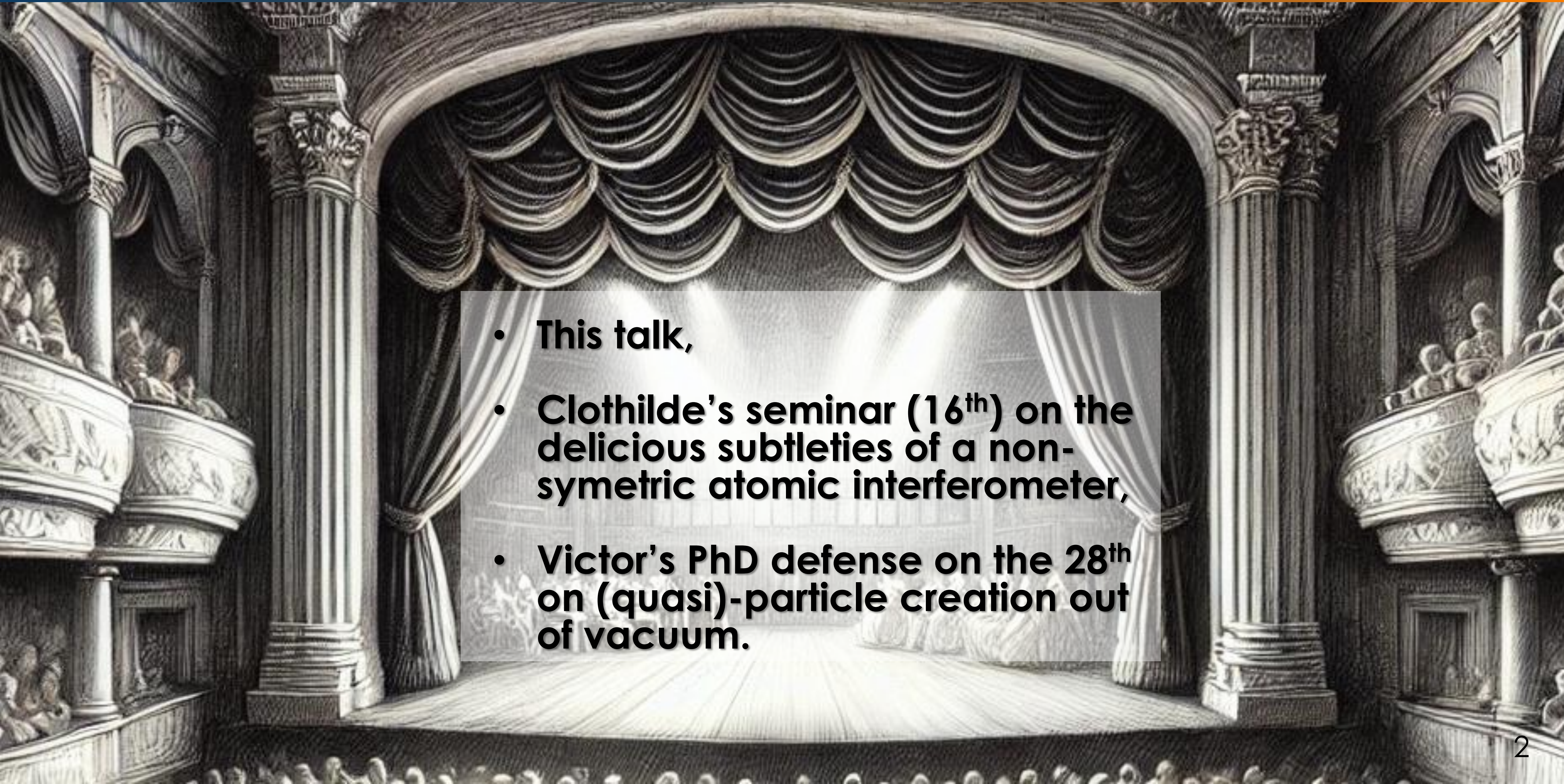
Quantifying the entanglement of two-mode Gaussian states *via* their full counting statistics

Victor Gondret

For the Quantum Atom Optics team

Quantum Gases group seminar – Monday, 13th of January, 2025

The Quantum Atom Optics team in the spotlight



- This talk,
- Clothilde's seminar (16th) on the delicious subtleties of a non-symetric atomic interferometer,
- Victor's PhD defense on the 28th on (quasi)-particle creation out of vacuum.

Outline of the talk

Part ④

Part ②

Quantifying the entanglement of two-mode Gaussian states *via* their full counting statistics

(and why)

Part ③

Part ①

Part ⑤ An entanglement criterion based on the one-, two-, ~~three-,~~ and four-body correlation functions.

Outline of the talk

① Full counting statistics and motivations

② Some notions on entanglement

③ Gaussian state formalism

④ Entanglement witness and quantifier

⑤ An entanglement criterion based on the one-, two-, and four-body correlation functions.

Why quantifying entanglement from FCS

Entanglement : fundamental resource of the 2nd quantum revolution.

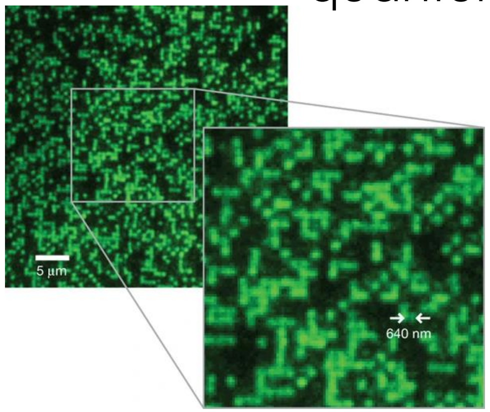
Quantifying entanglement

- How useful the state is for teleportation, communication…?
- Why and how entanglement is dissipated ? (thermal bath, gravity…)

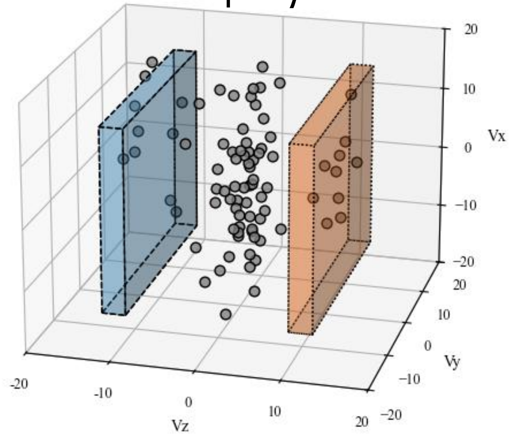
@LCFGQ: 3/6 experiments (and 5/8 permanents) !

FCS: full counting statistics (with a quantum gas microscope)

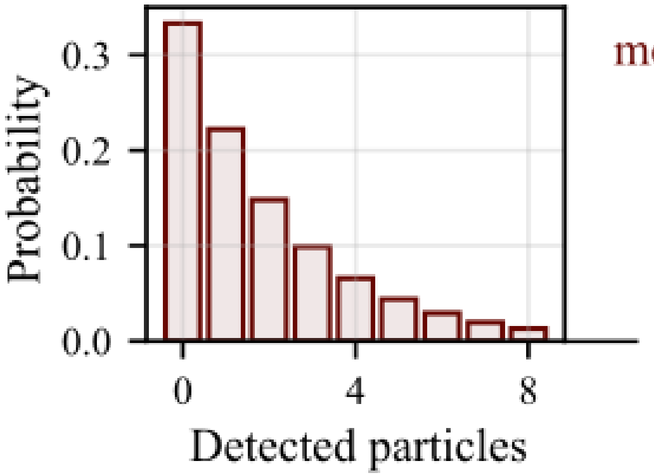
Bakr et al. *Nature* (2009)

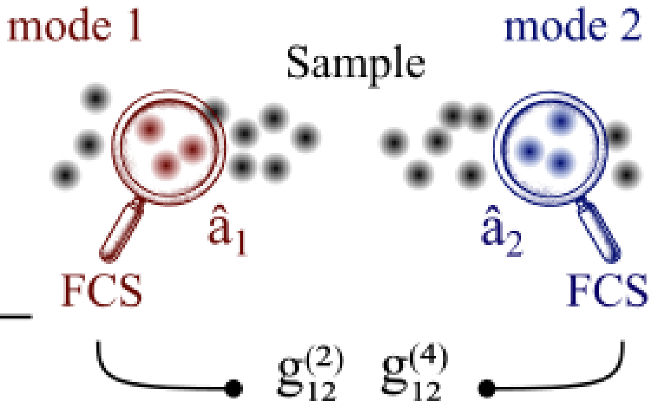


In situ (position space)



