On the entanglement of quasi-particles in a Bose-Einstein Condensate

From Faraday Waves to the Dynamical Casimir Effect

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PhD work under the supervision of Denis Boiron
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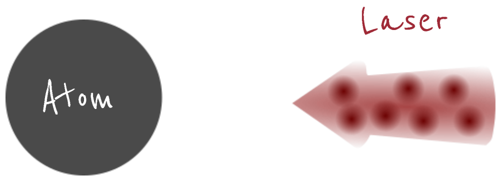
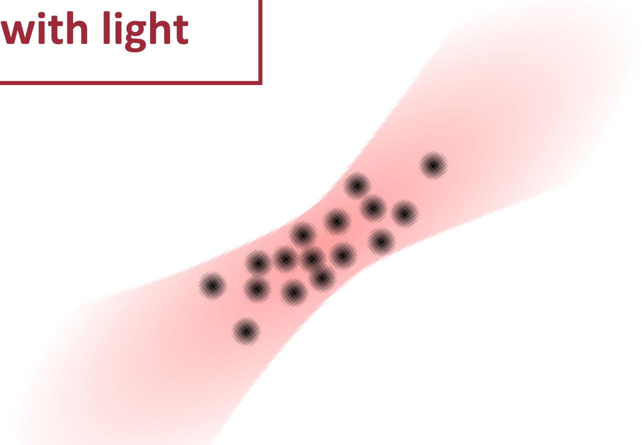
Frédéric Chevy
Nicolas Pavloff

Working with atoms @LCF

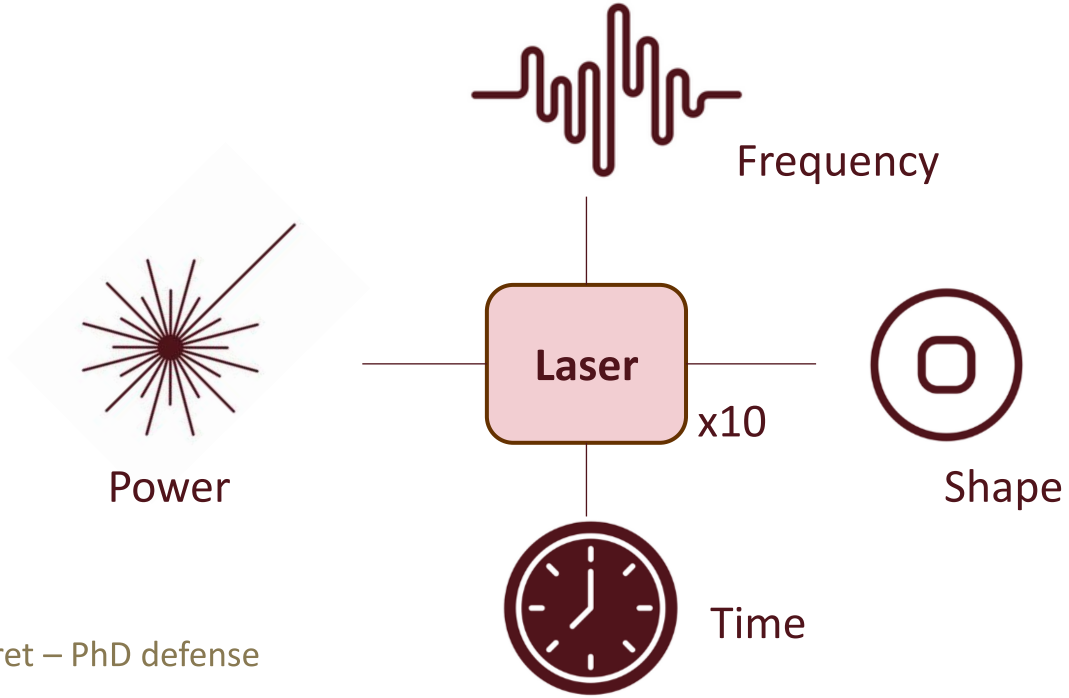
Interact with atoms with light

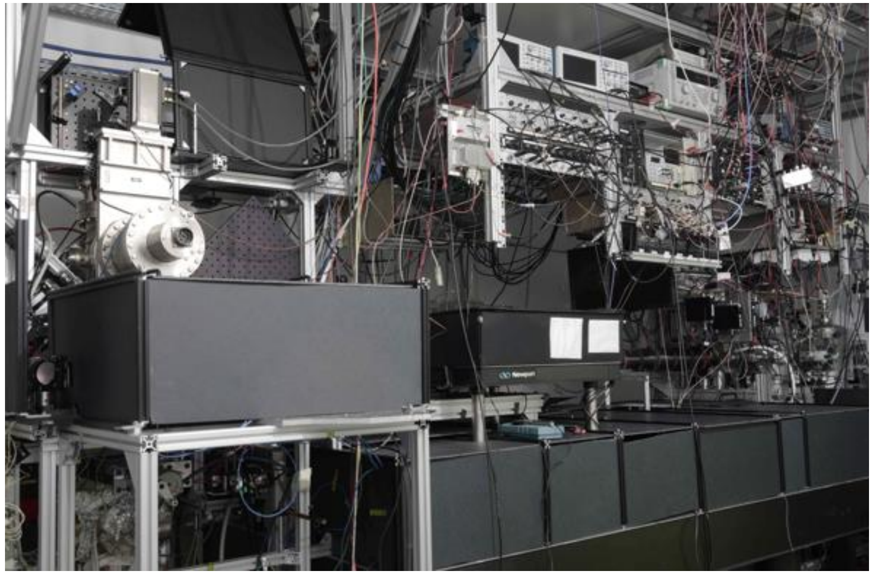


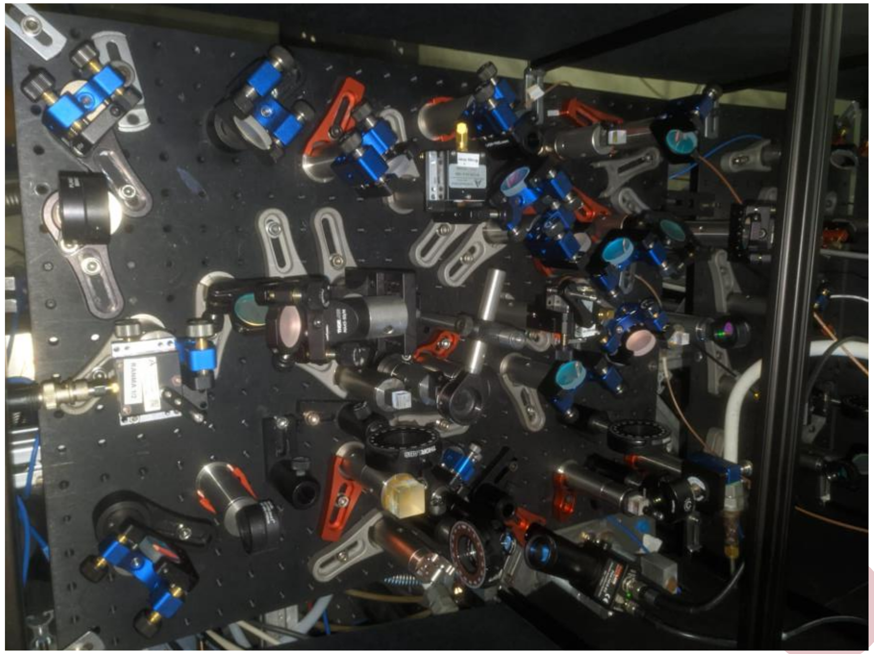
To change their speed



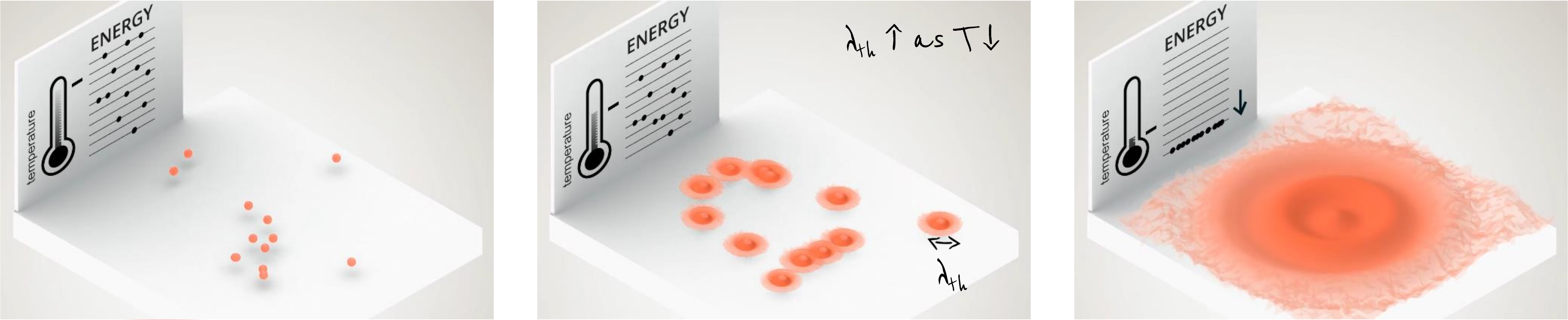
To trap them







Quantum Gas group: ultracold gases



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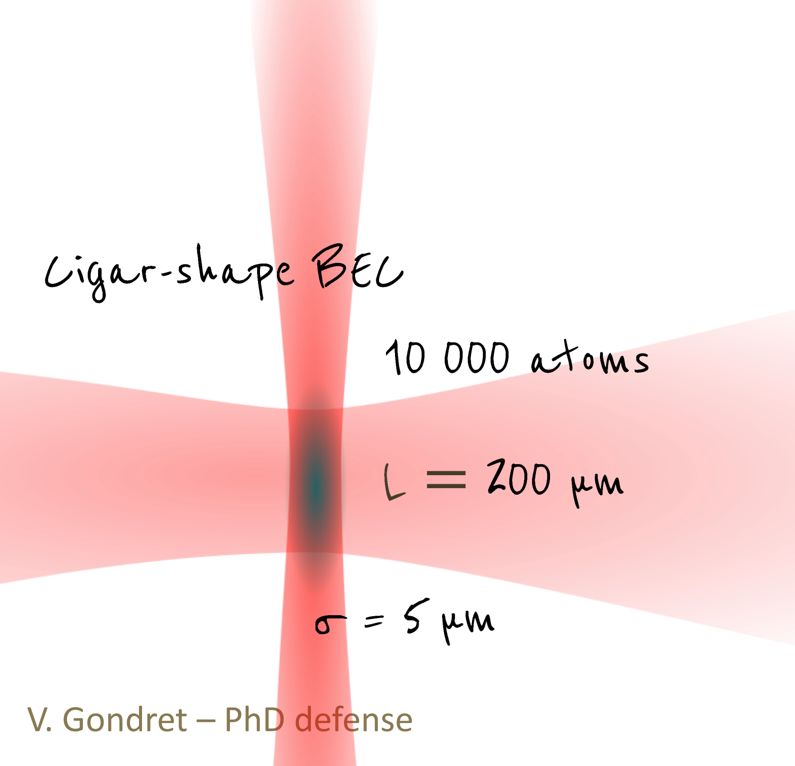


We decrease the temperature to reach a **Bose-Einstein condensate (BEC)**.

- Destructive measurement,
- We prepare a BEC in 10 s.
  Important to have statistics $\langle\hat{P}\rangle, \langle\hat{P}^2\rangle, \langle\hat{X}\rangle, \langle\hat{X}^2\rangle$….

In the experiment:

T = 40 nK

(0 K = −273.15 °C)

Quantum Atom Optics group

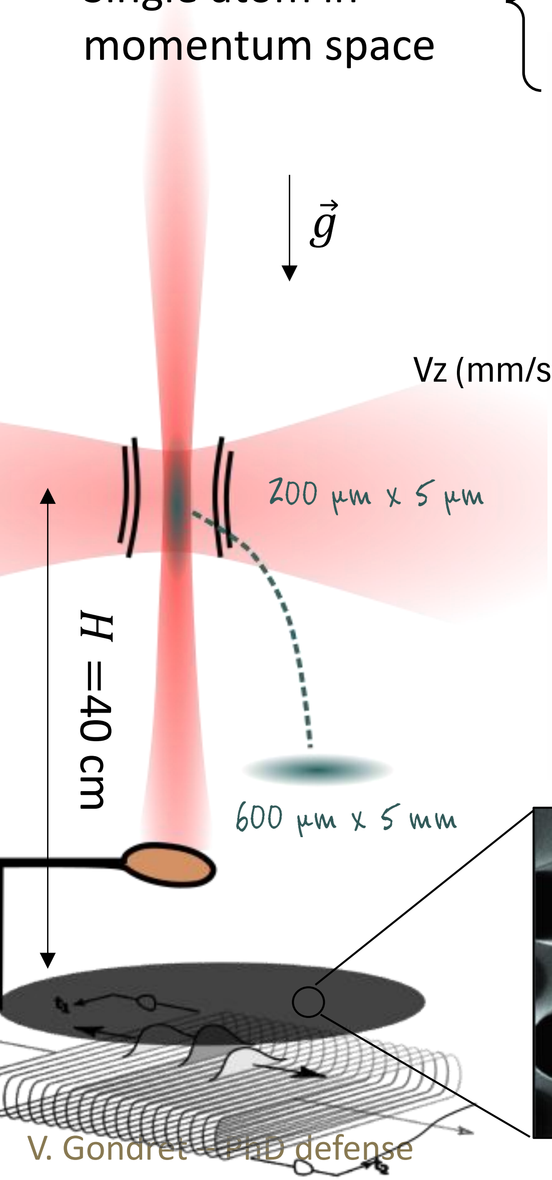
Single atom in momentum space

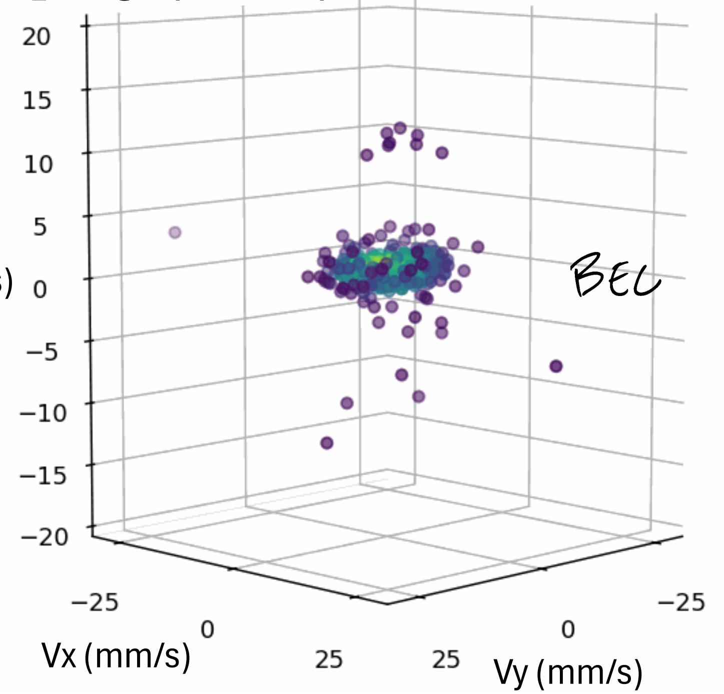
$$\begin{cases} v_x = X/T \\ v_y = Y/T \\ v_z = gT/2 - H/T \end{cases}$$

