CIRCULAR CHARACTERISTICS AND FIBRATIONS OF HYPERBOLIC CLOSED 3-MANIFOLDS.

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Abstract. This article provides sufficient conditions for a closed hyperbolic 3-manifold M with non zero first Betti number to fiber over the circle, and to find a fiber in M. Those conditions are formulated in terms of the behavior the circular characteristic in finite regular covers of M. We define the circular characteristic as an invariant associated to a non-trivial cohomology class α of M, using a Heegaard characteristic.

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Introduction

Thurston conjectured that every complete hyperbolic, connected and orientable 3-manifold of finite volume virtually fibers over the circle, i.e. such a manifold has a finite covering that is a surface bundle over the circle.

This conjecture received a great deal of attention during the past few years, culminating with the announcement of its proof by Ian Agol very recently (thanks to works of Daniel Wise, Jeremy Kahn and Vladimir Markovic, Frédéric Haglund, Nicolas Bergeron, and many other people). The proof is based on Daniel Wise's program (see [AGM], [Wi2] and [Wi1]).

With this result in mind, an interesting question is to find explicit criteria that are sufficient conditions for a closed hyperbolic 3-manifold M to fiber over the circle. A necessary condition for M to be fibered is that its first Betti number $b_1(M)$ is non zero.

The method is inspired by Lackenby's program to find surface bundles in towers of finite coverings of a given closed hyperbolic 3-manifold (see [L2] and [L1]).

The main idea of this article is to start with a non-trivial cohomology class α in $H^1(M, \mathbb{Z})$ and to study the behavior of a number associated to α called the circular characteristic. This is a kind of Heegaard characteristic, associated to a given non-trivial cohomology class.

Definition 0.1. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. If $\alpha \in H^1(M) = H^1(M, \mathbb{Z})$ is a non-trivial cohomology class, let us denote by $\|\alpha\|$ the Thurston norm of α. By definition,*

$$\|\alpha\| = \min\{\chi_-(R),\, [R] = \mathcal{P}(\alpha)\},$$

*where R is an embedded surface and $\mathcal{P}(\alpha)$ the Poincaré-dual class of α. We will call such a surface R realizing the Thurston norm of α an $\|\alpha\|$-**minimizing surface.***

*If R is a non-separating and $\|\alpha\|$-minimizing surface for a given non-trivial cohomology class $\alpha \in H^1(M)$, take $\mathcal{N}(R) \cong R \times (-1, 1)$ an open regular neighborhood of R in M, and denote by $M_R = M \setminus \mathcal{N}(R)$. For each non-trivial cohomology class $\alpha \in H^1(M)$, denote by $\chi_-^c(\alpha)$ the **circular characteristic** of α. It is the negative* *part of the Euler characteristic of a minimal genus Heegaard surface for M_R, such that this number is minimal over all choices of $\|\alpha\|$-minimizing surface R.*

Remark 0.2. *If α and R are as above and S is a Heegaard surface corresponding to a Heegaard splitting of $(M_R, R\times\{1\}, R\times\{-1\})$ such that $\chi_-(R) = \|\alpha\|$ and $\chi_-(S) = \chi_-^c(\alpha)$, then from the Heegaard decomposition of $(M_R, R\times\{1\}, R\times\{-1\})$, one can easily construct a Heegaard splitting of M by adding two small tubes connecting the surfaces R and S, each in one of the compression bodies of the decomposition of $(M_R, R\times\{1\}, R\times\{-1\})$. An easy calculation shows that*

$$\begin{aligned}\chi_-^h(M) &\leq \chi_-^c(\alpha) + \|\alpha\| + 2\\ &\leq 2\chi_-^c(\alpha) + 2.\end{aligned}$$

Lackenby developed the idea that a control on the growth of the Heegaard genus in a tower of finite covers in terms of the covering degree can lead to fibration results (see [L2] and [L1]). Here, using this number $\chi_-^c(\alpha)$ associated to a given cohomology class α, we get an explicit result. Studying the behavior of this circular characteristic when the class α lifts to finite regular covers of M, we adapted the proof of a theorem of Lackenby [L1, Theorem 1 (3)] to get a statement with explicit bounds and for a given finite cover instead of a tower. With a result of Maher [Mah] about minimal surfaces and explicit geometric constants and functions computed in the thesis [R2] (see also [R1]), we obtain the following theorem, which is the main result of this article.

Theorem 0.3. *Let M be a connected, oriented and closed hyperbolic 3-manifold, and set $\epsilon = \mathrm{Inj}(M)$, where $\mathrm{Inj}(M)$ is the injectivity radius of M.*

There exists an explicit constant $\ell = \ell(\epsilon, \mathrm{Vol}(M))$, depending only on ϵ and the volume of the manifold M, and satisfying the following properties.

Let $\alpha \in H^1(M)$ be a non-trivial cohomology class and R an $\|\alpha\|$-minimizing surface. Let $M' \to M$ be a ***regular*** *finite cover of M of degree d. Let R' be a component of the preimage of R in the cover M', and α' the cohomology class in $H^1(M', \mathbb{Z})$ that is Poincaré-dual to $[R']$.*

If $\ell\,\chi_-^c(\alpha') \leq \sqrt[4]{d}$, then the manifold M fibers over the circle and the surface R is a fiber.

Furthermore, with $a' = 6\left(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\frac{\epsilon}{4})}\right)$ and $D := \frac{8\epsilon \mathrm{Vol}(M)}{\pi(\sinh(2\epsilon)-2\epsilon)}$, one has

$$\ell := \sqrt[4]{\frac{117}{8}}\sqrt{a'\frac{\pi(\sinh(2D+2\epsilon) - 2D - 2\epsilon)}{\mathrm{Vol}(M)}}.$$

The constant ϵ can in fact be any number in $(0, \mathrm{Inj}(M)]$.

Remark 0.4. *The converse of Theorem 0.3 is true in some sense: if R is a fiber surface of M, then the quantity $\ell\,\chi_-^c(\alpha')$ will be constant in the collection of cyclic covers of M.*

Remark 0.5. *The explicit expression of the constant ℓ involved in Theorem 0.3 allows us to study its behavior. If the volume $\mathrm{Vol}(M)$ is fixed and $\mathrm{Inj}(M)$ tends to zero, or if $\mathrm{Inj}(M)$ is fixed and $\mathrm{Vol}(M)$ tends to infinity, ℓ tends to infinity. Thus, the sufficient condition given by the previous theorem becomes more and more difficult to satisfy when the injectivity radius decreases (which corresponds for example to a cusp opening), or if the volume grows (for instance if one passes to finite covers of M).*

The next corollary directly follows from Theorem 0.3.

