DETECTING SURFACE BUNDLES IN FINITE COVERS OF HYPERBOLIC CLOSED 3-MANIFOLDS.

CLAIRE RENARD.

ABSTRACT. The main theorem of this article provides sufficient conditions for a degree d finite cover M' of a hyperbolic 3-manifold M to be a surface-bundle. Let F be an embedded, closed and orientable surface of genus g, close to a minimal surface in the cover M', splitting M' into a disjoint union of q handlebodies and compression bodies. We show that there exists a fiber in the complement of F provided that d, q and g satisfy some inequality involving an explicit constant k depending only on the volume and the injectivity radius of M. In particular, this theorem applies to a Heegaard splitting of a finite covering M', giving an explicit lower bound for the genus of a strongly irreducible Heegaard splitting of M'. Applying the main theorem to the setting of a circular decomposition associated to a non trivial homology class of M gives sufficient conditions for this homology class to correspond to a fibration over the circle. Similar methods lead also to a sufficient condition for an incompressible embedded surface in M to be a fiber.

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INTRODUCTION

Thurston conjectured that every complete hyperbolic, connected and orientable 3-manifold of finite volume virtually fibers over the circle, i.e. such a manifold has a finite covering that is a surface bundle over the circle.

This conjecture received a great deal of attention during the past few years, culminating with the announcement of its proof by Ian Agol very recently (thanks to works of Daniel Wise, Jeremy Kahn and Vladimir Markovic, Frédéric Haglund, Nicolas Bergeron, and many other people). The proof is based on Daniel Wise's program.

The aim of this article is to provide explicit criteria for a given finite cover of a closed hyperbolic 3-manifold to be a surface bundle. More explicitly, given a cover $M' \to M$ of M with finite degree d, a natural question is to wonder whether M' contains an embedded surface that is a fiber, and to give an upper bound for its genus. The idea is to start with surfaces that already exist in M', like Heegaard surfaces.

The method is inspired by Lackenby's program to find surface bundles in towers of finite coverings of a given closed hyperbolic 3-manifold.

Let us be more precise. For a surface F, denote by $\chi_-(F) = \max\{0, -\chi(F)\}$ the negative part of the Euler characteristic of F. If C is a handlebody or a compression body, set $\chi_-(C) := \chi_-(\partial_+ C)$. If S is a union of connected components of $\partial_- C$, the definition implies that $\chi_-(S) \leq \chi_-(C)$.

Definition 0.1. *An embedded surface S in a Riemannian 3-manifold M is called* ***pseudo-minimal*** *if it is orientable, closed, and S is a minimal surface or the* *boundary of a regular neighborhood of a minimal non-orientable surface, possibly with a little tube attached vertically in the I-bundle structure.*

The main result of this article is the following theorem.

Theorem A. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. Denote by* $\mathrm{Inj}(M)$ *the injectivity radius of M and set $\epsilon = \frac{1}{2}\mathrm{Inj}(M)$.*

There exists an explicit constant $k = k(\epsilon, \mathrm{Vol}(M)) > 0$, depending only on ϵ and the volume $\mathrm{Vol}(M)$ *satisfying the following properties.*

Let $M' \to M$ be a cover of finite degree d which contains a closed, orientable, embedded and pseudo-minimal surface F, splitting M' into a disjoint union of q handlebodies and compression bodies $C_1, \ldots, C_q$. Suppose that:

1. *the union F^- of the components of F corresponding to the negative boundary components of C_j is a union of incompressible surfaces, and*
2. *the inequality $\boldsymbol{k\, c \ln c < \ln\ln \frac{d}{q}}$ holds, where $c = \max_{j=1,\ldots,q}\{\chi_-(C_j)\}$.*

Then one of the components of F^- is the fiber of a surface-bundle structure for M' (corresponding to a bundle over the circle or a twisted I-bundle).

The proof of this theorem leads to the following corollary.

Corollary 0.2. *Under the assumptions of theorem A, the volume of a handlebody C_j (i.e. such that $\partial_- C_j = \emptyset$), among the q compression bodies, must satisfy* $\mathrm{Vol}(C_j) < \mathrm{Vol}(M)d/q$.

The topological assumption (1) of theorem A may not be necessary. We conjecture that:

Conjecture (∗). *Theorem A is still true even if assumption* (1) *is not a priori satisfied.*

If N is a connected, compact and orientable 3-manifold, the Heegaard Euler characteristic $\chi_-^h(N)$ of N is the minimum over all Heegaard surfaces F of the negative part $\chi_-(F) = \min\{-\chi(F), 0\}$ of the Euler characteristic of F. Likewise, the strong Heegaard Euler characteristic $\chi_-^{sh}(N)$ is the minimum of $\chi_-(F)$ over all the strongly irreducible Heegaard surfaces F of N. By convention, if the manifold N does not contain any strongly irreducible Heegaard surface, $\chi_-^{sh}(N) = +\infty$. For further definitions and details about the theory of Heegaard splittings, see section 1.

As a Heegaard surface divides a 3-manifold into two compression bodies, after some work, this general theorem applies in the setting of Heegaard splittings. A consequence is the following result, which gives a stronger and explicit version of a theorem of J. Maher [Mah], stating that an infinite tower of finite coverings of M with a uniform bound on the Heegaard genus does contain surface bundles. This theorem of Maher and its proof were the starting point of this work.

Theorem 0.3. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. Denote by* $\mathrm{Inj}(M)$ *the injectivity radius of M and set $\epsilon = \frac{1}{2}\mathrm{Inj}(M)$.*

1. *There exists an explicit constant $\bar{k} = \bar{k}(\epsilon, \mathrm{Vol}(M))$ such that for every covering $M' \to M$ with finite degree d such that $\boldsymbol{\bar{k}\, \chi_-^h(M') \ln \chi_-^h(M') \leq \ln\ln d}$, M' is a surface bundle with fiber of genus at most $g(M')$.*

(2) *Moreover, for every covering $M' \to M$ with finite degree d, one always has* $\bar{\boldsymbol{k}}\, \boldsymbol{\chi_-^{sh}(M') \ln \chi_-^{sh}(M') > \ln\ln d}$.

Lackenby [L] in his program introduced the notion of Heegaard gradient.

Definition 0.4. [L, p. 319 and 320]

Let M be a compact, connected and orientable 3-manifold. One defines the ***infimal Heegaard gradient*** *of the collection of finite coverings $(M_i \to M)_{i\in\mathbb{N}}$ with degree d_i as:*

$$\nabla^h((M_i \to M)_{i\in\mathbb{N}}) = \inf_{i\in I} \left\{ \frac{\chi_-^h(M_i)}{d_i} \right\}.$$

Likewise, the ***infimal strong Heegaard gradient*** *of the collection $(M_i \to M)_{i\in\mathbb{N}}$ is:*

$$\nabla^{sh}((M_i \to M)_{i\in\mathbb{N}}) = \inf_{i\in I} \left\{ \frac{\chi_-^{sh}(M_i)}{d_i} \right\},$$

where $\chi_-^{sh}(M_i)$ is the strong Heegaard Euler characteristic of the finite covering $(M_i \to M)_{i\in\mathbb{N}}$.

If the family of finite covers is not specified, by convention it is the family of all finite covers of M. The corresponding gradients are called the Heegaard gradient of M, denoted by $\nabla^h(M)$, and the strong Heegaard gradient of M, denoted by $\nabla^{sh}(M)$.

Results of Lackenby show that those two quantities provide information about the existence of incompressible surfaces in finite covers of a manifold M with sufficiently large degrees. They led Lackenby to formulate the following conjectures.

Conjecture 0.1 (Heegaard gradient Conjecture)**.** [L, p. 320]

The Heegaard gradient of a compact, connected and orientable hyperbolic 3-manifold is zero if and only if the manifold M virtually fibers over the circle $\mathbb{S}^1$.

This conjecture would follow immediately from the announcement of Thurston's virtual fibration conjecture.

A second conjecture deals with the strong Heegaard genus, and remains open.

Conjecture 0.2 (Strong Heegaard gradient Conjecture)**.** [L, p. 320]

The strong Heegaard gradient of a closed, connected and orientable hyperbolic 3-manifold is always strictly positive.

Theorem 0.3 leads to a "sub-logarithmic" version of conjecture 0.1 (for given collections of finite coverings) and conjecture 0.2. As there exist infinite towers of non-fibered finite coverings of a hyperbolic 3-manifold M (see [BW]), it makes sense to asks for a condition to ensure that a given collection $(M_i \to M)_{i\in\mathbb{N}}$ of finite covers of M contains surface bundles.

Definition 0.5. *Let $\eta \in (0,1)$.*

The ***η-sub-logarithmic Heegaard gradient*** *associated to a sequence of finite covers $(M_i \to M)_{i\in\mathbb{N}}$ with finite degrees d_i is defined by :*

$$\nabla^h_{log,\eta}((M_i \to M)_{i\in\mathbb{N}}) = \inf \left\{ \frac{\chi_-^h(M_i)}{(\ln\ln d_i)^\eta} \right\}.$$

One can also define the ***strong η-sub-logarithmic Heegaard gradient*** *of M by*

$$\nabla^{sh}_{log,\eta}(M) = \inf \left\{ \frac{\chi_-^{sh}(M_i)}{(\ln\ln d_i)^\eta} \right\},$$

where the infimum is over the (countable) collection of all finite covers $(M_i \to M)_{i\in\mathbb{N}}$ of M.

Corollary 0.6. (1) *If the η-sub-logarithmic Heegaard gradient $\nabla^h_{log,\eta}((M_i \to M)_{i\in\mathbb{N}})$ is zero, then for infinitely many $i \in \mathbb{N}$ the finite covering M_i is a surface bundle.*
(2) *The strong η-sub-logarithmic Heegaard gradient of M is always positive: $\nabla^{sh}_{log,\eta}(M) > 0$.*

Theorem A also applies in the setting of circular decompositions associated to a non-trivial cohomology class, to give a sufficient condition for this class to be fibered.

Definition 0.7. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. If $\alpha \in H^1(M) = H^1(M,\mathbb{Z})$ is a non-trivial cohomology class, let us denote by $\|\alpha\|$ the Thurston norm of α. By definition,*

$$\|\alpha\| = \min\{\chi_-(R),\, [R] = \mathcal{P}(\alpha)\},$$

where R is an embedded surface and $\mathcal{P}(\alpha)$ the Poincaré-dual class of α. We will call such a surface R realizing the Thurston norm of α a ***$\|\alpha\|$-minimizing surface.***

If R is a non-separating and $\|\alpha\|$-minimizing surface for a given non-trivial cohomology class $\alpha \in H^1(M)$, take $\mathcal{N}(R) \cong R \times (-1,1)$ a regular neighborhood of R in M, and denote by $M_R = M \setminus \mathcal{N}(R)$. Set

$$h(M,\alpha,R) = \min\{\chi(R) - \chi(S)\},$$

where S is a Heegaard surface for $(M_R, R\times\{1\}, R\times\{-1\})$. Said differently, $\frac{1}{2}h(M,\alpha,R)$ is the minimal number of 1-handles we need to attach to a regular neighborhood of $R \times \{1\}$ in M_R to get the first compression body of a Heegaard splitting of $(M_R, R\times\{1\}, R\times\{-1\})$. Set

$$h(M,\alpha) = h(\alpha) = \min\{h(M,\alpha,R),\, [R] = \mathcal{P}(\alpha),\, \chi_-(R) = \|\alpha\|\}.$$

For each non-trivial cohomology class $\alpha \in H^1(M)$, let $\chi^c_-(\alpha) = \|\alpha\| + h(\alpha)$ be the ***circular characteristic*** *of α. It is the negative part of the Euler characteristic of a minimal genus Heegaard surface for M_R, where R is a $\|\alpha\|$-minimizing surface such that the number $h(M,\alpha,R)$ is minimal among all $\|\alpha\|$-minimizing surfaces.*

In this setting, theorem A leads to the following corollary, which is analogous to theorem 0.3 for circular decompositions associated to a non-trivial cohomology class.

Corollary 0.8. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. Set $\epsilon = \mathrm{Inj}(M)/2$. There exists an explicit constant $\ell' = \ell'(\epsilon, \mathrm{Vol}(M))$, depending only on ϵ and the volume of M, satisfying the following property. Let $M' \to M$ be a covering of M of finite degree d, and $\alpha' \in H^1(M')$ a non-trivial cohomology class.*

If $\ell'\, \chi^c_-(\alpha') \ln \chi^c_-(\alpha') \leq \ln\ln d$, then every $\|\alpha'\|$-minimizing surface R' in M' is a fiber.

Thus we have a criterion to ensure that a non-trivial homology class can be represented by a fiber. If R is an incompressible embedded surface in M, its homology class is trivial if and only if R is separating. We have also established a sufficient condition for an incompressible surface R to be a virtual fiber.

Definition 0.9. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. Suppose that R is an incompressible, orientable and connected embedded surface in M. If R is non-separating, the homology class $[R] \in H_2(M)$ is non-trivial. Let the* ***Heegaard characteristic of the surface*** *R be the minimum of $|\chi(S)|$, where S is a Heegaard surface for $(M_R := M \setminus \mathcal{N}(R), R \times \{1\}, R \times \{-1\})$.*

If the surface R is separating, the manifold $M_R := M \setminus \mathcal{N}(R)$ is the disjoint union of two connected components M_l and M_r. Let the ***Heegaard characteristic of the surface*** *R be the maximum of $\{\chi_-^h(M_l), \chi_-^h(M_r)\}$.*

In both cases, let us denote by $\chi_-^h(R)$ the Heegaard characteristic of the incompressible surface R.

In the following corollary, the surface R can either be separating or non-separating.

Corollary 0.10. *Let M be a hyperbolic, connected, oriented and closed 3-manifold, and set $\epsilon = \mathrm{Inj}(M)/2$. There exists an explicit constant $\ell'' = \ell''(\epsilon, \mathrm{Vol}(M))$, depending only on ϵ and $\mathrm{Vol}(M)$ and satisfying the following property. Let R be an incompressible, connected, orientable and closed embedded surface in M. Let $M' \to M$ be a covering of M of finite degree d. Let also R' be a connected component of the preimage of R in M'.*

If $\ell''\, \chi_-^h(R') \ln \chi_-^h(R') \leq \ln \ln d$, then the incompressible surface R is a fiber. Moreover, if R' is non-separating, R' is the fiber of a bundle over the circle, and the same holds for R if it is non-separating. Otherwise, it is the fiber of a twisted I-bundle.

Remark 0.11. *The explicit expression of constants k, $\bar{k}$, ℓ' and ℓ'' involved in theorem A and corollaries of this work allows us to study their behavior. If the volume $\mathrm{Vol}(M)$ is fixed and that ϵ tends to zero, or if ϵ is fixed and $\mathrm{Vol}(M)$ tends to infinity, all those constants tend to infinity. Thus, the sufficient conditions become more and more difficult to satisfy when the injectivity radius decreases, or if the volume grows.*

Outline of the paper: After some definitions and generalities about Heegaard splittings in the first section, we prove theorem A in the second section. The third section is dedicated to the application of theorem A to Heegaard splittings and the proof of theorem 0.3 and corollary 0.6. The last section deals with applications to circular decompositions and the proof of corollaries 0.8 and 0.10.

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1. Background on Heegaard splittings.

In this section, we briefly summarize the theory of Heegaard splittings. We also refer to [Sc] for a survey on the subject.

A **handlebody** is the regular neighborhood of a connected graph. Its boundary is a connected, orientable and closed surface. The genus g of this surface is called the **genus** of the handlebody. The original graph is called a **spine** for the handlebody. If an orientable 3-manifold M is closed, a **Heegaard splitting** of M is a decomposition of M as the union of two handlebodies with the same genus, glued together by a diffeomorphism of their boundaries. A **compression body** is a connected and orientable 3-manifold H with boundary, obtained from a regular neighborhood $S \times [0,1]$ of a closed surface S, not necessarily connected. One glues some 1-handles to the surface $S \times \{1\}$ to get the compression body H. The surface $S \times \{0\}$, denoted by $\partial_- H$, is called the **negative boundary** of the compression body H. The boundary of H minus the negative boundary $\partial_- H$ is a connected surface $\partial_+ H$, called the **positive boundary** of H. The genus of the closed surface $\partial_+ H$ is called the **genus** of the compression body H and denoted by $g(H)$. By convention, a handlebody as defined above is a compression body H for which $\partial_- H = \emptyset$. A **spine** for a compression body H is the union Γ of the negative boundary $\partial_- H$ together with a graph whose vertices lie on $\partial_- H$, such that H deformation retracts on Γ.

Definition 1.1 (Heegaard Splittings). *Let (M, N_0, N_1) be a cobordism of M, with possibly $N_0 = \emptyset$ or $N_1 = \emptyset$. In particular, if M is closed, $N_0 = N_1 = \emptyset$. A **Heegaard splitting of** M associated to the cobordism (M, N_0, N_1) is a decomposition of M into two compression bodies H_0 and H_1 such that:*

(1) $\partial_- H_0 = N_0$, $\partial_- H_1 = N_1$,
(2) $\partial_+ H_0 \cong \partial_+ H_1 \cong S$ *where S is a closed surface, and*
(3) $M = H_0 \cup_S H_1$ *is obtained from H_0 and H_1 by gluing their positive boundaries by a homeomorphism of S.*

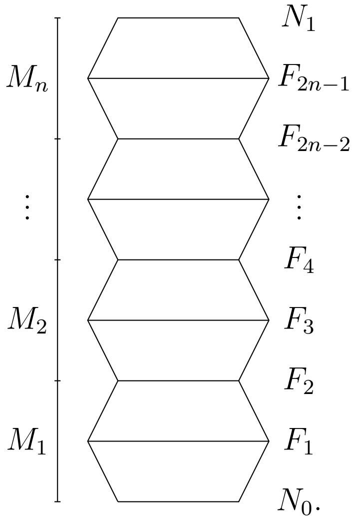
*The surface S is called a **Heegaard surface** for M and its genus is called the **genus** of the Heegaard splitting $M = H_0 \cup_S H_1$.*

Every compact and orientable 3-manifold M admits a Heegaard splitting. The **Heegaard genus** of the manifold M, denoted by $g(M)$, is the minimal genus of all Heegaard splittings of M. The **Heegaard Euler characteristic** of M is $\chi_-^h(M) = 2g(M) - 2$, the negative part of the Euler characteristic of a minimal genus Heegaard surface for M.

A **meridian disc** for a Heegaard splitting of M is a properly embedded disc in one of the compression bodies, which bounds an essential curve in the Heegaard surface. A Heegaard splitting (or a Heegaard surface) is said to be **strongly irreducible** if there does not exist any pair of disjoint meridian discs, one in each compression body. In other words, in a strongly irreducible Heegaard splitting, the boundaries of any two meridian discs each in one side of the Heegaard surface necessarily intersect. For any orientable 3-manifold M, one defines the **strong Heegaard Euler characteristic** $\chi_-^{sh}(M)$ of M as the minimum over all strongly irreducible Heegaard surfaces F of the negative part $\chi_-(F)$ of the Euler characteristic of F. If the manifold M does not have any strongly irreducible Heegaard splitting, then $\chi_-^{sh}(M) = +\infty$.

Note that in the case of hyperbolic 3-manifolds, the Heegaard Euler characteristics and the strong Heegaard Euler characteristics are always at least 2.

A Heegaard splitting can be seen as a handle decomposition for a closed 3-manifold M. Starting from a collection of 0-handles, one attaches some 1-handles to them, then a collection of 2-handles, to finish by 3-handles. The first handlebody corresponds to the 0- and 1-handles, to which the 2- and 3-handles that compose the second handlebody are attached. More generally, a **generalized Heegaard splitting** for a 3-manifold M corresponds to a handle decomposition: starting from 0-handles and possibly collars of some boundary components of M, one attaches some 1-handles, then a collection of 2-handles, then another collection of 1-handles, and so on, alternating 1- and 2-handles, to finish after the last collection of 2-handles with a collection of 3-handles. If one stops during the process, the object obtained after attaching the j-th batch of 1- or 2-handles is a 3-manifold embedded in M. Let F_j be its boundary, after discarding any 2-sphere component that bounds a 0- or a 3-handle. After a small isotopy to make all the surfaces F_j disjoint, one gets a collection of $2n-1$ disjoint surfaces F_j in M. The surfaces F_{2j}, called the **even surfaces**, separate the manifold M into n 3-manifolds, for which the surfaces F_{2j-1}, called the **odd surfaces**, form Heegaard surfaces.

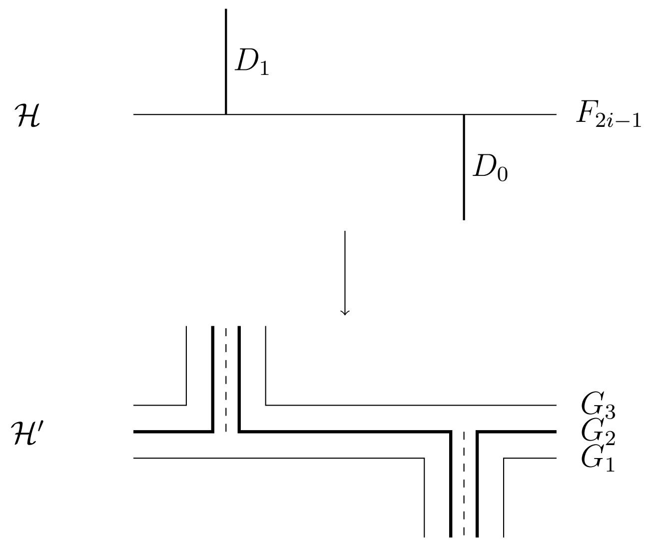


To each 1- and 2-handles of a generalized Heegaard splitting, one can associate a meridian disc. If the splitting of the region between two even surfaces is not strongly irreducible, two disjoint meridian discs can be used to change the order in which the handles are attached. A 2-handle corresponding to one of the meridian discs can be attached before a 1-handle corresponding to the other meridian disc. We will call this operation a **surgery of generalized Heegaard splitting**.

Let F be a closed and orientable surface. If F is connected, one defines the **complexity** of F as $c(F) = 0$ if F is the 2-sphere $\mathbb{S}^2$, and $c(F) = 2g(F) - 1 = 1 - \chi(F)$ otherwise. If F is not connected, the complexity of F is the sum over all components of F of the complexity of the component.