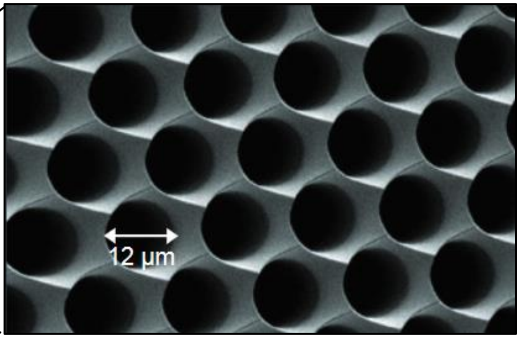
$p = mv = \hbar k$

$\hbar$ Planck constant

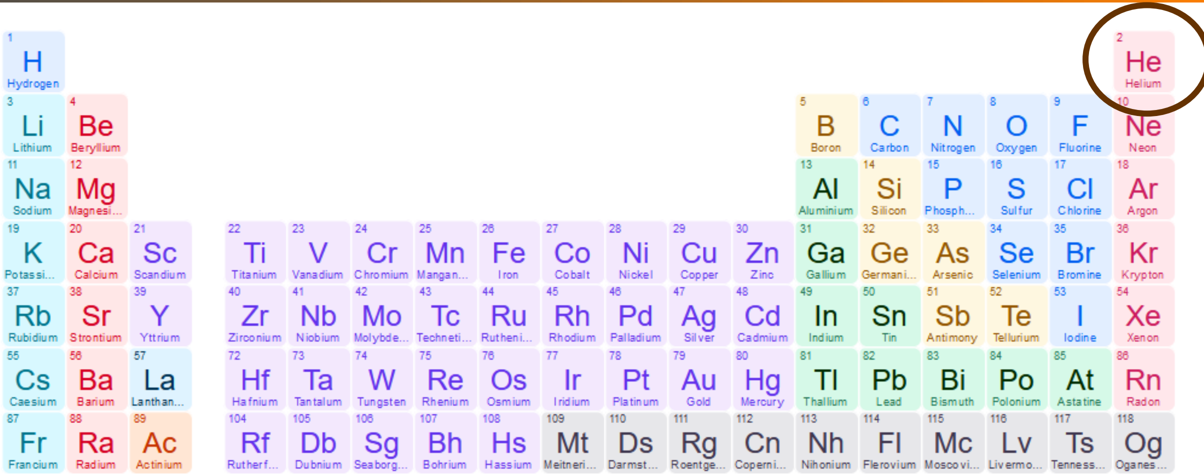
m mass



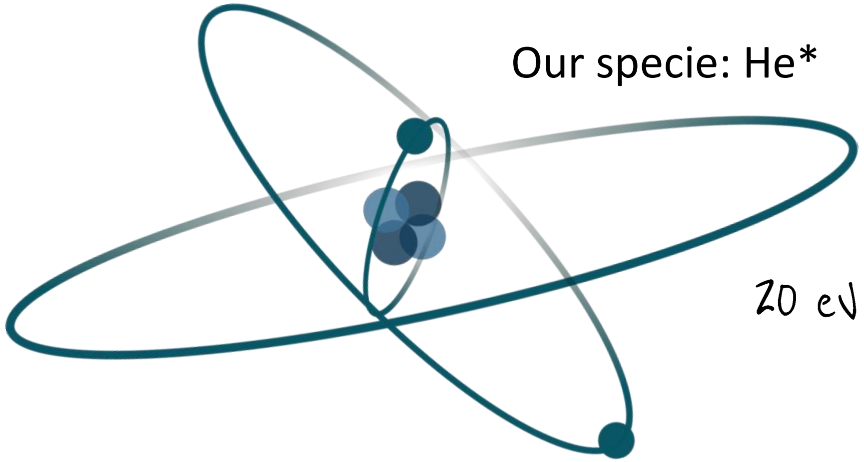




Electronic detection of individual atoms (X, Y, T)



Our specie: He*

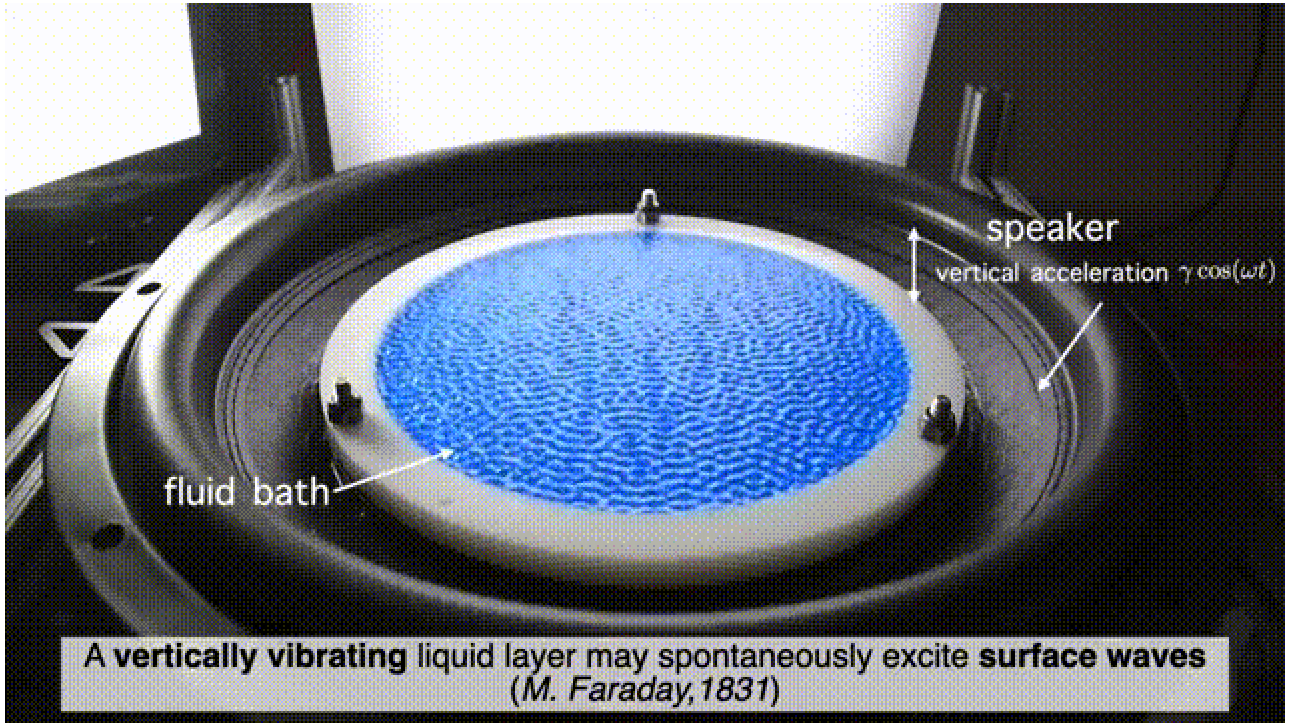


Study the correlations and quantum effects in momentum space.

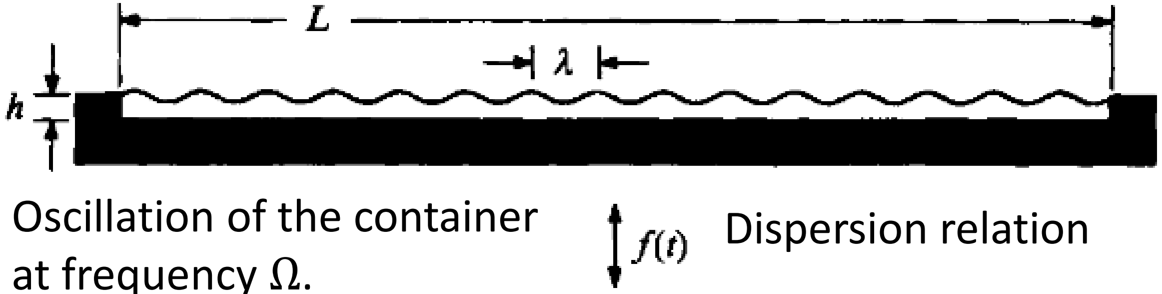
A parametrically excited BEC?

Outline

Faraday waves



Guan *et al.* PR Fluids (2023), Edwards & Fauve J. Fluid Mech. (1994)



$\downarrow \vec{g}$

Oscillation of the container at frequency Ω.

Dispersion relation

$$\omega_k = \sqrt{\tanh(hk)\,[gk + \gamma k^3]} = \Omega/2$$

Broughton Suspension Bridge collapsed in 1831

Parametric oscillation ≠ forced oscillation

$\Omega/2$	Ω
Variation of an internal parameter	External force

Parametric or forced excitation of a swing

Forced excitation

Parametric excitation

Parametric oscillation ≠ forced oscillation

$\Omega/2$	Ω
Variation of an internal parameter	External force

Parametric or forced excitation of a swing

Forced excitation

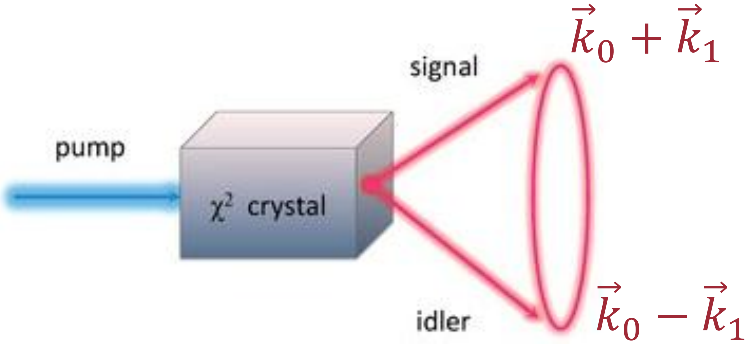
Parametric excitation

Parametric oscillation ≠ forced oscillation

Parametric oscillation	Forced oscillation
$\Omega/2$	Ω
Variation of an internal parameter	External force
Growth initialized by the force	Growth triggered by fluctuations

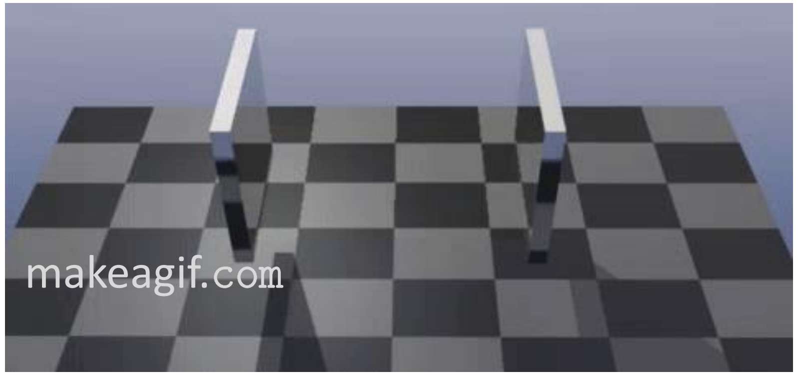
- experimental imperfections,
- thermal fluctuations,
- quantum fluctuations.

Parametric amplification across various scales



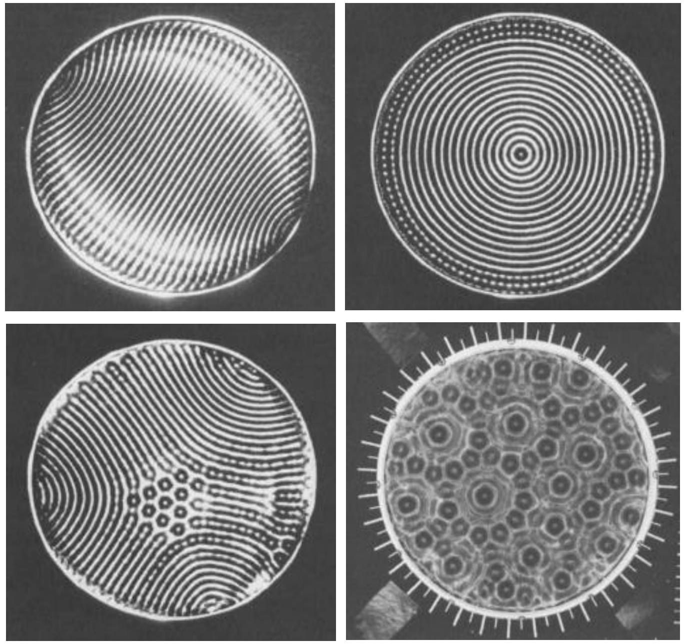
Quantum vacuum fluctuations trigger amplification which leads to entanglement.

Dynamical Casimir effect
Moore (1970)



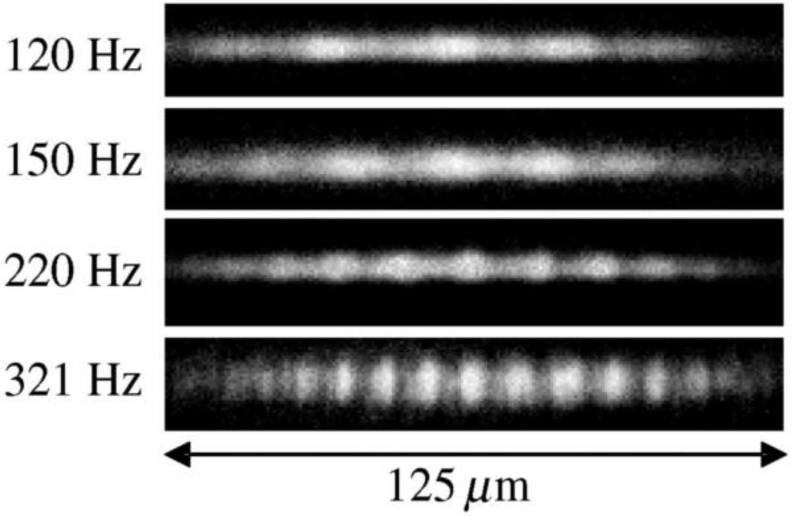
Photons

Fluid

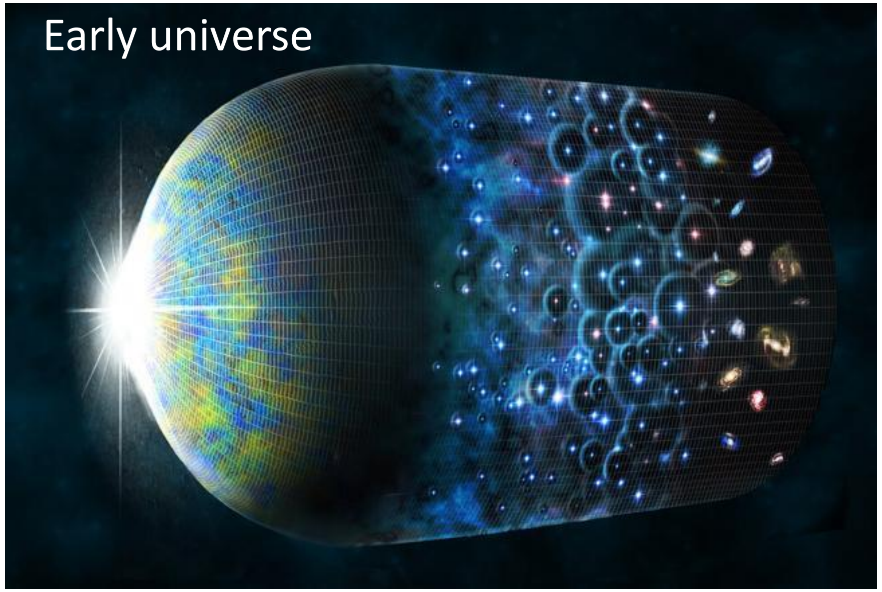


Edwards & Fauve J. Fluid Mech. (1994)

Engels *et al* PRL (2007)



BEC



Reheating after inflation

The **inflaton** *slowly rolls* from its initial false vacuum state. Its almost constant potential energy **drives the inflation**.

A. Linde, Phys. Lett. 129B, 177 (1983).

It starts to oscillate around its minimum and, coupled to matter fields, it creates particles through broad **parametric resonance**.

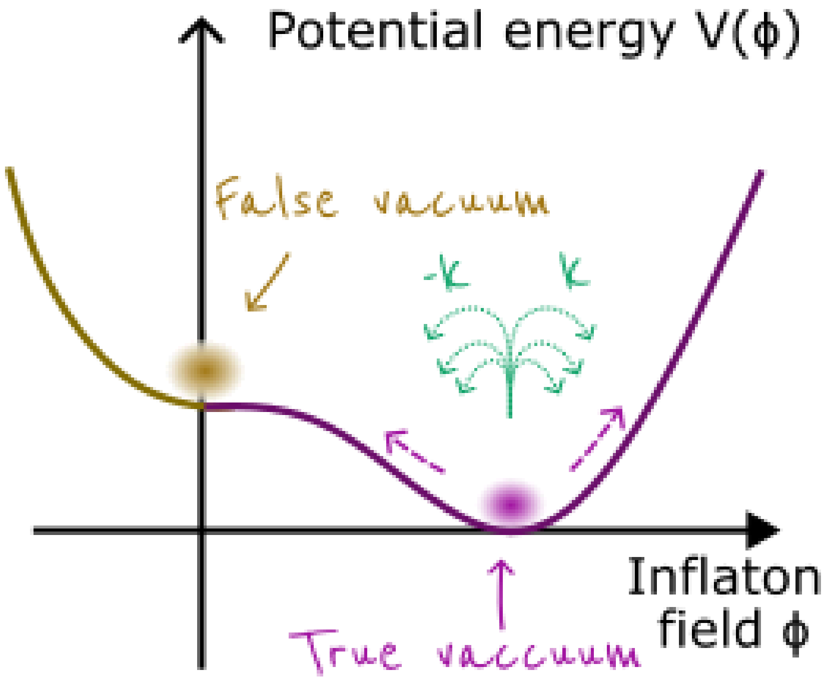
L. Kofman, A. Linde & A. Starobinsky, Phys. Rev. D 56, (1997).

Particles are created in **pairs** with **opposite momenta from vacuum** in a two modes squeezed state.

Interactions lead to decoherence and thermalization

D. Campo & R. Parentani, Phys. Rev. D **74**, 025001 (2006).

Analog gravity & cosmology: mimics quantum field theory in curved space time on table-top experiments. Jacquet *et al*. Phil.Trans.R.Soc.A (2020)



- Does parametric amplification of quasi-particles in a BEC lead to an entangled state?
- How these quasi-particles thermalize?

