Quantifying two-mode entanglement of bosonic Gaussian states from their full counting statistics

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arXiv:2503.09555

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Motivations

WHY?

Entanglement is a key signature of particle production by pairs from vacuum

HOW?

Just violate a Bell inequality

Bell *Physics* (1964)
CHSH *Phys. Rev. Lett.* (1969)

Entanglement $\Longleftrightarrow$ Bell inequalities
$\Longleftrightarrow$ Distillability
$\Longleftrightarrow$ Teleportation

EQUIVALENCE ONLY FOR PURE STATES

Gisin, *Phys. Lett. A* (1991)
Gisin & Peres, *Phys. Lett. A* (1992)
Popescu & Rohrlich, *Phys. Lett. A* (1992)

WHAT ABOUT *MIXED* STATES?

Teleportation $\not\Rightarrow$ Bell inequalities

Popescu Phys. Rev. Lett. (1994)

Define a partition 1-2 (two modes here). Any **separable** state can be written as

$$\rho = \sum_i \alpha_i \rho_{i,1} \otimes \rho_{i,2}$$

where $\alpha_i \geq 0$ are probabilities.

Other states are non-separable / entangled.

Werner *Phys. Rev. A* (1989)

Motivations

WHY?

Entanglement is a key signature of particle production by pairs from vacuum

SO HOW?

Many entanglement witnesses and criteria in the literature

PPT:

$$\widehat{\rho}^{t_2} \geq 0$$

Peres, *Phys. Rev. Lett.* (1996)

$$|\langle \hat{a}_1 \hat{a}_2 \rangle|^2 \leq n_1 n_2$$

Hillery & Zubairy *Phys. Rev. Lett.* (2006)

⋮

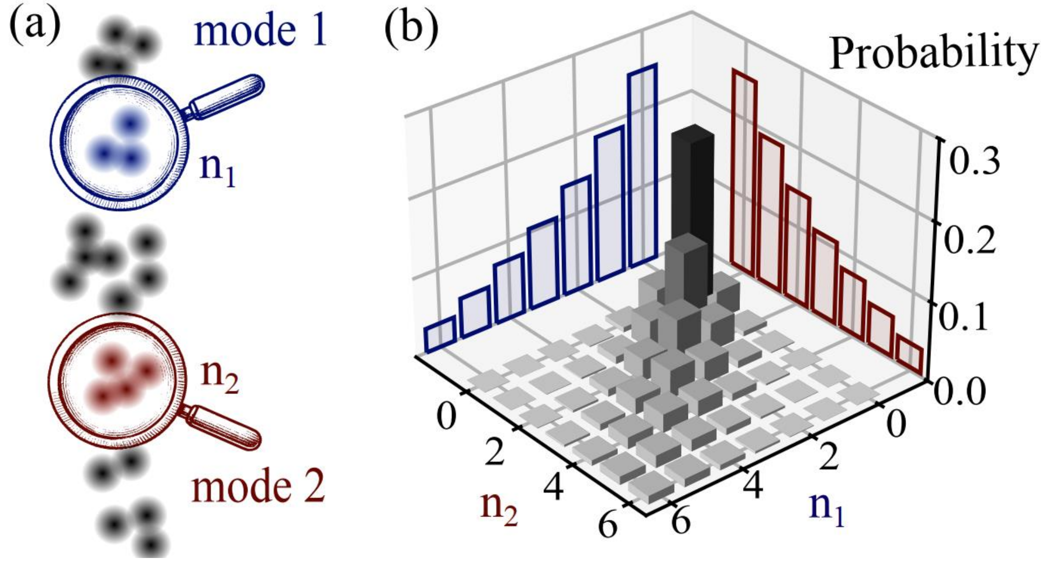
$\hat{a}_j$ annihilation operator for mode j

EXERIMENTAL TOOLS NEEDED

To measure the mean/variances of field operators, one needs homodyne-like detection scheme

Not always possible or easy

COULD WE USE THE FULL COUNTING STATISTICS?