Corollary 0.6. *Let M be a connected, oriented and closed hyperbolic 3-manifold. Let $\alpha \in H^1(M)$ be a non-trivial cohomology class and R an $\|\alpha\|$-minimizing surface. Let $(M_i \to M)_{i\in\mathbb{N}}$ be a collection of finite regular covers of M with degrees d_i. For each $i \in \mathbb{N}$, let R_i be a component of the preimage of R in M_i, and $\alpha_i \in H^1(M_i)$ the class that is Poincaré-dual to the class of R_i in $H_2(M_i)$. If*

$$\lim_{i\to+\infty} \frac{\chi_-^c(\alpha_i)}{\sqrt[4]{d_i}} = 0,$$

then the manifold M fibers over the circle, and the surface R is a fiber.

The expression $\lim_{i\to+\infty} \frac{\chi_-^c(\alpha_i)}{\sqrt[4]{d_i}}$ is very close to the definition of the infimal Heegaard gradient (see [L2, Definitions p. 319 and p. 339]). The idea is to replace the Heegaard characteristic by the circular characteristic and to study its asymptotic behavior to get results as [L1, Theorem 1 (3)], but related to a specific non-trivial cohomological class in M.

This corollary is true for any infinite collection of finite covers satisfying the given asymptotic condition.

Definition 0.7. *Set $h(M, \alpha, R) = \min\{\chi(R) - \chi(S)\}$, where S is a Heegaard surface for $(M_R, R \times \{1\}, R \times \{-1\})$. Said differently, $\frac{1}{2}h(M, \alpha, R)$ is the minimal number of 1-handles we need to attach to a regular neighborhood of $R \times \{1\}$ in M_R to get the first compression body of a Heegaard splitting of $(M_R, R \times \{1\}, R \times \{-1\})$. Set*

$$h(\alpha) = h(M, \alpha) = \min\{h(M, \alpha, R),\ [R] = \mathcal{P}(\alpha),\ \chi_-(R) = \|\alpha\|\}.$$

With this notation, note that the circular characteristic $\chi_-^c(\alpha)$ is equal to $\|\alpha\| + h(\alpha)$.

The number $h(\alpha)$ can also be viewed as the minimal number of critical points of a circular Morse function for M such that the regular level sets correspond to a surface the homology class of which is Poincaré dual to α. See section 1.

If one considers the tower of cyclic finite covers of M dual to the class α, Theorem 0.3 leads to the following corollary.

Corollary 0.8. *Let M be a connected, oriented and closed hyperbolic 3-manifold. Let $\alpha \in H^1(M)$ be a non-trivial cohomology class and R an $\|\alpha\|$-minimizing surface. Let $(M_i \to M)_{i\in\mathbb{N}}$ be the collection of cyclic finite covers of M dual to the class α, such that for every $i \in \mathbb{N}$, the cover $p_i : M_i \to M$ is regular, with degree i. For each $i \in \mathbb{N}$, let $\alpha_i := p_i^*(\alpha)$ be the cohomology class in $H^1(M_i, \mathbb{Z})$ corresponding to α.*

If there exists $i \geq i_0 = \lceil (2\ell\|\alpha\|)^4 \rceil$ such that

$$\frac{h(\alpha_i)}{\sqrt[4]{i}} \leq \frac{1}{4\ell},$$

then the manifold M fibers over the circle, and the surface R is a fiber .

Outline of the paper: In order to prove Theorem 0.3, we recall in the first section how the theory of generalized Heegaard splittings and thin decompositions can be adapted to circular decompositions (see also [MG]). Results like those of Lackenby [L1, Corollary 4] and [L2, Section 3] can be proven in this setting. With the assumptions and notations of Theorem 0.3, we prove that we can find in M' an embedded surface $\overline{F}$ very close to a minimal surface (what will be called a pseudo-minimal surface). This surface divides M' into a bounded number of compression bodies. The number of connected components of $\overline{F}$ is at most $3/2(\chi_-^c(\alpha') - \|\alpha\|)$ and each component has genus at most $\chi_-^c(\alpha')/2 - 1$.

In the second and last section, we prove Theorem 0.3 thanks to combinatorial arguments. The diameter of a pseudo-minimal surface in a hyperbolic 3-manifold is bounded from above by a constant depending only on ϵ times its Euler characteristic. Therefore, the diameter of $\overline{F}$ is controlled by a function of $\chi_-^c(\alpha')$. As the starting surface R' can also be isotoped to a pseudo-minimal surface, its diameter is bounded from above by a linear function of $-\chi(R') = \|\alpha'\| \leq \chi_-^c(\alpha')$. The idea is to consider the translates of R' under the group $G := \pi_1(M)/\pi_1(M')$ of deck transformations of the regular cover. As there are d such translates, if $\chi_-^c(\alpha')$ is very small compared to d, the diameter of those surfaces and of $\overline{F}$ is very small compared to the number of translates. Intuitively, there must be a lot of translates of R' that are disjoint and do not intersect the surface $\overline{F}$. Indeed, combinatorial arguments show that if $\ell\, \chi_-^c(\alpha') \leq \sqrt[4]{d}$, there are two copies of R' that are disjoint and parallel to the same component of $\overline{F}$, with coherent orientations. Then [L1, Lemma 14] applies to show that the covering M' is fibered, with fiber a copy of R'. Lemma 2.4 of [G] shows then that M is already fibered, with fiber R.

Proofs of Corollaries 0.6 and 0.8 at the end of the second section are then straightforward from Theorem 0.3.

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1. Circular decompositions and thin decomposition.

In order to prove Theorem 0.3, one needs to build a specific decomposition of a closed hyperbolic 3-manifold M'. Suppose that α' is a non-trivial cohomology class in $H^1(M', \mathbb{Z})$. Let R' be an $\|\alpha'\|$-minimizing surface, and such that a minimal-genus Heegaard surface S' for ${M'}_{R'} = M' \backslash \mathcal{N}(R')$ realizes the circular characteristic $\chi_-^c(\alpha')$, i.e. $\chi_-^h(M'_{R'}) = |\chi(S')| = \chi_-^c(\alpha')$.

Starting from the decomposition of M' cut along R' and S' into two compression bodies, we wish to find a way to build another decomposition by a bounded number of connected surfaces into compression bodies such that the diameter of the separating surfaces is bounded from above by a function depending only on $\chi_-^c(\alpha')$, $\epsilon \leq \mathrm{Inj}(M')$ and $\mathrm{Vol}(M')$. This control on the diameter can be obtained if the separating surfaces have their Euler characteristics at least $\chi(S')$ and if they are isotopic to minimal surfaces or to surfaces as closed as wanted to minimal surfaces. This is the case if those surfaces are obtained from S' by surgery and correspond to incompressible surfaces or strongly irreducible Heegaard surfaces. This occurs in a thin Heegaard decomposition. Thus, the aim is to find a kind of Heegaard decomposition adapted to this situation, keeping track of the given surface R' corresponding to the given non-trivial homology class. This is done by the notion of circular decomposition, and to build the desired decomposition from the decomposition of M' by R' and S'

corresponds to the construction of a thin circular decomposition. That is the object of this first section.

