PROPERTY T FOR LINEAR GROUPS OVER RINGS, AFTER

SHALOM

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Let $R$ be a ring (all rings here are supposed associative, with unity, but not
necessarily commutative). Recall that a vector $(r_1, \ldots, r_n) \in R^n$ is called unimodular
if there exist $t_1, \ldots, t_n \in R$ such that $\sum t_ir_i = 1$. Recall that the ring $R$ has stable
rank at most $n$, denoted $\text{s-rank}(R) \leq n$, if for every unimodular vector $(r_0, \ldots, r_n) \in R^{n+1}$, there exists $s_1, \ldots, s_n \in R$ such that the vector $(r_0 + s_0r_n, r_1 + s_1r_n, \ldots, r_{n-1} + s_{n-1}r_n) \in R^n$ is unimodular. For instance, $\text{s-rank}(\mathbb{Z}) = 2$, $\text{s-rank}(\mathbb{Z}[X]) = 3$, see
[HaOM89] for further examples.

Given any ring $R$, denote by $\text{EL}_n(R)$ the subgroup of $\text{GL}_n(R)$ generated by ele-
m mentary matrices (those matrices with 1’s on the diagonal, and at most one non-zero
entry outside the diagonal). When $R$ is commutative, it is contained in $\text{SL}_n(R)$.

We present here Shalom’s proof\footnote{The proof here follows a seminar talk given in Princeton on March 20, 2006. However I claim any error here is mine!} of the following result.

**Theorem 1** (Shalom). Fix $n \geq 3$ and a finitely generated ring $R$. If $n \geq \text{s-rank}(R) + 1$, then $\text{EL}_n(R)$ has Property T.

**Definition 2.** Let $G$ be a topological group and $X$ a subset. We say that $(G, \Omega)$ has corelative Property FH if every continuous Hilbert length function on $G$ which is bounded on $\Omega$ is bounded on all of $G$.

We say that $G$ is boundedly generated by a subset $\Omega$ if $\Omega$ generates $G$ so that the corresponding Cayley graph is bounded. The following lemma is trivial.

**Lemma 3.** If $G$ is boundedly generated by a subset $\Omega$, then $(G, \Omega)$ has corelative Property FH. \hfill $\blacksquare$

The first step for the proof of Theorem 1 is the following proposition. View $\text{GL}_{n-1}(R)$ as a subgroup of $\text{GL}_n(R)$, identifying it to the upper-left block, and set $H = \text{EL}_n(R) \cap \text{GL}_{n-1}(R)$.

**Proposition 4.** For every finitely generated ring $R$ and every $n \geq \text{s-rank}(R) + 1$, the pair $(\text{EL}_n(R), H)$ has corelative Property FH.

This proposition follows at once from the two ones below, independent of Property
T. Define $K_1$ as the subgroup of $\text{EL}_n(R)$ consisting of matrices whose entries differ from those in the identity matrix only on the $n$-th column. Define $K_2$ as its transpose.

**Proposition 5.** For every finitely generated ring $R$ and every $n \geq \text{s-rank}(R) + 1$, every $A \in \text{GL}_n(R)$ can be written $X_1X_2Y_1BY_2$ with $B \in \text{GL}_{n-1}(R)$, $X_1, Y_1 \in K_1$, $X_2, Y_2 \in K_2$. \hfill $\blacksquare$

Note that in particular, if $A \in \text{EL}_n(R)$, then $B \in H$. 

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**Proof:** We only sketch the elementary proof. Start from
\[ A = \begin{pmatrix} M & X \\ Y & r \end{pmatrix}. \]

1) Using the stable rank assumption, we can multiply \( A \) on the left by a matrix in \( K_1 \) so as to obtain a matrix \( A_1 = \begin{pmatrix} M' & X' \\ Y' & r \end{pmatrix} \) with \( X' \) unimodular.

2) Since \( X' \) is unimodular, we can multiply \( A_1 \) on the left by a matrix in \( K_2 \) so as to obtain a matrix \( A_2 = \begin{pmatrix} M' & X' \\ Y' & 1 \end{pmatrix} \).