If $\mathcal{H} = \{F_1, F_2, \ldots, F_{2n-1}\}$ is a generalized Heegaard splitting of M, the **width** of this decomposition is the set $w(\mathcal{H}) = \{c(F_1), \ldots, c(F_{2n-1})\}$ of the complexities of the odd surfaces, with repetitions and arranged in monotonically non-increasing order. Widths are compared using the lexicographic order.

Starting from a generalized Heegaard splitting $\mathcal{H} = \{F_1, F_2, \ldots, F_{2n-1}\}$ in which at least one of the surfaces F_{2i-1} is not strongly irreducible, one can do a surgery of generalized Heegaard splittings to change the order in which the 1- and 2-handles are attached, to get a new generalized Heegaard splitting $\mathcal{H}'$ with $w(\mathcal{H}') < w(\mathcal{H})$.



If $\mathcal{H}$ is a generalized Heegaard splitting for M, let $\mathcal{S}_{\mathcal{H}}$ be the set of all generalized Heegaard splittings obtained from $\mathcal{H}$ by surgery. An element $\mathcal{H}' \in \mathcal{S}_{\mathcal{H}}$ of minimal width is called an **$\mathcal{H}$-thin generalized Heegaard splitting**.

Proposition 1.2. *Let M be a connected, oriented and compact 3-manifold, and $\mathcal{H}$ a generalized Heegaard splitting for M.*

Every $\mathcal{H}$-thin generalized Heegaard splitting $\mathcal{H}' = (F_1, \dots, F_{2n-1})$ satisfies the following properties.

(1) *The odd surfaces F_{2i-1} correspond to strongly irreducible Heegaard surfaces.*
(2) *The even surfaces F_{2i} are incompressible surfaces in M.*
(3) *Furthermore, if the manifold M is irreducible, then no component of any even surface is a 2-sphere.*

□

The proof of this proposition is a consequence of the definition of a surgery of generalized Heegaard splittings. See for example [CG] and [ST].

A generalized splitting of minimal width among all generalized Heegaard splittings of M is called a **thin position** (see [ST]).

2. Finding a fibration.

2.1. **Main theorem.** The aim of this section is to prove theorem A.

If S is a surface, let us denote by $\chi_-(S) = \max\{0, -\chi(S)\}$ the negative part of the Euler characteristic of S.

If C is a compression body, set $\chi_-(C) := \chi_-(\partial_+ C)$. If S is a union of connected components of $\partial_- C$, the definition implies that $\chi_-(S) \leq \chi_-(C)$.

Definition 2.1. *An embedded surface S in a Riemannian 3-manifold M is called **pseudo-minimal** if it is orientable, closed, and S is a minimal surface or the boundary of a regular neighborhood of a minimal non-orientable surface, possibly with a little tube attached vertically in the I-bundle structure.*

<u>Proof of theorem A.</u>

Suppose that there exists a finite cover $M' \to M$ of degree d satisfying the hypotheses of theorem A. The proof relies on three key propositions, the proof of which we postpone to the three next subsections. Let us denote by $g = \frac{c}{2} + 1$. It is an upper bound for the genus of the compression bodies of M'.

Lemma 2.2. *There exists a compression body C among the q compression bodies $C_1, \ldots, C_q$ of M' such that*

$$\mathrm{Vol}(C) \geq \mathrm{Vol}(M)\,\frac{d}{q}.$$

Proof of lemma 2.2.

The proof is straightforward, as there are q compression bodies $C_1, \ldots, C_q$, and $\mathrm{Vol}(M') = d\,\mathrm{Vol}(M)$. □

Let C be a compression body as in lemma 2.2.

Lemma 2.3. *Let $k_0 = \max\left\{\frac{\ln(4(2\epsilon a'+1))}{2\ln 2}, 1 + \frac{\ln(1+\ln(12V_3/\mathrm{Vol}(M))}{2\ln 2}\right\}$, where V_3 is the maximal volume of an ideal hyperbolic tetrahedron in $\mathbb{H}^3$, and $a' = 6(21/4 + 3/4\pi + 3/4\epsilon + 2/\sinh^2(\epsilon/4))$.*

If $k_0\,\chi_-(C)\ln\chi_-(C) \leq \ln\ln d/q$ and $\mathrm{Vol}(M) \geq \pi/2$, then there is a way of replacing the boundary surfaces of C by simplicial surfaces, to obtain a new compression body C'' with:

$$\mathrm{Vol}(C'') \geq \frac{1}{4}\mathrm{Vol}(C) \geq \frac{\mathrm{Vol}(M)d}{4q}.$$

This lemma is proven in subsection 2.2.1. To simplify notations, this new compression body C'' will still be denoted by C.

Definition 2.4. *Let x be a point in C and S an immersed surface in C. We say that S **separates** x **from** ∂_+C if every oriented path from x to ∂_+C has its algebraic intersection number with ∂_+C equal to $+1$.*

*If two surfaces S and T immersed in C are such that S separates every point of T from ∂_+C, we say that T **separates** S **from** ∂_+C. In this case, the surfaces S and T are said to be **nested**.*

We will denote the ceil function of the real number x by $\lceil x \rceil$, i.e. the smallest integer not less than x. Similarly, $\lfloor x \rfloor$ is the floor function of x, and represents the largest integer no greater than x. By convention, we set $\lceil x \rceil$ and $\lfloor x \rfloor$ equal to zero if x is non-positive.

The following proposition is a step towards the construction of a certain amount of parallel surfaces in the compression body C. It is an adaptation of Lemma 4.5 p. 2251 of [Mah]. We postpone its proof to section 2.2.

Proposition B (of Nested Surfaces).

Let δ be the diameter of the compression body C of M', $\epsilon = \mathrm{Inj}(M)/2$, $K = 4\left(3 + \frac{1}{\sinh\frac{\epsilon}{8}^2}\right)g(C) - 10$ and $K' = 2a'\chi_-(C)$. Moreover, suppose that $k_0\,\chi_-(C)$ $\ln\chi_-(C) \leq \ln\ln\frac{d}{q}$.

Under those assumptions, there exist at least $n = \lceil\frac{\delta}{36\epsilon K} - \frac{2}{9} - \frac{K'}{3K}\rceil$ orientable, disjoint and nested surfaces, immersed in C. All of those surfaces are homotopic to components of surfaces obtained by compressing ∂_+C. Moreover, the ϵ-diameter of those surfaces in M' is bounded from above by K and they are separated from each other by a distance greater than or equal to $10\epsilon K$.

With this proposition, we obtain at least $n = \lceil \frac{\delta}{36\epsilon K} - \frac{2}{9} - \frac{K'}{3K} \rceil$ nested immersed surfaces in the handlebody C. Those surfaces are all disjoint and homotopic to components of surfaces obtained from $\partial_+ C$ by surgery. This implies that the the genus of those surfaces is between 0 and $g(C)$, the genus of C (which is, by assumption, less than or equal to g). We can thus find at least $\lfloor \frac{n}{g(C)+1} \rfloor$ such nested immersed surfaces of the same genus. The next step is then to replace those nested immersed surfaces by parallel embedded surfaces.

Proposition C (of Parallel Surfaces).
Let δ be the diameter of the compression body C in M', $\epsilon = \mathrm{Inj}(M)/2$, $K = 4\left(3 + \frac{1}{\sinh \frac{\epsilon}{8}^2}\right) g(C) - 10$ and $K' = 2a'\chi_-(C)$. Suppose that $k_0\, \chi_-(C) \ln \chi_-(C) \leq \ln \ln \frac{d}{q}$.

Under those assumptions, there exists at least $m = (\lfloor \frac{1}{g(C)+1} \lceil \frac{\delta}{36\epsilon K} - \frac{2}{9} - \frac{K'}{3K} \rceil \rfloor - 4)$ orientable, parallel and connected surfaces embedded in C, separated from each other by a distance greater than or equal to ϵK, and each of those surfaces can be covered by at most K embedded balls in M' of radius 2ϵ. In particular, their diameter in the manifold M' is uniformly bounded from above by $4\epsilon K$.

For the proof of this proposition, see section 2.3.
Let

$$\begin{aligned} a &= 2\left(3 + \frac{1}{\sinh^2(\frac{\epsilon}{8})}\right), \\ b &= 2\left(1 + \frac{2}{\sinh^2(\frac{\epsilon}{8})}\right) \text{ and} \\ a' &= 6\left(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\frac{\epsilon}{4})}\right). \end{aligned}$$

Lemma 2.5. *Under assumptions of theorem A, according to proposition C, there exist m parallel surfaces embedded in the compression body C of M', with*

$$m \geq \frac{2}{\chi_-(C) + 4}\left(\frac{\ln\left(\frac{d}{q}\right) + \ln\left(\frac{\mathrm{Vol}(M)}{2\pi}\right)}{72\epsilon(a\chi_-(C) + b)} - \frac{2}{9}(1 + \frac{3a'}{a})\right) - 5.$$

Proof of lemma 2.5.
The number m of embedded parallel surfaces in C obtained by proposition C is equal to:

$$m = \lfloor \frac{1}{g(C)+1} \lceil \frac{\delta}{36\epsilon K} - \frac{2}{9} - \frac{K'}{3K} \rceil \rfloor - 4,$$

where

$$\begin{aligned} K &= 4\left(3 + \frac{1}{\sinh \frac{\epsilon}{8}^2}\right) g(C) - 10 \\ &= a\chi_-(C) + b, \end{aligned}$$

and

$$K' = 2a'\chi_-(C).$$

The diameter δ of the compression body C and the ratio d/q are related. On the one hand,

$$\mathrm{Vol}(C) \le \mathrm{Vol}\left(\mathbb{B}_{\mathbb{H}^3}(\delta)\right) = \pi(\sinh(2\delta) - 2\delta) \le \frac{\pi}{2}e^{2\delta}.$$

Remark 2.6. *The second logarithm of the expression* $\ln\ln\frac{d}{q}$ *comes from this estimation linking the diameter with the volume of a hyperbolic 3-manifold.*

On the other hand, lemmas 2.2 and 2.3 give the lower bound

$$\mathrm{Vol}(C) \ge \mathrm{Vol}(M)\,\frac{d}{4q},$$

which leads to the inequality

$$\delta \ge \frac{1}{2}\ln\left(\frac{d}{q}\right) + \frac{1}{2}\ln\left(\frac{\mathrm{Vol}(M)}{2\pi}\right). \tag{1}$$

In particular, if d/q tends to infinity, δ tends also to infinity.

The expression of m involves the ratio $\frac{K'}{3K}$. Now,

$$\begin{aligned}\frac{K'}{3K} &= \frac{2a'\chi_-(C)}{3a\chi_-(C) + 3b}\\ &= \frac{2a'}{3a + 3b/\chi_-(C)}\\ &\le \frac{2a'}{3a}.\end{aligned}$$

Replacing the ratio $K'/3K$ by $2a'/3a$ and taking inequality (1) into account, one obtains

$$\begin{aligned} m &\ge \lfloor \frac{2}{\chi_-(C)+4}\lceil \frac{\ln\left(\frac{d}{q}\right) + \ln\left(\frac{\mathrm{Vol}(M)}{2\pi}\right)}{72\epsilon(a\chi_-(C)+b)} - \frac{2}{9} - \frac{2a'}{3a}\rceil\rfloor - 4\\ &\ge \frac{2}{\chi_-(C)+4}\left(\frac{\ln\left(\frac{d}{q}\right) + \ln\left(\frac{\mathrm{Vol}(M)}{2\pi}\right)}{72\epsilon(a\chi_-(C)+b)} - \frac{2}{9}(1 + \frac{3a'}{a})\right) - 5,\end{aligned}$$

which ends the proof of lemma 2.5. □

Those m parallel surfaces obtained by proposition C are candidates for a fiber. But we still have to select some of them to get a virtual fibration of the base manifold M.

Let $\mathcal{D}$ be a Dirichlet fundamental polyhedron for M in its universal cover $\widehat{M} \simeq \mathbb{H}^3$. Translates of $\mathcal{D}$ by the covering transformations give a tiling of the universal cover $\widehat{M}$. This tiling descends to a tiling of the finite cover M' by d copies of $\mathcal{D}$. Each of the m parallel, connected and embedded surfaces in M' obtained by proposition C intersects a finite and connected set of copies of $\mathcal{D}$. We call such a set a **pattern of fundamental domains**. We can suppose that each of the embedded surfaces is transverse to the 2-skeleton of the tiling. More precisely, we can suppose that each surface does not meet the vertices of the fundamental polyhedra, that it intersects the edges in isolated points and it is transverse to the 2-dimensional faces of the polyhedra. Thus a pattern of fundamental domains is a connected set that is the union of copies of $\mathcal{D}$ glued along some of their 2-dimensional faces.

Lemma 2.7. *Let $\mathcal{D}$ be a Dirichlet fundamental polyhedron for M in $\mathbb{H}^3$. Let α be the number of faces of $\mathcal{D}$ of dimension two.*

For each $\ell \in \mathbb{N}$, the number of possibilities to glue together at most ℓ copies of $\mathcal{D}$ to form a pattern of ℓ fundamental domains is less than or equal to $\left(\alpha\sqrt{2}\,\ell\right)^{\alpha\ell}$.

Proof of lemma 2.7.

For every $\ell \in \mathbb{N}$, let us denote by $B(\ell)$ the number of possibilities to glue together ℓ copies of $\mathcal{D}$ to form a pattern of ℓ fundamental domains. We have to find an upper bound for the number of possibilities to identify pairwise some 2-dimensional faces of at most ℓ Dirichlet polyhedra.

First, let us notice that there are at most $\alpha\ell$ such 2-dimensional faces. Thus, there are at most $(\alpha\ell)! \leq (\alpha\ell)^{\alpha\ell}$ ways to match pairwise those faces.

If (F_1, F_2) is a pair of such faces, we can choose to glue them together by an orientation-reversing isometry $h : F_1 \longrightarrow F_2$ (if such an isometry between those two faces exists). This isometry corresponds to a "pairing transformation" (see for example [Mar, Proposition 3.5.1 p. 117]). It is a reflection in $\mathbb{H}^3$ and its hyperplane contains one of the faces of $\mathcal{D}$. Thus, if such an isometry exists, it is unique. We can also decide not to glue those two faces together: by convention, we will say that we glue them by the empty gluing. Therefore, there are at most 2 ways to glue F_1 and F_2 together, including the empty gluing.

Thus there are at most $(\alpha\ell)!2^{\frac{\alpha\ell}{2}} \leq \left(\alpha\sqrt{2}\,\ell\right)^{\alpha\ell}$ ways to glue together at most ℓ copies of fundamental domains to form a pattern of fundamental domains, which ends the proof of lemma 2.7. □

The following lemma is a way to bound the number α of 2-faces of a fundamental polyhedron $\mathcal{D}$ and its diameter in $\mathbb{H}^3$ in terms of the volume of the manifold M and a lower bound for its injectivity radius.

Lemma 2.8. *Let $\mathcal{D}$ be a Dirichlet fundamental polyhedron for the manifold M, embedded in the universal cover $\widetilde{M} \simeq \mathbb{H}^3$. Let D be an upper bound for the diameter of $\mathcal{D}$ in $\mathbb{H}^3$ and α be the number of its 2-faces. We have the following estimates:*

$$\text{diam}(\mathcal{D}) \leq \frac{8\epsilon \text{Vol}(M)}{\pi(\sinh(2\epsilon) - 2\epsilon)} = D, \tag{2}$$

and

$$\alpha \leq \frac{\pi(\sinh(4D) - 4D)}{\text{Vol}(M)} - 1. \tag{3}$$

If S is an embedded surface in the finite cover M' of M, which can be covered by at most K embedded balls in M' of radius $\epsilon' \leq \text{Inj}(M)$, then S intersects at most L images of $\mathcal{D}$ in M', with

$$L = \lfloor \frac{\pi(\sinh(2D + 2\epsilon') - 2D - 2\epsilon')}{\text{Vol}(M)} K \rfloor. \tag{4}$$

Proof of lemma 2.8.

To prove inequality (2), first notice that $\text{diam}(\mathcal{D}) \leq 2\,\text{diam}(M)$. To prove it, recall that there exists $w \in \mathbb{H}^3$ such that $\mathcal{D} = \{x \in \mathbb{H}^3,\ \text{d}(\gamma(w), x) \geq \text{d}(w, x)\ \forall \gamma \in \pi_1(M)\}$. If x and $y \in \mathcal{D}$ satisfy $\text{d}(x, y) = \text{diam}(\mathcal{D})$, then

$$\text{diam}(\mathcal{D}) = \text{d}(x, y) \leq \text{d}(x, w) + \text{d}(y, w) \leq 2\,\text{diam}(M).$$

Take x and $y \in M$ such that $\text{d}(x, y) = \text{diam}(M)$, and let γ be a minimizing geodesic from x to y. By definition, $\text{length}\,(\gamma) = \text{diam}(M)$. Let $\mathcal{B}$ be a collection of points in γ which is maximal among collections of points of γ such that two balls of radius ϵ and whose centers are two distinct points of $\mathcal{B}$ have disjoint interiors. Then, by maximality of $\mathcal{B}$, the union of balls with centers in $\mathcal{B}$ and radius 2ϵ cover the geodesic γ.

Thus, $|\mathcal{B}| \geq \frac{\text{length}\,(\gamma)}{4\epsilon}$. As balls of centers in $\mathcal{B}$ and radius ϵ have disjoint interiors, considering volumes, we deduce:

$$\begin{aligned}
\text{Vol}(M) &\geq \sum_{u \in \mathcal{B}} \text{Vol}(B(u, \epsilon)) \\
&\geq \frac{\text{length}\,(\gamma)}{4\epsilon} \text{Vol}(B_{\mathbb{H}^3}(\epsilon)) \\
&\geq \frac{\text{diam}(M)}{4\epsilon} \pi(\sinh(2\epsilon) - 2\epsilon),
\end{aligned}$$

proving inequality (2).

Let us show inequality (3). To each 2-face of $\mathcal{D}$, one can associate a unique translate $g_F(\mathcal{D})$ of $\mathcal{D}$ adjacent to $\mathcal{D}$ along F. As the diameter of $g_F(\mathcal{D})$ is also $\text{diam}(\mathcal{D}) \leq D$, every point $x \in g_F(\mathcal{D})$ lies at distance at most $\text{dist}(x, F) + \text{diam}(\mathcal{D}) \leq 2D$ from $w \in \mathcal{D}$. Thus, the ball of center w and radius $2D$ contains the fundamental polyhedron $\mathcal{D}$ together with the union of all its translates $g_F(\mathcal{D})$, where F is a 2-face of $\mathcal{D}$. As those polyhedra have disjoint interiors, for volumes, we obtain:

$$(\alpha + 1)\text{Vol}(\mathcal{D}) \leq \text{Vol}(B_{\mathbb{H}^3}(w, 2D)),$$

and thus

$$\alpha \leq \frac{\pi(\sinh(4D) - 4D)}{\text{Vol}(M)} - 1.$$

The proof of inequality (4) is similar. Denote by $\mathcal{B}$ the set of the centers of a collection of K embedded balls in M' of radius ϵ' covering the surface S. Let $\mathcal{N} = \cup_{x \in \mathcal{B}} B(x, D + \epsilon')$. Those balls are not necessarily isometric to hyperbolic embedded balls in $\mathbb{H}^3$ as $D + \epsilon' > \text{Inj}(M)$. However, let us show that $\mathcal{N}$ contains every fundamental polyhedron of M' intersecting S.

To prove it, let x be a point in a fundamental polyhedron of M' intersecting S. Take $y \in S$ such that $\text{d}(x, y) = \text{dist}(x, S) \leq D$. As y is a point of S, there exists $x \in \mathcal{B}$ such that the ball $B(x, \epsilon')$ contains y. Therefore $\text{d}(z, x) \leq \text{d}(z, y) + \text{d}(y, x) \leq D + \epsilon'$, showing that $z \in B(x, \epsilon' + D) \subset \mathcal{N}$.

Comparing volumes, we get:

$$\begin{aligned} L\,\mathrm{Vol}(\mathcal{D}) &\leq \mathrm{Vol}(\mathcal{N}) \\ L\,\mathrm{Vol}(M) &\leq |\mathcal{B}|\,\mathrm{Vol}(B_{\mathbb{H}^3}(\epsilon'+D)) \\ L &\leq \frac{\pi(\sinh(2\epsilon'+2D)-2\epsilon'-2D)}{\mathrm{Vol}(M)}K, \end{aligned}$$

proving inequality (4), as L is a natural integer. □

The following key proposition is a quantitative version of Lemma 4.12 p. 2258 of [Mah]. We postpone its proof to section 2.4.

Proposition D (Pattern Proposition).
Assume that in the cover M' we have m connected, orientable, embedded, disjoint and parallel surfaces, at distance at least $r > 0$ from each other. Moreover, suppose that each of those surfaces can be covered by at most K embedded balls in M' of radius $\epsilon' \leq \mathrm{Inj}(M)$.

Let $\mathcal{D}$ be a Dirichlet fundamental domain for the manifold M in its universal cover $\widehat{M} \simeq \mathbb{H}^3$. Let us denote by D an upper bound for the diameter of $\mathcal{D}$ and α an upper bound for the number of its 2-dimensional faces.

For all $\ell \in \mathbb{N}$, let $B(\ell)$ be an upper bound for the number of possibilities of patterns obtained by gluing together at most ℓ fundamental domains that intersect a connected, orientable and embedded surface. Let $L = \lfloor \frac{\pi(\sinh(2D+2\epsilon')-2D-2\epsilon')}{\mathrm{Vol}(M)}K \rfloor$.

If $r/(2D+1) \geq 1$ and $\frac{m}{\alpha^2 L^2 B(L)} \geq 4$, or if $r/(2D+1) \leq 1$ and $\left(\frac{r}{2D+1}m - 1\right)\frac{1}{\alpha^2 L^2 B(L)} \geq 4$, then the manifold M virtually fibers over the circle $\mathbb{S}^1$, and the m parallel surfaces in M' are fibers of a bundle over the circle or of a twisted I-bundle.

Remark 2.9. *The first logarithm in the expression $\ln\ln\frac{d}{q}$ and the function of the complexity $c\ln(c)$ in the assumption $k\,c\ln(c) < \ln\ln\frac{d}{q}$ arise from the use of lemma 2.7 (providing an estimate of the number $B(\ell)$) in the proof of this proposition.*

We can now finish the proof of theorem A assuming propositions B, C and D, which will be proved in next sections.

