NOTES ON GROUPS WITH EXPONENTIAL DEHN FUNCTION (THIS IS NOT A PREPRINT)

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1. Lie groups have at most exponential Dehn function

Theorem 1 (Gromov). *Every connected Lie group has an at most exponential Dehn function.*

Sketch of proof. First, we use the classical fact that every connected Lie group is quasi-isometric to a simply connected solvable Lie group (when G is algebraic, start from a Levi decomposition $G = SU$ with S reductive and U the unipotent radical, take a minimal parabolic T in S to get an amenable real algebraic group. Decompose the latter as DKV with V the unipotent radical (possibly bigger that U), D a maximal split torus and K a maximal anisotropic torus: then the unit component of DV, in the real topology, is cocompact in G and simply connected solvable).

Second, we use that every simply connected solvable Lie group can be described as $(\mathbf{R}^k \times \mathbf{R}^\ell, *)$ with the law of the form

$$(u_1, v_1) * (u_2, v_2) = (u_1 + u_2, P(u_1, u_2, v_1, v_2)),$$

where P is a function each component of which, if we denote by (U_i) the $2k$ coordinates of (u_1, u_2) and by (V_j) the 2ℓ coordinates of (v_1, v_2), can be described as a real-valued polynomial in the variables U_i, V_j, and $e^{\lambda_k U_i}$, for some finite family of complex numbers λ_k. For instance, the law of $\mathrm{SOL}(\mathbf{R})$ can be described as

$$(u_1, x_1, y_1) * (u_2, x_2, y_2) = (u_1 + u_2, e^{u_2} x_1 + x_2, e^{-u_2} y_1 + y_2)$$

(here $(k, \ell) = (1, 2)$). It follows that the n-ball in G (with respect to a compact generating set, e.g. a compact neighbourhood of the identity) is contained in a product $B_1 \times B_2$, where B_1 is a Euclidean ball of linear radius in $\mathbf{R}^k$, and B_2 is a Euclidean ball of exponential radius in $\mathbf{R}^\ell$.

Let γ be a loop of length $\leq n$ in G. Then (translating if necessary), γ is contained in $B_1 \times B_2$. Consider a disc D of area $\leq n^2$, contained in $B_1 \times B_2$, whose boundary is γ. We have to estimate[1] the area of D in G, i.e. when $\mathbf{R}^{k+\ell}$ is endowed with a $*$-left-invariant Riemannian metric. If $x_0 \in B_1 \times B_2$, then the differential at 0 of the left multiplication $L_{x_0} : \mathbf{R}^{k+\ell} \to \mathbf{R}^{k+\ell}$ by x_0 has at

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[1]I thank S. Gouëzel for noticing a flaw in my initial argument.

most exponential norm. Therefore the area of D, in the $*$-left-invariant metric is bounded by $e^{Cn}n^2 \leq e^{C'n}$. □

Remark 2. Since $\mathbf{R}^n$ satifies polynomial isoperimetry inequalities in all dimensions, the same proof shows that every connected Lie group satisfies exponential isoperimetry inequalities in all dimensions.

Remark 3. The same proof shows that every *nilpotent* connected Lie group satisfies polynomial isoperimetric inequalities. Indeed, the law $*$ does not involve any exponential, in this case.

Remark 4 (linear isodiametric function). It is possible to refine the proof to prove that the disc filling a loop of linear length (or the k-ball filling a $(k-1)$-sphere of linear area and linear diameter) is contained in a ball of linear size.

To do this, we reduce to the case when G is a connected group of triangular real matrices, and using the descending central series we identify G with $\mathbf{R}_{k_1} \oplus \mathbf{R}_{k_2} \oplus \cdots \oplus \mathbf{R}_{k_d} \oplus \mathbf{R}_{k_\infty}$ ($\mathbf{R}_{k_\infty}$ is then the stable term of the descending central series, and is also the exponential radical), so that the n-ball $B_G(n)$ of G is comparable with $Q(n) = B(n) \times B(n^2) \times \cdots \times B(n^d) \times B(e^n)$ (i.e.

$$Q(C^{-1}n) \subset B_G(n) \subset Q(Cn)$$

for some suitable constant C).