Yields any order of *particle number* correlation function $G_{12}^{(m,p)} = \left\langle \left(\hat{a}_1^\dagger\right)^m \left(\hat{a}_2^\dagger\right)^p \hat{a}_1^m \hat{a}_2^p \right\rangle$

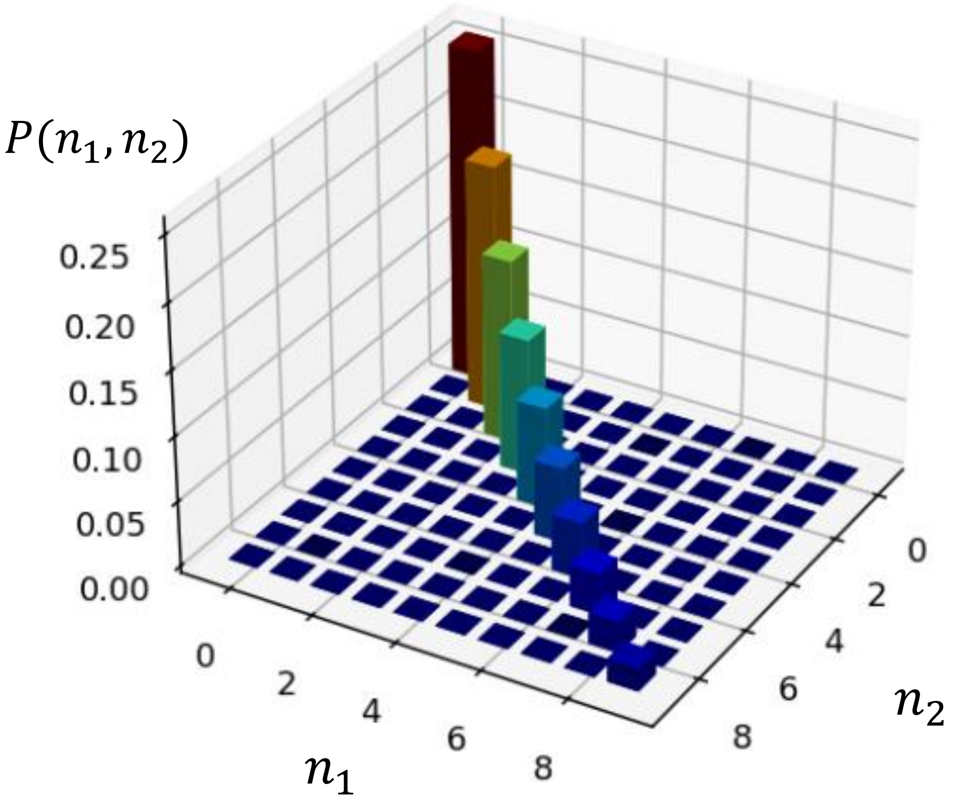
See also Barasiński *et al* Phys. Rev. Lett (2023)

Consider a two-mode squeezed vacuum state

$$|TMSv\rangle(r) \sim \sum_i \tanh^i r \,|i,i\rangle_{12}$$

$$\rho_{TMSv} \sim \sum_{i,k} \tanh^i r \tanh^k r \,|i,i\rangle_{12}\langle k,k|_{12}$$

ρ_{TMSv} is a non-separable state in the partition 1-2.



Can we prove the entanglement of this state from its FCS?

Ex: the state describe by

$$\rho_{sep} \sim \sum_i \tanh^i r \,|i,i\rangle\langle i,i|$$

is a separable state which has the same two-mode probability distribution as a TMSv.

Entanglement from the FCS

One cannot assess the entanglement of *any* quantum state from its full counting statistics.

It only measures the diagonal terms of the density matrix

THANK YOU FOR YOUR ATTENTION !

Wait a minute… Not true for *Gaussian* states!

Gaussian states

GAUSSIAN STATES

A Gaussian state: $G_C^{(n>2)}(\hat{a}_1^\dagger \dots \hat{a}_2) = 0.$

We can connect N-body correlation functions to 1- and 2-field correlation functions!

PROPERTIES

Any operator that involves more than 2 fields can be expressed with 1- and 2-field operators.

Ex:

$$G_{12}^{(2)} = \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle = n_1 n_2 + \underbrace{|\langle \hat{a}_1 \hat{a}_2 \rangle|^2}_{\text{Anomalous correlation}} + \underbrace{\left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2}_{\text{Coherence}}$$

