Quantifying mode entanglement of Gaussian states from 2- and 4-body correlation functions

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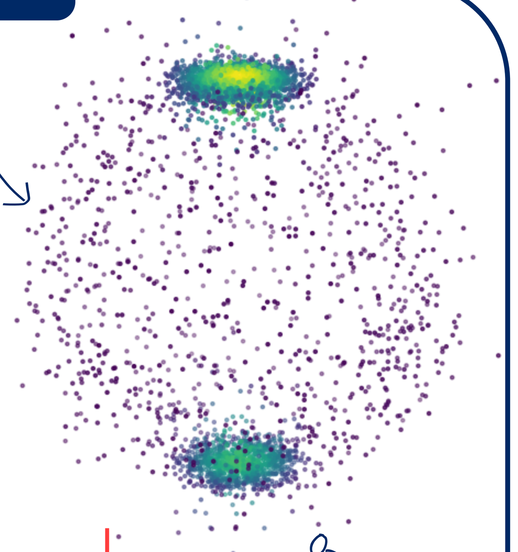
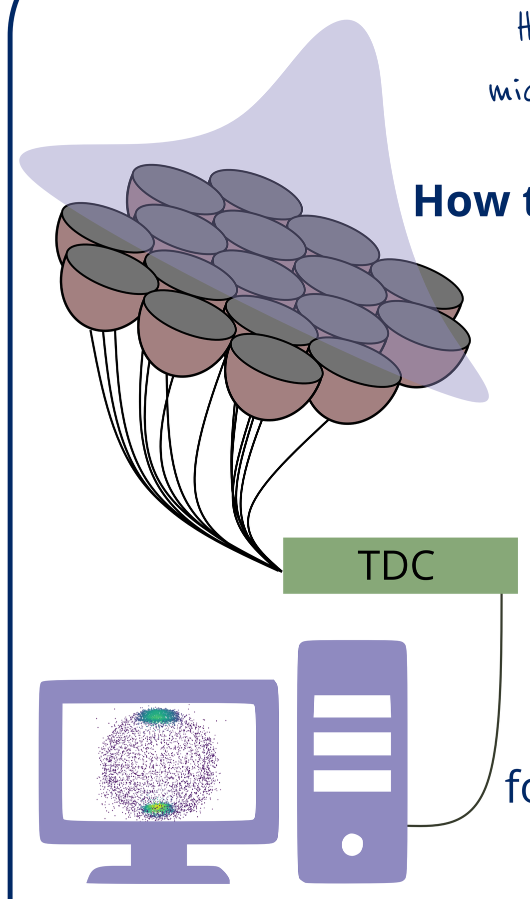
Context

He* internal energy of 20 eV ables the use of micro-channel plate to detect individual atoms



How to assess entanglement of CV with a single particle detector ?

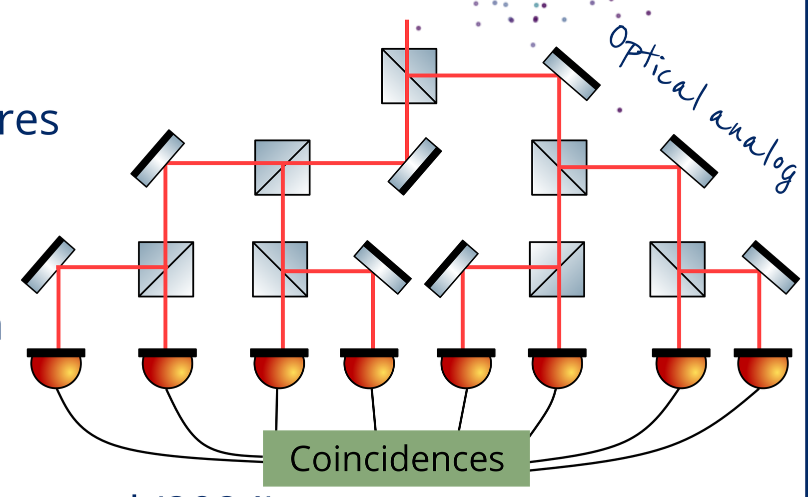


A micro-channel plate can be seen as N independant single particle detectors.

