{2m} + \left(\frac{\hbar^2 k^2}{2m}\right)^2}$$

What if g_1 is time dependant?

Robertson *et al* PRD (2017,2018)

Parametric excitation of an elongated BEC

Numerical simulation



Oscillation of $g_1 n_1$ at frequency Ω parametrically excites quasiparticles by pairs with $\omega_k = \Omega/2$

How to change $g_1 n_1$ (or gn) ?

- Feshbach resonance: Chicago, Rice, Heidelberg,
- Trap frequency modulation: NIST, Palaiseau, Mexico…

$g_1 \sim 1/\sigma^2$ 1D effective interaction with σ BEC radius.

Theoretical approach: Bogoliubov 1D

We study collective excitations:

- ○ $\hat{b}_k$ annihilates a quasiparticle at k
- ○ $\hat{\phi}_k = u_k \hat{b}_k + v_k \hat{b}^\dagger_{-k}$

$\hat{b}_k$ diagonalizes the Hamiltonian

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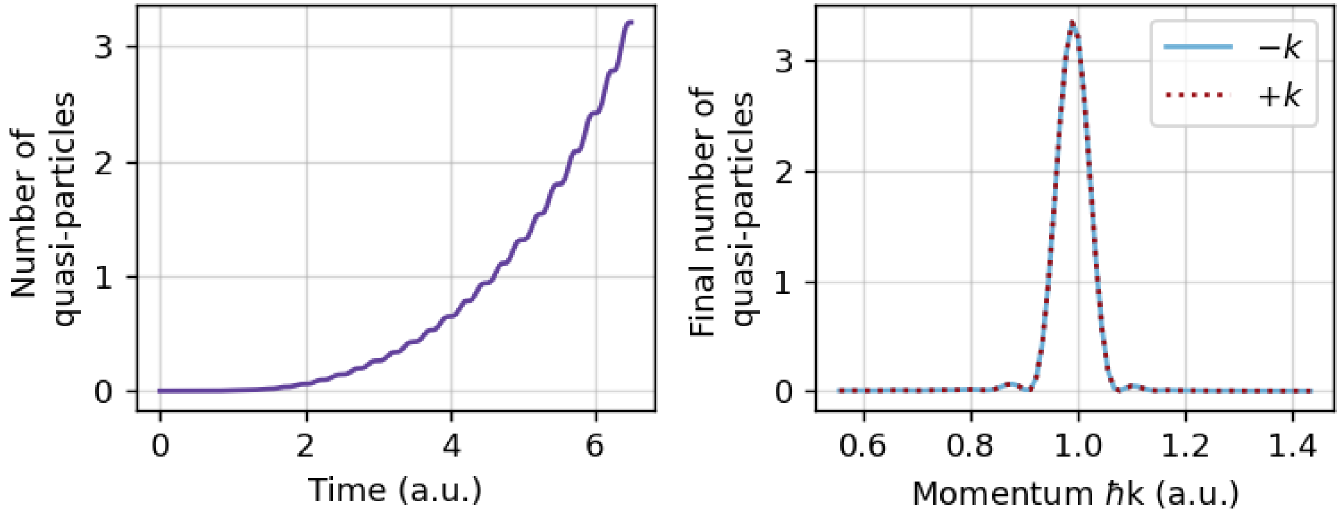
What if g_1 is time dependant?

Jaskula *et al.* PRL (2012)
Robertson *et al* PRD (2017,2018)

Parametric excitation of an elongated BEC

Numerical simulation



Oscillation of $g_1 n_1$ at frequency Ω parametrically excites quasiparticles by pairs with $\omega_k = \Omega/2$

If zero temperature, we expect a two-mode squeezed vacuum state

$$|\phi\rangle \sim \sum_i \tanh^i r \, |i, i\rangle_{-k,k}$$

i.e. vacuum fluctuations → entanglement.

Theoretical cheatsheet

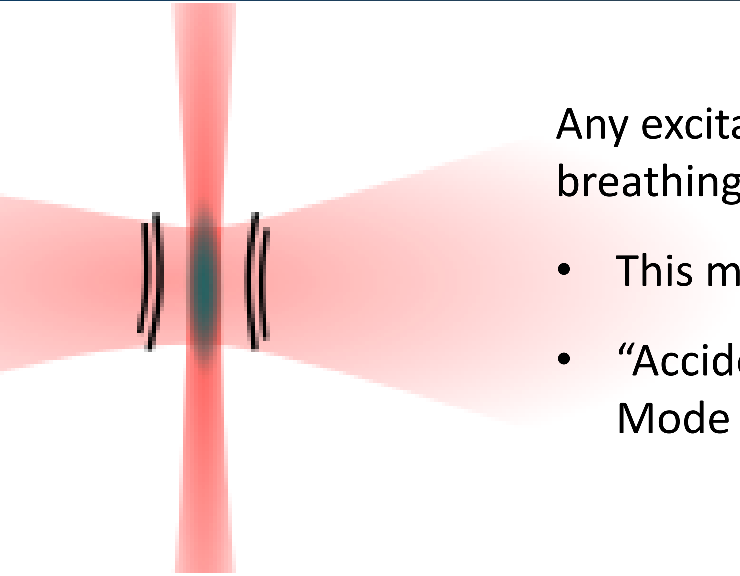
Carusotto *et al*. EPJD (2010), Busch *et al.* PRA (2014), Robertson *et al* PRD (2017,2018)

If non-zero temperature, both thermal and vacuum fluctuations trigger the growth.

Amplification of quantum fluctuations is witnessed by two-mode entanglement: $|\langle \hat{b}_k \hat{b}_{-k} \rangle|^2 > n_k n_{-k}$

Beyond Bogoliubov: quasiparticle interactions further destroy entanglement.

Outline



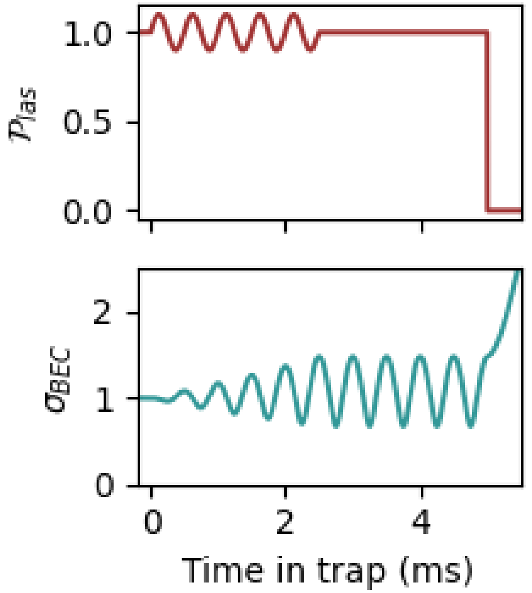
Any excitation frequency could work but we excite only the transverse breathing mode of the BEC at $2\omega_\perp$

Chevy *et al* PRL (2002)

- This mode is (almost) not damped
- "Accidental Suppression of Landau Damping of the Transverse Breathing Mode in Elongated Bose-Einstein Condensates"

Jackson & Zaremba PRL (2002)

Because both the BEC and the thermal cloud oscillate at $2\omega_\perp$



Excitation procedure does not heat the cloud.

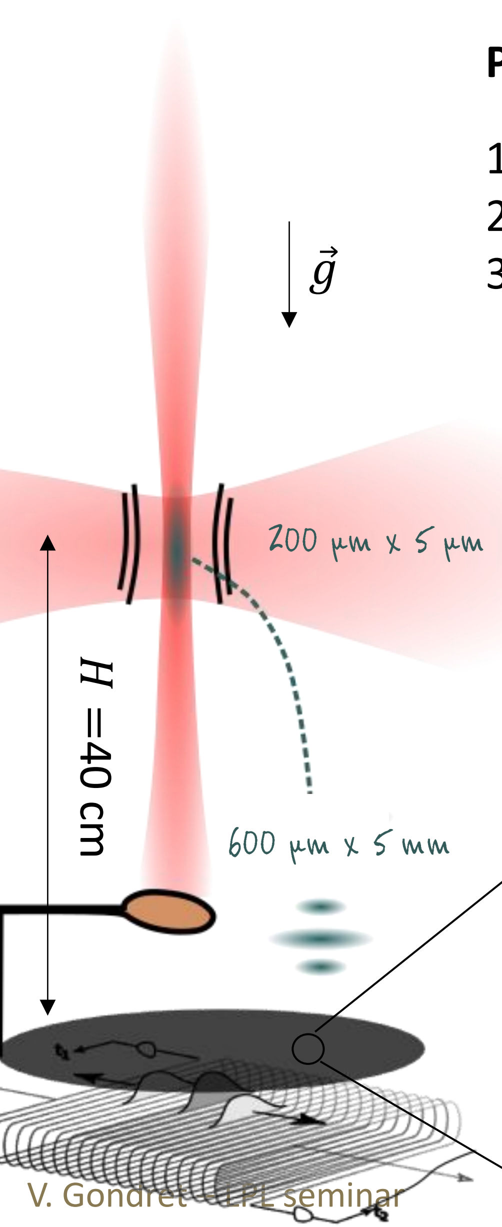
We can hope to get an entangled state!

Excitation at resonance of the transverse breathing mode → induces → Parametric excitation of the longitudinal modes

$p = mv = \hbar k$
$\hbar$ Planck constant
m mass

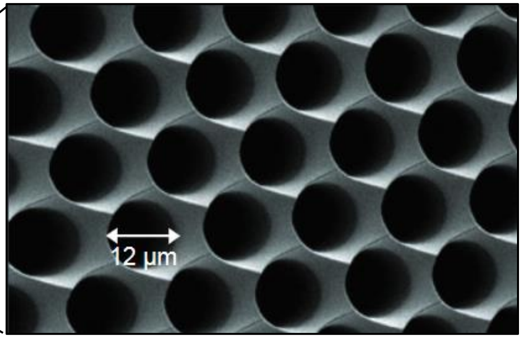
Protocol

1. Parametric excitation
2. Raman transfer (+kick)
3. Single particle detection

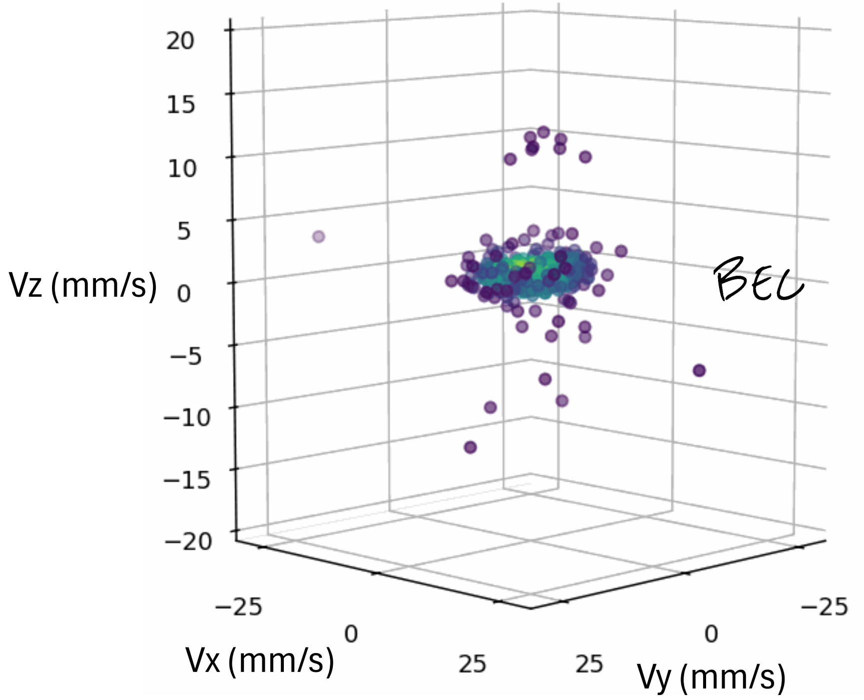


Single atom in momentum space

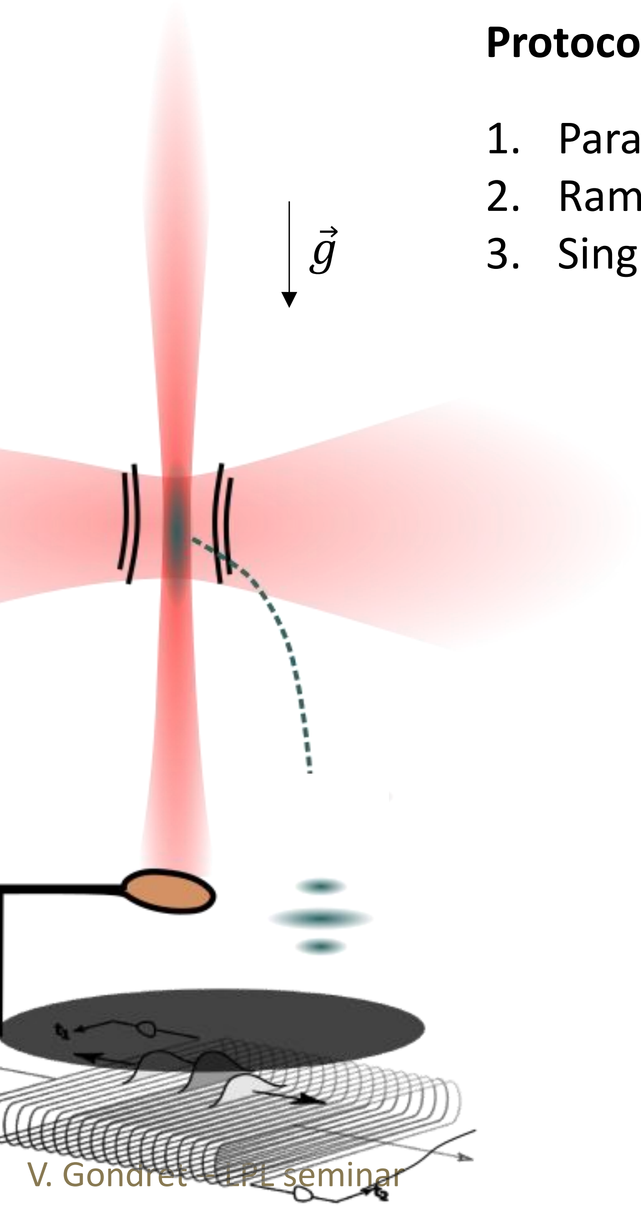
$$\begin{cases} v_x = X/T \\ v_y = Y/T \\ v_z = gT/2 - H/T \end{cases}$$