A circular decomposition is the equivalent of a Heegaard decomposition, but this decomposition is associated to a Morse function that no longer takes values in $I = [0,1]$ but in the circle $\mathbb{S}^1$. According to [MG], we have the following definitions.

Definition 1.1. *A* ***circular Morse function*** *is a Morse function $f : M \to \mathbb{S}^1$.*

If $f : M \to \mathbb{S}^1$ is a circular Morse function, the handle decomposition of M given by the function f is called the ***circular decomposition associated to*** *f.*

See F. Manjarrez-Gutiérrez [MG], Matsumoto [Mat] and Milnor [Mi] for further details about circular Morse functions. Let $f : M \to \mathbb{S}^1$ be a circular Morse function. If we remove a small open neighborhood of a regular value $x \in \mathbb{S}^1$, by restriction of f, we obtain a Morse function g of $M_R = M \setminus \mathcal{N}(R)$, which is the manifold M minus a small regular open neighborhood of the surface $R := f^{-1}(\{x\})$, on the interval I. Thus, the theory of Heegaard splittings and generalized Heegaard splittings applies to the function g.

Another viewpoint is to see a circular decomposition as a handle decomposition of the cobordism $(M \setminus \mathcal{N}(R), R \times \{1\}, R \times \{-1\})$. Starting with a Heegaard splitting of Heegaard surface S for $M_R = M \setminus \mathcal{N}(R)$, one can change the order in which 1- and 2-handles are attached to get a new generalized Heegaard splitting $(F_1 = R \times \{1\}, S_1, F_2, \ldots, S_n, F_{n+1} = R \times \{-1\})$ for $(M_R, R \times \{1\}, R \times \{-1\})$. Gluing back $R \times \{1\}$ to $R \times \{-1\}$, one obtains a circular decomposition for the manifold M. Denote it by $\mathcal{H} = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$, with $F_1 = F_{n+1} = R$. The surfaces F_j divide M into n 3-manifolds with boundary $W_1, \ldots, W_n$, and surfaces S_j are Heegaard surfaces for those manifolds. For $1 \leq j \leq n$, S_j divides the manifold W_j into two compression bodies A_j and B_j, such that $\partial_+ A_j = \partial_+ B_j = S_j$, $\partial_- A_j = F_j$ and $\partial_- B_j = F_{j+1}$.

Let S be a closed surface. If S is connected, recall that the **complexity** of S is $c(S) = \max(0, 2g(S) - 1)$. If S is the union of several connected components, the complexity of S is the sum of the complexities of the components of S. There is a definition of the complexity of a circular decomposition analogous to the complexity of a generalized Heegaard splitting.

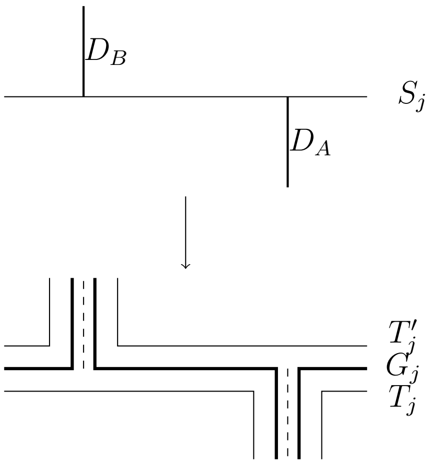
Definition 1.2. *The* ***circular width*** *of a circular decomposition $\mathcal{H} = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$ is the set of the n integers $(c(S_1), \ldots, c(S_n))$, with repetitions and arranged in monotonically non-increasing order. Widths are compared using the lexicographic order.*

The integer $n \geq 1$ is called the ***length*** *of the circular decomposition $\mathcal{H} = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$.*

If S is an embedded closed surface in a 3-manifold M which is not a collection of spheres, and c an essential simple closed curve on S bounding a disc D in M, one can cut the surface S open along c and glue two copies of D to cap off the holes and get a new closed surface S' of smaller complexity. This operation is called a **surgery**.

Take $\mathcal{H} = (F_1, S_1, \ldots, S_n, F_{n+1})$ a circular decomposition for M, such that for some index j, the Heegaard surface S_j for (A_j, B_j) is weakly reducible. As the Heegaard surface S_j is weakly reducible, there exists a pair of disjoint compression discs for S_j, say D_A embedded in A_j and D_B in B_j. Performing surgeries along those two discs, one gets a new circular decomposition $\mathcal{H}' := (F_1, \dots, F_j, T_j, G_j, T_j', F_{j+1}, \dots, F_{n+1})$, where the surface T_j is obtained from S_j by surgery along D_A, T_j' from S_j by surgery along D_B, and G_j from S_j by surgery along D_A and D_B.

Definition 1.3. *This operation will be called a **surgery** of circular decompositions.*



Proposition 1.4. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. Let R be an orientable, closed, non-separating, incompressible and embedded surface in M. Denote by S a Heegaard surface for $M \setminus \mathcal{N}(R)$. Starting from the circular decomposition $\mathcal{H} = (R, S, R)$ of M, there exists a finite number of surgeries to get a circular decomposition $\mathcal{H}' = (F_1, S_1, F_2, \dots, S_n, F_{n+1})$ with $F_1 = F_{n+1} = R$ and where parallel surfaces have been amalgamated, such that:*

1. *the circular width of $\mathcal{H}'$ is minimal among the widths of such circular decompositions obtained by a finite number of surgeries of $\mathcal{H}$,*
2. *each surface S_j is a strongly irreducible Heegaard surface for the Heegaard decomposition (A_j, B_j) of W_j and $g(S_j) \le g(S)$,*
3. *each surface F_j is incompressible, no component of F_j is a sphere, and $g(F_j) \le g(S)$,*
4. $n \le \frac{1}{2}(\chi(R) - \chi(S))$,
5. $\chi(R) - \chi(S) = \sum_{j=1}^{n}(\chi(F_j) - \chi(S_j))$.
6. *Furthermore, if the decomposition $\mathcal{H}'$ is of length at least 2, for every j, the surfaces F_j and F_{j+1} are not parallel.*

Definition 1.5. *Let $\mathcal{H}$ be a circular decomposition. A circular decomposition $\mathcal{H}' = (F_1, S_1, F_2, \dots, S_n, F_{n+1})$ that is circular-length-minimizing among all circular decompositions obtained from $\mathcal{H}$ by a finite number of surgeries and amalgamation of parallel components is said to be a **thin decomposition**. We will call such a decomposition a **thin circular decomposition associated to $\mathcal{H}$**.*

Proof of Proposition 1.4.

