SOFICITY OF CREMONA GROUPS AND SOFIC PROFILE

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ABSTRACT. We show that Cremona groups are sofic. We actually introduce of quantitative notion of soficity, called sofic profile, and show that the group of birational transformations of a d-dimensional variety has sofic profile at most polynomial of degree d.

CONTENTS

1. INTRODUCTION

Let K be a field. The *Cremona group* $\mathrm{Cr}_d(K)$ of K in dimension d is defined as the group of birational transformations of the d-dimensional K-affine space. It can also be described as the group of K-automorphisms of the field of rational functions $K(t_1, \dots, t_d)$.

The group of polynomial automorphisms of an arbitrary variety is known [Bas, BL] to be locally residually finite, i.e. every finitely generated subgroup is residually finite. The proof essentially consists in the following two steps. Consider a K-variety X and a finitely generated subgroup Γ of the group of automorphisms of X.

(1) Then for some finitely generated subdomain A of K, the variety X can be viewed as an A-scheme and the action of Γ is by automorphisms of A-schemes (A will depend on the structure constants of X and on the action of the generators of Γ).
(2) Using that A is a residually finite domain, the group of A-automorphisms of X is itself residually finite (this is very easy in case the variety X is affine).

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While Step (1) can be performed to a certain extent, Step (2) dramatically falls through when dealing with birational transformations. Indeed, because of singularities, there is no action on the set of points over finite fields.

Recall that a group Γ is *sofic* if it satisfies the following: for every finite subset F of Γ and every $\varepsilon > 0$, there exists n and a mapping $\phi : F \to \mathrm{Sym}_n$ satisfying

- $\mathsf{d}^n_{\mathrm{Ham}}(\phi(g)\phi(h), \phi(gh)) \le \varepsilon$ for all $g, h \in F$ such that $gh \in F$;
- $\mathsf{d}^n_{\mathrm{Ham}}(\phi(1), 1) \ge 1 - \varepsilon$;
- $\mathsf{d}^n_{\mathrm{Ham}}(\phi(u), \phi(v)) \ge 1/4$ for all $u \neq v$,

where $\mathsf{d}^n_{\mathrm{Ham}}$ is the normalized Hamming distance on the symmetric group Sym_n:

$$\mathsf{d}^n_{\mathrm{Ham}}(u, v) = \frac{1}{n}\#\{i : u(i) \neq v(i)\}.$$

Note that a group is sofic if and only if all its finitely generated subgroups are sofic. Sofic groups were independently introduced by B. Weiss [Wei] and Gromov [Gro]. Sofic groups notably include residually finite and amenable groups. For more, see also [ES2, Pe].

The purpose of this note is to prove

Theorem 1.1. *The Cremona group* $\mathrm{Cr}_d(K)$ *is sofic for all* d *and all fields* K. *More generally, for any absolutely irreducible variety* X *over a field* K, *the group of birational transformations* $\mathrm{Bir}_K(X)$ *is sofic.*

This result was only known for $n = 1$ since then $\mathrm{Cr}_1(K) = \mathrm{PGL}_2(K)$ has all its finitely generated subgroups residually finite.

Theorem 1.1 is proved in Section 2 in the case of Cremona groups, and in general in Section 4. Although the latter supersedes the former, the proof in the Cremona case is much less technical, so we include it. The main two steps are

(1) Reduction to finite fields;
(2) case of finite fields.

The second step uses the "quasi-action" on the set of points, using that the singular set being of positive codimension, its number of points over a given finite field can be bounded above in a quantitative way. The first step is fairly easy in the case of Cremona groups, and uses more elaborate (albeit standard) arguments in the general case.