In particular, it measures

$$\langle (\hat{a}_i^\dagger)^n (\hat{a}_j^\dagger)^m \hat{a}_i^n \hat{a}_j^m \rangle$$

for any reasonable n, m



Detector recently "tomographied", see Allemand (2024)

Allemand *et al.* Tomography of a spatially resolved single-atom detector in the presence of shot-to-shot number fluctuations, to appear in *PRX Quantum*, arXiv:2405.01211

Entanglement for bipartite Gaussian states

A quantum state is separable iff it can be written as

$$\hat{\rho} = \sum_j p_j \hat{\rho}_{jA} \otimes \hat{\rho}_{jB}$$

For Gaussian states, the gPPT is an entanglement **criterion** (Simon, 2001).
Iff the partial transpose of the state is negative, the state is entangled.

$$\boldsymbol{\sigma} = \begin{pmatrix} A & C \\ C^\intercal & B \end{pmatrix} \xrightarrow{\text{PT}} (\boldsymbol{\sigma})^{\mathrm{T}B} = \begin{pmatrix} A & C\sigma_z \\ (C\sigma_z)^\intercal & \sigma_z B \sigma_z \end{pmatrix}$$

A Gaussian state characterized by $\boldsymbol{\sigma}$ is positive iff

$$\det A \cdot \det B - \det A - \det B + \underbrace{(1-\det C)^2}_{(1+\det C)^2 \text{ for } (\boldsymbol{\sigma})^{TB}} - \mathrm{Tr}(A\Omega C \Omega B \Omega C^T \Omega) > 0$$

Is often called the bona fide condition

$$\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Simon, R. Peres-Horodecki Separability Criterion for Continuous Variable Systems. *Phys. Rev. Lett. 84, 2726–2729* (2000).
Adesso, G., Serafini, A. & Illuminati, F. Extremal entanglement and mixedness in continuous variable systems. *Phys. Rev. A 70, 022318* (2004).

$g^{(2)}$ and $g^{(4)}$ entanglement criterion

From $g^{(4)}$ and $g^{(2)}$, we access the two possible values for $|c|$ and $|d|$.

$$\beta_\pm^2 = \frac{n_1 n_2}{2}\left[g_{12}^{(2)} - 1 \pm \sqrt{2 + 8\left(g_{12}^{(2)}-1\right) + 3\left(g_{12}^{(2)}-1\right)^2 - \frac{g_{12}^{(4)}}{2}}\right]$$

gives the value of $|c|$ and $|d|$ but we do not know which is $|c|$ and which is $|d|$... However any quantum state must respect the *bona fide* condition that only depends on $|c|$ and $|d|$.

$$\begin{aligned}\mathcal{P}_+(n_1, n_2, c, d) &= (1+n_1)(1+n_2)(n_1 n_2 - |c|^2 - |d|^2) \\ &+ (|c|^2 - |d|^2)^2 - \tfrac{1}{2}(|d|^2 - |c|^2) \\ &+ \left(\tfrac{1}{2} - n_1 n_2\right)\left(|d|^2 + |c|^2\right)\end{aligned}$$

>0 for real quantum states

The states $(n_1, n_2, \beta_+, \beta_-)$ and $(n_1, n_2, \beta_-, \beta_+)$ are partial transpose of each other. If only one respects the *bona fide* condition, the state is known (up to a phase). But since its partial transpose is not positive, it is also **entangled**.

We have access to all the symplectic invariants of the covariance matrix and can compute the logarithm negativity, to *quantify* entanglement.

It does not apply if the local $g^{(2)}$ is not 2. In this case, the phase between *c*, *d* and α^2 matters.
If one measures $g^{(2)}$=1.8, then $g^{(4)}$ must lie within [19.36, 20.64]

EPoV: it is a very narrow intervall !

Conclusion

We can assess mode entanglement of some Gaussian states without measuring non-commuting observables.

Cauchy-Schwarz inequality is not a *mode* entanglement witness for too small populations. It is however a *particle* entanglement witness (Wasak, 2014).