The proof of the first three points of this proposition is based on the proof of [MG, Theorem 3.2], which is itself an adaptation of techniques of [ST2] to the case of circular decompositions. Recall here the arguments (see also [L2, section 3]).

Starting with the circular decomposition $\mathcal{H} = (R, S, R)$, the aim is to perform a certain number of surgeries to obtain a thin decomposition, i.e. of minimal complexity. Each surgery corresponds to a change on the order in which 1- and 2-handles are attached, such that a surgery strictly decreases the circular width of the decomposition. Thus, the number of necessary surgeries to get a thin decomposition is finite.

Lemma 1.6. *Let $\mathcal{H} = (F_1, S_1, \ldots, S_n, F_{n+1})$ be a circular decomposition for M, and suppose that for some index j, the Heegaard surface S_j for (A_j, B_j) is weakly reducible. If $\mathcal{H}'$ is the new circular decomposition obtained from $\mathcal{H}$ by surgery, the circular width of $\mathcal{H}'$ is strictly smaller.*

Proof of Lemma 1.6.
Set $\mathcal{H}' = (F_1, \ldots, F_j, T_j, G_j, T'_j, F_{j+1}, \ldots, F_{n+1})$ the new circular decomposition obtained from $\mathcal{H}$ by surgery as described above. As $|\chi(T_j)| = |\chi(T'_j)| = |\chi(S_j)| - 2$, the circular width of this new circular decomposition is strictly smaller than this of $\mathcal{H}$. □

As $\chi(T_j) = \chi(T'_j) = \chi(S_j) + 2$ and $\chi(G_j) = \chi(S_j) + 4$, one obtains $-\chi(S_j) = -\chi(T_j) + \chi(G_j) - \chi(T'_j)$. Thus, this surgery procedure does not modify the alternate sum $\sum(\chi(F_j) - \chi(S_j))$, proving point (5).

As this surgery procedure strictly decreases the circular width of the decomposition, there exists a finite number of such surgeries to get a circular decomposition $\mathcal{H}' = (F_1, S_1, \ldots, S_n, F_{n+1})$ of minimal circular width among the set of all decompositions obtained by surgeries from the starting circular decomposition $\mathcal{H} = (R, S, R)$.

To prove (2), recall [MG]. Just notice that if one of the Heegaard surfaces S_j is not strongly irreducible, from Lemma 1.6, one can perform another surgery to obtain a new circular decomposition of circular width strictly smaller than this of $\mathcal{H}'$, which is a contradiction if $\mathcal{H}'$ is a length-minimizing decomposition.

The proof of point (3) is done in [MG]. The surface $R = F_1 = F_{n+1}$ is incompressible. Suppose by contradiction that one of the surfaces F_j is compressible, for an index j between 2 and n. There exists then a compression disc D for F_j. Taking an innermost disc, one can furthermore assume that $D \cap (\cup_{k=1}^{n} F_k) = D \cap F_j = \partial D$. Thus, the disc D entirely lies in the region W_{j-1} bounded by the two surfaces F_{j-1} and F_j, or is entirely embedded in the region W_j bounded by F_j and F_{j+1}. Suppose for example that D is entirely embedded in W_j. From the boundary version of the Haken Lemma [H], as W_j is ∂-reducible, every Heegaard splitting of W_j is reducible, hence weakly reducible. This is a contradiction with point (2), proving the first part of point (3).

If one of the components of a surface F_j is a 2-sphere, as M is irreducible, this sphere bounds an embedded ball in M. Taking an innermost sphere, one obtains a sphere bounding the Heegaard splitting of a 3-ball. But this splitting, if not trivial, is reducible (see [Wa]), hence weakly reducible, contradicting point (2). This ends the proof of point (3).

To prove point (4), notice that the surgery process as described above is in fact a change on the order in which the handles are attached. More precisely, with the notations above, if we consider a handle decomposition associated to $\mathcal{H}$ where 1- and 2-handles correspond to meridian discs for the Heegaard splittings, a surgery is a handle reordering. The 2-handle corresponding to the meridian disc D_B is attached before the 1-handle corresponding to D_A. Thus, this process does not change the number of 1- and 2-handles. In the starting circular decomposition $\mathcal{H} = (R, S, R)$, the number of 1- and 2-handles is equal to $\chi(R) - \chi(S)$. So after each surgery, there are still $\frac{1}{2}(\chi(R)-\chi(S))$ 1-handles and $\frac{1}{2}(\chi(R)-\chi(S))$ 2-handles. As the number of regions of a circular decomposition $\mathcal{H}'$ is at most the number of 1- and 2-handles in this decomposition, there are at most $\chi(R)-\chi(S)$ regions in $\mathcal{H}$. Therefore, the number of even surfaces F_j is bounded above by $\frac{1}{2}(\chi(R)-\chi(S))$. In other words, $n \leq \frac{1}{2}(\chi(R)-\chi(S))$, which proves point (4).

Finally, for point (6) we recall the argument of [L2, Section 3]. If the length of the decomposition is just 1, this means that there is only one incompressible surface $F_1 = R = F_2$.

If the length of the decomposition $\mathcal{H}'$ is at least 2, parallel surfaces can be amalgamated. Indeed, suppose that there exists two parallel surfaces F_j and F_{j+1} for some j. From point (2), the surface S_j is a strongly irreducible Heegaard surface for the product region bounded by F_j and F_{j+1}. From the classification of Heegaard splittings for products (see [ST1]), this means that S_j is parallel to F_j. The two surfaces F_j and F_{j+1} can then be amalgamated to a single surface, forgetting the surface S_j, to obtain a new circular decomposition with complexity strictly smaller than this of $\mathcal{H}'$ and still verifying the other points of Proposition 1.4. □

Corollary 1.7. *Let M, R and S be as above, and $\mathcal{H}' = (F_1 = R, S_1, \dots, F_{n+1} = R)$ a thin circular decomposition associated to (R, S, R). Let $\overline{F} = \bigcup_j F_j \cup \bigcup_j S_j$. Then,*

(1) $\left|\chi(\overline{F})\right| \leq |\chi(S)-\chi(R)|\,|\chi(S)|$, *and*
(2) *the surface $\overline{F}$ has at most $\frac{3}{2}|\chi(S)-\chi(R)|$ connected components.*

Proof of Corollary 1.7.

We adapt here the proof of [L1, Corollary 4]. First, notice that no compression body in the complement of $\overline{F}$ is a punctured 3-ball, as no component of $\bigcup_j F_j \cup \bigcup_j S_j$ is a 2-sphere.

As M is hyperbolic, no compression body of the thin circular decomposition can be a solid torus.

