Abstract. We combine results of Vaserstein and Shalom to prove Kazhdan’s Property (T) for various topological groups of the form $\text{SL}(n, C(X, R))$ or $\text{SL}(n, C(X, C))$, for $n \geq 3$ and $X$ a topological subspace of a Euclidean space.

All the rings here are unitary and commutative. If $R$ is a ring, let $E(n, R)$ denote the subgroup of $\text{SL}(n, R)$ generated by elementary matrices. If $E(n, R)$ is normal in $\text{SL}(n, R)$, the quotient is denoted by $SK_{1,n}(R)$.

If $R$ is a topological ring, then $E(n, R)$ and $\text{SL}(n, R)$ are topological groups for the topology induced by the inclusion in $R^{n^2}$. We say that a topological ring is \textit{topologically finitely generated} if it has a finitely generated dense subring.

For any topological spaces $X, Y$, we denote by $C(X, Y)$ the set of all continuous functions $X \to Y$. If $K$ denotes $R$ or $C$, $C(X, K)$ is a topological ring for the compact-open topology, which coincides with the topology of uniform convergence on compact subsets.

It is known [Vas86] that, if $n \geq 3$,

$$E(n, C(X, K)) = \{u : X \to \text{SL}(n, K) \text{ homotopically trivial}\}$$

(this is immediate if $X$ is compact, for all $n \geq 2$).

If $G$ is a topological group, a continuous unitary representation of $G$ on a Hilbert space $H$ almost has invariant vectors if, for every compact (not necessarily Hausdorff) subset $K \subset G$ and every $\varepsilon > 0$, there exists $\xi \in H$, of norm one, such that $\sup_{g \in K} \|g \cdot \xi - \xi\| \leq \varepsilon$. The topological group $G$ has Kazhdan’s Property (T) or Property (T) if every continuous unitary representation of $G$ almost having invariant vectors actually has nonzero invariant vectors; see [BHV05] for more about Property (T).

**Theorem 1.** Let $n \geq 3$, and $X \subset R^d$ be a topological subspace of a Euclidean space. Let $K = R$ or $C$. Endow $C(X, K)$ with the topology of uniform convergence on compact subsets. Then $E(n, C(X, K))$ has Kazhdan’s Property (T).

**Corollary 2.** Let $n \geq 3$, and $X \subset R^d$ be a compact subset. Endow $X$ with the topology of uniform convergence. Then $\text{SL}(n, C(X, K))$ has Kazhdan’s Property (T) if and only if the discrete group $SK_{1,n}(C(X, K)) = \text{SL}(n, C(X, K))/E(n, C(X, K))$ does.

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**Proof:** Since $X$ is compact\(^1\), $E(n, \mathcal{C}(X, K))$, is open (hence closed) in $\text{SL}(n, \mathcal{C}(X, K))$. The corollary follows from the trivial fact that Property (T) is stable under quotients and extensions.

*Example 3.* Fix $k \geq 1$ and $n \geq 3$. Then $SK_{1,n}(\mathcal{C}(S^k, K)) = \pi_k(\text{SL}(n, K))$, which is an abelian group. It follows that $\text{SL}(n, \mathcal{C}(S^k, K))$ has Kazhdan’s Property (T) if and only if $\pi_k(\text{SL}(n, K))$ is finite; it is known (see [MiTo91]) that it is infinite if and only if:

- $K = \mathbb{C}$, $k$ is odd, and $3 \leq k \leq 2n - 1$, or
- $K = \mathbb{R}$, $(k \equiv -1 \pmod{4})$ and $3 \leq k \leq 2n - 1$ or $(n$ is even and $k = n - 1$)

In particular, $\pi_k(\text{SL}(n, K))$ is finite for $k = 1$, $k$ even, or $k \geq 2n$.

*Example 4.* Let $W$ denote a Cantor set. It is straightforward to show that, for every connected manifold $M$, all maps $W \to M$ are homotopic. Thus, $SK_{1,n}(\mathcal{C}(W, K))$ is trivial, and accordingly, for all $n \geq 3$, $\text{SL}(n, \mathcal{C}(W, K))$ has Kazhdan’s Property (T).

Theorem 1 rests on two results: a $K$-theoretic result of Vasertstein (Theorem 8), and the work of Shalom on Kazhdan’s Property (T) (Theorem 7).

In [Sha99], Shalom introduces new methods to establish Kazhdan’s Property (T) for the special linear groups over certain rings. This leads, for instance, to the first proof that $\Gamma = \text{SL}(n, \mathbb{Z})$, $n \geq 3$, has Property (T), that does not use the embedding of $\Gamma$ into $\text{SL}(n, \mathbb{R})$ as a lattice.

Before stating his main result, let us begin with a definition.

**Definition 5.** If $G$ is a group and $S \subset G$ is a subset, we say that $G$ is boundedly generated by $S$ if there exist $m < \infty$ such that every $g \in G$ is a product of at most $m$ elements in $S$.

**Theorem 6** (Shalom, [Sha99]). Let $n \geq 3$, and let $R$ be a topologically finitely generated ring. Suppose that $\text{SL}(n, R)$ is boundedly generated by elementary matrices. Then $\text{SL}(n, R)$ has Kazhdan’s Property (T).

