Gyroscopic Instability of a Drop Trapped Inside an Inclined Circular Hydraulic Jump

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A drop of moderate size deposited inside a circular hydraulic jump remains trapped at the shock front and does not coalesce with the liquid flowing across the jump. For a small inclination of the plate on which the liquid is impacting, the drop does not always stay at the lowest position and oscillates around it with a sometimes large amplitude, and a frequency that slightly decreases with flow rate. We suggest that this striking behavior is linked to a gyroscopic instability in which the drop tries to keep constant its angular momentum while sliding along the jump.

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There is presently growing interest in the dynamics of levitating liquids such as in the Leidenfrost effect [1], or in the situation of delayed coalescence between liquids [2–6]. These conditions arise when a very thin layer of air or vapor remains trapped at the interface, thus preventing the liquid from spreading on a solid, or from coalescing with a free surface of the same liquid. These unusual situations of nonwetting give rise to surprising liquid dynamics, of great fundamental interest. One can refer, for instance, to Poincaré's figures of equilibrium observed by Aussillous and Quéré for drops coated with hydrophobic grains rolling down a solid substrate [7], or to the striking particle-wave duality evidenced by Couder *et al.*, on drops walking on a vibrated liquid [3–5].

In the present Letter we consider a different system. A drop deposited inside a circular hydraulic jump [8–10] is pushed by the flow against the jump and remains trapped if its size is not too large [6]. Owing to the liquid motion inside the jump and inside the drop, a thin layer of air trapped in between prevents coalescence when the two liquids are the same. We show that this situation also leads to a striking dynamical behavior of the drop, that tends to rotate around the jump. In the special case (considered here) of a slightly inclined jump, with respect to horizontal, the drop does not stay static at the lowest equilibrium point but rather oscillates in a self-sustained way around this position, the oscillation reaching very high amplitudes (nearly 180°) without loosing harmonicity. A model describing this behavior is proposed, based on a gyroscopic instability: the drop both slides and rotates above the liquid surface, exchanging friction with it against the air film, while trying to keep constant its large angular momentum. Although the connections with this kind of problem are not obvious, this instability is reminiscent of others encountered by rotating systems when a slight amount of dissipation is added to a situation in apparent equilibrium [11].

A picture of the experiment is reproduced in Fig. 1. A jet of silicone oil (viscosity 20 cS) issued from a vertical tube of diameter 4 mm, hits the surface of a plate placed 3 cm below the outlet. The plate inclination α is fixed to $1.0 \pm 0.1°$. For this low inclination, the hydraulic jump is observed to be of type I (i.e., unidirectional surface flow), and to remain nearly circular, in a way comparable to what occurs for jumps obtained for slightly oblique jets [12]. The radius of the jump increases with flow rate [Fig. 2(a)] and follows a law similar to those reported in earlier works for a horizontal plate [6,8,9]. Millimeter sized drops of the same fluid as the bath are deposited directly inside the jump. As reported in [6], small enough droplets do not cross the jump and remain trapped at its contact [Fig. 1], without coalescing with the flowing liquid. Qualitative tests were performed with different silicone oils to investigate the influence of viscosity. For high enough viscos-

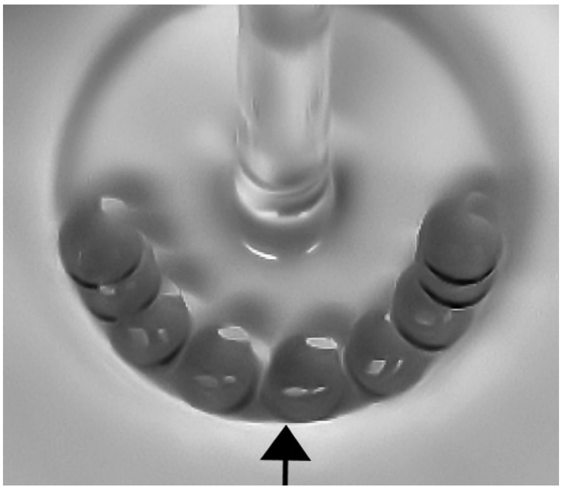

FIG. 1. Nearly circular hydraulic jump formed by a vertical liquid jet of silicone oil impacting a slightly inclined plate. In the appropriate range of parameters, a drop levitating above the jump does not stay stationary at the lowest position (arrow). It enters an oscillatory motion around this position along the jump perimeter, shown here by superimposing successive frames.