Outline

① Theoretical considerations

a) Parametric amplification & motivations

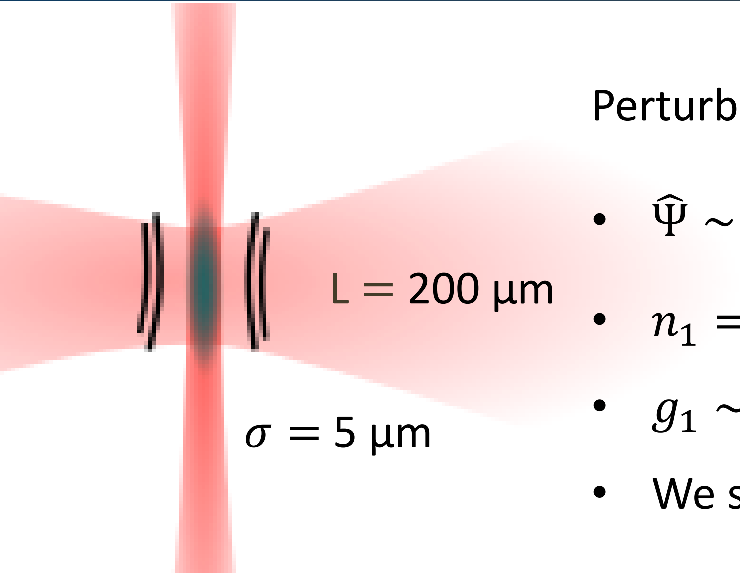
b) Parametric amplification of quasi-particles in a Bose-Einstein condensate.

② Experimental result

a) Growth of the quasi-particle number

b) Correlations & entanglement of the two-mode state.

Parametric excitation of an elongated BEC



Perturbative approach: Bogoliubov 1D

- $\hat{\Psi} \sim \frac{\sqrt{n_1}}{\sigma} e^{-r^2/2\sigma^2}[1+\hat{\phi}(z)]$
- $n_1 = N/L$
- $g_1 \sim 1/\sigma^2$ 1D effective interaction
- We study collective excitations:
  - $\hat{b}_k$ annihilates a quasi-particle at k
  - $\hat{b}_k^\dagger$ creates a quasi-particle at k

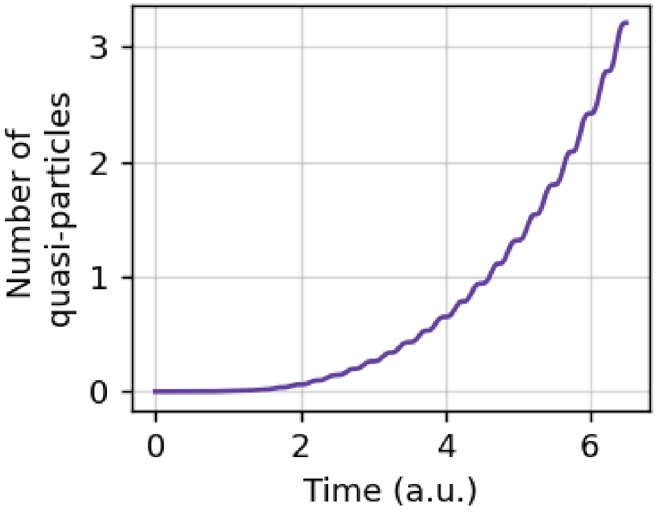
- $\hat{b}_k$ diagonalizes the Hamiltonian

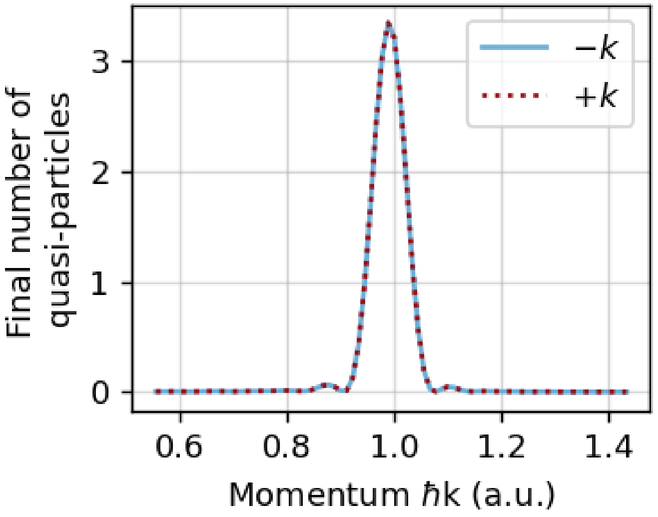
$$i\hbar\partial_t \hat{b}_k = \omega_k \hat{b}_k + i\frac{\dot{\omega}_k}{2\omega_k}\hat{b}^\dagger_{-k}$$

- Dispersion relation

$$\omega_k = \sqrt{2 g_1 n_1 \frac{\hbar^2 k^2}{2m} + \left(\frac{\hbar^2 k^2}{2m}\right)^2}$$

g_1 depends on $\sigma(t)$





σ oscillates at Ω parametrically excites quasi-particles by pairs with $\omega_k = \Omega/2$.

TMSv if zero temperature,

$$|\phi\rangle \sim \sum_i \alpha^i \, |i, i\rangle_{-k,k}$$

Mode -k, Mode +k

i.e. an entangled state.

Theoretical predictions & experimental challenges

Theoretical cheatsheet

Carusotto *et al.* EPJD (2010),
Busch *et al.* PRA (2014),
Robertson *et al* PRD (2017,2018),
Micheli & Robertson, PRD (2022)

Perturbative approach (linear equation of motion)

Excitation at Ω produce opposite moment quasi-particles with frequency $\omega_k = \Omega/2$

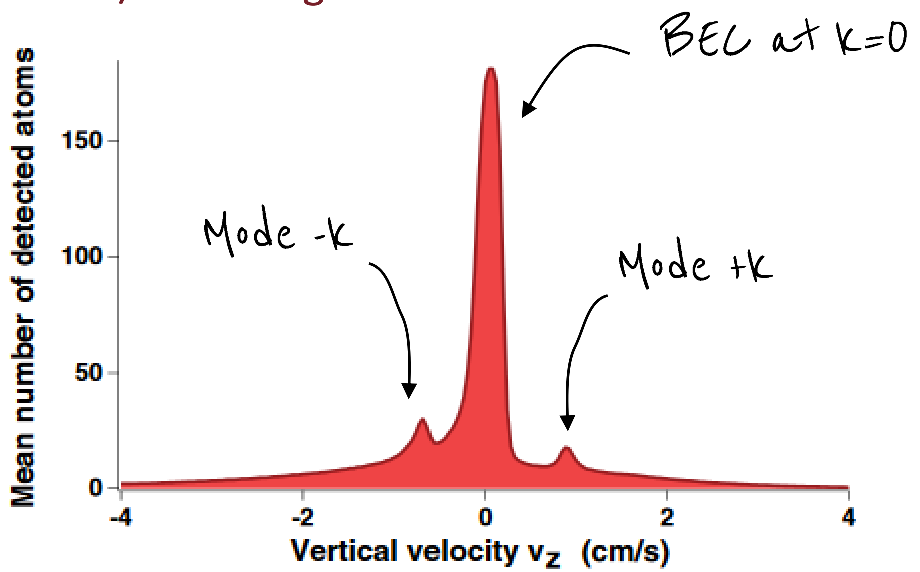
Exponential growth of the quasi-particle number trigger by fluctuations

Entanglement of the (k, -k) state only if the temperature is small enough.

Non-linear effects lead to decoherence and thermalization.

Experimental challenges

- Observation of quasi-particle creation,
- Quasi-particle frequency is $\omega_k = \Omega/2$,
- Study the exponential creation process,
- Observation of correlations, non-classical effects and/or entanglement.



Jaskula *et al* PRL (2012)

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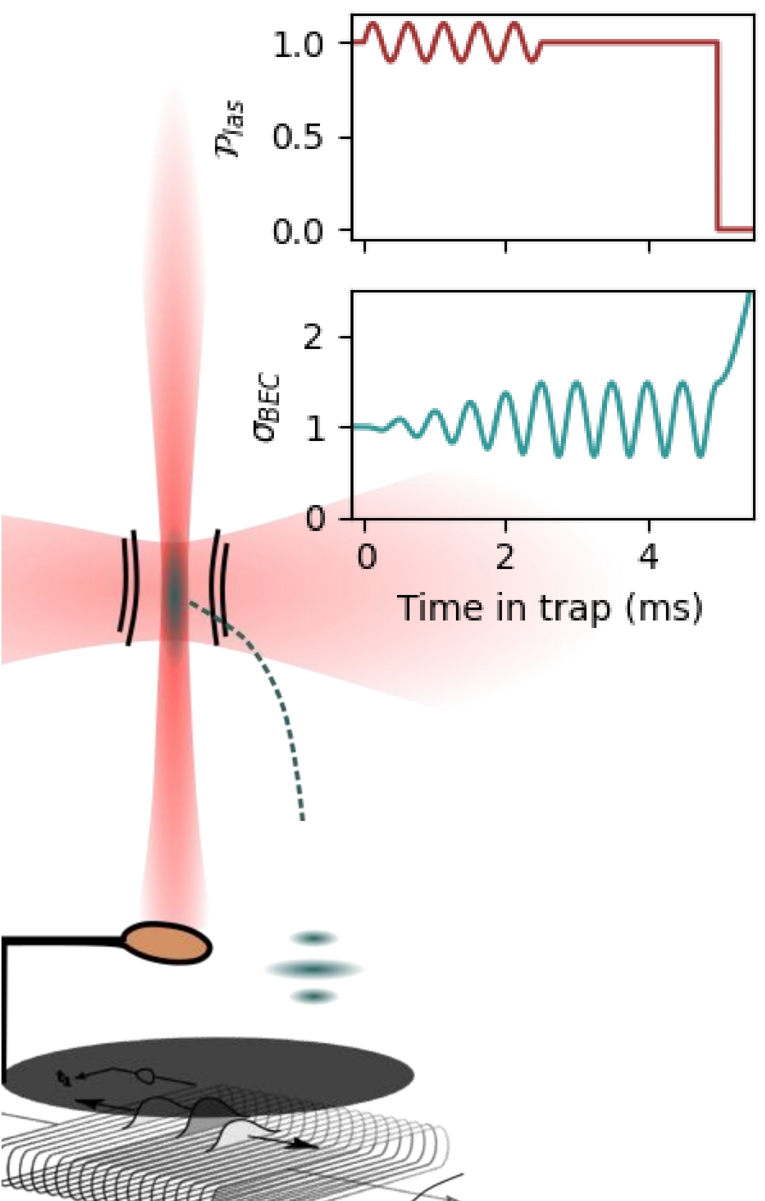
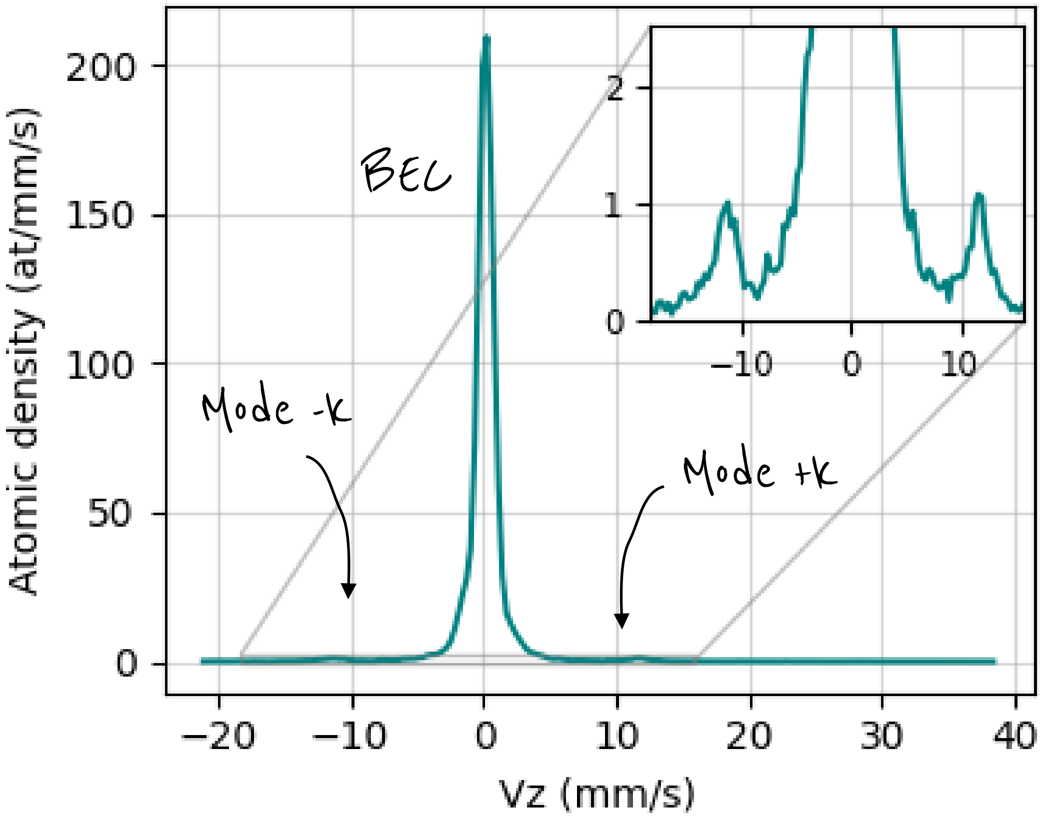
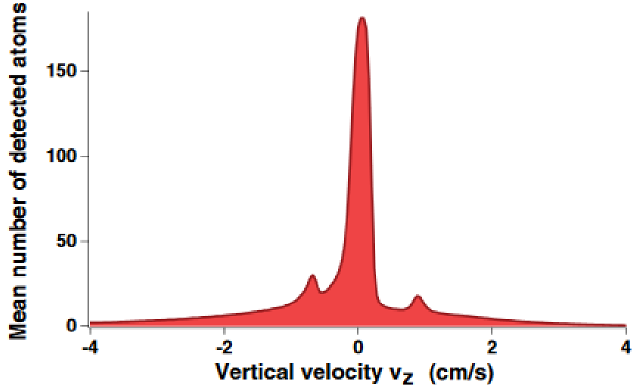
Experimental preparation



We excite the breathing mode of the BEC.



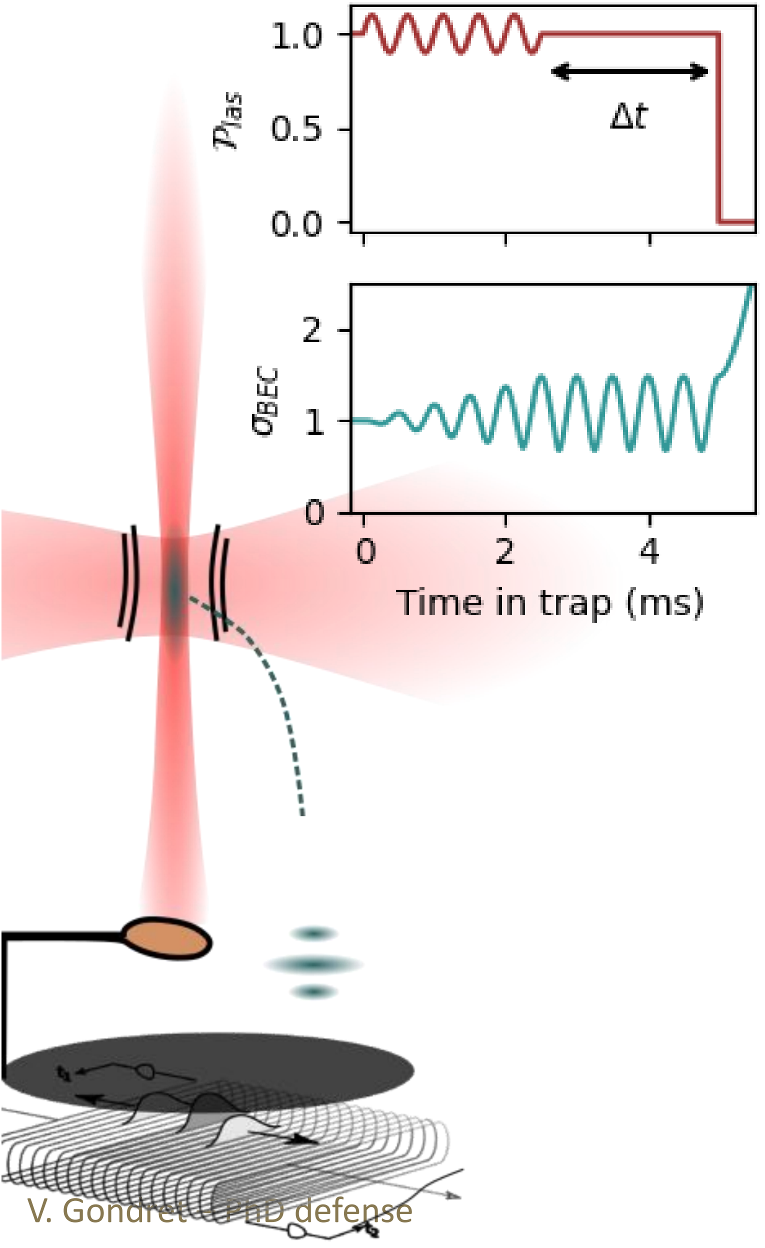


Jaskula *et al* PRL (2012)

- ✓ Breathing mode keeps low temperature,
- ✓ Small excitation to stay in perturbative regime.