Metastable He^4 : electronic detection of individual atoms (X, Y, T)



Single shot images



Protocol

1. Parametric excitation
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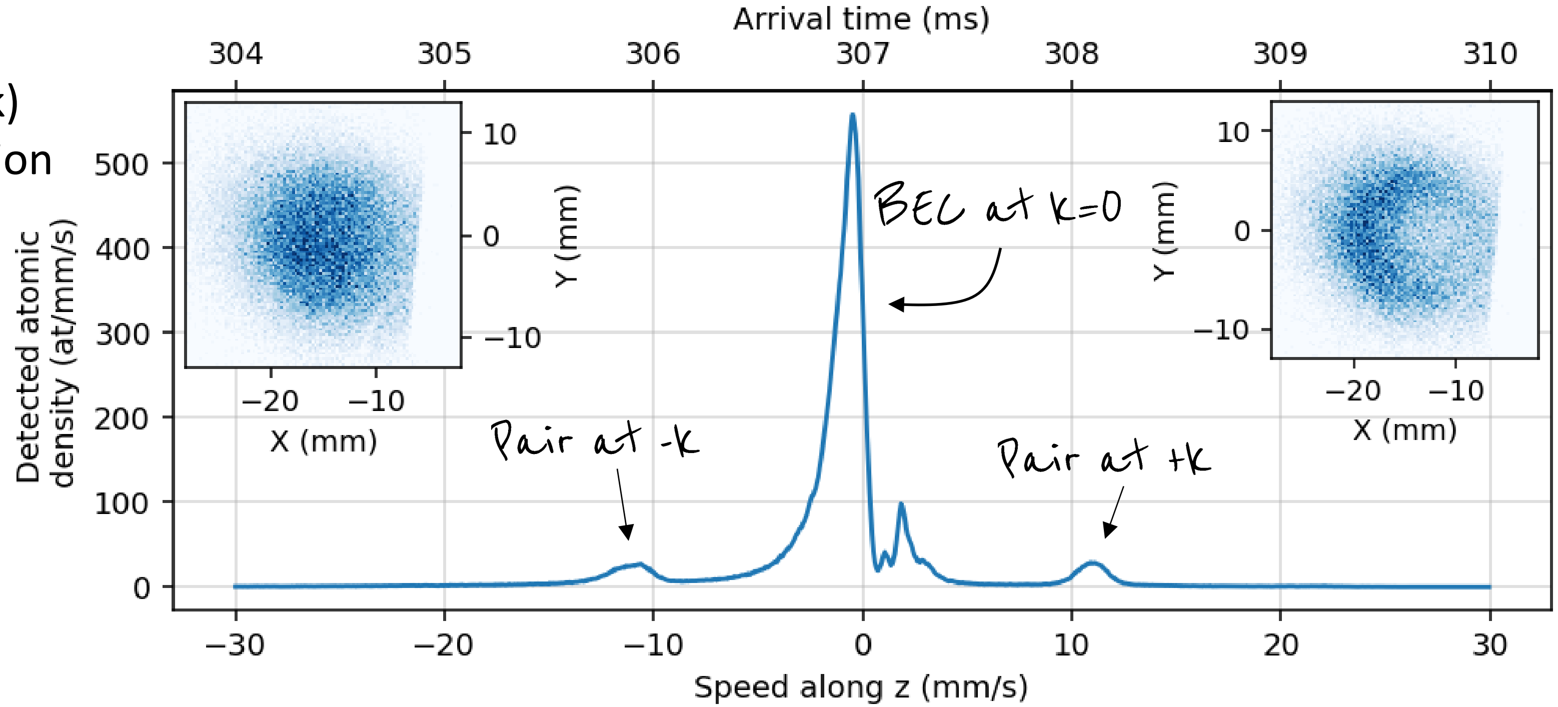
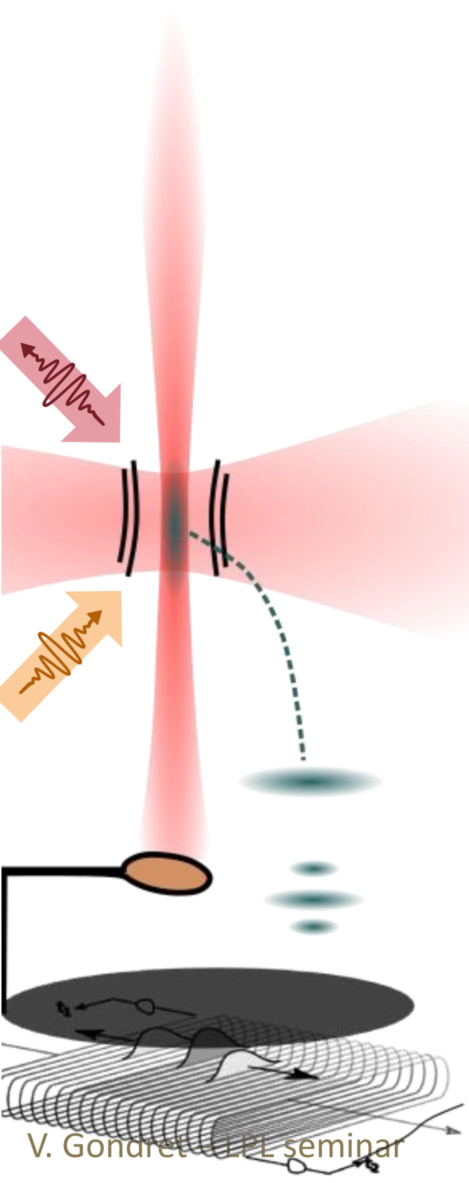


The BEC saturation affects the 2nd pair detectivity….

Use a velocity selective two-photon process to deflect only the BEC.

Pulse shaping techniques



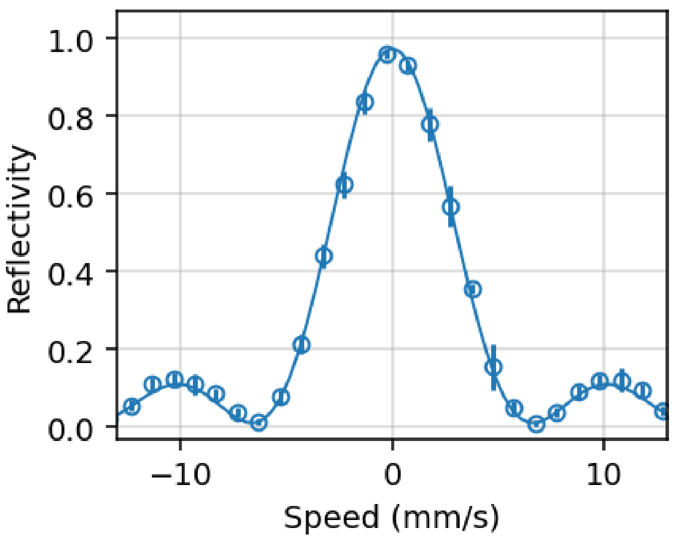
Protocol

1. Parametric excitation
2. Raman transfer (+kick)
3. Bragg deflection of the BEC
4. Single particle detection

Two-photon transition couples two momenta $|k\rangle \leftrightarrow |k + k_B\rangle$

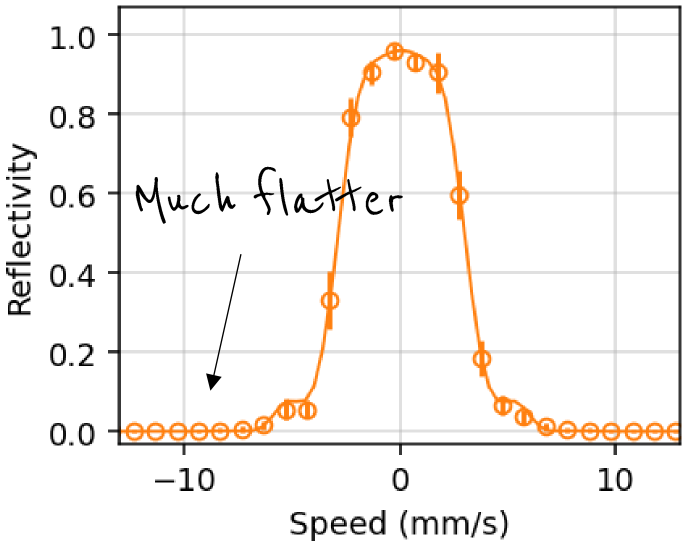
Use a velocity selective two-photon process to deflect only the BEC.

π pulse with constant Rabi frequency



(which looks like a |sinc| function)

Time dependant Rabi freq as a sinc function



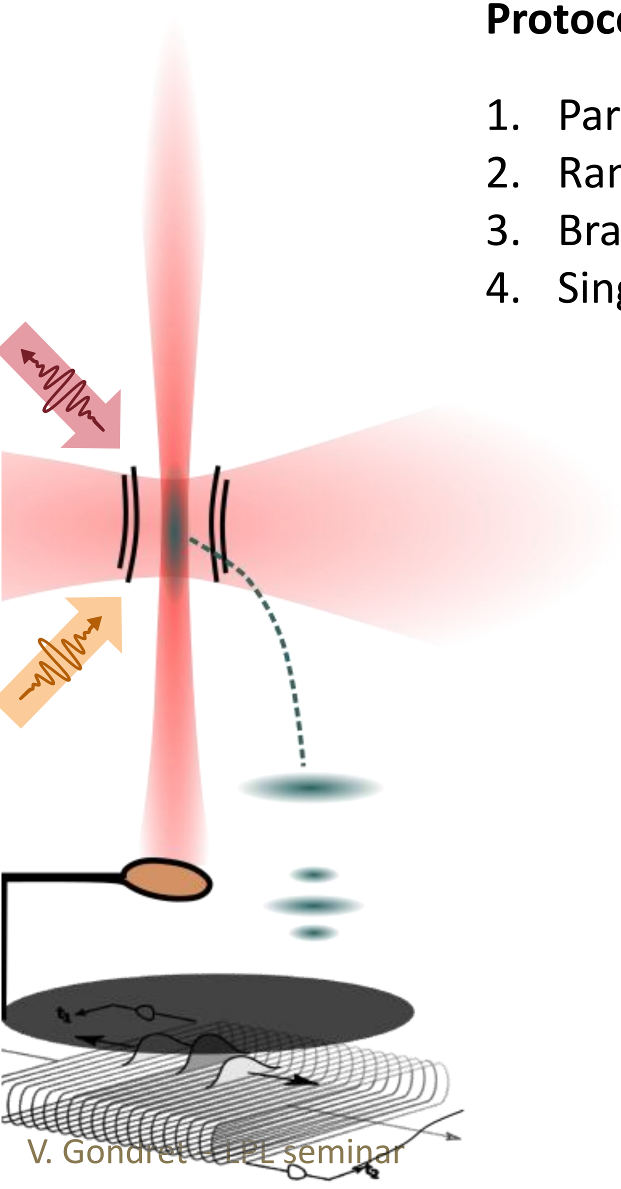
(which looks more like a square)

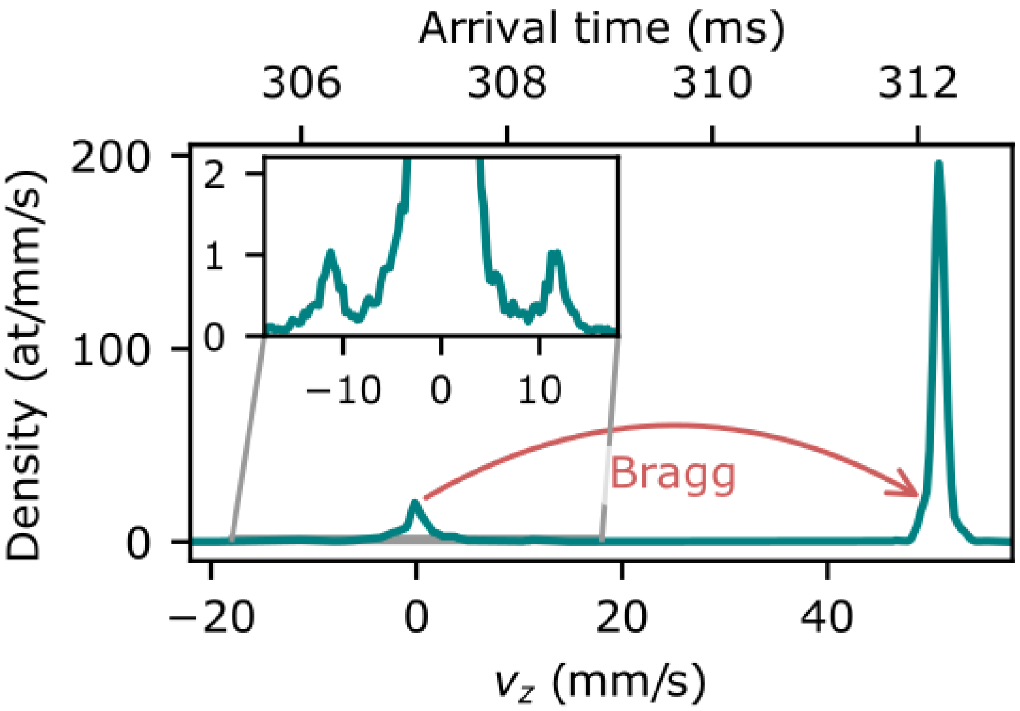
Leprince *et al* Phys. Rev. A **111**, 063304 (2025)

Bragg deflection of the BEC

Protocol

1. Parametric excitation
2. Raman transfer (+kick)
3. Bragg deflection of the BEC
4. Single particle detection analytical