The aim is to apply Proposition D to the m parallel surfaces obtained in Proposition C, with

$$\begin{aligned} K &= a\chi_-(C) + b, \text{ and} \\ r &= \epsilon K = \epsilon(a\chi_-(C)+b). \end{aligned}$$

Set

$$\begin{aligned} D &:= \frac{8\epsilon\mathrm{Vol}(M)}{\pi(\sinh(2\epsilon)-2\epsilon)}, \\ \alpha &:= \frac{\pi(\sinh(4D)-4D)}{\mathrm{Vol}(M)} - 1, \text{ and} \\ \sigma &:= \frac{\pi(\sinh(2D+4\epsilon)-2D-4\epsilon)}{\mathrm{Vol}(M)}. \end{aligned}$$

From lemma 2.8, D is an upper bound for the diameter of the fundamental polyhedron $\mathcal{D}$, and the number of 2-faces of $\mathcal{D}$ is at most α.

In addition, from lemma 2.8 again, $L = \lfloor \frac{\pi(\sinh(2D+4\epsilon)-2D-4\epsilon)}{\mathrm{Vol}(M)} K \rfloor$. In particular,

$$L \leq \frac{\pi(\sinh(2D+4\epsilon)-2D-4\epsilon)}{\mathrm{Vol}(M)}(a\chi_-(C)+b) = \sigma(a\chi_-(C)+b).$$

Claim 1. *There exist* $c_1 \geq 2$ *and* $k_1 > 0$*, depending only on* ϵ *and* $\mathrm{Vol}(M)$*, such that if* $\chi_-(C) \leq c_1$ *and* $k_1\,\chi_-(C)\ln\chi_-(C) \leq \ln\ln d/q$*, then assumptions of Proposition D are satisfied. In particular,* M *virtually fibers over the circle and the* m *embedded surfaces in* M' *are fibers.*

Furthermore, one can take $c_1 := \frac{1}{a}\left(\frac{16\mathrm{Vol}(M)}{\pi(\sinh(2\epsilon)-2\epsilon)} + \frac{1}{\epsilon} - b\right)$ *and*

$$\begin{aligned} k_1 &:= \frac{1}{2\ln 2}\ln\Big(72(2D+1)(c_1+4)\left(3+2(\alpha\sigma)^2(ac_1+b)^2(\sqrt{2}\alpha\sigma(ac_1+b))^{\alpha\sigma(ac_1+b)}\right)\\ &+ 16\epsilon(1+\frac{3a'}{a})(ac_1+b) - \ln\left(\frac{\mathrm{Vol}(M)}{2\pi}\right)\Big). \end{aligned}$$

Proof of claim 1.

Recall that $r = \epsilon(a\chi_-(C)+b)$ and $2D+1 = \frac{16\epsilon\mathrm{Vol}(M)}{\pi(\sinh(2\epsilon)-2\epsilon)} + 1$. Thus, if $\chi_-(C) \leq c_1 = \frac{1}{a}\left(\frac{16\mathrm{Vol}(M)}{\pi(\sinh(2\epsilon)-2\epsilon)} + \frac{1}{\epsilon} - b\right)$, then $r \leq 2D+1$. Assumptions of the second case of Proposition D are then satisfied if $\left(\frac{r}{2D+1}m - 1\right)\frac{1}{\alpha^2L^2B(L)} \geq 4$.

Taking lemma 2.5 and the expression of r into account, one obtains the sufficient condition:

$$\left(\frac{2\epsilon(a\chi_-(C)+b)}{(2D+1)(\chi_-(C)+4)}\left(\frac{\ln\left(\frac{d}{q}\right)+\ln\left(\frac{\mathrm{Vol}(M)}{2\pi}\right)}{72\epsilon(a\chi_-(C)+b)} - \frac{2}{9}(1+\frac{3a'}{a})\right) - 6\right)\frac{1}{\alpha^2L^2B(L)} \geq 4.$$

Replace L by its upper bound $\sigma(a\chi_-(C)+b)$. From lemma 2.7, one can chose for $B(L)$ the function $B(L) = (\sqrt{2}\alpha L)^{\alpha L} \leq (\sqrt{2}\alpha\sigma(a\chi_-(C)+b))^{\alpha\sigma(a\chi_-(C)+b)}$.

Thus one obtains the sufficient condition

$$\left(\frac{2\epsilon(a\chi_-(C)+b)}{(2D+1)(\chi_-(C)+4)}\left(\frac{\ln\left(\frac{d}{q}\right)+\ln\left(\frac{\mathrm{Vol}(M)}{2\pi}\right)}{72\epsilon(a\chi_-(C)+b)} - \frac{2}{9}(1+\frac{3a'}{a})\right) - 6\right) \geq 4(\alpha\sigma)^2(a\chi_-(C)+b)^2(\sqrt{2}\alpha\sigma(a\chi_-(C)+b))^{\alpha\sigma(a\chi_-(C)+b)}.$$

Under assumptions of claim 1, $2 \leq \chi_-(C) \leq c_1$. One can then easily check that if $k_1\,\chi_-(C)\ln\chi_-(C) \leq \ln\ln d/q$, the sufficient condition above is satisfied. □

Claim 2. *Suppose that* $\mathrm{Vol}(M) \geq 2\pi$*. There exist* $c_2 \geq c_1$ *and* $k_2 > 0$*, depending only on* ϵ *and* $\mathrm{Vol}(M)$*, such that if* $\chi_-(C) \geq c_2$ *and* $k_2\,\chi_-(C)\ln\chi_-(C) \leq \ln\ln d/q$*, then assumptions of Proposition D are satisfied. In particular,* M *virtually fibers over the circle and the* m *embedded surfaces in* M' *are fibers.*

Furthermore, one can take $k_2 := 4\alpha\sigma a$*, and*

$$c_2 = \max\{c_1, \frac{1}{a}\left(\frac{\ln 5 - \ln(4\alpha^2\sigma^2(2a+b)^2)}{\alpha\sigma\ln(2\sqrt{2}\alpha\sigma a)} - b\right),$$
$$\frac{1}{a}\left(\frac{\ln(1+\frac{3a'}{a}) - \ln(108\alpha^2\sigma^2(2a+b)^2)}{\alpha\sigma\ln(\sqrt{2}\alpha\sigma(2a+b))} - b\right), b/a, 4, 2\sqrt{2}\alpha\sigma a,$$
$$\frac{b}{a} + \frac{4}{\alpha\sigma a}, \frac{\ln(18432\epsilon\alpha^2\sigma^2a^3(2\sqrt{2}\alpha\sigma a)^{\alpha\sigma b})}{\alpha\sigma a\ln 2},$$
$$\frac{1}{a}\left(\frac{1}{\alpha\sigma\ln(\sqrt{2}\alpha\sigma(2a+b))}\ln\left(\frac{1}{4\alpha^2\sigma^2(2a+b)^2}(\frac{\left|-\ln\frac{\mathrm{Vol}(M)}{2\pi} - \frac{2}{9}(1+\frac{3a'}{a})\right|}{216\epsilon(2a+b)} - 5)\right) - b\right)\}.$$

<u>Proof of claim 2.</u>

As $\chi_-(C) \geq c_2 \geq c_1$, from the proof of the first claim, $r \geq 2D+1$. Assumptions of the first case of Proposition D are then satisfied if $\frac{m}{\alpha^2L^2B(L)} \geq 4$. Now, taking lemma 2.5 into account, together with the inequalities $L \leq \sigma(a\chi_-(C)+b)$ and $B(L) \leq (\sqrt{2}\alpha\sigma(a\chi_-(C)+b))^{\alpha\sigma(a\chi_-(C)+b)}$, one obtains the following sufficient condition:

$$\frac{2}{\chi_-(C)+4}\left(\frac{\ln\left(\frac{d}{q}\right) + \ln\left(\frac{\mathrm{Vol}(M)}{2\pi}\right)}{72\epsilon(a\chi_-(C)+b)} - \frac{2}{9}(1+\frac{3a'}{a})\right) - 5 \geq$$
$$4\alpha^2\sigma^2(a\chi_-(C)+b)^2(\sqrt{2}\alpha\sigma(a\chi_-(C)+b))^{\alpha\sigma(a\chi_-(C)+b)},$$

which can also be written

$$\begin{aligned}\ln\left(\frac{d}{q}\right) \geq\ & 72\epsilon(a\chi_-(C)+b)(\frac{\chi_-(C)+4}{2}(4\alpha^2\sigma^2(a\chi_-(C)+b)^2 \\ & (\sqrt{2}\alpha\sigma(a\chi_-(C)+b))^{\alpha\sigma(a\chi_-(C)+b)} + 5) + \frac{2}{9}(1+\frac{3a'}{a})) - \ln\frac{\mathrm{Vol}(M)}{2\pi},\end{aligned}$$

or also

$$\begin{aligned}\ln\ln\left(\frac{d}{q}\right) \geq\ & \ln\big(72\epsilon(a\chi_-(C)+b)(\frac{\chi_-(C)+4}{2}(4\alpha^2\sigma^2(a\chi_-(C)+b)^2 \\ & (\sqrt{2}\alpha\sigma(a\chi_-(C)+b))^{\alpha\sigma(a\chi_-(C)+b)} + 5) + \frac{2}{9}(1+\frac{3a'}{a})) - \ln\frac{\mathrm{Vol}(M)}{2\pi}\big).\end{aligned}$$

When $\chi_-(C)$ becomes very large, the dominant expression in the right hand side of last inequality behaves like $\alpha\sigma a\chi_-(C)\ln\chi_-(C)$. In fact, an explicit calculation shows that if $\chi_-(C) \geq c_2$ and $\frac{\mathrm{Vol}(M)}{2\pi} \geq 1$, then

$$\begin{aligned}& \ln(72\epsilon(a\chi_-(C)+b)(\frac{\chi_-(C)+4}{2}(4\alpha^2\sigma^2(a\chi_-(C)+b)^2 \\ & (\sqrt{2}\alpha\sigma(a\chi_-(C)+b))^{\alpha\sigma(a\chi_-(C)+b)} + 5) + \frac{2}{9}(1+\frac{3a'}{a})) - \ln\frac{\mathrm{Vol}(M)}{2\pi}) \\ \leq\ & 4\alpha\sigma a\chi_-(C)\ln\chi_-(C).\end{aligned}$$

(see [R2, Chapter 1] for explicit details and calculations).

Thus, if $k_2 := 4\alpha\sigma a$, if $\chi_-(C) \geq c_2$, assuming that $k_2\,\chi_-(C)\ln\chi_-(C) \leq \ln\ln d/q$ implies that the sufficient condition above is satisfied, hence conclusions of the Pattern Proposition D. □

Claim 3. *If $c_1 \leq \chi_-(C) \leq c_2$, conclusions of Proposition D still hold if $k_3\,\chi_-(C)$ $\ln\chi_-(C) \leq \ln\ln\frac{d}{q}$, with*

$$k_3 = \frac{1}{c_1\ln c_1}\ln\Big(36\epsilon(ac_2+b)(c_2+4)\left(4(\alpha\sigma)^2(ac_2+b)^2(\sqrt{2}\alpha\sigma(ac_2+b))^{\alpha\sigma(ac_2+b)}+5\right)$$
$$+16\epsilon(ac_2+b)(1+\frac{3a'}{a})-\ln\frac{\mathrm{Vol}(M)}{2\pi}\Big).$$

Proof of claim 3.

As $\chi_-(C) \geq c_1$, it is the case where $r \geq 2D+1$, and we proceed as above, using like during the proof of claim 1 that one has the bounds $c_1 \leq \chi_-(C) \leq c_2$. □

Set $k := \max\{k_0, k_1, k_2, k_3\}$. It follows from the last three claims that if $k\,\chi_-(C)$ $\ln\chi_-(C) \leq \ln\ln\frac{d}{q}$, then conclusions of Proposition D hold. In particular, M virtually fibers over the circle and the m embedded surfaces in M' are fibers. Furthermore, the constant $k = k(\epsilon, \mathrm{Vol}(M))$ depends only on $\epsilon = \mathrm{Inj}(M)/2$ and the volume $\mathrm{Vol}(M)$, and its expression is explicit. This ends the proof of theorem A. □

Proof of corollary 0.2.

If C_j is a handlebody and $\mathrm{Vol}(C_j) \geq \mathrm{Vol}(M)d/q$, the proof of theorem A shows that one can construct in C_j surfaces that are fibers. In particular, the handlebody C_j contains incompressible surfaces, which is a contradiction. □

2.2. **Proof of Proposition B: finding nested surfaces.**

Let C be the compression body of M' obtained in lemma 2.2. The boundary of C is a union of pseudo-minimal surfaces, the genus of each boundary component is at most g, and $\mathrm{Vol}(C) \geq \mathrm{Vol}(M)\frac{d}{q}$.

2.2.1. *Some modifications of the compression body.*

Instead of the manifold with boundary C, we need to work in a complete Riemannian manifold of sectional curvature at most -1. This is the aim of the following lemma.

Lemma 2.10. *Up to modifying the compression body C without significant changes of volume, one can add collars to boundary components of C to obtain a (non compact) Riemannian 3-manifold homeomorphic to the interior of C. Let $\rho > 0$ as small as desired. This manifold is equipped with a complete metric of sectional curvature at most $-1+\rho$, which coincides on C with the induced metric given by the embedding of C in M'.*

Proof of lemma 2.10.

We start with the compression body C embedded in M' and its non complete induced hyperbolic metric. If necessary, we need to modify slightly the compression body C in order that each boundary component of C has its intrinsic sectional curvature at most -1.

That is not a problem for boundary components which are minimal surfaces, as their sectional curvature is always at most this of the ambient hyperbolic manifold, i.e. -1.

If a boundary component of C is the boundary of a small neighborhood of a non-orientable minimal surface, we can choose this neighborhood small enough in order that the sectional curvature of this pseudo minimal surface is bounded from above by $-1+\rho$, with $\rho > 0$ as small as desired. This is a consequence of the continuity of the intrinsic sectional curvature in a neighborhood of the minimal surface (because of the continuity of the Gauss curvature).

If $\partial_+ C$ is the boundary of a regular neighborhood $N(S)$ of a non orientable minimal surface S with a small tube attached vertically in the I-bundle structure, we have to consider two cases. If this tube $\mathbb{D}^2 \times I$ belongs to the compression body C, we can remove it. More precisely, we compress C along the disc $\mathbb{D}^2 \times \{1/2\}$ to get a new compression body of lower genus. We lose the tube $\mathbb{D}^2 \times I$ during this process, but as we can make this tube as small as we like, this compression does not change significantly the volume of the compression body. As the positive boundary of this new compression body C' is then the boundary of a small regular neighborhood of the minimal non orientable surface S, the previous argument shows that we can suppose that the intrinsic curvature of $\partial_+ C'$ is at most $-1+\rho$.

Otherwise, in the second case the tube $\mathbb{D}^2 \times I$ lies outside C, meaning that C is contained in $N(S)$. We can then collapse the small tube to an arbitrarily small geodesic arc γ in the regular neighborhood of the minimal non orientable surface S. The positive boundary $\partial_+ C$ becomes the union of the boundary of $N(S)$ and the arc γ. As before, we can suppose that the sectional curvature of the surface $\partial N(S)$ is at most $-1+\rho$.

For each boundary component F of C, we glue a copy of $F \times [0, +\infty)$ equipped with a warped product metric. A computation of the sectional curvature of a warped product (see for example Bishop and O'Neil [BO, p. 26]) shows that as we start from a surface F with sectional curvature at most $-1+\rho$, there exists a warped product metric on $S \times [0, +\infty)$ such that this Riemannian manifold is complete of sectional curvature at most $-1+\rho$. If we are in the last case where F is the boundary of a regular neighborhood $N(S)$ of a minimal non orientable surface S with a small tube attached, and this tube is lying outside C, then we forget the arc γ for this construction and we just glue a copy of $\partial N(S) \times [0, +\infty)$ with a Riemannian metric of curvature at most $-1+\rho$. We perturb slightly this metric to make it smooth, and we obtain thus a complete Riemannian metric for the interior of C (union γ if we are in this last case) with sectional curvature at most $-1+\rho$, which extends the induced metric given by the embedding of C in M'. □

The boundary surfaces of C are pseudo minimal surfaces. This fact is crucial as one can homotop a minimal surface of genus g to a simplicial surface not too far away in C. This can be done by the following lemmas.

Definition 2.11. *Let $\epsilon > 0$. The (intrinsic) ϵ-diameter of a Riemannian surface S is the minimal number of balls of radius ϵ for the metric of S needed to cover the surface S.*

Lemma 2.12. *Suppose S is a pseudo minimal surface in a closed Riemannian 3-manifold N of sectional curvature at most -1. Let $\epsilon \le \mathrm{Inj}(N)$ and*

$$a' = 6\left(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\frac{\epsilon}{4})}\right).$$

Then the surface S has ϵ-diameter at most $a'\,|\chi(S)|$, and it admits a one-vertex triangulation in which each edge has length at most $2\epsilon a'\,|\chi(S)|$.

Proof of lemma 2.12.

This lemma is a direct consequence of [Mah, Lemma 4.2 p. 2249] and [L, Proposition 6.1] in the case the surface S is minimal and orientable, and we can take $a'/6$ instead of a'. If S is minimal, but not orientable, its homology class $[S]$ is non zero in $H_2(N, \mathbb{Z}/2\mathbb{Z})$. By Poincaré's duality, it corresponds to a non-trivial element $\alpha \in H^1(N, \mathbb{Z}/2\mathbb{Z})$. As the homology class of the double cover of S can be represented by the boundary of a small regular neighborhood of the non-orientable surface S, we have $2[S] = 0$ in $H_2(N, \mathbb{Z})$. If we take the double cover N' of N corresponding to the kernel of α, the surface S lifts to a minimal orientable surface S'. We can apply the stronger version of lemma 2.12, and bound the ϵ-diameter of S' by $a'/6\,|\chi(S')| = a'/6 \times 2\,|\chi(S)| = a'/3\,|\chi(S)|$, and the length of a one-vertex triangulation for S' by $2\epsilon a'/3\,|\chi(S)|$. As those numbers bound also from above the ϵ-diameter and the length of a one-vertex triangulation of S, this proves the lemma for a minimal non orientable surface, with $a'/3$ instead of a'.

If the surface S is just pseudo minimal, it is the boundary of an arbitrarily small regular neighborhood of a minimal surface S'. As the diameter and the length of the edges of a one-vertex triangulation are at most $a'/3\,|\chi(S')|$ and $2\epsilon a'/3\,|\chi(S')|$, with $|\chi(S)| \le 2\,|\chi(S')|$, this ends the proof of lemma 2.12. □

As from lemma 2.10, the boundary components of C are pseudo minimal surfaces, lemma 2.12 applies to bound from above the ϵ-diameter and the length of the edges of a one-vertex triangulation those surfaces. Furthermore, if some geodesic arcs need to be added, they can be made as small as necessary.

Recall some definitions and results of [Mah, Sections 2 et 3].

Definition 2.13. *A **coned n-simplex** in a compact Riemannian manifold N of sectional curvature at most -1 is defined inductively as follows. A **coned 1-simplex** $\Delta^1 = (v_0, v_1)$ is a constant speed geodesic from v_0 to v_1. The speed is allowed to be zero, and in this case the 1-simplex degenerates to the point v_0. A **coned n-simplex** is a map $\phi : \Delta^n \to N$ such that $\phi_{|\Delta^{n-1}}$ is a coned $(n-1)$-simplex and for all $x \in \Delta^{n-1}$, $\phi_{|\{tx+(1-t)v_n\,|\,t\in[0,1]\}}$ is a constant speed geodesic. The map ϕ depends on the order of the vertices $(v_0, \ldots, v_n)$ and its image may not be embedded in N, just immersed.*

*A **simplicial surface** is a continuous map $\phi : S \to N$ where S is a triangulated surface, such that the restriction of the map ϕ to each triangle Δ of S is a coned 2-simplex.*

Lemma 2.14. *Let N be a complete Riemannian manifold with sectional curvature at most -1. Suppose that T is a connected and orientable pseudo-minimal surface in N with diameter bounded from above by $\mathcal{N}$ and admitting a one-vertex triangulation in which the length of the edges is at most $\mathcal{N}'$. Then T can be homotoped to a* *simplicial surface T' with diameter at most $2\mathcal{N}'$ and such that any point $x \in T$ and $x' \in T'$ are at distance at most $\mathcal{N} + \mathcal{N}'$ from each other. Furthermore, every point of T' is at distance at most $\mathcal{N}'$ from the vertex of the one-vertex triangulation of T.*

Proof of lemma 2.14.

Let v be the vertex of the one-vertex triangulation of T. First, we homotop each edge e of the triangulation of T to its closed length-minimizing geodesic representative e' in $\pi_1(N, v)$. If the homotopy class of e is zero (meaning that the surface T is compressible in N), we homotop e to the degenerate constant speed geodesic $\{v\}$.

Let $\mathcal{T}$ be a triangle in T. If all edges of $\mathcal{T}$ are null-homotopic, $\mathcal{T}'$ is the degenerate 2-simplex corresponding to $\{v\}$. If at least one edge of $\mathcal{T}$ corresponds to a null-homotopic curve, then we build a simplicial triangle $\mathcal{T}'$ containing the closed geodesic at v corresponding to this edge, coned from v. More precisely, the 1-skeleton of $\mathcal{T}'$ is the union of closed geodesics corresponding to its non homotopically trivial edges. To build the 2-skeleton, we choose one of those non trivial edges and we cone v to this edge with constant speed geodesics. In this case, each point in $\mathcal{T}'$ is at distance at most $\mathcal{N}'/2$ from the vertex v (as it is on a closed geodesic of length at most $\mathcal{N}'$).

If all the edges of $\mathcal{T}$ are non zero in $\pi_1(N, v)$, they correspond to three non trivial closed geodesics c_1, c_2 and c_3, starting and ending at the point v. In the universal cover $\widetilde{N}$ of N, we can choose lifts a_1, a_2 and a_3 of c_1, c_2 and c_3 that bound a totally geodesic triangle $\mathbb{T}$. By definition, the covering projection maps a_i to c_i for $i = 1, 2, 3$. The simplicial triangle $\mathcal{T}'$ corresponding to $\mathcal{T}$ is the image under the covering projection of the totally geodesic triangle $\mathbb{T}$ in $\widetilde{N}$. As the covering projection is an isometry from the interior of $\mathbb{T}$ to the interior of $\mathcal{T}'$, and as each point in the interior of $\mathbb{T}$ lies at distance at most $\mathcal{N}'$ (which is an upper bound for the maximum of the lengths of the sides a_1, a_2 and a_3), each point x' in the interior of $\mathcal{T}'$ lies at distance at most $\mathcal{N}'$ from the vertex v.

