Uniform convexity and related metric geometric properties of Banach spaces

by
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Abstract. A Banach space B is called uniformly convex if for any $0 < \varepsilon \le 2$ there is a $\delta > 0$ such that for any pair x, y in B the following implication holds

$$(\|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon) \Rightarrow \left\| \frac{x+y}{2} \right\| \le 1 - \delta.$$

The modulus of uniform convexity $\delta_B(\varepsilon)$ is defined as the "best possible" δ i.e.

$$\delta_B(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| \;\middle|\; \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \varepsilon \right\}.$$

We will present the notions of cotype, type and super-reflexivity of Banach spaces and describe their relation to uniform convexity (and the dual property: uniform smoothness). Whenever possible, we will describe the more recent work done by Assaf Naor and his collaborators on the non-linear versions of these notions. Our emphasis will be on the concepts that may have some relevance for the problem of finding obstructions for embedding expanding graphs into Banach spaces with "special" properties such as finite cotype.