Remark 5. Gromov's argument in [3] is by embedding a simply connected solvable Lie group into $\mathrm{SL}_n(\mathbf{R})$, and proving that there is an "exponentially Lipschitz retraction" of $\mathrm{SL}_n(\mathbf{R})$ onto G, starting from the fact we have polynomial isoperimetric inequalities in $\mathrm{SL}_n(\mathbf{R})$ (because it is quasi-isometric to a CAT(0)-space). He omits details (as usually) and I checked his arguments only in special cases.

2. Some Lie groups with exponential Dehn function

Proposition 6. *Consider a central extension of connected Lie groups*

$$1 \to Z \to \tilde{G} \to G \to 1;$$

consider a path $\tilde{\gamma}$ of size $\leq n$ in $\tilde{G}$, joining 1 to an element z_0 of Z. Suppose that z_0 has size $\geq a_0$ in Z. Let γ be the projection of $\tilde{\gamma}$ on G; this is a loop. Then the area of γ is $\geq a_0$.

Proof. Consider a compact generating subset S of $\tilde{G}$. We can write G as the quotient of the free group F_S generated by S, by a set of relators $\mathcal{R}$, where for some k, $\mathcal{R}$ is the set of elements in the k-ball F_S which map to the identity element in G. View γ as a word (think of the loop as a sequence of points, each one at distance ≤ 1 from the preceding one in the Cayley graph). If a is its area, we can write, in F_S

$$\gamma = \prod_{i=1}^{a} g_i r_i g_i^{-1},$$

where $r_i \in \mathcal{R}$. Push this equality forward to $\tilde{G}$ (if we push it forward to G, we get an elegant proof of 1=1); since r_i maps to the identity element in G, it is central in $\tilde{G}$, so we obtain

$$\gamma = \prod_{i=1}^{a} r_i,$$

but γ represents by definition the element z_0 in $\tilde{G}$. Since r_i are bounded and the above relation holds in Z, we obtain (always forgetting the constants)

$$a_0 = |z_0| \leq a. \quad \square$$

As an application, we have the following corollary. Let L be any closed subgroup of $\mathrm{SL}_2(\mathbf{R})$ containing the matrix $u = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Corollary 7. *In the group $\mathbf{R}^2 \rtimes L$ (or $\mathbf{Z}^2 \rtimes L$ if $L \subset \mathrm{SL}_2(\mathbf{Z})$), and letting (x, y) denote the standard basis of $\mathbf{R}^2$, the loop*

$$\gamma = u^n x u^{-n} y u^n x^{-1} u^{-n} y^{-1} = [u^n x u^{-n}, y],$$

of linear length, has exponential area.

Proof. We can lift the action of $\mathrm{GL}_2(\mathbf{R})$ on $\mathbf{R}^2$ to an action on the 3-dimensional Heisenberg group H_3, so that an element $v \in \mathrm{GL}_2(\mathbf{R})$ acts on the center of H_3 by multiplication by $\det(v)^2$ (GL_2 thus appears as a Levi factor in $\mathrm{Aut}(H_3)$, of the unipotent radical which consists of inner automorphisms). In particular, the action of $\mathrm{SL}_2(\mathbf{R})$ on H_3 centralizes the center of H_3. So we have a central extension

$$1 \to \mathbf{R} \to H_3 \rtimes L \to \mathbf{R}^2 \rtimes L \to 1.$$

We lift u to $H_3 \rtimes L$ as the same element in L, and lift x, y to the elements

$$\tilde{x} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \tilde{y} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then for all n,

$$[u^n \tilde{x} u^{-n}, \tilde{y}] = \begin{pmatrix} 1 & 0 & c\alpha^n + O(1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

with c some explicit nonzero constant and $\alpha = (3 + \sqrt{5})/2$. This element has exponential size inside $Z = \mathbf{R}$, so by Proposition 6, the area of γ is exponential. $\square$

3. $\mathrm{SL}_3(\mathbf{Z})$ has at least exponential Dehn function

Theorem 8. *The Dehn function of* $\mathrm{SL}_3(\mathbf{Z})$ *has an exponential lower bound.*

