Quantifying two-mode entanglement of bosonic Gaussian states from their full counting statistics

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We study the entanglement properties of two-mode bosonic Gaussian states from their multi-mode counting statistics. We introduce the idea that measuring high-order correlations of particle numbers can reveal entanglement without making any assumptions about the coherence of the fields. In particular, we show that the two- and four-body number correlations are sufficient to fully characterize the entanglement of two-mode bosonic Gaussian states for which each mode exhibits a thermal distribution. In addition, we derive an entanglement witness based on two-body correlations alone. Our findings are of great importance because it becomes possible to reveal entanglement in a series of recent experiments.

Entanglement of bosonic Gaussian states is a key ingredient of continuous-variable quantum information theory [1, 2] that has found many applications in Quantum Optics. It is also expected to emerge in a variety of other situations described by Hamiltonians that are quadratic in field operators, ranging from Bogoliubov pairing in superfluids [3], Hawking emission at the black hole horizon [4] to the dynamical Casimir effect [5]. Existing criteria which detect such entanglement are often expressed in terms of inequalities involving the expectation values, products, and variances of *field* operators [6–11]. Measuring these quantities requires some variant of homodyne or heterodyne detection in which the states of interest are made to interfere with coherent local oscillators [12, 13]. Although these methods have been successfully implemented experimentally in some specific configurations [14–16], they require a large set of measurements and can be difficult to apply in general. This is the case, for example, for massive particles entangled in their external degrees of freedom [17–19], when the spatial mode to be probed does not match that of the local oscillator, or simply when the number of modes in the system is large [20]. Characterization of quantum states would benefit from finding entanglement criteria that do not require such homodyne techniques.

On the other hand, bosonic particle number resolved detection methods [21, 22] allow quantum states to be characterized via their multi-mode counting statistics [17, 23–31], as sketched in Fig. 1. The information extracted about the quantum state yields any order of normally ordered correlation functions of the *particle number* operators *i.e.* $\langle : \hat{n}_1^{\alpha_1} \hat{n}_2^{\alpha_2} : \rangle$ where the α_i's are integers, $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$ and $\hat{a}_i$ is the annihilation operator of mode i [32, 33]. These observables thus involve the same number of annihilation and creation operators of a given mode. As such, the multi-mode correlations of the *field*, such as $\langle \hat{a}_1 \hat{a}_2^\dagger \rangle$ or $\langle \hat{a}_1 \hat{a}_2 \rangle$, cannot be directly obtained from the counting statistics. In general, this lack of information hinders the ability to probe entanglement properties using existing criteria, and thus requires homodyne or heterodyne detection.

By contrast, we introduce the idea that in certain cases,
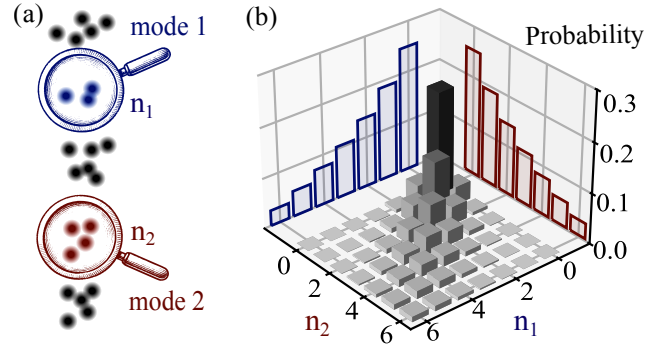

FIG. 1. (a) Two modes in a system are defined (shown by the lens here) and the number of particles n_1 and n_2 and in each mode is counted. (b) These counts can be histogrammed to obtain the full counting statistics of each mode. In addition, the joint probability distribution is also accessible. This information yields the *n*-body correlation functions of particle number operators of any order.

the full counting statistics offers a simpler alternative to homodyning or heterodyning for the detection of entanglement [34]. We demonstrate this possibility on two-mode bosonic Gaussian states [35] for which the single-mode statistics follow a thermal distribution. Such two-mode Gaussian states have been produced in various cold-atom experiments, ranging from the parametric amplification of non-interacting quasi-particles [36–39] and the Bogoliubov pairing in interacting Bose gases [40, 41], to analog Hawking radiation [42] and the quasi-particle creation in an expanding metric [43]. Thus far, observations of positive two-mode correlations were related to two-mode entanglement only upon *assuming* the absence of coherence between the two modes [38, 40, 42]. A similar assumption is made when evaluating entanglement from the time evolution of the density-density correlation function within Bogoliubov theory [44–46].