We count the mean number of atoms n_k, n_{-k}

Exponential growth of the phonon number



Fit function:

Theoretical quasi-particle growth dynamics

Empirical oscillation

$$n_k(t) = \left[n_k^{(in)} + \left(n_k^{(in)} + n_{-k}^{(in)} + 1\right) \sinh^2(G_k \Delta t/2)\right] \times (1 + A_k \cos(2\omega_k \Delta t + \varphi))$$

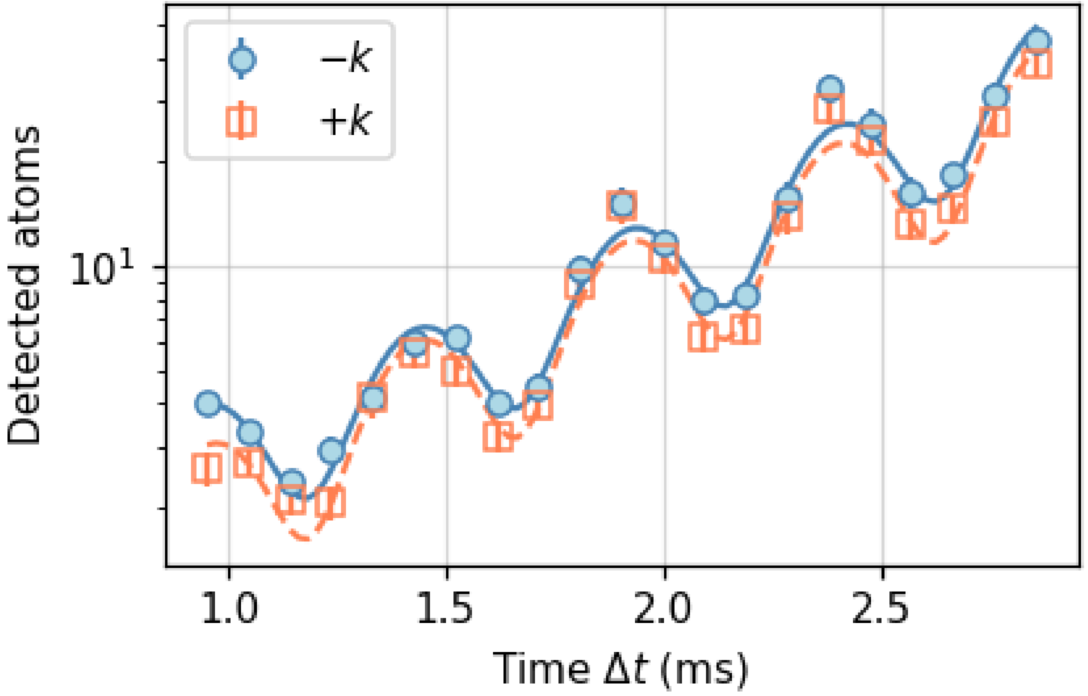
thermal vacuum

Fluctuations

Growth rate

Oscillation amplitude

Quasi-particle frequency



Measuring the growth rate

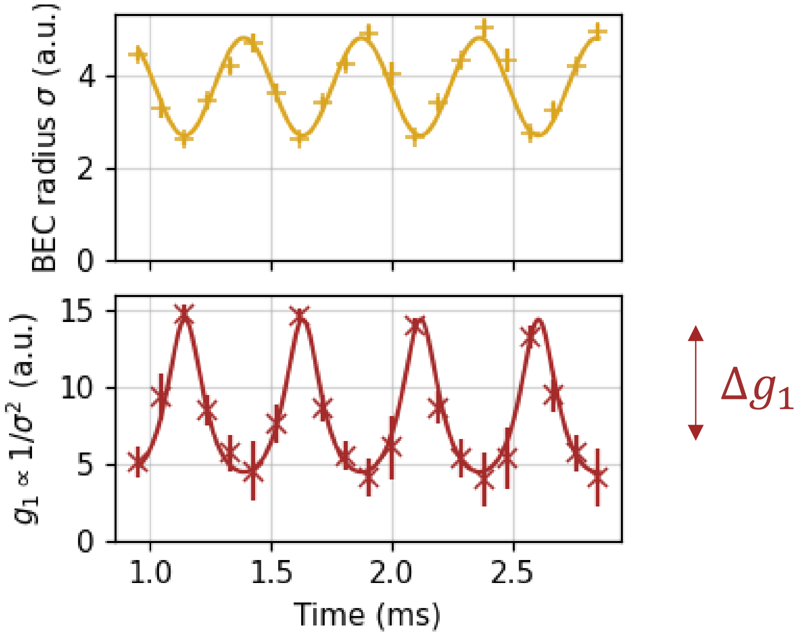
We fit the growth rate $G_k^{(exp)}$ from the population growth.

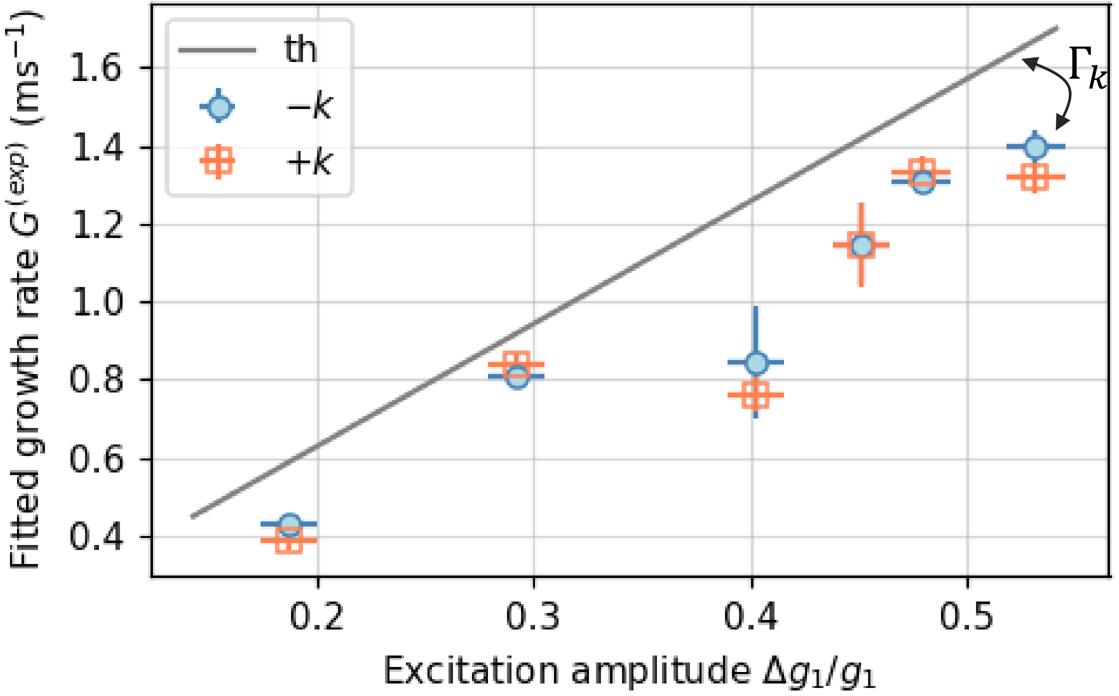
Measure the theoretical growth rate from the BEC

$$G_k^{(th)} = \frac{\omega_k}{2} \frac{\Delta g_1/g_1}{1+k^2\xi^2}$$

BEC healing length

Busch *et al.* PRA (2014)





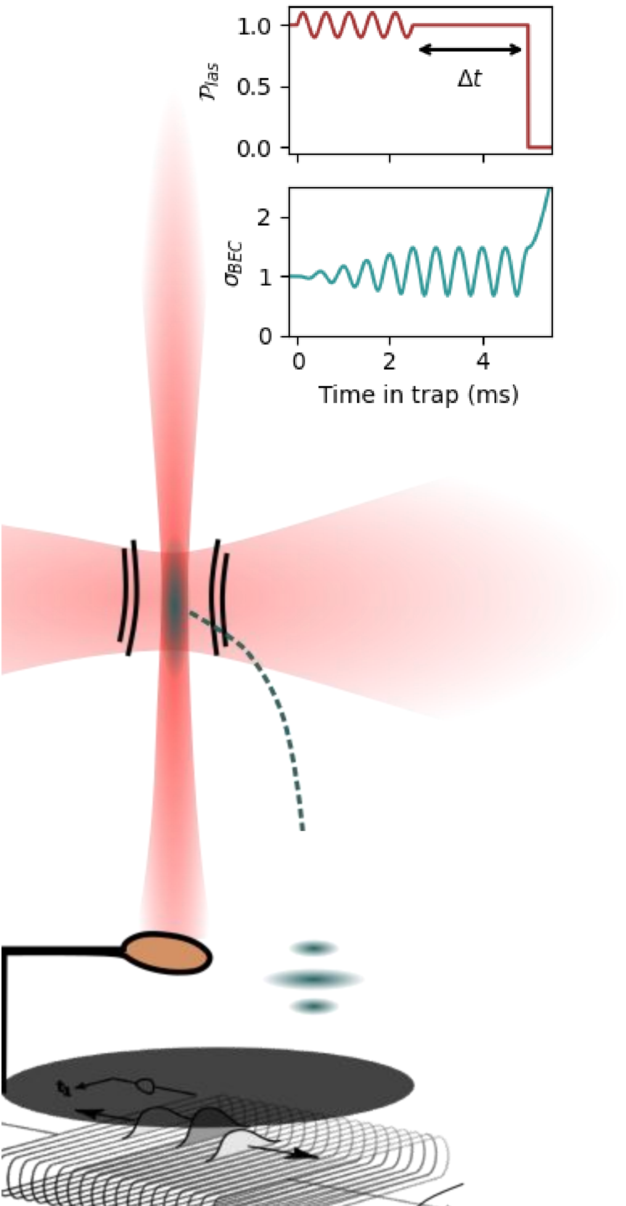
Higher-order quasi-particle interactions decrease the growth rate $\Gamma_k = G_k^{(th)} - G_k^{(exp)}$.

The slowing of the growth (*i.e.* the decay rate) we measure is compatible with theoretical predictions*.

* Within not so small error bars.

Micheli & Robertson PRD (2022)

Exponential growth of the phonon number

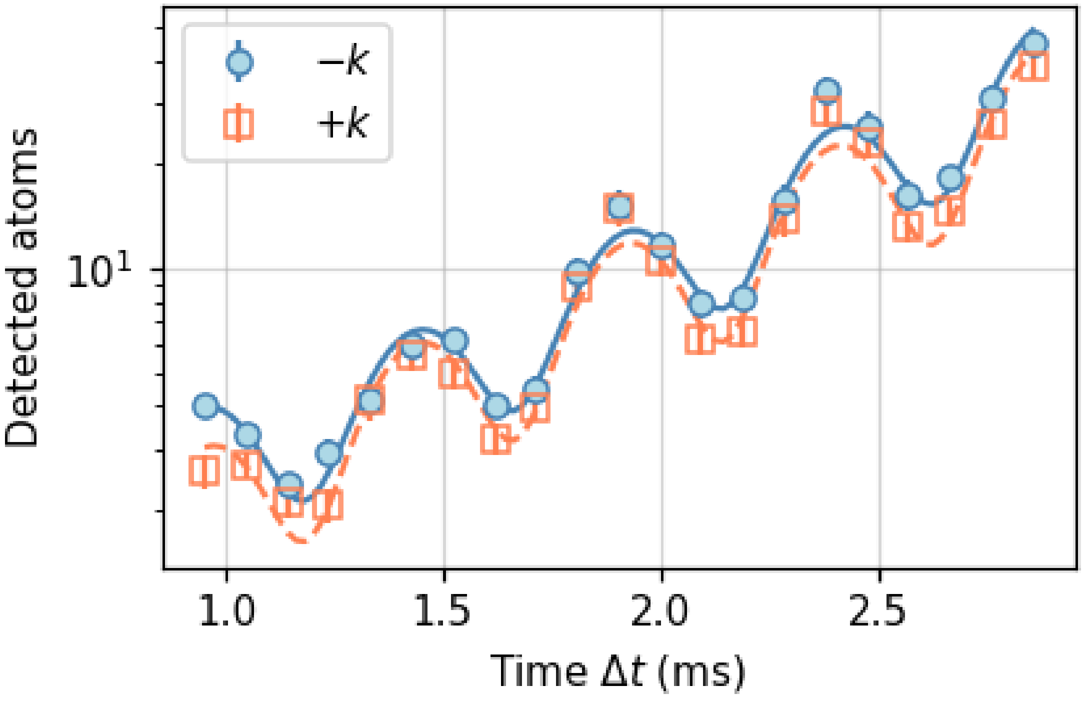


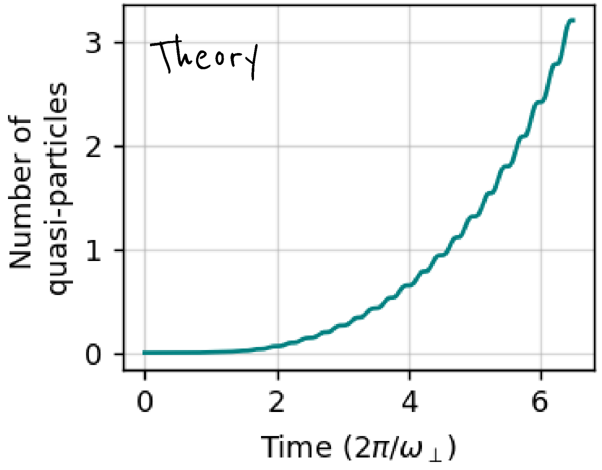
Fit function: Theoretical quasi-particle growth dynamics — Empirical oscillation

$$n_k(t) = \left[n_k^{(in)} + \left(n_k^{(in)} + n_{-k}^{(in)} + 1\right)\sinh^2(G_k\Delta t/2)\right] \times (1 + A_k\cos(2\omega_k\Delta t + \varphi))$$

thermal — vacuum — Growth rate — Oscillation amplitude — Quasi-particle frequency

Fluctuations





Such oscillation in the quasi-particle growth is not expected…

Adiabatic mapping from the collective excitation basis to the atomic basis

We measure atoms and not quasi-particles : how does the *collective excitations* state $\hat{b}_k$ maps to the *atomic* state $\hat{\phi}_k$?

$$\hat{\phi}_k \sim \hat{b}_k + \hat{b}^\dagger_{-k}$$

The detected atom number:

$$n_k = \langle \hat{\phi}_k^\dagger \hat{\phi}_k \rangle \sim \langle \hat{b}_k^\dagger \hat{b}_k \rangle + \cdots + |\langle \hat{b}_{-k} \hat{b}_k \rangle| \cos 2\omega_k \Delta t$$

Cool — Not cool

BUT

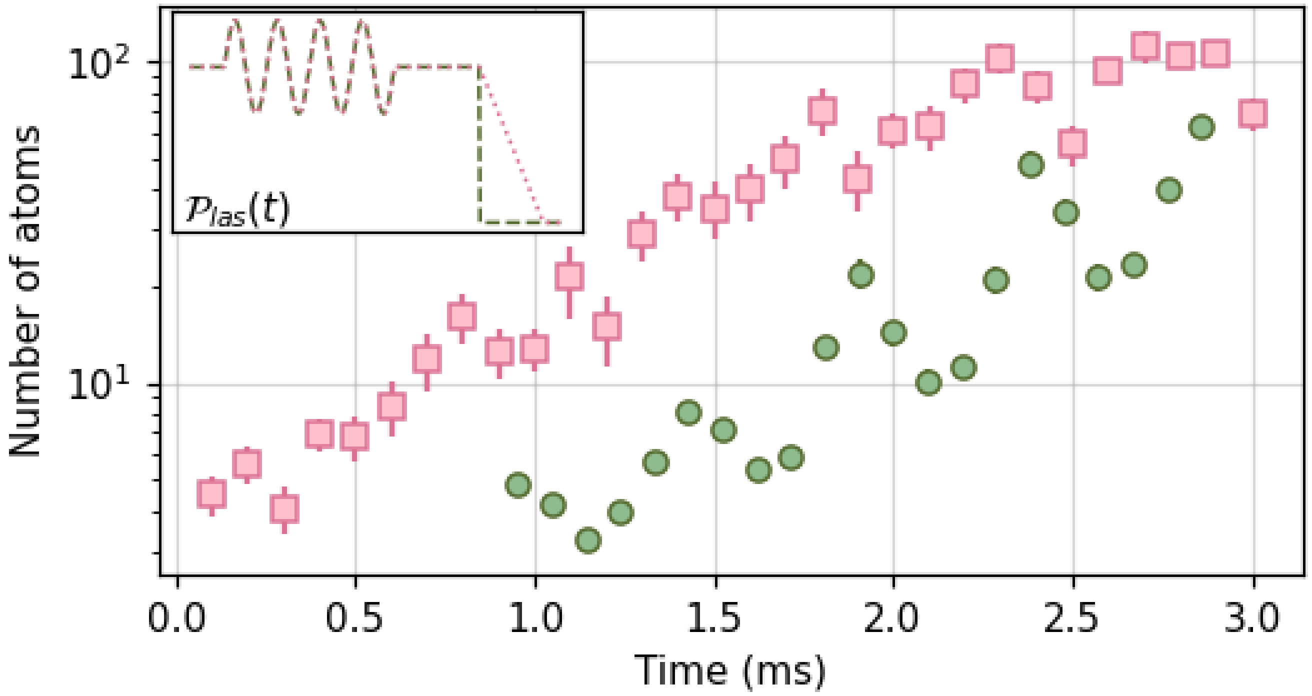
$$\hat{b}_k \Rightarrow \hat{\phi}_k^{det}$$

if ω_k changes adiabatically w.r.t. ω_k^{-1}.

$$\omega_k = \sqrt{2 g_1 n_1 \frac{\hbar^2 k^2}{2m} + \left(\frac{\hbar^2 k^2}{2m}\right)^2}$$

g_1 depends on $\sigma(t)$

We turn off slowly the laser power so that σ changes slowly.