In the following, we use a pulse-shaped Bragg deflector

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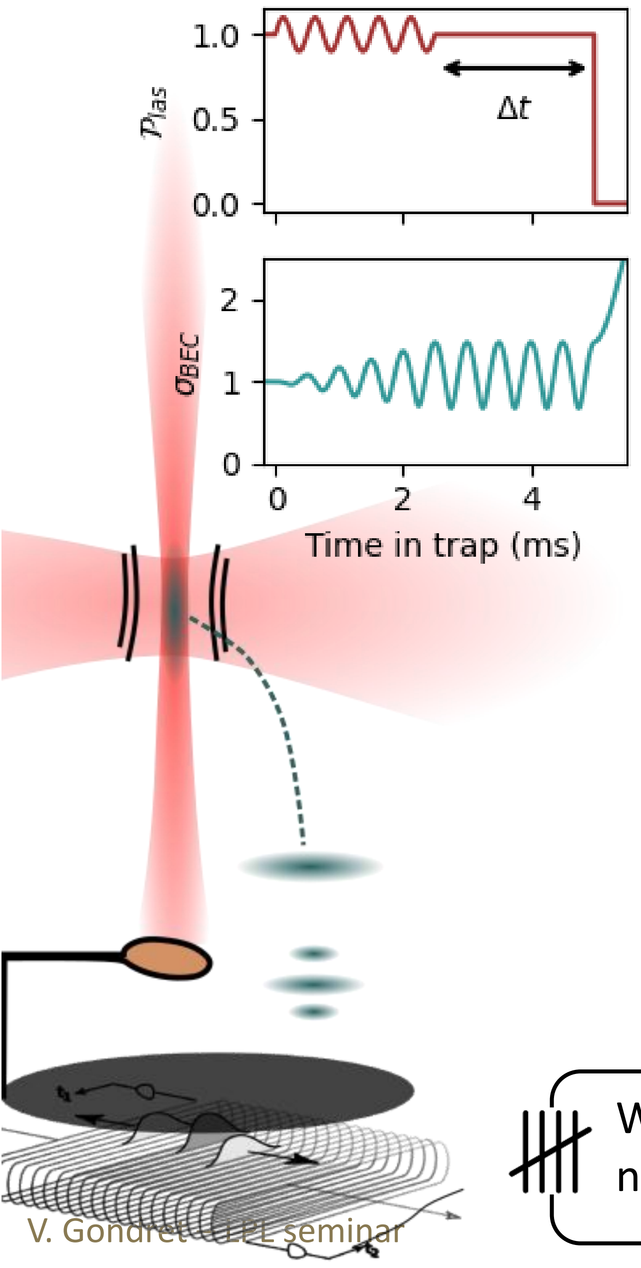
Exponential growth of the phonon number



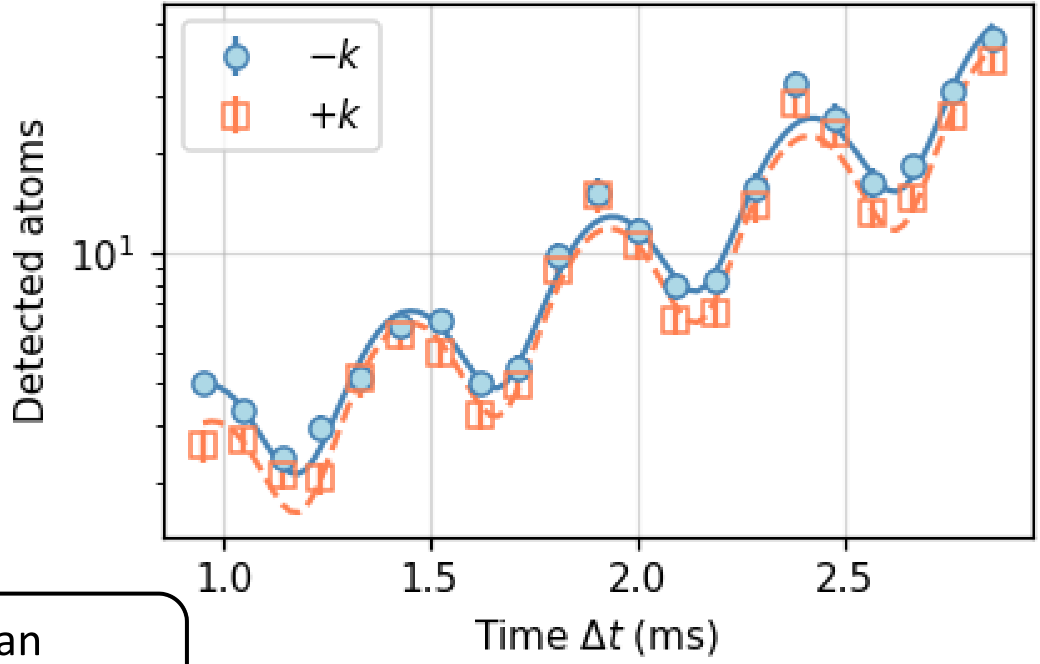
Fit function:

Theoretical quasi-particle growth dynamics

Empirical oscillation

$$n_k(t) = \eta\left[n_k^{(in)} + \left(n_k^{(in)} + n_{-k}^{(in)} + 1\right)\sinh^2(G_k(\Delta t - t_0)/2)\right] \times (1 + A_k\cos(2\omega_k\Delta t + \varphi))$$

Efficiency — thermal — vacuum — Fluctuations — Growth rate — Oscillation amplitude — Quasi-particle frequency



Fit parameters:

$G_k, n_k^{(in)}, t_0, A_k, \omega_k$

We count the mean number of atoms n_k, n_{-k}

Measuring the growth rate

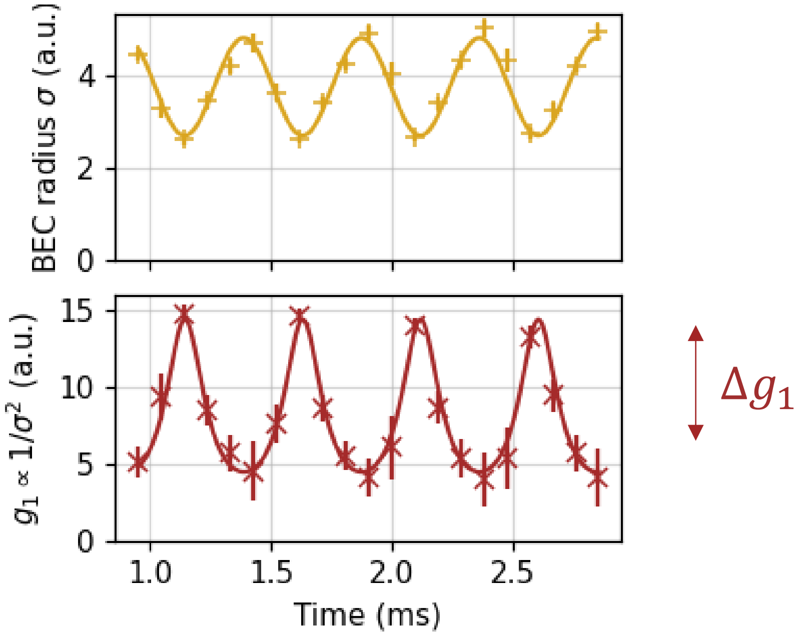
We fit the growth rate $G_k^{(exp)}$ from the population growth.

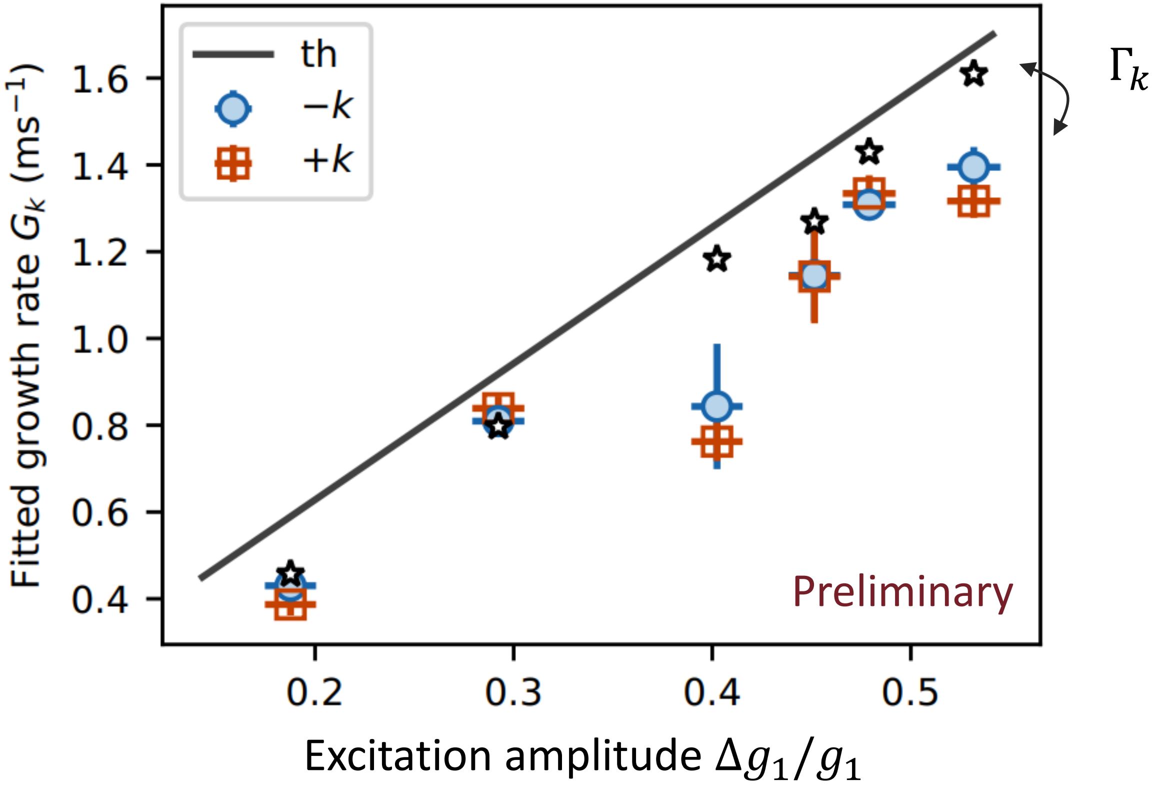
Measure the theoretical growth rate from the BEC

$$G_k^{(th)} = \frac{\omega_k}{2}\frac{\Delta g_1/g_1}{1+k^2\xi^2}$$

BEC healing length

Busch *et al.* PRA (2014)



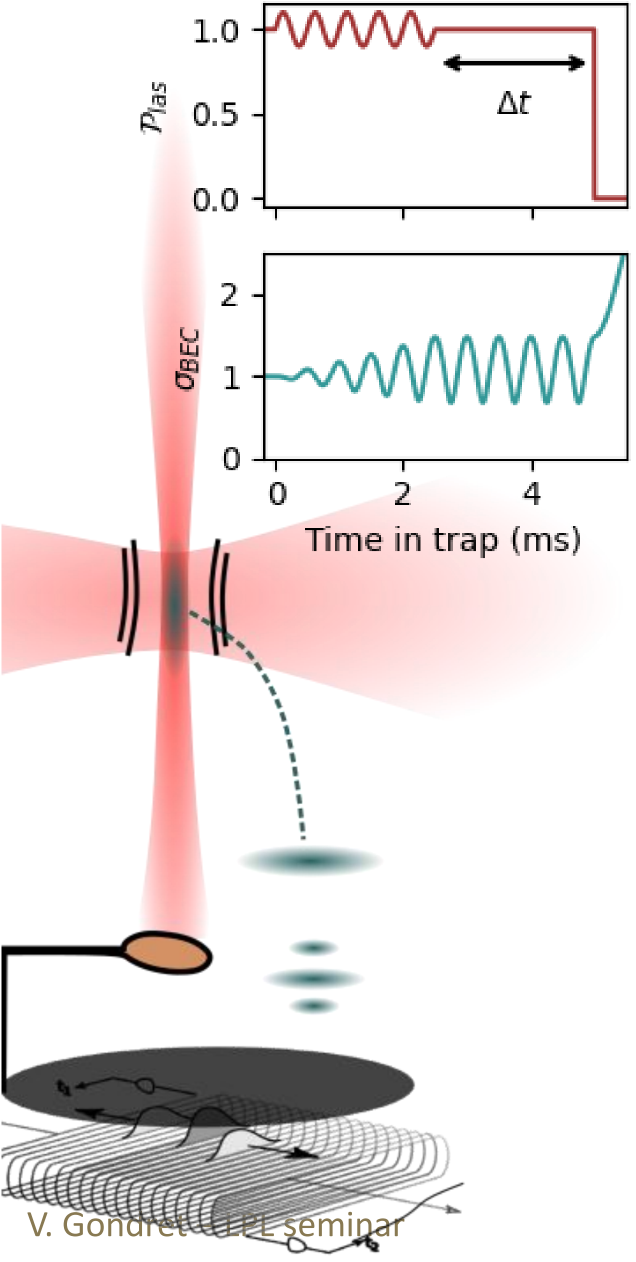


Beyond Bogoliubov quasiparticle interactions decrease the growth rate $\Gamma_k = G_k^{(th)} - G_k^{(exp)}$.

The slowing of the growth (*i.e.* the decay rate) we measure in qualitative agreement with theoretical predictions (black stars, large error bars not shown).