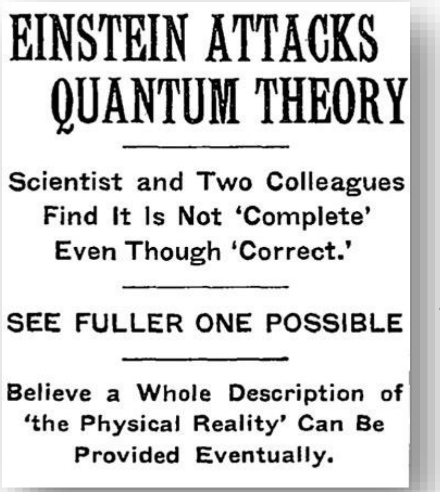
Time-of-flight momentum space





Outline of the talk

Definition of entanglement



EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

EPR (1935) paradox

$$|\Psi^{(\pm)}\rangle \sim \frac{|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle}{\sqrt{2}}$$

Entanglement is "not *one* but rather *the* characteristic trait of quantum mechanics", Schrödinger (1935)

Pure states

Any non-product pure state violates a Bell inequality.

Entanglement $\Leftrightarrow$ Bell inequalities

$\Leftrightarrow$ Distillability

$\Leftrightarrow$ Teleportation

Gisin, Phys. Lett. A (1991)
Gisin & Peres, Phys. Lett. A (1992)
Popescu & Rohrlich, Phys. Lett. A (1992)

Manifestation of entanglement through the violation of a Bell inequality

Bell (1964)
CHSH (1969)

Mixed states

Teleportation $\not\Rightarrow$ Bell inequalities

$\rho = \frac{1}{8}\mathbb{I} + \frac{1}{2}|\Psi^{(+)}\rangle\langle\Psi^{(+)}|$, Popescu PRL, (1994)

Mathematically, any **separable** state can be written as

$$\rho = \sum_i \alpha_i \rho_{i,1} \otimes \rho_{i,2}$$

where 1 and 2 refer to the two subsystems (the partition) and $\alpha_i \geq 0$ are probabilities.

Entanglement $\Leftarrow$ Bell inequalities

Werner *Phys. Rev. A* (1989)

Non-separability exemples

Mathematically, any **separable** state can be written as

$$\rho = \sum_i \alpha_i \rho_{i,1} \otimes \rho_{i,2}$$

where 1 and 2 refer to the two subsystems (the partition) and $\alpha_i \geq 0$ are probabilities.

Consider two modes 1 & 2 in a partition A with $\hat{a}_1, \hat{a}_2$ annihilation operators.

A two-mode squeezed vacuum state $$|TMSv\rangle(r) \sim \sum_i \tanh^i r \, |i,i\rangle$$

$$\rho_{TMSv} \sim \sum_{i,k} \tanh^i r \tanh^k r \, |i,i\rangle\langle k,k|$$

✓ ρ_{TMSv} is a non-separable state in the partition A

Werner *Phys. Rev. A* **40**, 4277 (1989)

Non-separability exemples

Mathematically, any **separable** state can be written as

$$\rho = \sum_i \alpha_i \rho_{i,1} \otimes \rho_{i,2}$$

where 1 and 2 refer to the two subsystems (the partition) and $\alpha_i \geq 0$ are probabilities.

① A 2-mode, 1-particle state

$$|u\rangle = \frac{|0,1\rangle + |1,0\rangle}{\sqrt{2}} = \frac{\hat{a}_1^\dagger + \hat{a}_2^\dagger}{\sqrt{2}} |vac\rangle$$

$$\rho_u \sim |0,1\rangle\langle 0,1| + |1,0\rangle\langle 0,1| + |0,1\rangle\langle 1,0| + |1,0\rangle\langle 1,0|$$

✓ ρ_u is **a non-separable** state (in the partition A)

Sperling *et al.* Phys. Rev. A **100**, 062129 (2019)

Consider two modes 1 & 2 in a partition A with $\hat{a}_1, \hat{a}_2$ annihilation operators.

Does the non-separability of a bi-partite state depend on the partition?

Consider now the partition E= $(\hat{e}_1, \hat{e}_2)$ where

$$\begin{pmatrix} \hat{e}_1 \\ \hat{e}_2 \end{pmatrix} = U \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

$$|u\rangle = \hat{e}_1^\dagger |vac\rangle$$

$$\rho_u \sim |1,0\rangle_E \langle 1,0|_E$$

× Is ρ_u is a **separable** state (in the partition E)

Partition: chose the basis $e^{\pm ikx}$ or the cosine and sine basis.

Non-separability exemples

Mathematically, any **separable** state can be written as

$$\rho = \sum_i \alpha_i \rho_{i,1} \otimes \rho_{i,2}$$

where 1 and 2 refer to the two subsystems (the partition) and $\alpha_i \geq 0$ are probabilities.

Does the non-separability of a bi-partite state depend on the partition?

Consider now a 2-mode, 2-particle state

✓ Both particles in the same mode:

$$|u\rangle = \frac{1}{2}\left(\hat{a}_1^\dagger + \hat{a}_2^\dagger\right)^2 |vac\rangle$$

ρ_u is non-separable in the partition A.

$\rho_u \sim |2{,}0\rangle_E\langle 2{,}0|_E$ is separable in the partition E.

✓ Both particles in orthogonal modes:

$$|u\rangle = \frac{1}{2}\left(\hat{a}_1^\dagger + \hat{a}_2^\dagger\right)\left(\hat{a}_1^\dagger - \hat{a}_2^\dagger\right)|vac\rangle$$

ρ_u is non-separable in the partition A.

$\rho_u \sim |1{,}1\rangle_I$ … the partit…

× Take each particle in modes which are nonparallel and nonorthogonal

$$|u\rangle = \hat{a}_1^\dagger \frac{\hat{a}_1^\dagger + \hat{a}_2^\dagger}{\sqrt{2}} |vac\rangle = \frac{|2{,}0\rangle_A + |1{,}1\rangle_A}{\sqrt{2}}$$

Theorem: Sometimes, the non-separability of a bi-partite state depends on the partition. But there are states that are entangled no matter the partition.

Sperling *et al.* Phys. Rev. A **100**, 062129 (2019)

Probing the non-separability of a TMSv state from its FCS

Mathematically, any **separable** state can be written as

$$\rho = \sum_i \alpha_i \rho_{i,1} \otimes \rho_{i,2}$$

where 1 and 2 refer to the two subsystems (the partition) and $\alpha_i \geq 0$ are probabilities.

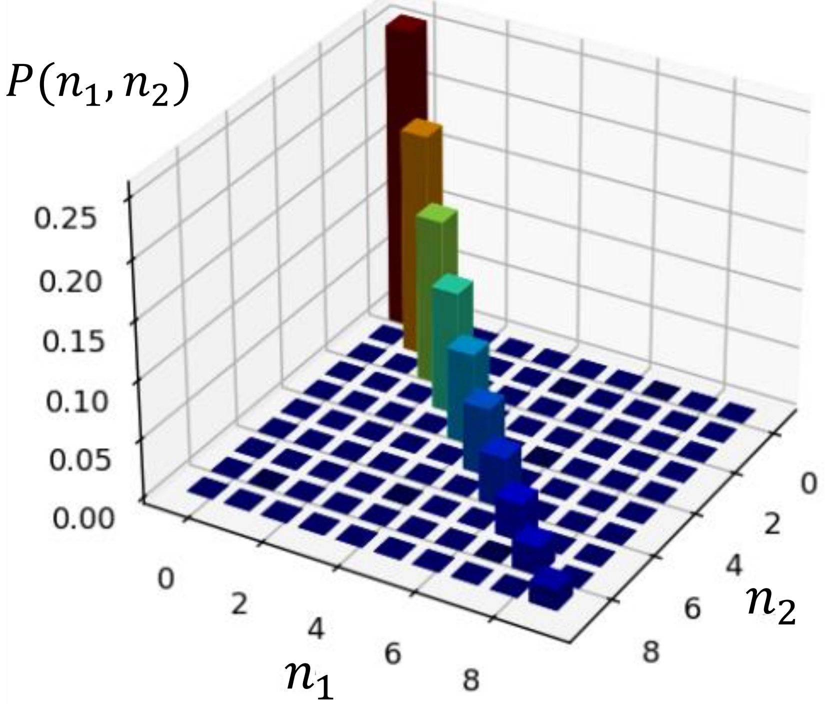
Consider a two-mode squeezed vacuum state

$$|TMSv\rangle(r) \sim \sum_i \tanh^i r \, |i,i\rangle_A$$

$$\rho_{TMSv} \sim \sum_{i,k} \tanh^i r \tanh^k r \, |i,i\rangle_A \langle k,k|_A$$

ρ_{TMSv} is a non-separable state in the partition A.