No example is known of a non-sofic group; in particular, so far Theorem 1.1 provides no example of groups that cannot be embedded into any Cremona group. However, the proof provides a property stronger than soficity, namely that $\mathrm{Cr}_d(K)$ (or more generally $\mathrm{Bir}_K(X)$ when X is d-dimensional) has its "*sofic profile*" in $O(n^d)$ (see Corollary 4.5), which might result in explicit examples of groups not embedding into Cremona groups, without exhibiting non-sofic groups. See Section 3, in which the sofic profile is defined, and related to the classical isoperimetric profile (or Følner function), and to the sofic dimension recently introduced by Arzhantseva and Cherix.

Nevertheless, the sofic property is interesting because of its various positive consequences. For instance, if G is a group and K is a domain, a conjecture by Kaplansky asserts that the group algebra KG is directly finite, i.e. satisfies $xy = 1 \Rightarrow yx = 1$. This conjecture is known to hold when G is sofic, by a result of Elek and Szabo [ES1]. Also another conjecture, by Gottschalk, is that if M is a finite set, any G-equivariant continuous injective map $M^G \to M^G$ is surjective; Gromov [Gro] proved that this is true when G is sofic.

Outline of the paper. Section 2 contains the proof of soficity of the Cremona group $\mathrm{Cr}_d(K)$. Section 3 introduces the notion of sofic profile, yielding various examples. Then Section 4 proves Theorem 1.1 in full generality. This actually makes Section 2 (precisely: Proposition 2.2) redundant. A good reason to keep this plan is that the proof of Proposition 2.2 is much simpler than its general version, Proposition 4.1; although the latter uses only basic commutative algebra that are extensively used by algebraic geometers (generic flatness, openness conditions), these notions are not of the utmost common background for readers in geometric group theory, who can stick to Section 2 and 3. Section 3 can also be read independently, without reference to Cremona groups.

Finally, Section 5 contains two complementary observations about the Cremona group. The first is an example of a non-linear finitely generated subgroup of $\mathrm{Cr}_2(\mathbf{C})$. The existence of such a subgroup is not new, for instance it follows from an unpublished construction of S. Cantat; our example has the additional feature of being 3-solvable. Its non-linearity follows from the fact it contains nilpotent subgroups of arbitrary large nilpotency length. We also show there that $\mathrm{Cr}_2(K)$ has no nontrivial linear representation over any field, extending a result of Cerveau and Déserti.

We end this introduction by a few questions.

(1) for $d \geq 2$, and any field K, is $\mathrm{Cr}_d(K)$ locally residually finite (i.e. is every finitely generated subgroup residually finite)? approximable by finite groups (see Definition 2.1)? (I heard the question about local residual finiteness for $\mathrm{Cr}_d(\mathbf{C})$ from S. Cantat.)
(2) Does there exist a finitely generated subgroup of $\mathrm{Aut}(\mathbf{C}^2)$ with no faithful linear representation (see Remark 5.3)?

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2. Soficity of Cremona groups

We begin by the notion of approximation, which is classical in model theory.

Definition 2.1. Let $\mathcal{C}$ be a class of groups. We say that a group G is *approximable by the class* $\mathcal{C}$ (or *initially sub-*$\mathcal{C}$ in Gromov's terminology [Gro]) if for every finite symmetric subset F of G containing 1, there exists a group $H \in \mathcal{C}$ and an abstract injective map $\phi : F \to H$ such that $\phi(1) = 1$ and for all $x, y, z \in F$ we have $\phi(x)\phi(y) = \phi(z)$ whenever $xy = z$ (in particular $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in F$). Equivalently, G is approximable by the class $\mathcal{C}$ if and only if it is isomorphic to a subgroup of an ultraproduct of groups of the class $\mathcal{C}$.

Note that plainly, if a group is approximable by $\mathcal{C}$ then so are all its subgroups, and conversely if all its finitely generated subgroups are approximable by $\mathcal{C}$, then so is the whole group. A residually finite group is always approximable by finite groups, and the converse holds for finitely presented groups, but not for general finitely generated groups (see [St, VG]). It is also straightforward from the definition that if a group is approximable by sofic groups, then it is sofic as well. Therefore the first part of Theorem 1.1 follows from the following two propositions.