Micheli & Robertson PRD (2022)

Exponential growth of the phonon number



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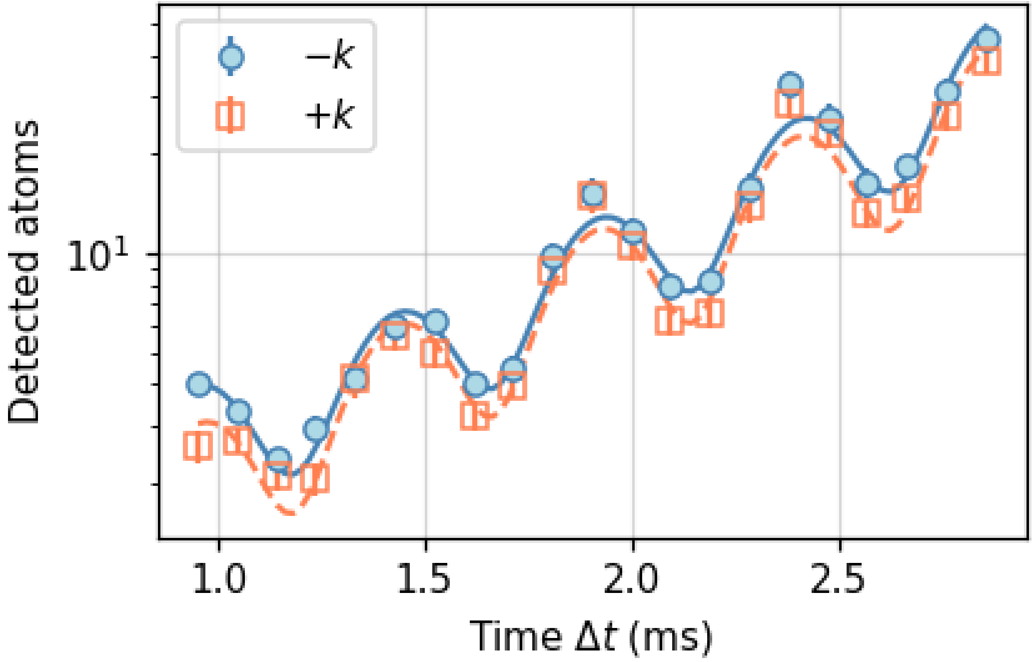
thermal vacuum

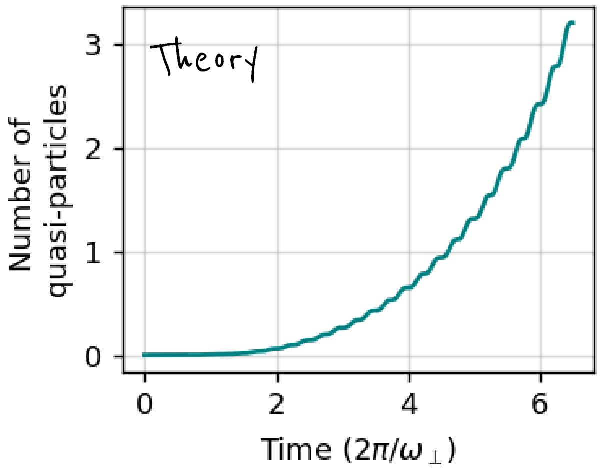
Fluctuations

Growth rate

Oscillation amplitude

Quasi-particle frequency





Such oscillation in the quasi-particle growth is not expected…

Adiabatic mapping from the collective excitation basis to the atomic basis

We measure atoms and not quasi-particles : how does the *collective excitations* state $\hat{b}_k$ maps to the *atomic* state $\hat{\phi}_k$?

In situ: $\hat{b}_k$

$$\omega_k = \sqrt{2 g_1 n_1 \frac{\hbar^2 k^2}{2m} + \left(\frac{\hbar^2 k^2}{2m}\right)^2}$$

At the detector: $\hat{\phi}_k^{det}$

$$\omega_k = \frac{\hbar^2 k^2}{2m}$$

Atomic field: $\hat{\phi}_k \sim u_k \hat{b}_k + v_k \hat{b}_{-k}^\dagger$

If ω_k changes adiabatically w.r.t. ω_k^{-1}:

$$\hat{b}_k \sim \hat{\phi}_k^{det}$$

The detected atom number:

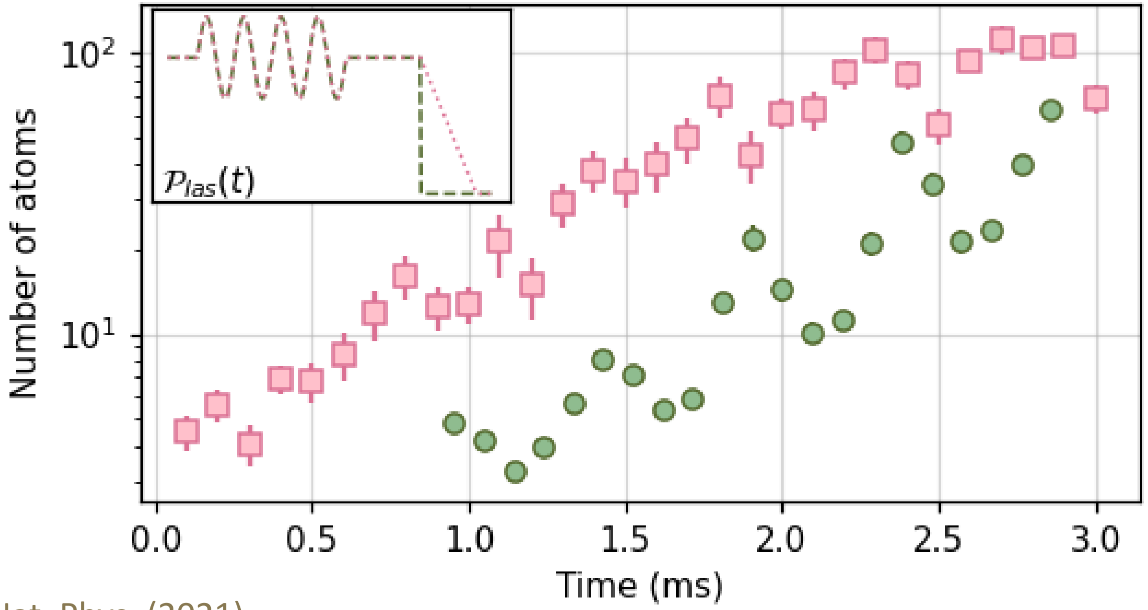
At equilibrium: thermal and quantum depletion[1,2]

$$n_k = \langle \hat{\phi}_k^\dagger \hat{\phi}_k \rangle = |u_k|^2 \langle \hat{b}_k^\dagger \hat{b}_k \rangle + |v_k|^2 \left(\langle \hat{b}_{-k}^\dagger \hat{b}_{-k} \rangle + 1 \right)$$
$$+ 2\,\mathrm{Re}\left(u_k v_k^* \langle \hat{b}_{-k} \hat{b}_k \rangle\right)$$

with $\hat{b}_k \sim \hat{b}_k^{(out)} e^{-i\omega_k t}$

It oscillates

- $\langle \hat{b}_{-k} \hat{b}_k \rangle \neq 0 \Rightarrow$ pair creation process
- We don't measure quasiparticles



Robertson *et al* PRD (2017)

[1] Lopes *et al.* PRL (2017), [2] Ténart *et al.* Nat. Phys. (2021)

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What is entanglement?

HOW?

Just violate a Bell inequality

Bell *Physics* (1964)
CHSH *Phys. Rev. Lett*. (1969)

Entanglement $\Leftrightarrow$ Bell inequalities
$\Leftrightarrow$ Distillability
$\Leftrightarrow$ Teleportation

EQUIVALENCE ONLY FOR PURE STATES

Gisin, *Phys. Lett. A* (1991)
Gisin & Peres, *Phys. Lett. A* (1992)
Popescu & Rohrlich, *Phys. Lett. A* (1992)

WHAT ABOUT *MIXED* STATES?

Teleportation $\not\Rightarrow$ Bell inequalities

Popescu Phys. Rev. Lett. (1994)

Define a partition 1-2 (two modes here). Any **separable** state can be written as

$$\rho = \sum_i \alpha_i \rho_{i,1} \otimes \rho_{i,2}$$

where $\alpha_i \geq 0$ are probabilities.

Other states are non-separable / entangled.

Werner *Phys. Rev. A* (1989)

How to probe entanglement?

SO HOW?

Many entanglement witnesses and criteria in the literature

PPT:

$$\hat{\rho}^{t_2} \geq 0$$

Peres, *Phys. Rev. Lett.* (1996)

$$|\langle \hat{a}_1 \hat{a}_2 \rangle|^2 \leq n_1 n_2$$

Hillery & Zubairy *Phys. Rev. Lett.* (2006)

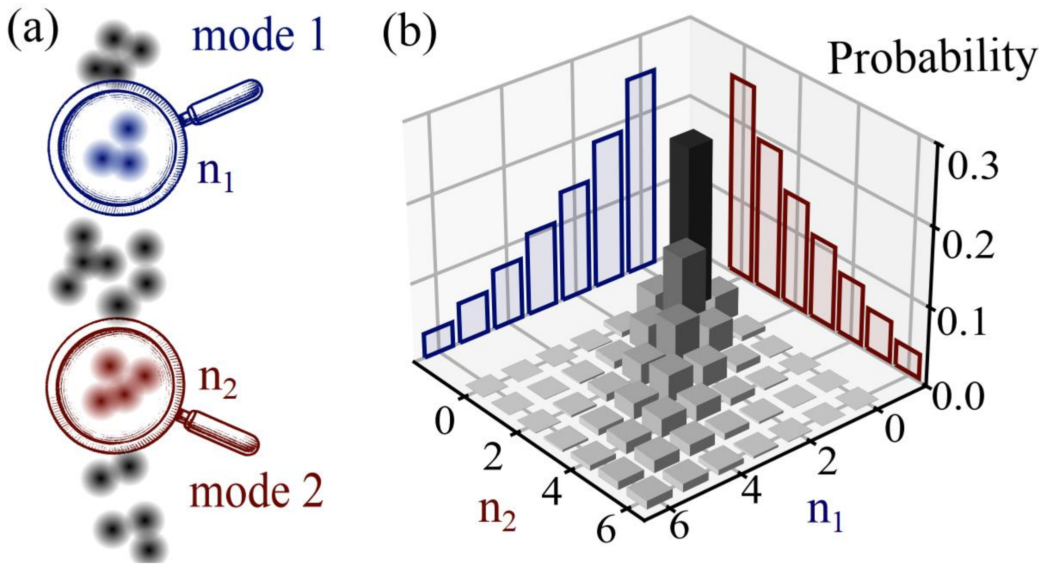
⋮

EXERIMENTAL TOOLS NEEDED

To measure the mean/variances of field operators, one needs homodyne-like detection schemes[1] or to reconstruct the state measuring non-commuting operators[2] (e.g. $\hat{x}$ and $\hat{p}$)

[1] Gross *et al.* Nature (2011)
[2] Bergschneider *et al.* Nat. Phys. (2019)

COULD WE USE THE FULL COUNTING STATISTICS?



Yields any order of *particle number* correlation function $G_{12}^{(m,p)} = \left\langle \left(\hat{a}_1^\dagger\right)^m \left(\hat{a}_2^\dagger\right)^p \hat{a}_1^m \hat{a}_2^p \right\rangle$

See also Barasiński *et al* PRL (2023)

Probing the entanglement of a TMSv state from its FCS

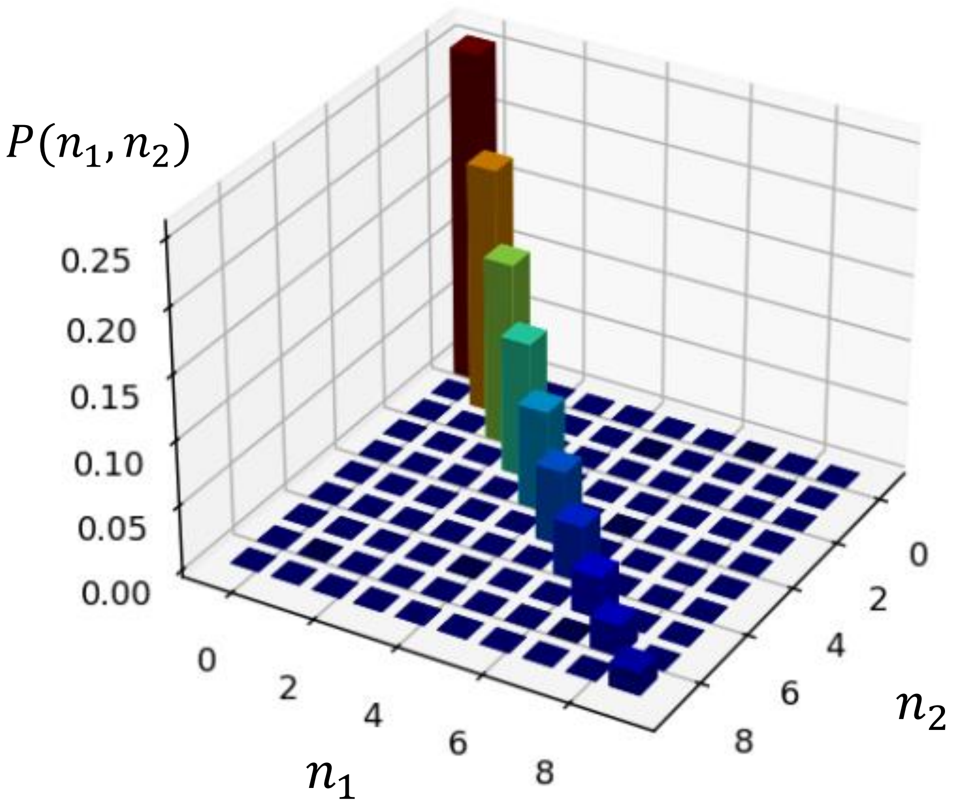
Consider a two-mode squeezed vacuum state

$$|TMSv\rangle(r) \sim \sum_i \tanh^i r\, |i,i\rangle_{12}$$

$$\rho_{TMSv} \sim \sum_{i,k} \tanh^i r \tanh^k r\, |i,i\rangle_{12}\langle k,k|_{12}$$

ρ_{TMSv} is a non-separable state in the partition 1-2.

Can we prove the entanglement of this state from its FCS?



Ex: the state describe by

$$\rho_{sep} \sim \sum_i \tanh^i r\, |i,i\rangle\langle i,i|$$

is a separable state which has the same two-mode probability distribution as a TMSv.

Entanglement from the FCS

One cannot assess the entanglement of *any* quantum state from its full counting statistics.

It only measures the diagonal terms of the density matrix

THANK YOU FOR YOUR ATTENTION !

Wait a minute… Not true for *Gaussian* states!

Probing entanglement

We can connect N-body correlation functions to 1- and 2-field correlation functions!

GAUSSIAN STATES

A Gaussian state: $G_C^{(n>2)}(\hat{a}_1^\dagger \dots \hat{a}_2) = 0.$

[Gaussianity is preserved under evolution of 2nd order Hamiltonian (including Bogoliubov theory).]