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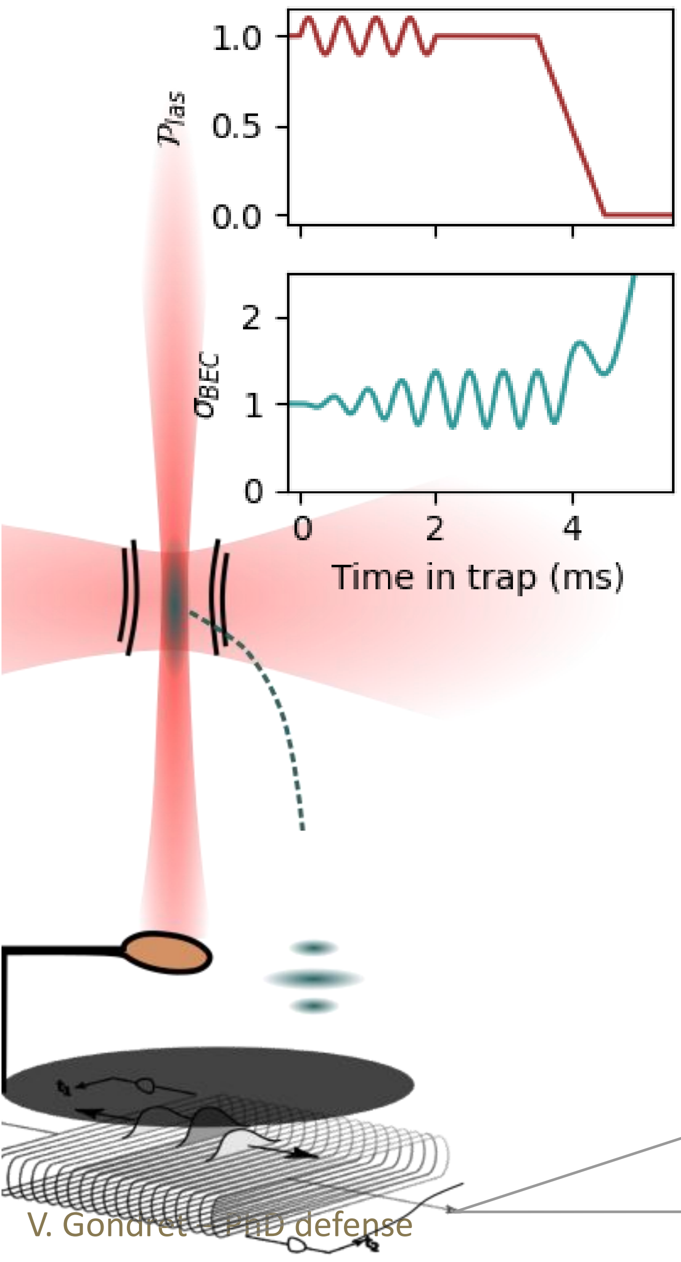
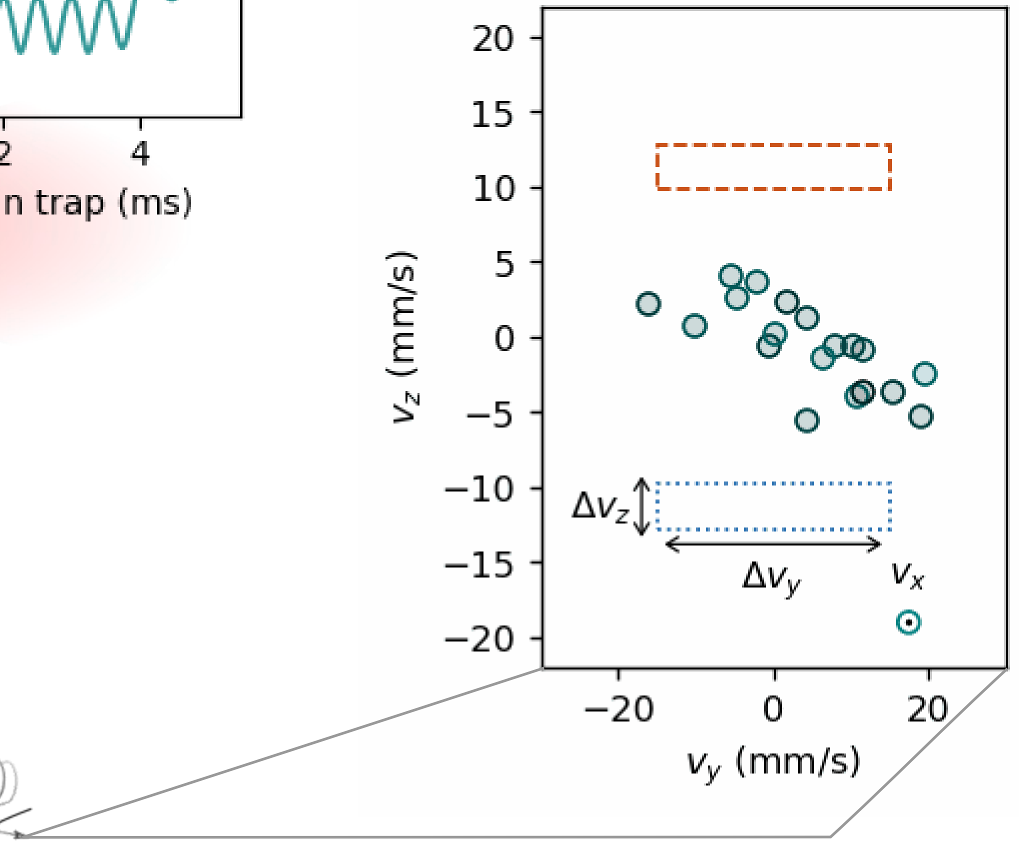
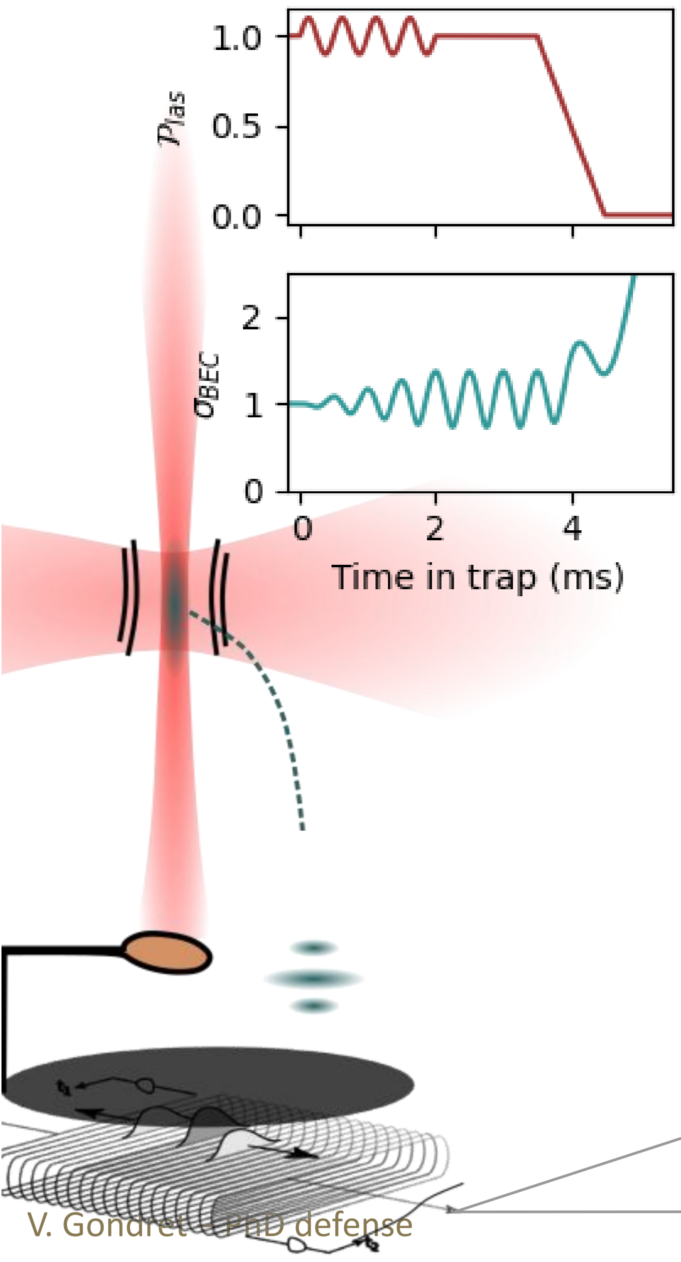


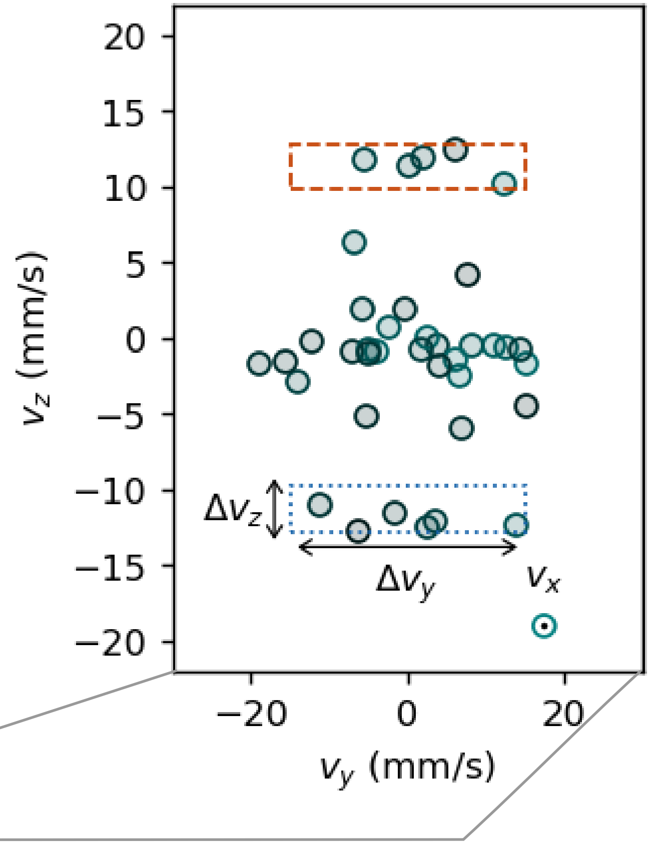
We study correlations between $\hat{n}_{-k}$ & $\hat{n}_k$.



Correlations and relative number squeezing

We study correlations between $\hat{n}_{-k}$ & $\hat{n}_k$.





We observe relative number squeezing

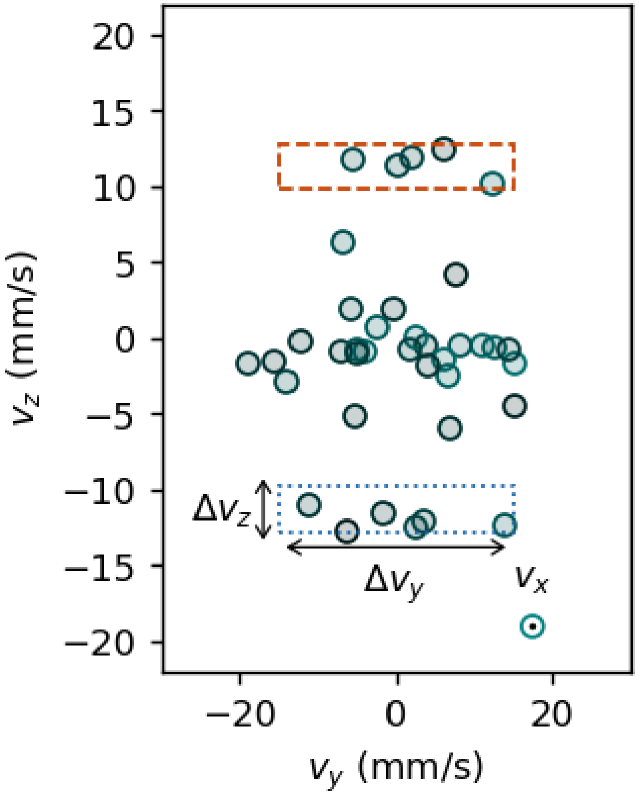
$$\frac{\mathrm{Var}(n_k - n_{-k})}{n_k + n_{-k}} = 0.91(4) < 1$$

Non-classical effect!

But how to witness entanglement ?

How to witness entanglement?

$\hat{\phi}_k$ annihilation operator for an atom



What we need:

$$\underbrace{n_k n_{-k}}_{\text{Populations}} < \left|\underbrace{\langle \hat{\phi}_k \hat{\phi}_{-k} \rangle}_{\text{Anomalous correlation}}\right|^2 \implies \text{entanglement}$$

What we measure:

- Populations: $n_k = \langle \hat{\phi}_k^\dagger \hat{\phi}_k \rangle$, $n_{-k} = \langle \hat{\phi}_{-k}^\dagger \hat{\phi}_{-k} \rangle$
- **N-body** correlation functions

$$G^{(2)}_{k,-k} = \langle \hat{\phi}^\dagger_{-k} \hat{\phi}^\dagger_k \hat{\phi}_{-k} \hat{\phi}_k \rangle,$$

$$G^{(4)}_{k,-k} = \left\langle \hat{\phi}^{\dagger\,2}_{-k} \hat{\phi}^{\dagger\,2}_k \hat{\phi}^2_{-k} \hat{\phi}^2_k \right\rangle,$$

*We need to connect **N-body** correlation functions to **field** correlation functions.*

Hillery & Zubairy PRL (2006), Campo & Parentani PRD (2005)

Single mode statistics

$n_k n_{-k} < |\langle \hat{\phi}_k \hat{\phi}_{-k} \rangle|^2 \implies$ entanglement

Hillery & Zubairy PRL (2006)

Theoretical consideration

- weakly interacting gas,
- linear equation of motion,
- small depletion (1/10 000) to stay perturbative.

} Gaussian hypothesis

Robertson *et al* PRD (2018)

Any correlation functions solely depend on correlation functions between one- and two-field operators $\langle \hat{\phi}_q \rangle$, $\langle \hat{\phi}_q \hat{\phi}_\kappa^{(\dagger)} \rangle$.

Gaussian hypothesis

← We can connect N-body to two-field correlation functions !