Can we prove the NS from the FCS?



But the state describe by

$$\rho_{classical} \sim \sum_i \tanh^i r \, |i,i\rangle\langle i,i|$$

Is a separable state which has the same two-mode probability distribution as a TMSv).

Non-separability from the FCS

One cannot assess the non-separability of any quantum state from their full counting statistics.

THANK YOU FOR YOUR ATTENTION !

Wait a minute… Not true for Gaussian states!

Outline of the talk

For mixed states, Werner's definition of entanglement (non-separability)

① Full counting statistics and motivations

② Some notions on entanglement

③ Gaussian state formalism

④ Entanglement witness and quantifier

⑤ An entanglement criterion based on the one-, two-, and four-body correlation functions.

Gaussian states

Definition
Any operator that involves more than 2 fields can be expressed with 1- and 2-field operators.

Exemple:

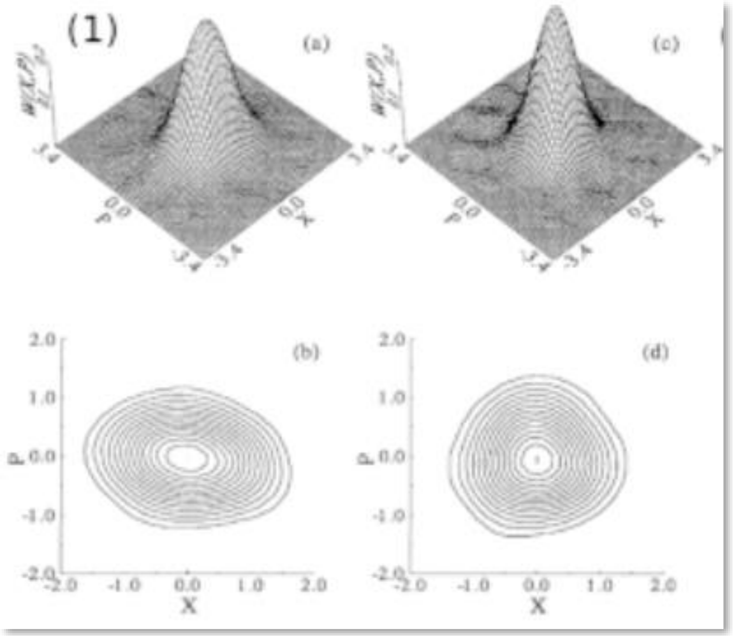
$$\langle \hat{a}_1^\dagger \hat{a}_1 \hat{a}_2^\dagger \rangle = \langle \hat{a}_1^\dagger \hat{a}_1 \rangle \langle \hat{a}_2^\dagger \rangle + \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \rangle \langle \hat{a}_1 \rangle + \langle \hat{a}_1 \hat{a}_2^\dagger \rangle \langle \hat{a}_1^\dagger \rangle - 2 \langle \hat{a}_1^\dagger \rangle \langle \hat{a}_1 \rangle \langle \hat{a}_2^\dagger \rangle$$

Demonstration of Gaussianity by showing that all cumulants higher than 2 vanish.

Leonhardt, *Essential of Quantum Optics* (2010)
Leibfried *et al.* Phys. Rev. Lett. **77**, 4281 (1996)

Definition
A Gaussian states has a Gaussian Wigner quasi-probability distribution of the quadratures.

$$W(x,p) = \frac{1}{2\pi} \int e^{ipyp/\hbar} \langle x - y/2 | \hat{\rho} | x + y/2 \rangle dy$$



Measure the Wigner function and show it is Gaussian

Covariance matrix of a Gaussian state

A Gaussian state $\hat{\rho}$ is defined by its first and second moments

$$\hat{\boldsymbol{r}} = \left(\hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2, \hat{a}_2^\dagger\right)^T$$

Mean $\langle \hat{\boldsymbol{r}}_j \rangle$

Covariance matrix

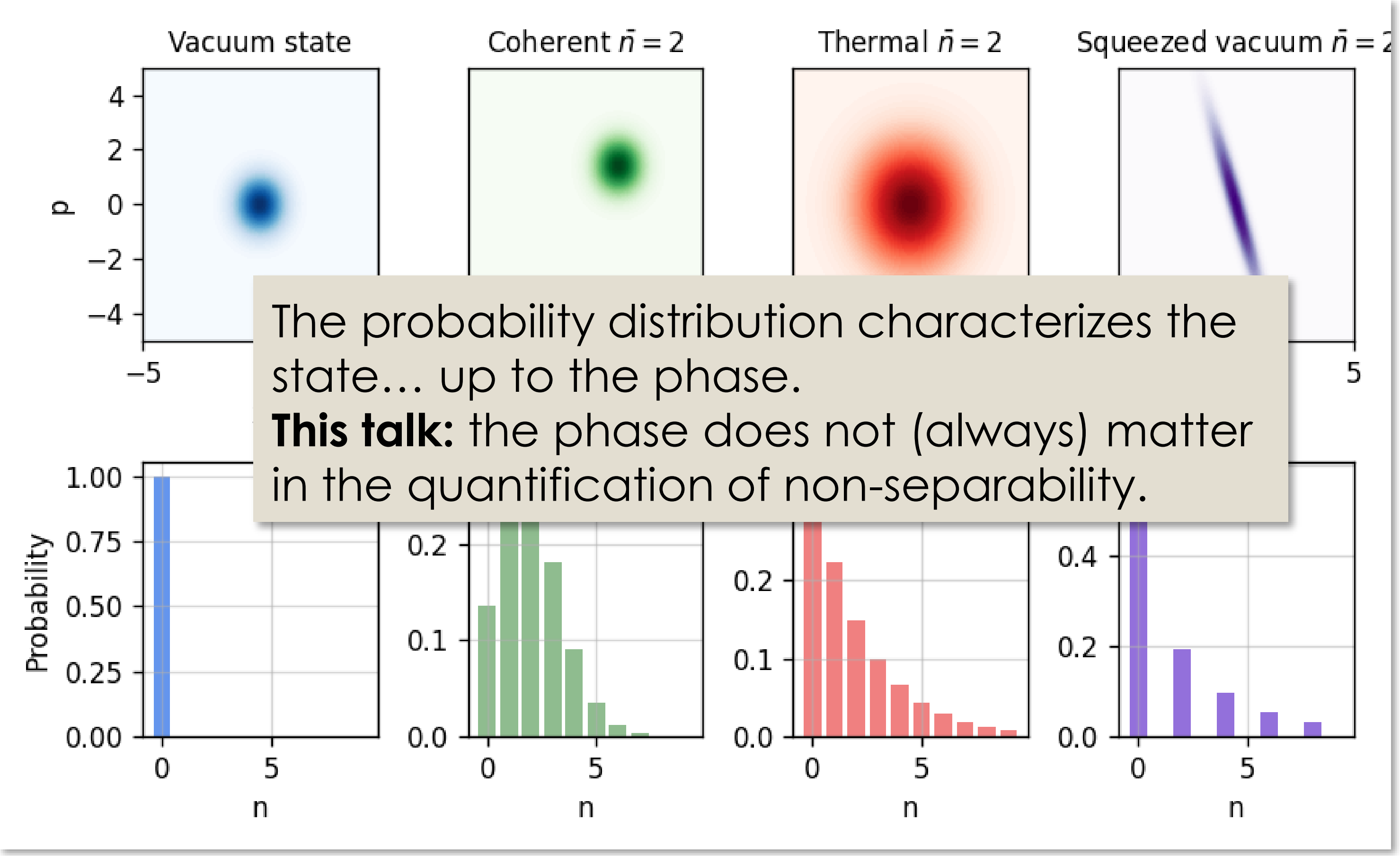
$$\boldsymbol{\sigma}_{ji} = \langle \{ \hat{\boldsymbol{r}}_j - \langle \hat{\boldsymbol{r}}_j \rangle, \hat{\boldsymbol{r}}_i^\dagger - \langle \hat{\boldsymbol{r}}_i^\dagger \rangle \} \rangle$$

$\boldsymbol{\sigma}$ is hermitian (but is often defined as real symmetric).