PROPERTIES

Any operator that involves more than 2 fields can be expressed with 1- and 2-field operators.

$$G_{12}^{(2)} = \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle = n_1 n_2 + \underbrace{|\langle \hat{a}_1 \hat{a}_2 \rangle|^2}_{\text{Anomalous correlation}} + \underbrace{\left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2}_{\text{Coherence}}$$

LINK TO ENTANGLEMENT

If $\langle \hat{a}_1^\dagger \hat{a}_2 \rangle = 0$, observation of

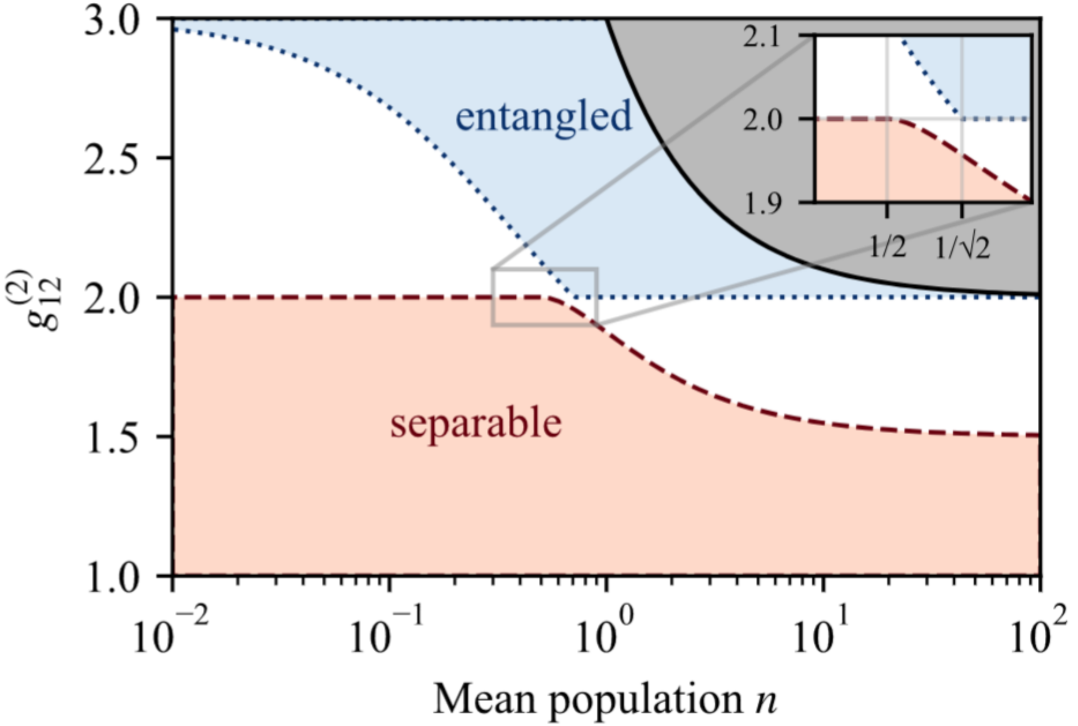
$$g_{12}^{(2)} = G_{12}^{(2)}/n_1 n_2 > 2$$

implies entanglement because $n_1 n_2 < |\langle \hat{a}_1 \hat{a}_2 \rangle|^2$

Hillery & Zubairy *Phys. Rev. Lett.* (2006)

Our $g^{(2)}$ WITNESS

For two-mode Gaussian states with each mode having thermal statistics, $g_{12}^{(2)}$ is an entanglement witness



+ the measurement of $g_{12}^{(2)}$ **quantifies** entanglement.

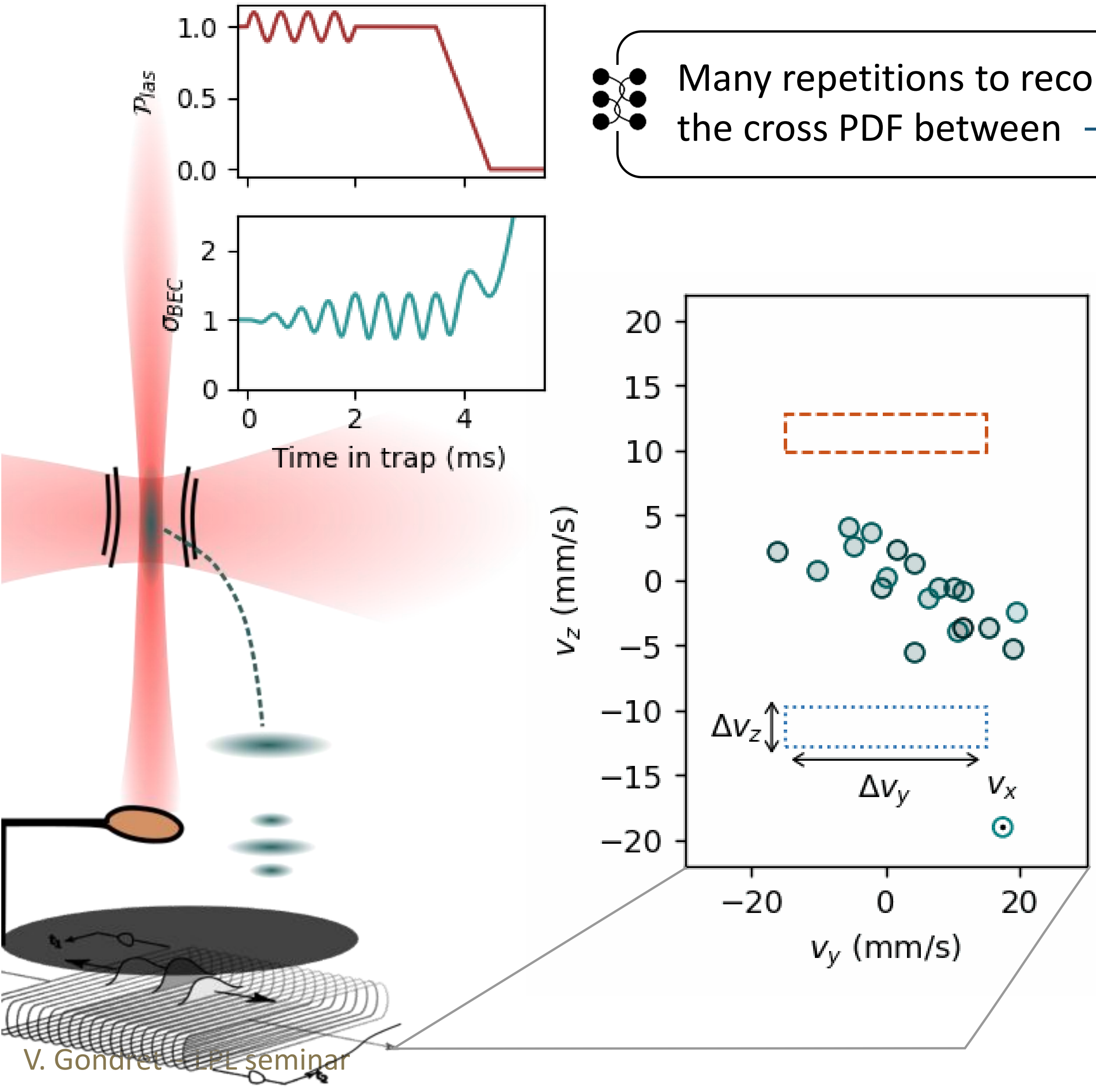
See arXiv 2503.09555

Outline

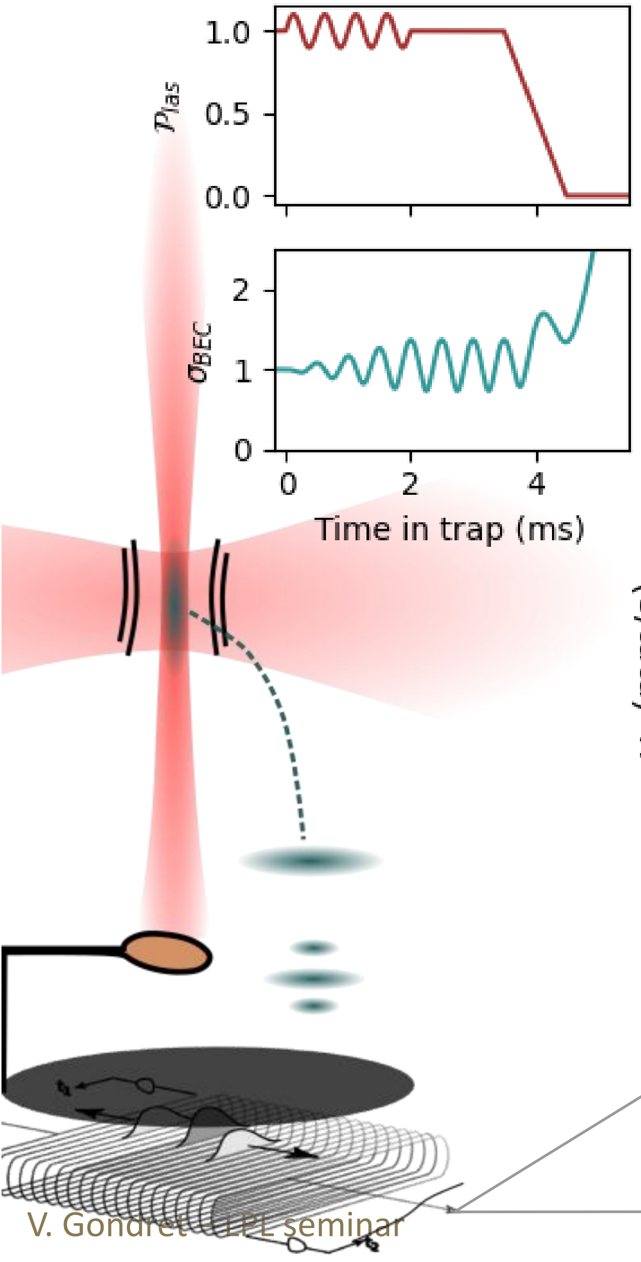
1. Parametric amplification of quasiparticles in an elongated BEC
2. Experimental setup and protocol
3. Observation of the growth and decay of quasiparticles
4. Quantifying entanglement from number correlation functions
5. Observation of quasiparticle entanglement

Experimental procedure

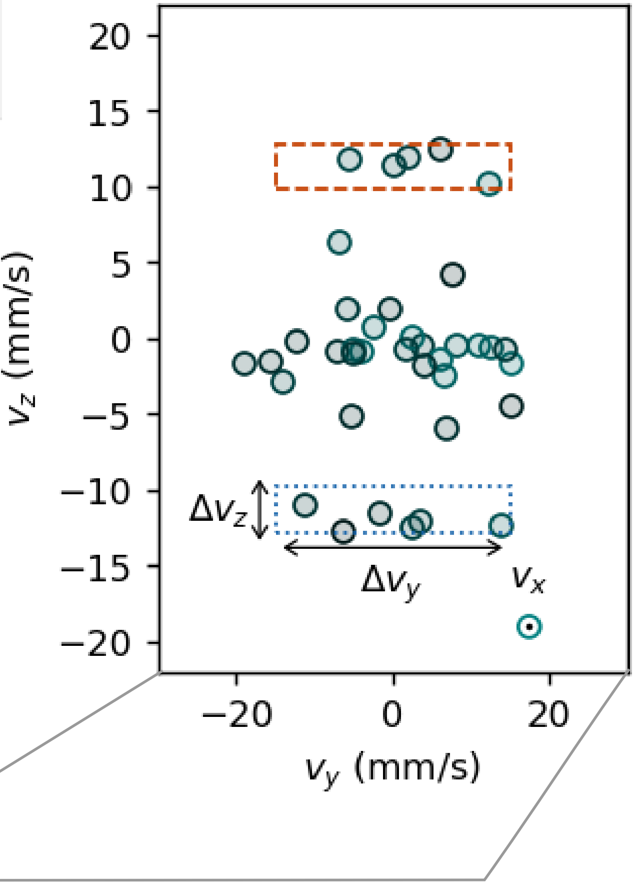
Many repetitions to reconstruct the cross PDF between $-k$ & k.



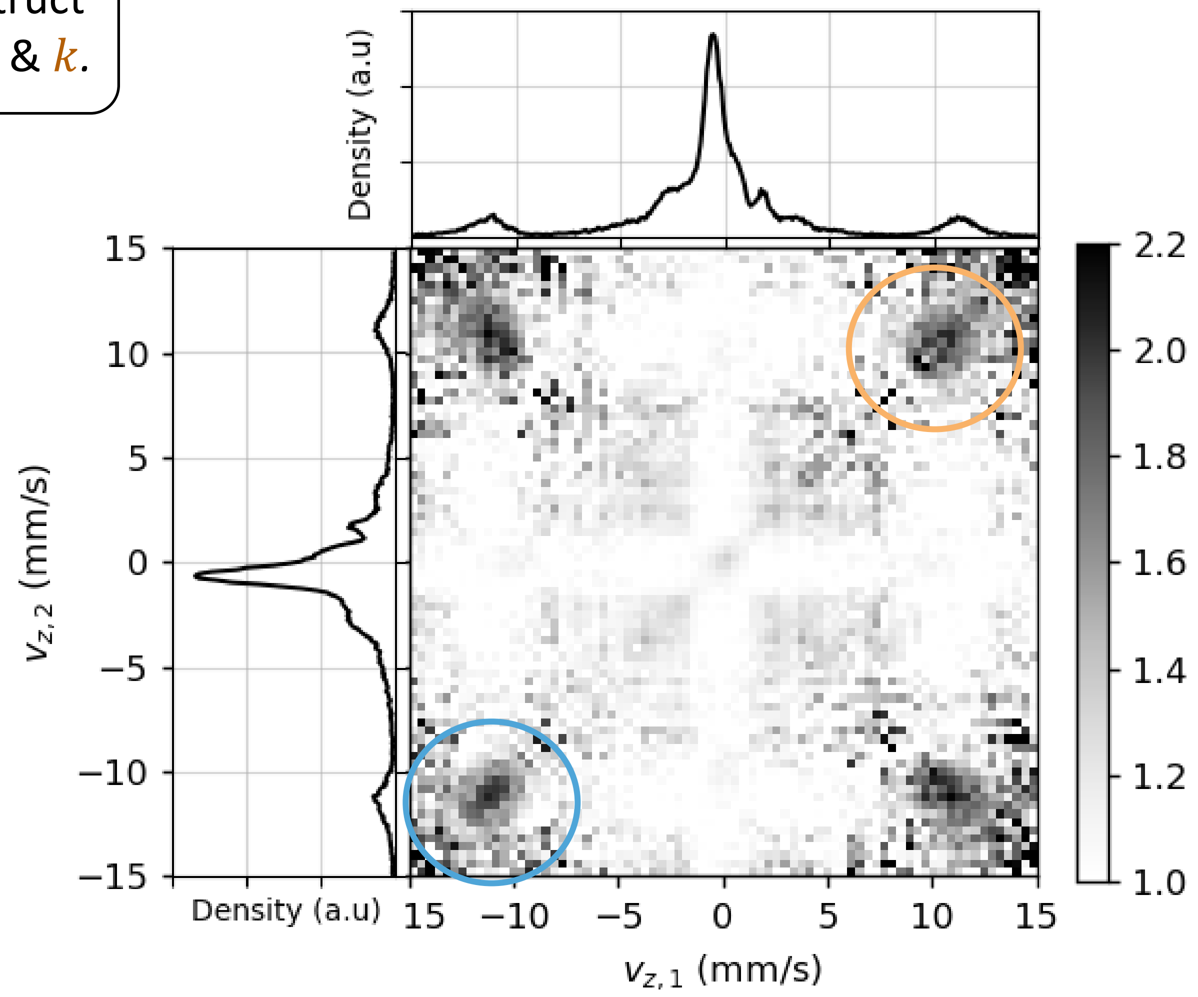
Experimental procedure



Many repetitions to reconstruct the cross PDF between $-k$ & k.