If $\langle \hat{\phi}_k \rangle = \langle \hat{\phi}_{-k} \rangle = 0$, the normalized cross two-body correlation function is given by:

$$g^{(2)}_{-k,k} = \frac{G^{(2)}_{-k,k}}{n_k n_{-k}} = 1 + \underbrace{\frac{|\langle \hat{\phi}_k \hat{\phi}_{-k} \rangle|^2}{n_k n_{-k}}}_{\text{Anomalous correlation}} + \underbrace{\frac{|\langle \hat{\phi}_k^\dagger \hat{\phi}_{-k} \rangle|^2}{n_k n_{-k}}}_{\text{Coherence}}$$

Easily checked!

if $\begin{cases} \langle \hat{\phi}_k \rangle = \langle \hat{\phi}_{-k} \rangle = 0 \\ \langle \hat{\phi}_k^2 \rangle = \langle \hat{\phi}_{-k}^2 \rangle = 0 \\ \langle \hat{\phi}_k^\dagger \hat{\phi}_{-k} \rangle = 0 \end{cases}$

$g^{(2)}_{-k,k} > 2 \iff$ entanglement

Busch & Parentani, PRD (2014)

Entanglement and many-body correlation functions

$$g^{(2)}_{-k,k} = \frac{G^{(2)}_{-k,k}}{n_k n_{-k}} = 1 + \underbrace{\frac{\left|\langle\hat{\phi}_k\hat{\phi}_{-k}\rangle\right|^2}{n_k n_{-k}}}_{\text{Anomalous correlation}} + \underbrace{\frac{\left|\langle\hat{\phi}_k^\dagger\hat{\phi}_{-k}\rangle\right|^2}{n_k n_{-k}}}_{\text{Coherence}}$$

$n_k n_{-k} < \left|\langle\hat{\phi}_k\hat{\phi}_{-k}\rangle\right|^2 \Rightarrow$ entanglement

Hillery & Zubairy PRL (2006)

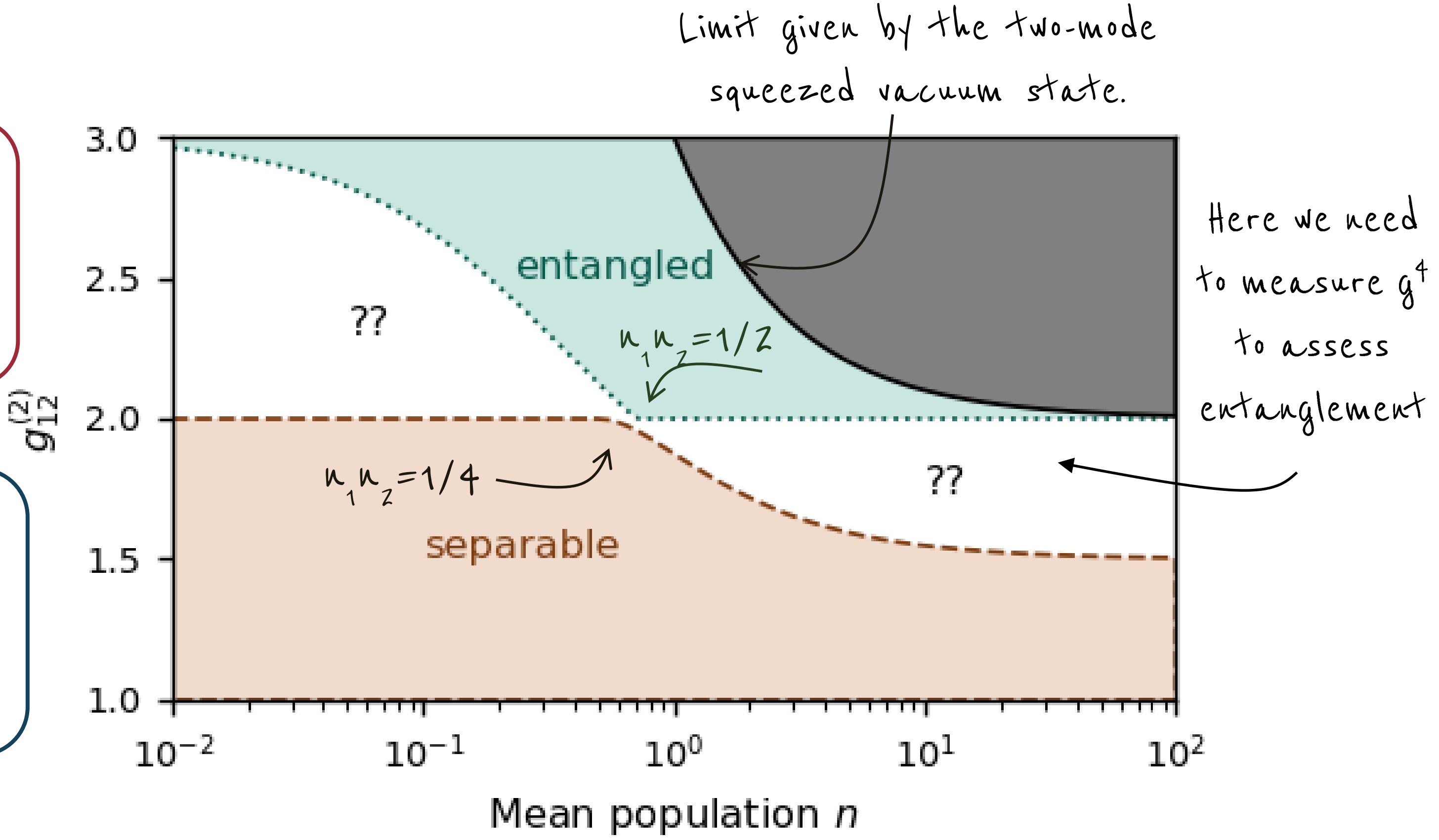
$g^{(2)}/g^{(4)}$ entanglement criterion

For a two-mode thermal Gaussian state*, the measurement of the $n_k, n_{-k}, g^{(2)}_{-k,k}$ & $g^{(4)}_{-k,k}$ quantifies entanglement.

$g^{(2)}$ entanglement witness

For a two-mode thermal Gaussian state*, the measurement of n_k, n_{-k} & $g^{(2)}_{-k,k}$ are sufficient to detect entanglement.

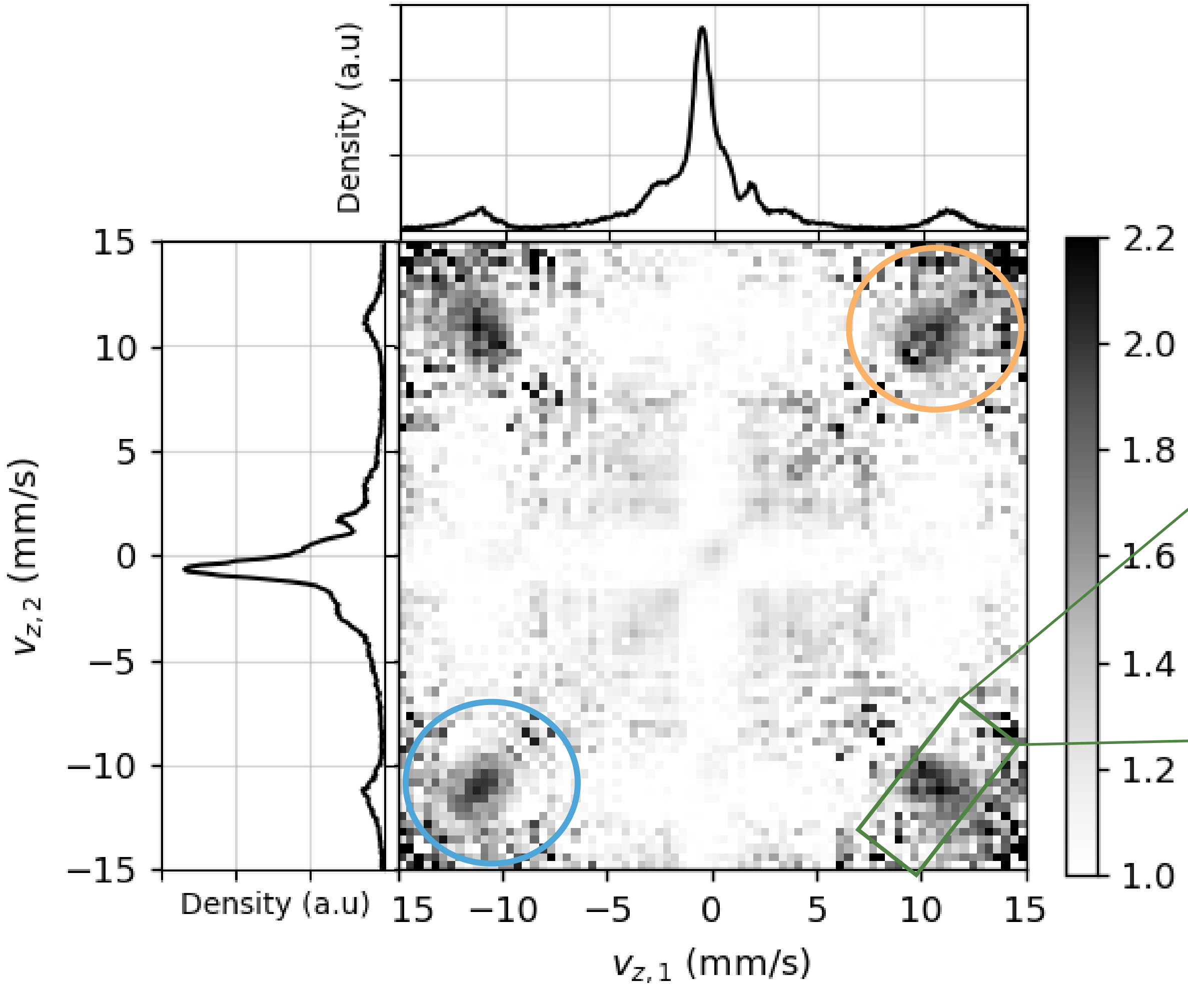
$$* \begin{cases} \langle\hat{\phi}_k\rangle = \langle\hat{\phi}_{-k}\rangle = 0 \\ \langle\hat{\phi}_k^2\rangle = \langle\hat{\phi}_{-k}^2\rangle = 0 \end{cases}$$



Let's check on the experimental side!

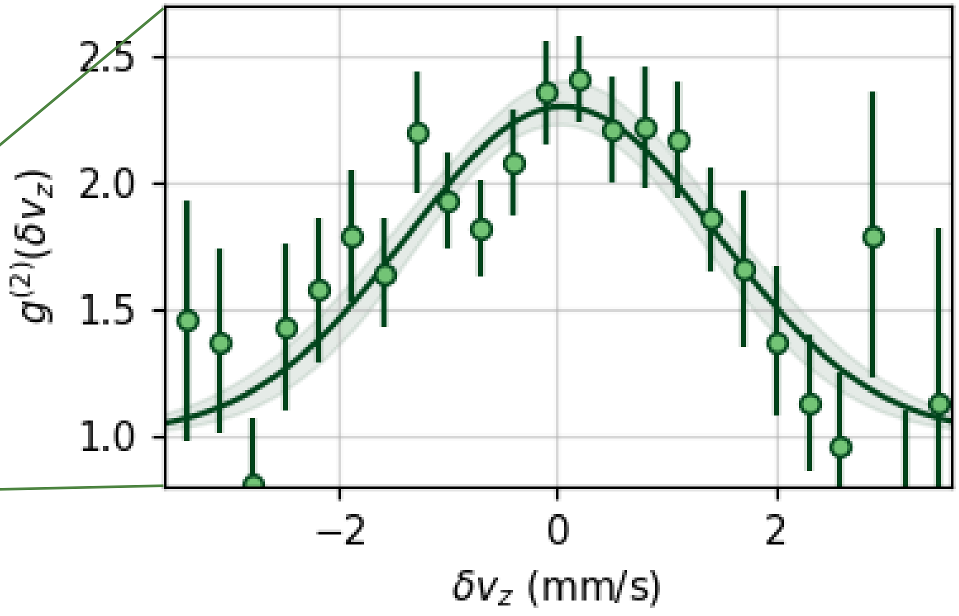
Measuring the two- and four-body correlation function

Map of $g^{(2)}(v_{z,1}, v_{z,2}) = \frac{\langle \hat{n}_1 \hat{n}_2 \rangle}{\langle \hat{n}_1 \rangle \langle \hat{n}_2 \rangle}$



Bosonic bunching:
- $g^{(2)}_{-k,-k} = 1.94(9)$
- $g^{(2)}_{k,k} = 1.98(12)$

As expected for thermal statistics!



We measure $g^{(2)}_{k,-k} = 2.2(1)$

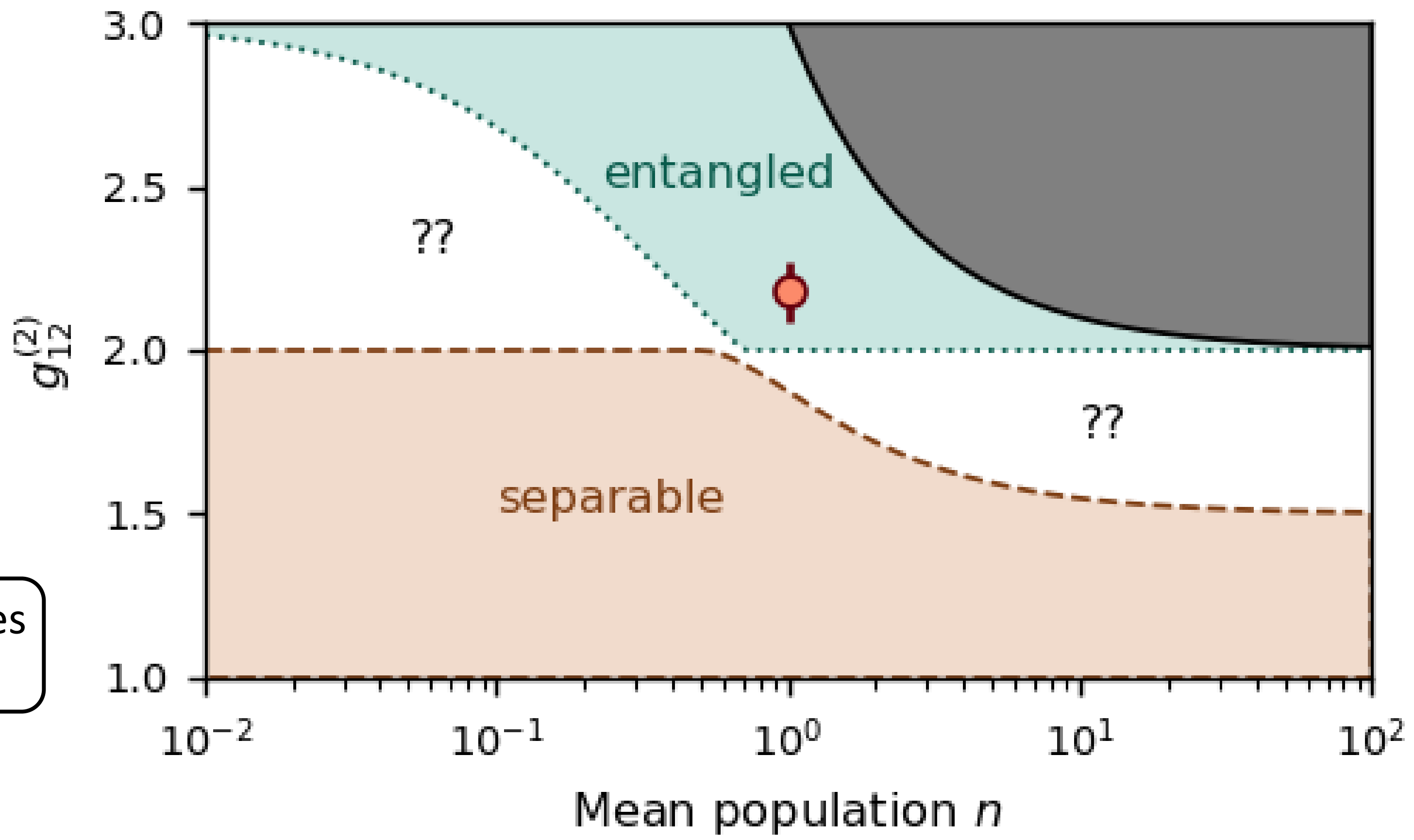
Revealing entanglement!

$g^{(2)}$ entanglement witness

The entanglement witness reveals

that the state is entangled!

Parametric amplification of quasi-particles in a BEC leads to an entangled state.