LINK TO ENTANGLEMENT

If $\langle \hat{a}_1^\dagger \hat{a}_2 \rangle = 0$, observation of

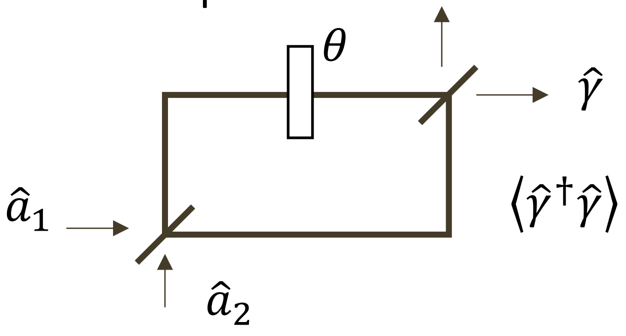
$$g_{12}^{(2)} = G_{12}^{(2)}/n_1 n_2 > 2$$

implies entanglement as $n_1 n_2 < |\langle \hat{a}_1 \hat{a}_2 \rangle|^2$

Campo & Parentani, *Phys. Rev D* (2005)
Hillery & Zubairy *Phys. Rev. Lett.* (2006)

HOW TO MEASURE THE COHERENCE?

Sol. 1: set up an interferometer



$$\langle \hat{\gamma}^\dagger \hat{\gamma} \rangle \propto \left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2 \sin 2\theta$$

Sol. 2: use higher-order correlation functions.

ASSUMPTIONS

- ❑ Gaussian state,
- ✓ Zero-mean,
- ✓ Neither mode is squeezed $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

The single-mode probability distribution is thermal.

Experimentally testable as Chris showed!

The g2/g4 criterion (1)

TWO- AND FOUR-BODY CORRELATION FUNCTIONS

$$g_{12}^{(2)} = 1 + \left(|\langle \hat{a}_1 \hat{a}_2 \rangle|^2 + \left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2\right) / n_1 n_2$$

$$g_{12}^{(4)} = \left\langle \left(\hat{a}_1^\dagger \hat{a}_2^\dagger\right)^2 (\hat{a}_1 \hat{a}_2)^2 \right\rangle / n_1^2 n_2^2$$

$$= f\left(G_{12}^{(2)}, n_1 n_2\right) + 8\, |\langle \hat{a}_1 \hat{a}_2 \rangle|^2 \times \left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2 / n_1^2 n_2^2$$

Symmetric system to find $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ and $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle|$ yields two solutions $\beta_\pm$

TWO SOLUTIONS

We have two possible solutions

- "State" μ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_+$ & $\left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right| = \beta_-$,
- "State" γ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_-$ & $\left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right| = \beta_+$.

GAUSSIAN STATES FORMALISM

From A. Brady's lecture, *Benasque 2023*

Gaussian states respect a *bona fide* condition based on their symplectic spectrum $\{\nu_\pm\}$:

$$\nu_- \geq 1$$

A 2-mode Gaussian state is entangled iff its partial transpose (PT) is not a *bona fide* Gaussian state

$$\nu_-^{t_2} < 1$$

The g2/g4 criterion (2)

TWO SOLUTIONS

We have two possible solutions

- "State" μ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_+$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle| = \beta_-$,
- "State" γ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_-$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle| = \beta_+$.

i. The *bona fide* condition for μ and γ only depends on $n_1, n_2, \beta_\pm, \beta_\mp \Rightarrow \{\nu_\pm^{(\gamma)}\}, \{\nu_\pm^{(\mu)}\}$

ii. "States" μ and γ are PT of each other

iii. We measure λ_-= Min $\left\{\nu_\pm^{(\mu)}, \nu_\pm^{(\gamma)}\right\}$

THE g2/g4 CRITERION

The measurement of $n_1, n_2, g_{12}^{(2)}, g_{12}^{(4)}$ yields λ_-, the smallest symplectic eigenvalue of the state and its PT. If $\lambda_- < 1$, the state is entangled.

LN = Max$(-\log_2(\lambda_-), 0)$

GAUSSIAN STATES FORMALISM

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$$\nu_-^{t_2} < 1$$

Formulas in arXiv:2503.09555

The g2/g4 criterion (3)