Therefore, starting from the triangulated surface T, we can build a simplicial surface T' such that v is the only vertex of the simplicial structure of T' and each point x' in T' is at distance at most $\mathcal{N}'$ from v. In particular, the diameter of T' is at most $2\mathcal{N}'$. As the diameter of T is at most $\mathcal{N}$ and that v is also a point of T, for any points $x' \in T'$ and $x \in T$, we have:

$$\begin{aligned} d(x, x') &\leq d(x, v) + d(x', v) \\ &\leq \operatorname{diam}(T) + \mathcal{N}' \\ &\leq \mathcal{N} + \mathcal{N}', \end{aligned}$$

which proves lemma 2.14. □

Given a spine Γ for the compression body C which is a union of simplicial surfaces corresponding to $\partial_- C$ joined by geodesic arcs, there exists a simplicial surface homotopic to this spine, by a homotopy that does not sweep out too much volume. More precisely, this follows from the next lemma, proven in [Mah, Lemma 4.3 p. 2250].

Lemma 2.15. [Mah, Lemma 4.3]

Let $\sigma_1, \ldots, \sigma_n$ be a collection of simplicial surfaces, with basepoints v_i in N, a complete Riemannian 3-manifold of sectional curvature at most -1. Join the basepoint v_1 to each of the other basepoints by at least one geodesic arc to obtain a geodesic 2-complex Γ homotopic to a surface of genus g. Then, there exists a homotopy of Γ to *a simplicial surface* Σ_0 *of genus* g*, and this homotopy sweeps out a volume of at most* $3(2g+2)V_3$*, where* V_3 *is the maximal volume of an ideal hyperbolic tetrahedron.* □

Recall that $\epsilon \leq \mathrm{Inj}(M)/2$, and $a' = 6\left(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\epsilon/4)}\right)$. The constant 2ϵ is a uniform lower bound for the injectivity radius of any finite cover of M. In particular, $\mathrm{Inj}(M') \geq 2\epsilon$.

Lemma 2.3. *Let* $k_0 = \max\left\{\frac{\ln(4(2\epsilon a'+1))}{2\ln 2}, 1 + \frac{\ln(1+\ln(12V_3/\mathrm{Vol}(M)))}{2\ln 2}\right\}$.

If $k_0\, \chi_-(C) \ln \chi_-(C) \leq \ln\ln d/q$ *et* $\mathrm{Vol}(M) \geq \pi/2$*, then applying lemmas 2.14 and 2.15 to replace the boundary surfaces of* C *to simplicial surfaces, one obtains a new compression body* C'' *corresponding to the region between the simplicial surfaces that is swept out in a degree one manner. The volume of* C'' *satisfies:*

$$\mathrm{Vol}(C'') \geq \frac{1}{4}\mathrm{Vol}(C) \geq \frac{\mathrm{Vol}(M)d}{4q}.$$

Proof of lemma 2.3.

Let $\partial_- C = T_1 \cup \ldots \cup T_n$ be the components of $\partial_- C$, with $g(T_1) + \ldots + g(T_n) \leq g(\partial_+ C)$. As in lemma 2.14, replace $\partial_+ C =: S_0$ and $\partial_- C = S_1 \cup \ldots \cup S_n$ by simplicial surfaces S_0' and $T_1' \cup \ldots \cup T_n'$, close to the previous surfaces. If $v_j \in T_j$ is the vertex of the one-vertex triangulation of T_j, then lemmas 2.12 and 2.14 show that every point of T_j' lies at distance at most $\mathcal{N}' = 2\epsilon a' \left|\chi(T_j)\right| \leq 2\epsilon a' \chi_-(C)$ from v_j. Thus, each new surface T_j' is contained in the ball of center v_j and radius $2\epsilon a' \chi_-(C)$. Let C' is the new compression body obtained by taking the region between the simplicial surfaces T_j' swept out in a degree one manner. The modification of volume is at most

$$\begin{aligned}
\mathrm{Vol}(C') &\geq \mathrm{Vol}(C) - \sum_{j=0}^{n} \mathrm{Vol}(B(v_j, 2\epsilon a' \chi_-(C))) \\
&\geq \mathrm{Vol}(C) - (g(C)+1)\mathrm{Vol}(B_{\mathbb{H}^3}(2\epsilon a' \chi_-(C))) \\
&\geq \mathrm{Vol}(C)\left(1 - \frac{\pi(\chi_-(C)+4)(\sinh(4\epsilon a' \chi_-(C)) - 4\epsilon a' \chi_-(C))}{2\mathrm{Vol}(M)d/q}\right)
\end{aligned}$$

Let us show that $\mathrm{Vol}(C') \geq \mathrm{Vol}(C)/2$, which is the same as proving that

$$\frac{\pi(\chi_-(C)+4)(\sinh(4\epsilon a' \chi_-(C)) - 4\epsilon a' \chi_-(C))}{2\mathrm{Vol}(M)d/q} \leq \frac{1}{2}.$$

It suffices to prove that $\ln \frac{\pi(\chi_-(C)+4)(\sinh(4\epsilon a' \chi_-(C)) - 4\epsilon a' \chi_-(C))}{\mathrm{Vol}(M)d/q} \leq 0$. But

$$\begin{aligned}
\ln\left(\frac{\pi}{\mathrm{Vol}(M)d/q}(\chi_-(C)+4)(\sinh(4\epsilon a' \chi_-(C)) - 4\epsilon a' \chi_-(C))\right) &\leq \\
\ln\left(\frac{\pi}{2\mathrm{Vol}(M)}\frac{(\chi_-(C)+4)\exp(4\epsilon a' \chi_-(C))}{d/q}\right) &\leq \\
\ln\left(\frac{\pi}{2\mathrm{Vol}(M)}\right) + \ln\left((\chi_-(C)+4)\exp(4\epsilon a' \chi_-(C))\right) - \ln(d/q) &\leq \\
\ln(\chi_-(C)+4) + 4\epsilon a' \chi_-(C) &- \ln(d/q),
\end{aligned}$$

as by assumption, $\mathrm{Vol}(M) \geq \pi/2$.

As for every $x \geq 2$, $\ln(x+4) \leq 2x$, it suffices to prove that

$$(2+4\epsilon a')\chi_-(C) \leq \ln(d/q),$$

which is the same as

$$\ln(2+4\epsilon a') + \ln \chi_-(C) \leq \ln\ln(d/q).$$

Now by assumption, $\frac{\ln\ln d/q}{\chi_-(C)\ln\chi_-(C)} \geq k_0 \geq \frac{\ln(4(2\epsilon a'+1))}{2\ln 2}$. Thus,

$$\begin{aligned} \ln\ln d/q &\geq \frac{\ln(4(2\epsilon a'+1))}{2\ln 2}\chi_-(C)\ln\chi_-(C) \\ &\geq \frac{\ln 2 + \ln(2+4\epsilon a')}{2\ln 2}\chi_-(C)\ln\chi_-(C) \\ &\geq \frac{\chi_-(C)\ln\chi_-(C)}{2} + \frac{\ln(2+4\epsilon a')\chi_-(C)\ln\chi_-(C)}{2\ln 2} \\ &\geq \ln\chi_-(C) + \ln(2+4\epsilon a') \end{aligned}$$

as $\chi_-(C) \geq 2$, showing that $\mathrm{Vol}(C') \geq \mathrm{Vol}(C)/2$.

From lemma 2.15, the volume swept out by the homotopy between Γ and Σ_0 is at most $3(\chi_-(C)+4)V_3$. As the volume of C is at least $\mathrm{Vol}(M)d/q$ by lemma 2.2, the volume of what remains after cutting the metric completion of C' along Σ_0 and throwing off components containing the infinite products to obtain a new compression body C'' is at least

$$\mathrm{Vol}(C') - 3(\chi_-(C)+4)V_3 \geq \mathrm{Vol}(C)/2\left(1 - 3V_3(\chi_-(C)+4)\frac{2q}{\mathrm{Vol}(M)d}\right).$$

Therefore, it suffices to prove that $3V_3(\chi_-(C)+4)\frac{2q}{\mathrm{Vol}(M)d} \leq \frac{1}{2}$, or $\ln(\frac{12V_3(\chi_-(C)+4)}{\mathrm{Vol}(M)d/q}) \leq 0$, or also

$$\ln\left(\ln\frac{12V_3}{\mathrm{Vol}(M)} + \ln(\chi_-(C)+4)\right) \leq \ln\ln(d/q).$$

As $\chi_-(C) \geq 2$, $\ln(\chi_-(C)+4) \geq \ln 6 > 1$. Thus,

$$\begin{aligned} \ln\left(\ln\frac{12V_3}{\mathrm{Vol}(M)} + \ln(\chi_-(C)+4)\right) &= \\ \ln\left(\ln(\chi_-(C)+4)\left(1+\frac{\ln\frac{12V_3}{\mathrm{Vol}(M)}}{\ln(\chi_-(C)+4)}\right)\right) &= \\ \ln\ln(\chi_-(C)+4) + \ln\left(1+\frac{\ln\frac{12V_3}{\mathrm{Vol}(M)}}{\ln(\chi_-(C)+4)}\right) &\leq \ln\ln(\chi_-(C)+4) + \ln\left(1+\ln\frac{12V_3}{\mathrm{Vol}(M)}\right). \end{aligned}$$

As soon as $c \geq 2$, $\frac{\ln\ln(c+4)}{c\ln c} \leq 1$. Then,

$$\begin{aligned} \frac{\ln\left(\ln\frac{12V_3}{\mathrm{Vol}(M)} + \ln(\chi_-(C)+4)\right)}{\chi_-(C)\ln\chi_-(C)} &\leq \frac{\ln\ln(\chi_-(C)+4)}{\chi_-(C)\ln\chi_-(C)} + \frac{\ln\left(1+\ln\frac{12V_3}{\mathrm{Vol}(M)}\right)}{\chi_-(C)\ln\chi_-(C)} \\ &\leq 1 + \frac{\ln\left(1+\ln\frac{12V_3}{\mathrm{Vol}(M)}\right)}{2\ln 2}. \end{aligned}$$

Now, as $\frac{\ln\ln d/q}{\chi_-(C)\ln\chi_-(C)} \geq k_0 \geq 1+\frac{\ln(1+\ln(12V_3/\mathrm{Vol}(M))}{2\ln 2}$,

$$\frac{\ln\ln d/q}{\chi_-(C)\ln\chi_-(C)} \geq \frac{\ln\left(\ln\frac{12V_3}{\mathrm{Vol}(M)}+\ln(\chi_-(C)+4)\right)}{\chi_-(C)\ln\chi_-(C)},$$

and so $\mathrm{Vol}(C'') \geq \mathrm{Vol}(C)/4 \geq \mathrm{Vol}(M)\frac{d}{4q}$, which ends the proof of lemma 2.3. □

In the sequel, to simplify notations, we will still denote by C the new compression body C'' and we work in the closure of the region of C bounded by the two connected simplicial surfaces Σ_0 (corresponding to $\partial_- C$ union some arcs, and forming a spine for C), and Σ_1 corresponding to $\partial_+ C$.

2.2.2. *Sweepouts.*

Definition 2.16. *Let C be a compression body. Set $S=\partial_+C$. A* ***sweepout*** *of the compression body C is a 1-parameter family of surfaces $\{S_t\}_{t\in[0,1]}$ such that S_0 is a spine of C, $S_1=S=\partial_+C$, for all $t\in(0,1]$ the surface S_t is homeomorphic to S, and the application $\Phi : S\times I\to C$ is of homological degree one.*

There exists a sweepout $\{S_t\}_{t\in[0,1]}$ of the compression body C such that $S_0=\Sigma_0$ and $S_1=\Sigma_1$. The sweepout surfaces S_t for $t>0$ are of interest in order to construct a long product in the compression body C. But, if we can control the diameter of a minimal surface in terms of its genus and the injectivity radius of the ambient manifold, we cannot control uniformly the diameter of all the sweepout surfaces S_t: there may appear some long and thin Margulis tubes, containing a closed geodesic of the surface with length less than the injectivity radius of M'.

To face this problem, we work with the notion of ϵ-diameter, for which non-connected surfaces with small diameter components are considered as ”small”. Recall the definition.

Definition 2.17. *Let $\epsilon>0$. The* ***(intrinsic) ϵ-diameter*** *of a non-necessarily connected surface F is the minimal number of balls of radius ϵ for the metric of F required to cover the surface F.*

If F is immersed in a Riemannian 3-manifold N, the ***ϵ-diameter*** *of F in N is the minimal number of 3-balls in N of radius ϵ for the metric of N required to cover F.*

Remark 2.18. *If F is immersed in a Riemannian 3-manifold N, the ϵ-diameter of F in N is always at most the intrinsic ϵ-diameter of F with respect to the induced metric.*

At this point, we recall the technique of Maher to construct from the original sweepout $\{S_t\}_{t\in I}$ of C what he calls a ”generalized sweepout” $\{\widehat{S}_t\}_{t\in I}$ in which the ϵ-diameter of good sweepout surfaces is uniformly bounded from above (see [Mah, Sections 2 and 3]).

The first step is to straighten the sweepout $\{S_t\}_{t\in I}$ to a simplicial sweepout, using results of Bachman, Cooper and White [BCW]. We recall terminology and results stated in [Mah, Sections 2 and 3].

Definition 2.19. *A* ***simplicial sweepout*** *is a sweepout $\Phi : S\times I\to N$ such that each surface S_t is mapped to a simplicial surface with at most $4g(S)$ triangles, and at most one vertex of angle sum less than 2π.*

The following lemma ensures that we can homotop the sweepout $\{S_t\}_{t\in[0,1]}$ between Σ_0 and Σ_1 to a simplicial sweepout. It is an improvement of [BCW, Theorem 2.3], and is proven by Maher [Mah, Lemma 2.5 p. 2236].

Lemma 2.20. [Mah, Lemma 2.5]
Let N be a closed orientable Riemannian manifold of sectional curvature at most -1. If Σ_0 and Σ_1 are simplicial surfaces with one-vertex triangulations, which are homotopic by a homotopy $\Phi : S \times I \to N$, then there exists a simplicial sweepout $\Phi' : S \times I \to N$ homotopic to Φ relative to $S \times \partial I$. □

Therefore, we can suppose that the sweepout in the compression body C is simplicial between the simplicial surfaces $\Sigma_0 = S_0$ and $\Sigma_1 = S_1$.

After getting this simplicial sweepout in the compression body C, the next step will be to get rid of the long and thin tubes in the sweepout surfaces to get a "generalized sweepout" in which the ϵ-diameter of all sweepout surfaces is uniformly bounded from above.

Definition 2.21. [Mah, Definition 3.2 p. 2237]
*Let N be a compact, connected and oriented 3-manifold. A **generalized sweepout** of N is given by a triple (Σ, f, h), where Σ is an orientable and compact 3-manifold, the map $h : \Sigma \to \mathbb{R}$ is a Morse function, constant on each boundary component of Σ and such that for all but finitely many $t \in \mathbb{R}$, the set $f^{-1}(\{t\})$ is an immersed surface. Moreover, it is required that $f : (\Sigma, \partial\Sigma) \to (N, \partial N)$ is of homological degree one.*

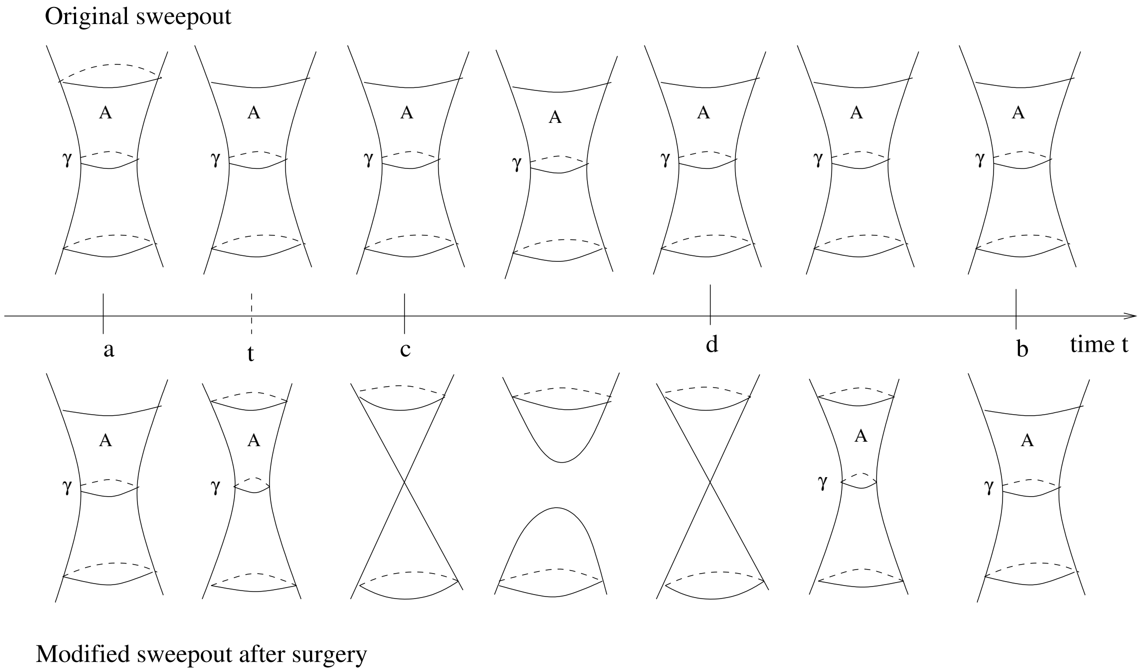
Of course, an ordinary sweepout $\Phi : S \times I \to N$ is an example of generalized sweepout: the Morse function $h : S \times I \to \mathbb{R}$ is given by the projection to the factor I, and for all $t \in (0,1)$, $h^{-1}(\{t\}) = S_t$ is an immersed surface in N. By definition of a sweepout, $\Phi : (S \times I, S \times \partial I) \to (N, \partial N)$ is of homological degree one.

For all $x \in \Sigma$, we think of $h(x) = t$ as the time coordinate. A generalized sweepout can be seen as a one-parameter family of immersed surfaces S_t with singular times t where the genus or the number of components of those surfaces change.

Starting from the simplicial sweepout $\{S_t\}_{t\in I}$ of C, we wish to obtain a generalized sweepout in which each sweepout surface has bounded ϵ-diameter. To this aim, we follow Maher and introduce the notion of **surgery** of a generalized sweepout.

Definition 2.22. *One can obtain from a generalized sweepout given by (Σ, f, h) a new generalized sweepout (Σ', f', h') by an operation called a **surgery of generalized sweepouts**, as described below. (In fact, it is a special case of a more general construction called a modification of generalized sweepout, described by Maher and Rubinstein in* [MR]*.)*

Let (Σ, f, h) be a generalized sweepout of a 3-manifold N. Take a submanifold in Σ of the form $A \times [a,b]$ where $0 < a < b < 1$ and A is an annulus in the surfaces S_t for $t \in [a,b]$. We do $(0,1)$ surgery to this solid torus $A \times [a,b]$ in the following way: choose two times c and d such that $a < c < d < b$. Take a core geodesic γ for the annulus A in the surface S_a. Shrink this geodesic : it gets shorter and shorter, until it collapses to a point in a modification S'_c of the surface S_c. For all $t \in (c,d)$, replace the surface S_t by the surface S'_t obtained from S_t by surgering along γ, i.e. we cut S_t along γ and cap off the resulting surface with two discs. Do this in a smooth way, such that the two discs of S'_t get closer and shrink to a single point at time d. The new surface S'_d is then singular, with a singular point corresponding to the two former discs. This point becomes again the geodesic γ that gets larger when $t \in (d, b]$ increases. Do this in such a way that you do not modify S_a nor S_b nor $\partial A \times [a, b]$. In this way, we get a new generalized sweepout (Σ', f', h'), where Σ' is obtained by replacing $A \times [a, b] \subset \Sigma$ by the new manifold where S_t is replaced by S'_t for all $t \in [a, b]$. Let us denote by T the small tube in N bounded by A, where the surgeries take place. The new maps (f', h') coincide with (f, h) outside $T \times [a, b]$ and in $\partial(T \times [a, b])$. As the modification of the sweepout takes place in a proper compact subset of N, there exists a point x in the interior of $N \setminus (T \times [a, b])$. As the map f is not modified in a neighborhood of $f^{-1}(\{x\})$, the homological degree of f' is the same as the homological degree of f, so it is still equal to one. Thus the triple (Σ', f', h') is still a generalized sweepout.



Set $\boldsymbol{\mathcal{N}_+ = \mathcal{N}(\partial_+ C)} = \{x \in C \, , \, d(x, \Sigma_1) \leq \epsilon/2\}$.
Let $K = 4\left(3 + 1/\sinh^2(\epsilon/8)\right) g(C) - 10$ and $K' := 2a' \, |\chi(\partial_+ C)|$.

Proposition 2.23. *Let $\mu > 0$. There exists a constant $\eta > 0$ as small as wanted, depending only on the simplicial sweepout $\{S_t\}_{t \in I}$ and μ, and a finite sequence of surgeries of the simplicial sweepout giving a generalized sweepout $\{\widehat{S}_t\}_{t \in I}$ of C and satisfying the following properties.*

For every regular time $t \in [\eta, 1 - \eta]$, the intrinsic ϵ-diameter of every component of $\widehat{S}_t$ disjoint from $\mathcal{N}_+$ is less than or equal to K. In every case, the diameter of any connected component of $\widehat{S}_t$ in the compression body C is at most $\epsilon(1 + 2K' + 2K)$. For $t \geq 1 - \eta$, each point on the surface $\widehat{S}_t$ lies at distance at most $\epsilon K'$ from Σ_1. For $t \leq \eta$, any point on one of the original surfaces S_t is at distance at most $\mu/2$ from Σ_0. Furthermore, for each regular time t, the surface $\widehat{S}_t$ is homotopic to an embedded surface obtained from $\partial_+ C$ by surgeries.

Proof of proposition 2.23.