Summary

Experimental challenges

- Fast production of a stable ultra-cold BEC,
- Observation of really all quasi-particles,
- Quasi-particle frequency is $\omega_k = \Omega/2$,
- **Study the exponential creation process,**
- **Slow-down of the growth in agreement with theoretical predictions,**
- **Observation of entanglement between two modes of opposite momentum massive particles.**

Theoretical proposal

Theoretical studies

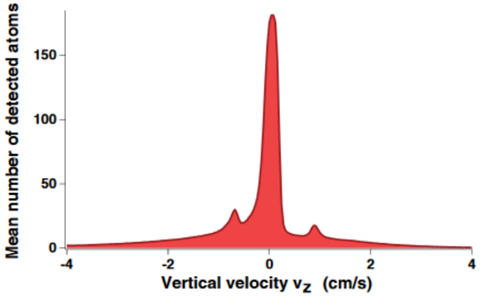
2012

2025

Experimental upgrades

Experimental input

These results



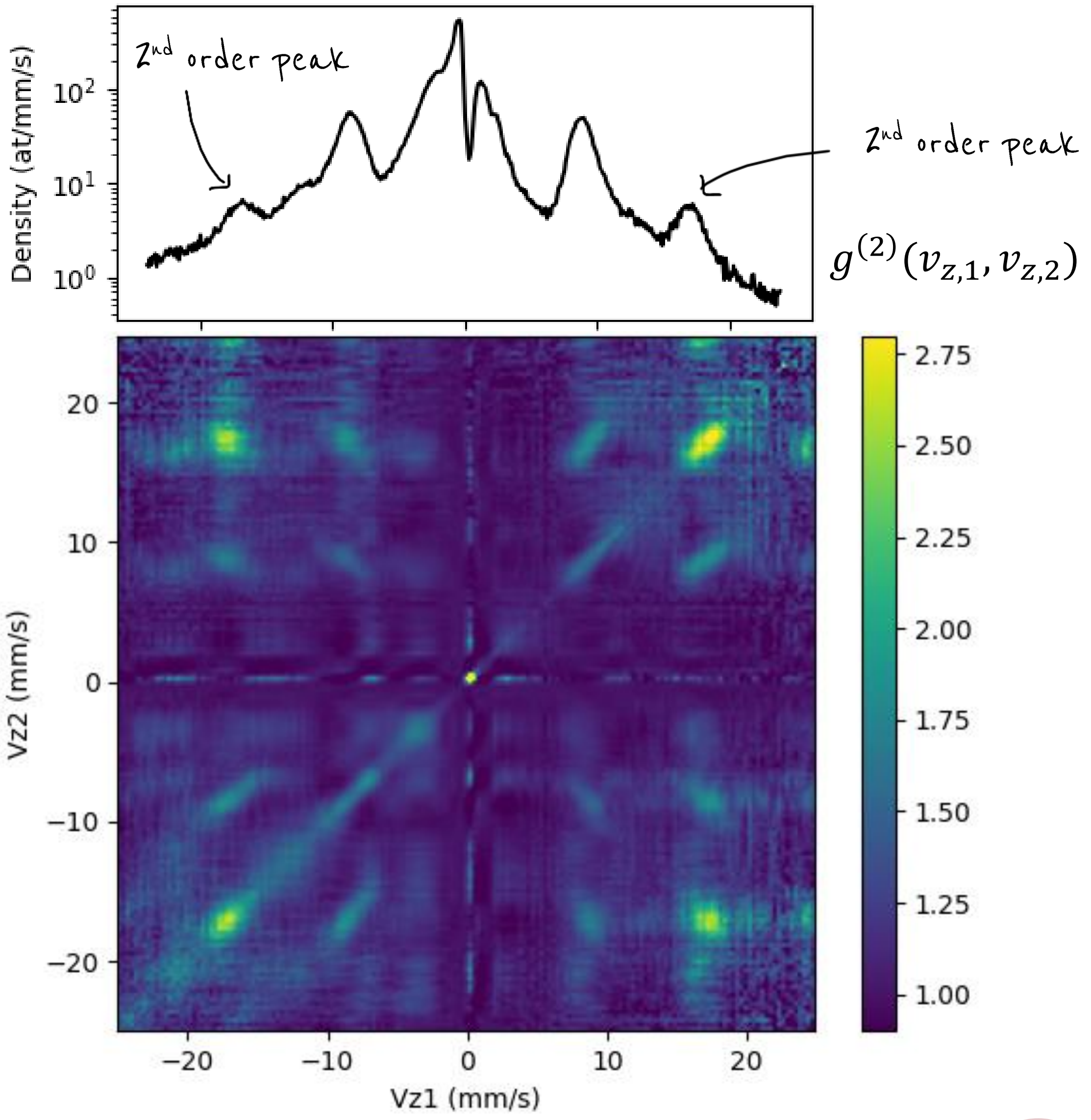
Research perspectives

Entanglement evolution

How entanglement evolves, disappear? Study thermalization of quasi-particles.

Higher order excitations

Higher order resonance with rich correlations properties.



Research perspectives

Research perspectives

Entanglement evolution

How entanglement evolves, disappear? Study thermalization of quasi-particles.

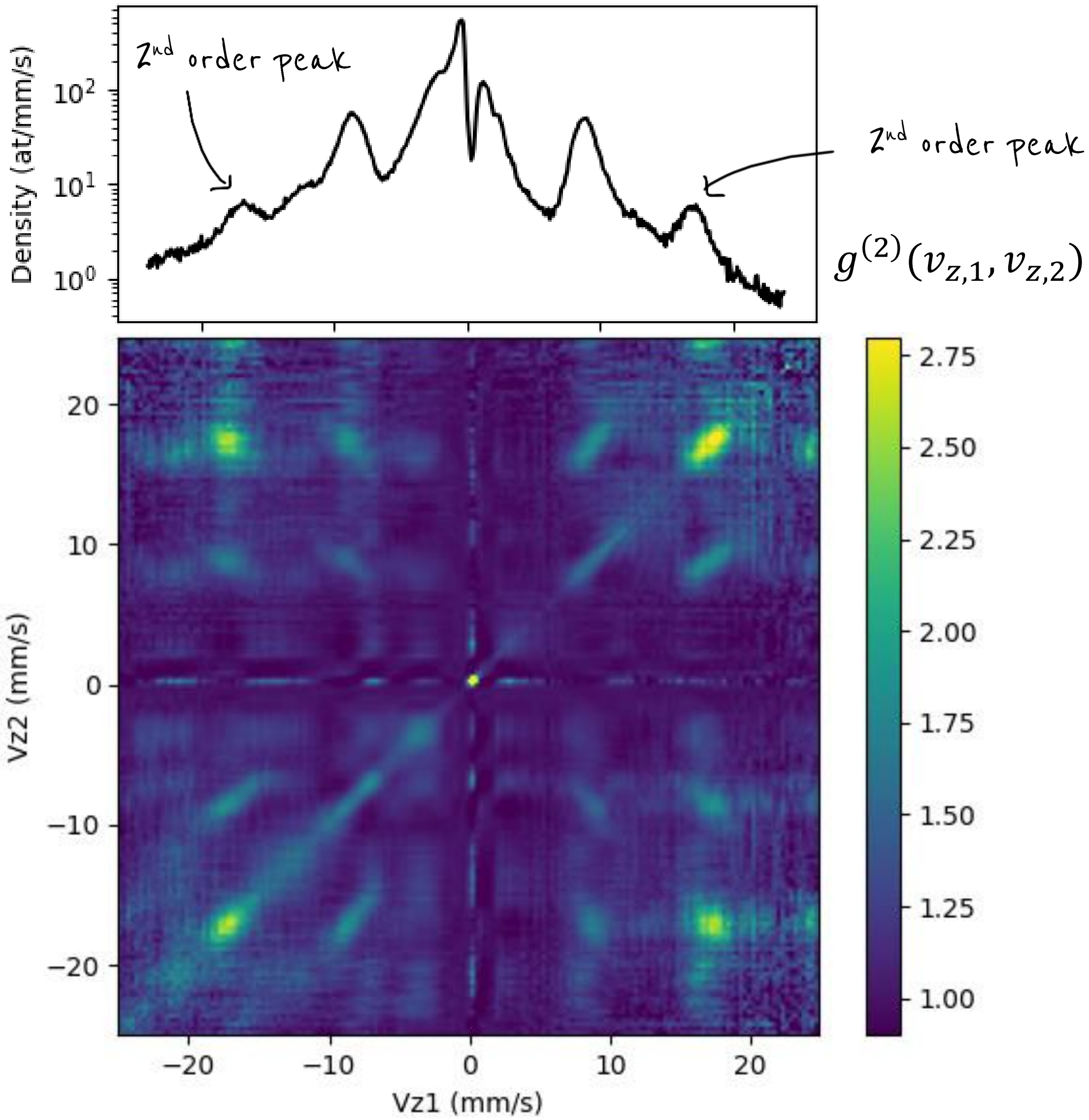
Higher order excitations

Higher order resonance with rich correlations properties.

Beyond the Gaussian hypothesis

Violation of Bell inequalities

- Need for 2x2 modes,
- Low population regime.



Thank you!

Serment

En présence de mes pairs.
Parvenu à l'issue de mon doctorat en physique, et ayant ainsi pratiqué, dans ma quête du savoir, l'exercice d'une recherche scientifique exigeante, en cultivant la rigueur intellectuelle, la réflexivité éthique et dans le respect des principes de l'intégrité scientifique, je m'engage, pour ce qui dépendra de moi, dans la suite de ma carrière professionnelle quel qu'en soit le secteur
ou le domaine d'activité, à maintenir une conduite intègre dans mon rapport au savoir, mes méthodes et mes résultats.

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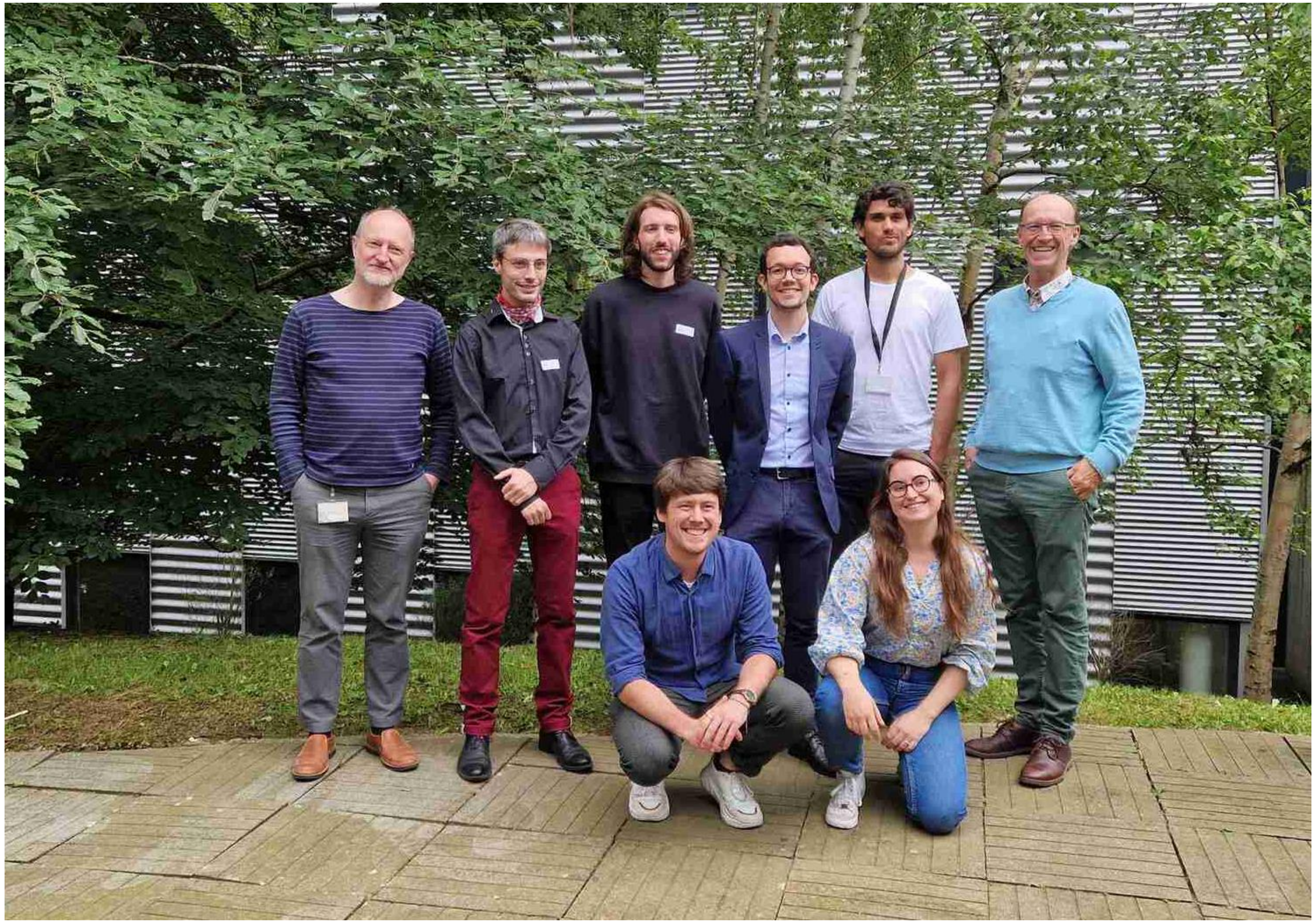
Quentin Marolleau
Charlie Leprince
Clothilde Lamirault
Paul Paquiez
Rui Dias
Léa Camier

Cosqua

Scott Robertson
Amaury Micheli



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Léa, Guillaume, Maxime…

LCF

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Sophie Coumar & l'atelier d'Optique
Fabien Siffritt & le service infrastructure
Enseignants de l'IOGS

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Enseignants de l’IOGS

Soutiens

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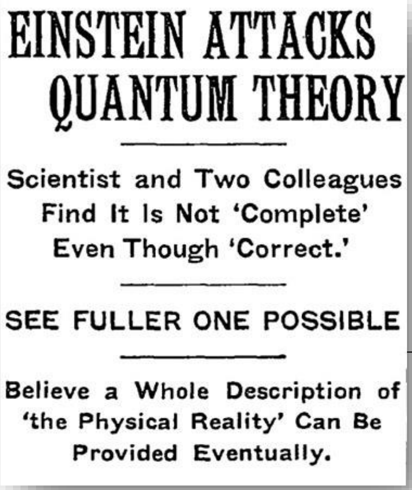
Amis
La coloc’ de la Plata
Famille
Alice

Pôt

Marcelo pour les empanadas
Catherine pour la mousse au chocolat
Mes parents, grands-mères & Alice pour tout le reste.

Appendix

We want more!


EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

EPR Paradox

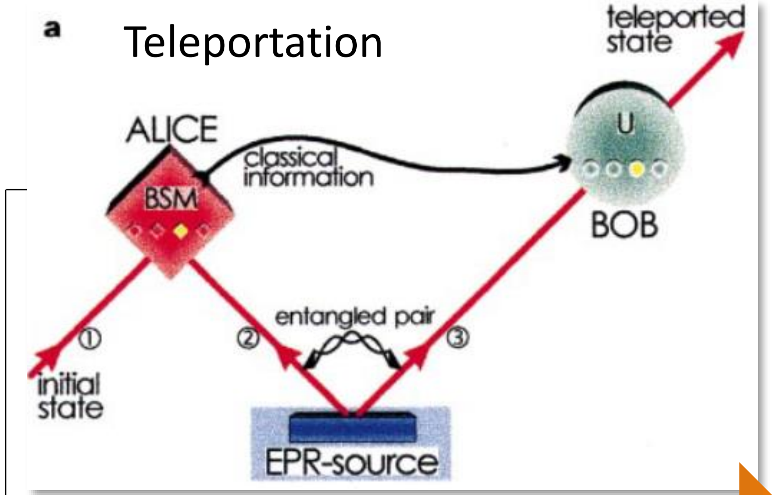
$$|\Psi^{(\pm)}\rangle \sim \frac{|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle}{\sqrt{2}}$$

"Entanglement"

"Steerability"

From 1935 to 2025

Bell's inequality
Bell, *Physics* (1964)
Aspect *et al*, PRL (1981-82)



Bennett *et al*, *PRL* (1993)
Popescu, *PRL*, (1994)
Bouwmeester *et al. Nature* (1997)

Equivalence for pure states
Gisin, Phys. Lett. A (1991)

Non-separability
Werner, *PRA* (1989)

PPT criterion
Peres, PRL (1996)
Horodecki[3], *Phys. Lett. A*(1996)

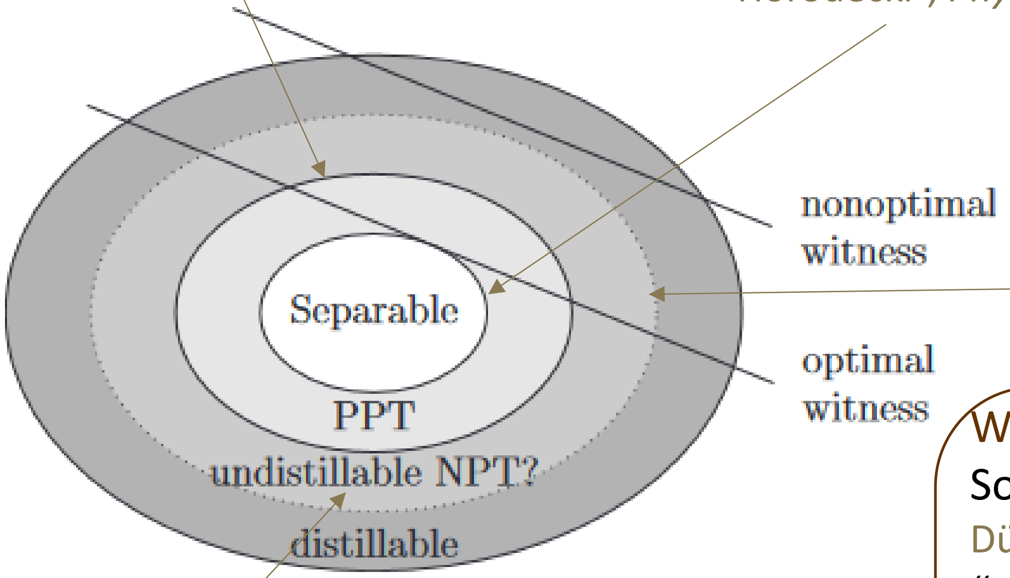
Entanglement distillation
Bennett *et al*, PRL, (1993)
Horodecki[3], *PRL* (1998)
Dür, PRL (1998)

A bipartite state ρ is distillable, iff - by means of LOCC – we can create $|\Psi^{(\pm)}\rangle$ out of n identical copies of ρ.

What is entanglement?