Proof. For convenience, we consider, only in this proof, right-invariant distances. Set $G = \mathrm{SL}_d(\mathbf{R})$, $\Gamma = \mathrm{SL}_d(\mathbf{Z})$, and $K = \mathrm{SO}_d(\mathbf{R})$. Consider the function $h : G \to \mathbf{R}$ given by $h(A) = \|Ae_1\|$, with $\|\cdot\|$ the Euclidean norm on $\mathbf{R}^d$. Define $W = \{h = 1\}$ and $H \simeq \mathbf{R}^{d-1} \rtimes \mathrm{SL}_{d-1}(\mathbf{R})$ the stabilizer of e_1. (Note that W is not a subgroup.) Clearly, the inclusion of H into W is a quasi-isometry, because $W = KH$ (recall that we work with *right*-invariant distances).

The function h factors through a function on $K\backslash G \to \mathbf{R}$; we admit that $\log \circ h$ is a Busemann function on the symmetric space $K\backslash G$. In particular, $C = \{h \leq 1\}$ is convex. Let p be the projection $K\backslash G \to C$. Then p is well-defined and 1-Lispchitz, because $K\backslash G$ is CAT(0). Moreover, the image of Γ in G/K is contained in the exterior $\{h \geq 1\}$ of C, so its image by p is contained in the boundary $K\backslash W = \{h = 1\}$ of C.

Now let $\gamma = \gamma_n$ be the loop in H given in the statement of Corollary 7. Suppose that is has a filling of area $a = a_n$ in $\mathrm{SL}_3(\mathbf{Z})$. Taking the image in $K\backslash G$ and applying the projection p, we map γ the a loop $\bar{\gamma}$, with a filling of area a (because the projection is one-Lipschitz), lying inside the boundary of $K\backslash W$ C. Now observe that $G \to K\backslash G$ is a quasi-isometry, and restricts to a quasi-isometry $H \to K\backslash W$. Taking the "inverse image", we obtain a loop γ', at bounded distance from γ, and a filling of area $\leq a$. So γ itself has area a. By Corollary 7, there is an exponential lower bound on $a = a_n$. ☐

Remark 9. There is an alternate (and more common) proof that the loop given in SOL has exponential area, using Stokes' theorem on some reasonable differential form (see Epstein et al [2]). This is slightly more complicated, but has a double advantage:

- it generalizes to deformations of SOL, defined as semidirect products $\mathbf{R}^2 \rtimes \mathbf{R}$ with an action $t \cdot (x, y) = (e^t x, e^{-\lambda t} y)$; these groups have no non-trivial central extension as Lie groups;
- it generalizes to higher Dehn functions, to groups of the form $\mathbf{R}^k \rtimes L$, L containg $k-1$ multiplicatively independent $\mathbf{R}$-diagonalizable matrices with integer coefficients, showing that this has (explicit) $(k-1)$-spheres with not better than exponential filling by k-balls. Then arguing as in the proof of Theorem 8, a consequence is that $\mathrm{SL}_d(\mathbf{Z})$ has (at least) exponential filling of $(d-2)$-spheres. This is done in (Epstein et al [2]) but their redaction is very obscure to me (unlike in the case of SOL mentioned above).

Remark 10. Following the same lines as for Theorem 8, replacing $\mathbf{R}$ by $\mathbf{F}_q((t^{-1}))$, we obtain that the group $\mathrm{SL}_3(\mathbf{F}_q[t])$ is not finitely presented, a result due to Behr [1] (I don't know if his proof is similar). Note that $\mathrm{SL}_3(\mathbf{F}_q[t])$ is finitely generated as a group, because $\mathbf{F}_q[t]$ is a finitely generated P.I.D.: more precisely it is generated by the elementary matrices whose non-zero non-diagonal coefficient is 1 or t).

References

[1] H. Behr. $\mathrm{SL}_3(\mathbf{F}_q[t])$ is not finitely presentable, in: "Homological group theory", Proceedings Durham 1997, London Math. Soc. Lect. Notes 36 (1977), 213-224.

[2] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, William. "Word processing in groups". Jones and Bartlett Publishers, Boston, 1992. xii+330 pp.

[3] M. Gromov. *Asymptotic invariants of infinite groups.*Geometric Group Theory, London Math. Soc. Lecture Note Ser. (G. Niblo and M. Roller, eds.), no. 182, 1993.

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