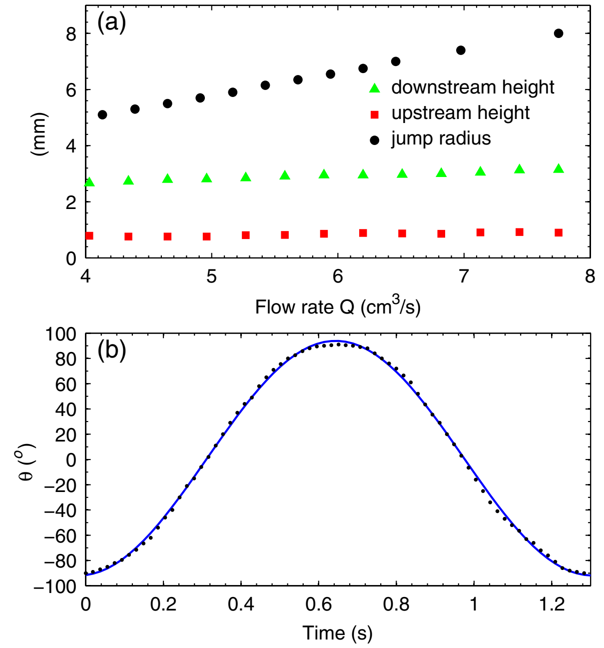

FIG. 2 (color online). Measured for a plate inclination of $\alpha \approx 1.0 \pm 0.1°$: (a) Radius of the hydraulic jump (circle), upstream (square), and downstream (triangle) fluid heights around the jump, versus flow rate. (b) Typical law $\theta(t)$ for the drop motion, θ being the drop angular position defined with respect to the lowest point of the jump (unstable equilibrium position). The line is a fit with a cosine function.

ities (above 35 cS), the lowest location of the drop in the circular jump is a stable equilibrium position, but it turns out that for lower viscosities this position becomes unstable. The drop, after a transient, does not remain stationary at the lowest point but enters a regime of oscillations around this equilibrium position as suggested in Fig. 1, with a time period of order 1 s. A typical oscillatory motion is plotted in Fig. 2(b). It is nearly harmonic, amplitude and frequency being functions of the flow rate. The amplitude of the angular motion increases dramatically when flow rate is reduced and can even reach 2π for small enough flow rates.

For the sake of simplicity, it will be assumed thereafter that the drop, assimilated to a solid sphere, has two contact points a and b with the jump as shown in Fig. 3(a), and that the contact areas s are both of order πr^2, with r the radius of the drop. As a first stage, only the point a will be taken into account. The corresponding thickness of the air layer d_a, that separates the drop from the flowing liquid entering the jump, is a key point of the dynamics. This thickness was not measured, but, as explained in [6], reasonable orders of magnitude can be obtained by drawing an analogy with standard bearing theory. Here a scaling argument is given, that simply replaces this approach. If U_a designates the liquid velocity when it enters the jump, air is trapped between the drop and the flowing liquid, and should flow at a typical velocity U_a inside a gap d_a. This will develop a pressure gradient of order $\eta U_a/d_a^2$ in the air film, and thus a pressure $\eta r U_a/d_a^2$ where η is the air dynamic viscosity. Balancing the expected levitation force of order $\eta r^3 U_a/d_a^2$ with the drop weight leads to

$$d_a \propto \left(\frac{\eta U_a}{\rho g}\right)^{1/2}, \qquad (1)$$

where ρ is the mass density of liquid. For a velocity $U_a = 15$ cm/s deduced from the fluid height in front of the jump, this estimate is of order 18 μm and is consistent with the estimate of 14 μm obtained in [6], while the minimum thickness expected to observe coalescence is approximately 200 nm [3].