In this Letter, we show how the measurement of higher order correlation functions can assess and quantify entanglement, *without assuming the absence of mode coherence* and *without implementing a homodyne detection*. We consider two-mode Gaussian states of bosons for which single modes exhibit thermal statistics, a property which can be verified

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in experiments. Elaborating on the positivity of the partial transpose (PPT) criterion [47] and on the properties of Gaussian states, we introduce an entanglement criterion based on the mode populations and the two- and four-body correlation functions as obtained from measuring the full counting statistics. We also derive an entanglement witness based on the two-body correlation function alone.

Hypothesis – In the following, we consider two-mode bosonic Gaussian states [48] satisfying $\langle \hat{a}_i \rangle = 0$ and $\langle \hat{a}_i^2 \rangle = 0$. In this case, each single mode distribution is thermal. Note that "thermal" refers here to the statistical properties of a mode and does not imply a thermodynamic temperature nor thermal equilibrium [49]. Thermal single-mode distributions are expected in many situations [31, 36–43] and can be verified experimentally as shown in Refs. [23, 25, 26].

The normalized two-body correlation function of such states can be expanded using Wick's theorem [50]

$$g_{12}^{(2)} := \frac{\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_1 \hat{a}_2 \rangle}{n_1 n_2} = 1 + \frac{|\langle \hat{a}_1 \hat{a}_2 \rangle|^2}{n_1 n_2} + \frac{|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|^2}{n_1 n_2}. \quad (1)$$

Any separable state must satisfy $|\langle \hat{a}_1 \hat{a}_2 \rangle| \leq n_1 n_2$ [8]. Thus, the observation of $g_{12}^{(2)} > 2$ can be considered a signature of entanglement only *under the assumption* that the coherence between the two modes vanishes, *i.e.*, $\langle \hat{a}_1 \hat{a}_2^\dagger \rangle = 0$. This is the assumption that we aim to drop by considering higher-order correlations.

Within our hypothesis, the four-body correlation function between the two modes is given by

$$g_{12}^{(4)} := \frac{\langle (\hat{a}_1^\dagger \hat{a}_2^\dagger)^2 (\hat{a}_1 \hat{a}_2)^2 \rangle}{n_1^2 n_2^2} = 4 \left[1 + \frac{\left(g_{12}^{(2)} - 1 \right)^2}{n_1^2 n_2^2} \right.$$
$$\left. + 4 \frac{g_{12}^{(2)} - 1}{n_1 n_2} + 2 \frac{|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|^2 |\langle \hat{a}_1 \hat{a}_2 \rangle|^2}{n_1^2 n_2^2} \right]. \quad (2)$$

Importantly, while $g_{12}^{(2)}$ involves the quadratic sum of $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|$ and $|\langle \hat{a}_1 \hat{a}_2 \rangle|$, the normalized four-body correlation function $g_{12}^{(4)}$ includes their product. Therefore, Eqs. (1) and (2) constitute a symmetric system to determine the values of $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ and $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|$ and can characterize entanglement without further assumptions.

Entanglement criterion - A two-mode Gaussian state with thermal single-mode statistics is entangled if and only if the smallest value of the set of the symplectic eigenvalues of its covariance matrix and its partial transpose is strictly smaller than one. This value, λ_-, is defined in Eq. (5) and solely depends on the measurement of the populations n_1 and n_2 and of the two- and four-body correlation functions $g_{12}^{(2)}$ and $g_{12}^{(4)}$.

Proof. Solving the system consisting of Eqs. (1) and (2) for $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ and $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|$ yields two solutions $\beta_\pm$ given by

$$\beta_\pm^2 = n_1 n_2 (g_{12}^{(2)} - 1) \frac{1 \pm \sqrt{1 - \theta}}{2}, \quad (3)$$

where θ is defined by

$$\theta = \frac{g_{12}^{(4)} + 12 - 16 g_{12}^{(2)} - 4 \left(g_{12}^{(2)} - 1 \right)^2}{2 \left(g_{12}^{(2)} - 1 \right)^2}. \quad (4)$$

The quantity $\beta_\pm$ in Eq. (3) corresponds to the value of $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ and $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|$ but, because the system is symmetric, the distinction between the two solutions is not *a priori* possible. However, the covariance matrix of a Gaussian state must satisfy a so-called *bona fide* condition which imposes an inequality on its symplectic eigenvalues [35, 51]. These constraints identify the physical solutions. In appendix A we show that, within our hypotheses:

(i) the symplectic eigenvalues of a two-mode Gaussian state only depend on n_1, n_2, $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ and $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|$ (see Eq. (A3)) and must be greater or equal to one to describe a physical state,

(ii) the partial transpose operation simply interchanges the roles of $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ and $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|$. According to the PPT criterion [7], the state is entangled if and only if its partial transpose is not a *bona fide* Gaussian state, *i.e.*, if at least one of the symplectic eigenvalues of the partially transposed state is smaller than one. These eigenvalues are found as in (i), with $|\langle \hat{a}_1 \hat{a}_2 \rangle|$ and $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|$ interchanged.