Proposition 2.2. *For any field K and d, the Cremona group $\mathrm{Cr}_d(K)$ is approximable by the family*

$$\{\mathrm{Cr}_d(F) : \ F \text{ finite field}\}.$$

Proposition 2.3. *For any finite field F and d, the Cremona group $\mathrm{Cr}_d(F)$ is sofic.*

Proof of Proposition 2.2. Since any field extension $K \subset L$ induces a group embedding $\mathrm{Cr}_d(K) \subset \mathrm{Cr}_d(L)$, it is enough to prove the proposition when K is algebraically closed.

Consider $f = (f_1, \dots, f_d)$, where $f_i \in K(t_1, \dots, t_d)$. To such a d-tuple corresponds to the regular map defined outside the zero set of the denominators of the f_i, mapping $(x_1, \dots, x_d) \in K^d$ to $(f_1(x_1, \dots, x_d), \dots, f_d(x_1, \dots, x_d))$. We say that f is non-degenerate if f has Zariski-dense image. If g is another d-tuple and f is non-degenerate, we can define the composition $g \circ f$ by

$$(x_1, \dots, x_d) \mapsto (g_1(f_1(x_1, \dots, x_d), \dots, f_d(x_1, \dots, x_d)), \dots, g_d(\dots)).$$

The non-degenerate d-tuples thus form a semigroup under composition, and by definition the Cremona group $\mathrm{Cr}_d(K)$ is the set of invertible elements of this semigroup.

Let W be a finite symmetric subset of $\mathrm{Cr}_d(K)$ containing 1. Write each coordinate of every element of W as a quotient of two polynomials. Let c_1 be the product in K of all nonzero coefficients of denominators of coordinates of elements of WW; let c_2 be the product of all nonzero coefficients of numerators of coordinates of elements of the form $u - v$ for distinct $u, v \in W$. Let A be the domain generated by all coefficients of elements of W, so $c = c_1c_2 \in A - \{0\}$. Since the ring A is residually a finite field [Mal], there exists a finite quotient field F of A in which $\bar{c} \neq 0$, where $x \mapsto \bar{x}$ is the natural projection $A \to F$. If $u \in F$, u can be viewed as a element of $F(t_1, \dots, t_d)^d$ as above (the denominator does not vanish because $\bar{c}_1 \neq 0$. Also, the condition $\bar{c}_1 \neq 0$ implies that whenever $uv = w$, we also have $\bar{u}\bar{v} = \bar{w}$. In particular, since W is symmetric with 1, it follows that the elements $\bar{u}$ are invertible, i.e. belong to $\mathrm{Cr}_d(F)$. Finally, whenever $u \neq v$, since $\bar{c}_2 \neq 0$, we have $\bar{u} \neq \bar{v}$. $\square$

Remark 2.4. It follows from the proof that $\mathrm{Cr}_d(K)$ is approximable by some suitable subclasses of the class of d-Cremona groups over finite fields: if K has characteristic p it is enough to restrict to finite fields of characteristic p, and if K has characteristic p it is enough to restrict to the class of finite fields of characteristic $p \geq p_0$ for any fixed p_0. Also, if $K = \mathbf{Q}$, it is enough to restrict to the class cyclic fields $\mathbf{Z}/p\mathbf{Z}$ (for $p \geq p_0$).

Proof of Proposition 2.3. Write $F = F_q$. Let W be a finite symmetric subset of $\mathrm{Cr}_d(F_q)$ containing 1.

For any $u \in \mathrm{Cr}_d(K)$, let Z_u be the singular set of u; view it as a closed, F_q-defined subvariety of the affine d-space; for every F_q-field L, u induces a bijection from $L^d - Z_u$ to $L^d - Z_{u^{-1}}$. We extend it arbitrarily (for each given L) to a permutation $\hat{u}$ of L^d.

Note that $\hat{u}\hat{v}$ and $\widehat{uv}$ coincide on the complement of $Z_v \cup v^{-1}(Z_u)$.