Remark 1.8. *Another way to prove point (4) of Proposition 1.4 starting from point (5) is the following.*

Recall that $F_1 = R = F_{n+1}$. Point (5) of Proposition 1.4 can also be written:
(1)
$$\chi(R)-\chi(S) = \frac{\chi(F_1)-\chi(S_1)}{2} + \frac{\chi(F_2)-\chi(S_1)}{2} + \frac{\chi(F_2)-\chi(S_2)}{2} + \ldots + \frac{\chi(F_{n+1})-\chi(S_n)}{2}.$$

If H is a compression body that is not a punctured 3-ball, nor a solid torus, nor a product, then $\chi(\partial_- H) - \chi(\partial_+ H) > 0$ and this integer is even. As the $2n$ components of the complementary of $\bigcup_j F_j \cup \bigcup_j S_j$ are such compression bodies, the right hand side of equality (1) is bounded from below by $2n$. Therefore, $2n \leq \chi(R)-\chi(S)$. It is exactly point (4) of Proposition 1.4.

Thus,

$$\begin{aligned} \left|\chi(\overline{F})\right| = \left|\chi\Big(\bigcup_j F_j \cup \bigcup_j S_j\Big)\right| &= \sum_{j=1}^{n} |\chi(F_j)| + \sum_{j=1}^{n} |\chi(S_j)| \\ &\leq 2n\,|\chi(S)| \\ &\leq |\chi(R)-\chi(S)|\,|\chi(S)|\,. \end{aligned}$$

If H is a compression body that is not a punctured 3-ball, nor a solid torus, nor a product, one can check that $|\partial H| \leq \frac{3}{2}(\chi(\partial_- H) - \chi(\partial_+ H))$. The sum over all compression bodies H in the complement of $\bigcup_j F_j \cup \bigcup_j S_j$ of $\chi(\partial_- H) - \chi(\partial_+ H)$ is equal to $\sum_H (\chi(\partial_- H) - \chi(\partial_+ H)) = 2\sum_{j=1}^n (\chi(F_j) - \chi(S_j)) = 2(\chi(R) - \chi(S))$. Now, the number of components of $\overline{F}$ is at most $\frac{1}{2}\sum_H |\partial H|$, where H describes all compression bodies that are the components of $M \setminus \overline{F}$ which are not product regions. But

$$\begin{aligned}\frac{1}{2}\sum_H |\partial H| &\leq \frac{1}{2}\sum_H \frac{3}{2}(\chi(\partial_- H) - \chi(\partial_+ H)) \\ &= \frac{3}{2}\sum_{j=1}^n (\chi(F_j) - \chi(S_j)) \\ &= \frac{3}{2}\left|\chi(R) - \chi(S)\right|.\end{aligned}$$

Therefore, $\left|\overline{F}\right| \leq \frac{3}{2}\left|\chi(R) - \chi(S)\right|$, which ends the proof of Corollary 1.7. $\square$

Thus, from the starting circular decomposition (R, S, R) of M, one can build a decomposition of M into compression bodies by a surface $\overline{F}$ of bounded complexity. The number of connected components of $\overline{F}$ is also bounded, and each component corresponds to an incompressible surface, or to a strongly irreducible Heegaard surface for some submanifold of M.

The proof of Theorem 0.3 will require us to control the diameter of the surface $\overline{F} = \bigcup_j F_j \cup \bigcup_j S_j$. A way to do this is to control the metric of its components, for example by showing that up to isotopy, they are very closed to minimal surfaces.

Definition 1.9. *An embedded surface S in a Riemannian 3-manifold M is called **pseudo-minimal** if it is orientable, closed, and S is a minimal surface or the boundary of a regular neighborhood of a minimal non-orientable surface, possibly with a little tube attached vertically in the I-bundle structure.*

Part (1) of the following theorem comes from results of Frohman, Freedman, Hass and Scott about incompressible surfaces ([FHS] and [FH]). Part (2) is a result of Pitts and Rubinstein ([PR], see also [S, Existence Theorem of minimal surfaces], [CDL] and [P]).

Theorem 1.10. *Let N be a connected, oriented and closed hyperbolic 3-manifold.*

(1) *Any incompressible surface in N can be isotoped to a minimal surface or the boundary of a small neighborhood of a non-orientable minimal surface.*

(2) *Any embedded surface corresponding to a strongly irreducible Heegaard surface for a region of N lying between two (possibly empty) embedded, incompressible and pseudo-minimal surfaces as above can be isotoped to a minimal surface, or to the boundary of a small regular neighborhood of a non-orientable minimal surface, with a small tube attached vertically in the I-bundle structure.* $\square$

The next corollary directly follows from Theorem 1.10 combined with Proposition 1.4.

Corollary 1.11. *Let M be a hyperbolic, connected, oriented and closed 3-manifold. Take $\mathcal{H} = (F_1, S_1, F_2, \ldots, S_n, F_{n+1})$ a thin circular decomposition of M. Then, up to isotopy, one can assume that all surfaces F_j and S_j are pseudo-minimal.* $\square$

2. Homology classes and fibration of finite regular covers.

The aim of this section is to prove Theorem 0.3 and Corollaries 0.6 and 0.8.

Proof of Theorem 0.3.

The proof is an adaptation of the proof of [L1, Theorem 1 (3)] for a given cover M' of M and a given non-trivial homology class, together with some calculations of explicit constants.

Let M be a connected, oriented and closed hyperbolic 3-manifold as in the assumptions of Theorem 0.3, and $\epsilon \leq \mathrm{Inj}(M)$. Let $\alpha \in H^1(M)$ be a non-trivial cohomology class and R an $\|\alpha\|$-minimizing surface. Let $M' \to M$ be a **regular** finite cover of M with degree d. Let R' be a connected component of the preimage of R in the cover M', and α' the cohomology class in $H^1(M', \mathbb{Z})$ that is Poincaré-dual to $[R']$.

Claim . *To prove Theorem 0.3, one can assume in addition that the surface R' is $\|\alpha'\|$-minimizing and that a minimal-genus Heegaard surface S' for $M'_{R'}$ realizes the circular characteristic $\chi_-^c(\alpha')$, i.e. $\chi_-^h(M'_{R'}) = |\chi(S')| = \chi_-^c(\alpha')$, where ${M'}_{R'} = M' \setminus \mathcal{N}(R')$.*

Proof of claim.

Let us show that this is not a problem to furthermore assume that the surface R' is $\|\alpha'\|$-minimizing and $\chi_-^c(\alpha')$-minimizing. Suppose that the theorem has been proven in this particular case, and take R' a component of the preimage of R in M' such that $[R'] = \alpha'$. If R'' is an $\|\alpha'\|$-minimizing and $\chi_-^c(\alpha')$-minimizing surface, as we suppose that the theorem is proven in this case, M' fibers over the circle and R'' is a fiber. But as the surface R' is a component of the preimage of R, it is incompressible and in the same homology class as R''. Thus it is also a fiber. By a result of Gabai [G, Lemma 2.4], the homology class of R is also fibered in M. As R is an embedded and incompressible surface (as it is $\|\alpha\|$-minimizing), this means that the manifold M fibers over the circle, and that R is a fiber. □

It remains to show that under the assumptions of Theorem 0.3, and if in addition R' is $\|\alpha'\|$-minimizing and $\chi_-^c(\alpha')$-minimizing, then it is a fiber of a fibration of M' over the circle.

