Géométrie ℓ_1 et processus localement linéaire :

Preuve directe de NSP par la méthode de Stephen O. Rice

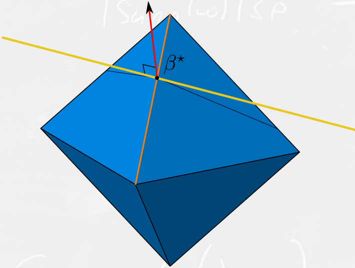
J.-M. Azaïs (IMT), Y. de Castro (LM-Orsay) and S. Mourareau (IMT)

Département de Mathématiques d'Orsay

Séminaire Parisien de Statistiques, 13/10/14

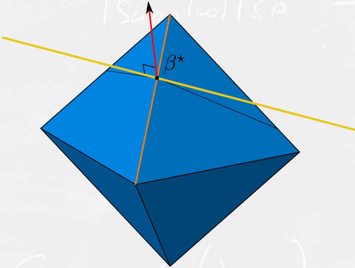
Outline

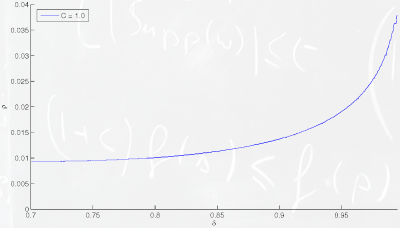
1. ℓ_1-geometry, sparse regression and polytopes,
2. Null-Space Property (NSP)
3. Rice's formulas



Outline

1. ℓ_1-geometry, sparse regression and polytopes,
2. Null-Space Property (NSP)
3. Rice's formulas





4. Sketch of the proof
5. Lower bound on the phase transition
6. Restricted Isometry Property

The ℓ_1-recovery problem:

- Given $\mathbf{X} \in \mathbb{R}^{n\times p}$ and $\beta \in \mathbb{R}^p$, do we have:

$$\{\beta\} = \arg\min_{\gamma\in\mathbb{R}^p} \{\|\gamma\|_1\,,\ \mathbf{X}(\gamma-\beta)=0\}\,?$$

The ℓ_1-recovery problem:

- Given $\mathbf{X} \in \mathbb{R}^{n \times p}$ and $\beta \in \mathbb{R}^p$, do we have:

$$\{\beta\} = \arg \min_{\gamma \in \mathbb{R}^p} \{\|\gamma\|_1 \,,\ \mathbf{X}(\gamma - \beta) = 0\}\ ?$$

- Yes, if $\forall h \in \ker(\mathbf{X}) \setminus \{0\}\,,\ \|h_{I^c}\|_1 > |\langle \mathrm{sgn}(\beta_I), h_I \rangle|$ where $I = \mathrm{Support}(\beta)$.

The ℓ_1-recovery problem:

- Given $\mathbf{X} \in \mathbb{R}^{n \times p}$ and $\beta \in \mathbb{R}^p$, do we have:

$$\{\beta\} = \arg\min_{\gamma \in \mathbb{R}^p} \{\|\gamma\|_1 \,,\ \mathbf{X}(\gamma - \beta) = 0\}\,?$$

- Yes, if $\forall h \in \ker(\mathbf{X}) \setminus \{0\}\,,\ \|h_{I^c}\|_1 > |\langle \mathrm{sgn}(\beta_I), h_I \rangle|$ where $I = \mathrm{Support}(\beta)$.

Sparse vectors:

- Assume $\beta \in \Sigma_s := \{\gamma \in \mathbb{R}^p\,,\ \|\gamma\|_0 \leq s\}$.
- Applications: signal processing, medical imaging, genetics...

Compressed Sensing problem

Given $\mathbf{X} \in \mathbb{R}^{n \times p}$ and $s > 0$, do we have:

$$\forall\, \beta \in \Sigma_s\,, \quad \{\beta\} = \arg\min_{\gamma \in \mathbb{R}^p} \{\|\gamma\|_1 \,,\ \mathbf{X}(\gamma - \beta) = 0\}\,? \tag{CS Problem}$$

The projection of the cross-polytope problem:

- Consider the cross-polytope $C^p = \{\gamma \in \mathbb{R}^p\,,\ \|\gamma\|_1 \leq 1\}$.
- Let F be a face of C^p whose extreme points are $(\varepsilon_i e_i)_{i\in I}$ with $\varepsilon_i = \pm 1$.

