PROPERTY T FOR LINEAR GROUPS OVER RINGS, AFTER SHALOM

YVES DE CORNULIER

Let R be a ring (all rings here are supposed associative, with unity, but not necessarily commutative), Recall that a vector $(r_1, \dots, r_n) \in R^n$ is called unimodular if there exist $t_1, \dots, t_n \in R$ such that $\sum t_i r_i = 1$. Recall that the ring R has stable rank at most n, denoted $\text{s-rank}(R) \leq n$, if for every unimodular vector $(r_0, \dots, r_n) \in R^{n+1}$, there exists $s_1, \dots, s_n \in R$ such that the vector $(r_0 + s_0 r_n, r_1 + s_1 r_n, \dots, r_{n-1} + s_{n-1} r_n) \in R^n$ is unimodular. For instance, $\text{s-rank}(\mathbf{Z}) = 2$, $\text{s-rank}(\mathbf{Z}[X]) = 3$, see [HaOM89] for further examples.

Given any ring R, denote by $\mathrm{EL}_n(R)$ the subgroup of $\mathrm{GL}_n(R)$ generated by elementary matrices (those matrices with 1's on the diagonal, and at most one non-zero entry outside the diagonal). When R is commutative, it is contained in $\mathrm{SL}_n(R)$.

We present here Shalom's proof[1] of the following result.

Theorem 1 (Shalom)**.** *Fix $n \geq 3$ and a finitely generated ring R. If $n \geq \text{s-rank}(R) + 1$, then $\mathrm{EL}_n(R)$ has Property T.*

Definition 2. Let G be a topological group and X a subset. We say that (G, Ω) has corelative Property FH if every continuous Hilbert length function on G which is bounded on Ω is bounded on all of G.

We say that G is boundedly generated by a subset Ω if Ω generates G so that the corresponding Cayley graph is bounded. The following lemma is trivial.

Lemma 3. *If G is boundedly generated by a subset Ω, then (G, Ω) has corelative Property FH.* ■

The first step for the proof of Theorem 1 is the following proposition. View $\mathrm{GL}_{n-1}(R)$ as a subgroup of $\mathrm{GL}_n(R)$, identifying it to the upper-left block, and set $H = \mathrm{EL}_n(R) \cap \mathrm{GL}_{n-1}(R)$.

Proposition 4. *For every finitely generated ring R and every $n \geq \text{s-rank}(R) + 1$, the pair $(\mathrm{EL}_n(R), H)$ has corelative Property FH.*

This proposition follows at once from the two ones below, independent of Property T. Define K_1 as the subgroup of $\mathrm{EL}_n(R)$ consisting of matrices whose entries differ from those in the identity matrix only on the n-th column. Define K_2 as its transpose.

Proposition 5. *For every finitely generated ring R and every $n \geq \text{s-rank}(R) + 1$, every $A \in \mathrm{GL}_n(R)$ can be written $X_1 X_2 Y_1 B Y_2$ with $B \in \mathrm{GL}_{n-1}(R)$, $X_1, Y_1 \in K_1$, $X_2, Y_2 \in K_2$.* ■

Note that in particular, if $A \in \mathrm{EL}_n(R)$, then $B \in H$.

Date: February 6, 2007.

[1] The proof here follows a seminar talk given in Princeton on March 20, 2006. However I claim any error here is mine!

Proof: We only sketch the elementary proof. Start from $A = \begin{pmatrix} M & X \\ Y & r \end{pmatrix}$.

1) Using the stable rank assumption, we can multiply A on the left by a matrix in K_1 so as to obtain a matrix $A_1 = \begin{pmatrix} M' & X' \\ Y & r \end{pmatrix}$ with X' unimodular.

2) Since X' is unimodular, we can multiply A_1 on the left by a matrix in K_2 so as to obtain a matrix $A_2 = \begin{pmatrix} M' & X' \\ Y' & 1 \end{pmatrix}$.

3) Finally take the product $B = \begin{pmatrix} I_{n-1} & -Y' \\ 0 & 1 \end{pmatrix} A_2 \begin{pmatrix} I_{n-1} & 0 \\ -X' & 1 \end{pmatrix}$, which belongs to $\mathrm{GL}_{n-1}(R)$. ■

The following result is crucial; it is due to Shalom [Sha99] when R is commutative, and Kassabov [Kas05] subsequently observed that the argument also works for non-commutative R.