Then there exists a constant $C > 0$ such that for all $u \in W$ and all m we have $\#Z_u(F_{q^m}) \leq Cq^{m(d-1)}$ (this is a standard consequence, for instance, of the Lang-Weil estimates [LW] but can be checked directly).

So, when $L = F_{q^m}$ the Hamming distance in $\mathrm{Sym}(L^d)$ between $\hat{u}\hat{v}$ and $\widehat{uv}$ is $\leq 2Cq^{-m}$, which tends to 0.

Also, by considering the zero set D_{uv} of the numerator of $u - v$, we obtain that if $u \neq v$, the Hamming distance from $\hat{u}$ and $\hat{v}$ is $\geq 1 - 2C'q^{-m}$, for some fixed constant C' and for all m. We thus proved that $\mathrm{Cr}_d(K)$ is sofic. $\square$

Remark 2.5. We actually proved that for every field K, the group $\mathrm{Cr}_d(K)$ satisfies the following property: for every finite subset $W \subset \mathrm{Cr}_d(K)$ there is a constant $c > 0$ such that every integer n there exists $k \leq n$ and a map $W \to \mathrm{Sym}_k$ satisfying

- $\mathsf{d}^k_{\mathrm{Ham}}(\phi(g)\phi(h), \phi(gh)) \leq cn^{-1/d}$ for all $g, h \in F$ such that $gh \in F$;
- $\phi(1) = 1$ and $\phi(g) = \phi(g^{-1})$ for all $g \in F$;
- $\mathsf{d}^k_{\mathrm{Ham}}(\phi(u), \phi(v)) \geq 1 - cn^{-1/d}$ for all $u \neq v$.

where $\mathsf{d}^k_{\mathrm{Ham}}$ is the normalized Hamming distance on the symmetric group Sym_k. (In Section 3, we will interpret this by saying that the "sofic profile" of $\mathrm{Cr}_d(K)$ is in $O(n^d)$.) Note that for every integer $m \geq 1$ there exists a distance-preserving homomorphism $(\mathrm{Sym}_k, \mathsf{d}^k_{\mathrm{Ham}}) \to (\mathrm{Sym}_{mk}, \mathsf{d}^{mk}_{\mathrm{Ham}})$; in particular k can be chosen so that $k \geq n/2$.

3. Sofic profile

A notion of sofic dimension of a finitely generated group was recently introduced by Arzhantseva and Cherix (see Remark 3.9 for the precise definition and comments). We introduce here a very similar, but different, notion of sofic profile, not for groups, but for its finite pieces, or "chunks".

Definition 3.1. Let us call *chunk* a finite set F, endowed with a basepoint 1_F and a subset D of $F \times F \times F$ satisfying the condition $(x, y, z), (x, y, z') \in D$ implies $z = z'$. So we can view it as a partially defined function $(x, y) \mapsto z$ and we write $xy = z$ to mean that $(x, y, z) \in D$.

If F is an abstract chunk and G is a group, we call representation of F into G an *injective* mapping $f : F \to G$ such that $f(1) = 1$ and $f(x)f(y) = f(z)$ whenever $xy = z$.

If F is a subset of a group G with $1 \in F$, it is naturally a chunk by setting $xy = z$ whenever this holds in the group G. We call it a chunk of G (symmetric chunk if F is symmetric in G).

In this setting, if $\mathcal{C}$ is a class of groups, to say that G is approximable by the class $\mathcal{C}$ means that every chunk of G has a representation in a group in $\mathcal{C}$.

Definition 3.2. Let F be a chunk. If n is an integer and $\varepsilon > 0$, define a ε-morphism from F to Sym_n to be a mapping $f : F \to \mathrm{Sym}_n$ such that $f(1) = 1$ and $\mathsf{d}^n_{\mathrm{Ham}}(f(xy), f(x)f(y)) \le \varepsilon$ for all $x, y \in F$. A mapping from F to the symmetric group Sym_n is said to be $(1 - \varepsilon)$-expansive if $\mathsf{d}^n_{\mathrm{Ham}}(x, y) \ge 1 - \varepsilon$ whenever x, y are distinct points of F.