Is $\mathbf{X}F$ a face of $\mathbf{X}C^p$ whose extreme points are $(\varepsilon_i X_i)_{i\in I}$?

The projection of the cross-polytope problem:

- Consider the cross-polytope $C^p = \{\gamma \in \mathbb{R}^p\,,\ \|\gamma\|_1 \leq 1\}$.
- Let F be a face of C^p whose extreme points are $(\varepsilon_i e_i)_{i\in I}$ with $\varepsilon_i = \pm 1$.

Is $\mathbf{X}F$ a face of $\mathbf{X}C^p$ whose extreme points are $(\varepsilon_i X_i)_{i\in I}$?

- Yes, if $\forall h \in \ker(\mathbf{X}) \setminus \{0\}\,,\ \|h_{I^c}\|_1 > |\langle \varepsilon_I, h_I\rangle|$.

Cross-Polytope problem

Given $\mathbf{X} \in \mathbb{R}^{n\times p}$ and $s > 0$, do C^p and $\mathbf{X}C^p$ share the same number of s-faces?

- Let $I \subset \{1,\ldots,p\}$ and $K_I := \{h \in \mathbb{R}^p,\ \|h_I\|_1 \geq \|h_{I^c}\|_1\}$. For $s > 0$, set:

$$\mathcal{K}_s := \bigcup_{|I| \leq s} K_I\,.$$

Null-Space Property (NSP)

$G \subset \mathbb{R}^p$ satisfies $\mathrm{NSP}(s,1)$ iff

$$G \cap \mathcal{K}_s = \{0\}\,.$$

Null Space Property

- Let $I \subset \{1,\ldots,p\}$ and $K_I := \{h \in \mathbb{R}^p,\ \|h_I\|_1 \geq \|h_{I^c}\|_1\}$. For $s > 0$, set:

$$\mathcal{K}_s := \bigcup_{|I| \leq s} K_I\,.$$

Null-Space Property (NSP)

$G \subset \mathbb{R}^p$ satisfies $\mathrm{NSP}(s,1)$ iff

$$G \cap \mathcal{K}_s = \{0\}\,.$$

[CDD06] and [D05] Theorems

[CS problem]
$\Leftrightarrow$ [Cross-Polytope problem]
$\Leftrightarrow$ [$\ker(X)$ satisfies $\mathrm{NSP}(s,1)$]

Some references

- Implicit expression: Donoho and Tanner (2009), Xu and Hassibi (2011).
- Based on convex integral geometry.

Null Space Property

- Let $I \subset \{1,\ldots,p\}$ and $K_I := \{h \in \mathbb{R}^p,\ \|h_I\|_1 \geq \|h_{I^c}\|_1\}$. For $s > 0$, set:

$$\mathcal{K}_s := \bigcup_{|I|\leq s} K_I\,.$$

Null-Space Property (NSP)

$G \subset \mathbb{R}^p$ satisfies $\mathrm{NSP}(s,1)$ iff

$$G \cap \mathcal{K}_s = \{0\}\,.$$

[CDD06] and [D05] Theorems

[CS problem]
$\Leftrightarrow$ [Cross-Polytope problem]
$\Leftrightarrow$ [$\ker(X)$ satisfies $\mathrm{NSP}(s,1)$]

Some references

- Implicit expression: Donoho and Tanner (2009), Xu and Hassibi (2011).
- Based on convex integral geometry.
- Sub-exponential rows: Lecué and Mendelson (2014).

Null Space Property

- Let $I \subset \{1,\ldots,p\}$ and $K_I := \{h \in \mathbb{R}^p,\ \|h_I\|_1 \geq \|h_{I^c}\|_1\}$. For $s > 0$, set:

$$\mathcal{K}_s := \bigcup_{|I|\leq s} K_I\,.$$

Null-Space Property (NSP)

$G \subset \mathbb{R}^p$ satisfies $\mathrm{NSP}(s,1)$ iff

$$G \cap \mathcal{K}_s = \{0\}\,.$$

[CDD06] and [D05] Theorems

[CS problem]
$\Leftrightarrow$ [Cross-Polytope problem]
$\Leftrightarrow$ [$\ker(X)$ satisfies $\mathrm{NSP}(s,1)$]

Some references

- Implicit expression: Donoho and Tanner (2009), Xu and Hassibi (2011).
- Based on convex integral geometry.
- Sub-exponential rows: Lecué and Mendelson (2014).
- Explicit expression: AdCM14.