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}^\dagger & \boldsymbol{B} \end{pmatrix}$$

Single mode properties (A,B) is obtained by tracing out other modes.

$$\boldsymbol{A} = \begin{pmatrix} 2n_1 + 1 & 2\langle \hat{a}_1^2 \rangle \\ 2\left\langle \hat{a}_1^{\dagger 2} \right\rangle & 2n_1 + 1 \end{pmatrix} \quad \text{if } \langle \hat{\boldsymbol{r}}_j \rangle = 0$$



The probability distribution characterizes the state… up to the phase.
This talk: the phase does not (always) matter in the quantification of non-separability.

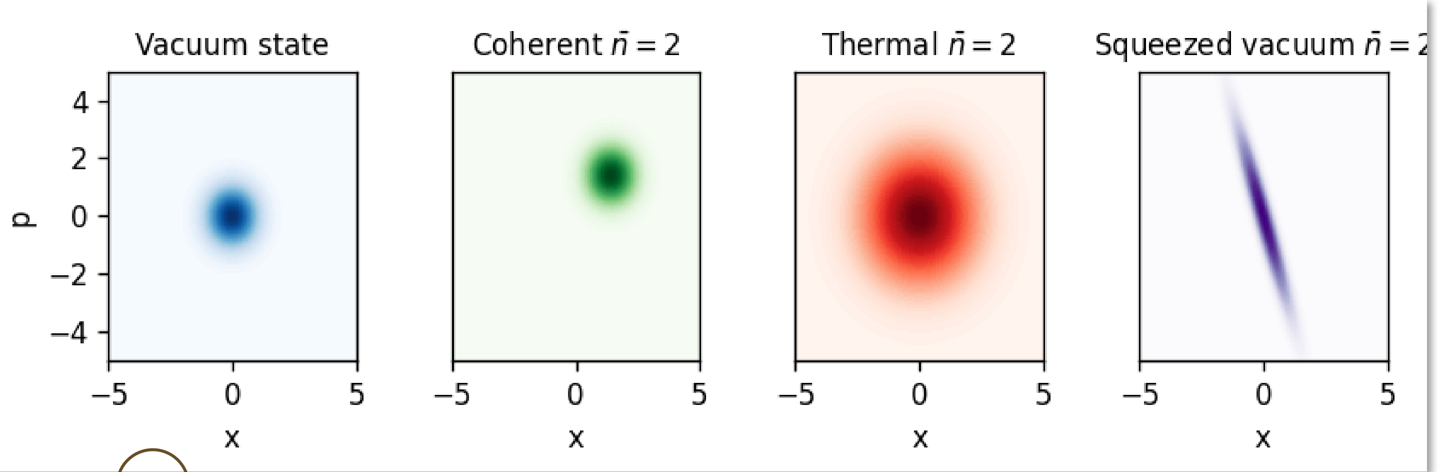
$$g_{ii}^{(n)} = \frac{\left\langle \hat{a}_i^{\dagger n} \hat{a}_i^n \right\rangle}{\left\langle \hat{a}_i^\dagger \hat{a}_i \right\rangle^n}$$

$g_{ii}^{(2)} = 1$ $\quad g_{ii}^{(2)} = 2$ $\quad g_{ii}^{(2)} > 2$

Brask *arXiv*:2102.05748 (2022)

Bona fide condition of a Gaussian state

An arbitrary Hermitian matrix does not necessary correspond to a covariance matrix of a '*bona fide*' quantum state $\hat{\rho}$:



$\boldsymbol{\sigma}$ must **respect** a generalized Heisenberg inequality: a ***bona fide* condition.**

All its eigenvalues must be bigger or equal to 1.

Otherwise, the state is unphysical.

(necessary for the positivity of any quantum state, sufficient for Gaussian)

Two-mode Gaussian state:

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}^\dagger & \boldsymbol{B} \end{pmatrix}$$

The eigenvalues of $\boldsymbol{\sigma}$ are given by

$$\nu_\pm = \Delta \pm \sqrt{\Delta - \det \boldsymbol{\sigma}}$$

where $\Delta = \det \boldsymbol{A} + \det \boldsymbol{B} - 2 \det \boldsymbol{C}$.

Serafini *et al.* J. of Phys. B **37**, L21 (2004)

Arvind *et al.* Pramana **45** (1995)
Serafini, *Quantum continuous variable* (2017)

Outline of the talk

A Gaussian state must satisfy a *bona fide* condition based on the eigenvalues of its covariance matrix.

Detecting entanglement

Gühne & Tóth, Phys. Rep. **474**,1-6 (2009)

Entanglement witness: provides a sufficient condition.

Entanglement criterion: is a necessary and sufficient condition.

PPT criterion (which is not always a criterion…)

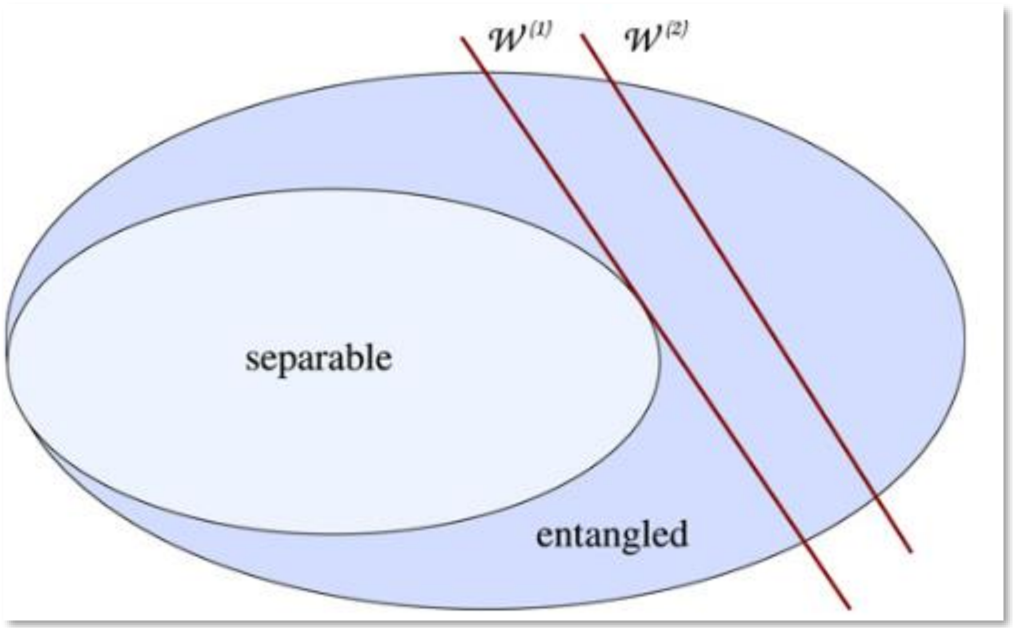
- Consider a quantum state $\hat{\rho}$,
- Take a partition (A, B).
- Compute the partial transpose operation

$$\hat{\rho} \overset{PT}{\rightarrow} \hat{\rho}^{t_B}$$

$$\hat{\rho}_{n\mu,m\nu} \overset{PT}{\rightarrow} \hat{\rho}_{n\nu,m\mu}$$

- Is $\hat{\rho}^{t_B}$ a valid quantum state ? (Positive semidefinite density matrix)

Peres, Phys. Rev. Lett. **77**, 8 (1996)



PPT criterion (witness)

- Separable state $\Longrightarrow \hat{\rho}^{t_B} \geq 0$
- Non $(\hat{\rho}^{t_B} \geq 0) \Longrightarrow$ entangled state

Horodecki, Phys. Lett. A **223**, 1-2 (1996)

Entanglement and logarithmic negativity for Gaussian states

Simon (2000) shows that

- the Wigner distribution of a Gaussian state remains Gaussian under partial transpose operation,

(not true for fermions)

$$\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}^\dagger & \boldsymbol{B} \end{pmatrix} \overset{PT}{\rightarrow} \boldsymbol{\sigma}^{t_B} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C}\boldsymbol{\sigma}_{\boldsymbol{x}} \\ (\boldsymbol{C}\boldsymbol{\sigma}_{\boldsymbol{x}})^\dagger & \boldsymbol{\sigma}_{\boldsymbol{x}}\boldsymbol{B}\boldsymbol{\sigma}_{\boldsymbol{x}} \end{pmatrix}$$

$$\nu_\pm^{t_B} = \Delta^{\mathrm{t_B}} \pm \sqrt{\Delta^{\mathrm{t_B}} - \det \boldsymbol{\sigma}}$$

where $\Delta^{\mathrm{t_B}} = \det \boldsymbol{A} + \det \boldsymbol{B} + 2 \det \boldsymbol{C}$.