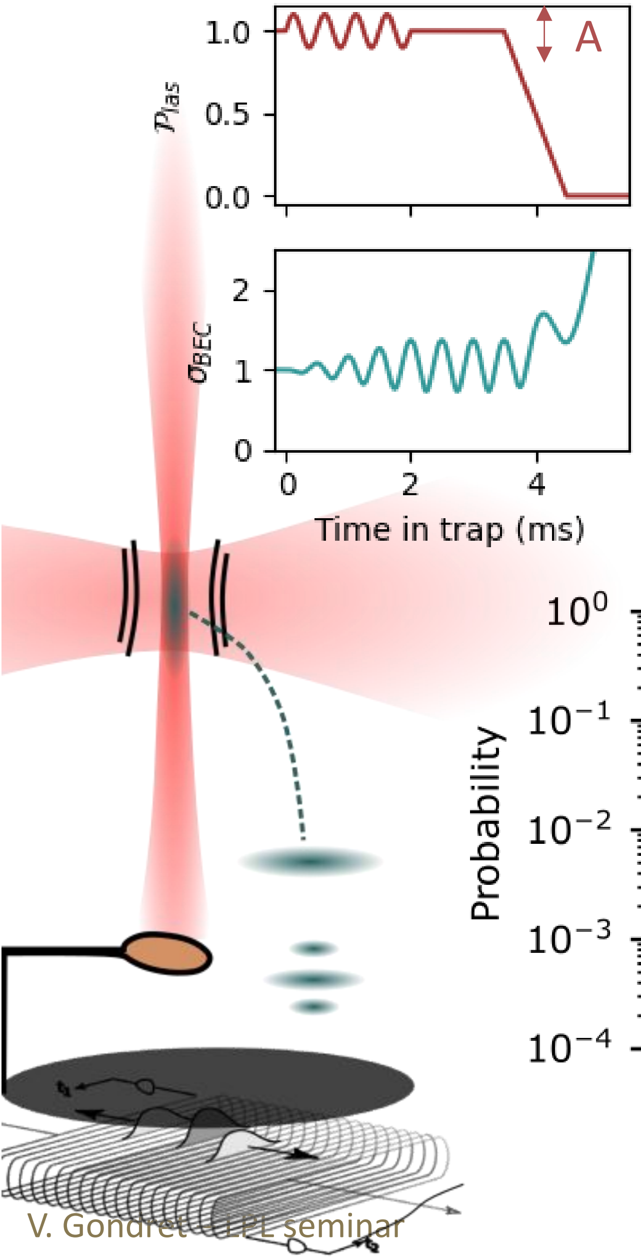
Map of $g^{(2)}(v_{z,1}, v_{z,2}) = \frac{\langle \hat{n}_1 \hat{n}_2 \rangle}{\langle \hat{n}_1 \rangle \langle \hat{n}_2 \rangle}$



$g^{(2)}_{-k,-k} = g^{(2)}_{+k,+k} = 2.0(1)$ bosonic bunching

Set the size of a mode $\Delta k = 2\pi/L$

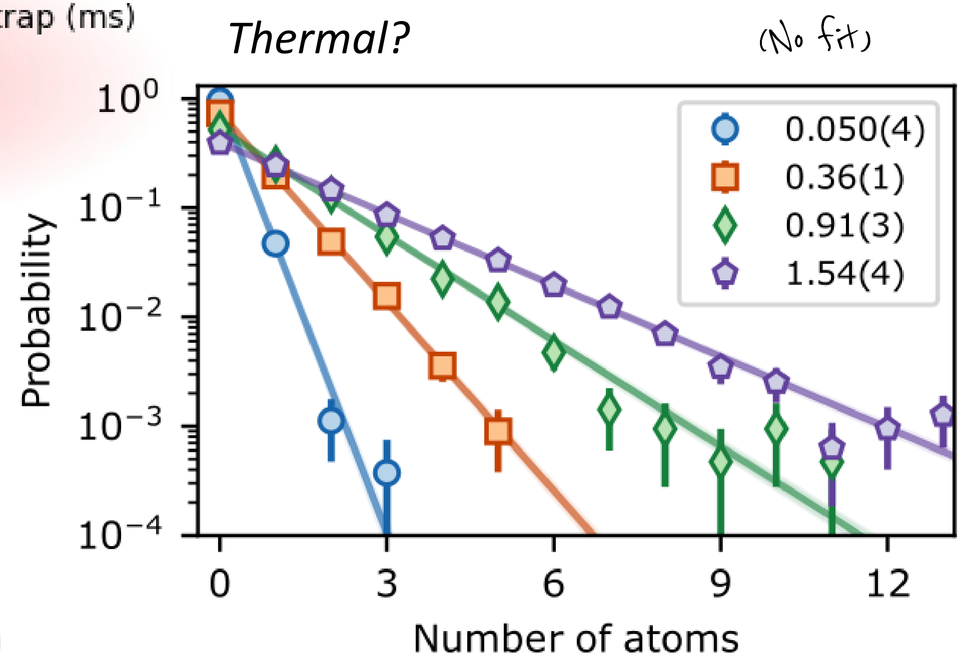
Entanglement between quasiparticles



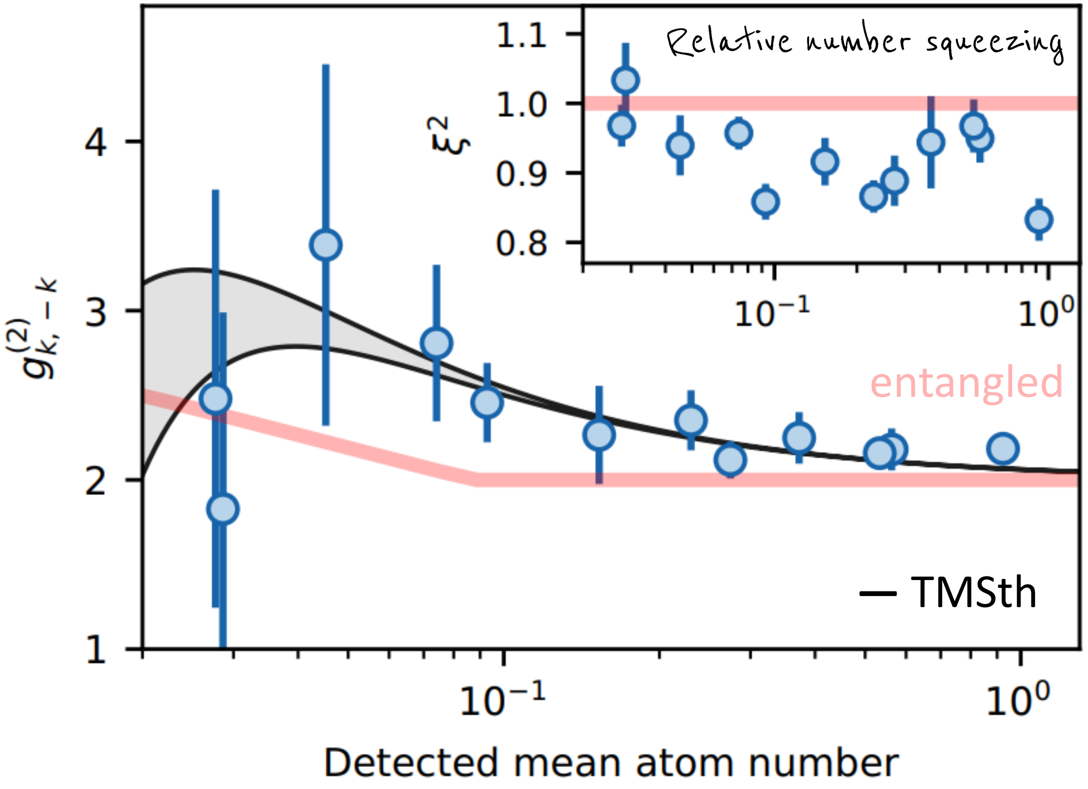
Protocol: repeat the experiment for various values of A.

Gaussian?
Numerical checks that (some) connected correlation functions vanish for $n \in [3,8]$.

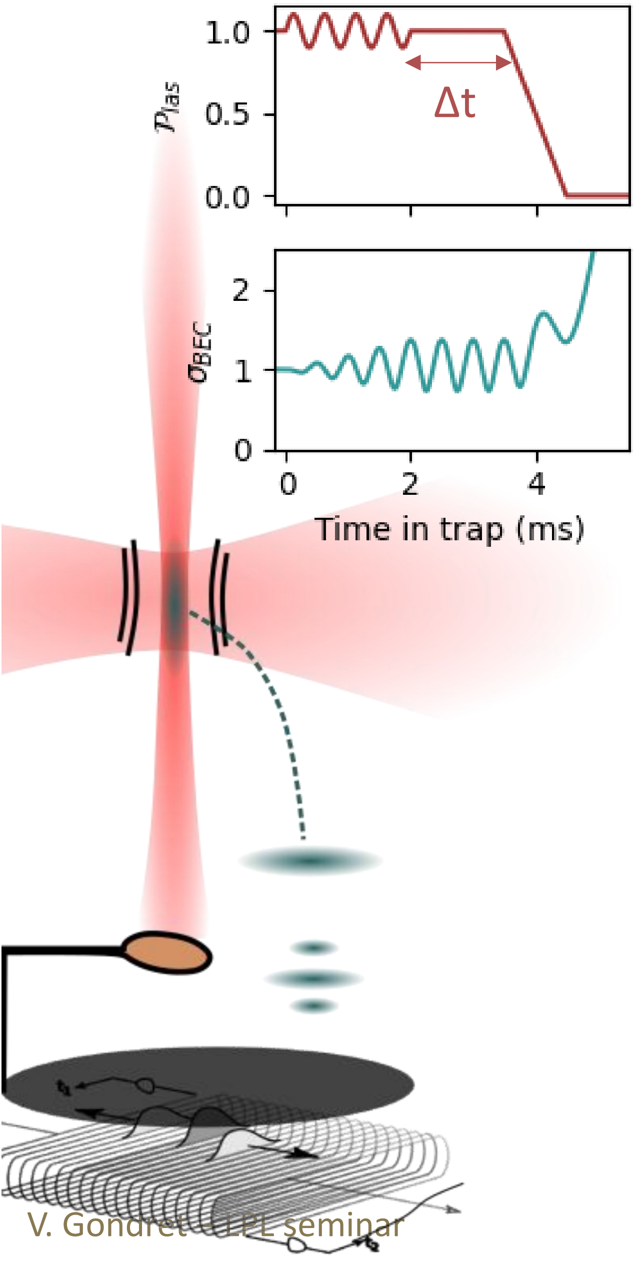
Thermal?



☒ Gaussian state,
✓ Each mode has a PDF which is thermal

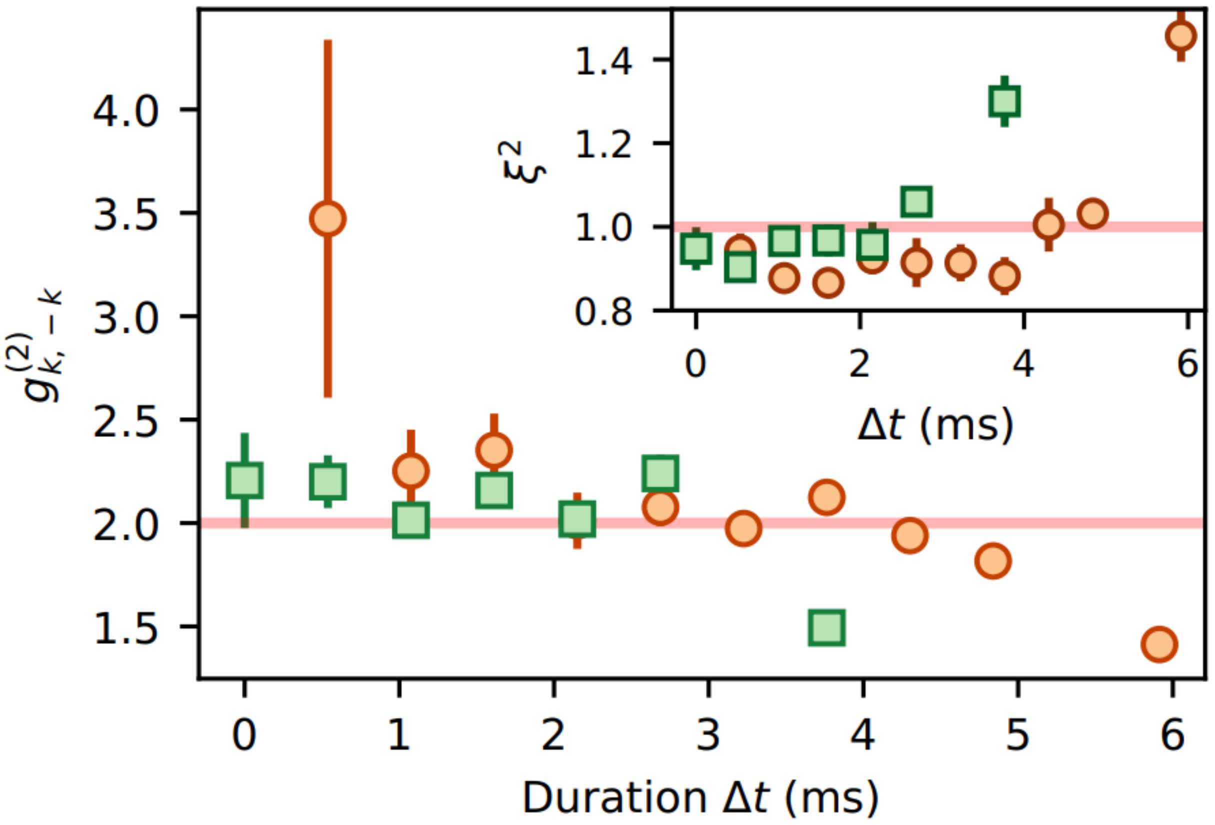


How does ~~entanglement~~ correlations evolve in time?