Rice's formulas

Tools for extrema of random processes

Let X be a real-valued Gaussian process defined on $I \subset \mathbb{R}$ with C^1 trajectories.

- Let $N_u = N_u(X, I)$ be the number of crossings of level u then

$$\mathbb{E}(N_u) = \int_I \mathbb{E}\Big[|X'(t)| \Big| X(t) = u\Big] p_{X(t)}(u) dt \,.$$

Rice's formulas

Tools for extrema of random processes

Let X be a real-valued Gaussian process defined on $I \subset \mathbb{R}$ with C^1 trajectories.

- Let $N_u = N_u(X, I)$ be the number of crossings of level u then

$$\mathbb{E}(N_u) = \int_I \mathbb{E}\Big[|X'(t)|\Big|X(t) = u\Big]p_{X(t)}(u)dt\,.$$

- Estimation of the tail

$$\mathbb{P}[\sup_{t\in[0,T]} X_t > u] \leq T\frac{e^{-\frac{u^2}{2}}}{2\pi} + 1 - \Phi(u)\,.$$

Tools for extrema of random processes

Let X be a real-valued Gaussian process defined on $I \subset \mathbb{R}$ with C^1 trajectories.

- Let $N_u = N_u(X, I)$ be the number of crossings of level u then

$$\mathbb{E}(N_u) = \int_I \mathbb{E}\Big[|X'(t)|\Big|X(t) = u\Big] p_{X(t)}(u)dt\,.$$

- Estimation of the tail

$$\mathbb{P}[\sup_{t\in[0,T]} X_t > u] \leq T\frac{e^{-\frac{u^2}{2}}}{2\pi} + 1 - \Phi(u)\,.$$

- Generalization in higher dimension. Let $Z : U \longrightarrow \mathbb{R}^d$ be a Gaussian random field, $U \subset \mathbb{R}^d$ open subset, then under suitable conditions

$$\mathbb{E}(N_u(Z,B)) = \int_B \mathbb{E}\big[|\det(Z'(t))|\big|Z(t) = u\big] p_{Z(t)}(u)dt\,,$$

  where B is a Borel set contained in U.
- Generalization to non Gaussian processes.

Link with locally linear processes

- Consider a uniform random plane of co-dimension n

$$G = \mathrm{Span}\{g_i\,,\ i = 1,\ldots,p-n\}\,,$$

where $g_i \overset{i.i.d}{\sim} \mathcal{N}(0, \mathrm{Id}_p)$.

Link with locally linear processes

- Consider a uniform random plane of co-dimension n

$$G = \mathrm{Span}\{g_i\,,\ i = 1, \ldots, p-n\}\,,$$

where $g_i \overset{i.i.d}{\sim} \mathcal{N}(0, \mathrm{Id}_p)$.

- Set $m := p - n$. Consider direction $t \in \mathbb{S}^{m-1}$ and set

$$Z(t) := \sum_{i=1}^{m} t_i g_i = (Z_1(t), \ldots, Z_p(t))\,,$$

a random point of G.

Link with locally linear processes

- Consider a uniform random plane of co-dimension n

$$G = \mathrm{Span}\{g_i\,,\ i = 1, \ldots, p-n\}\,,$$

where $g_i \overset{i.i.d}{\sim} \mathcal{N}(0, \mathrm{Id}_p)$.
- Set $m := p - n$. Consider direction $t \in \mathbb{S}^{m-1}$ and set

$$Z(t) := \sum_{i=1}^{m} t_i g_i = (Z_1(t), \ldots, Z_p(t))\,,$$

a random point of G.
- Consider the stochastic process $\{X(t)\,,\ t \in \mathbb{S}^{m-1}\}$ defined by

$$X(t) = |Z_{(1)}(t)| + \cdots + |Z_{(s)}(t)| - \big[|Z_{(s+1)}(t)| + \cdots + |Z_{(p)}(t)|\big],$$

where $|Z_{(k)}(t)|$ denotes the order statistic.