3) Finally take the product \( B = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 1 \end{pmatrix} A_2 \begin{pmatrix} I_{n-1} & 0 \\ -X' & 1 \end{pmatrix} \), which belongs to \( \text{GL}_{n-1}(R) \). ■

The following result is crucial; it is due to Shalom [Sha99] when \( R \) is commutative, and Kassabov [Kas05] subsequently observed that the argument also works for non-commutative \( R \).

**Theorem 6.** For every finitely generated ring \( R \), the pair \((\text{EL}_2(R) \ltimes R^2, R^2)\) has relative Property T. In particular, for every \( n \geq 3 \), and \( i = 1, 2 \), the pair \((\text{EL}_n(R), K_i)\) has Property T. ■

The second step for the proof of Theorem 1 is the following theorem.

**Theorem 7.** Suppose that a group \( G \) contains three subgroups \( H, K_1 \) and \( K_2 \) satisfying the five following assumptions.

1) \( H \) normalizes both \( K_1 \) and \( K_2 \);
2) \( K_1 \cup K_2 \) generates \( G \);
3) \( G \) is finitely generated;
4) Hom\((G, R) = \{0\} \)
5) \((G, H)\) has corelative property FH;
6) \((G, K_1)\) and \((G, K_2)\) have relative Property T.

Then \( G \) has Property T.

**Remark 8.** Actually Assumption (4) is redundant as it follows from (2) and (6). However we leave it for the following reasons:

- (4) is in general much easier to check than (6);
- it might be tempting to change slightly the hypotheses of the theorem, in such a way that this implication fails to hold.

Observe that these assumptions are satisfied in the example with the assumptions of Theorem 1: (1) is trivial, (5) is Proposition 4, and (6) is contained in Theorem 6. For (2), (3), and (4), write the identity \([e_{ik}(y), e_{nk}(-x)]\) (where \([a, b] = aba^{-1}b^{-1}\)), for \( i, j, k \) pairwise distincts, which has the following easy consequences:

- If \( n \geq 3 \), then \( G = \text{EL}_n(R) \) is perfect, so that (4) is satisfied.
- If \( R \) is a finitely generated ring and \( n \geq 3 \), then \( G \) is finitely generated.
- In particular, if \( i, j, n \) are pairwise distincts, then \( e_{ij}(x) = [e_{in}(1), e_{nj}(-x)] \).

Thus if \( n \geq 3 \), then \( \text{EL}_n(R) \) is generated by \( K_1 \cup K_2 \).

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\(^2\)The explicit statement of this theorem is mine; the however the proof follows Shalom’s one for \( \text{EL}_n \) without changes.
Let us finally prove Theorem 7, completing the proof of Theorem 1. Using Assumptions (2) and (3), we can fix a finite generating subset $S \subset K_1 \cup K_2$ of $G$. Consider the set $\mathcal{A}$ of all (equivalence classes of) affine isometric actions $(\alpha, \mathcal{H})$ on Hilbert spaces such that for every $x \in \mathcal{H}$, we have $\sup_{s \in S} \|\alpha(s)x - x\| \geq 1$. Suppose by contradiction that $G$ does not have Property (T). By a result of Shalom [Sha00] (see also Gromov [Gro03]), it follows that $\mathcal{A} \neq \emptyset$.

For every $(\alpha, \mathcal{H}) \in \mathcal{A}$, define

$$d_\alpha = \inf \{\|v_1 - v_2\| : v_1 \in \mathcal{H}^{\alpha(K_1)}, v_2 \in \mathcal{H}^{\alpha(K_2)}\}.$$

Assumption (6) implies that $d_\alpha < \infty$ for every $\alpha \in \mathcal{A}$. Now define $d = \inf_{\alpha \in \mathcal{A}} d_\alpha$. As $\mathcal{A} \neq \emptyset$, we have $d < \infty$. We claim that this infimum is attained:

**Lemma 9.** There exists $\alpha \in \mathcal{A}$ such that $d_\alpha = d$. Moreover, we can choose it so that the linear part has no invariant vector.