Based on (i) and (ii), we conclude that the measurement of $\beta_\pm$ allows us to compute the set of the symplectic eigenvalues of the state and of its partial transpose. In particular, λ_-, the smallest symplectic eigenvalue of this set, is given by [52]

$$2\lambda_-^2 = \Delta - \sqrt{\Delta^2 - 4 \det \boldsymbol{\sigma}} \quad (5)$$

where

$$\det \boldsymbol{\sigma} = 16 \left(\beta_+^2 - \beta_-^2 \right)^2 + (1 + 2n_1)^2 (1 + 2n_2)^2$$
$$- 8 \left(\beta_+^2 + \beta_-^2 \right) (1 + 2n_1)(1 + 2n_2) \quad (6)$$

is the determinant of the covariance matrix $\boldsymbol{\sigma}$ and

$$\Delta = (2n_1 + 1)^2 + (2n_2 + 1)^2 + 8(\beta_+^2 - \beta_-^2). \quad (7)$$

If $\lambda_- \geq 1$, all the symplectic eigenvalues of the state and its partial transpose are greater than one. Based on (ii), the state is therefore separable.

If $\lambda_- < 1$, λ_- cannot be a symplectic eigenvalue of the state, based on (i). Thus, λ_- must be a symplectic eigenvalue of the partially transposed state, which means that the state is entangled from (ii). In this case, we can also conclude $|\langle \hat{a}_1 \hat{a}_2 \rangle| = \beta_+$ and $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle| = \beta_-$. □

Logarithmic negativity - The logarithmic negativity is useful to quantify entanglement [53, 54]. For instance, it increases with the squeezing strength [55, 56], and provides an upper bound for the number of distillable Bell states [53]. It
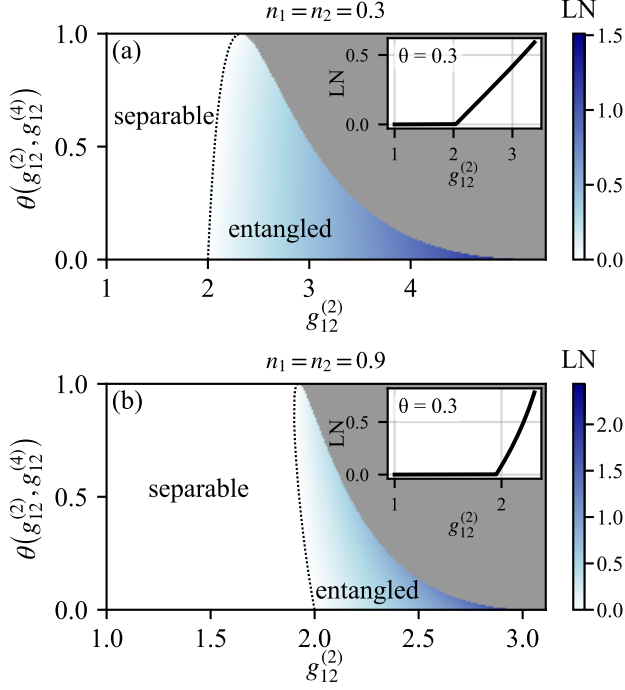
FIG. 2. Two-mode entanglement in the $(g_{12}^{(2)}, \theta)$ plane for equal mean populations $n_1 = n_2 = 0.3$ (a) and 0.9 (b). The parameter θ is defined in Eq. (4) and varies linearly with $g_{12}^{(4)}$. The grey region corresponds to unphysical states or states not fulfilling our hypotheses and the color-scale to the logarithmic negativity: the bluer, the more entangled. The entanglement border (dotted black line) splits the plane between separable (left) and entangled states (right). The inset shows logatithmic negativity as a function of $g_{12}^{(2)}$ for $\theta = 0.3$.

can be computed from the smallest symplectic eigenvalue of the partially transposed state. Within our assumptions (see above), the logarithmic negativity is

$$\mathrm{LN} = \max(-\log_2(\lambda_-), 0). \tag{8}$$

When the state is separable ($\lambda_- \geq 1$), the logarithmic negativity is zero while it is strictly positive when the state is entangled, ($\lambda_- < 1$).