Applying Proposition 1.4 to the circular decomposition (R', S', R'), one obtains a thin circular decomposition $\mathcal{H}' = (F_1, S_1, \ldots, F_{n+1})$ associated to (R', S', R'), where $F_1 = F_{n+1} = R'$. Moreover, $\overline{F} = \bigcup F_j \cup \bigcup S_j$ is a pseudo-minimal surface, and it follows from Corollary 1.7 that $\left|\chi(\overline{F})\right| \leq |\chi(R') - \chi(S')|\,|\chi(S')| \leq \chi_-^c(\alpha')^2$.

The interest of pseudo-minimal surfaces is that their metric is not far from the metric of a minimal surface. Lackenby and Maher have shown that the diameter of a minimal surface can be bounded from above by a constant depending only on ϵ times its Euler characteristic (see [L2, Proposition 6.1] and [Mah, Lemma 4.2 p. 2249]). We establish the same kind of result for pseudo-minimal surfaces.

As $\overline{F}$ and R' are pseudo-minimal, their diameter is then bounded from above by a constant times respectively $\left|\chi(\overline{F})\right| \leq \chi_-^c(\alpha')^2$ and $|\chi(R')| = \|\alpha'\| \leq \chi_-^c(\alpha')$. The idea is to consider the translates of R' into the action of the group $G := \pi_1(M)/\pi_1(M')$ of deck transformations of the regular cover $M' \to M$. There are d such copies. If d is big enough and $\chi_-^c(\alpha')$ rather small, there are plenty of such copies, with bounded diameter as they are isometrical to R'. As the diameters of $\overline{F}$ and R' are bounded and small compared to d, there are lots of copies of R' that are disjoint and can not intersect $\overline{F}$. A copy of R' not intersecting $\overline{F}$ is an incompressible surface lying in the complement of $\overline{F}$, a union of compression bodies. Therefore, it must be parallel to a negative component of the boundary of the compression body, which is an incompressible component of $\overline{F}$.

As the number of components of $\overline{F}$ is bounded, if the number d of copies of R' is really big, there must be a large number of copies of R' that are disjoint and parallel to the same component of $\overline{F}$. Eventually, we show that if the inequality of Theorem 0.3 is satisfied, then there are at least two such surfaces with coherent orientation in the product region they bound. This means that they are in fact fibers of a fibration of M' over the circle. This is the idea of the proof of [L1, Theorem 1 (3)], together with explicit calculations of geometric constants and combinatorial arguments.

Let us detail and make explicit calculations. First, let us try and estimate the number of copies of R' that are disjoint and in the complement of $\overline{F}$.

As the surface $\overline{F}$ is not connected, it is more interesting to work with the notion of ϵ-diameter instead of diameter for surfaces. Indeed, with this definition, small components that are far apart are still considered "small".

Definition 2.1. *Let $\epsilon > 0$. The **ϵ-diameter** of a non-necessarily connected surface F is the minimal number of balls of radius ϵ for the metric of F required to cover the surface F.*

The following lemma is an adaptation of Lackenby and Maher's results for pseudo-minimal surfaces: there is still an explicit bound on their ϵ-diameter.

Lemma 2.2. *Let S be an embedded pseudo-minimal surface in N, a Riemannian closed 3-manifold, whose sectional curvature is at most -1. Let $\epsilon \leq \mathrm{Inj}(N)$ and*

$$a' = 6\left(\frac{21}{4} + \frac{3}{4\pi} + \frac{3}{4\epsilon} + \frac{2}{\sinh^2(\frac{\epsilon}{4})}\right).$$

Then the ϵ-diameter of the surface S is bounded from above by $a'\,|\chi(S)|\,/3$. Furthermore, the ϵ-diameter of a pseudo-minimal surface Σ homotopic to S and close enough is at most $a'\,|\chi(\Sigma)|$.

Proof of Lemma 2.2.

This lemma is a direct consequence of [Mah, Lemma 4.2 p. 2249] and [L2, Proposition 6.1] in the case the surface S is minimal and orientable, and we can take $a'/6$ instead of a'. If S is minimal, but not orientable, its homology class $[S]$ is non zero in $H_2(N, \mathbb{Z}/2\mathbb{Z})$. By Poincaré's duality, it corresponds to a non-trivial element $\alpha \in H^1(N, \mathbb{Z}/2\mathbb{Z})$. As the homology class of the double cover of S can be represented by the boundary of a small regular neighborhood of the non-orientable surface S, we have $2[S] = 0$ in $H_2(N, \mathbb{Z})$. If we take the double cover N' of N corresponding to the kernel of α, the surface S lifts to a minimal orientable surface S'. We can apply the orientable version of Lemma 2.2, and bound the ϵ-diameter of S' by $a'/6\,|\chi(S')| = a'/6 \times 2\,|\chi(S)| = a'/3\,|\chi(S)|$. As this number bounds also from above the ϵ-diameter of S, this proves the lemma for a minimal non orientable surface, with $a'/3$ instead of a'.

If the surface S is just pseudo-minimal, it is the boundary of an arbitrarily small regular neighborhood of a minimal surface S'. As the diameter is at most $a'/3\,|\chi(S')|$, with $|\chi(S)| \leq 2\,|\chi(S')|$, this ends the proof of Lemma 2.2. $\square$

With this lemma, the ϵ-diameter of the surfaces $\overline{F}$ and R' is explicitly bounded from above. To study the geometric behavior of the translates of R', an idea is to look for an explicit control on the geometry of fundamental polyhedra for M in M', which are also translated under the action of G.

Let $\mathcal{D}$ be a Dirichlet fundamental polyhedron for the manifold M in its universal cover $\mathbb{H}^3$. The union of the translates of $\mathcal{D}$ under the action of the fundamental group of M composes a tiling of $\mathbb{H}^3$. By the covering map $\mathbb{H}^3 \to M'$, this tiling projects to a tiling of M' by d copies of $\mathcal{D}$. Let $\mathcal{D}'$ be one of those polyhedra. As the cover $M' \to M$ is regular, the tiling of M' is the union of the translates of $\mathcal{D}'$ under the action of the group $G = \pi_1(M)/\pi_1(M')$.

The following lemma is a way to bound the diameter of a fundamental polyhedron $\mathcal{D}$ in $\mathbb{H}^3$ in terms of the volume of the manifold M and a lower bound for its injectivity radius. The second part allows us to explicitly estimate the number of such polyhedra a surface of given ϵ-diameter can intersect.