Theorem 6. *For every finitely generated ring R, the pair $(\mathrm{EL}_2(R) \ltimes R^2, R^2)$ has relative Property T. In particular, for every $n \geq 3$, and $i = 1, 2$, the pair $(\mathrm{EL}_n(R), K_i)$ has relative Property T.* ■

The second step for the proof of Theorem 1 is the following theorem.

Theorem 7. [2] *Suppose that a group G contains three subgroups H, K_1 and K_2 satisfying the five following assumptions.*

(1) *H normalizes both K_1 and K_2;*
(2) *$K_1 \cup K_2$ generates G;*
(3) *G is finitely generated;*
(4) $\mathrm{Hom}(G, \mathbf{R}) = \{0\}$
(5) *(G, H) has corelative property FH;*
(6) *(G, K_1) and (G, K_2) have relative Property T.*

Then G has Property T.

Remark 8. Actually Assumption (4) is redundant as it follows from (2) and (6). However we leave it for the following reasons:

- (4) is in general much easier to check than (6);
- it might be tempting to change slightly the hypotheses of the theorem, in such a way that this implication fails to hold.

Observe that these assumptions are satisfied in the example with the assumptions of Theorem 1: (1) is trivial, (5) is Proposition 4, and (6) is contained in Theorem 6. For (2), (3), and (4), write the identity $[e_{ik}(y), e_{nk}(-x)]$ (where $[a, b] = aba^{-1}b^{-1}$), for i, j, k pairwise distincts, which has the following easy consequences:

- If $n \geq 3$, then $G = \mathrm{EL}_n(R)$ is perfect, so that (4) is satisfied.
- If R is a finitely generated ring and $n \geq 3$, then G is finitely generated.
- In particular, if i, j, n are pairwise distincts, then $e_{ij}(x) = [e_{in}(1), e_{nj}(-x)]$. Thus if $n \geq 3$, then $\mathrm{EL}_n(R)$ is generated by $K_1 \cup K_2$.

[2]The explicit statement of this theorem is mine; the however the proof follows Shalom's one for EL_n without changes.

Let us finally prove Theorem 7, completing the proof of Theorem 1. Using Assumptions (2) and (3), we can fix a finite generating subset $S \subset K_1 \cup K_2$ of G. Consider the set $\mathcal{A}$ of all (equivalence classes of) affine isometric actions $(\alpha, \mathcal{H})$ on Hilbert spaces such that for every $x \in \mathcal{H}$, we have $\sup_{s \in S} \|\alpha(s)x - x\| \geq 1$. Suppose by contradiction that G does not have Propety (T). By a result of Shalom [Sha00] (see also Gromov [Gro03]), it follows that $\mathcal{A} \neq \emptyset$.

For every $(\alpha, \mathcal{H}) \in \mathcal{A}$, define

$$d_\alpha = \inf\{\|v_1 - v_2\| \ : v_1 \in \mathcal{H}^{\alpha(K_1)}, v_2 \in \mathcal{H}^{\alpha(K_2)}\}.$$

Assumption (6) implies that $d_\alpha < \infty$ for every $\alpha \in \mathcal{A}$. Now define $d = \inf_{\alpha \in \mathcal{A}} d_\alpha$. As $\mathcal{A} \neq \emptyset$, we have $d < \infty$. We claim that this infimum is attained:

Lemma 9. *There exists $\alpha \in \mathcal{A}$ such that $d_\alpha = d$. Moreover, we can choose it so that the linear part has no invariant vector.*

Proof: Consider a sequence $(\alpha_n, \mathcal{H}_n)$ such that $d_{\alpha_n} \to d$. In $\mathcal{H}_n$, choose points $x_n \in \mathcal{H}^{\alpha_n(K_1)}, y_n \in \mathcal{H}^{\alpha_n(K_2)}$ such that $\|x_n - y_n\| \to d$. Changing the origin in $\mathcal{H}_n$, we can suppose that $y_n = 0$ for all n. Now fix a non-principal ultrafilter ω on $\mathbf{N}$, and define $\mathcal{H}_*$ as the ultralimit of all $\mathcal{H}_n$: this is constructed as follows: take all bounded sequences (v_n) with $v_n \in \mathcal{H}_n$, kill all sequences (v_n) such that $\lim_\omega \|v_n\| = 0$, define the scalar product $\langle (v_n), (w_n) \rangle = \lim_\omega \langle v_n, w_n \rangle$, and finally take the completion.

