Planche 25 (ENSAM 2011 - élève 11)

1) a) $r\circ r = (p+q-q\circ p)\circ(p+q-q\circ p) = \underbrace{p\circ p}_{p} + \underbrace{p\circ q}_{0} - \underbrace{p\circ q}_{0}\circ p + \cancel{q\circ p} + \underbrace{q\circ q}_{q}$
$- \underbrace{q\circ q\circ p}_{\cancel{q\circ p}} - \underbrace{q\circ p\circ p}_{q\circ p} - q\circ \underbrace{p\circ q}_{0} + q\circ \underbrace{p\circ q}_{0}\circ p$

Donc $r\circ r = p+q-q\circ p = r \Longrightarrow$ $r\circ r = r$ donc r est un projecteur

b) En suppose $x \in \mathrm{Ker}\, p \cap \mathrm{Ker}\, q \Rightarrow p(x) = q(x) = 0$

$r(x) = p(x) + q(x) - q\circ p(x) = -q(0) \underset{q \text{ linéaire}}{=} 0 \Longrightarrow r(x) = 0 \Rightarrow x \in \mathrm{Ker}\, r$

Donc $\mathrm{Ker}\, p \cap \mathrm{Ker}\, q \subset \mathrm{Ker}\, r$

• $x \in \mathrm{Ker}\, r \Rightarrow r(x) = 0 \Rightarrow p(x) + q(x) - q\circ p(x) = 0$

$\Rightarrow \underbrace{p\circ p}_{p}(x) + \underbrace{p\circ q}_{0}(x) - \underbrace{p\circ q}_{0}\circ p(x) = p(0) \underset{p \text{ linéaire}}{=} 0$

$\Rightarrow p(x) = 0 \Rightarrow x \in \mathrm{Ker}\, p$

et $q\circ p(x) + q\circ q(x) - q\circ q\circ p(x) = q(0) \underset{q \text{ linéaire}}{=} 0$

$\Rightarrow q\circ \cancel{p(x)} + q(x) - q\circ \cancel{p(x)} = 0 \Rightarrow q(x) = 0 \Rightarrow x \in \mathrm{Ker}\, q$

Donc $x \in \mathrm{Ker}\, p \cap \mathrm{Ker}\, q$

$\Rightarrow \mathrm{Ker}\, r \subset \mathrm{Ker}\, p \cap \mathrm{Ker}\, q$

$$\boxed{\mathrm{Ker}\, r = \mathrm{Ker}\, p \cap \mathrm{Ker}\, q}$$

c) • En suppose $x \in \mathrm{Im}\, r \Rightarrow \exists y \,/\, r(y) = x$

$\Rightarrow p(y) + q(y) - q\circ p(y) = x \Rightarrow x = \underbrace{p(y)}_{x_1} + \underbrace{q(y - p(y))}_{x_2}$

$\Rightarrow x = x_1 + x_2$ avec $x_1 \in \mathrm{Im}\, p$ et $x_2 \in \mathrm{Im}\, q$

Donc $\mathrm{Im}\, r \subset \mathrm{Im}\, p + \mathrm{Im}\, q$

• $x \in \mathrm{Im}\, p + \mathrm{Im}\, q \Rightarrow \exists x_1 \in \mathrm{Im}\, p$ et $x_2 \in \mathrm{Im}\, q \,/\, x = x_1 + x_2$
et $\exists y_1 \,/\, p(y_1) = x_1$ et $y_2 \,/\, q(y_2) = x_2$

$r(x) = r(x_1) + r(x_2) = p(x_1) + q(x_1) - q\circ p(x_1) + p(x_2) + q(x_2) - q\circ p(x_2)$
$= \underbrace{p\circ p(y_1)}_{p(y_1)} + q\circ \cancel{p(y_1)} - q\circ \underbrace{p\circ p(y_1)}_{q\circ \cancel{p(y_1)}} + \underbrace{p\circ q(y_2)}_{0} + \underbrace{q\circ q(y_2)}_{q(y_2)} - q\circ \underbrace{p\circ q(y_2)}_{0}$

$= p(y_1) + q(y_2) = x_1 + x_2 = x$

Donc $r(x) = x$ Or $r(x) \in \operatorname{Im} r \Rightarrow \underline{x \in \operatorname{Im} r}$

Donc $\underline{\operatorname{Im} p + \operatorname{Im} q \subset \operatorname{Im} r}$

$$\boxed{\operatorname{Im} r = \operatorname{Im} p + \operatorname{Im} q}$$