NPT implies entanglement
Peres, *PRL* (1996)

Some states are entangled but with PPT (*bound* entanglement)
Horodecki[3], *Phys. Lett. A*(1996)



Any distillable state must be NPT.
Horodecki[3], *PRL* (1998)
However, it is equivalent for two-modes Gaussian states (also 1xN)
Duan *et al*, *PRL* (2001), Werner and Wolf, *PRL* (2001)

If a bounded NPT state σ exists, then it also exists ρ PPT such that $\rho \otimes \sigma$ is distillable (*superactivation*).
Shor, Smolin & Terhal, *PRL* (2001)

What about Bell's inequalities?
Some states violate a Bell inequalities but are not distillable (*bounded*).
Dür, *PRL* (2001)
"Violation of a Bell inequality implies *bipartite* distillability".
Acin, *PRL* (2001)
Any bipartite entangled state σ exhibits a hidden nonlocality which can be activated ($\exists\ \rho$ entangled that does not violate CHSH inequality but $\rho \otimes \sigma$ violates it)
Acin, *PRL* (2001)

Horodecki[4], *RMP* (2009)

Probing the entanglement of Gaussian states from its FCS

The state is characterized by $n_1,\ n_2,\ \langle\hat{a}_1\hat{a}_2\rangle$ & $\langle\hat{a}_1^\dagger\hat{a}_2\rangle$

Lemma 1: Measurement of $n_1,\ n_2$, $g_{12}^{(2)}$ & $g_{12}^{(4)}$ yields a symmetric system for $|\langle\hat{a}_1\hat{a}_2\rangle|$ & $|\langle\hat{a}_1^\dagger\hat{a}_2\rangle|$

- $g_{12}^{(2)}$ involves their quadratic sum,
- $g_{12}^{(4)}$ also involves their product.

We find two solutions $\beta_\pm$

$$\beta_\pm^2 = n_1 n_2\left(g_{12}^{(2)} - 1\right)\frac{1 \pm \sqrt{1-\theta}}{2}$$

where

$$\theta = \frac{g_{12}^{(4)} + 12 - 16 g_{12}^{(2)} - 4\left(g_{12}^{(2)} - 1\right)^2}{\left(g_{12}^{(2)} - 1\right)^2}$$

Hypothesis

- ✓ Gaussian state
- ✓ Zero mean
- ✓ $\langle\hat{a}_1^2\rangle = \langle\hat{a}_2^2\rangle = 0$

$\theta \in [0,1]$ so that $\beta_\pm^2 \geq 0$ as a supplementary check for the consistency of the hypothesis.

We have two possible solutions

- "State" μ: $|\langle\hat{a}_1\hat{a}_2\rangle| = \beta_+$ & $|\langle\hat{a}_1^\dagger\hat{a}_2\rangle| = \beta_-$,
- "State" γ: $|\langle\hat{a}_1\hat{a}_2\rangle| = \beta_-$ & $|\langle\hat{a}_1^\dagger\hat{a}_2\rangle| = \beta_+$.

$g_{12}^{(4)} = \langle\hat{a}_1^{\dagger 2}\hat{a}_2^{\dagger 2}\hat{a}_1^2\hat{a}_2^2\rangle / n_1^2 n_2^2$

Probing the entanglement of Gaussian states from its FCS

We have two possible solutions

- "State" μ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_+$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle| = \beta_-$,
- "State" γ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_-$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle| = \beta_+$.

Check they can exist!

$$f(n_1, n_2, x, y) = \frac{\Delta - \sqrt{\Delta^2 - \det \boldsymbol{\sigma}}}{2}$$

where

$$\det \boldsymbol{\sigma} = 16(x^2 - y^2)^2 + (1 + 2n_1)^2(1 + 2n_2)^2 - 9(x^2 + y^2)(1 + 2n_1)(1 + 2n_2)$$

and

$$\Delta = (2n_1 + 1)^2 + (2n_2 + 1)^2 - 8(x^2 - y^2)$$

Hypothesis

- ✓ Gaussian state
- ✓ Zero mean
- ✓ $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

Lemma 2:

The *bona fide* condition does not depend on the phase of $\langle \hat{a}_1 \hat{a}_2 \rangle$ and $\langle \hat{a}_1^\dagger \hat{a}_2 \rangle$.

The (smallest) eigenvalue is given by

$$\nu_\mu = f(n_1, n_2, \beta_+, \beta_-) \quad \& \quad \nu_\gamma = f(n_1, n_2, \beta_-, \beta_+)$$

Lemma 3:

'States' μ and γ are partial transpose of each other.

We have 3 possibilities

- $\nu_\gamma \leq \nu_\mu < 1$: unphysical states (wrong hypothesis)
- $\nu_\gamma < 1 \leq \nu_\mu$: only one solution (we found it), → The state is entangled
- $1 \leq \nu_\gamma \leq \nu_\mu$: two solutions and we cannot distinguish the states → The state is separable

Probing the entanglement of Gaussian states from its FCS

Entanglement criterion

- Measure $n_1,\ n_2, g_{12}^{(2)}$ & $g_{12}^{(4)}$ and deduce $\beta_\pm$,
- Compute $\nu_\gamma = f(n_1, n_2, \beta_-, \beta_+)$
- The state is entangled iff $\nu_\gamma < 1$, (*criterion*)
- Quantify entanglement $\mathrm{LN} = \mathrm{Max}(-\log_2 \nu_\gamma, 0)$

Without $g_{12}^{(4)}, g_{12}^{(2)}$ is still a witness!

Hypothesis

- ✓ Gaussian state
- ✓ Zero mean
- ✓ $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

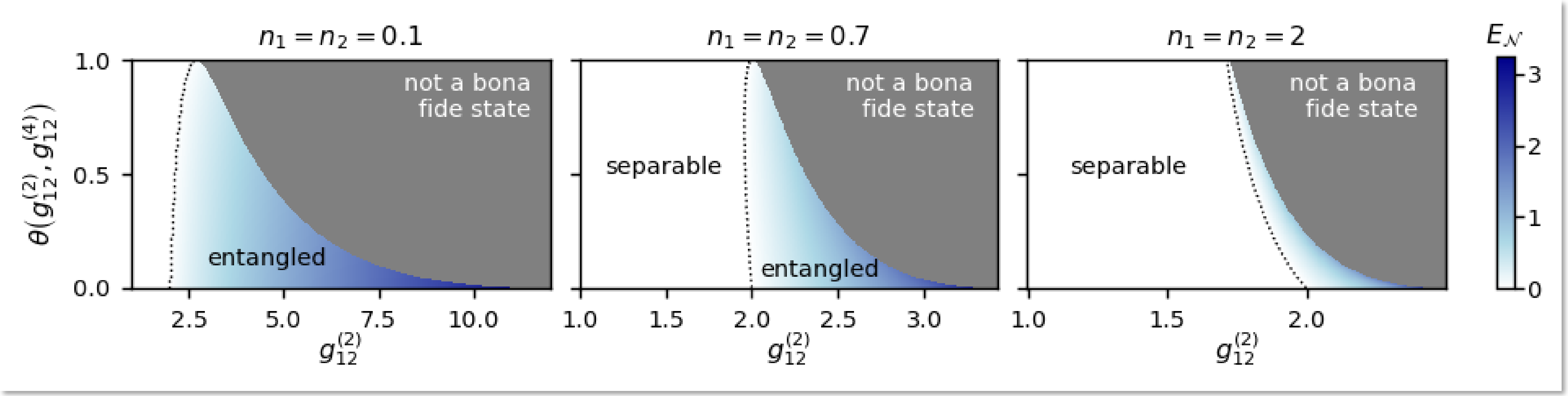


Fig: Entanglement in the $(g_{12}^{(2)}, \theta)$ plane for three populations.

Probing the entanglement of Gaussian states from its FCS

Entanglement criterion

- Measure $n_1, \ n_2, g_{12}^{(2)}$ & $g_{12}^{(4)}$ and deduce $\beta_{\pm}$,
- Compute $\nu_\gamma = f(n_1, n_2, \beta_-, \beta_+)$
- The state is entangled iff $\nu_\gamma < 1$, (*criterion*)
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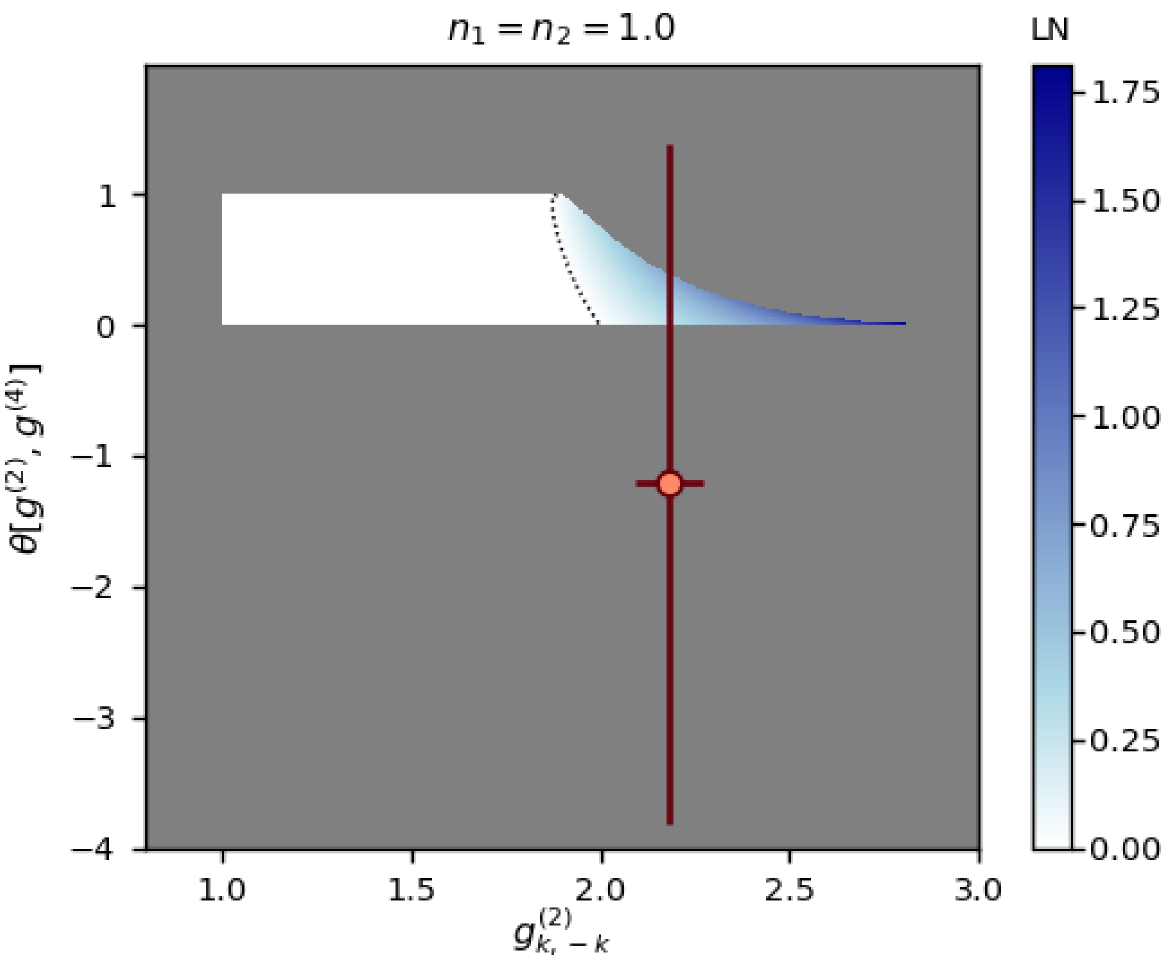

Fig: Trying to use the g4/g2 criterion with experimental data

Particle *versus* mode entanglement

The debate

Consider $\hat{a}_\uparrow^\dagger \hat{a}_\downarrow^\dagger |vac\rangle = |1,1\rangle$ in 2nd quantization.

In the 1st quantized picture, labelling particles by A and B, we have

$$\frac{|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B}{\sqrt{2}}$$

which is entangled?

For some, this 'entanglement' is unphysical and the labels A and B are meaningless.

No consensus on the nature of this correlation due to exchange symmetry, sometime referred to as *particle entanglement*.

Nevertheless, particle entanglement is a useful and consistence resource"

Morris *et al.* PRX **10** (2020)

"Identical particle entanglement can be transferred, with unit probability, onto independent modes using elementary operations. Thus, symmetrization entanglement is a fundamental, ubiquitous, and readily extractable resource for standard quantum information tasks."

Killoran, Cramer and Plenio, PRL **112** (2014)

New definitions of entanglement have been proposed but only Werner's definition based on the *mode entanglement* is satisfying

Benatti *et al.* Phys. Rep. (2020).

Violation of the Cauchy-Schwarz inequality is a particle entanglement witness

Wasak *et al.* PRA. (2014).

(I strongly recommend to read the introduction of Morris *et al* and Killoran *et al*.)

$g_{12}^{(2)}$ entanglement witness

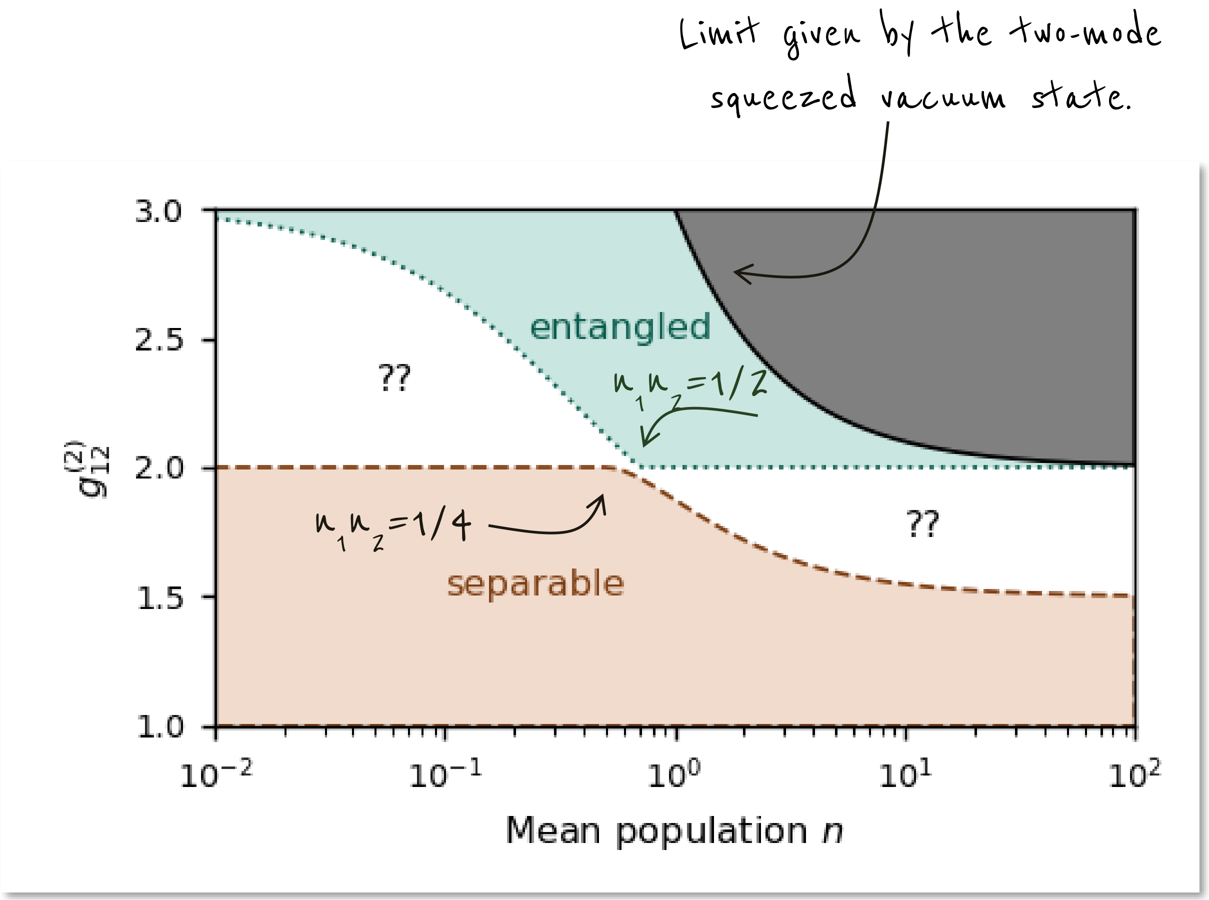

Fig: Entanglement witness based on the value of $g_{12}^{(2)}$.

- The $g_{12}^{(2)}$ entanglement witness depends on the populations,
- The value *of* $g_{12}^{(4)}$ is needed to determine the entanglement in the '??'.
- Taking into account the quantum efficiency of the detector can 'help' to witness entanglement,

Hypothesis

- ✓ Gaussian state
- ✓ Zero mean
- ✓ $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

If $\langle \hat{a}_j^2 \rangle \neq 0$, the phases matter in the state's non-separability.