Define the *sofic profile* of F as the non-decreasing function

$$\begin{aligned}\sigma_F(r) = \inf\{n :\ &\exists f : F \to (\mathrm{Sym}_n, \mathsf{d}^n_{\mathrm{Ham}}),\\ &f \text{ is a } (1 - r^{-1})\text{-expansive } r^{-1}\text{-morphism}\} \quad (r \ge 1),\end{aligned}$$

where $\inf \emptyset = +\infty$. Say that the chunk F is *sofic* if $\lim_{n\to\infty} \sigma_F(r) < \infty$ for all $r \ge 1$. Say that a group G is *sofic* if every chunk in G is sofic. If S is a finite subset of G, write $\sigma_{G,S}(r) = \sigma_F(r)$, where $F = S$ endowed with its chunk structure.

Observe that this definition of soficity is exactly the same as the previous one. Also, it is immediate that the sofic profile of G is bounded (in the sense that σ_S is bounded for every finite $S \subset G$). It also equivalent to the statement that it is sublinear: indeed if F is a chunk and $\sigma_F(r) = n < r$ for some $r > 1$, then the chunk is representable into Sym_n.

This notion is related to the classical notion of isoperimetric profile (or Følner function) of a group G. If S is a finite subset of G and $X \subset G$, define $\partial_S X = SX - X$. Following Vershik [V], define the isoperimetric profile of (G, S) as the nondecreasing function

$$\alpha_{G,S}(r) = \inf\{n \ge 1 : \exists F \subset G, \#(F) = n, \#(\partial_S(F))/\#(F) < r^{-1}\}.$$

By definition, G is amenable[1] if $\alpha_{G,S}(r) < +\infty$ for every finite subset S and $r \geq 1$. Note that the isoperimetric profile is bounded if and only if G is locally finite.

If $u, v : [1, \infty[\to [0, \infty]$ are functions, we say that $u \preceq v$ if there exist positive constants such that $u(r) \leq Cv(C'r) + C''$ for all $r \geq 1$, and $u \simeq v$ if $u \preceq v \preceq u$.

If G is a group u is a function, we say that the sofic profile of G is $\preceq u(r)$ if for every finite subset F in G we have $\sigma_{G,S}(r) \preceq u(r)$, it is $\succeq u(r)$ if for some finite subset F in G we have $\sigma_{G,S}(r) \succeq u(r)$; it is $\simeq u(r)$ if both hold. We have similar definitions for the isoperimetric profile (in this case, if G is generated by a finite symmetric generating subset S, the isoperimetric profile of G is $\simeq \alpha_{G,S}$ in the sense above).

Remark 3.3. The main advantage of this definition is that for a group it depends only on its chunks, and therefore, tautologically, if any group in $\mathcal{C}$ has the property that its sofic profile is $\preceq u(r)$, then it still holds for any group approximable by the class $\mathcal{C}$. In particular, for any u, to have sofic profile $\preceq u(r)$ is a closed property in the space of marked groups.

In contrast to the definition given by Arzhantseva-Cherix (see Remark 3.9), its main drawback however is that I am unable to define, even for a finitely presented group, its sofic profile as a certain (asymptotic class of) function. It might indeed, in principle, happen that larger and larger chunks have larger and larger sofic profile, and this is why we can only define the sofic profile to be asymptotically bounded by a certain function.

By a result of Coulhon and Saloff-Coste [CS], the isoperimetric profile grows at least as fast as the volume growth.

If $G = \mathbf{Z}^d$, the isoperimetric profile is $\simeq n^d$ and this is optimal; the same estimate holds for groups of polynomial growth of degree d. If G is amenable of exponential growth, then the isoperimetric profile is $\succeq 1/\log(n)$ and this is optimal for polycyclic groups.