A possible interpretation of the fact that the lowest drop position becomes unstable, and of the occurrence of this oscillatory motion is as follows. First, because of the friction between the liquid bath and the drop (through the thin air film), the drop will never remain static and should develop a sizable rotational motion. This effect is suggested in Fig. 3(a), in which the angular velocity of the drop defined in a plane containing the axis of symmetry Oz reads Ω_θ. Assuming that the drop and the flowing film, of velocity U_a, exchange a classical shear stress of approximate value $\sigma = -\eta[U_a + r\Omega_\theta]/d_a$, the evolution equation for Ω_θ reads $J(d\Omega_\theta/dt) = -\eta(rs/d_a)[U_a + r\Omega_\theta]$,

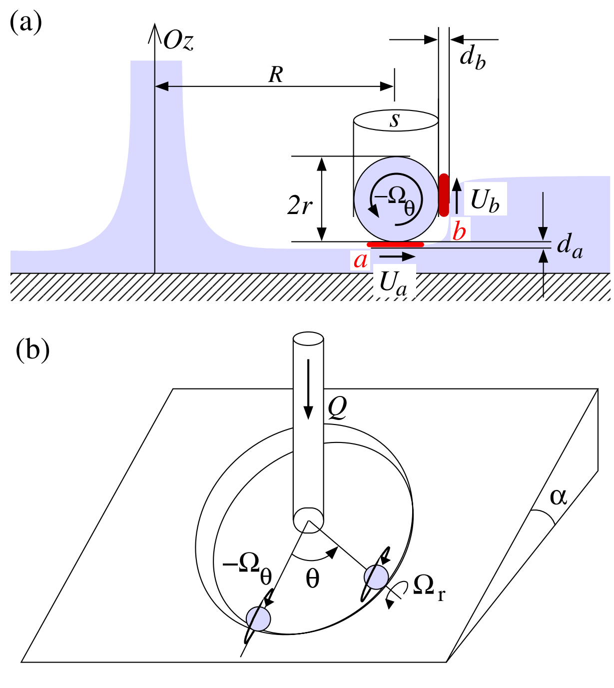

FIG. 3 (color online). (a) Sketch of the drop and fluid motion expected in a radial vertical plane containing the jump central axis. The drop is rotating very fast because of the shear stress transmitted across the air film. (b) Principle of the instability viewed in three dimensions. The drop, when pushed from the lowest point of the jump, tries to keep constant its angular momentum, which develops an active radial component of the rotation rate vector that tends to amplify the drop displacement.

where $J = (2/5)mr^2$ is the moment of inertia of the drop of mass m. This equation can be rewritten as

$$\tau_a \frac{d\Omega_\theta}{dt} + \Omega_\theta = -\frac{U_a}{r}, \tag{2}$$

where the time constant reads $\tau_a = (8/15)\rho r d_a/\eta$ for a spherical drop. Typically, a drop of radius $r = 1.3$ mm, levitating above a jump of velocity $U_a = 16$ cm/s, on an air film of thickness $d_a = 18.5$ μm, reaches a rate of 20 rotations per second in a typical time of order τ_a that is comparable to the period of the oscillations. If for some reason the rotating drop is moved from the lowest position in the jump to an out-of-equilibrium position, the angular displacement being called θ [see Fig. 3(b)], the drop will try to keep constant its angular momentum $\mathbf{L} = J\mathbf{\Omega}$. This will develop a radial component of the rotation rate vector suggested in Fig. 3(b), of amplitude $\Omega_r = \Omega_\theta \sin\theta$, that will be propulsive, and that will tend to amplify the drop displacement. The propulsive force $f = \eta(s/d_a)(r\Omega_r)$ has to be compared with a gravity restoring force $p = mg\sin\alpha\sin\theta$. The lowest drop position is unstable when

$$\eta \frac{s}{d_a} U_a > mg\sin\alpha \tag{3}$$

or, equivalently with the relationships $s \approx \pi r^2$ and $m = (4/3)\pi\rho r^3$, when

$$\frac{\eta U_a}{\rho g r d_a \sin\alpha} \gtrsim 1. \tag{4}$$

With the orders of magnitude introduced in the foregoing, this ratio is of order 1 for an air film of $d_a = 18.5$ μm and even 10 for a thinner film of thickness 1 μm, indicating that the viewpoint adopted here seems to be consistent though it is quite sensitive to the air film thickness.