Lemma 2.3. *Let $\mathcal{D}$ be a Dirichlet fundamental polyhedron for the manifold M, embedded in the universal cover $\widetilde{M} \simeq \mathbb{H}^3$. Let D be an upper bound for the diameter of $\mathcal{D}$ in $\mathbb{H}^3$. We have the following estimate:*

$$\mathrm{diam}(\mathcal{D}) \leq \frac{8\epsilon \mathrm{Vol}(M)}{\pi(\sinh(2\epsilon) - 2\epsilon)} = D. \tag{2}$$

If S is an embedded surface in the finite cover M' of M, which can be covered by at most λ embedded balls in M' of radius $\epsilon \leq \mathrm{Inj}(M)$, then S intersects at most L images of $\mathcal{D}$ in M', with

$$L = \lfloor \frac{\pi(\sinh(2D + 2\epsilon) - 2D - 2\epsilon)}{\mathrm{Vol}(M)} \lambda \rfloor. \tag{3}$$

$\square$

<u>Proof of Lemma 2.3.</u>

To prove inequality (2), first notice that $\mathrm{diam}(\mathcal{D}) \leq 2\,\mathrm{diam}(M)$. To prove it, recall that there exists $w \in \mathbb{H}^3$ such that $\mathcal{D} = \{x \in \mathbb{H}^3\,,\ \mathrm{d}(\gamma(w), x) \geq \mathrm{d}(w, x)\ \forall \gamma \in \pi_1(M)\}$. If x and $y \in \mathcal{D}$ satisfy $\mathrm{d}(x, y) = \mathrm{diam}(\mathcal{D})$, then

$$\mathrm{diam}(\mathcal{D}) = \mathrm{d}(x, y) \leq \mathrm{d}(x, w) + \mathrm{d}(y, w) \leq 2\,\mathrm{diam}(M).$$

Take x and $y \in M$ such that $\mathrm{d}(x, y) = \mathrm{diam}(M)$, and let γ be a minimizing geodesic from x to y. By definition, $\mathrm{length}\,(\gamma) = \mathrm{diam}(M)$. Let $\mathcal{B}$ be a collection of points in γ which is maximal among collections of points of γ such that two balls of radius ϵ and whose centers are two distinct points of $\mathcal{B}$ have disjoint interiors. Then, by maximality of $\mathcal{B}$, the union of balls with centers in $\mathcal{B}$ and radius 2ϵ cover the geodesic γ.

Thus, $|\mathcal{B}| \geq \frac{\mathrm{length}\,(\gamma)}{4\epsilon}$. As balls of centers in $\mathcal{B}$ and radius ϵ have disjoint interiors, considering volumes, we deduce:

$$
\begin{aligned}
\mathrm{Vol}(M) &\geq \sum_{u\in\mathcal{B}} \mathrm{Vol}(B(u,\epsilon)) \\
&\geq \frac{\text{length}\,(\gamma)}{4\epsilon}\mathrm{Vol}(B_{\mathbb{H}^3}(\epsilon)) \\
&\geq \frac{\mathrm{diam}(M)}{4\epsilon}\pi(\sinh(2\epsilon)-2\epsilon),
\end{aligned}
$$

proving inequality (2).

To prove inequality (3), denote by $\mathcal{B}$ the set of the centers of a collection of K embedded balls in M' of radius ϵ' covering the surface S. Let $\mathcal{N} = \cup_{x\in\mathcal{B}} B(x, D+\epsilon')$. Those balls are not necessarily isometric to hyperbolic embedded balls in $\mathbb{H}^3$ as $D+\epsilon' > \mathrm{Inj}(M)$. However, let us show that $\mathcal{N}$ contains every fundamental polyhedron of M' intersecting S.

To prove it, let x be a point in a fundamental polyhedron of M' intersecting S. Take $y \in S$ such that $\mathrm{d}(x,y) = \mathrm{dist}(x,S) \leq D$. As y is a point of S, there exists a ball $B(x,\epsilon')$ with $x \in \mathcal{B}$ containing y. Therefore $\mathrm{d}(z,x) \leq \mathrm{d}(z,y)+\mathrm{d}(y,x) \leq D+\epsilon'$, showing that $z \in B(x,\epsilon'+D) \subset \mathcal{N}$.

Comparing volumes, we get:

$$
\begin{aligned}
L\,\mathrm{Vol}(\mathcal{D}) &\leq \mathrm{Vol}(\mathcal{N}) \\
L\,\mathrm{Vol}(M) &\leq |\mathcal{B}|\,\mathrm{Vol}(B_{\mathbb{H}^3}(\epsilon'+D)) \\
L &\leq \frac{\pi(\sinh(2\epsilon'+2D)-2\epsilon'-2D)}{\mathrm{Vol}(M)}K,
\end{aligned}
$$

proving inequality (3), as L is an integer. □

In the sequel, set $a' = 6\left(\frac{21}{4}+\frac{3}{4\pi}+\frac{3}{4\epsilon}+\frac{2}{\sinh^2(\frac{\epsilon}{4})}\right)$ and $D := \frac{8\epsilon\mathrm{Vol}(M)}{\pi(\sinh(2\epsilon)-2\epsilon)}$ as in Lemmas 2.2 and 2.3. As D is an upper bound for the diameter of $\mathcal{D}$ in $\mathbb{H}^3$, it is also an upper bound for the diameter of $\mathcal{D}'$ in M'.

As the surface $\overline{F}$ and the translates of R' are pseudo-minimal surfaces, we then obtain explicit bounds for the number of copies of $\mathcal{D}'$ they intersect.

Lemma 2.4. *Set $\kappa := a'\frac{\pi(\sinh(2D+2\epsilon)-2D-2\epsilon)}{\mathrm{Vol}(M)}$. If Σ is a pseudo-minimal surface in M', Σ intersects at most $\kappa\,|\chi(\Sigma)|$ translates of $\mathcal{D}'$ under the action of the group $G = \pi_1(M)/\pi_1(M')$ of deck transformations of the regular cover. From another viewpoint, for a given translate of $\mathcal{D}'$ in M', there exist at most $\kappa\,|\chi(\Sigma)|$ copies of Σ under the action of G which intersect it.*

Proof of Lemma 2.4.

Lemma 2.4 is straightforward from inequality (3) of Lemma 2.3. The embedded surface Σ in M' can be covered by at most $a'\,|\chi(\Sigma)|$ embedded balls in M' of radius ϵ. Therefore, this surface cannot intersect more than $\lfloor\frac{\pi(\sinh(2D+2\epsilon)-2D-2\epsilon)}{\mathrm{Vol}(M)}a'\,|\chi(\Sigma)|\rfloor \leq \frac{\pi(\sinh(2D+2\epsilon)-2D-2\epsilon)}{\mathrm{Vol}(M)}a'\,|\chi(\Sigma)|$ translates of $\mathcal{D}'$ in M'. □

Lemma 2.4 applies to the pseudo-minimal surface $\overline{F}$. Thus, this surface intersects at most $\kappa\,\left|\chi(\overline{F})\right| \leq \kappa\,\chi_-^c(\alpha')^2$ translates of $\mathcal{D}'$ in M'. Let B be the subset of the corresponding elements of G.