The logarithmic negativity is plotted in Fig. 2 as a function of θ and $g_{12}^{(2)}$ for two populations, assuming that $n_1 = n_2$. On the left of the black dotted curve lie separable states while entangled states lie on the right. The shape of this entanglement border depends on the population. On the x-axis, $\theta = 0$ which gives $\beta_- = 0$. The entangled states then correspond to two-mode squeezed thermal states for which $\langle \hat{a}_1 \hat{a}_2^\dagger \rangle = 0$. The two-mode squeezed vacuum state is the one having the maximal value of $g_{12}^{(2)}$. The parameter θ should lie in the interval $[0, 1]$ since $\beta_\pm^2$ are by definition positive, see Eq. (3). This imposes bounds on the possible values of $(g_{12}^{(2)}, g_{12}^{(4)})$. This is also the case for the gray region in which states do not satisfy condition *(i)*. The case of non-equal populations $n_1 \neq n_2$ is qualitatively similar, but will not be discussed further.

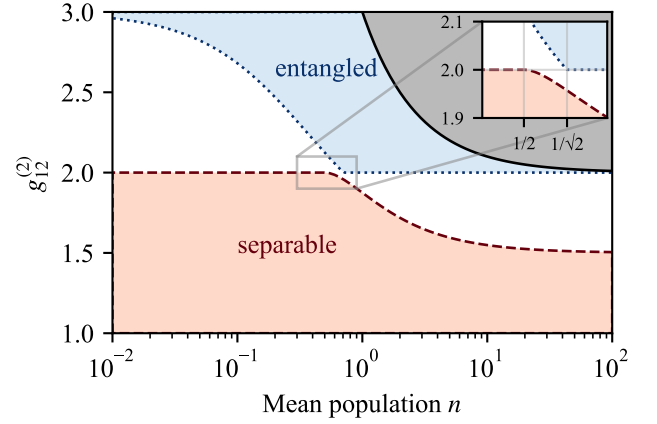*Detector efficiency* - On the one hand, $g_{12}^{(2)}$, $g_{12}^{(4)}$ and θ are

FIG. 3. Entanglement $g_E^{(2)}$ (dotted blue) and separability $g_S^{(2)}$ (dashed red) thresholds for the two-body correlation function as a function of the mean population of the state $n_1 = n_2 = n$, in log scale. The four-body correlation function is needed to discriminate entanglement in the white region located in between the separable and entangled regions. The unphysical gray region is above the line $2 + 1/n$ which is the value of $g_{12}^{(2)}$ for a two-mode squeezed vacuum state with population n.

normalized quantities that do not depend on the detection efficiency η. On the other hand, the populations (n_1, n_2) depend linearly on η [57]. As a result, one must scale the measured mean populations by $1/\eta$ to use the results of Fig. 2 with a imperfect detector ($\eta < 1$). Consider a state with ($\theta = 0.5, g_{12}^{(2)} = 2.03$) and a measured mean population $n_1 = n_2 = 0.3$: with a unit detection efficiency $\eta = 1$ one concludes from Fig. 2 that the state is separable; in contrast, if $\eta = 0.3$, one concludes that the state is entangled.

Two-body witnesses - To witness entanglement or separability, the measurement of the four-body correlation function is not necessary. From Fig. 2(a), it is evident that $g_{12}^{(2)} > 2.3$ (entanglement border for $\theta = 1$) guarantees entanglement, whereas Fig. 2(b) reveals that $g_{12}^{(2)} > 2$ is enough. The entanglement threshold thus depends on the curvature of the entanglement border which in turn depends on the population of the state. In Appendix B, we demonstrate that a state is entangled if $g_{12}^{(2)} > g_E^{(2)}$ where

$$\text{if } n_1 n_2 < 1/2, \quad g_E^{(2)} = 2 + \frac{1/2 - n_1 n_2}{2 n_1 n_2 + n_1 + n_2 + 1/2},$$
$$\text{if } n_1 n_2 \geq 1/2, \quad g_E^{(2)} = 2. \tag{9}$$

The entanglement threshold $g_E^{(2)}$ is plotted in Fig. 3 as a blue dotted line in the case of equal populations, $n_1 = n_2 = n$. When the mean populations are sufficiently high ($n_1 n_2 \geq 1/2$), a value of $g_{12}^{(2)} > 2$ is sufficient to witness entanglement. When the mean population is smaller, the entanglement threshold $g_E^{(2)}$ exceeds 2. It reaches 3 in the limit of vanishing populations arising from the general inequality $|\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|^2 < n_1 n_2$ (see Appendix B).

Likewise, the quantum states located on the left of the entanglement border in Fig. 2 are separable, allowing the derivation of a separability threshold based solely on the value of $g_{12}^{(2)}$. We demonstrate in Appendix B that, if $g_{12}^{(2)} \leq g_S^{(2)}$, the state is separable where

$$
\begin{aligned}
&\text{if } n_1 n_2 \leq 1/4, \quad g_S^{(2)} = 2, \\
&\text{if } n_1 n_2 > 1/4, \quad g_S^{(2)} = 2 - \frac{(1 - 4n_1 n_2)^2}{8 n_1 n_2 (1 + 2n_1)(1 + 2n_2)}.
\end{aligned}
\tag{10}
$$

This separability threshold, depicted as a red dashed line in Fig. 3, equals 2 when the populations fulfill $n_1 n_2 < 1/4$ and approaches 1.5 at large values of the mean populations.