THE $g^{(2)}/g^{(4)}$ CRITERION

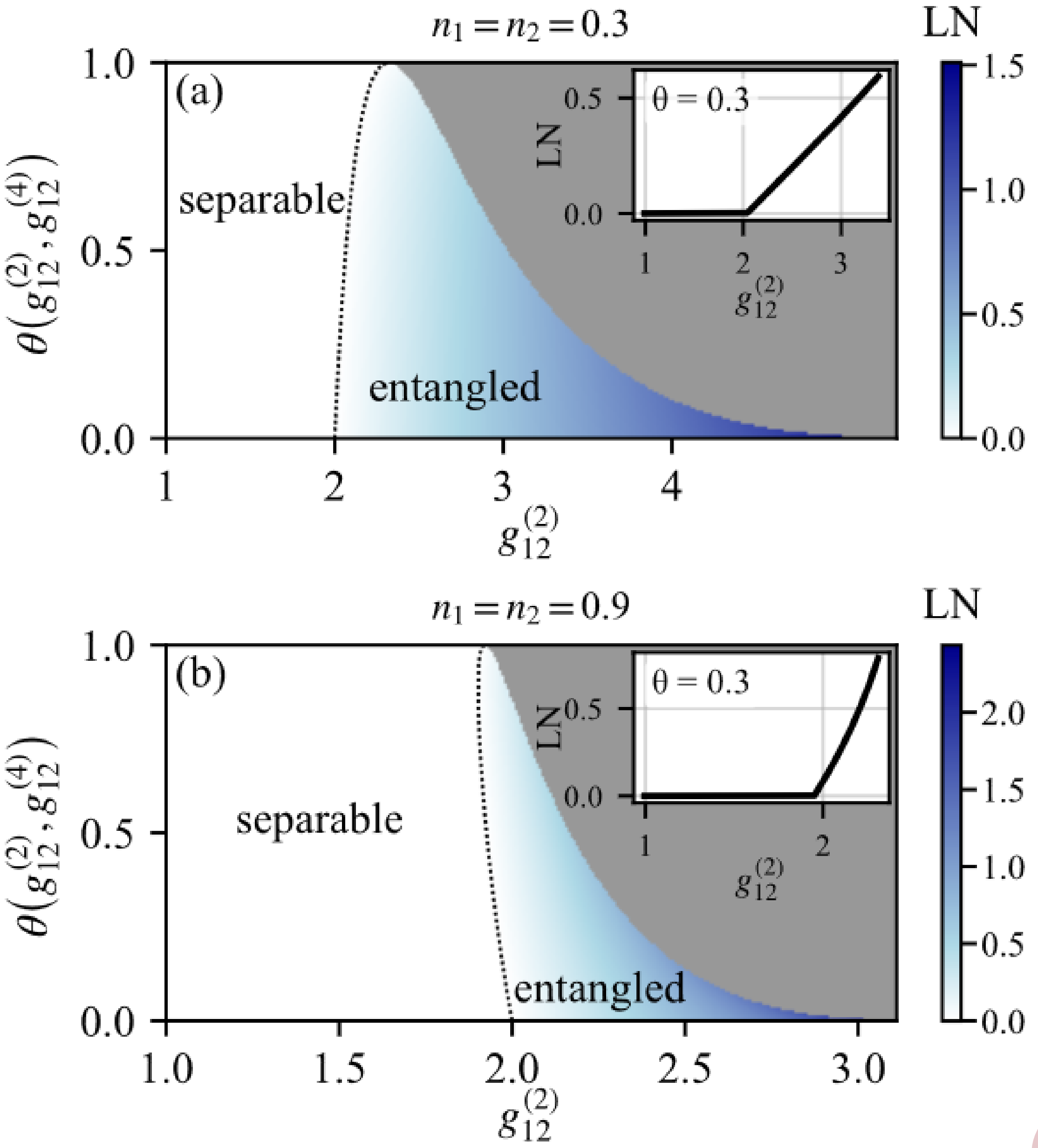
The measurement of $n_1, n_2, g_{12}^{(2)}, g_{12}^{(4)}$ yields λ_-, the smallest symplectic eigenvalue of the state and its PT. If $\lambda_- < 1$, the state is entangled.

$$\mathrm{LN} = \mathrm{Max}(-\log_2(\lambda_-), 0)$$

$$\theta = \frac{g_{12}^{(4)} + 12 - 16 g_{12}^{(2)} - 4\left(g_{12}^{(2)} - 1\right)^2}{\left(g_{12}^{(2)} - 1\right)^2} \in [0,1]$$