In order to prove this proposition, the general idea is to cut the simplicial surfaces S_t along curves that are too short, namely of length less than or equal to ϵ, and to replace them by ruled discs, to get rid of long and thin tubes. This is described by Maher in the third section of [Mah, p. 2238 to p. 2245]. We recall here the proof, and we bring some precisions when they appear to be necessary.

Let t be a regular time. The simplicial surface S_t is composed of ruled triangles with at most one vertex of angle sum less than 2π, denoted by v_t. Let $\overline{S}_t$ be the completion of the universal cover $\widetilde{S}_t$ of $S_t \setminus \{v_t\}$. As it is a metric 2-complex composed of triangles of curvature at most -1 and with vertices whose cone angles are all greater than or equal to 2π, $\overline{S}_t$ is a complete CAT(-1) geodesic metric space. Those spaces satisfy some useful properties, see [BH] and [Mah, p. 2239].

Let α be a homotopy class in $S_t \setminus \{v_t\}$. To α, we can associate a covering transformation of the universal cover of $S_t \setminus \{v_t\}$, which can be extended to an isometry of $\overline{S}_t$. As the completion of a fundamental domain for $S_t \setminus \{v_t\}$ is compact, this isometry cannot be parabolic. Thus it is hyperbolic or elliptic. Let $\overline{\gamma}_t$ be the set of points in $\overline{S}_t$ which are moved the least distance by the isometry. This is a geodesic if the isometry is hyperbolic, or isolated points if the isometry is elliptic. We denote by γ_t the projection of $\overline{\gamma}_t$ under the covering map, in the sense that if $\overline{\gamma}_t$ is a geodesic and does not meet $\overline{S}_t \setminus \widetilde{S}_t$, γ_t is a closed piecewise geodesic homotopic to α in $S_t \setminus \{v_t\}$. If $\overline{\gamma}_t$ is a geodesic meeting $\overline{S}_t \setminus \widetilde{S}_t$, then we perturb it slightly and in an equivariant way such that it avoids $\overline{S}_t \setminus \widetilde{S}_t$ and its projection γ_t in $S_t \setminus \{v_t\}$ is an embedded closed curve in the homotopy class of α. Finally, if $\overline{\gamma}_t$ is a set of points, it corresponds to the constant loop γ_t of length zero and equal to the point v_t. By extension, in any case we will call γ_t **the geodesic representative of** α. Notice that γ_t is an embedded curve or a point in C.

As the negatively curved triangles that compose the surfaces S_t vary continuously with the time t, we can expect the geodesic representatives γ_t to vary also continuously. This is proven by Maher [Mah, Lemma 3.4 p. 2240].

Lemma 2.24. [Mah, Lemma 3.4]

Let γ be a simple closed curve in $S \setminus \{v\}$ where v is a point of S mapping to the point v_t for each time t. Then the geodesic representatives γ_t of γ vary continuously with t. □

Definition 2.25. *A geodesic representative γ_t is said to be* ***short*** *if its length is less than or equal to ϵ.*

For all t, let Γ_t be the set of short geodesic representatives of S_t. This is a finite set and it is not empty, as the geodesic representative of the homotopy class of the loop around v_t has length zero.

Let γ_t be a short geodesic representative. Pick up a connected component $\widetilde{\gamma}_t$ of $\overline{\gamma}_t$, the preimage of γ_t in $\overline{S}_t$. Choose an orientation for $\widetilde{\gamma}_t$ so that the distance function from $\widetilde{\gamma}_t$ has a well defined sign. In the special case where the length of γ_t is zero, the distance from $\widetilde{\gamma}_t$ will always be non negative. If $[p, q]$ is an interval of $\mathbb{R}$, let $\widetilde{N}_{[p,q]}(\widetilde{\gamma}_t)$ be the set of points $x \in \overline{S}_t$ such that $p \leq d(x, \widetilde{\gamma}_t) \leq q$. Let $N_{[p,q]}(\gamma_t)$ be the image in S_t of $\widetilde{N}_{[p,q]}(\widetilde{\gamma}_t)$ under the covering projection. If the interval is the single point $\{r\}$, we will denote this neighborhood by $N_{[r]}(\gamma_t)$.

Definition 2.26. *Let $\mathcal{A}(\gamma_t)$ be the maximal neighborhood $N_{[p,q]}(\gamma_t)$ such that for every length $r \in [p, q]$, $N_{[r]}(\gamma_t)$ is an embedded simple curve of length at most ϵ. The set $\mathcal{A}(\gamma_t)$ is called the* ***annular neighborhood*** *of γ_t.*

Define $\mathcal{E}(\gamma_t) = N_{[p+\epsilon/2,q-\epsilon/2]}(\gamma_t)$ to be the ***surgery neighborhood*** *corresponding to γ_t, with the convention that $\mathcal{E}(\gamma_t)$ is the empty set if $q-p<\epsilon$. This neighborhood is the subset of $\mathcal{A}(\gamma_t)$ corresponding to the union of all curves $N_{[r]}(\gamma_t)$ lying at distance at least $\frac{\epsilon}{2}$ from the boundary of $\mathcal{A}(\gamma_t)$.*

As the annular neighborhood $\mathcal{A}(\gamma_t)$ contains $\gamma_t = N_{[0]}(\gamma_t)$, it is not empty. The annular neighborhood and the surgery neighborhood vary continuously with t, but the surgery neighborhood $\mathcal{E}(\gamma_t)$ can be empty, and it does not necessarily contain the geodesic representative γ_t.

The following lemma is proven by Maher in [Mah, Lemma 3.7 p. 2242].

Lemma 2.27. [Mah, Lemma 3.7]

If α_t and β_t are short geodesic representative of distinct homotopy classes in $S_t\backslash v_t$, then their surgery neighborhoods $\mathcal{E}(\alpha_t)$ and $\mathcal{E}(\beta_t)$ are disjoint. □

We notice that this lemma implies that for each time t, there are at most $2g-1$ surgery neighborhoods, where g is the genus of the sweepout surface S_t.

Lemma 2.27 allows us to do surgeries on the sweepout surfaces S_t to get a generalized sweepout where the diameter of the thin tubes can be controlled. More precisely, the idea is to remove the surgery neighborhoods from the sweepout surfaces each time it is possible, to get a new generalized sweepout $(\widehat{S}_t)_{t\in I}$. We describe this construction in detail.

Let $\mathcal{E}(\gamma_t)$ be a surgery neighborhood, and $[a,b]$ a maximal time interval on which $\mathcal{E}(\gamma_t)$ is not empty. We have $0\le a\le b\le 1$. First, suppose that $0<a\le b<1$, i.e. that $[a,b]$ is contained in the interior of I. Then $\mathcal{E}(\gamma_a)$ and $\mathcal{E}(\gamma_b)$ are two embedded simple curves $N_{[r_a]}(\gamma_a)$ and $N_{[r_b]}(\gamma_b)$ for lengths r_a and r_b, and the union of the surgery neighborhoods $\mathcal{E}(\gamma_{[a,b]}) = \{\mathcal{E}(\gamma_t)\,,\, t\in[a,b]\}$ is a solid torus in Σ, on which we wish to do a surgery of generalized sweepouts. When it is possible, we follow Maher's construction.

There is a difficulty here, as the surgery of generalized sweepouts described above is possible only if the geodesic γ_t bounds an immersed disc in the compression body C. Therefore, we need to make a distinction between two cases of surgery neighborhoods.

Definition 2.28. *A* ***persistent surgery neighborhood*** *of S_t is a surgery neighborhood $\mathcal{E}(\gamma_t)$ for which the corresponding geodesic γ_t is not homotopically trivial in C.*

Lemma 2.29. *Let $t\in(0,1)$ and $\mathcal{E}(\gamma_t)$ be a persistent surgery neighborhood for S_t.*

If the corresponding surgery curve γ_t is homotopically trivial in C, then $\mathcal{E}(\gamma_t)$ is entirely contained in $\mathcal{N}_+$.

In particular, for every points x and y in the union of the persistent surgery neighborhoods of S_t, the distance in C between x and y is at most $\epsilon + \operatorname{diam}(\Sigma_1) \le \epsilon(1+2K')$.

Proof of lemma 2.29.

For each $r\in[p,q]$, as the curve $N_{[r]}(\gamma_t)$ is of length at most $\epsilon < \operatorname{Inj}(M')$, it is null-homotopic in M' and is contained in a hyperbolic 3-ball B, isometrically embedded in M' and of diameter $\epsilon/2$.

The curve γ_t is not homotopically trivial in C and $N_{[r]}(\gamma_t)$ is homotopic to γ_t, so $B\cap\partial C\neq\emptyset$. By assumption (1) of theorem A, the negative boundary $\partial_- C$

is a union of incompressible surfaces. Necessarily, $B \cap \partial_+ C \neq \emptyset$, which, with the simplicial surfaces, implies $B \cap \Sigma_1 \neq \emptyset$. Thus, as the curve $N_{[r]}(\gamma_t)$ is contained in B and that B intersects Σ_1, each point of $N_{[r]}(\gamma_t)$ is at distance at most $\epsilon/2$ from Σ_1. It follows that the surgery neighborhood $\mathcal{E}(\gamma_t)$ is entirely contained in $\mathcal{N}_+$.

Noticing then that the diameter of $\mathcal{N}_+$ in the manifold C is at most $\epsilon + \mathrm{diam}(\Sigma_1) \leq \epsilon(1+2K')$ suffices to prove lemma 2.29. □

Remark 2.30. *Let $\mathcal{E}(\gamma_{t_0})$ be a surgery neighborhood for a given time $t_0 \in (0,1)$ and $[a,b]$ a maximal time interval on which $\mathcal{E}(\gamma_t)$ is not empty. If one of the surgery neighborhoods $\mathcal{E}(\gamma_{t_1})$ is persistent, the all surgery neighborhoods $\mathcal{E}(\gamma_t)$ are persistent for every time t in $[a,b]$.*

Indeed, for $t \in [a,b]$, the curves γ_t are homotopic, so if one of them is homotopically non trivial in C, it is the case for each of them. □

We can now carry on with the proof of proposition 2.23. Suppose that $\mathcal{E}(\gamma_t)$ is a non persistent surgery neighborhood. We describe the operation of surgery on $\mathcal{E}(\gamma_t)$, following Maher [Mah, p. 2242 and 2243].

Choose a continuous family of basepoints on the boundary of $\mathcal{E}(\gamma_t)$ such that the two basepoints agree at times a and b. We modify the sweepout by expanding times a and b to short intervals I_a and I_b on which the map is constant for the moment.

On the interval I_a, the curve $N_{[r_a]}(\gamma_a)$ is an embedded simple curve homotopic to the geodesic γ_t. As by assumption, γ_t is null-homotopic in C, the curve $N_{[r_a]}(\gamma_a)$ bounds an immersed disc in C.

In the interval I_a, we replace in a continuous way the curve $N_{[r_a]}(\gamma_a) = \mathcal{E}(\gamma_a)$ by a pair of ruled discs in C coned from the basepoint x_a. More precisely, as the metric of C is complete, the ruled disc is the union of all minimizing geodesics between each point of $N_{[r_a]}(\gamma_a)$ and the basepoint x_a. Its curvature is then at most -1. In the interval (a,b), we remove the surgery neighborhood $\mathcal{E}(\gamma_t)$ and we replace it by a pair of such ruled discs in C coned from the basepoints of the boundary of the surgery neighborhood. Finally, in the interval I_b we paste the discs together to come back to the original surface. This is a surgery of a generalized sweepout as defined above.

The following lemma directly follows from Lemmas 3.8 to 3.10 and is proven in [Mah, p. 2243 to 2246].

Lemma 2.31. *Suppose that all surgery neighborhoods $\mathcal{E}(\gamma_t)$ of S_t can be replaced by pairs of ruled discs as described above and let $\widehat{S}_t$ be the resulting surface. Then the ϵ-diameter of $\widehat{S}_t$ is at most $K = 4\left(3 + 1/\sinh^2(\epsilon/8)\right) g(C) - 10$.* □

The construction is now the following. Let $t \in [\eta, 1-\eta]$. From remark 2.30, the fact that a given surgery neighborhood $\mathcal{E}(\gamma_t)$ of S_t is persistent or not depends only on the maximal time interval $[a,b]$ on which it exists. If it is not persistent, then apply the surgery procedure described above. If it is persistent, leave it unchanged. Let $\widehat{S}_t$ be the new generalized sweepout surface obtained.

If none of the surgery neighborhoods $\mathcal{E}(\gamma_t)$ of S_t are persistent, they have been removed by the surgery procedure. From lemma 2.31, the intrinsic ϵ-diameter of $\widetilde{S}_t$ is at most K.

Otherwise, lemma 2.29 ensures that the diameter of the union of all persistent surgery neighborhoods in C is at most $\epsilon(1+2K')$, as they are contained in $\mathcal{N}_+$. As the intrinsic ϵ-diameter of each component of $\widehat{S}_t$ cut along persistent surgery neighborhoods is at most K from lemma 2.31, the diameter in the compression body C of each connected component of $\widehat{S}_t$ is at most $\epsilon(1+2K'+2K)$.

Furthermore, if a component of $\widehat{S}_t$ does not intersect $\mathcal{N}_+$, it does not contain any persistent surgery neighborhood, and its intrinsic ϵ-diameter is at most K from lemma 2.31.

An other difficulty is that Maher's construction does not take the boundaries of the time interval I into account. However, it may happen that $a = 0$ or $b = 1$, and in this case we might be obliged to modify the starting and finishing simplicial sweepout surfaces $S_0 = \Sigma_0$ and $S_1 = \Sigma_1$, which we want to avoid. Therefore, if this case occurs, we need to refine the construction to modify the simplicial sweepout in a small regular neighborhood of $S_0 \cup S_1$ in such a way that we do not modify the surfaces S_0 and S_1. As we will lose control on the diameter of the sweepout surfaces in this regular neighborhood, we have to choose it small enough in order that the sweepout surfaces we will pick up later to be some of the nested surfaces are not in this neighborhood. Thus we can control their diameter well. The constant μ has been introduced in assumptions of proposition 2.23 in order to take care of that, and its value will be defined later.

To finish to modify the original simplicial sweepout to get the desired generalized sweepout, there remains to consider the case when $a = 0$ or $b = 1$. If $\mathcal{E}(\gamma_0)$ is a non persistent surgery neighborhood and just a single closed curve, we can apply the previous construction, replacing the time 0 by an interval I_0 and doing surgery on this interval, without modifying the starting boundary surface $S_0 = \Sigma_0$. It works similarly if $\mathcal{E}(\gamma_1)$ is a single closed curve. The problem is when $\mathcal{E}(\gamma_0)$ or $\mathcal{E}(\gamma_1)$ have non empty interior and are non persistent surgery neighborhoods. As the two cases are similar, let us suppose for instance that the interior of $\mathcal{E}(\gamma_0)$ is not empty. As everything is continuous, there exists a maximal time $b \in (0, 1]$ such that $\mathcal{E}(\gamma_t)$ is a non empty and non persistent surgery neighborhood for all $t \in [0, b]$.

As the sweepout surfaces $(S_t)_{t\in I}$ vary continuously with t, there exists a constant $\eta > 0$ as small as we like, depending only on the original simplicial sweepout $(S_t)_{t\in I}$ and the choice of the point x_0 and the geodesic arc c, such that for every $t \in [0, \eta]$, each point of S_t lies at distance at most $\mu/2$ from $\Sigma_0 = S_0$, and that for every $t \in [1 - \eta, 1]$, each point in S_t is at distance at most $\epsilon K'/2$ from $\Sigma_1 = S_1$. If $b \leq \eta$, we do not modify the sweepout. Otherwise, if $\eta < b < 1$, we apply the surgery construction for all $t \in [\eta, b]$: we replace the surgery neighborhoods $\mathcal{E}(\gamma_t)$ by a pair of ruled discs coned from basepoints in the boundary of $\mathcal{E}(\gamma_t)$ in a continuous way. On the interval $[0, \eta]$, we replace the surgery neighborhoods by a pair of discs for t near η, that get pasted to the initial surgery neighborhood $\mathcal{E}(\gamma_0)$ as the time is decreasing to 0, not too far from the original surface S_t and in a continuous way. We can do this in such a way that it is still a modification of a generalized sweepout. If $b = 1$, do the same for all $t \in [1 - \eta, 1]$. As the diameter of the ruled discs is less than ϵ and $K'/2 \geq 1$, one can suppose that every point in $\widehat{S}_t$ is at distance at most $\epsilon K'$ from Σ_1 for all $t \in [1 - \eta, 1]$.

This ends the proof of proposition 2.23. □

2.2.3. *Sweepout surfaces and nested surfaces.*

To go on with the proof of proposition B, we now need a lemma to precisely determine the constant μ, which corresponds to the size of the collar neighborhood of S_0 one has to take into consideration. Set $K' := 2a'\chi_-(C)$. From lemma 2.12, the number $2\epsilon K'$ is an upper bound for the diameter of the simplicial surface Σ_1, identified with $\partial_+ C$. Let δ be the diameter of the compression body C.

Lemma 2.32. *There exists a point x_0 in the interior of C and lying at distance at least $(\frac{\delta}{2} - 2\epsilon K')$ from $\partial_+ C$.*

Proof of lemma 2.32.

Suppose that the lemma is false: for every point z in the interior of C, $\text{dist}(z, \partial_+ C) < \frac{\delta}{2} - 2\epsilon K'$. For every point z of C, the following inequality remains true: $\text{dist}(z, \partial_+ C) \leq \frac{\delta}{2} - 2\epsilon K'$. Take two points x and y in C such that $d(x, y) = \text{diam}(C) = \delta$. Then,

$$\begin{aligned} d(x,y) = \delta &\leq \text{dist}(x, \partial_+ C) + \text{diam}(\partial_+ C) + \text{dist}(y, \partial_+ C) \\ &\leq (\frac{\delta}{2} - 2\epsilon K') + 2\epsilon K' + (\frac{\delta}{2} - 2\epsilon K') \\ &\leq \delta - 2\epsilon K' < \delta, \end{aligned}$$

which is a contradiction, proving lemma 2.32. □

Let c be a length-minimizing geodesic arc between x_0 and $\partial_+ C$. Let μ be the distance between the geodesic c and Σ_0. As c is embedded in the interior of C (excepted for one extremity which belongs to $\partial_+ C = \Sigma_1$), the constant μ is strictly positive.

Now, for completeness of the proof of proposition B, we state and prove a few lemmas which are implicit in [Mah, proof of Lemma 4.5 p. 2251].

We recall from definition 2.4 that if x is a point in C and S an immersed surface of C, we say that S **separates** x **from** $\partial_+ C$ if every oriented path from x to $\partial_+ C$ has its algebraic intersection number equal to $+1$.

If two surfaces S and T immersed in C are such that S separates every point of T from $\partial_+ C$, we say that S **separates** T **from** $\partial_+ C$. In this case, the surfaces S and T are said to be **nested**.

Lemma 2.33. *A point x lying in the interior of C is separated from $\partial_+ C$ by $\widehat{S}_t$ if and only if there exists a path γ from x to $\partial_+ C$ intersecting the surface $\widehat{S}_t$ with algebraic intersection number $+1$.*

Proof of lemma 2.33.

It suffices to show that if there exists a path γ from x to $\partial_+ C$ with algebraic intersection number with $\widehat{S}_t$ equal to $+1$, then every path γ' from x to $\partial_+ C$ intersects $\widehat{S}_t$ with algebraic intersection number $+1$.

Let γ' be another path from x to $\partial_+ C$ in C. As the immersed surface $\widehat{S}_t$ is homologous to $\partial_- C$, the homology class of $[\widehat{S}_t]$ is equal to zero in $H_2(C, \partial_- C)$. The composition $\alpha = \gamma^{-1} \cdot \gamma'$ is a 1-cycle in $H_1(C, \partial_+ C)$.

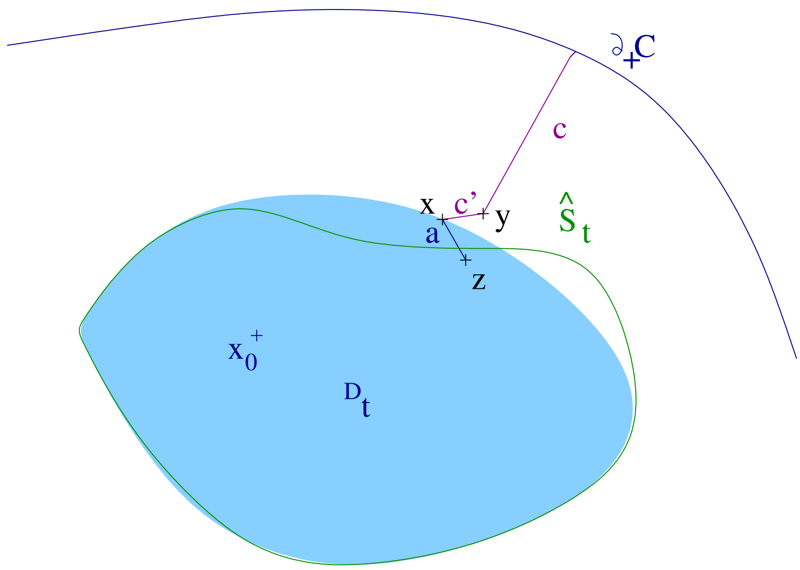
As $\partial C = \partial_- C \cup \partial_+ C$, $[\alpha] \cdot [\widehat{S}_t] = [\alpha] \cdot 0 = 0$, and thus $\gamma' \cdot \widehat{S}_t = \gamma \cdot \widehat{S}_t = +1$, proving lemma 2.33. □

For all $t \in [0, 1]$, let D_t be the closure of the set of points $x \in C$ separated from $\partial_+ C$ by $\widehat{S}_t$. As the immersed surfaces $\left(\widehat{S}_t\right)_{t \in [0,1]}$ are generalized sweepout surfaces of the compression body C, D_0 is the starting sweepout surface $\widehat{S}_0$, and D_1 is equal to the whole compression body C. Let E_t be the component of D_t containing x_0. As before, E_0 is a complex of dimension at most 2, and $E_1 = C$.

Lemma 2.34. *The boundary of the set D_t is a subset of the surface $\widehat{S}_t$.*

Proof of lemma 2.34.