These two thresholds act as entanglement and separability witnesses and not as criteria [58]: they cannot certify the (non-)separability of the state in the question mark region of Fig. 3, where a measurement of the four-body correlation function is required.

Discussion - In this work, we focus on entanglement between *modes*, following the non-separability definition of Werner [59]. This is in contrast to the so-called *particle* entanglement that emerges from the indistinguishability of particles [60–62]. Defining the ratio $C_S = g_{12}^{(2)}/\left(g_1^{(2)} g_2^{(2)}\right)^{1/2}$ where $g_i^{(2)} = \langle (\hat{a}_i^\dagger)^2 \hat{a}_i^2 \rangle / n_i^2$, Ref. [63] shows that the violation of the classical Cauchy-Schwarz inequality $C_S \leq 1$ is a sufficient condition for particle entanglement. For the single-mode thermal statistics we consider here, one has $g_1^{(2)} = g_2^{(2)} = 2$ and $C_S = g_{12}^{(2)}/2$. Mode entanglement ($g_{12}^{(2)} > g_E^{(2)} \geq 2$) therefore necessarily implies particle entanglement. Our witness for mode entanglement is thus more stringent than the one on particle entanglement.

Our results apply to Gaussian states for which single-mode distributions are thermal, which is not always true. For example, recent work [64, 65] has suggested using a single-mode squeezed vacuum state at an analog black-hole horizon to boost the effect of the two-mode squeezer. The output state is expected to exhibit super-thermal statistics with $g_i^{(2)} > 2$. In this scenario, the symplectic spectrum depends on the relative phase between the modes [66] and the four-body correlation function contains additional terms with respect to those of Eq. (2). An interesting question for follow-up studies is whether an entanglement criterion could be established in such situations as well, by extending our approach to higher orders of the counting statistics (*e.g.* using $g_{12}^{(6)}$).

Finally, the entanglement criterion introduced using the values of $g_{12}^{(2)}$ and $g_{12}^{(4)}$ does not require the measurement of non-commuting observables. While this may appear puzzling at first, we emphasize that this is the consequence of the assumption that the states are Gaussian states. Proving unambiguously that a state is indeed Gaussian would require the measurement of its quadratures [67, 68] *i.e.* the implementation of a homodyne scheme.

Conclusion - In this Letter, we have introduced an entanglement criterion for two-mode Gaussian states with single-mode thermal statistics based on the PPT criterion and the measurement of the two-body and four-body correlation functions. In addition, we derive a lower bound above which the two-mode state is entangled, relying solely on the value of two-body correlations. Based on our results, the experimental data associated with four-wave mixing induced by the collision of Bose-Einstein condensates reported by Ref. [69] and to the Bogoliubov quantum depletion observed by Ref. [40] are sufficient to claim entanglement, without any additional assumptions on the field coherence $\langle \hat{a}_1 \hat{a}_2^\dagger \rangle$.

Appendix A: Gaussian state formalism in a nutshell

We consider the reduced density matrix of a quantum state and focus only on two modes. We do not require the quantum state to be Gaussian but only the two-mode reduced density matrix of the state to be Gaussian. This (reduced) state is thus fully characterized by its first and second moments: the mean vector (null based on our hypothesis) and the covariance matrix [35]. Usually, the covariance matrix of the state is defined within the quadrature operator basis rather than using the creation and annihilation operator basis. Here, it is more natural to work in the basis $\hat{\boldsymbol{r}} = (\hat{a}_1, \hat{a}_1^\dagger, \hat{a}_2, \hat{a}_2^\dagger)^\top$ and define the covariance matrix as $\boldsymbol{\sigma}_{ij} = \langle \hat{r}_i \hat{r}_j^\dagger + \hat{r}_j^\dagger \hat{r}_i \rangle$, which is a complex hermitian matrix. Within our hypothesis, this matrix is given by [35]

$$
\boldsymbol{\sigma} = \begin{pmatrix} \boldsymbol{A} & \boldsymbol{C} \\ \boldsymbol{C}^\dagger & \boldsymbol{B} \end{pmatrix},
\tag{A1}
$$