Let also C be the subset of G corresponding to the translates of $\mathcal{D}'$ that intersect R'. Applying Lemma 2.4 again, $|C| \leq \kappa\,|\chi(R')| = \kappa\|\alpha'\|$.

The fundamental polyhedra are somehow intermediates to estimate the number of disjoint translates of R' that in addition do not intersect $\overline{F}$. This is the crucial following lemma. This lemma and its proof are adapted from the proof of [L1, Lemma 13], but with explicit bounds.

Lemma 2.5. *Set* $\ell := \sqrt[4]{117\kappa^2/8}$.

If $\ell\,\chi_-^c(\alpha') \leq \sqrt[4]{d}$, *under the action of* G, *there are at least* $m' = 9\chi_-^c(\alpha')/2$ *translates of* R' *that are disjoint and do not intersect* $\overline{F}$.

Proof of Lemma 2.5.

By contradiction, suppose that the lemma is false. Then, for each m'-tuple $(g_1R', \dots, g_{m'}R')$ of translates of R', at least two of them intersect, or at least one of them intersects $\overline{F}$. There exist j and k, with $1 \leq j < k \leq m'$, c_1 and $c_2 \in C$ such that $g_jc_1 = g_kc_2$, or there exist $b \in B$, $c_1 \in C$ and s such that $g_sc_1 = b$. In the first case, $g_k^{-1}g_j \in CC^{-1}$, and in the second case, $g_s \in BC^{-1}$. This means that $G^{m'}$ is the union of the sets $q_{jk}^{-1}(CC^{-1})$ and $p_s^{-1}(BC^{-1})$, where for $1 \leq j < k \leq m'$ and $1 \leq s \leq m'$, q_{jk} and p_s are the maps

$$\begin{array}{rcl} q_{jk} \,:\, G^{m'} & \to & G \\ (g_1, \dots, g_{m'}) & \mapsto & g_k^{-1}g_j \\ p_s \,:\, G^{m'} & \to & G \\ (g_1, \dots, g_{m'}) & \mapsto & g_s. \end{array}$$

The cardinality of $q_{jk}^{-1}(CC^{-1})$ is $|G|^{m'-1}\,|CC^{-1}|$, and the cardinality of $p_s^{-1}(BC^{-1})$ is $|G|^{m'-1}\,|BC^{-1}|$. Thus,

$$\begin{array}{rcl} |G|^{m'} & \leq & \binom{m'}{2}|G|^{m'-1}\,|C|^2 + m'\,|G|^{m'-1}\,|C|\,|B| \\ d^{m'} & \leq & \binom{m'}{2}d^{m'-1}(\kappa\|\alpha'\|)^2 + m'd^{m'-1}\kappa\|\alpha'\|\kappa\,\chi_-^c(\alpha')^2. \end{array}$$

As $\|\alpha'\| = |\chi(R')| \leq |\chi(S')| = \chi_-^c(\alpha')$, one has

$$d \leq \frac{\kappa^2}{2}m'(m'-1)\chi_-^c(\alpha')^2 + \kappa^2m'\chi_-^c(\alpha')^3. \tag{4}$$

As $m' = 9\chi_-^c(\alpha')/2$, this leads to

$$\begin{array}{rcl} d & \leq & \frac{9\kappa^2}{4}\chi_-^c(\alpha')(\frac{9\chi_-^c(\alpha')}{2} - 1)\chi_-^c(\alpha')^2 + \frac{9\kappa^2}{2}\chi_-^c(\alpha')^4 \\ & \leq & \frac{117\kappa^2}{8}\chi_-^c(\alpha')^4 - \frac{9\kappa^2}{4}\chi_-^c(\alpha')^3 \\ & < & \frac{117\kappa^2}{8}\chi_-^c(\alpha')^4 = \ell^4\,\chi_-^c(\alpha')^4 \leq d, \end{array}$$

which is a contradiction. Therefore, the lemma holds. □

By Lemma 2.5, there exist at least $9\chi_-^c(\alpha')/2$ translates of R' such that any two of them are disjoint, and which do not intersect the surface $\overline{F}$ either. As each of those $9\chi_-^c(\alpha')/2$ incompressible surfaces is in the complement of $\overline{F}$, which is a disjoint union of compression bodies, this surface is in fact parallel to a component of $\overline{F}$. From Corollary 1.7, $\overline{F}$ has at most $3\chi_-^c(\alpha')/2$ components. Therefore, there are at least three disjoint translates of R' that are parallel to the same component of $\overline{F}$. Thus, those three translates are parallel. If the surface R' is arbitrarily given an orientation, each of the translates of R' is oriented, and its orientation is given by the orientation of R'. With those conventions, there are at least two of those parallel translates whose orientations are coherent in the product region they bound. Thus, there exists an incompressible surface R'' in M' and $h \in G$ an orientation preserving homeomorphism such that R'' and $h(R'')$ are parallel and disjoint in M'. As R'' is incompressible, Lemma 14 of [L1] applies: the cover M' fibers over the circle, with fiber R''. But as R'' is a translate of the surface R' under the action of G, if $p : M' \to M$ is the covering map, the homology class of $p^{-1}(R)$ is fibered. This ends the proof of Theorem 0.3. □

Proof of Corollary 0.6.

The proof is straightforward from Theorem 0.3. If $\lim_{i\to+\infty} \frac{\chi_-^c(\alpha_i)}{\sqrt[4]{d_i}} = 0$, for i large enough, $\ell\,\chi_-^c(\alpha_i) \leq \sqrt[4]{d_i}$, and Theorem 0.3 applies. □

Proof of Corollary 0.8.

As the cover $M_i \to M$ is the i-sheeted cyclic cover associated to the class α and $\alpha_i = p_i^*(\alpha)$, $\|\alpha_i\| = \|\alpha\|$. Thus, $\chi_-^c(\alpha_i) = \|\alpha_i\| + 2h(\alpha_i) = \|\alpha\| + 2h(\alpha_i)$. If there exists $i \geq i_0 = \lceil (2\ell\|\alpha\|)^4 \rceil$ such that $\frac{h(\alpha_i)}{\sqrt[4]{i}} \leq \frac{1}{4\ell}$, then

$$\ell\chi_-^c(\alpha_i) = \ell(\|\alpha\| + 2h(\alpha_i)) \leq \ell\|\alpha\| + \sqrt[4]{i}/2 \leq \sqrt[4]{i_0}/2 + \sqrt[4]{i}/2 \leq \sqrt[4]{i}.$$

Theorem 0.3 then applies. □

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