Let us assume that there exists a point x in the boundary of D_t which does not belong to the surface $\widehat{S}_t$, and seek for a contradiction. The distance $d = \mathrm{dist}(x, \widehat{S}_t)$ is then strictly positive. As the point x belongs to the boundary of D_t, there exists a point y in the complement of D_t in C such that $d(x, y) \leq \frac{d}{2}$. As y is in the complement of D_t, there is a path c from y to the boundary $\partial_+ C$ with algebraic intersection number with $\widehat{S}_t$ different from $+1$. Let c' be a minimizing geodesic from x to y: as the length of c', which is equal to the distance between x and y, is strictly less than the distance of x to $\widehat{S}_t$, the geodesic c' does not intersect the surface $\widehat{S}_t$. If $c'' = c' \cup c$, c'' is a path from x to $\partial_+ C$ with algebraic intersection number with $\widehat{S}_t$ not equal to $+1$. Therefore, the point x is not separated from $\partial_+ C$ by $\widehat{S}_t$.



But as the point x belongs also to D_t, there exists a point z in C separated from $\partial_+ C$ by $\widehat{S}_t$ and such that the distance between z and x is less than $\frac{d}{2}$. Take a minimizing geodesic a from z to x. Let us denote by $b = a \cup c''$. The path b is linking z to $\partial_+ C$, which implies that the algebraic intersection number of b with $\widehat{S}_t$ is equal to $+1$. From the other hand, the distance between z and x is at most $\frac{d}{2} < \mathrm{dist}(x, \widehat{S}_t)$, which implies that the minimizing geodesic a does not intersect the surface $\widehat{S}_t$. But then, the algebraic intersection number of the path $b = a \cup c''$ with the surface $\widehat{S}_t$ is not equal to $+1$, which contradicts the fact that z is separated from $\partial_+ C$ by $\widehat{S}_t$. Thus, the point x necessarily belongs to the surface $\widehat{S}_t$, which ends the proof of lemma 2.34. □

Lemma 2.35. *For every time t, the boundary of E_t is connected.*

Proof of lemma 2.35.

Indeed, if the boundary of E_t were not connected, it would have at least two components S and T of $\widehat{S}_t$. But then, S and T would be two disjoint and separating surfaces in the compression body C. If they are not nested, the set of the points separated from $\partial_+ C$ by S is disjoint to the set of points separated from $\partial_+ C$ by T, which contradicts the fact that E_t is connected. Therefore, the surfaces S and T are nested. But the surface $\widehat{S}_t$ is homotopic to a surface obtained from $\partial_+ C$ by surgeries and as surgeries preserve the algebraic intersection number in homology, two components of the same surface $\widehat{S}_t$ cannot be nested, which ends the proof of lemma 2.35. □

To prove proposition B, we will pick up the desired nested surfaces among the family of connected surfaces $(\partial E_t)_{t\in[0,1]}$.

End of proof of proposition B.

Let c be the length-minimizing geodesic arc from the point x_0 obtained in lemma 2.32 to $\partial_+ C$. As before, denote by μ the distance between the geodesic c and Σ_0. Let L be the length of c. One has $L \geq \frac{\delta}{2} - 2\epsilon K'$. Take $\ell \mapsto c(\ell)$ an arc-length parameterization of c, such that $c(0) = x_0$ and $c(L) = y_0 \in \partial_+ C$.

First, let us show that $E_t = \emptyset$ for $t \in [0, \eta]$, where η is the constant given by proposition 2.23. As every original sweepout surface S_t is contained in a $\mu/2$-neighborhood of $S_0 = \widehat{S}_0$ for all $t \leq \eta$, and that the distance between c and S_0 is at least μ, the geodesic c does not meet the sweepout surfaces S_t for every $t \leq \eta$. As the new sweepout surfaces $\widehat{S}_t$ are obtained from the surfaces S_t by surgery, the intersection number between c and $\widehat{S}_t$ is the same as the intersection number between c and S_t, so it is zero for $t \leq \eta$. Therefore, the geodesic c is an arc joining x_0 to $\partial_+ C$ with intersection number with $\widehat{S}_t$ equal to zero for $t \leq \eta$. By definition, the surfaces $\widehat{S}_t$ do not separate x_0 from $\partial_+ C$ for $t \leq \eta$, showing that there is no component of D_t containing x_0. Thus, $E_t = \emptyset$ for every $t \in [0, \eta]$.

Let us assume that $\frac{\delta}{2} - 6\epsilon K' \geq 5\epsilon K$. As the sets E_t vary continuously with the time t, the function $\mathcal{L}$ which maps the time t to the length of $c \cap E_t$ is a continuous map. From the fact that $\mathcal{L}(\eta) = 0$ and $\mathcal{L}(1) = L$ the length of c, we deduce that there is a time $t_1 \in (\eta, 1)$ such that $\mathcal{L}(t_1) = L - 2\epsilon(1 + K + K')$. Let S_1 be the boundary of E_{t_1}. From lemma 2.35, the immersed surface S_1 is a connected component of $\widehat{S}_{t_1}$. As c is a minimizing arc-length parametrized geodesic, for every a and $b \in [0, L]$, we have $d(c(a), c(b)) = |b - a|$. Thus, the intersection point $c(\mathcal{L}(t_1))$ between S_1 and c is lying at distance $2\epsilon(1 + K + K')$ from $\partial_+ C$. Since by construction every point in the surface $\widehat{S}_t$ for $t \geq 1 - \eta$ is at distance at most $\epsilon K'$ from $\partial_+ C$, necessarily $t_1 < 1 - \eta$. As the sets E_t are empty for $t \leq \eta$, in fact $\eta < t_1 < 1 - \eta$. By definition of E_{t_1}, the surface S_1 separates x_0 from $\partial_+ C$. By proposition 2.23, S_1 is connected. Let us show that its ϵ-diameter is at most K. By proposition 2.23, the diameter in C of a component of $\widehat{S}_t$ is at most $\epsilon(1 + 2K + 2K')$. Furthermore, if S_1 contains a persistent surgery neighborhood, it means that S_1 intersects $\mathcal{N}_+$. That implies that every point of S_1 is at distance at most $\epsilon(1 + 2K + 2K') + \epsilon/2$ of $\partial_+ C$, contradicting the fact that the intersection point between S_1 and c is at distance $2\epsilon(1 + K + K') > \epsilon(1 + 2K + 2K') + \epsilon/2$ of $\partial_+ C$. Thus, S_1 does not contain any persistent surgery neighborhood. Proposition 2.23 ensures that its intrinsic ϵ-diameter is at most K and its diameter in C is at most $2\epsilon K$. Therefore, the surface S_1 cannot meet $\{c(\ell)\,,\, 0 \leq \ell < L - 2\epsilon(1 + K + K') - 2\epsilon K)\} \cup \{c(\ell)\,,\, L - 2\epsilon(K + K') < \ell \leq L\}$. Let ℓ_1 be the smallest value of ℓ such that $c(\ell) \in S_1$. We have $L - 2\epsilon(1 + 2K + K') \leq \ell_1 \leq L - 2\epsilon(1 + K')$. As $K' > 1$, this implies that $L - 2\epsilon(1 + 2K + K') \geq L - 4\epsilon(K + K') \geq \frac{\delta}{2} - 4\epsilon K - 6\epsilon K' \geq \epsilon K > 0$.

Let $c_1 = \{c(\ell)\,,\, 0 \leq \ell \leq \ell_1 - 14\epsilon K\}$. Replacing c by c_1, we can iterate the previous process. If K is small enough compared to δ, there exists a time t_2 such that the length of $c_1 \cap E_{t_2}$ is equal to: length $(c_1) - 2\epsilon K = \ell_1 - 16\epsilon K \geq L - 20\epsilon K - 2\epsilon(1+K') \geq L - 20\epsilon K - 4\epsilon K'$. For the same reasons as before, the boundary of E_{t_2} is a surface S_2 which is a connected component of $\widehat{S}_{t_2}$ separating x_0 from $\partial_+ C$, and it intersects c_1 only on the set $\{c_1(\ell)\,,\, (\ell_1 - 14\epsilon K) - 4\epsilon K \leq \ell \leq \ell_1 - 14\epsilon K\}$. Furthermore, the surface S_2 is too far from the boundary $\partial_+ C$ to contain a persistent surgery neighborhood, and its intrinsic ϵ-diameter is at most K by proposition 2.23.

Let us prove that the distance between the surfaces S_1 and S_2 is less than or equal to $10\epsilon K$. Let ℓ_2 be the smallest real number ℓ such that $c(\ell) \in S_2$. From the former discussion, $\ell_2 \leq \ell_1 - 14\epsilon K$. As $c(\ell_1) \in S_1$ and $c(\ell_2) \in S_2$, we have:

$$\begin{aligned} \operatorname{dist}(S_1, S_2) &\geq \operatorname{dist}(c(\ell_1), c(\ell_2)) - \operatorname{diam}(S_1) - \operatorname{diam}(S_2) \\ &\geq (\ell_1 - \ell_2) - 4\epsilon K \\ &\geq 14\epsilon K - 4\epsilon K = 10\epsilon K. \end{aligned}$$

We can iterate the process with $c_2 = \{c(\ell)\,,\, 0 \leq \ell \leq \ell_2 - 14\epsilon K\}$, on condition that $\ell_2 - 14\epsilon K > 4\epsilon K$, so for example if $L - 2 \times 18\epsilon K - 4\epsilon K' > 4\epsilon K$.

The iteration process stops when $L - 18\epsilon K(n-1) - 4\epsilon K' > 4\epsilon K$ but $L - 18\epsilon K n - 4\epsilon K' \leq 4\epsilon K$, so for $n = \lceil \frac{L - 4\epsilon(K+K')}{18\epsilon K} \rceil$. As $L \geq \frac{\delta}{2} - \epsilon K'$, $n \geq \lceil \frac{\delta}{36\epsilon K} - \frac{2}{9} - \frac{K'}{3K} \rceil$, which proves proposition B. □

2.3. Proof of Proposition C: from nested to parallel surfaces.

With proposition B, we know that we can find $n = \lceil \frac{\delta}{36\epsilon K} - \frac{2}{9} - \frac{K'}{3K} \rceil$ immersed surfaces in the compression body C of the cover M'. All those surfaces are nested, their ϵ-diameter is at most K and they are at distance at least $10\epsilon K$ from each other, where $K = 4\left(3 + 1/\sinh^2(\epsilon/8)\right) g(C) - 10$. Furthermore, all those surfaces are homotopic to embedded surfaces obtained from $\partial_+ C$ by surgery.

Thus the genus of those immersed surfaces is between 0 and $g(C) = g(\partial_+ C)$. So there are at least $n' = \lfloor n/(g(C)+1) \rfloor$ surfaces $S_1, \ldots, S_{n'}$ with the same genus, and this genus is at most $g(C)$. We take the indices j such that S_{j+1} separates S_j from $\partial_+ C$.

We then follow the proof of Maher [Mah, p. 2252–2257]. Let $S = S_j$ be one of the previous immersed and nested surfaces with the same genus. A **collection Δ_S of compression discs of $\partial_+ C$ to get** S is a finite set of properly embedded discs in C, such that the sweepout gives a homotopy from S to a subset of $\partial_+ C \cup \Delta_S$. The first step is to show that for two connected and nested sweepout surfaces, one can choose collections of compression discs such that one of them is a subset of the other one. This is done in [Mah, Lemma 4.6 p. 2252]. In particular, if the two surfaces have the same genus, they are homotopic.

Lemma 2.36. [Mah, Lemma 4.6]

Let S_1 and be two of the immersed surfaces obtained in proposition B. Suppose for example that S_2 separates S_1 from $\partial_+ C$. Then we can choose a collection of compression discs of $\partial_+ C$, say Δ_{S_1} to get S_1 and Δ_{S_2} to get S_2, such that Δ_{S_2} is a subset of Δ_{S_1}. In particular, if the two surfaces S_1 and S_2 have the same genus, $\Delta_{S_1} = \Delta_{S_2}$. □

This lemma shows that all the nested surfaces $S_1, \ldots, S_{n'}$ are homotopic, as they have the same genus.

The following lemma is crucial: we wish to replace the nested immersed surfaces by embedded surfaces of the same genus in an arbitrarily small neighborhood of the original immersed surfaces. This lemma is proven in [Mah, Lemma 4.7 p. 2253].

Lemma 2.37. [Mah, Lemma 4.7]
Let S be one of the surfaces obtained in proposition B. Let T be a least genus, connected and embedded surface, separating S from $\partial_+ C$. Then T is incompressible in $C \setminus S$ and the genus of T is greater than or equal to the genus of S.

Proof of lemma 2.37.

We recall here Maher's proof.

If the surface T were compressible in $C \setminus S$, it could be compressed along embedded discs in $C \setminus S$ to obtain a new surface T' embedded in $C \setminus S$. But one component of T' would be an embedded surface in C separating S from $\partial_+ C$, with genus strictly less than the genus of T, which is a contradiction. So the surface T is incompressible in $C \setminus S$.

The surface S is homotopic to $\partial_+ C$ compressed along a collection Δ_S of embedded discs. Thus, if C' is the component of $C \setminus \Delta_S$ containing the surface S, C' is a compression body and we can find for it a spine Γ that is homotopic to the immersed surface S. The map on first homology $H_1(\Gamma) \to H_1(C)$ induced by the inclusion of Γ in C is injective.

The surface T is an embedded surface in the compression body C, so it is separating and there exists a set D_T of embedded compression discs for T such that T compressed along D_T is parallel to some components of $\partial_- C$ (c.f. [B, Lemma 2.3]). As T is incompressible in $C \setminus S$, the compression discs of D_T for the surface T are only in one side of T. So the surface T bounds a compression body C'' in C. As the composition of the maps induced by the inclusions $H_1(\Gamma) \to H_1(C'') \to H_1(C)$ is injective, the map $H_1(\Gamma) \to H_1(C'')$ is injective. Thus the rank of $H_1(C'')$ is greater than or equal to the rank of $H_1(\Gamma)$, and necessarily the genus of T is greater than or equal to the genus of S. □

A consequence of lemma 2.36 is that all the nested and immersed surfaces $S_1, \ldots, S_{n'}$ are homotopic. We want a little more: we need to find for all j between 1 and $(n'-1)$ a homotopy between S_j and $S_{n'}$ that is disjoint from S_k for all $k < j$. We follow the arguments of the proof of [Mah, Lemma 4.8 p. 2254], but we compute precise upper bounds.

Lemma 2.38. *From the surfaces $S_1, \ldots, S_{n'}$, one can construct a collection of immersed surfaces $S'_1, \ldots, S'_{n'-1}, S'_{n'}$ which are disjoint, nested and homotopic, and the homotopy from $S'_{n'}$ to S'_j is disjoint from S'_k for $1 \leq k < j$. Furthermore, the diameter of the surfaces S'_j is at most $8\epsilon K$, they are at distance at least $2\epsilon K$ from each other, and the ϵ-diameter of $S'_2, \ldots, S'_{n'-1}$ is at most K.*

Proof of lemma 2.38

Each surface S_j admits a one-vertex triangulation with edge-length bounded by $4\epsilon K$, and its diameter is at most $2\epsilon K$. Therefore, by lemma 2.14 the surfaces S_1 and $S_{n'}$ are homotopic to simplicial surfaces S'_1 and $S'_{n'}$ with diameter at most $4\epsilon K$ and such that for every points $x \in S_j$ and $x' \in S'_j$ (where $j = 1$ and n'), the distance between x and x' is at most $6\epsilon K$. In fact, by construction of S'_j, each point of S'_j is at distance at most $4\epsilon K_i$ from the original surface S_j.

The homotopy between the two simplicial surfaces S'_1 and $S'_{n'}$ can be modified into a simplicial sweepout as in section 2.2. By proposition 2.23, there exists a finite sequence of surgeries of generalized sweepouts, starting from this simplicial sweepout and ending to a generalized sweepout in which all the sweepout surfaces S'_t for $t \in [\eta, 1-\eta]$ have ϵ-diameter bounded above by K. We can use the same constant K as before since the genus of the surfaces S_j is at most $g(C)$. Moreover, the surfaces S'_t are homotopic to the surface $S_{n'}$ after some compressions if necessary. For j between 2 and $(n'-1)$, let S'_j be the first sweepout surface S'_t intersecting S_j. As S'_j is a generalized sweepout surface, its ϵ-diameter is at most K.

We know from the construction of a generalized sweepout that the genus of the surface S'_j is at most the genus of the surface S_j. In fact, we show that those two genera are equal.

Claim . *For all $1 \leq j \leq n'-1$, the genus of the surface S'_j is the same as the genus of the original sweepout surface S_j.*

Assuming the claim, since the modified sweepout surfaces S'_j have the same genus as the original sweepout surfaces S_j, in fact there is no compression to obtain the surfaces S'_j and they were already sweepout surfaces of the original simplicial sweepout between S'_1 and $S'_{n'}$. So the surfaces S'_j are homotopic to the surface $S'_{n'}$, and by definition of a sweepout, this homotopy is disjoint from the surfaces S'_k for every $k < j$.

Proof of claim.

Suppose that there exists some j such that the genus of S'_j is strictly less than the genus of S_j. By a result of Gabai, we can then replace our simplicial surface S'_j by an embedded surface T'_j in an arbitrarily small neighborhood of the immersed surface S'_j. More precisely, take a small regular neighborhood $N(S'_j)$ of the immersed surface S'_j. This neighborhood contains embedded surfaces in the same homology class as S'_j in $H_2(N(S'_j), \partial N(S'_j))$. Gabai showed that the singular norm on homology is the same as the embedded Thurston norm [G1], hence there exists an embedded surface T'_j in $N(S'_j)$ with the same homology class as S'_j and of genus less than or equal to the genus of S'_j. If we choose sufficiently small neighborhoods $N(S'_j)$, we can ensure that the diameter of the embedded surface T'_j is less than $3\epsilon K$. In particular, as the surfaces S'_1 and $S'_{n'}$ are too far away, the embedded surface T'_j is disjoint from S'_1 and $S'_{n'}$, and it is separating S'_1 from $S'_{n'}$. Applying lemma 2.37, we see that the genus of T'_j must be at least the genus of S'_1: $g(T_j) \geq g(S'_1)$. But as the genus of S'_1 is the same as the genus of S_j, and that the genus of T'_j is at most the genus of S'_j, which we have supposed strictly less than the genus of S_j, we have $g(T'_j) < g(S'_1)$, which is a contradiction. $\square$

As the surfaces S_j were at distance at least $10\epsilon K$ from each other and that every point of S'_j is at distance at most $4\epsilon K$ from the original surface S_j for all $j = 1, \ldots, n'$, the new surfaces S'_j are at distance at most $2\epsilon K$ from each other (which also shows that the surfaces S'_j are all disjoint). Furthermore, their diameter is bounded from above by $8\epsilon K$ and the ϵ-diameter of $S'_2, \ldots, S'_{n'-1}$ is at most K.

There remains to show that the surfaces $S'_1, \dots, S'_{n'}$ are nested. In the spirit of the proof of proposition B, let us denote by $D_{n'}$ the closure of the subset of the points of C separated from $\partial_+ C$ by $S'_{n'}$. For all $j < n'$, the surface S'_j intersects the surface S_j, which lies in $D_{n'}$. As S'_j is at distance at least $2\epsilon K$ from $S'_{n'} = \partial D_{n'}$, S'_j is contained in the interior of $D_{n'}$. So it is separated from $\partial_+ C$ by $S'_{n'}$. Therefore, if we denote by D_j the closure of the points of C separated from $\partial_+ C$ by S'_j, $D_j \subset D_{n'}$. Let $1 \leq k < j < n'$. If we take a point x in D_k, as $D_k \subset D_{n'}$, every path γ from x to $\partial_+ C$ has its algebraic intersection number with $\partial_+ C$ equal to $+1$. As the surface S'_j is homotopic to $S'_{n'}$ by a homotopy that is disjoint from S'_k, this homotopy does not change the intersection number, so the intersection number of γ with S'_j is still equal to $+1$, and x is in D_j. Thus $D_k \subset D_j$ for $1 \leq k < j \leq n'$, showing that the surfaces $S'_1, \dots, S'_{n'}$ are nested. This ends the proof of lemma 2.38. □

In the sequel, we replace the family $S_1, \dots, S_{n'}$ by the new family $S'_1, \dots, S'_{n'-1}, S'_{n'}$ of surfaces obtained by lemma 2.38, and for simplicity, we will still denote this family by $S_1, \dots, S_{n'}$.

We then wish to replace our immersed surfaces by embedded surfaces in an arbitrarily small neighborhood of the immersed surfaces. It is the aim of the following lemma.

Lemma 2.39. *For every j from 1 to n', there exists an embedded surface T_j in a small regular neighborhood of S_j, with the same genus as S_j, and which can be covered by at most $\mathrm{diam}_\epsilon(S_j) \leq K$ embedded balls in M' of radius 2ϵ. Furthermore, two surfaces T_j and T_k for $j \neq k$ are at distance at least ϵK from each other.*

Proof of lemma 2.39.

Take a small regular neighborhood $N(S_j)$ of one of the immersed and nested surfaces S_j. As in the proof of the claim, by Gabai [G1], this neighborhood contains an embedded surface T_j in the same homology class as S_j in $H_2(N(S_j), \partial N(S_j))$ and of genus less than or equal to the genus of S_j. If we choose sufficiently small neighborhoods $N(S_j)$, we can ensure that the diameter of the embedded surfaces T_j in the ambient manifold M' is less than $9\epsilon K$, and two embedded surfaces T_j and T_k are at distance at least ϵK. Furthermore, if we take a set $\mathcal{B}$ of $\mathrm{diam}_\epsilon(S_j)$ embedded balls of radius ϵ and centers on the surface S_j, one can choose $N(S_j)$ small enough such that it is contained in the union of corresponding balls with the same center and radius 2ϵ. Thus, the surface T_j can be covered by at most $\mathrm{diam}_\epsilon(S_j)$ embedded balls of M' with radius 2ϵ.

The genus of T_j is at most the genus of S_j, but we wish to show that in fact, the genus of T_j is the same as the genus of S_j.

With lemma 2.37, we know that the genus of the embedded surface T_j for $j = 2, \dots, n'$ is greater than or equal to the genus of the immersed surface S_1 that it separates from $\partial_+ C$. But as the genus of T_j is at most the genus of S_j, which is equal to the genus of S_1, in fact the genus of T_j is equal to the genus of S_j: the surfaces $T_2, \dots, T_{n'}$ have the same genus as the immersed surfaces $S_2, \dots, S_{n'}$. This proves lemma 2.39. □

The final step in the proof of proposition C is to show that some of the embedded surfaces are actually parallel.

