A remark on Convex Relaxation and Signed Measure recovery

Yohann de Castro

École d'été de Saint-Flour, 3 - 16 Juillet 2011

Outline

Goal: Faithful recovery of a signed measure from "generalized" moments

Outline

Goal: Faithful recovery of a signed measure from "generalized" moments

1. Observation of generalized moments of a signed measure σ
   - real polynomials,
   - Müntz polynomials,
   - trigonometric functions,
   - Characteristic function,
   - Stieljes transformation,
   - Laplace transform, ...

Outline

Goal: Faithful recovery of a signed measure from "generalized" moments

1. Observation of generalized moments of a signed measure σ
   - real polynomials,
   - Müntz polynomials,
   - trigonometric functions,
   - Characteristic function,
   - Stieljes transformation,
   - Laplace transform, ...
2. *TV*-minimization finds the simplest (i.e. discrete) solution
   - Convex relaxation in Banach spaces
   - Dual Certification

Outline

Goal: Faithful recovery of a signed measure from "generalized" moments

1. Observation of generalized moments of a signed measure σ
    - real polynomials,
    - Müntz polynomials,
    - trigonometric functions,
    - Characteristic function,
    - Stieljes transformation,
    - Laplace transform, ...
2. *TV*-minimization finds the simplest (i.e. discrete) solution
    - Convex relaxation in Banach spaces
    - Dual Certification
3. Results
    - Nonnegative measure recovery
    - Combinatorial geometry
    - Signed measure recovery

Observation of $n+1$ generalized moments of a signed measure σ

The target σ has **finite support**.

Observation of $n+1$ generalized moments of a signed measure σ

The target σ has **finite support**.

- I - fixed bounded set,
- $\mathcal{M}(I)$ - finite signed measures,
- $\sigma = \sum_{i=1}^{s} \sigma_i \, \delta_{x_i}$,
- $\mathcal{S} = \{x_1, \ldots, x_s\}$ - support of σ.

Observation of $n+1$ generalized moments of a signed measure σ

The target σ has **finite support**.

- I - fixed bounded set,
- $\mathcal{M}(I)$ - finite signed measures,
- $\sigma = \sum_{i=1}^{s} \sigma_i \, \delta_{x_i}$,
- $\mathcal{S} = \{x_1, \ldots, x_s\}$ - support of σ.

The support and the weights are **unknowns**!

Observation of $n+1$ generalized moments of a signed measure σ

The target σ has **finite support**.

- I - fixed bounded set,
- $\mathcal{M}(I)$ - finite signed measures,
- $\sigma = \sum_{i=1}^{s} \sigma_i \, \delta_{x_i}$,
- $\mathcal{S} = \{x_1, \ldots, x_s\}$ - support of σ.

The support and the weights are **unknowns**!

We know $n+1$ generalized moments.

- $\mathcal{F} := \{u_0, u_1, \ldots, u_n\}$ is a given family of continuous functions on $\bar{I}$,
- $c_k(\mu) := \int_I u_k \, \mathrm{d}\mu$,
- $\mathcal{K}_n(\sigma) := (c_0(\sigma), \ldots, c_n(\sigma))$ is the observation.

Observation of $n+1$ generalized moments of a signed measure σ

The target σ has **finite support**.

- I - fixed bounded set,
- $\mathcal{M}(I)$ - finite signed measures,
- $\sigma = \sum_{i=1}^{s} \sigma_i \, \delta_{x_i}$,
- $\mathcal{S} = \{x_1, \ldots, x_s\}$ - support of σ.

The support and the weights are **unknowns**!

We know $n+1$ generalized moments.

- $\mathcal{F} := \{u_0, u_1, \ldots, u_n\}$ is a given family of continuous functions on $\bar{I}$,
- $c_k(\mu) := \int_I u_k \, \mathrm{d}\mu$,
- $\mathcal{K}_n(\sigma) := (c_0(\sigma), \ldots, c_n(\sigma))$ is the observation.

Can we recover σ from $\mathcal{K}_n(\sigma)$?

Convex optimization leads to "simple" solutions

Basis Pursuit [*Donoho et al.* 1998]

$$x^\star \in \arg\min_{y\in\mathbb{R}^p} \|y\|_1 \quad \text{s.t. } Ay = Ax_0\,,$$

where $A \in \mathbb{R}^{n\times p}$ is the design matrix and $x_0 \in \mathbb{R}^p$ is the target vector.