where $\boldsymbol{A} = (2n_1 + 1)\mathbb{I}_2$ and $\boldsymbol{B} = (2n_2 + 1)\mathbb{I}_2$ are the covariance matrices of the respective reduced states composed of modes 1 and 2 and $\mathbb{I}_2$ is the identity matrix [70]. $\langle \hat{a}_j^2 \rangle \neq 0$ would imply the presence of non-zero off-diagonal terms in these matrices. Sub-matrix $\boldsymbol{C}$ encodes the correlation between the two modes and is given by

$$
\boldsymbol{C} = 2 \begin{pmatrix} \langle \hat{a}_1 \hat{a}_2^\dagger \rangle & \langle \hat{a}_1 \hat{a}_2 \rangle \\ \langle \hat{a}_1 \hat{a}_2 \rangle^* & \langle \hat{a}_1 \hat{a}_2^\dagger \rangle^* \end{pmatrix}.
\tag{A2}
$$

Because the covariance matrix is a positive semi-definite matrix, it can be diagonalized through symplectic transformations [71] *i.e.* transformations that preserve the canonical commutation relations [51]. The symplectic eigenvalues $\nu_\pm$ of this matrix are given by [52]

$$2\nu_\pm^2 = \Gamma \pm \sqrt{\Gamma^2 - 4\det\boldsymbol{\sigma}} \qquad (A3)$$

where $\Gamma = \det\boldsymbol{A} + \det\boldsymbol{B} + 2\det\boldsymbol{C}$. From Eq. (A2), $\det\boldsymbol{C}$ does not depend on the relative phase between $\langle\hat{a}_1\hat{a}_2^\dagger\rangle$ and $\langle\hat{a}_1\hat{a}_2\rangle$. The value of $\det\boldsymbol{\sigma}$ is given by Eq. (6) of the main text, where $\beta_\pm$ should be replaced by the values of $|\langle\hat{a}_1\hat{a}_2\rangle|$ and $|\langle\hat{a}_1\hat{a}_2^\dagger\rangle|$, although the assignment is ambiguous. As a result, the values of $\nu_\pm$ do not depend on the relative phase between $\langle\hat{a}_1\hat{a}_2^\dagger\rangle$ and $\langle\hat{a}_1\hat{a}_2\rangle$. This fact is a consequence of the $\langle\hat{a}_j^2\rangle = 0$ hypothesis.

With our convention, the *bona fide* condition on the covariance matrix is equivalent to $\nu_\pm \geq 1$ [56]. Positivity of the partial transpose (PT) state is a necessary and sufficient condition for the separability of the state [7]. In the creation and annihilation operator basis, the partial transpose operation on mode 2 transforms the covariance matrix as

$$\boldsymbol{A} \xrightarrow{\text{PT}} \boldsymbol{A}, \quad \boldsymbol{B} \xrightarrow{\text{PT}} \boldsymbol{\sigma}_x\boldsymbol{B}\boldsymbol{\sigma}_x, \quad \boldsymbol{C} \xrightarrow{\text{PT}} \boldsymbol{C}\boldsymbol{\sigma}_x \qquad (A4)$$

where $\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Within our hypothesis, the partial transpose only exchanges the roles of $\langle\hat{a}_1\hat{a}_2^\dagger\rangle$ and $\langle\hat{a}_1\hat{a}_2\rangle$.

The symplectic eigenvalues of the partial transpose covariance matrix $\tilde{\nu}_\pm$ are thus given by Eq. (A3) where Γ is replaced by $\tilde{\Gamma} = \det\boldsymbol{A} + \det\boldsymbol{B} - 2\det\boldsymbol{C}$. The determinant of the covariance matrix is preserved under partial transpose operation [52], which also explains why Eq. (6) is invariant under the exchange of $\beta_\pm$. Thus we conclude that the two-mode Gaussian state is entangled iff $\tilde{\nu}_- < 1$. The expression $\lambda_- = \min(\nu_-, \tilde{\nu}_-)$ defined in Eq. (5) in the main text is given by Eq. (A3) replacing Γ by $\Delta = \det\boldsymbol{A}+\det\boldsymbol{B}+2|\det\boldsymbol{C}|$. As discussed by Simon [7], an entangled state must have $\det\boldsymbol{C} < 0$ to respect the *bona fide* condition. Thus, when the state is entangled, inequality $|\langle\hat{a}_1\hat{a}_2\rangle| > |\langle\hat{a}_1\hat{a}_2^\dagger\rangle|$ must hold and because $\beta_+ \geq \beta_-$, we can identify $|\langle\hat{a}_1\hat{a}_2\rangle| = \beta_+$ and $|\langle\hat{a}_1\hat{a}_2^\dagger\rangle| = \beta_-$.