- PPT is an entanglement **criterion** (also sufficient for separability).

Entanglement $\Leftrightarrow \nu_-^{t_B} < 1$

Simon, Phys. Rev. Lett, **84**, 2726 (2000)

Outline of the talk

The PT of the Wigner function of a Gaussian state is still Gaussian.

A Gaussian is entangled iff its PT is not a *bona fide* Gaussian state

① Full counting statistics and motivations

② Some notions on entanglement

③ Gaussian state formalism

④ Entanglement witness and quantifier

⑤ An entanglement criterion based on the one-, two-, and four-body correlation functions.

Probing the entanglement of Gaussian states from $g_{12}^{(2)}$

Particle detectors can measure the two-body correlation function

$$G_{12}^{(2)} = \langle \hat{n}_1 \hat{n}_2 \rangle$$

Wick expansion (*Gaussian* + *centered*)

$$G_{12}^{(2)} = \langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle = n_1 n_2 + \underbrace{|\langle \hat{a}_1 \hat{a}_2 \rangle|^2}_{\text{Anomalous correlation}} + \underbrace{\left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2}_{\text{Coherence}}$$

If $\langle \hat{a}_1^\dagger \hat{a}_2 \rangle = 0$, observation of

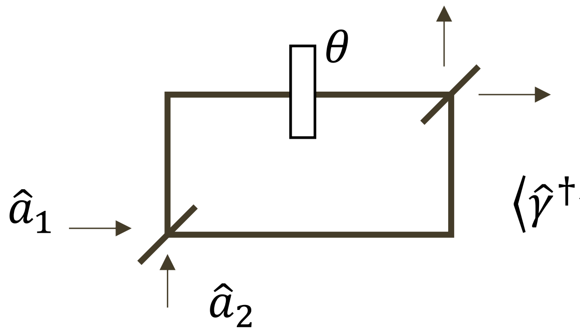
$$g_{12}^{(2)} = G_{12}^{(2)} / n_1 n_2 > 2$$

implies entanglement

Hillery-Zubairy, Phys. Rev. Lett. **96**, 050503 (2006)

How to measure the coherence?

Sol. 1: set up an interferometer



$$\langle \hat{\gamma}^\dagger \hat{\gamma} \rangle \propto \left|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle\right|^2 \sin 2\theta$$

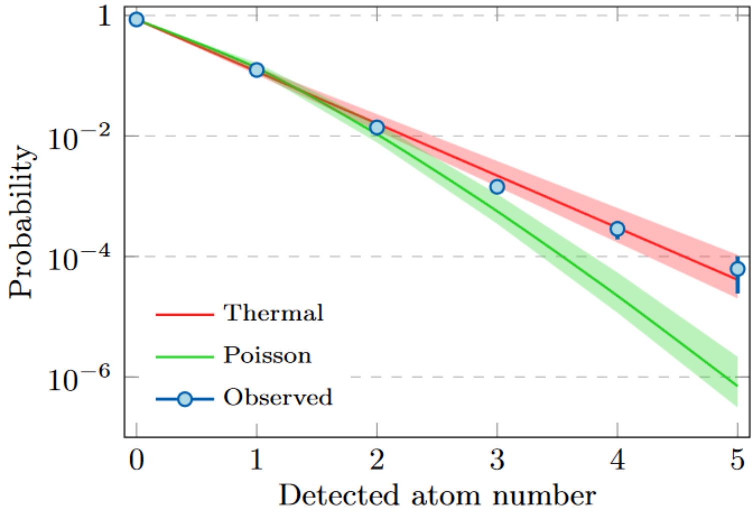
Sol. 2: use the four-body correlation function and the tools of this talk.

Observation of $n_1 n_2 < |\langle \hat{a}_1 \hat{a}_2 \rangle|^2$ implies entanglement (HZ06).

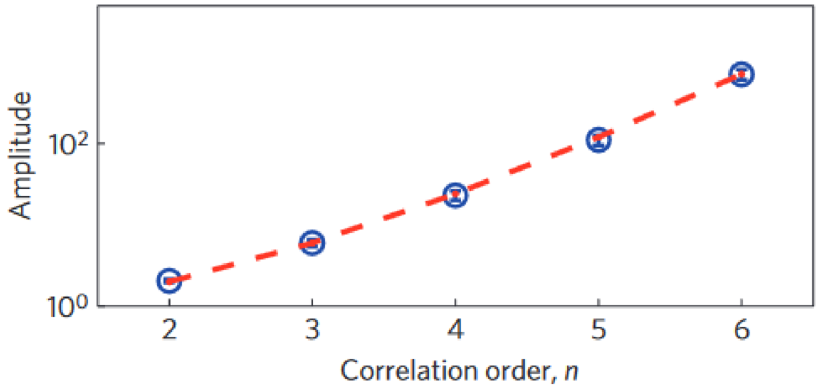
An additional hypothesis

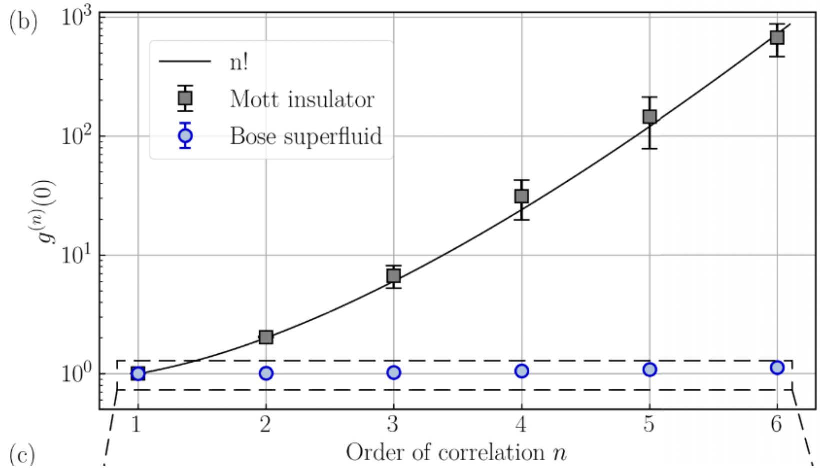
We further assume that $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$ (neither mode is squeezed)

… but this hypothesis can be verified probing the single mode statistics which must be purely thermal.



Perrier *et al.* Scipost, **7**, 002 (2019)





Dall *et al.* Nat. Phys. 9(6) (2013)
Hercé *et al.* Phys. Rev. Res. **5**, L012037 (2023)

Hypothesis
- ✓ Gaussian state
- ✓ Zero mean
- ✓ $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

For a thermal state,

$$g_i^{(n)} = n!$$

If $\langle \hat{a}_i^2 \rangle \neq 0$:

$$g_i^{(2)} = 2 + \left|\langle \hat{a}_i^2 \rangle\right|^2$$
$$g_i^{(3)} = 6 + 9\left|\langle \hat{a}_i^2 \rangle\right|^2$$
$$g_i^{(4)} = 24 + 72\left|\langle \hat{a}_i^2 \rangle\right|^2 + 9\left|\langle \hat{a}_i^2 \rangle\right|^4$$

Probing the entanglement of Gaussian states from its FCS

The state is characterized by $n_1,\ n_2,\ \langle \hat{a}_1 \hat{a}_2 \rangle$ & $\langle \hat{a}_1^\dagger \hat{a}_2 \rangle$

Lemma 1: Measurement of $n_1,\ n_2, g_{12}^{(2)}$ & $g_{12}^{(4)}$ yields a symmetric system for $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle|$

- $g_{12}^{(2)}$ involves their quadratic sum,
- $g_{12}^{(4)}$ also involves their product.