Link with locally linear processes

- Consider a uniform random plane of co-dimension n

$$G = \text{Span}\{g_i\,,\ i = 1, \ldots, p-n\}\,,$$

where $g_i \overset{i.i.d}{\sim} \mathcal{N}(0, \text{Id}_p)$.

- Set $m := p - n$. Consider direction $t \in \mathbb{S}^{m-1}$ and set

$$Z(t) := \sum_{i=1}^{m} t_i g_i = (Z_1(t), \ldots, Z_p(t))\,,$$

a random point of G.

- Consider the stochastic process $\{X(t)\,,\ t \in \mathbb{S}^{m-1}\}$ defined by

$$X(t) = |Z_{(1)}(t)| + \cdots + |Z_{(s)}(t)| - \big[|Z_{(s+1)}(t)| + \cdots + |Z_{(p)}(t)|\big],$$

where $|Z_{(k)}(t)|$ denotes the order statistic.

$$\{G \text{ satisfies NSP}(s,1)\} = \{\forall t \in \mathbb{S}^{m-1},\ X(t) < 0\} = \{\max_{t \in \mathbb{S}^{m-1}} X(t) < 0\}\,.$$

The Rice method

Goal: Upper bound on $\Pi := \mathbb{P}(\max_{t \in S} X(t) > 0)$ (with $S \subset \mathbb{S}^{m-1}$ and $S = -S$).

The Rice method

Goal: Upper bound on $\Pi := \mathbb{P}(\max_{t \in S} X(t) > 0)$ (with $S \subset \mathbb{S}^{m-1}$ and $S = -S$).

Stephen O. Rice

- $\mathcal{M}_S$: number of local maxima of $X(t)$ along $\mathbb{S}^{m-1}$ satisfying $X(t) > 0$ and $t \in S$.

The Rice method

Goal: Upper bound on $\Pi := \mathbb{P}(\max_{t\in S} X(t) > 0)$ (with $S \subset \mathbb{S}^{m-1}$ and $S = -S$).

Stephen O. Rice

- $\mathcal{M}_S$: number of local maxima of $X(t)$ along $\mathbb{S}^{m-1}$ satisfying $X(t) > 0$ and $t \in S$.
- Kac type formula (Adler):

$$\mathcal{M}_S = \lim_{\delta\to 0} \frac{1}{V_\delta} \int_{\mathbb{S}^{m-1}} \mathbb{E}(|\det X''(t)|\ \mathbf{I}_{|X'(t)|<\delta}\ \mathbf{I}_{t\in S}\ \mathbf{I}_{X(t)>0})\sigma(\mathrm{d}t),$$

  where σ is the surfacic measure of $\mathbb{S}^{m-1}$ and V_δ the volume of the ball B_δ with radius δ.
- Symmetry: $\mathbb{P}\{\mathcal{M}_S > 0\} \leq \frac{1}{2}\mathbb{E}(\mathcal{M}_S)$.

The Rice method

Goal: Upper bound on $\Pi := \mathbb{P}(\max_{t\in S} X(t) > 0)$ (with $S \subset \mathbb{S}^{m-1}$ and $S = -S$).

Stephen O. Rice

- $\mathcal{M}_S$: number of local maxima of $X(t)$ along $\mathbb{S}^{m-1}$ satisfying $X(t) > 0$ and $t \in S$.
- Kac type formula (Adler):

$$\mathcal{M}_S = \lim_{\delta\to 0} \frac{1}{V_\delta} \int_{\mathbb{S}^{m-1}} \mathbb{E}(|\det X''(t)|\ \mathbf{I}_{|X'(t)|<\delta}\ \mathbf{I}_{t\in S}\ \mathbf{I}_{X(t)>0})\sigma(\mathrm{d}t),$$

where σ is the surfacic measure of $\mathbb{S}^{m-1}$ and V_δ the volume of the ball B_δ with radius δ.

- Symmetry: $\mathbb{P}\{\mathcal{M}_S > 0\} \le \frac{1}{2}\mathbb{E}(\mathcal{M}_S)$.