Having recognized a possible instability mechanism, a more complete investigation is needed, to check whether or not this instability may lead to stationary oscillations, and in that event, to explore the nature of these oscillations. It is assumed that the viscous force due to the velocity gradients reads

$$\mathbf{F} = \frac{\eta s}{d_a}\left(\mathbf{U} - D\frac{d\theta}{dt}\hat{\theta} + \mathbf{r}\times\mathbf{\Omega}\right),$$

where the first and the second right-hand-side terms come from the relative motion of the drop with respect to the fluid, and the third one to its own rotational motion. Considering now both contact points (a, below the drop, and b, on the jump side), and using that the torque $\tau = \mathbf{r}\times\mathbf{F} = Jd\mathbf{\Omega}/dt$, it can be shown that the components of the rotation introduced in Fig. 3(b) are linked by the following evolution equations:

$$\begin{aligned} \frac{d\Omega_r}{dt} &= -\frac{1}{\tau_a}\left(\frac{R}{r}\frac{d\theta}{dt} + \Omega_r\right) + \Omega_\theta \frac{d\theta}{dt} \\ \frac{d\Omega_\theta}{dt} &= -\frac{1}{\tau_a}\left(\frac{U_a}{r} + \Omega_\theta\right) - \frac{1}{\tau_b}\left(\frac{U_b}{r} + \Omega_\theta\right) - \Omega_r\frac{d\theta}{dt}, \\ \frac{d\Omega_z}{dt} &= -\frac{1}{\tau_b}\left(\frac{R}{r}\frac{d\theta}{dt} + \Omega_z\right), \end{aligned} \tag{5}$$

where R is the radius of the circular trajectory drawn by the center of the drop. These equations must be coupled with the evolution equation for θ that can be deduced from the fundamental principles of the dynamics. Balancing inertia $mRd^2\theta/dt^2$ with the sum of gravity $-mgR\sin\theta\sin\alpha$ and friction $\eta(s/d_{a,b})[r\Omega_r - R(d\theta/dt)]$ forces yields

$$\frac{d^2\theta}{dt^2} + \frac{2}{5}\left(\frac{1}{\tau_a} + \frac{1}{\tau_b}\right)\frac{d\theta}{dt} + \left(\frac{g}{R}\sin\alpha\right)\sin\theta = \frac{2}{5}\frac{r}{R}\left(\frac{\Omega_r}{\tau_a} + \frac{\Omega_z}{\tau_b}\right). \tag{6}$$

Equation (6) completed by Eqs. (5) is simply that of a harmonic oscillator, in which the damping term is balanced by a gyroscopic effect. As a result, the time period of the oscillation is close to the estimate

$$T = 2\pi\left(\frac{R}{g\sin\alpha}\right)^{1/2}. \tag{7}$$

This value increases with flow rate, in view of the evolution of the jump radius $(R + r)$.

Characteristic plots of the obtained behaviors are shown in Fig. 4, for the experimental data introduced above and $\alpha = 1.1°$, which implies $\tau_a = 0.63$ s. Estimating that U_b is of order $U_a/10$, τ_b is obtained the same way as τ_a by using the friction force developed at point a instead of the drop weight. This leads to $\tau_b = 1.7$ s. In the plane (θ, $d\theta/dt$), a well-defined limit cycle is reached regardless of initial conditions (not shown). Figure 4(a) shows the corresponding oscillations $\theta(t)$ at long time scales. As in the experiment, a nearly perfect harmonic oscillation is obtained, even at large amplitude, which is due to the $\sin\theta$ dependence of the restoring force. Plots of the rotation frequencies $\Omega_r(t)$, $\Omega_\theta(t)$, and $\Omega_z(t)$ are reproduced in Figs. 4(b)–4(d). The frequency of oscillations of $\Omega_\theta(t)$ is twice that of $\theta(t)$ and $\Omega_r(t)$. It can be noted that the typical spin-up time of the drop remains smaller than the period of oscillations $[\tau_a/T = 4r/15\pi(\rho U_a\sin\alpha/\eta R)^{1/2} < 0.65]$, which supports the consistency of our modeling. Experimental results and theoretical predictions for time period and amplitude are plotted in Fig. 4(e) for the range of flow rates investigated. The model reproduces quantitatively the observed features (oscillations, harmonicity, time period, and amplitude) for $\alpha = 1.1°$. Nevertheless, the model is quite sensitive to the plate inclination, and using the experimental value of $\alpha = 1.0°$ as input leads to a shift (dashed lines). Moreover, the period increases slightly with flow rate while it seems almost independent of flow rate in the experiment.