We find two solutions $\beta_\pm$

$$\beta_\pm^2 = n_1 n_2 \left(g_{12}^{(2)} - 1\right) \frac{1 \pm \sqrt{1-\theta}}{2}$$

where

$$\theta = \frac{g_{12}^{(4)} + 12 - 16 g_{12}^{(2)} - 4\left(g_{12}^{(2)} - 1\right)^2}{\left(g_{12}^{(2)} - 1\right)^2}$$

Hypothesis

- ✓ Gaussian state
- ✓ Zero mean
- ✓ $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

$\theta \in [0,1]$ so that $\beta_\pm^2 \geq 0$ as a supplementary check for the consistency of the hypothesis.

We have two possible solutions

- "State" μ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_+$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle| = \beta_-$,
- "State" γ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_-$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle| = \beta_+$.

$$g_{12}^{(4)} = \langle \hat{a}_1^{\dagger 2} \hat{a}_2^{\dagger 2} \hat{a}_1^2 \hat{a}_2^2 \rangle / n_1^2 n_2^2$$

Probing the entanglement of Gaussian states from its FCS

We have two possible solutions

- "State" μ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_+$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle| = \beta_-$,
- "State" γ: $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_-$ & $|\langle \hat{a}_1^\dagger \hat{a}_2 \rangle| = \beta_+$.

Check they can exist!

Lemma 2:

The *bona fide* condition does not depend on the phase of $\langle \hat{a}_1 \hat{a}_2 \rangle$ and $\langle \hat{a}_1^\dagger \hat{a}_2 \rangle$.

The (smallest) eigenvalue is given by

$$\nu_\mu = f(n_1, n_2, \beta_+, \beta_-) \quad \& \quad \nu_\gamma = f(n_1, n_2, \beta_-, \beta_+)$$

We have 3 possibilities

- $\nu_\gamma \leq \nu_\mu < 1$: unphysical states (wrong hypothesis)
- $\nu_\gamma < 1 \leq \nu_\mu$: only one solution (we found it), → The state is entangled
- $1 \leq \nu_\gamma \leq \nu_\mu$: two solutions and we cannot distinguish the states → The state is separable

$$f(n_1, n_2, x, y) = \frac{\Delta - \sqrt{\Delta^2 - \det \boldsymbol{\sigma}}}{2}$$

where

$$\det \boldsymbol{\sigma} = 16(x^2 - y^2)^2 + (1 + 2n_1)^2(1 + 2n_2)^2 - 9(x^2 + y^2)(1 + 2n_1)(1 + 2n_2)$$

and

$$\Delta = (2n_1 + 1)^2 + (2n_2 + 1)^2 - 8(x^2 - y^2)$$

Hypothesis

- ✓ Gaussian state
- ✓ Zero mean
- ✓ $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

Lemma 3:

'States' μ and γ are partial transpose of each other.

Probing the entanglement of Gaussian states from its FCS

Entanglement criterion

- Measure $n_1,\ n_2\,, g_{12}^{(2)}$ & $g_{12}^{(4)}$ and deduce $\beta_\pm$,
- Compute $\nu_\gamma = f(n_1, n_2, \beta_-, \beta_+)$
- The state is entangled if $\nu_\gamma < 1$, *(criterion)*
- Quantify entanglement $\mathrm{LN} = \mathrm{Max}(-\log_2 \nu_\gamma\,, 0)$

Without $g_{12}^{(4)}$, $g_{12}^{(2)}$ is still a witness!

Hypothesis

- ✓ Gaussian state
- ✓ Zero mean
- ✓ $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

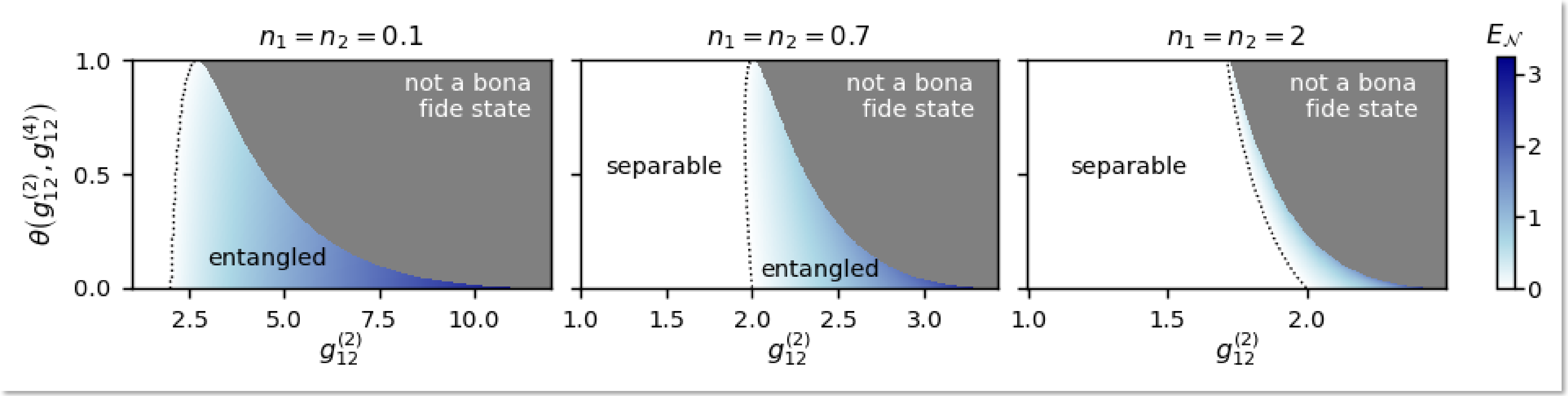


Fig: Entanglement in the $(g_{12}^{(2)}, \theta)$ plane for three populations.

$g_{12}^{(2)}$ entanglement witness

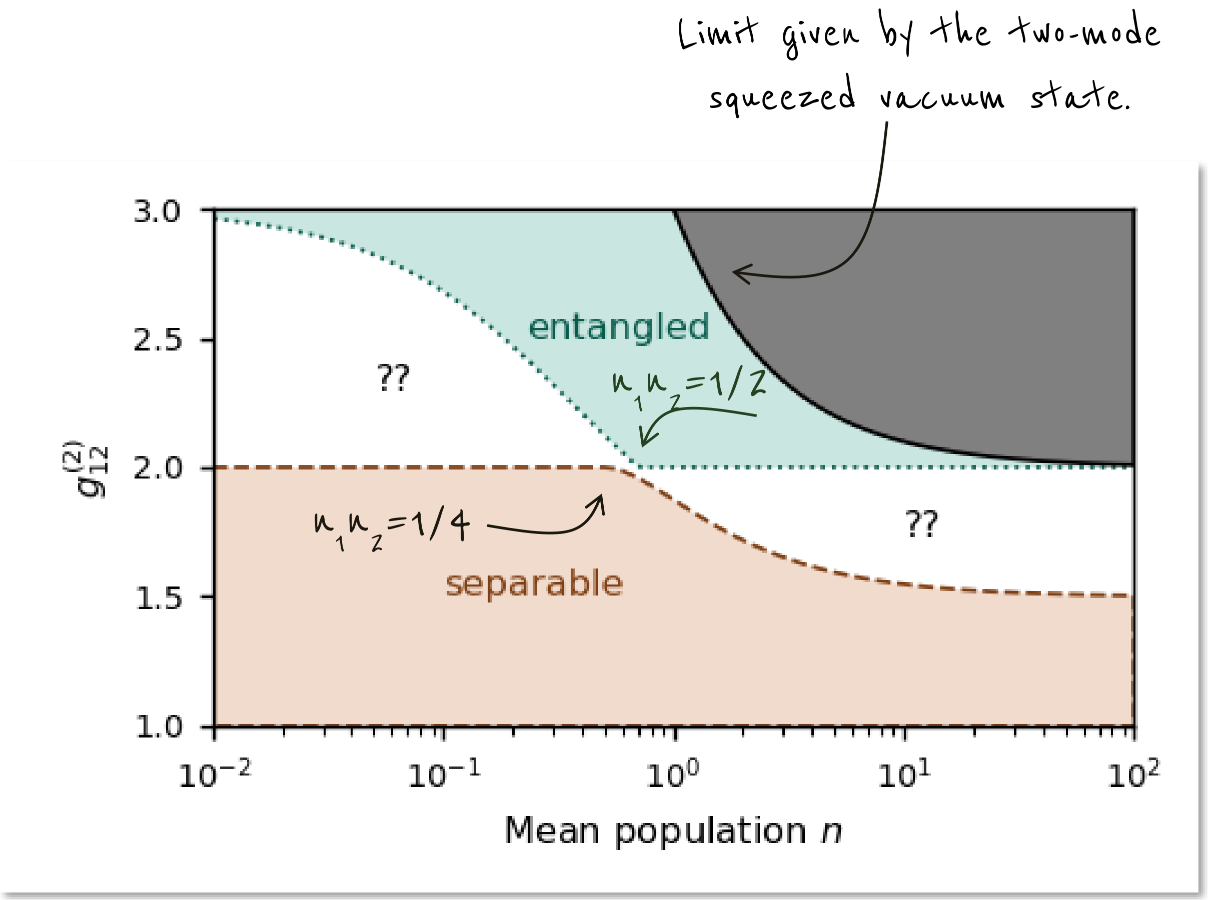


Fig: Entanglement witness based on the value of $g_{12}^{(2)}$.