Difference between particle entanglement and mode entanglement: a twin fock state $|n,n\rangle$ is not mode entangled but outperforms the mode entangled TMS for interferometry purpose (Marolleau, 2024)....

Wasak, T *et al.* Cauchy-Schwarz inequality and particle entanglement. *Phys. Rev. A 90, 033616* (2014).
Marolleau, Q. *et. al.* Sub-shot-noise interferometry with two-mode quantum states. *Phys. Rev. A 109, 023701* (2024).

Gaussian states 1.0.1

Gaussian states are fully determined by their first and second moments

Quadrature: $\hat{a}_j = (\hat{x}_j + i\hat{p}_j)/\sqrt{2}$ $\qquad \hat{r} = (\hat{x}_1, \hat{p}_1, ..., \hat{x}_N, \hat{p}_N)$

Displacement $\quad \mu_i = \langle \hat{r}_i \rangle$; Covariance matrix $\quad \sigma_{i,j} = \langle \{\Delta\hat{r}_i, \Delta\hat{r}_j\} \rangle$

Symplectic group size: N(2N+1). Much smaller than proba in Fock space !

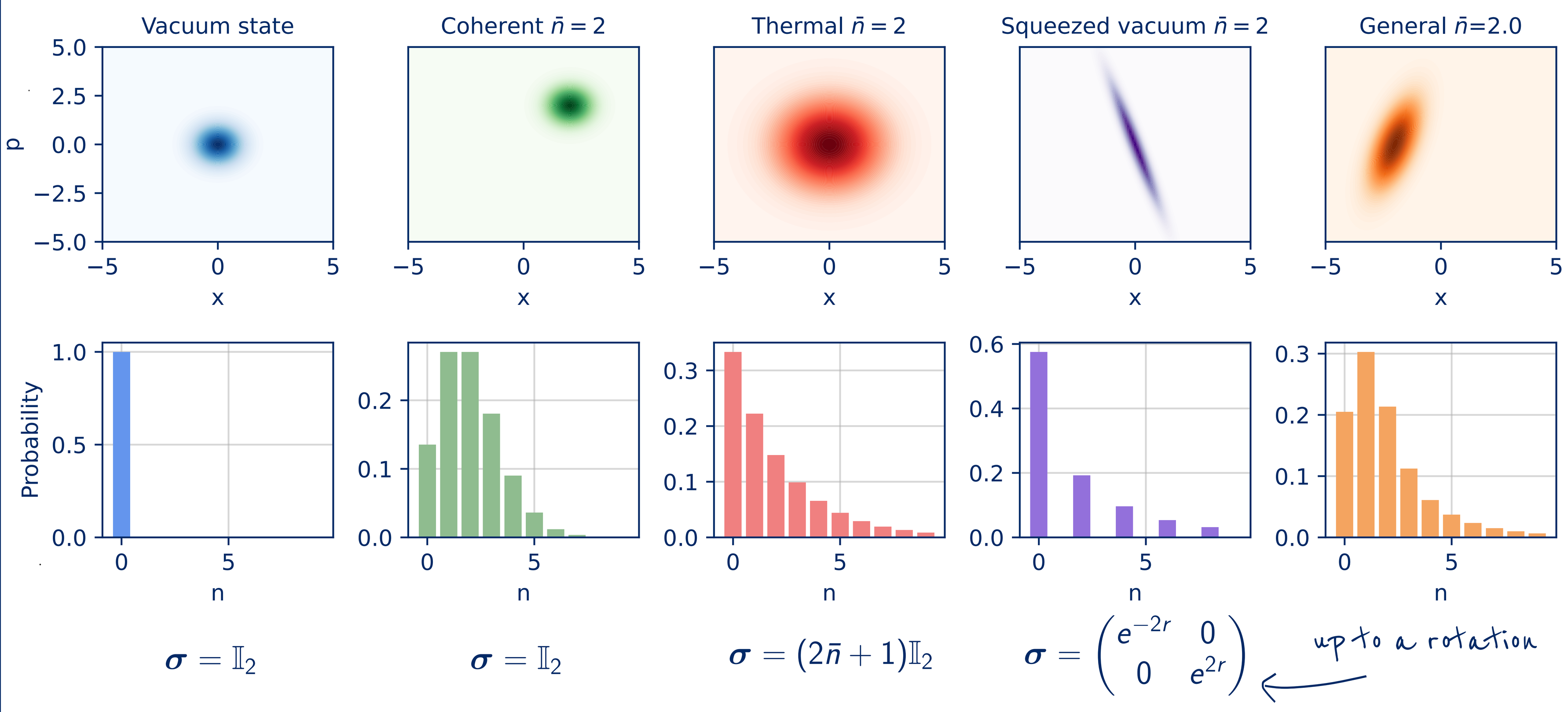