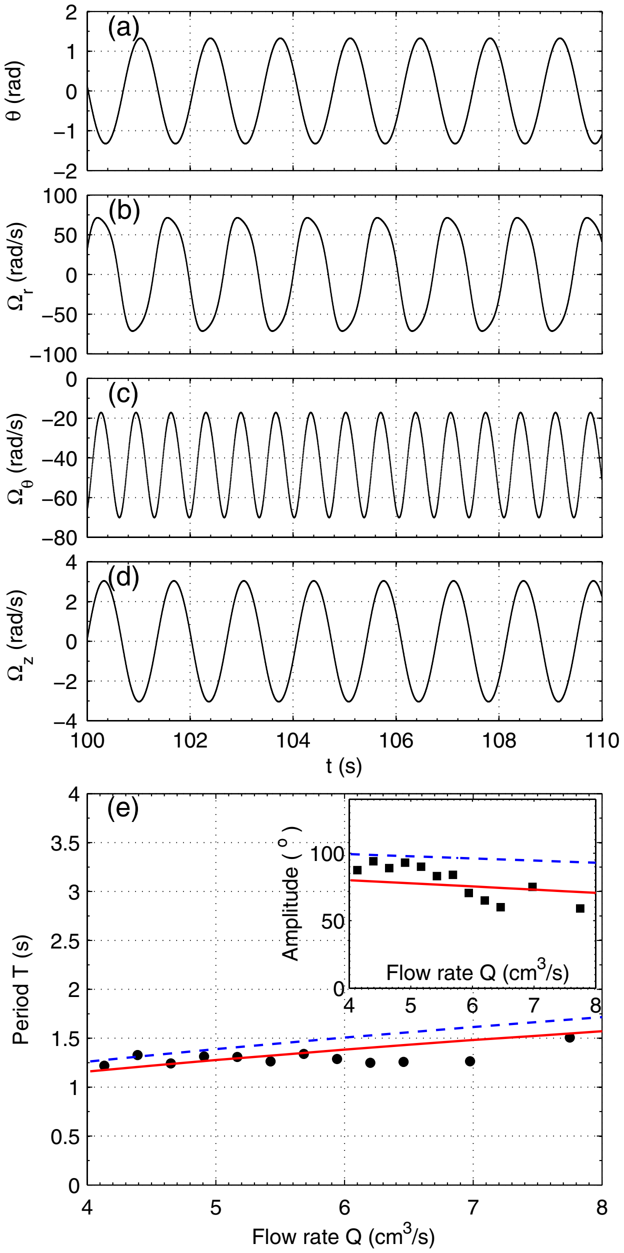

FIG. 4 (color online). Typical solutions of Eqs. (5) and (6) numerically integrated with $\tau_a = 0.63$ s, $\tau_b = 1.70$ s, $r/R = 0.25$, $U_a/r = 123\ \mathrm{s}^{-1}$, and $\Omega_{r0}^2 = (g/R)\sin\alpha = 36.9\ \mathrm{s}^{-2}$; (a) oscillatory motion $\theta(t)$ obtained at long time scales, (b)–(d) related oscillations of the angular velocities $\Omega_r(t)$, $\Omega_\theta(t)$, and $\Omega_z(t)$. (e) Time period of the drop oscillations versus flow rate. Inset: amplitude versus flow rate. Lines show theoretical predictions integrating Eqs. (5) and (6). Straight line $\alpha = 1.1°$, dashed line $\alpha = 1.0°$.

The relative discrepancy could be attributed to an additional restoring force and/or a lessening of friction not included in the analysis:

(i) The loss in mechanical energy linked to viscous dissipation inside the drop has been neglected.

(ii) The model ignores the interactions of the drop with the jump itself. The presence of the drop should affect the jump structure, with respect to both velocity field and capillary effects [9], which could in turn modify the drop dynamics.

(iii) The jump inclination implies an extra draining superimposed on the classical radial flow. This should give rise to a tangential component of the flow along the jump, which will exert an extra restoring force on the drop with the same symmetries as the tangential gravity force considered here (i.e., proportional to $\sin\theta\sin\alpha$), and thus certainly modify the time period. Moreover, the inclination of the plate for a vertical impinging jet implies an inclination of the jet with respect of the plate itself, which is known to affect the jump structure [12].

Nontrivial oscillatory dynamics of a single drop trapped inside a circular hydraulic jump have been considered, in the case of a jump developing on a slightly inclined plate. A model has been proposed, based on the idea that the drop is rapidly rotating while exchanging friction with the liquid of the jump, which can lead to a gyroscopic instability. This interpretation provides quantitative agreements, both for frequency and amplitude selections. The problem of the inclined circular jump itself must be investigated, owing to its probable influence on the drop dynamics. Further studies on this fascinating phenomenon are under way.

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