- The $g_{12}^{(2)}$ entanglement witness depends on the populations,
- The value of $g_{12}^{(4)}$ is needed to determine the entanglement in the '??'.
- Taking into account the quantum efficiency of the detector can 'help' to witness entanglement,

Hypothesis

✓ Gaussian state
✓ Zero mean
✓ $\langle \hat{a}_1^2 \rangle = \langle \hat{a}_2^2 \rangle = 0$

If $\langle \hat{a}_j^2 \rangle \neq 0$, the phases matter in the state's non-separability.

Conclusion

We can quantify the entanglement of thermal Gaussian states from their full counting statistics.

(and it is useful, see the 28th)

Take home message

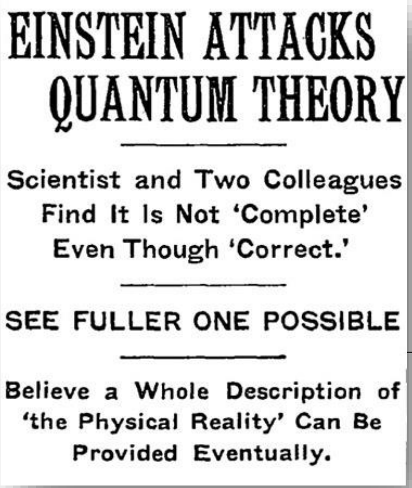
- Gaussian states must satisfy a *bona fide* condition (generalized Heisenberg),
- Entangled Gaussian states have an un-physical partial transposed (PPT criterion),
- The spectrum of the PT state quantifies the state' entanglement (LN),
- This spectrum can be measure *via* the FCS for thermal Gaussian states.

Thank you for your attention.

Appendix

We want more!

What is entanglement?



EINSTEIN ATTACKS QUANTUM THEORY

Scientist and Two Colleagues Find It Is Not 'Complete' Even Though 'Correct.'

SEE FULLER ONE POSSIBLE

Believe a Whole Description of 'the Physical Reality' Can Be Provided Eventually.

EPR Paradox

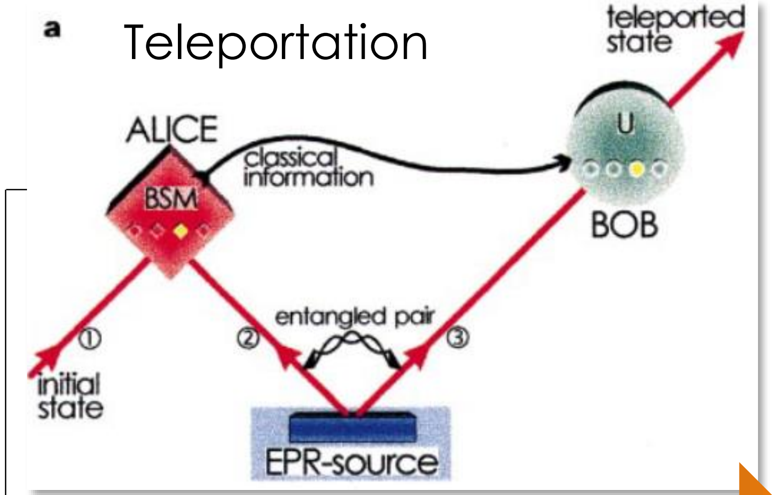
$$|\Psi^{(\pm)}\rangle \sim \frac{|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle}{\sqrt{2}}$$

"Entanglement"

"Steerability"

From 1935 to 2025

Bell's inequality
Bell, *Physics* (1964)
Aspect *et al*, PRL (1981-82)



Bennett *et al*, *PRL* (1993)
Popescu, *PRL*, (1994)
Bouwmeester *et al*. *Nature* (1997)

Equivalence for pure states
Gisin, Phys. Lett. A (1991)

Non-separability
Werner, *PRA* (1989)
PPT criterion
Peres, PRL (1996)
Horodecki³, *Phys. Lett. A* (1996)

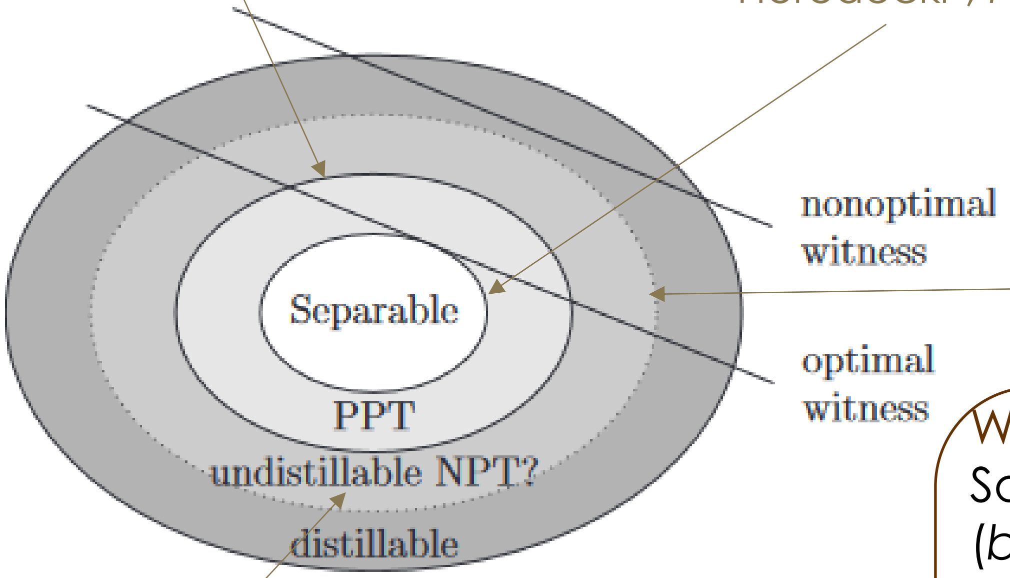
Entanglement distillation
Bennett *et al*, PRL, (1993)
Horodecki³, *PRL* (1998)
Dür, PRL (1998)

A bipartite state ρ is distillable, iff - by means of LOCC – we can create $|\Psi^{(\pm)}\rangle$ out of n identical copies of ρ.

What is entanglement?

NPT implies entanglement
Peres, *PRL* (1996)

Some states are entangled but with PPT (*bound* entanglement)
Horodecki[3], *Phys. Lett. A* (1996)



Any distillable state must be NPT.
Horodecki[3], *PRL* (1998)
However, it is equivalent for two-modes Gaussian states (also 1xN)
Duan *et al*, *PRL* (2001), Werner and Wolf, *PRL* (2001)

If a bounded NPT state σ exists, then it also exists ρ PPT such that $\rho \otimes \sigma$ is distillable (*superactivation*).
Shor, Smolin & Terhal, *PRL* (2001)

What about Bell's inequalities?
Some states violate a Bell inequalities but are not distillable (*bounded*).
Dür, *PRL* (2001)
"Violation of a Bell inequality implies *bipartite* distillability".
Acin, *PRL* (2001)
Any bipartite entangled state σ exhibits a hidden nonlocality which can be activated ($\exists\, \rho$ entangled that does not violate CHSH inequality but $\rho \otimes \sigma$ violates it)
Acin, *PRL* (2001)

Horodecki[4], *RMP* (2009)

Quantifying entanglement with logarithmic negativity

Logarithmic negativity:

$$\mathrm{LN} = \log\|\hat{\rho}^{t_B}\|_1$$

i.e. the sum of the absolute eigenvalues of $\hat{\rho}^{t_B}$.