The grey region in Fig. 2 corresponds to states which do not satisfy the *bona fide* condition. For these states, the two solutions $(|\langle\hat{a}_1\hat{a}_2\rangle|, |\langle\hat{a}_1\hat{a}_2^\dagger\rangle|) = (\beta_\pm, \beta_\mp)$ are unphysical: both ν_- and $\tilde{\nu}_-$ are smaller than 1. Thus, $\lambda'_- = \max(\nu_-, \tilde{\nu}_-)$ is smaller than 1 where λ'_- is given by Eq. (5) where Δ should be replaced by

$$\Delta' = \det\boldsymbol{A} + \det\boldsymbol{B} - 2|\det\boldsymbol{C}|$$
$$= (2n_1 + 1)^2 + (2n_2 + 1)^2 - 8(\beta_+^2 - \beta_-^2). \qquad (A5)$$

Detection efficiency - The effect of non-unit efficiency η can be modeled by a pure loss channel that mixes each mode with the vacuum through a beam-splitter with a transmitivity η. Such an operation does not alter the Gaussianity of the state and transforms the covariance matrix $\boldsymbol{\sigma}$ according to $\eta\boldsymbol{\sigma}+(1-\eta)\mathbb{I}_4$ [72, 73]. With our parametrization, the values n_1, n_2, $\langle\hat{a}_1\hat{a}_2^\dagger\rangle$ and $\langle\hat{a}_1\hat{a}_2\rangle$ are divided by $\sqrt{\eta}$. In the main text, the

values of $\beta_\pm$ are proportional to the populations and should be rescaled accordingly when taking into account the non-unit efficiency.

Appendix B: Entanglement and separability thresholds for $g^{(2)}$

As we saw in the last section, a two-mode Gaussian state must satisfy the *bona fide* condition $\nu_\pm \geq 1$ and such a state is entangled if and only if $\tilde{\nu}_- < 1$. As a result, a two-mode Gaussian state is entangled if only if $\min(\nu_-, \tilde{\nu}_-) < 1$, or equivalently $\mathcal{P}_- < 0$ where [7, 52, 56]

$$\mathcal{P}_- = 1 + \det\boldsymbol{\sigma} - \det\boldsymbol{A} - \det\boldsymbol{B} - 2|\det\boldsymbol{C}|. \qquad (B1)$$

This condition being necessary and sufficient, such a state is separable if and only if $\mathcal{P}_- \geq 0$. As the second order correlation function gives access to $|\langle\hat{a}_1\hat{a}_2\rangle|^2 + |\langle\hat{a}_1\hat{a}_2^\dagger\rangle|^2$, it is natural to introduce

$$\delta = \left||\langle\hat{a}_1\hat{a}_2\rangle|^2 - |\langle\hat{a}_1\hat{a}_2^\dagger\rangle|^2\right|/n_1n_2 \qquad (B2)$$

so that $\mathcal{P}_-$ is given by

$$\mathcal{P}_- = 16n_1n_2\left[(1+n_1)(1+n_2)(2 - g_{12}^{(2)})\right.$$
$$+ \left(\frac{1}{2} - n_1n_2\right)\left(g_{12}^{(2)} - 1\right) \qquad (B3)$$
$$\left. + \delta\left(n_1n_2\delta - \frac{1}{2}\right)\right].$$

The value of δ cannot take arbitrarily large or small values. In particular, since the density matrix of the state is positive semi-definite, the following Cauchy-Schwarz inequality for any operators $\hat{\phi}$ and $\hat{\psi}$ must be satisfied [74]

$$|\langle\hat{\phi}^\dagger\hat{\psi}\rangle|^2 \leq \langle\hat{\phi}^\dagger\hat{\phi}\rangle\langle\hat{\psi}^\dagger\hat{\psi}\rangle. \qquad (B4)$$

Note that this inequality cannot be violated and that the so-called violation of the classical Cauchy-Schwarz inequality refers to a normally ordered version of Eq. (B4) [74]. This inequality yields

$$|\langle\hat{a}_1\hat{a}_2\rangle|^2 \leq n_1n_2 + n_1,$$
$$|\langle\hat{a}_2\hat{a}_1\rangle|^2 \leq n_1n_2 + n_2, \qquad (B5)$$
$$|\langle\hat{a}_1\hat{a}_2^\dagger\rangle|^2 \leq n_1n_2.$$

Assuming that $n_1 \leq n_2$ without loss of generality, we see that δ must lie within $[0, 1 + 1/n_2]$.

Lemma: If $\delta > 1$, the state is entangled.