- Fatou lemma and invariance:

$$2\Pi \le (2\pi p)^{\frac{1-m}{2}} V(S) \int_0^\infty \mathbb{E}\big[|\det(X''(t))|\ \big| X(t) = x, X'(t) = 0\big] p_{X(t)}(x)\mathrm{d}x\,.$$

Sketch of the proof

- **Major issue:** Lack of differentiability at points with 0 entries (due to the ℓ_1-norm).

Sketch of the proof

- **Major issue:** Lack of differentiability at points with 0 entries (due to the ℓ_1-norm).
- **Solution:** Cut the sphere:

$$\mathbb{S}^{m-1} \overset{a.s.}{=} \bigcup_{|A|\leq m-1} \dot{\mathbb{S}}_A\,,$$

  where $\dot{\mathbb{S}}_A = \{t \in \mathbb{S}^{m-1}\,;\ \ Z_A(t) = 0 \text{ and } \forall j \notin A,\ Z_j(t) \neq 0\}$.
- Invoke the Rice method on each sphere.

Sketch of the proof

- **Major issue:** Lack of differentiability at points with 0 entries (due to the ℓ_1-norm).
- **Solution:** Cut the sphere:

$$\mathbb{S}^{m-1} \overset{a.s.}{=} \bigcup_{|A|\leq m-1} \dot{\mathbb{S}}_A\,,$$

  where $\dot{\mathbb{S}}_A = \{t \in \mathbb{S}^{m-1}\,;\; Z_A(t) = 0 \text{ and } \forall j \notin A,\; Z_j(t) \neq 0\}$.
- Invoke the Rice method on each sphere.
- Compute the Hessian and upper bound $\mathbb{P}(X(t) > 0)$ for a given t using the $\ell_1 \setminus \ell_2$**-distortion:**

Sketch of the proof

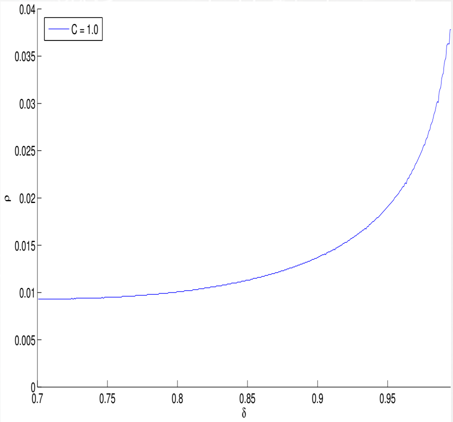
- **Major issue:** Lack of differentiability at points with 0 entries (due to the ℓ_1-norm).
- **Solution:** Cut the sphere:
$$\mathbb{S}^{m-1} \overset{a.s.}{=} \bigcup_{|A|\leq m-1} \dot{\mathbb{S}}_A\,,$$
where $\dot{\mathbb{S}}_A = \{t \in \mathbb{S}^{m-1}\,;\ \ Z_A(t) = 0 \text{ and } \forall j \notin A,\ Z_j(t) \neq 0\}$.
- Invoke the Rice method on each sphere.
- Compute the Hessian and upper bound $\mathbb{P}(X(t) > 0)$ for a given t using the $\ell_1 \setminus \ell_2$**-distortion:** except with a probability smaller than $\psi := 2(-\frac{s}{C_0 p}\log(e\frac{s}{C_0 p}))^{\frac{p}{2}}$ with $C_0 = \frac{\pi}{16e^2}$, a standard Gaussian vector $g \in \mathbb{R}^p$ enjoys:
$$\mathbb{R}g \cap \mathcal{K}_s = \{0\}\,. \qquad \text{(NSP}(s,1)\text{ for }\mathbb{R}g)$$

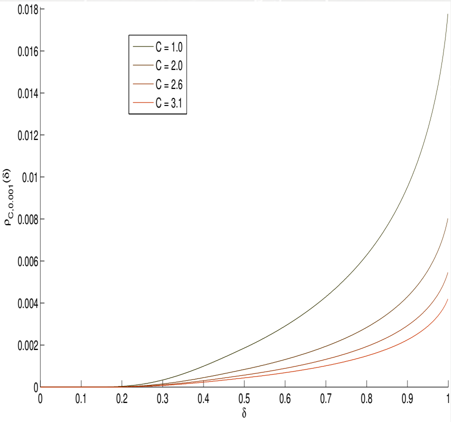
[AdCM14] NSP using Rice formula

It holds $\mathbb{P}[\ker(X(n,p)) \text{ enjoys NSP}(s,1)] = 1-\alpha$, where:

