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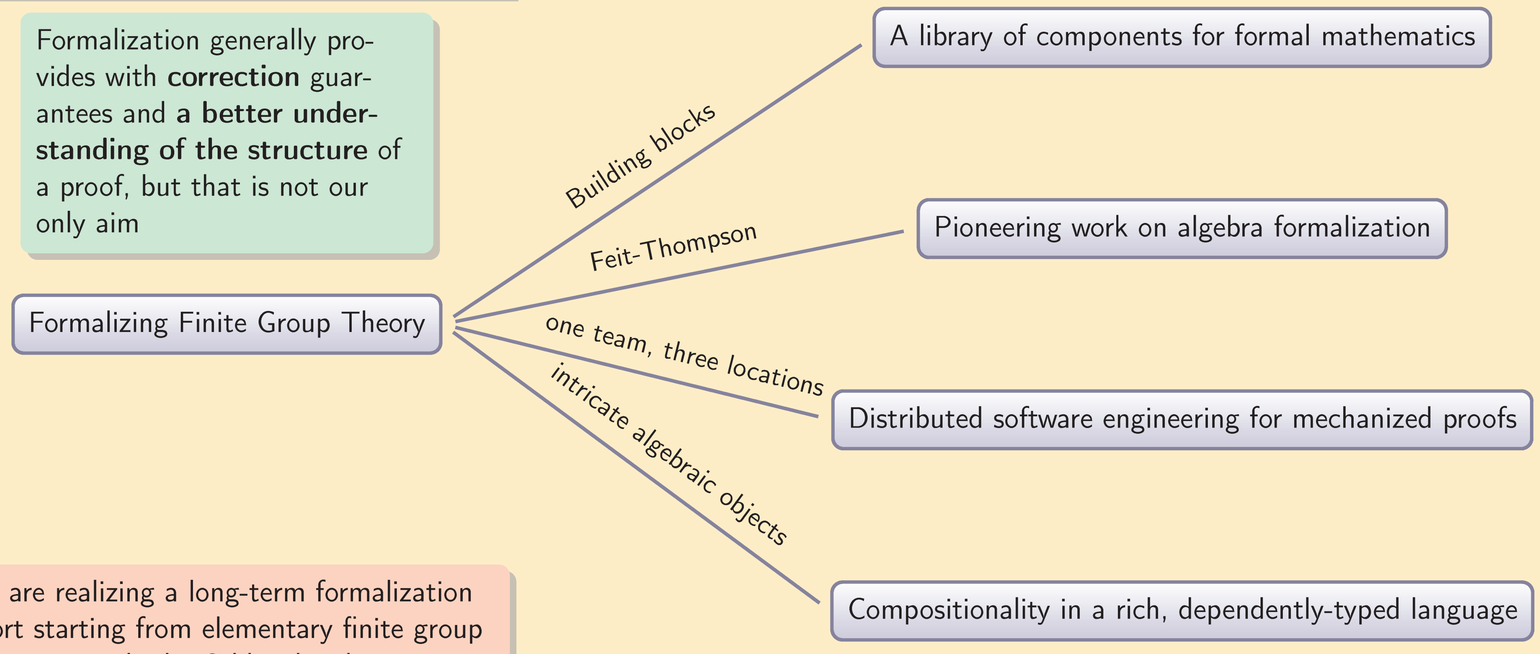
Mechanized foundations of finite group theory

The work of the *Mathematical Components* team at Microsoft Research and INRIA.

Motivation

Why formalize finite group theory ?

Formalization generally provides with **correction** guarantees and **a better understanding of the structure** of a proof, but that is not our only aim



We are realizing a long-term formalization effort starting from elementary finite group theory, towards the Odd order theorem.