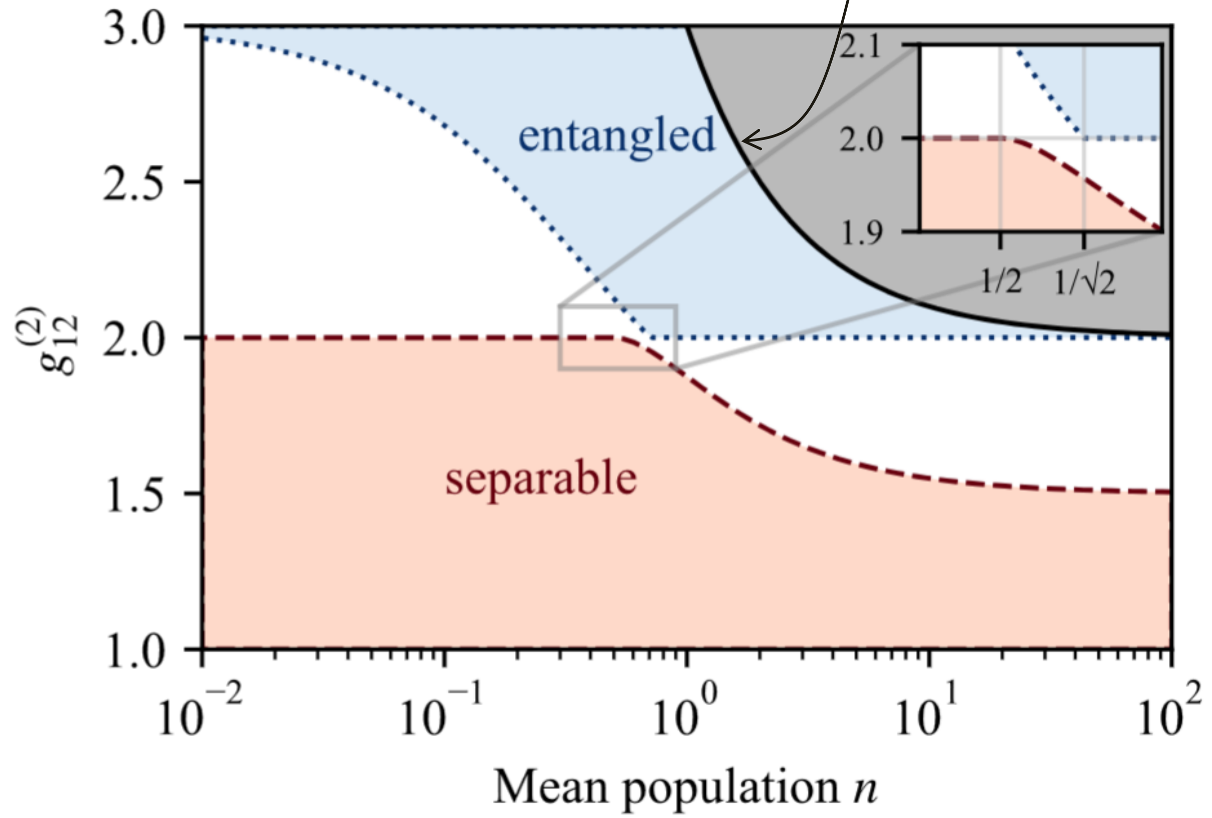
THE $g^{(2)}$ WITNESS

- Low population: if $g_{12}^{(2)} \leq 2$ separable state,
- High population : if $g_{12}^{(2)} \leq 2$ entangled state.



G2 witness

Limit given by the two-mode squeezed vacuum state.



- The $g_{12}^{(2)}$ entanglement witness depends on the populations,
- The value of $g_{12}^{(4)}$ is needed to determine the entanglement in the middle region.
- Taking into account the quantum efficiency of the detector can reveal entanglement,

Formulas in arXiv:2503.09555

Conclusion

ASSUMPTIONS MADE

- ❑ Gaussian state,
- ✓ Zero-mean,
- ✓ Neither mode is squeezed $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

Thermal single-mode probability distribution.

Take home message

Measurement of $n_1, n_2, g_{12}^{(2)}, g_{12}^{(4)}$ yields LN (entanglement criterion)

Measurement of $n_1, n_2, g_{12}^{(2)}$ can witness entanglement.

No commutating measurement needed because the state is *assumed* Gaussian. Homodyne-like scheme needed to check this assumption.

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Thank you for your attention!

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Take home message

Measurement of $n_1, n_2, g_{12}^{(2)}, g_{12}^{(4)}$ yields LN (entanglement criterion)

Measurement of $n_1, n_2, g_{12}^{(2)}$ can witness entanglement.

No commutating measurement needed because the state is assumed Gaussian. Homodyne-like scheme needed to check this assumption.

All the details in arXiv:2503.09555