**Proof:** Consider a sequence $(\alpha_n, \mathcal{H}_n)$ such that $d_{\alpha_n} \to d$. In $\mathcal{H}_n$, choose points $x_n \in \mathcal{H}^{\alpha_n(K_1)}$, $y_n \in \mathcal{H}^{\alpha_n(K_2)}$ such that $\|x_n - y_n\| \to d$. Changing the origin in $\mathcal{H}_n$, we can suppose that $y_n = 0$ for all $n$. Now fix a non-principal ultrafilter $\omega$ on $\mathbb{N}$, and define $\mathcal{H}_\omega$ as the ultralimit of all $\mathcal{H}_n$: this is constructed as follows: take all bounded sequences $(v_n)$ with $v_n \in \mathcal{H}_n$, kill all sequences $(v_n)$ such that $\lim_{\omega} \|v_n\| = 0$, define the scalar product $\langle (v_n), (w_n) \rangle = \lim_{\omega} \langle v_n, w_n \rangle$, and finally take the completion.

If $g \in G$ and $z_n$ is a bounded sequence, we claim that the sequence $(\alpha_n(g)z_n)$ is bounded. It suffices to check this for $g \in S$. As we have chosen $S \subset K_1 \cup K_2$, every $g \in S$ fixes a point at bounded (independently of $n$) distance from zero (observing that $x_n$ is bounded as $\|x_n\|$ tends to $d$).

Therefore $\alpha(g)((z_n)) = (\alpha(g)z_n)$ defines an isometric action on $\mathcal{H}$, where $K_2$ fixes $0$ and $K_1$ fixes $(x_n)$ which has norm $d$. Finally observe that $\alpha \in \mathcal{A}$. Indeed, fix a bounded sequence $(z_n)$ with each $z_n \in \mathcal{H}_n$. For every $n$ there exists $s_n \in S$ such that $\|\alpha_n(s_n)z_n - z_n\| \geq 1$. The sets $N_s = \{n \in \mathbb{N} : s_n = s\}$ make up a finite partition of $\mathbb{N}$, so that one of them satisfies $\omega(N_s) = 1$. Therefore we obtain that $\|\alpha(s)((z_n)) - (z_n)\| \geq 1$, proving that $\alpha \in \mathcal{A}$.

It remains to check the last statement about the linear action. Let $\pi$ denote the linear part of the action $\alpha$. Denote by $\mathcal{H}_\omega = V_1 \oplus V_2$, where $V_1$ denote the $\pi(G)$-invariant vectors and $V_2$ its orthogonal. As by Assumption (4) $G$ has no non-trivial action by translations, the action writes as $\alpha(g)(v_1, v_2) = v_1 + \pi_2(g)v_2 + b_2(g)$. In particular, the orthogonal of the invariant vectors is invariant under $\alpha(G)$, and the induced action $\alpha'$ is thus in $\mathcal{A}$. On the other hand, it clearly satisfies $d_{\alpha'} = d$. $\blacksquare$

Now consider $\alpha$ as provided by the lemma, with points $x_1$ and $x_2$ fixed by $\alpha(K_1)$ and $\alpha(K_2)$ respectively, at distance $d$; let $\pi$ be the linear part of $\alpha$. As $H$ normalizes both $K_1$ and $K_2$ by Assumption (1), for some $g \in H$, if we define $y_i = \alpha(g)x_i$, then $y_i$ is also fixed by $\alpha(K_i)$.

By Assumption (5), we can choose $g$ so that $y_1 \neq x_1$. It is easy to check that the function $f(t) = t \mapsto \|(1-t)x_1 + ty_1 - (1-t)x_2 - ty_2\|^2$ is strictly convex unless $x_1 - x_2 = y_1 - y_2$. As $f(0) = f(1) = d \leq f$, this implies that $x_1 - x_2 = y_1 - y_2$. Observe now that this vector is fixed by both $\pi(K_1)$ and $\pi(K_2)$, and hence by all of $\pi(G)$ by Assumption (2). Thus $x_1 = x_2$, a contradiction.
References