$$\alpha \leq \psi \times \Big[\sum_{k=0}^{p-n-2} \binom{p}{k} \sqrt{\pi}\Big(\frac{p-k}{s}\Big)^{\frac{n+k+1-p}{2}} \frac{\Gamma(\frac{2p-2k-n-1}{2})}{\Gamma(\frac{p-k}{2})\Gamma(\frac{p-n-k}{2})} + \binom{p}{n+1}\Big]$$

Lower bound on the phase transition





[AdCM14] Explicit lower bound on the phase transition

If $s \leq [1 - (1 - e^{-D}(\frac{1}{2})^{\frac{4p-n}{n}})^{\frac{1}{2}}]n$ then:

$$\mathbb{P}[\ker(X(n,p)) \text{ enjoys } \mathrm{NSP}(s,1)] \geq 1 - 9.7p^{\frac{3}{2}} \exp(-Dn/2)\,.$$

Sufficient condition for reconstruction

Restricted Isometry Property

A matrix $\mathbf{X} \in \mathbb{R}^{n \times p}$ satisfies RIP if

$$\forall \gamma \in \Sigma_s\,, \quad (1-\delta_0)\|\gamma\|_2^2 \leq \|X\gamma\|_2^2 \leq (1+\delta_0)\|\gamma\|_2^2$$

where $0 < \delta_0 < \sqrt{2}-1$ is a constant called the restricted isometry constant.

Sufficient condition for reconstruction

Restricted Isometry Property

A matrix $\mathbf{X} \in \mathbb{R}^{n \times p}$ satisfies RIP if

$$\forall \gamma \in \Sigma_s \,, \quad (1-\delta_0)\|\gamma\|_2^2 \leq \|X\gamma\|_2^2 \leq (1+\delta_0)\|\gamma\|_2^2$$

where $0 < \delta_0 < \sqrt{2} - 1$ is a constant called the restricted isometry constant.

Proof in the Gaussian case

- Exponential bound for $\mathbb{P}(RIP^c) =$

$$\mathbb{P}\left(\bigcup_{S \subseteq \{1,\ldots,p\}, dim(S) \leq s} \left\{ \min \frac{\|X\gamma_S\|_2^2}{\|\gamma_S\|_2^2} < n(1-\delta_0) \right\} \cup \left\{ \max \frac{\|X\gamma_S\|_2^2}{\|\gamma_S\|_2^2} > n(1+\delta_0) \right\} \right)$$

Restricted Isometry Property

Sufficient condition for reconstruction

Restricted Isometry Property

A matrix $\mathbf{X} \in \mathbb{R}^{n\times p}$ satisfies RIP if

$$\forall \gamma \in \Sigma_s\,, \quad (1-\delta_0)\|\gamma\|_2^2 \leq \|X\gamma\|_2^2 \leq (1+\delta_0)\|\gamma\|_2^2$$

where $0 < \delta_0 < \sqrt{2}-1$ is a constant called the restricted isometry constant.

Proof in the Gaussian case

- Exponential bound for $\mathbb{P}(RIP^c) =$

$$\mathbb{P}\left(\bigcup_{S\subseteq\{1,\dots,p\},dim(S)\leq s}\left\{\min\frac{\|X\gamma_S\|_2^2}{\|\gamma_S\|_2^2} < n(1-\delta_0)\right\}\cup\left\{\max\frac{\|X\gamma_S\|_2^2}{\|\gamma_S\|_2^2} > n(1+\delta_0)\right\}\right)$$

- Union bound $\Rightarrow \binom{p}{s}$ possibilities
- Compute $\mathbb{P}(\lambda_1(W_{n,s}) > u)$ and $\mathbb{P}(\lambda_s(W_{n,s}) < u)$ where $\lambda_1(W_{n,s}) < \cdots < \lambda_s(W_{n,s})$ denote the ordered eigenvalues of a $n\times s$ Wishart matrix.