Relative number squeezing

$$\xi^2 = \frac{\mathrm{Var}(n_k - n_{-k})}{n_k + n_{-k}}$$



Circles: A = 18%   Squares: A = 25%

Late time: decrease of non-classical correlation.

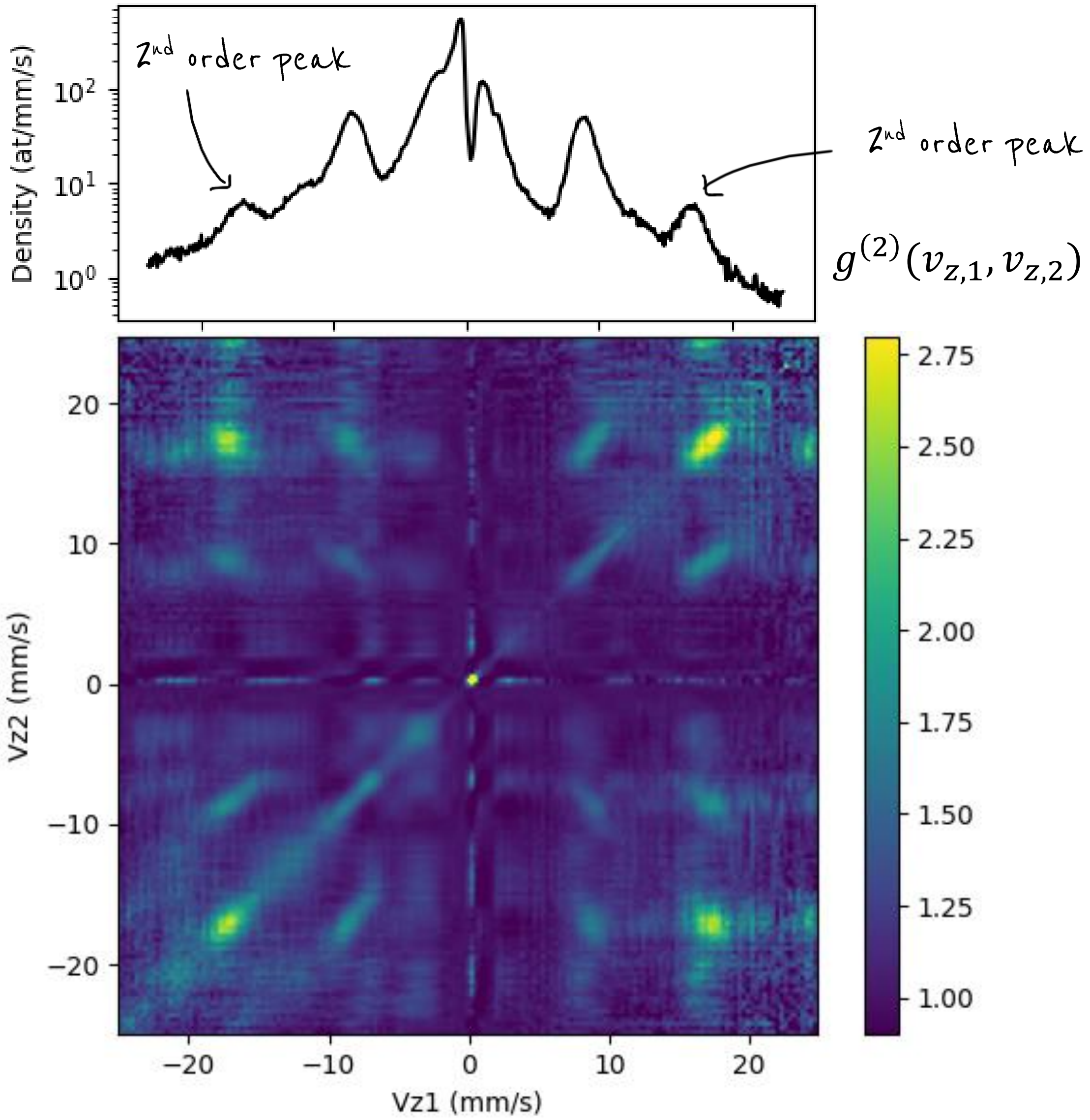
Hard to speak about entanglement, we certainly loose our ability to detect it:

- too many particles per mode (saturation for > 60)
- Beyond Bogoliubov physics: failure of the Gaussian hypothesis? Thermalization

Outlooks: beyond Bogoliubov

Correlation between non-opposite quasiparticles modes: beyond Bogoliubov model.

- Numerically observed by Robertson *et al* Phys. Rev. D **98**, 056003 (2018)
- Similar experiment in water tanks by Gregory *et al* arXiv:2410.08842 (2024)



Noun Project

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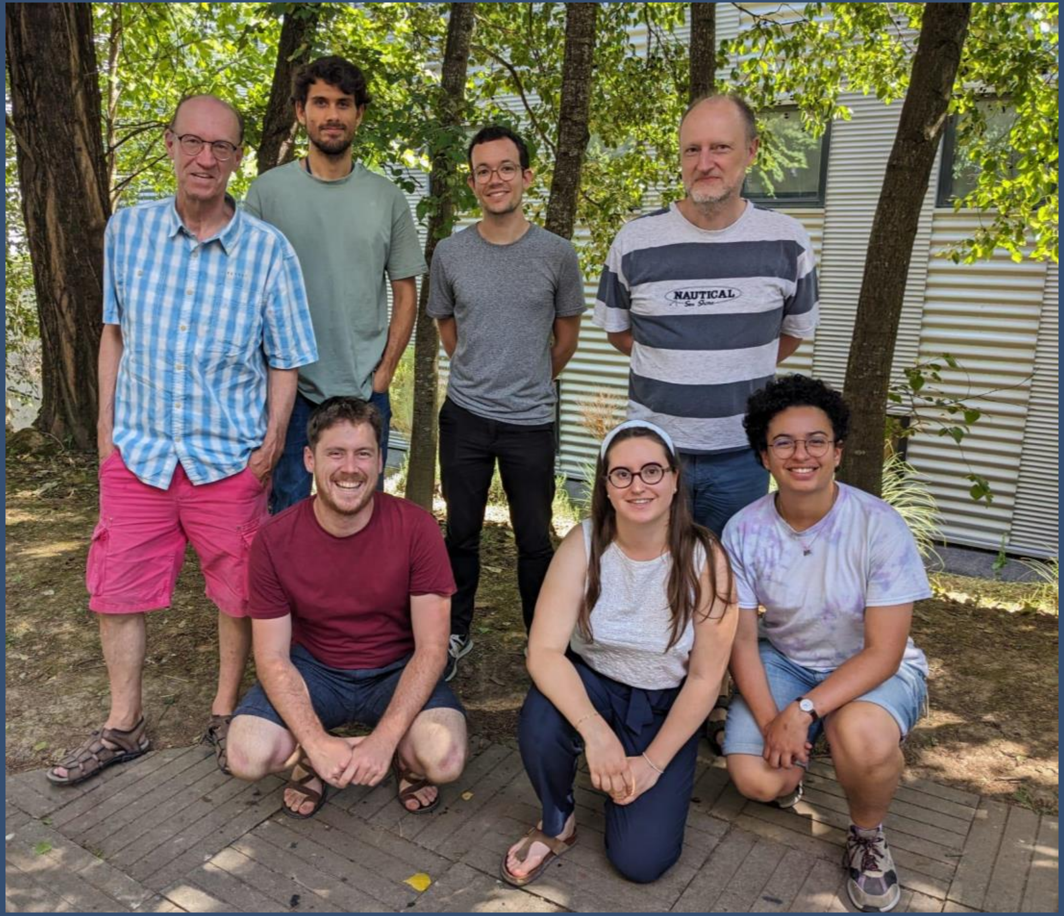
Thank you for your attention!

Chris Westbrook, Rui Dias, Charlie Leprince, Denis Boiron
Victor Gondret, Clothilde Lamirault, Léa Camier



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Gaussian states

We can connect N-body correlation functions to 1- and 2-field correlation functions!

GAUSSIAN STATES

A Gaussian state: $G_C^{(n>2)}(\hat{a}_1^\dagger \ldots \hat{a}_2) = 0.$

[Gaussianity is preserved under evolution of 2nd order Hamiltonian (including Bogoliubov theory).]

LINK TO ENTANGLEMENT

If $\langle \hat{a}_1^\dagger \hat{a}_2 \rangle = 0$, observation of

$$g_{12}^{(2)} = G_{12}^{(2)}/n_1 n_2 > 2$$

implies entanglement because $n_1 n_2 < |\langle \hat{a}_1 \hat{a}_2 \rangle|^2$

Campo & Parentani, *Phys. Rev D* (2005)
Hillery & Zubairy *Phys. Rev. Lett.* (2006)

PROPERTIES

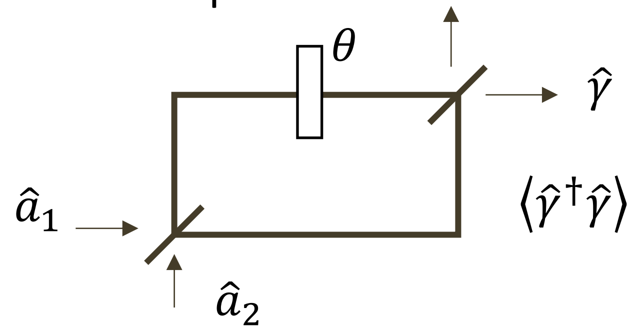
Any operator that involves more than 2 fields can be expressed with 1- and 2-field operators.

Ex:

$$G_{12}^{(2)} = \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle = n_1 n_2 + \underbrace{|\langle \hat{a}_1 \hat{a}_2 \rangle|^2}_{\text{Anomalous correlation}} + \underbrace{\left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2}_{\text{Coherence}}$$

HOW TO MEASURE THE COHERENCE?

Sol. 1: set up an interferometer



$$\langle \hat{\gamma}^\dagger \hat{\gamma} \rangle \propto \left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2 \sin 2\theta$$

Sol. 2: use higher-order correlation functions.

Entanglement criterion

ASSUMPTIONS

- ❑ Gaussian state,
- ✓ Each mode has a PDF which is thermal

→ As a result, $\langle \hat{a}_i \rangle = \langle \hat{a}_j^2 \rangle = 0$

Avagyan *et al* J. of Phys. B (2023)

TWO- AND FOUR-BODY CORRELATION FUNCTIONS

$$g_{12}^{(2)} = 1 + \left(|\langle \hat{a}_1 \hat{a}_2 \rangle|^2 + \left| \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \right|^2 \right) / n_1 n_2$$

$$g_{12}^{(4)} = \left\langle \left(\hat{a}_1^\dagger \hat{a}_2^\dagger \right)^2 (\hat{a}_1 \hat{a}_2)^2 \right\rangle / n_1^2 n_2^2$$

$$= f\left(G_{12}^{(2)}, n_1 n_2 \right) + 8\, |\langle \hat{a}_1 \hat{a}_2 \rangle|^2 \times \left| \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \right|^2 / n_1^2 n_2^2$$

Symmetric system to find $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ and $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle|$ yields two solutions $\beta_\pm$

TWO SOLUTIONS

We have two possible solutions

- "State" μ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_+$ & $\left| \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \right| = \beta_-$,
- "State" γ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_-$ & $\left| \langle \hat{a}_1^\dagger \hat{a}_2 \rangle \right| = \beta_+$.

A "physical" Gaussian state must respect an inequality based on the symplectic eigenvalues of its covariance matrix $\{\nu_\pm\} : \nu_- \geq 1$

Lemma 1. We can compute $\nu_-^{(\mu)}, \nu_-^{(\gamma)}$ from $n_1, n_2, \beta_\pm$

If only one is physical, we "know" the state

Lemma 2. States μ and γ are PT of each other

If only one is unphysical, the other is entangled!