As an application, Shalom proves bounded elementary generation for the loop group $\text{SL}(n, \mathcal{C}(S^1, C))$ ($n \geq 3$), and deduces that it has Property (T). He asks if the same holds for $\mathbb{R}$ instead of $\mathbb{C}$, noting that $\text{SL}(n, \mathcal{C}(S^1, \mathbb{R}))$ is not generated by elementary matrices (since $\pi_1(\text{SL}(n, \mathbb{R})) \neq 1$). This is answered positively by Theorem 1; see Example 3 above.

Actually, without modification, the proof of Theorem 6 ([Sha99], see also [BHV05]) gives a stronger statement.

**Theorem 7.** Let $n \geq 3$, let $R$ be a topologically finitely generated commutative ring, and suppose that $E(n, R)$ is boundedly generated by elementary matrices. Then $E(n, R)$ has Kazhdan’s Property (T).

Theorem 6 is the particular case of Theorem 7 when $E(n, R) = \text{SL}(n, R)$.

Theorem 7 is a strong motivation for studying bounded elementary generation for the group $E(n, R)$. For instance, this is an open question for $R = F_p[X, Y]$ or

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\(^1\)If $X$ is not compact, we do not know whether $E(n, \mathcal{C}(X, K))$ is closed in $\text{SL}(n, \mathcal{C}(X, K))$ for the topology of uniform convergence on compact subsets. However, we can restate the corollary as follows: $\text{SL}(n, \mathcal{C}(X, K))$ has Kazhdan’s Property (T) if and only if the group $SK_{1,n}(\mathcal{C}(X, K)) = \text{SL}(n, \mathcal{C}(X, K))/E(n, \mathcal{C}(X, K))$ does.
\[ R = \mathbb{Z}[X], \text{ for all } n \geq 3. \] We now focus on the case when \( R = \mathcal{C}(X, K) \), where \( K = \mathbb{R} \) or \( \mathbb{C} \).

The notion of dimension of a topological space involved here is defined in [Vas71], and it will be sufficient for our purposes to know that \( \dim(X) \) is finite for every topological subspace of a Euclidean space.

**Theorem 8** (Vaserstein, [Vas88]). Let \( X \) be a finite dimensional topological space, and let \( K = \mathbb{R} \) or \( \mathbb{C} \), and fix \( n \geq 3 \). Then \( \mathcal{E}(n, \mathcal{C}(X, K)) \) is boundedly generated by elementary matrices.

Now we show how Theorems 7 and 8 imply Theorem 1. Since \( X \subset \mathbb{R}^d \) for some \( d \), \( X \) is finite dimensional, so that Theorem 8 applies: \( \mathcal{E}(n, \mathcal{C}(X, K)) \) is boundedly generated by elementary matrices.

It remains to show that Theorem 7 applies, that is, \( \mathcal{C}(X, K) \) is topologically finitely generated for the compact-open topology (that is, the topology of uniform convergence on compact subsets).

Let \( p_1, \ldots, p_d \) be the projections of \( X \) on the \( d \) coordinates of \( \mathbb{R}^d \), and let \( A \) be the (unital) \( K \)-subalgebra of \( \mathcal{C}(X, K) \) generated by \( p_1, \ldots, p_d \). By the Stone-Weierstrass Theorem, \( A \) is dense in \( \mathcal{C}(X, K) \) for the topology of uniform convergence on compact subsets. Then the finite family \( \{(p_j), \sqrt{2}\} \) (resp. \( \{(p_j), \sqrt{2}, i\} \)) generates a dense subring in \( \mathcal{C}(X, K) \) if \( K = \mathbb{R} \) (resp. if \( K = \mathbb{C} \)). \( \blacksquare \)

**Remark 9.**

1. The hypothesis in Theorem 1 that \( X \) is homeomorphic to a subset of an Euclidean space is close to being necessary in order to apply Theorem 7. Indeed, suppose that \( \mathcal{C}(X, K) \) is endowed with a topology such that the evaluation functions \( f \mapsto f(x) \) are continuous. Besides, suppose that \( \mathcal{C}(X, K) \) is topologically finitely generated as a ring by \( p_1, \ldots, p_d \). Then there exists a continuous injection of \( X \) in some Euclidean space, given by \( x \mapsto (p_1(x), \ldots, p_d(x)) \).
2. If \( X \) is metrizable and non-compact, and \( \mathcal{C}(X, K) \) is endowed with the uniform convergence topology, then an easy growth argument shows that \( \mathcal{C}(X, K) \) is not topologically finitely generated.

It would be interesting to generalize Theorem 8 to general semisimple Lie groups without compact factors, and Theorem 7 to semisimple groups without compact factors and with Property (T), or at least to higher rank ones. Theorem 6 is extended to symplectic groups in [Neu03]. On the other hand, if \( G \) is a connected compact simple Lie group, \( \mathcal{C}(S^1, G) \) does not have Kazhdan’s Property (T) [BHV05, Exercise 4.4.5].

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**References**


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