St-Flour lecture of E.J. Candès:

- Dual certificate,
- Combinatorial geometry (Donoho-Tanner).

Convex optimization leads to "simple" solutions

Basis Pursuit [Donoho et al. 1998]

$$x^\star \in \arg\min_{y \in \mathbb{R}^p} \|y\|_1 \quad \text{s.t. } Ay = Ax_0 \,,$$

where $A \in \mathbb{R}^{n \times p}$ is the design matrix and $x_0 \in \mathbb{R}^p$ is the target vector.

St-Flour lecture of E.J. Candès:

- Dual certificate,
- Combinatorial geometry (Donoho-Tanner).

TV-minimization

$$\sigma^\star \in \arg\min_{\mu \in \mathcal{M}(I)} \|\mu\|_{TV} \text{ s.t. } \mathcal{K}_n(\mu) = \mathcal{K}_n(\sigma) \,,$$

where $\| \, . \, \|_{TV}$ is the *total variation* norm.

- Dual Certificate,
- Reconstruction of nonnegative vectors

Dual Certification as an interpolation problem

The generalized dual polynomials

If there exists a linear combination $P = \sum_{k=0}^{n} a_k u_k$, a sign s-tuple $(\varepsilon_1, \ldots, \varepsilon_s) \in \{\pm 1\}^s$, and points $\{x_1, \ldots, x_s\} \subset I$ such that

(*i*) the generalized Vandermonde system

$$\begin{pmatrix} u_0(x_1) & u_0(x_2) & \ldots & u_0(x_s) \\ u_1(x_1) & u_1(x_2) & \ldots & u_1(x_s) \\ \vdots & \vdots & & \vdots \\ u_n(x_1) & u_n(x_2) & \ldots & u_n(x_s) \end{pmatrix}$$

has full column rank.

Dual Certification as an interpolation problem

The generalized dual polynomials

If there exists a linear combination $P = \sum_{k=0}^{n} a_k u_k$, a sign s-tuple $(\varepsilon_1, \dots, \varepsilon_s) \in \{\pm 1\}^s$, and points $\{x_1, \dots, x_s\} \subset I$ such that

(*i*) the generalized Vandermonde system

$$\begin{pmatrix} u_0(x_1) & u_0(x_2) & \dots & u_0(x_s) \\ u_1(x_1) & u_1(x_2) & \dots & u_1(x_s) \\ \vdots & \vdots & & \vdots \\ u_n(x_1) & u_n(x_2) & \dots & u_n(x_s) \end{pmatrix}$$

has full column rank.

(*ii*) $P(x_i) = \varepsilon_i, \ \forall\, i = 1, \dots, s,$

(*iii*) $|P(x)| < 1, \ \forall x \in I \setminus \{x_1, \dots, x_s\},$

Dual Certification as an interpolation problem

The generalized dual polynomials

If there exists a linear combination $P = \sum_{k=0}^{n} a_k u_k$, a sign s-tuple $(\varepsilon_1, \dots, \varepsilon_s) \in \{\pm 1\}^s$, and points $\{x_1, \dots, x_s\} \subset I$ such that

(*i*) the generalized Vandermonde system

$$\begin{pmatrix} u_0(x_1) & u_0(x_2) & \dots & u_0(x_s) \\ u_1(x_1) & u_1(x_2) & \dots & u_1(x_s) \\ \vdots & \vdots & & \vdots \\ u_n(x_1) & u_n(x_2) & \dots & u_n(x_s) \end{pmatrix}$$

has full column rank.

(*ii*) $P(x_i) = \varepsilon_i, \ \forall\, i = 1, \dots, s,$

(*iii*) $|P(x)| < 1, \ \forall x \in I \setminus \{x_1, \dots, x_s\},$

Then every measure $\sigma = \sum_{i=1}^{s} \sigma_i \delta_{x_i}$, such that $\mathrm{sgn}(\sigma_i) = \varepsilon_i$, is the **unique** solution of support pursuit given the observation $\mathcal{K}_n(\sigma)$.

Homogeneous *M*-systems

Chebyshev systems of order k

The family $\mathcal{F} = \{u_0, u_1, \ldots, u_k\}$ is a T-system of degree k **if and only if** every generalized polynomial

$$P = \sum_{i=0}^{k} a_i u_i \,,$$

where $(a_0, \ldots, a_k) \neq (0, \ldots, 0)$, has at most k zeros in I.