Figure: Representation of the Wigner function of the state and its probability distribution.

Serafini, A. *Quantum Continuous Variables: A Primer of Theoretical Methods*. (2017).

How to probe it

We need to measure the terms of the covariance matrix. $\quad \boldsymbol{\sigma} = \begin{pmatrix} A & C \\ C^\intercal & B \end{pmatrix} \quad c := \langle \hat{a}_1 \hat{a}_2 \rangle \quad d := \langle \hat{a}_1 \hat{a}_2^\dagger \rangle$

For a non-displaced Gaussian state

$$A = \begin{pmatrix} 2n_1 + 1 + 2\Re(\langle \hat{a}_1^2 \rangle) & 2\Im(\langle \hat{a}_1^2 \rangle) \\ 2\Im(\langle \hat{a}_1^2 \rangle) & 2n_1 + 1 - 2\Re(\langle \hat{a}_1^2 \rangle) \end{pmatrix} \qquad C = \begin{pmatrix} 2\Re(c+d) & 2\Im(c-d) \\ 2\Im(c+d) & 2\Re(-c+d) \end{pmatrix}$$

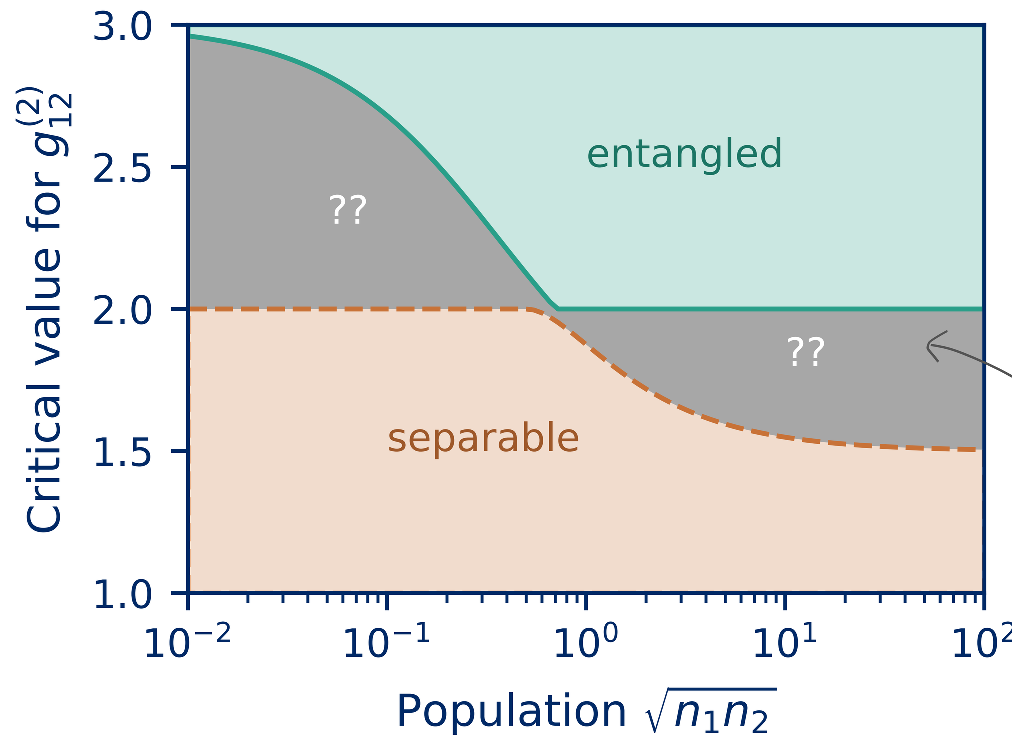
What we measure

- ✓ n_1 ✓ $g_{ii}^{(2)} = 2 + \frac{|\langle \hat{a}_i^2 \rangle|}{n_i^2}$
- ✓ n_2 ✓ $g_{12}^{(2)} = \frac{n_1 n_2 + |c|^2 + |d|^2}{n_1 n_2}$
- + $g^{(4)}$ that involves $|c|^2|d|^2$