Lemma 2.40. *The embedded surfaces $T_4, \dots, T_{n'-1}$ are parallel.*

Proof of lemma 2.40.

This lemma relies on homological arguments, see [Mah, Lemmas 4.9 to 4.11]. For completeness, we give here a shorter proof, based on classical 3-manifold topological results.

Let V be the 3-complex in C bounded by the immersed surfaces S_1 and $S_{n'}$. There is a sweepout ϕ between S_1 and $S_{n'}$ such that for each $1 \leq j \leq n'$, the surface S_j is a sweepout surface. In other words, the application $\phi : S \times I \to V$ induces in homology an isomorphism $\phi_* : H_3(S \times I, \partial(S \times I)) \to H_3(V, \partial V)$, and for each j, there exists a time $t_j \in I$ such that $S_j = \phi(S \times \{t_j\})$. Moreover, we have $0 = t_1 < t_2 < \ldots < t_{n'} = 1$.

By a classical construction (see [St, point 3. p. 96] for example), we can homotop the sweepout ϕ to a map ϕ' which is still degree one, and such that for every $2 \leq j \leq n'$, $\phi'^{-1}(T_j)$ is an embedded incompressible surface (not necessarily connected) in $S \times I$.

Take $3 < j < k \leq n' - 1$. As the homology class of the surfaces T_j and T_k is the same as the homology class of S_3, the homology class of the preimages $\phi'^{-1}(T_j)$ and $\phi'^{-1}(T_k)$ in $H_2(S \times [t_3, 1], \partial(S \times [t_3, 1]))$ is the same as the homology class of the fiber $S \times \{t\}$. As those preimages are incompressible embedded surfaces, $\phi'^{-1}(T_j)$ and $\phi'^{-1}(T_k)$ are each composed of an odd number of connected surfaces isotopic to the fiber $S \times \{t\}$ with total algebraic intersection number with any path from $S \times \{t_3\}$ to $S \times \{1\}$ equal to $+1$. Up to isotopy, we can suppose that there exist times $t_3 < t_1^j < \ldots < t_{2n_j+1}^j$ and $t_3 < t_1^k < \ldots < t_{2n_k+1}^k$ such that $\phi'^{-1}(T_j) = \cup_{\ell=1}^{2n_j+1} \epsilon_\ell^j (S \times \{t_\ell^j\})$ and $\phi'^{-1}(T_k) = \cup_{\ell=1}^{2n_k+1} \epsilon_\ell^k (S \times \{t_\ell^k\})$, with ϵ_ℓ^j and ϵ_ℓ^k equal to $+1$ or -1, depending on the orientation of the component of $\phi'^{-1}(T_j)$ or $\phi'^{-1}(T_k)$ corresponding to the fiber $S \times \{t_\ell^j\}$ or $S \times \{t_\ell^k\}$. As $\sum_{\ell=1}^{n_j+1} \epsilon_\ell^j = +1$ and $\sum_{\ell=1}^{n_k+1} \epsilon_\ell^k = +1$, there exists ℓ and ℓ' such that $\epsilon_\ell^j = +1 = \epsilon_{\ell'}^k$. Suppose for example that $t_\ell^j < t_{\ell'}^k$. Then $\phi' : S \times [t_\ell^j, t_{\ell'}^k] \to V$ is a homotopy between the embedded surfaces T_j and T_k contained in the region in V bounded by S_3 and $S_{n'}$. As the embedded surface T_2 is not in this region, if we denote by Y the submanifold of C bounded by T_2 and $\partial_+ C$, the two embedded surfaces T_j and T_k are homotopic in the interior of Y.

By lemma 2.37, the surfaces T_j and T_k are incompressible in $C \setminus S_1$. As they are contained in the interior of Y and Y is included in the component of $C \setminus S_1$ containing T_j and T_k, the surfaces T_j and T_k are incompressible in Y. Thus, by a result of Waldhausen [W, Corollary 5.5 p. 76], they are in fact isotopic in Y. Therefore, T_j and T_k are parallel in C, for $3 < j < k \leq n' - 1$. Thus we have $m = n' - 4$ embedded surfaces $T_4, \ldots, T_{n'-1}$ parallel in the compression body C, which ends the proof lemma 2.40. As the ϵ-diameter of $S_2, \ldots, S_{n'-1}$ is at most K and the surfaces $T_4, \ldots, T_{n'-1}$ can be covered by at most K embedded balls in M' of radius 2ϵ, this ends also the proof of proposition C. □

□

2.4. Proof of Proposition D: from patterns of fundamental domains to virtual fibration.

This part is dedicated to the proof of Proposition D, which is based on [Mah, Lemma 4.12 p.2258]. This proof is more involved than the one of Lemma 4.12 in [Mah], which is too quick for our purpose since we need explicit bounds and precise constants.

Assume that there are m connected, orientable, embedded and disjoint parallel surfaces in M'. Furthermore, suppose that each of those surfaces can be covered by at most K embedded balls in M' of radius 2ϵ and that any two surfaces are at distance at least $r > 0$ from each other. In particular, there exists an embedded product $T \times [0, m-1]$ in the manifold M' in which the surface T_j coincides with the fiber $T \times \{j\}$ for all j from 0 to $m-1$.

Let $\mathcal{D}$ be a Dirichlet fundamental domain for the manifold M in its universal cover $\widehat{M} \simeq \mathbb{H}^3$. The translates of $\mathcal{D}$ by the covering maps form a tiling of the universal cover $\widehat{M}$. This tiling descends to a tiling of the cover M' by d copies of $\mathcal{D}$. Each of the m embedded and parallel surfaces $T_1, \dots, T_m$ in M' intersects some copies of $\mathcal{D}$.

Definition 2.41. *The union in M' of copies intersected by one of the surfaces S_j is called a* ***pattern (of fundamental domains) for*** *S_j and denoted by P_j.*

As the surface is connected, a pattern is a connected 3-complex. We can suppose that each of the embedded surfaces intersects the 2-skeleton of the tiling transversally. More precisely, we can suppose that each surface does not meet the vertices of the fundamental polyhedra, that it intersects the edges in isolated points and it is transverse to the 2-dimensional faces of the polyhedra. Therefore, a pattern is a connected union of some copies of $\mathcal{D}$ glued along their 2-dimensional faces. Let D be an upper bound for the diameter of $\mathcal{D}$, and α an upper bound for the number of its 2-dimensional faces.

For all $\ell \in \mathbb{N}$, we recall that $B(\ell)$ is an upper bound for the number of possibilities of patterns obtained by gluing together at most ℓ fundamental domains. Let $L = \lfloor \frac{\pi(\sinh(2D+4\epsilon)-2D-4\epsilon)}{\mathrm{Vol}(M)} K \rfloor$ as in lemma 2.8. The integer L is an upper bound for the number of fundamental domains a given surface can intersect. Thus, a pattern is the union of at most L fundamental domains.

Suppose that $r/(2D+1) \geq 1$ and $\frac{m}{\alpha^2 L^2 B(L)} \geq 4$ (which will be called condition (a)), or that $r/(2D+1) < 1$ and $\left(\frac{r}{2D+1} m - 1\right) \frac{1}{\alpha^2 L^2 B(L)} \geq 4$ (called condition (b)).

Lemma 2.42. *If conditions (a) or (b) are satisfied, there are at least $4\alpha^2 L^2 B(L)$ surfaces for which the corresponding patterns of fundamental domains are disjoints.*

Proof of lemma 2.42.

If two surfaces T_j and T_k are at distance strictly more than $2D$, the patterns of fundamental domains associated to T_j and T_k are necessarily disjoint, as the diameter of a fundamental domain is at most D.

If $r/(2D+1) \geq 1$ as in condition (a), any pair of surfaces T_j and T_k with $j \neq k$ are at distance strictly more than $2D$, and all the m patterns associated to the parallel surfaces are disjoint.

Otherwise, $r/2D < 1$. In this case, there are at least $\lfloor \frac{r}{2D+1} m \rfloor \geq \frac{r}{2D+1} m - 1$ surfaces T_j which are separated from each other by a distance at least $2D+1 > 2D$. Thus, every corresponding patterns of fundamental domains are disjoint.

As in condition (a), $m \geq 4\alpha^2 L^2 B(L)$, or in condition (b), $\frac{r}{2D+1} m - 1 \geq 4\alpha^2 L^2 B(L)$, there are at least $4\alpha^2 L^2 B(L)$ surfaces whose corresponding patterns are disjoint. □

Lemma 2.43. *There exist an "abstract" pattern of fundamental domains P and at least $4\alpha^2 L^2$ patterns of fundamental domains P_j, which are disjoint and homeomorphic to P. More precisely, for at least $4\alpha^2 L^2$ of the previous indices j for which the corresponding patterns of fundamental domains are disjoint, there exists a* *homeomorphism* $\varphi_j : P_j \to P$ *which preserves polyhedral decomposition and gluing isometries between the faces of the fundamental domains belonging to the patterns.*

Proof of lemma 2.43.

The proof is straightforward. Indeed, as a pattern is the union of at most L fundamental domains, there are at most $B(L)$ possible patterns. Among the $4\alpha^2 L^2 B(L)$ disjoint previous patterns, there are at least $4\alpha^2 L^2$ of them corresponding to the same "abstract" pattern P. □

From now on, we only consider $4\alpha^2 L^2$ indices j satisfying the conclusions of last lemma.

Lemma 2.44. *The number of boundary components of the pattern* P *is between* 2 *and* αL*.*

Proof of lemma 2.44.

Each fundamental polyhedron in the pattern P has α 2-faces. As P is the union of at most L polyhedra, it has at most αL 2-faces. It is an upper bound for the number of boundary components of P.

To see that there is at least two boundary components in P, it suffices to show that for example P_1 has at least two boundary components. But as the surface T_1 is contained in the interior of the pattern P_1, $P_1 \cap (T \times [0,1]) \neq \emptyset$ and $P_1 \cap (T \times [1,2]) \neq \emptyset$. The pattern P_1 is disjoint to T_0 and T_2, so the product regions $T \times [0,1]$ and $T \times [1,2]$ are not contained in P_1. By connectivity of $T \times [0,1]$ and $T \times [1,2]$, the boundary of the pattern P_1 has at least two components, one as a subset of $T \times (0,1)$ and the other one in $T \times (1,2)$. This proves lemma 2.44. □

Set $\partial P = E_1 \cup E_2 \cup \ldots \cup E_s$, where the immersed surfaces E_j are the boundary components of the pattern P, with $2 \leq s \leq \alpha L$.

Definition 2.45. *For every index* j *between 1 and* $4\alpha^2 L^2 - 2$*, the pattern* P_j *intersects* $T \times (j-1, j)$ *and* $T \times (j, j+1)$*. At least one component of the boundary of* P_j *is in the boundary of the component of* $(T \times [j-1, j]) \setminus (T \times [j-1, j]) \cap P_j$ *containing the fiber* $T \times \{j-1\}$*, which we will call a* ***"left" component of the boundary of the pattern*** P_j*. Similarly, at least one component of the boundary of* P_j *is in the boundary of the connected component of* $(T \times [j, j+1]) \setminus (T \times [j, j+1]) \cap P_j$ *containing the fiber* $T \times \{j+1\}$*. We will call this component a* ***"right" component for the boundary of*** P_j*.*

Lemma 2.46. *For every index* j *between* 1 *and* $4\alpha^2 L^2 - 2$*, choose a left and a right component for the pattern* P_j *(arbitrarily if there exist at least two such components). Those two component correspond to components* E_j^- *and* E_j^+ *in the boundary of the abstract pattern* P*. There are at least two indices* j *and* k *for which the pairs of left and right components corresponding to the patterns* P_j *and* P_k *coincide in* ∂P*.*

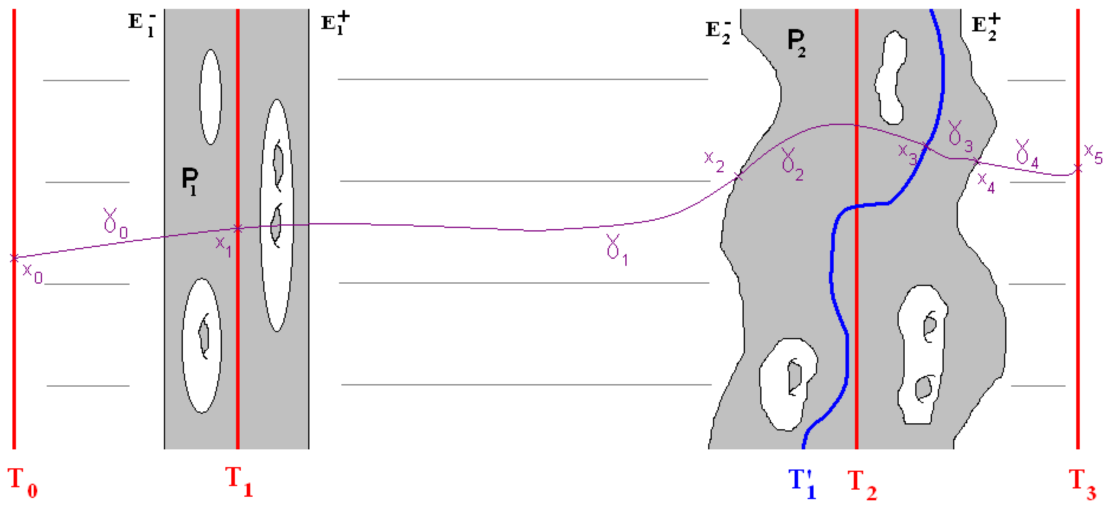
Proof of lemma 2.46.

As there are at most $s(s-1) \leq \alpha L(\alpha L - 1) < \alpha^2 L^2$ pairs of left and right boundary components of P, there are at least $(4\alpha^2 L^2 - 2)/(\alpha^2 L^2) \geq 2$ surfaces T_j and T_k with $1 \leq j < k \leq 4\alpha^2 L^2 - 2$, for which the pairs of left and right components corresponding to the patterns P_j and P_k coincide. □

In the sequel, in order to simplify notations, let us denote by T_1 the surface T_j, T_2 the surface T_k and T_3 the last surface $T_{4\alpha^2L^2-1}$. The surfaces T_0 and T_3 bound a product $T \times [0,3]$ in M', such that $T_1 = T \times \{1\}$ and $T_2 = T \times \{2\}$. The two patterns P_1 and P_2 are contained in the interior of the product $T \times [0,3]$. Denote by $\psi := \varphi_2^{-1} \circ \varphi_1$ the composed homeomorphism between patterns P_1 and P_2. Let T_1' be the image of the surface T_1 in the interior of the pattern P_2 under the action of ψ : $T_1' = \varphi_2^{-1} \circ \varphi_1(T_1) = \psi(T_1)$. It is an embedded surface in the product $T \times [0,3]$. Clearly, the surfaces T_1 and T_2 are parallel, but they may not be embedded in their patterns in the same way. However, the surfaces T_1 and T_1' are embedded in the patterns P_1 and P_2 in exactly the same way, but there is no evidence to say a priori that those two surfaces are parallel. It is in fact true, thanks to the following lemma.

Lemma 2.47. *The surfaces T_1 and T_1' are parallel in M'.*

Proof of lemma 2.47.



Claim . *The homology class of T_1' in the product $T \times [0,3]$ is equal to the homology class of the fiber $[T] = [T_1] = [T_2]$.*

Proof of claim.

By choice of the surfaces T_1 and T_2, the left component E_1^- of the boundary of the pattern P_1 and the left component E_2^- of the boundary of the pattern P_2 have the same image in the pattern P: $\varphi_1(E_1^-) = \varphi_2(E_2^-)$, so $E_2^- = \varphi_2^{-1} \circ \varphi_1(E_1^-)$. By definition, E_2^- is a boundary component of the connected component of $(T \times [1,2]) \setminus (T \times [1,2]) \cap P_2$ containing the fiber T_1, and the component E_1^- is a boundary component of the pattern P_1 in the boundary of the component of $(T \times [0,1]) \setminus (T \times [0,1]) \cap P_1$ containing the fiber T_0. As $P_1 \cap (T \times [0,1])$ is connected, there exists a path γ_2', properly embedded in $P_1 \cap (T \times [0,1])$ and joining the component E_1^- to the surface T_1. The image by the homeomorphism $\varphi_2^{-1} \circ \varphi_1$ between the patterns P_1 and P_2 of the path γ_2' is a path $\gamma_2 = \varphi_2^{-1} \circ \varphi_1(\gamma_2')$ in P_2 from the boundary component E_2^- to the surface T_1'. The interior of the path γ_2 is contained in the interior of the component of $P_2 \setminus T_1'$ containing E_2^-. Let x_2 be the extremity of γ_2 belonging to the boundary component E_2^-, and x_3 the other one, on the surface T_1'.

Similarly, there exists a path γ_3 from x_3 to a point x_4 lying on the right component E_2^+ of the boundary of P_2, and such that its interior is contained in the interior of the component of $P_2 \setminus T_1'$ containing E_2^+.

As E_2^- is in the boundary of the connected component of $(T\times[1,2])\setminus P_2\cap(T\times[1,2])$ containing the fiber T_1, there exists a path γ_1 with its interior contained in the interior of this component, and joining a point x_1 of the fiber T_1 to the point x_2 of E_2^-. Similarly, by choice of E_2^+, there exists a path γ_4 with interior contained in the interior of the component of $(T\times[2,3])\setminus P_2\cap(T\times[2,3])$ containing the fiber T_3 and linking the point x_4 of E_2^+ to a point x_5 of T_3. Eventually, as the product $T\times[0,1]$ is connected, there exists a path γ_0 with interior contained in $T\times(0,1)$ joining the point x_1 of T_1 to a point x_0 of T_0.

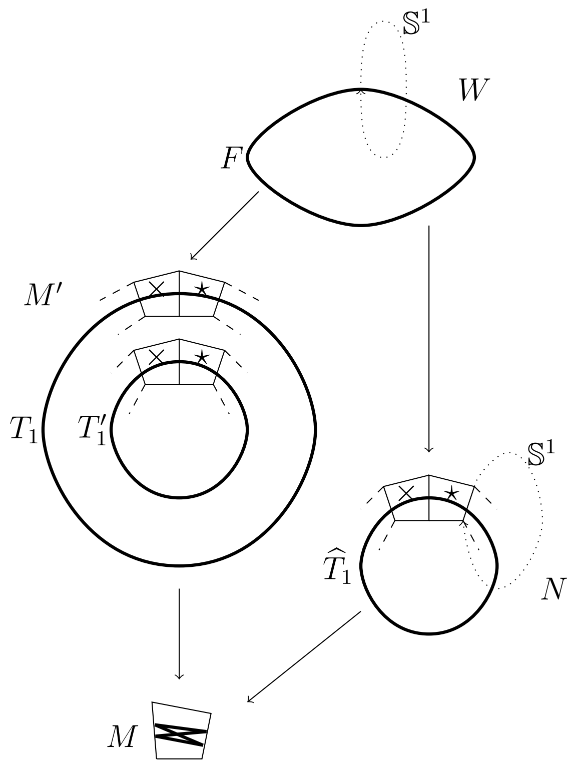
Let γ be the path obtained by concatenating the paths γ_0, γ_1, γ_2, γ_3 and γ_4. The path γ joins the point x_0 of T_0 to the point x_5 of T_3 and intersects the surface T_1' only once, at the point x_3. As the orientations of the patterns P_1 and P_2 coincide, the intersection number of γ with the surface T_1' is $+1$. So it is the same as the intersection number of γ with the fiber T. By Poincaré duality, as the homology group $H_2(T\times[0,3],\mathbb{Z})$ is of rank one, generated by the class of the fiber $[T]$, the class of the surface T_1' is equal to the class of the fiber in the homology of the product, showing the claim. □

As the surface T_1' is embedded in the product $T\times[0,3]$, by a result of Waldhausen [W], it follows that the surface T_1' is parallel to the fiber T_1, possibly after performing a finite number of compressions on T_1'. But as the surface T_1' is homeomorphic to T_1, it is of the same genus as the fiber T_1. So in fact there is no compression. Therefore, those two surfaces bound a product in M'. □

Lemma 2.48. *The manifold M admits a cover N of finite degree at most d which fibers over the circle, and the embedded surface T_1 in M' is an (incompressible) virtual fiber.*

<u>Proof of lemma 2.48.</u>

One can cut the manifold M' open along those two disjoint surfaces T_1 and T_1'. We keep only the component corresponding to the product region between the two parallel surfaces, and we identify the two surfaces via the homeomorphism $\psi = (\varphi_2^{-1}\circ\varphi_1)_{|T_1}$ to obtain a manifold N fibering over the circle, with fiber $\widehat{T_1} = (T_1 \sim T_1')$. The homeomorphism $\varphi_2^{-1}\circ\varphi_1$ identifies the "left" part of the pattern P_2 with the "left" part of the pattern P_1, so we get a pattern $\widehat{P_1}$ corresponding to $\widehat{T_1}$ in N homeomorphic to the pattern P: the "left" part of this pattern corresponds to the left part of the pattern P_2 via the homeomorphism φ_2, and the "right" part of the pattern corresponds to the right part of P_1 via the homeomorphism φ_1. As those homeomorphisms preserve the gluings between the 2-dimensional faces of the fundamental domains, the gluings between the fundamental domains in the pattern $\widehat{P_1}$ are the same as the gluings in the model pattern P. Therefore, we obtain a tiling of N by finitely many copies of fundamental domains homeomorphic to $\mathcal{D}$ and with matching gluings. Thus, N is a finite cover of the original manifold M, and N is fibered over the circle, with fiber $\widehat{T_1}$.



The two covers M' and N admit a common regular finite cover W, which fibers over the circle as it is a finite cover of N. A component of the preimage of $\widehat{T}_1$ by the covering projection $W \to N$ is a fiber F for the fibration of W over the circle. As the diagram is commutative, it is also a component the preimage of the embedded surface T_1 in M', as T_1 and $\widehat{T}_1$ have the same image in M, which is an immersed surface. As F is incompressible in W, the surface T_1 embedded in M' we started from is also incompressible. Thus the embedded surface T_1 is a virtual fiber in M', and is incompressible. □

Therefore, the m initial parallel surfaces are virtual fibers for the manifold M'. In fact, they are fibers of a bundle over the circle or of a twisted I-bundle. Indeed, if T is one of those surfaces, the complement M'_T of an open neighborhood of T in M' admits a finite cover that is the product of a T' by an interval I. In particular, the fundamental group of the compact manifold M'_T contains a finite index surface subgroup. By [H, Theorem 10.6], it is a I-bundle, possibly twisted. This ends the proof of the Pattern Proposition D. □

3. Heegaard genera and fibration.

The proof of theorem 0.3 is the starting point for the proof of the main theorem A. The aim was to establish a virtual fibration criterion standing between Lackenby's conjecture 0.1 and Maher's theorem. Maher himself suggested in [Mah] the possibility to get explicit constants and upper bounds at each stage of the proof of Theorem 1.1 of [Mah], but without precise statements.