- There exists a **unique** generalized polynomial P such that

$$P(x_i) = y_i,$$

given $x_i \in I$, $y_i \in \mathbb{R}$, for $i = 1, \ldots, k$.

- The Vandermonde system,

$$\begin{pmatrix} u_0(x_0) & \ldots & u_0(x_k) \\ u_1(x_0) & \ldots & u_1(x_k) \\ \vdots & & \vdots \\ u_k(x_0) & \ldots & u_k(x_k) \end{pmatrix}$$

has full rank for all $x_0, \ldots, x_k$ distinct points of I.

Moments derived from homogeneous M-systems

$\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ is a homogeneous M-systems **if and only if**

- $\forall k = 0, \ldots, n$, $\mathcal{F}$ is a T-system of order k (M-systems),
- $u_0 = 1$ (homogeneous).

Moments derived from homogeneous M-systems

$\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ is a homogeneous M-systems **if and only if**

- $\forall k = 0, \ldots, n$, $\mathcal{F}$ is a T-system of order k (M-systems),
- $u_0 = 1$ (homogeneous).

Standard polynomials

$\mathcal{F} = \{1, x, x^2, \ldots\}$

Moments derived from homogeneous M-systems

$\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is a homogeneous M-systems **if and only if**

- $\forall k = 0, \dots, n$, $\mathcal{F}$ is a T-system of order k (M-systems),
- $u_0 = 1$ (homogeneous).

Standard polynomials

$\mathcal{F} = \{1, x, x^2, \dots\}$

Müntz polynomials

$\mathcal{F} = \{1, x^{\alpha_1}, x^{\alpha_2}, \dots\}$ where
$0 < \alpha_1 < \alpha_2 < \cdots$ and $I = [0, +\infty)$

Moments derived from homogeneous M-systems

$\mathcal{F} = \{u_0, u_1, \dots, u_n\}$ is a homogeneous M-systems **if and only if**

- $\forall k = 0, \dots, n$, $\mathcal{F}$ is a T-system of order k (M-systems),
- $u_0 = 1$ (homogeneous).

Standard polynomials

$\mathcal{F} = \{1, x, x^2, \dots\}$

Müntz polynomials

$\mathcal{F} = \{1, x^{\alpha_1}, x^{\alpha_2}, \dots\}$ where
$0 < \alpha_1 < \alpha_2 < \cdots$ and $I = [0, +\infty)$

Trigonometric functions

$\mathcal{F} = \{1, \cos(\pi x), \cos(2\pi x), \dots\}$ on
$I = [0, 1]$

Moments derived from homogeneous M-systems

$\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ is a homogeneous M-systems **if and only if**

- $\forall k = 0, \ldots, n$, $\mathcal{F}$ is a T-system of order k (M-systems),
- $u_0 = 1$ (homogeneous).

Standard polynomials

$\mathcal{F} = \{1, x, x^2, \ldots\}$

Laplace transform

$\mathcal{F} = \{1, \exp(-x), \exp(-2x), \ldots\}$

Müntz polynomials

$\mathcal{F} = \{1, x^{\alpha_1}, x^{\alpha_2}, \ldots\}$ where $0 < \alpha_1 < \alpha_2 < \cdots$ and $I = [0, +\infty)$

Trigonometric functions

$\mathcal{F} = \{1, \cos(\pi x), \cos(2\pi x), \ldots\}$ on $I = [0, 1]$

Moments derived from homogeneous M-systems

$\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ is a homogeneous M-systems **if and only if**

- $\forall k = 0, \ldots, n$, $\mathcal{F}$ is a T-system of order k (M-systems),
- $u_0 = 1$ (homogeneous).