Proof. If $\delta > 1$, then at least one of the two terms $|\langle\hat{a}_1\hat{a}_2\rangle|^2/n_1n_2$ and $|\langle\hat{a}_1\hat{a}_2^\dagger\rangle|^2/n_1n_2$ is strictly greater than 1. Since the Cauchy-Schwarz inequality (B5) gives $|\langle\hat{a}_1\hat{a}_2^\dagger\rangle|^2 \leq n_1n_2$, then we must have $|\langle\hat{a}_1\hat{a}_2\rangle|^2 > n_1n_2$ which implies entanglement from Ref. [8]. $\qquad\square$

Entanglement threshold: A two-mode Gaussian state for which each mode is thermal and which satisfies $g_{12}^{(2)} > g_E^{(2)}$ is entangled. $g_E^{(2)}$ depends on the population and is given in Eq. (9).

Proof. $\mathcal{P}_-$ is a strictly decreasing function of $g_{12}^{(2)}$, therefore if $\mathcal{P}_-(g_E^{(2)}, \delta) \leq 0$ for the allowed values of δ, then any state satisfying $g_{12}^{(2)} > g_E^{(2)}$ is entangled. We therefore look for $g_E^{(2)}$ which is the highest value of the normalized two-body correlation function for a separable state *i.e.* the "last one". Fixing the populations n_1 and n_2, $g_E^{(2)}$ is found solving

$$\text{Max}_\delta \left\{ \mathcal{P}_-(n_1, n_2, g_E^{(2)}, \delta) \right\} = 0. \qquad \text{(B6)}$$

We now seek an upper bound for the expression $\delta(n_1 n_2 \delta - 1/2)$ in Eq. (B3) and subsequently solve $\mathcal{P}_- = 0$. From the above lemma, any state with $\delta > 1$ is entangled ($\mathcal{P}_- < 0$). As we aim to find the "last" *separable* state ($\mathcal{P}_- = 0$), we restrict our analysis to $\delta \in [0, 1]$. We are left to distinguish two cases.

- If $n_1 n_2 < 1/2$, the last term of equation (B3) is always negative, and an upper bound for this term is obtained for $\delta = 0$. Solving $\mathcal{P}_- = 0$ leads to $g_E^{(2)}$ given in Eq. (9).

- If $n_1 n_2 > 1/2$, an upper bound is found for $\delta = 1$. In this scenario, the second and third terms of Eq. (B3) simplify, and solving $\mathcal{P}_-=0$ yields $g_E^{(2)} = 2$.

$\square$

We can interpret these two cases from the behavior of the entanglement border in Fig. 2. When the population is small,

the value of $g_E^{(2)}$ is given by the point on the entanglement border for which $\theta = 1$ in Fig. 2(a). When $\theta = 1$, we have $\beta_+ = \beta_-$ thus $|\langle \hat{a}_1 \hat{a}_2 \rangle| = |\langle \hat{a}_1 \hat{a}_2^\dagger \rangle|$ and $\delta = 0$. In this case, the threshold is greater than 2. When the population is larger, the entanglement border bends to the left (see Fig. 2(b)) and the $g^{(2)}$ threshold is given by the point where $\theta = 0$, which corresponds to $\delta = 1$ and the limit is 2.

Separability threshold: A two-mode Gaussian state for which each mode is thermal and which satisfies $g_{12}^{(2)} \leq g_S^{(2)}$ is separable. $g_S^{(2)}$ depends on the population and is given in Eq. (10).

Proof. $\mathcal{P}_-$ is a strictly decreasing function of $g_{12}^{(2)}$, therefore if $\mathcal{P}_-(g_S^{(2)}, \delta) \geq 0$ for all allowed values of δ, then any state satisfying $g_{12}^{(2)} \leq g_S^{(2)}$ is separable. From Eq. (B3), the minimum of the $\mathcal{P}_-(\delta)$ polynomial is reached for $\delta = 1/4n_1 n_2$. But here again, δ cannot take arbitrarily large values: it must lie within $[0, 1]$, as we show below.

When the coherence is null, we know that separable states satisfy $g_{12}^{(2)} \leq 2$ thus this specific case means that $g_S^{(2)}$ satisfies $g_S^{(2)} \leq 2$. However, $\delta > 1$ implies that $g_{12}^{(2)} > 2$. As we require $g_{12}^{(2)} \leq g_S^{(2)} \leq 2$, we conclude that $\delta \leq 1$ and thus $\delta \in [0, 1]$. We are left to distinguish two cases.

- If $n_1 n_2 \leq 1/4$, the minimum of the $\mathcal{P}_-$ polynomial on $[0, 1]$ is 1, and solving $\mathcal{P}_-(\delta = 1) = 0$ yields $g_S^{(2)} = 2$.

- If $n_1 n_2 > 1/4$, the minimum of $\mathcal{P}_-$ is reached for $\delta_{\min} = 1/4n_1 n_2$. Solving $\mathcal{P}_-(\delta_{\min}) = 0$ yields $g_S^{(2)}$ given in Eq. (9).

$\square$

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