Informally, soficity of G means that points in G are well separated by "quasi-actions" of G on finite sets, and amenability is the additional requirement that these finite sets lie inside G with the action by the left multiplication.

It is elementary to check that the sofic profile is asymptotically bounded above by the isoperimetric profile. Precisely we have the following lemma.

Lemma 3.4. *For any finite subset F of G, we have $\sigma_F(n) \leq 3I_{G,F}(n)$.*

Proof. Suppose that $I_{G,F}(n) < \varepsilon$ and let us show that $\sigma_F(n) < 3\varepsilon$. By assumption there exists $X \subset G$ with $0 < \#(X) \leq n$ and $\#(FX - X)/\#(X) < \varepsilon$. For $s \in S$,

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[1] The isoperimetric profile is also often defined as the non-increasing function

$$I_{G,S}(n) = \inf\{\#(\partial_S(F))/\#(F) : F \subset G, 0 < \#(F) \leq n\}.$$

We check immediately that for all $r \geq 1$, $n \geq 1$ we have $\alpha(r) \leq n \Leftrightarrow r < I(n)^{-1}$. Thus α and $1/I$ are essentially inverse functions to each other.

define $\phi(s) : X \to X$ to map $x \mapsto sx$ if $sx \in X$, and extend it arbitrarily to a bijection. By assumption, the proportion of x such that $\phi(s)(x) = sx$ is $> 1-\varepsilon$. It follows that the Hamming distance of $\phi(s)$ and $\phi(s')$ is $> 1-2\varepsilon$ whenever $s, s' \in F$ and $s \neq s'$, and the Hamming distance between $\phi(st)$ and $\phi(s)\phi(t)$ is 3ε whenever $s, t, st \in F$. So $I_{G,F}(n) < 3\varepsilon$. □

It is known [ES2] that any sofic-by-amenable group (i.e. lying in an extension with sofic kernel and amenable quotient) is still sofic. The proof given there is an explicit construction, yielding without any change the following.

Proposition 3.5. *Let $u, v : [1, \infty[\to [1, \infty]$ be functions. Let G be a group in a short exact sequence $1 \to N \to G \to Q \to 1$. If the sofic profile of N is $\preceq u(r)$ and the isoperimetric profile of Q is $\preceq v(r)$, then the sofic profile of G is $\preceq u(r)v(r)$.* □

Example 3.6. Say that a group G has *polynomial sofic profile* if every chunk in G has its sofic profile bounded above by a polynomial (no uniformity on the degree is assumed). This amounts to say that for every suprapolynomial function $u(r)$ (in the sense that $\log u(r)/\log r \to \infty$), the sofic profile of G is $\preceq u(r)$. It follows from Proposition 3.5 that the class of groups with polynomial sofic profile is stable under extension with virtually abelian quotients. Since it is also stable under taking direct limits, it follows that every elementary amenable group has polynomial sofic profile. (Recall that the class of elementary amenable groups is the smallest class containing the trivial group and stable under direct limits and extensions with finitely generated virtually abelian quotients.) In particular, any solvable group has polynomial sofic profile.

Example 3.7. For $k, \ell \in \mathbf{Z} - \{0\}$, the sofic profile of the Baumslag-Solitar group

$$\Gamma = \mathrm{BS}(k, \ell) = \langle t, x | tx^k t^{-1} \rangle$$

is at most linear (i.e. is $\preceq r$). (This group is not approximable by finite groups unless $|k| = 1$, $|\ell| = 1$, or $|k| = |\ell|$.)