This section is dedicated to the proof of theorem 0.3 and corollary 0.6.

3.1. **Proof of theorem 0.3 (1) and corollary 0.6 (1): Heegaard genus.**

Proof of theorem 0.3 (1).

Suppose that $M' \to M$ is a cover of M with finite degree d. Let $S \subset M'$ be a minimal genus Heegaard surface for M': $g(S) = g(M')$. The aim is to construct from S a pseudo-minimal surface which satisfies assumptions of theorem A. We start with the following lemma.

Lemma 3.1. *Let N be a connected, oriented and closed hyperbolic 3-manifold. Let S be a minimal genus Heegaard surface for N and $\mathcal{H}$ the Heegaard splitting of N with Heegaard surface S. Let F be the union of the even and odd surfaces of a $\mathcal{H}$-thin generalized Heegaard splitting for N. Then F is a pseudo-minimal surface, which divides the manifold N in $q \leq \chi_-^h(N) + 2$ compression bodies $C_1, \ldots, C_q$ with $\chi_-(C_j) \leq \chi_-^h(N)$ for all j between 1 and q.*

Furthermore, if F^- is the union of the negative boundary components $\partial_- C_j$, then it is a union of incompressible surfaces.

Proof of lemma 3.1.

The topological part (1) of the following theorem is a consequence of works of Casson and Gordon, Scharlemann and Thompson ([CG] and [ST]). The metric part (2) comes from results of Frohman, Freedman, Hass and Scott about incompressible surfaces ([FHS] and [FH]). The last part (3) is a result of Pitts and Rubinstein ([PR], see also [So], [CDL] and [P]).

Theorem 3.2. *Let N be a connected, oriented and closed hyperbolic 3-manifold, and $\mathcal{H}$ a $\mathcal{H}'$-thin generalized Heegaard splitting for some Heegaard decomposition $\mathcal{H}'$. Then $\mathcal{H}$ satisfies the following properties.*

(1) *Each of the even surfaces is incompressible in N and the odd surface are strongly irreducible Heegaard surfaces for the components of the manifold N cut along the even surfaces.*
(2) *Each even surface can be isotoped to a minimal surface or the boundary of a small neighborhood of a non-orientable minimal surface.*
(3) *each odd surface can be isotoped to a minimal surface, or to the boundary of a small regular neighborhood of a non-orientable minimal surface, with a small tube attached vertically in the I-bundle structure.*

□

Thanks to theorem 3.2, up to isotopy, one can assume that the surface F is pseudo-minimal, and it is immediate that F^- is a union of incompressible surfaces.

As described in section 1, surgeries of generalized Heegaard splittings are a modification in the order of attachment of the 1- and 2-handles of a corresponding handle decomposition of the manifold. Therefore, surgeries do not change the number of 1- and 2-handles. As it is equal in the starting Heegaard splitting to $2g(S) = \chi_-^h(N)+2$, there are also $(\chi_-^h(N) + 2)$ 1- and 2-handles in a handle decomposition associated to the surface F. As this number is an upper bound for the number of compression bodies in the complement of F, the inequality $q \leq \chi_-^h(N) + 2$ holds.

Furthermore, as each component of F is obtained from S by surgery, the characteristic $\chi_-(C)$ of each compression body is at most $|\chi(S)| = \chi_-^h(N)$. □

End of the proof of theorem 0.3 (1).

Recall that S is a minimal genus Heegaard surface for the cover $M' \to M$ of finite degree d. Let F be the pseudo-minimal surface obtained in lemma 3.1. The aim is to apply the main theorem A to F. With notations of theorem A and this choice of surface F, one has $c = \chi_-^h(M')$ and $q = \chi_-^h(M') + 2$.

Set $\epsilon = \mathrm{Inj}(M)/2$ and let $k = k(\epsilon, \mathrm{Vol}(M))$ be the constant obtained in theorem A. To satisfy assumptions of theorem A, one needs to have

$$k\,\chi_-^h(M')\ln\chi_-^h(M') \leq \ln\ln\frac{d}{\chi_-^h(M')+2}.$$

If the ratio $\chi_-^h(M')\ln\chi_-^h(M')/\ln\ln d$ tends to zero, then the ratio $\chi_-^h(M')/\sqrt{d}$ tends also to zero. Therefore, there exists an explicit constant $\overline{k_1} > 0$ such that if $\overline{k_1}\,\chi_-^h(M')\ln\chi_-^h(M') \leq \ln\ln d$, then $\chi_-^h(M') + 2 \leq \sqrt{d}$. Under this assumption, one has

$$\begin{aligned}\ln\ln\frac{d}{\chi_-^h(M')+2} &\geq \ln\ln\sqrt{d}\\ &= \ln\left(\frac{1}{2}\ln d\right)\\ &= \ln\ln d - \ln 2 \geq \frac{1}{2}\ln\ln d\end{aligned}$$

if $\ln\ln d \geq 2\ln 2$, which is the fact for example if $\ln\ln d \geq \chi_-^h(M')\ln\chi_-^h(M')$ as $\chi_-^h(M') \geq 2$.

Therefore, if $\chi_-^h(M')\ln\chi_-^h(M') \leq \ln\ln d$, $\overline{k_1}\,\chi_-^h(M')\ln\chi_-^h(M') \leq \ln\ln d$ and $2k\,\chi_-^h(M')\ln\chi_-^h(M') \leq \ln\ln d$, then

$$k\,\chi_-^h(M')\ln\chi_-^h(M') \leq \ln\ln\frac{d}{\chi_-^h(M')+2}$$

and assumptions of theorem A are satisfied. This proves theorem 0.3 with $\overline{k} = \max\{1, 2k, \overline{k_1}\}$. □

Proof of corollary 0.6 (1).

It is obvious that if M virtually fibers over the circle, then the η-sub-logarithmic Heegaard gradient of M is zero for every $\eta \in (0,1)$, as M admits an infinite family of finite degree covers with bounded Heegaard genus.

If the η-sub-logarithmic Heegaard gradient of M is zero for some $\eta \in (0,1)$, this means that M admits an infinite family of covers $(M_i \to M)_{i\in\mathbb{N}}$ with finite degrees d_i, and such that

$$\lim_{i\to+\infty}\frac{\chi_-^h(M_i)}{(\ln\ln d_i)^\eta} = 0,$$

which can be also written

$$\lim_{i\to+\infty}\frac{\chi_-^h(M_i)^{1/\eta}}{\ln\ln d_i} = 0.$$

As $1/\eta > 1$, this implies that

$$\lim_{i\to+\infty}\frac{\chi_-^h(M_i)\ln\chi_-^h(M_i)}{\ln\ln d_i} = 0.$$

Thus, for i large enough, on has $\overline{k}\,\chi_-^h(M_i)\ln\chi_-^h(M_i) \leq \ln\ln d_i$ and the assumptions of theorem 0.3 are satisfied. In particular, the manifold M virtually fibers over the circle, which proves corollary 0.6 (1). □

3.2. **Proof of theorem 0.3 (2) and corollary 0.6 (2): strong Heegaard genus.**

Proof of theorem 0.3 (2).

Suppose by contradiction that in a finite cover $M' \to M$ of degree d, one has $\bar{k}\,\chi_-^{sh}(M')\ln\chi_-^{sh}(M') \leq \ln\ln d$. Let F be a strongly irreducible Heegaard surface for M' such that $\chi_-^{sh}(M') = \chi_-(F)$.

Thanks to theorem 3.2, up to isotopy, one can assume that F is pseudo-minimal. This surface separates M' into two handlebodies, so the volume of one of those handlebodies C must be at least $\mathrm{Vol}(M)d/2$. But as $\bar{k}\,\chi_-(F)\ln\chi_-(F) \leq \ln\ln d$, the proof of theorem 0.3 (1) shows that the surface F satisfies the assumptions of theorem A. This is in contradiction with corollary 0.2. This proves theorem 0.3 (2). □

Proof of corollary 0.6 (2).

To prove corollary 0.6 (2), just notice that as $\bar{k}\,\chi_-^{sh}(M')\ln\chi_-^{sh}(M') > \ln\ln d$, for $\theta \in (0,1)$, there is a constant $\bar{k}_\theta > 0$ such that $\chi_-^{sh}(M')/(\ln\ln d)^\theta \geq \bar{k}_\theta$, proving that the strong η-sub-logarithmic Heegaard gradient of M is strictly positive. □

4. Circular decomposition and fibered homology classes.

The aim of this section is to consider the case of circular decompositions, and to prove corollaries 0.8 and 0.10.

4.1. **Circular decomposition and thin position.**

A circular decomposition is the equivalent of a Heegaard decomposition, but this decomposition is associated to a Morse function that no longer takes values in $I = [0,1]$ but in the circle $\mathbb{S}^1$.

Definition 4.1. *A* ***circular Morse function*** *is a Morse function $f : M \to \mathbb{S}^1$.*

If $f : M \to \mathbb{S}^1$ is a circular Morse function, the handle decomposition of M given by the function f is called the ***circular decomposition associated to*** *f.*

See F. Manjarrez-Gutiérrez [MG], Matsumoto [Mat] and Milnor [Mi] for further details about circular Morse functions. Let $f : M \to \mathbb{S}^1$ be a circular Morse function. If we remove a small open neighborhood of a regular value $x \in \mathbb{S}^1$, by restriction of f, we obtain a Morse function g of $M_R = M \setminus \mathcal{N}(R)$, which is the manifold M minus a small regular open neighborhood of the surface $R := f^{-1}(\{x\})$, on the interval I. Thus, the theory of Heegaard splittings and generalized Heegaard splittings applies to the function g, as recalled in section 1.

An other viewpoint is to see a circular decomposition as a handle decomposition of the cobordism $(M \setminus \mathcal{N}(R), R \times \{1\}, R \times \{-1\})$. Starting with a Heegaard splitting of Heegaard surface S for $M_R = M \setminus \mathcal{N}(R)$, one can change the order in which 1- and 2-handles are attached to get a new generalized Heegaard splitting $(F_1 = R \times \{1\}, S_1, F_2, \ldots, S_n, F_{n+1} = R \times \{-1\})$ for $(M_R, R \times \{1\}, R \times \{-1\})$. Gluing back $R \times \{1\}$ to $R \times \{-1\}$, one obtains a circular decomposition for the manifold M. Denote it by $\mathcal{H} = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$, with $F_1 = F_{n+1} = R$. The surfaces F_j divide M into n 3-manifolds with boundary $W_1, \ldots, W_n$, and surfaces S_j are Heegaard surfaces for those manifolds. For $1 \leq j \leq n$, S_j divides the manifold W_j

into two compression bodies A_j and B_j, such that $\partial_+ A_j = \partial_+ B_j = S_j$, $\partial_- A_j = F_j$ and $\partial_- B_j = F_{j+1}$.

Let S be a closed surface. If S is connected, recall that the **complexity** of S is $c(S) = \max(0, 2g(S) - 1)$. If S is the union of several connected components, the complexity of S is the sum of the complexities of the components of S. There is a definition of the complexity of a circular decomposition analogous to the complexity of a generalized Heegaard splitting.

Definition 4.2. *The* ***circular width*** *of a circular decomposition* $\mathcal{H} = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$ *is the set of the* n *integers* $(c(S_1), \ldots, c(S_n))$*, with repetitions and arranged in monotonically non-increasing order. Widths are compared using the lexicographic order.*

The integer $n \geq 1$ *is called the* ***length*** *of the circular decomposition* $\mathcal{H} = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$*.*

Proposition 4.3. *Let* M *be a hyperbolic, connected, oriented and closed 3-manifold. Let* R *be an orientable, closed, non-separating, incompressible and embedded surface in* M*. Denote by* S *a Heegaard surface for* $M \setminus \mathcal{N}(R)$*. Starting from the circular decomposition* $\mathcal{H} = (R, S, R)$ *of* M*, there exists a finite number of surgeries to get a circular decomposition* $\mathcal{H}' = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$ *with* $F_1 = F_{n+1} = R$*, such that:*

1. *the circular width of* $\mathcal{H}'$ *is minimal among the widths of such circular decompositions obtained by a finite number of surgeries of* $\mathcal{H}$*,*
2. *each surface* F_j *is incompressible, no component of* F_j *is a sphere, and* $g(F_j) \leq g(S)$*,*
3. *each surface* S_j *is a strongly irreducible Heegaard surface for the Heegaard decomposition* (A_j, B_j) *of* W_j *and* $g(S_j) \leq g(S)$*,*
4. $n \leq \frac{1}{2}(\chi(R) - \chi(S))$*,*

Definition 4.4. *Let* $\mathcal{H}$ *be a circular decomposition. A circular decomposition* $\mathcal{H}' = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$ *that is circular-length-minimizing among all circular decompositions obtained from* $\mathcal{H}$ *by a finite number of surgeries is said to be a* ***thin position****. We will call such a decomposition a* ***thin circular decomposition associated to*** $\mathcal{H}$*.*

Proof of proposition 4.3.

The proof of this proposition is essentially the same as the proof of [MG, Theorem 3.2], which is itself an adaptation of techniques of [ST] to the case of circular decompositions (see also [L]). See [R1], Proposition 1.1 and its proof. The proof is based on an operation called a **surgery** of circular decompositions, which is analogous to the surgery of generalized Heegaard splittings described in section 1. Again, the crucial fact is that a surgery procedure strictly decreases the complexity of the circular decomposition. □

Corollary 4.5. *Let* M *be a hyperbolic, connected, oriented and closed 3-manifold. Take* $\mathcal{H} = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$ *a thin circular decomposition of* M*. Then, up to isotopy, one can assume that all surfaces* F_j *and* S_j *are pseudo-minimal.*

Proof of corollary 4.5.

From proposition 4.3 points (2) and (3), the surfaces F_j are incompressible for each j and the surfaces S_j correspond to strongly irreducible Heegaard surfaces. The proof is then the same as for theorem 3.2 (2) and (3) of section 3. □

4.2. **Circular characteristic and fibered homology classes.**
Recall definition 0.7 from the introduction. Corollary 0.8 is analogous to theorem 0.3 for circular decompositions associated to a non trivial cohomology class.

Proof of corollary 0.8.
Let $M' \to M$ be a cover of M with finite degree d, and a non-trivial cohomology class $\alpha' \in H^1(M')$. The aim is to show that if the ratio $\chi_-^c(\alpha') \ln \chi_-^c(\alpha') / \ln \ln d$ is small enough, then the assumptions of theorem A are satisfied.

Let R' be an embedded surface in M' and $\|\alpha'\|$-minimizing. First, suppose that in addition $h(M', \alpha') = h(M', \alpha', R')$. Take S' a minimal genus Heegaard surface for $M'_{R'}$. By construction, $\chi_-^c(\alpha') = |\chi(S')|$.

From proposition 4.3, starting from the circular decomposition (R', S', R') of M', we can construct a thin circular decomposition $\mathcal{H} = (F_1, S_1, F_2, \dots, S_n, F_{n+1})$. Set $F := \bigcup_j F_j \cup \bigcup_j S_j$. From corollary 4.5, one can assume that F is a pseudo-minimal surface.

Still from proposition 4.3, the surface F separates the manifold M' into $q \leq \frac{1}{2}(\chi(R') - \chi(S')) \leq \chi_-^c(\alpha')/2$ compression bodies $C_1, \dots, C_q$, with $\chi_-(C_j) \leq |\chi(S')| = \chi_-^c(\alpha')$ for every j.

As the surfaces F_j are incompressible, assumption (1) of theorem A is satisfied.

Let $k = k(\epsilon, \mathrm{Vol}(M))$ be the constant given by theorem A. To satisfy assumptions of theorem A, there remains to show that $k\, \chi_-^c(\alpha') \ln \chi_-^c(\alpha') \leq \ln \ln \frac{2d}{\chi_-^c(\alpha')}$. But as in section 3, one can find a constant $\ell' = \ell'(\epsilon, \mathrm{Vol}(M))$ such that if $\ell'\, \chi_-^c(\alpha') \ln \chi_-^c(\alpha') \leq \ln \ln d$, then $k\, \chi_-^c(\alpha') \ln \chi_-^c(\alpha') \leq \ln \ln \frac{2d}{\chi_-^c(\alpha')}$ and all the assumptions of theorem A are satisfied.

Therefore, if $\ell'\, \chi_-^c(\alpha') \ln \chi_-^c(\alpha') \leq \ln \ln d$, then from theorem A, the manifold M' contains an embedded surface that is a virtual fiber.

Furthermore, all the constructions take place in fact in ${M'}_{R'} = M' \setminus \mathcal{N}(R')$. Thus, the virtual fiber built in theorem A is in the complement of R' in M'. This virtual fiber lifts to a connected fiber $\overline{T}$ in a fibered finite cover $\overline{M'} \to M'$ of M'. In this cover, the incompressible surface R' lifts to a surface $\overline{R'}$ in the complement of the fiber. Cutting along $\overline{T}$, this shows that the connected components of $\overline{R'}$, which are all incompressible, are parallel in the product to the fiber $\overline{T}$ (see [W]). Thus, the homology class of $\overline{R'}$ is fibered. Still from Gabai [G2, Lemma 2.4], this implies that the homology class of R' is fibered. As the surface R' minimizes Thurston's norm, it is also a fiber.

To end the proof of corollary 0.8, there remains to show that if R' be an embedded surface in M', $\|\alpha'\|$-minimizing, but that does not necessarily satisfy $h(M', \alpha') = h(M', \alpha', R')$, then R' is still a fiber. But if one takes an embedded surface R'' such that R'' is $\|\alpha'\|$-minimizing, and satisfies $h(M', \alpha') = h(M', \alpha', R'')$, the proof above shows that R'' is a fiber. As R' is norm-minimizing, it is an incompressible surface in the homology class of R'', hence also a fiber. This ends the proof of corollary 0.8. $\square$

The following corollary is immediate from corollary 0.8.

Corollary 4.6. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. Suppose that there exists an infinite family of covers $(M_i \to M)_{i \in \mathbb{N}}$ with finite degrees*

d_i, and for each $i \in \mathbb{N}$, a non-trivial cohomology class $\alpha_i \in H^1(M_i)$ such that:

$$\inf_{i\in\mathbb{N}} \frac{\chi_-^c(\alpha_i)\ln\chi_-^c(\alpha_i)}{\ln\ln d_i} = 0.$$

Then, for infinitely many indices $i \in \mathbb{N}$, every embedded surface R_i in M_i, $\|\alpha_i\|$-minimizing and such that $h(M_i,\alpha_i) = h(M_i,\alpha_i,R_i)$ is a fiber. In particular, the manifold M virtually fibers over the circle $\mathbb{S}^1$.

4.3. **Incompressible surfaces and fibrations.**

In section 4.2, we have established criteria in order to show that a non-trivial cohomology class of a hyperbolic 3-manifold M lifts to fibered classes in finite covers. Now, if R is a non separating embedded surface in M, there is a dual cohomology class associated to R. In some cases, we have seen that R could then be a fiber. But the question can be asked for any embedded, incompressible and connected surface R in M, separating or not.

Recall definition 0.9 from the introduction. Corollary 0.10 is different from last section as the surface R is a priori not supposed to be non-separating.

Proof of corollary 0.10.

In the case where the surface R' is not separating, it is a generalization of corollary 0.8. Indeed, if S' is a minimal genus Heegaard surface for $M'_{R'}$, $\chi_-^h(R') = |\chi(S')|$ and (R',S',R') is a circular decomposition of M'. As the starting surface R' is incompressible, we apply then proposition 4.3 to build a thin circular decomposition. From the proof of corollary 0.8, assumptions of theorem A are satisfied if $\ell'\chi_-^h(R')\ln\chi_-^h(R') \leq \ln\ln d$, and in this case, the surface R' is a virtual fiber. But as R' belongs to the preimage of R, the surface R is also a virtual fiber. Furthermore, if the surface R is not separating, its homology class is non zero, and the same argument applies to prove that this class is fibered. As the surface R is incompressible, it is itself a fiber.

In the case where the surface R' separates the manifold M' into two connected components M_l and M_r, let S_l and S_r be minimal genus Heegaard surfaces for M_l and M_r respectively. By definition, $\chi_-^h(R') = \max\{|\chi(S_l)|, |\chi(S_r)|\}$.

In each side of R', we can then build a generalized Heegaard decomposition for M_l and M_r in thin position starting from surfaces S_l and S_r. We get then a surface F with one component which is the incompressible surface R', separating the manifold M' into $q \leq 2g(S_l) + 2g(S_r) \leq 2\chi_-^h(R') + 4$ compression bodies $C_1,\ldots,C_q$ with $\chi_-(C_j) \leq \chi_-^h(R')$ for all j.

As F is the union of incompressible surfaces and strictly irreducible Heegaard surfaces, we may assume that F is pseudo-minimal, and F^- is the union of incompressible surfaces.

Thus, to satisfy assumptions of theorem A, it suffices to show that $k\,\chi_-^h(R')$ $\ln\chi_-^h(R') \leq \ln\ln\frac{d}{2\chi_-^h(R')+4}$. But as before, one can find a constant $\ell'' \geq \ell'$ such that if $\ell''\,\chi_-^h(R')\ln\chi_-^h(R') \leq \ln\ln d$, then $k\,\chi_-^h(R')\ln\chi_-^h(R') \leq \ln\ln\frac{d}{2\chi_-^h(R')+4}$. From theorem A, in this case, the manifold M' contains an incompressible surface that is a virtual fiber. But as in the proof of corollary 0.8, this incompressible surface is built in the complement of the incompressible surface R'. Thus, the surface R' is also a virtual fiber. As R' is a lift of R, the starting surface R is a virtual fiber, hence the fiber of a twisted I-bundle, which ends the proof of corollary 0.10. $\square$

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Claire Renard,
École Normale Supérieure de Cachan,
Centre de Mathématiques et de Leurs Applications.
61 avenue du président Wilson
F-94235 CACHAN CEDEX.
claire.renard@normalesup.org