Theorem: For a non-displaced Gaussian state, and if one *measures* that the local correlation function is 2, the measurement of the populations and the second and fourth order correlation functions assesses the separability of the Gaussian state.

$g^{(2)}$ as entanglement witness

What if I do not have a (good) measurement of $g^{(4)}$?

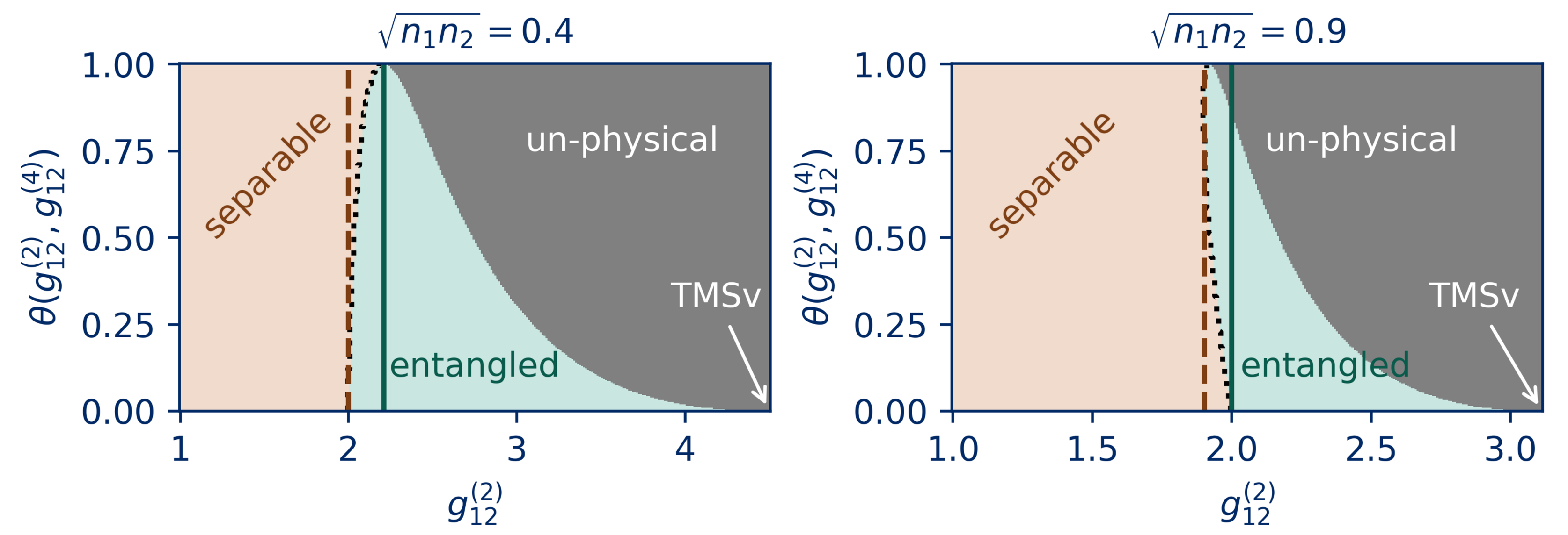


We provide a threshold to witness entanglement using only the second order correlation function.

We need $g^{(4)}$ to assess entanglement in this question mark region.

Same "terms of service" apply as before however

Demonstration: Separable states cannot have a too large second order correlation function



θ gives the range of $g^{(4)}$ defined as: $\beta_\pm^2 = n_1 n_2 (g_{12}^{(2)} - 1)\frac{1 \pm \sqrt{1-\theta}}{2}$

Fundings

SIRTEQ — quantum PARIS-SACLAY — anr