Criterion: If "state" γ is unphysical, the state is entangled. (and entanglement is quantified with log neg)

Gondret et al arXiv (2025)

The $g^{(2)}/g^{(4)}$ entanglement criterion

THE $g^{(2)}/g^{(4)}$ CRITERION

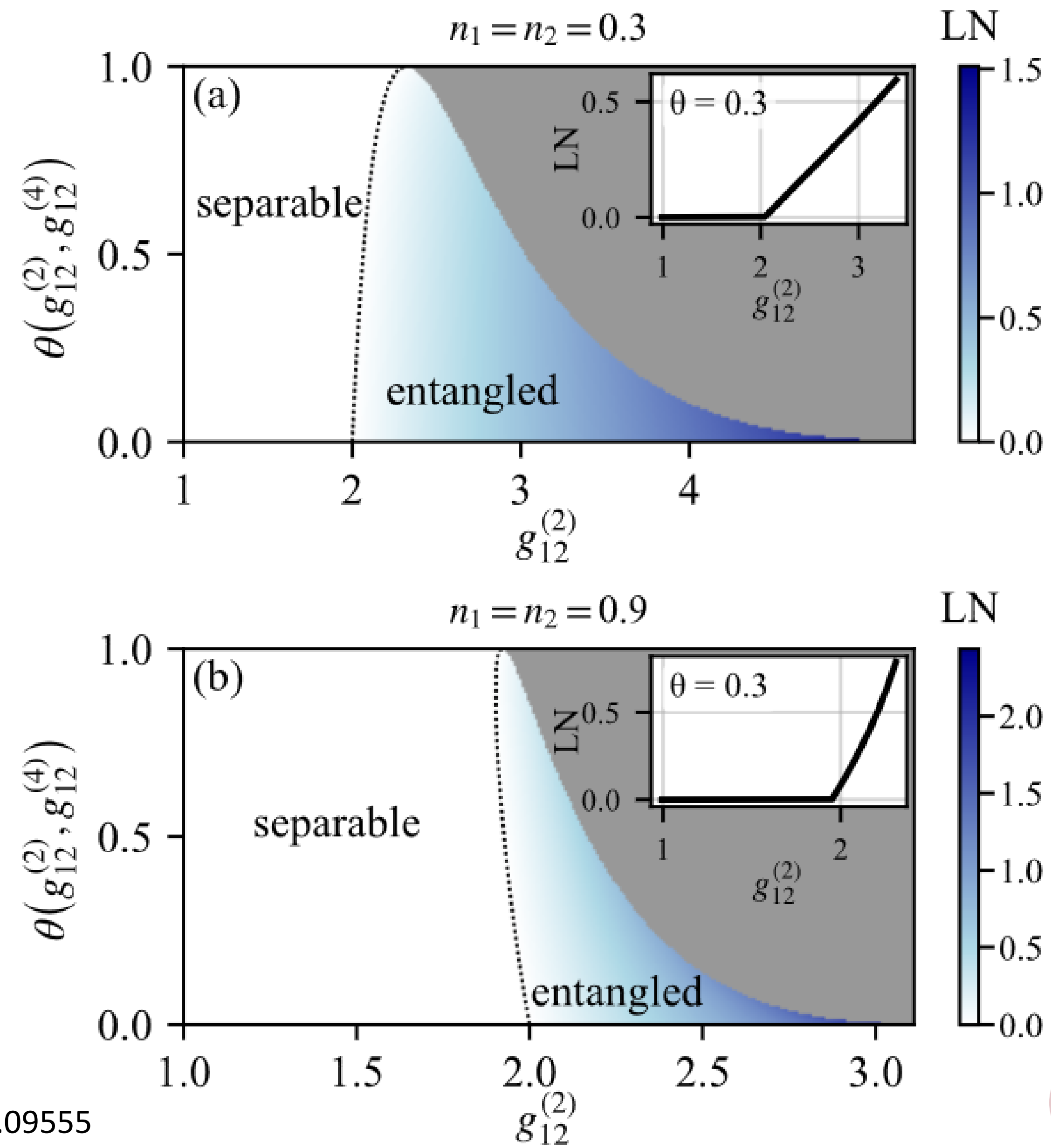
The measurement of $n_1, n_2, g_{12}^{(2)}, g_{12}^{(4)}$ yields λ_-, the smallest symplectic eigenvalue of the state and its PT. If $\lambda_- < 1$, the state is entangled.

$$\mathrm{LN} = \mathrm{Max}(-\log_2(\lambda_-), 0)$$

$$\theta = \frac{g_{12}^{(4)} + 12 - 16 g_{12}^{(2)} - 4\left(g_{12}^{(2)} - 1\right)^2}{\left(g_{12}^{(2)} - 1\right)^2} \in [0,1]$$

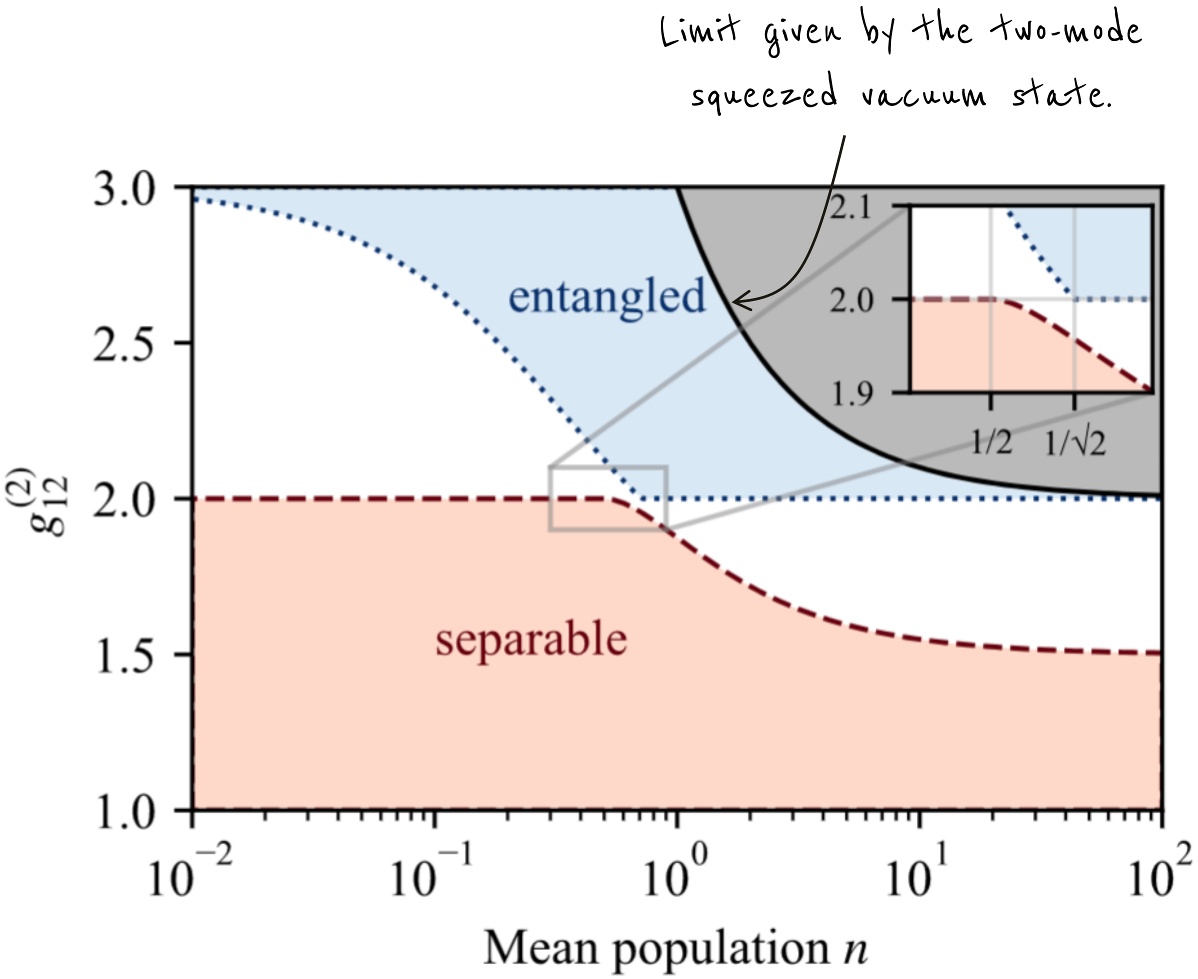
THE $g^{(2)}$ WITNESS

- Low population: if $g_{12}^{(2)} \leq 2$ separable state,
- High population : if $g_{12}^{(2)} \leq 2$ entangled state.



Formulas in Gondret *et al* arXiv:2503.09555

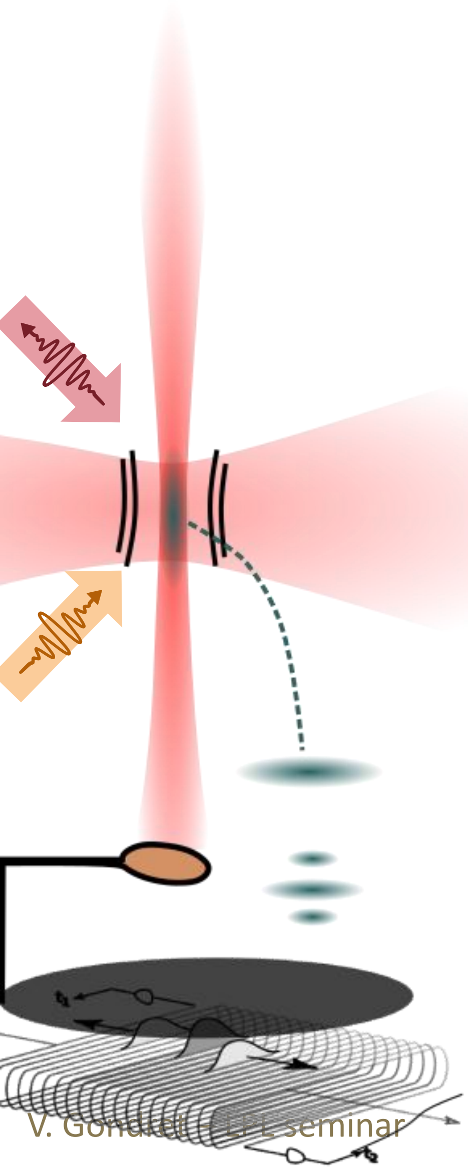
The $g^{(2)}$ entanglement witness



- The $g_{12}^{(2)}$ entanglement witness depends on the populations,
- The value of $g_{12}^{(4)}$ is needed to determine the entanglement in the middle region.
- Taking into account the quantum efficiency of the detector can reveal entanglement,

So what is the experimental result ??...

Some advertising on pulse shaping techniques



Protocol

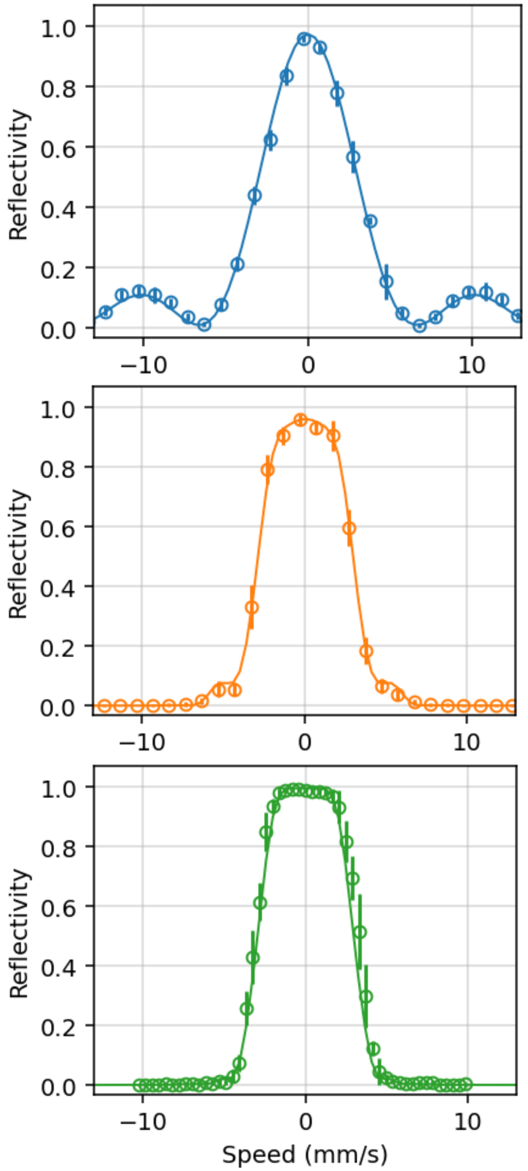
1. Parametric excitation
2. Raman transfer (+kick)
3. Bragg deflection of the BEC
4. Single particle detection analytical

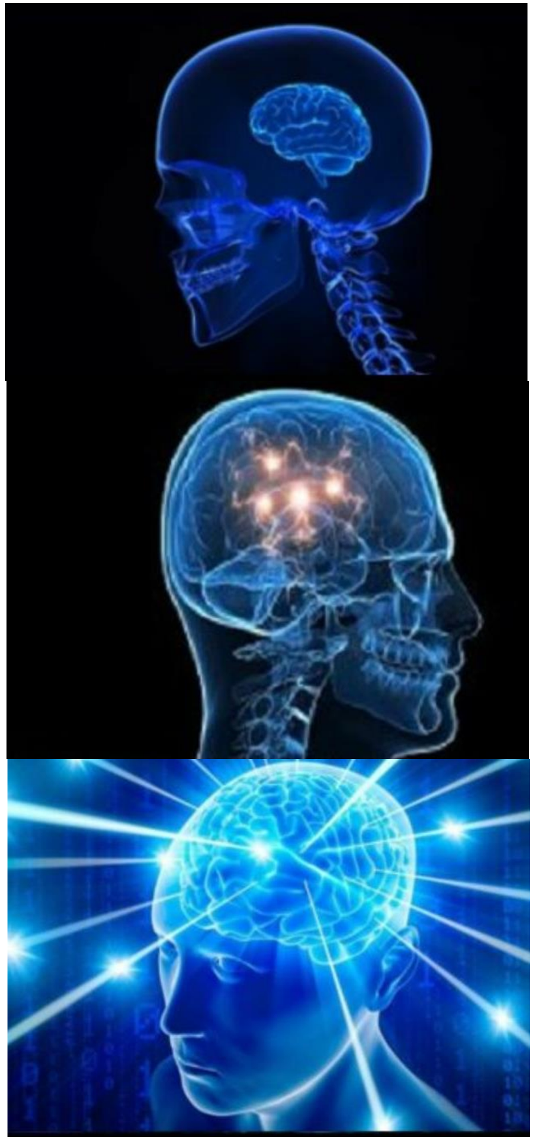
More information in

Leprince *et al* Coherent coupling of momentum states: selectivity and phase control, Phys. Rev. A **111**, 063304 (2025)

→ Analytical (smart) functions as good as optimal control but simpler.

+ applied to interferometry techniques





Constant pulse

Sinc pulse

Reburp pulse