Exercice n°7 : ENSAM - OT 2010 , Olivier LESIMPLE

2) Soit $I(x,y) = \displaystyle\int_x^y \frac{t^2-1}{(t^2+1)\sqrt{(t^4+1)}}\,dt$

<u>Montrons que $\forall (x,y) \in \mathbb{R}^2$, $I(-x,-y) = -I(x,y)$</u>

On pose $t = -T$ d'où $dt = -dT$

donc $$I(-x,-y) = \int_{-x}^{-y} \frac{t^2-1}{(t^2+1)\sqrt{t^4+1}}\,dt$$

$$= -\int_x^y \frac{T^2-1}{(T^2+1)\sqrt{T^4+1}}\,dT$$

$\boxed{I(-x,-y) = -I(x,y)}$, Il semblerait y avoir une erreur d'énoncé.

<u>Montrons que</u> $I(x,y) = \displaystyle\int_{x+\frac{1}{x}}^{y+\frac{1}{y}} \frac{du}{u\sqrt{u^2-2}}$

On pose le changement de variable $u = t + \dfrac{1}{t}$

d'où $du = \left(1 - \dfrac{1}{t^2}\right)dt = \dfrac{t^2-1}{t^2}\,dt$

et $u^2 = t^2 + 2 + \dfrac{1}{t^2}$

On a $$I(x,y) = \int_x^y \frac{t^2-1}{(t^2+1)\sqrt{t^4+1}}\,dt = \int_x^y \left(\frac{t^2-1}{t^2}\,dt\right)\left(\frac{1}{\left(1+\frac{1}{t^2}\right)\sqrt{t^4+1}}\right)$$

$$= \int_x^y \left(\frac{t^2-1}{t^2}\,dt\right)\left(\frac{1}{\left(t+\frac{1}{t}\right)\sqrt{t^2+\frac{1}{t^2}}}\right)$$

$$I(x,y) = \int_{x+1/x}^{y+\frac{1}{y}} \frac{du}{u\sqrt{u^2-2}}$$

• on a désormais $\int_a^b \frac{du}{u\sqrt{u^2-2}}$ on pose $v = \sqrt{u^2-2}$ d'où $u = \sqrt{v^2+2}$

$$du = \frac{v\,dv}{\sqrt{v^2+2}}$$

donc $\displaystyle\int_a^b \frac{du}{u\sqrt{u^2-2}} = \int_{\sqrt{a^2-2}}^{\sqrt{b^2-2}} \frac{dv}{v^2+2}$

$$= \frac{1}{2}\int_{\sqrt{a^2-2}}^{\sqrt{b^2-2}} \frac{dv}{1+\left(\frac{v}{\sqrt{2}}\right)^2}$$

$$= \frac{\sqrt{2}}{2}\left[\operatorname{Arctan}\left(\frac{v}{\sqrt{2}}\right)\right]_{\sqrt{a^2-2}}^{\sqrt{b^2-2}}$$

donc $\displaystyle\int_a^b \frac{du}{u\sqrt{u^2-2}} = \frac{\sqrt{2}}{2}\operatorname{Arctan}\left(\sqrt{\frac{b^2-2}{2}}\right) - \frac{\sqrt{2}}{2}\operatorname{Arctan}\left(\sqrt{\frac{a^2-2}{2}}\right)$

<u>Calculons $I(x,y)$</u>: On a $\displaystyle I(x,y) = \int_x^y \frac{t^2-1}{(t^2+1)\sqrt{t^4+1}}\,dt$

$$= \int_{x+\frac{1}{x}}^{y+\frac{1}{y}} \frac{du}{u\sqrt{u^2-2}}$$

donc $\displaystyle I(x,y) = \frac{\sqrt{2}}{2}\operatorname{Arctan}\left(\sqrt{\frac{x^2+\frac{1}{x^2}}{2}}\right) - \frac{\sqrt{2}}{2}\operatorname{Arctan}\left(\sqrt{\frac{y^2+\frac{1}{y^2}}{2}}\right)$