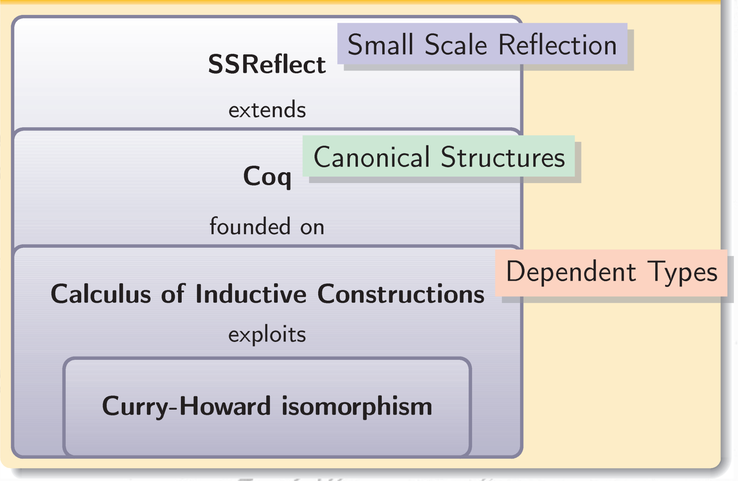
The Feit-Thompson Theorem

Group theory ≡ the study of reversible composition laws.
A finite group can be decomposed in *simple groups*.

Those simple groups can all be exactly described in a classification, achived in 1983. It is **widely recognized as one of the greatest achievements of twentieth century mathematics**. The proof of this *enormous theorem* is disseminated in hundreds of journal articles.
The foundational breakthrough of this classification came in 1962, when Feit & Thompson showed that
Every finite group of odd order is solvable.

The proof is **255 pages long**, and **one of the reasons J.G. Thompson has been awarded the Abel prize in 2008.**

Tools and Theories

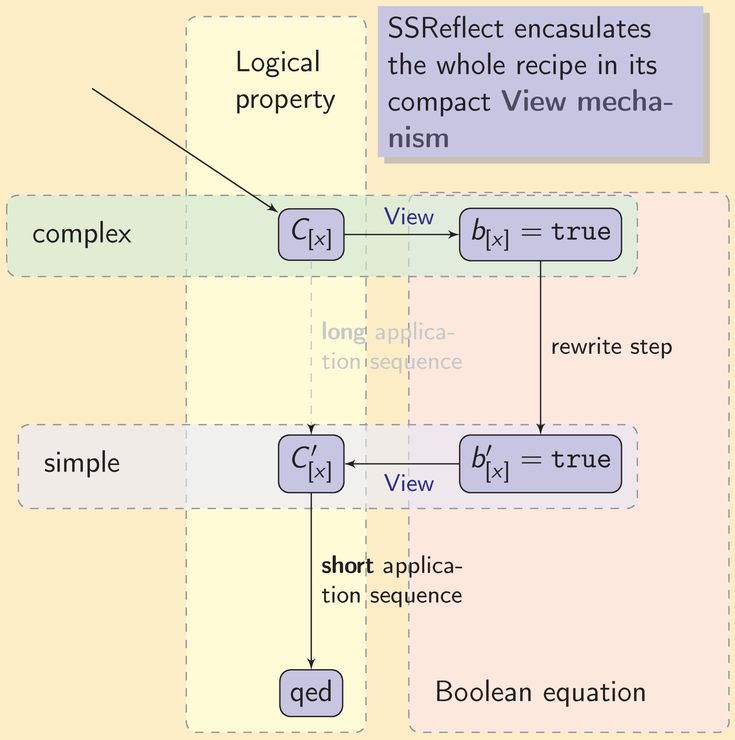


Small Scale Reflection

Recipe.
To prove a goal that uses a predicate P.

- Implement in Coq a partial decision procedure for P that **reflects** the truth of P with an algorithm returning booleans.
- Prove that P holds if and only if the procedure returns `true`.
- Prove individual instances by applying that soundness theorem with reflexivity proofs.