LN is an entanglement monotone

- ✓ LN is zero for separable states,
- ✓ LN does not increase under LOCC,
- ✓ LN provides an upper bound for distillable entanglement

Demonstration: (PT preserves the trace)

separable

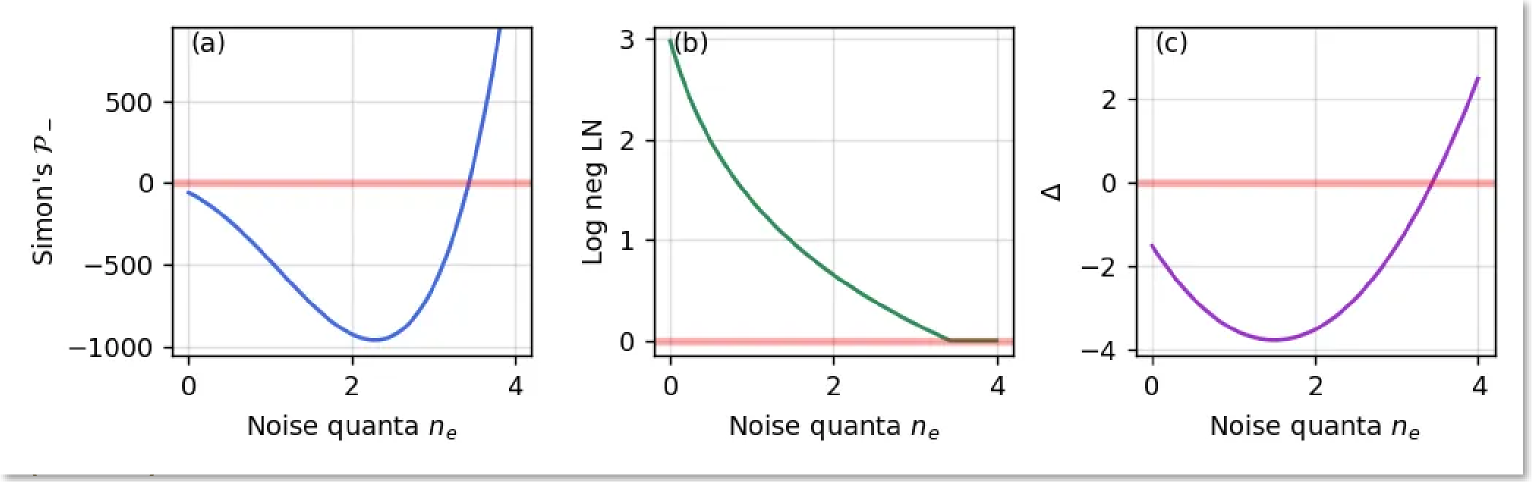
$$Tr(\hat{\rho}^{t_B}) = 1, \quad \hat{\rho}^{t_B} \geq 0 \longrightarrow \sum_i \nu_i = 1, \quad \forall i, \nu_i \geq 0 \longrightarrow \sum_i |\nu_i| = 1 \longrightarrow LN = 0$$

NPT

$$Tr(\hat{\rho}^{t_B}) = 1$$

~~$\hat{\rho}^{t_B} \geq 0$~~

$$\longrightarrow \sum_i \nu_i = 1, \quad \exists\, i, \nu_i \geq 0 \longrightarrow \sum_i |\nu_i| > 1 \longrightarrow LN > 0$$



Comparison of 3 entanglement witnesses: LN decreases with noise

Vidal & Werner, Phys. Rev. A, **65**, 032314
Plenio, Phys. Rev. Lett., **95** 090503 (2005)

Particle *versus* mode entanglement

The debate

Consider $\hat{a}^\dagger_\uparrow \hat{a}^\dagger_\downarrow |vac\rangle = |1,1\rangle$ in 2nd quantization.

In the 1st quantized picture, labelling particles by A and B, we have

$$\frac{|\uparrow\rangle_A |\downarrow\rangle_B + |\downarrow\rangle_A |\uparrow\rangle_B}{\sqrt{2}}$$

which is entangled?

For some, this 'entanglement' is unphysical and the labels A and B are meaningless.

No consensus on the nature of this correlation due to exchange symmetry, sometime referred to as *particle entanglement*.

Nevertheless, particle entanglement is a useful and consistence resource"

Morris *et al.* PRX **10** (2020)

"Identical particle entanglement can be transferred, with unit probability, onto independent modes using elementary operations. Thus, symmetrization entanglement is a fundamental, ubiquitous, and readily extractable resource for standard quantum information tasks."

Killoran, Cramer and Plenio, PRL **112** (2014)

New definitions of entanglement have been proposed but only Werner's definition based on the *mode entanglement* is satisfying

Benatti *et al.* Phys. Rep. (2020).

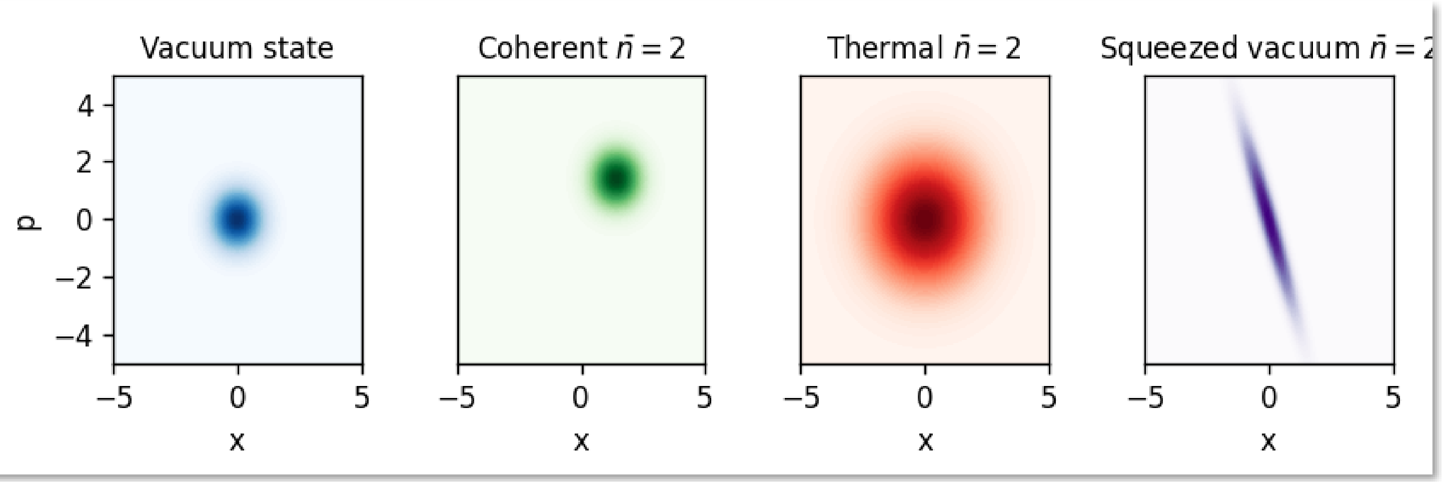
Violation of the Cauchy-Schwarz inequality is a particle entanglement witness

Wasak *et al.* PRA. (2014).

(I strongly recommend to read the introduction of Morris *et al* and Killoran *et al*.)

Bona fide condition of a Gaussian state

An arbitrary Hermitian matrix does not necessary correspond to a covariance matrix of a '*bona fide*' quantum state $\hat{\rho}$:



$\boldsymbol{\sigma}$ must **respect** a generalized Heisenberg inequality: a ***bona fide* condition.**

All its eigenvalues must be bigger or equal to 1.

Otherwise, the state is unphysical.

(necessary for the positivity of any quantum state, sufficient for Gaussian)

The symplectic group: all transformations of $\boldsymbol{\sigma}$ that preserve the canonical commutation relations.

Exemples: Bogoliubov transformations, a displacement, rotation….

Arvind *et al.* Pramana **45** (1995)
Serafini, *Quantum continuous variable* (2017)