If $g \in G$ and z_n is a bounded sequence, we claim that the sequence $(\alpha_n(g)z_n)$ is bounded. It suffices to check this for $g \in S$. As we have chosen $S \subset K_1 \cup K_2$, every $g \in S$ fixes a point at bounded (independently of n) distance from zero (observing that x_n is bounded as $\|x_n\|$ tends to d).

Therefore $\alpha(g)((z_n)) = (\alpha(g)z_n)$ defines a isometric action on $\mathcal{H}$, where K_2 fixes 0 and K_1 fixes (x_n) which has norm d. Finally observe that $\alpha \in \mathcal{A}$. Indeed, fix a bounded sequence (z_n) with each $z_n \in \mathcal{H}_n$. For every n there exists $s_n \in S$ such that $\|\alpha_n(s_n)z_n - z_n\| \geq 1$. The sets $\mathbf{N}_s = \{n \in \mathbf{N} : s_n = s\}$ make up a finite partition of $\mathbf{N}$, so that one of them satisfies $\omega(\mathbf{N}_s) = 1$. Therefore we obtain that $\|\alpha(s)((z_n)) - (z_n)\| \geq 1$, proving that $\alpha \in \mathcal{A}$.

It remains to check the last statement about the linear action. Let π denote the linear part of the action α. Denote by $\mathcal{H}_* = V_1 \oplus V_2$, where V_1 denote the $\pi(G)$-invariant vectors and V_2 its orthogonal. As by Assumption (4) G has no non-trivial action by translations, the action writes as $\alpha(g)(v_1, v_2) = v_1 + \pi_2(g)v_2 + b_2(g)$. In particular, the orthogonal of the invariant vectors is invariant under $\alpha(G)$, and the induced action α' is thus in $\mathcal{A}$. On the other hand, it clearly satisfies $d_{\alpha'} = d$. ■

Now consider α as provided by the lemma, with points x_1 and x_2 fixed by $\alpha(K_1)$ and $\alpha(K_2)$ respectively, at distance d; let π be the linear part of α. As H normalizes both K_1 and K_2 by Assumption (1), for some $g \in H$, if we define $y_i = \alpha(g)x_i$, then y_i is also fixed by $\alpha(K_i)$.

By Assumption (5), we can choose g so that $y_1 \neq x_1$. It is easy to check that the function $f(t) = t \mapsto \|(1-t)x_1 + ty_1 - (1-t)x_2 - ty_2\|^2$ is strictly convex unless $x_1 - x_2 = y_1 - y_2$. As $f(0) = f(1) = d \leq f$, this implies that $x_1 - x_2 = y_1 - y_2$. Observe now that this vector is fixed by both $\pi(K_1)$ and $\pi(K_2)$, and hence by all of $\pi(G)$ by Assumption (2). Thus $x_1 = x_2$, a contradiction.

References

[Gro03] Misha Gromov. *Random walk in random groups.* Geom. Funct. Anal. **13**(1), 73-146, 2003.

[HaVa89] Pierre de la Harpe, Alain Valette. "La propriété (T) de Kazhdan pour les groupes localement compacts". Astérisque **175**, Soc. Math. France, 1989.

[HaOM89] Alexander J. Hahn, O. Timothy O'Meara. *The classical groups and K-theory.* Grundlehren Math. Wiss. 291, Springer-Verlag, Berlin, 1989.

[Kas05] Martin Kassabov. *Universal lattices and unbounded rank expanders.* Preprint 2005, arXiv math.GR/0502237.

[Sha99] Yehuda Shalom. *Bounded generation and Kazhdan's property (T).* Publ. Math. Inst. Hautes Études Sci. **90**, 145-168, 1999.

[Sha00] Yehuda Shalom. *Rigidity of commensurators and irreducible lattices.* Invent. Math. **141**, no. 1, 1-54, 2000.