SSReflect encasulates the whole recipe in its compact **View mechanism**



Canonical Structures

Type classes are, essentially, **implicitly passed dictionaries**.
In **Coq**, they correspond to **Canonical Structures** and become **implicitly passed proofs**, providing us with

> *"the shorthand that makes mathematics usable" (Bourbaki)*

```
Structure monoid : Type := Monoid {
  sort :> Type;
  add : sort -> sort -> sort;
  unit: sort;
}

Variable (mT: monoid A).

Fixpoint sum (s:list (mT)) :=
  match s with
  | h::t -> add h (sum t)
  | _ -> unit
  end;;
```

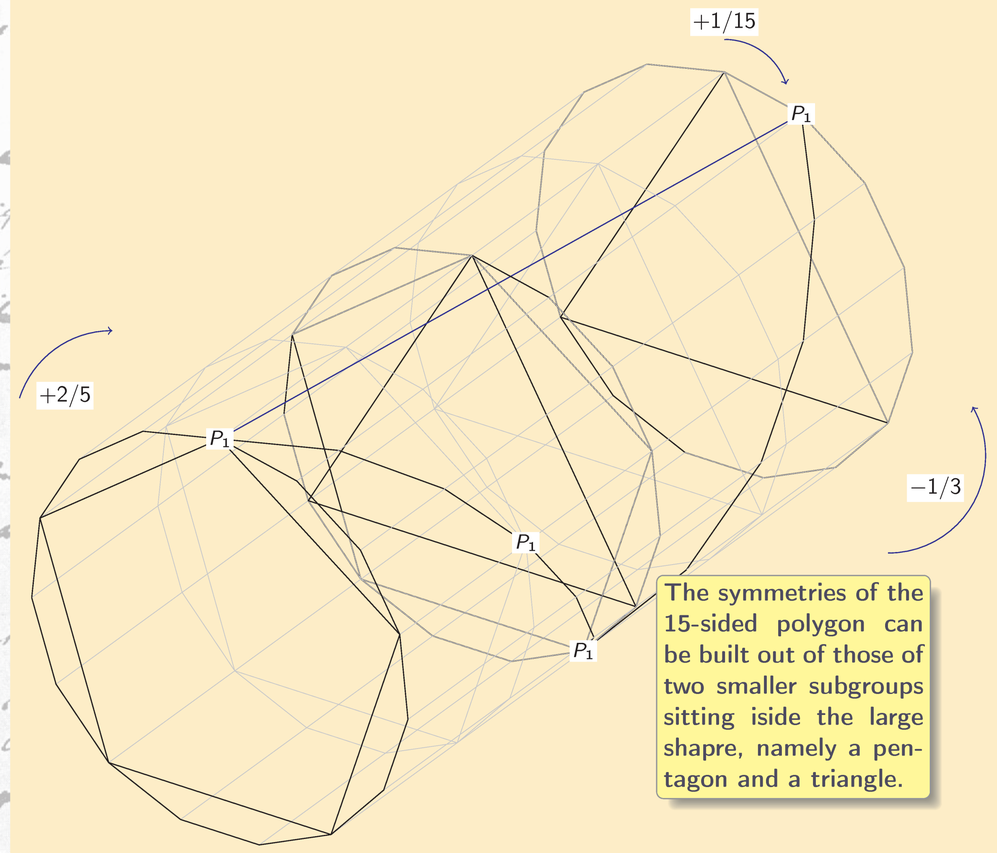
These structures allow-us to thin-slice group properties, offering many levels of abstraction for our properties.
They form **narrowly-focused definitions**, on which we can **interface** our developments.

SSReflect

New release 1.1 !

- A renewed tactic shell for Coq that accelerates proof development using Small Scale Reflection
- Property bookkeeping is eliminated or shortened;
- A set of reusable **components** that structures proofs on finite domains;
- try it:
  `http://www.msr-inria.inria.fr/`

Decomposition example : the pentadecagon



The symmetries of the 15-sided polygon can be built out of those of two smaller subgroups sitting iside the large shapre, namely a pentagon and a triangle.

Results obtained, future steps

Most group formalizations stop at the Lagrange theorem. We have formalised:

- the Sylow theorems
- the Cayley-Hamilton theo
- the Frobenius-Cauchy len
- the simplicity of the alter
- the Schur-Zassenhaus theorem
- the Jordan-Hölder theorem
- ... and counting

We already have one of the most advanced formalizations of finite algebra !

We now expect:

- to continue using programming language constructs to express theory (notations, phantom types, etc ...)
- to find a way to quickly relate properties on an object to those of one of its isomorphic images, doing **property transfer** on a well-behaved mapping

and pursue our development of group theory!

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