Standard polynomials

$\mathcal{F} = \{1, x, x^2, \ldots\}$

Laplace transform

$\mathcal{F} = \{1, \exp(-x), \exp(-2x), \ldots\}$

Müntz polynomials

$\mathcal{F} = \{1, x^{\alpha_1}, x^{\alpha_2}, \ldots\}$ where $0 < \alpha_1 < \alpha_2 < \cdots$ and $I = [0, +\infty)$

Stieltjes transformation

$\mathcal{F} = \{1, \frac{1}{z_1 - x}, \frac{1}{z_2 - x}, \ldots\}$, where none of the z_k's belongs to I

Trigonometric functions

$\mathcal{F} = \{1, \cos(\pi x), \cos(2\pi x), \ldots\}$ on $I = [0, 1]$

Moments derived from homogeneous M-systems

$\mathcal{F} = \{u_0, u_1, \ldots, u_n\}$ is a homogeneous M-systems **if and only if**

- $\forall k = 0, \ldots, n$, $\mathcal{F}$ is a T-system of order k (M-systems),
- $u_0 = 1$ (homogeneous).

Standard polynomials

$\mathcal{F} = \{1, x, x^2, \ldots\}$

Laplace transform

$\mathcal{F} = \{1, \exp(-x), \exp(-2x), \ldots\}$

Müntz polynomials

$\mathcal{F} = \{1, x^{\alpha_1}, x^{\alpha_2}, \ldots\}$ where $0 < \alpha_1 < \alpha_2 < \cdots$ and $I = [0, +\infty)$

Stieltjes transformation

$\mathcal{F} = \{1, \frac{1}{z_1 - x}, \frac{1}{z_2 - x}, \ldots\}$, where none of the z_k's belongs to I

Trigonometric functions

$\mathcal{F} = \{1, \cos(\pi x), \cos(2\pi x), \ldots\}$ on $I = [0, 1]$

Characteristic function

$\mathcal{F} = \{1, \exp(\imath\pi x), \exp(\imath 2\pi x), \ldots\}$ on $I = [-1, 1)$

Exact reconstruction of the nonnegative measures

Theorem [d.C. and Gamboa, 2011]

Let $\mathcal{F}$ be an homogeneous M-system on I. Consider a nonnegative measure σ with finite support included in I. **Then** the measure σ is the **unique** solution to support pursuit given the observation $\mathcal{K}_n(\sigma)$ where n is not less than twice the size of the support of σ.

- Only $n \geq 2s + 1$,
- "homogeneous" is only sufficient **but** for all target σ with finite support, it is possible to find "non-homogeneous" M-systems that go against this theorem,
- Unique solution among **all** signed measures such that $\mathcal{K}_n(\sigma) = \mathcal{K}_n(\mu)$.

Deterministic matrices for compressed sensing

Theorem

Let n, p, s be integers such that

$$s \leq \min(n/2, p).$$

Let $\{1, u_1, \dots, u_n\}$ be a homogeneous M-system on $\{t_1, \dots, t_p\}$. Let A be the generalized vandermonde system defined by

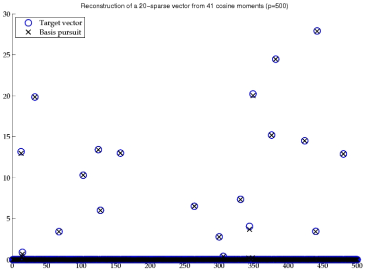
$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ u_1(t_1) & u_1(t_2) & \dots & u_1(t_p) \\ \vdots & \vdots & & \vdots \\ u_n(t_1) & u_n(t_2) & \dots & u_n(t_p) \end{pmatrix}.$$

Then basis pursuit exactly recovers **all nonnegative** s-sparse vectors $x_0 \in \mathbb{R}^p$ from the observation Ax_0.

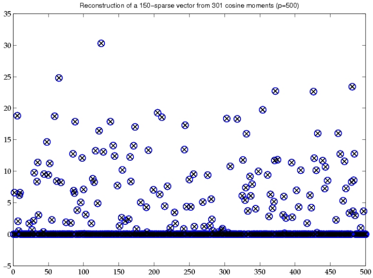
Go to Basis Pursuit

- Donoho and Tanner, 2005
- dC. and Gamboa, 2011

Numerical experiments



- $p = 500$,
- $s = 20$,
- $n = 41$.



- $p = 500$,
- $s = 150$,
- $n = 301$.

Convex polytopes neighborliness

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \end{bmatrix} \qquad A_i \in \mathbb{R}^m, \quad i = 1, \dots, n$$

k-neighborly polytopes

$A \in \mathbb{R}^{m \times n}$ is k-neighborly **iff**

- every A_i is a vertex of $\mathrm{Conv}(A)$,
- every pairs (A_i, A_j) is an edge of $\mathrm{Conv}(A)$,
- $\vdots$
- every k-tuple spans a $(k-1)$-dimensional face of $\mathrm{Conv}(A)$.