Goal: Prove or improve Edelman's bound for $\mathbb{P}(\lambda_1 < u)$ and $\mathbb{P}(\lambda_s > u)$

- Invoke Rice formula for $\chi(t) = t' W_{n,s} t$, $t \in \mathbb{S}^{s-1}$

Restricted Isometry Property

Goal: Prove or improve Edelman's bound for $\mathbb{P}(\lambda_1 < u)$ and $\mathbb{P}(\lambda_s > u)$

- Invoke Rice formula for $\chi(t) = t' W_{n,s} t$, $t \in \mathbb{S}^{s-1}$
- Use invariance by isometry

Restricted Isometry Property

Goal: Prove or improve Edelman's bound for $\mathbb{P}(\lambda_1 < u)$ and $\mathbb{P}(\lambda_s > u)$

- Invoke Rice formula for $\chi(t) = t' W_{n,s} t$, $t \in \mathbb{S}^{s-1}$
- Use invariance by isometry
- Compute χ'' and χ' at the point $e_1 = (1, 0, \dots, 0)$ give

$$\mathbb{P}(\lambda_1 > u) = \frac{\sqrt{\pi}}{\Gamma(s/2)\Gamma(n/2)} \int_0^u (x/2)^{\frac{n-s-1}{2}} \exp(-x/2) \mathbb{E}(\det(W_{n-1,s-1} - x I_{s-1})^+) dx \,.$$

Restricted Isometry Property

Goal: Prove or improve Edelman's bound for $\mathbb{P}(\lambda_1 < u)$ and $\mathbb{P}(\lambda_s > u)$

- Invoke Rice formula for $\chi(t) = t' W_{n,s} t$, $t \in \mathbb{S}^{s-1}$
- Use invariance by isometry
- Compute χ'' and χ' at the point $e_1 = (1, 0, \dots, 0)$ give

$$\mathbb{P}(\lambda_1 > u) = \frac{\sqrt{\pi}}{\Gamma(s/2)\Gamma(n/2)} \int_0^u (x/2)^{\frac{n-s-1}{2}} \exp(-x/2) \mathbb{E}(\det(W_{n-1,s-1} - xI_{s-1})^+) dx \,.$$

- Projection argument imply

$$\mathbb{E}(\det(W_{n-1,s-1} - xI_{s-1})^+) \leq \frac{(n-1)!}{(n-s)!}$$

Goal: Prove or improve Edelman's bound for $\mathbb{P}(\lambda_1 < u)$ and $\mathbb{P}(\lambda_s > u)$

- Invoke Rice formula for $\chi(t) = t' W_{n,s} t$, $t \in \mathbb{S}^{s-1}$
- Use invariance by isometry
- Compute χ'' and χ' at the point $e_1 = (1, 0, \dots, 0)$ give

$$\mathbb{P}(\lambda_1 > u) = \frac{\sqrt{\pi}}{\Gamma(s/2)\Gamma(n/2)} \int_0^u (x/2)^{\frac{n-s-1}{2}} \exp(-x/2) \mathbb{E}(\det(W_{n-1,s-1} - xI_{s-1})^+) dx \,.$$

- Projection argument imply

$$\mathbb{E}(\det(W_{n-1,s-1} - xI_{s-1})^+) \leq \frac{(n-1)!}{(n-s)!}$$

- Conclusion: Same result as Edelman

References

- Azaïs, J.-M., de Castro, Y., Mourareau, S. "*A Rice method proof of the Null-Space Property over the Grassmannian*", arXiv:1405.6417 (2014).
- Azaïs, J.-M., Wschebor, M. (2009). "*Level sets and extrema of random processes and fields*", John Wiley & Sons.
- Candes, E.J., Romberg, J., Tao, T., "*Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information*" Information Theory, IEEE Transactions on , vol.52, no.2, pp.489,509, Feb. 2006
- Cohen, A., Dahmen, W., DeVore, R. (2009). "*Compressed sensing and best k-term approximation*", Journal of the American Mathematical Society, 22(1), 211-231.
- Donoho, D. L., Tanner, J. (2009). "*Counting faces of randomly projected polytopes when the projection radically lowers dimension*", Journal of the American Mathematical Society, 22(1), 1-53.
- Edelman, A. (1988). "*Eigenvalues and condition numbers of random matrices*", Journal on Matrix Analysis and Applications, 9, pp. 543-560.