Convex polytopes neighborliness

$$A = \begin{bmatrix} A_1 & A_2 & \dots & A_n \end{bmatrix} \qquad A_i \in \mathbb{R}^m, \quad i = 1, \dots, n$$

k-neighborly polytopes

$A \in \mathbb{R}^{m \times n}$ is k-neighborly **iff**

- every A_i is a vertex of $\mathrm{Conv}(A)$,
- every pairs (A_i, A_j) is an edge of $\mathrm{Conv}(A)$,
- $\vdots$
- every k-tuple spans a $(k-1)$-dimensional face of $\mathrm{Conv}(A)$.

Donoho and Tanner, '05

Let $A \in \mathbb{R}^{m \times n}$ such that $m < n$. These two properties of A are equivalent:

1. A is k-neighborly,
2. whenever $y = Ax$ has a nonnegative solution x_0 having at most k nonzeros, x_0 is the unique solution to Basis Pursuit with nonnegative constraints.

Chebyshev Measures

$$T_k(x) = \cos(k \arccos(x)), \quad \forall x \in [-1, 1].$$

- there exists $1 = \zeta_0 > \zeta_1 > \cdots > \zeta_k = -1$ such that

$$T_k(\zeta_i) = (-1)^i \, \| T_k \|_\infty = (-1)^i ,$$

Theorem

We have

$$\min_{p \in \mathcal{P}_{k-1}^{\mathbb{C}}} \left\| x^k - p(x) \right\|_\infty = \left\| 2^{1-k} T_k \right\|_\infty = 2^{1-k} ,$$

where $\mathcal{P}_{k-1}^{\mathbb{C}}$ denotes the set of the complex polynomials of degree less than $k-1$, and the supremum norm is taken over $[-1, 1]$. Moreover, the minimum is uniquely attained by $p(x) = x^k - 2^{1-k} T_k(x)$.

Chebyshev Measures

- $\mathfrak{T}_k \in \mathrm{Span}\{u_0, u_1, \ldots, u_k\}$,
- there exists $x_0 < x_1 < \cdots < x_k$ such that

$$\mathrm{sgn}(\mathfrak{T}_k(x_{i+1})) = -\mathrm{sgn}(\mathfrak{T}_k(x_i)) = \pm \|\mathfrak{T}_k\|_\infty \,, \tag{1}$$

- $$\|\mathfrak{T}_k\|_\infty = 1 \quad \text{with} \quad \mathfrak{T}_k(\max I) > 0 \,. \tag{2}$$

Theorem (Borwein and Edérlyi, '85)

The k-th generalized Chebyshev polynomial $\mathfrak{T}_k$ exists and can be written as

$$\mathfrak{T}_k = c\left(u_k - \sum_{i=0}^{k-1} a_i u_i\right),$$

where $a_0, a_1, \ldots, a_{k-1} \in \mathbb{R}$ are chosen to minimize

$$\left\| u_k - \sum_{i=0}^{k-1} a_i u_i \right\|_\infty ,$$

and the normalization constant $c \in \mathbb{R}$ can be chosen so that $\mathfrak{T}_k$ satisfies property (2).

Stieljes transformation

$$\widetilde{\mathcal{F}}_s^n := \left\{1, \frac{1}{z_1 - x}, \frac{1}{z_2 - x}, \ldots, \frac{1}{z_n - x}\right\},$$

Theorem (Borwein and Edérlyi, '85)

$$\mathfrak{T}_k(x) = \frac{1}{2}(f_k(z) + f_k(z)^{-1}), \quad \forall x \in [-1, 1],$$

where z is uniquely defined by

1. $x = \frac{1}{2}(z + z^{-1})$,
2. $|z| < 1$.

and

1. *f_k is a known analytic function in a neighborhood of the closed unit disk*
2. *f_k can be expressed in terms of only $(z_i)_{i=1}^k$.*

Open problems

1. Given a sign sequence, What is the smallest degree of the dual certificates?

Open problems

1. Given a sign sequence, What is the smallest degree of the dual certificates?
2. Who is going to win the Ping-Pong tournament?

Thank You!

www.math.univ-toulouse.fr∼decastro

A bit more...