Proof. Let N be the kernel of homomorphism onto $Q = \mathbf{Z}$ mapping (t, x) to $(1, 0)$. The assertion follows from Proposition 3.5 the fact that the isoperimetric profile of $\mathbf{Z}$ is linear, and that N is approximable by finite groups (so its sofic profile is bounded). Let us check the latter fact: using that Γ is the HNN-extension of $\mathbf{Z}$ by the two embeddings of $\mathbf{Z}$ into itself by multiplication by k and ℓ respectively, the group N is an iterated free product with amalgamation $\cdots \mathbf{Z} *_{\mathbf{Z}} \mathbf{Z} *_{\mathbf{Z}} \mathbf{Z} *_{\mathbf{Z}} \cdots$, where each embedding of $\mathbf{Z}$ to the left, resp. to the right, is given by multiplication by k, resp. by ℓ [Se, I.1.4, Prop. 6]. This group is locally residually finite, i.e. every such finite iteration $\mathbf{Z} *_{\mathbf{Z}} \mathbf{Z} *_{\mathbf{Z}} \cdots *_{\mathbf{Z}} \mathbf{Z}$ is residually finite; this follows, for instance, from [Ev]. (In case k, ℓ are coprime, R. Campbell [Ca] checked that N itself is not residually finite, and even that all its finite quotients are abelian.)

Since Γ is finitely presented, the assertion of non-approximability means that Γ is not residually finite except in the excluded cases, this is a result of Meskin [Me] (correcting a error in [BS]). $\square$

Note that the fact that $\mathrm{BS}(k, \ell)$ is residually solvable (indeed, free-by-metabelian) immediately implies its soficity, but yields a much worse upper bound on its sofic profile.

Problem 3.8. Develop methods to compute lower bounds for the sofic profile of explicit groups. Is there any group for which the sofic profile is not $\preceq r$?

This problem only concerns groups not approximable by finite groups, since otherwise the sofic profile is bounded. Otherwise the sofic profile grows at least linearly as we observed above, but we have no example with a better lower bound. Here are some examples of finitely generated groups not approximable by finite groups, which could be looked over.

- Infinite isolated groups. A group G is by definition isolated if it has a chunk S such that any representation of S into a group H extends to an injective homomorphism $G \to H$. (This clearly implies that G is generated by S and actually is presented with the set of conditions $st = u$, $s, t, u \in S$ as a set of relators.) These include finitely presented simple groups. Many more examples are given in [CGP], e.g. Thompson's group F of the interval. It includes several examples that are amenable (solvable or not) and therefore sofic. We can also find in [CGP] examples of non-amenable isolated groups but their soficity is not known; however an example of a non-amenable isolated group that is known to be sofic, is given in [C].
- Other finitely presented non-residually finite groups. This includes Baumslag-Solitar groups mentioned above, as well as various other one-relator groups [Bau, BMT]. Another example is Higman's group [Se, I.1.4, Prop. 5]
$$\langle x_1, x_2, x_3, x_4 | \; x_{i-1} x_i x_{i-1}^{-1} = x_i^2 \; (i = 1, 2, 3, 4 \mod 4) \rangle,$$
which has no proper subgroup of finite index. Its soficity is not known.
- Direct products of the above groups. For instance, $\mathrm{BS}(2, 3)^d$ has sofic profile $\preceq n^d$.

Remark 3.9. The notion of sofic dimension previously introduced by Arzhantseva and Cherix is the following. Let G be generated by a finite subset S. The sofic dimension $\phi(n)$ is, in the language introduced here, $\phi_S(n) = \sigma_{S^n}(n)$. Arzhantseva and Cherix showed that its asymptotics only depend on G and not on the choice of S, and related it to the isoperimetric profile. However, it is quite different in spirit to the sofic profile, because it takes into account the shape of balls. In particular, the sofic dimension is bounded only for finite groups.

I am not able to adapt the specification process used to estimate the sofic profile of Cremona groups (Proposition 2.2) to give any upper bound on the sofic dimension of their finitely generated subgroups. This is probably doable, but at the cost of some tedious estimates on the degrees of singular subvarieties arising in the proof, which would not give better than an exponential upper bound for the sofic dimension.

Note that the knowledge of the function of two variables $\Phi(m,n) = \sigma_{S^m}(n)$ encompasses both the sofic dimension $\phi_S(n) = \Phi(n,n)$ and the sofic profile (asymptotic behavior of $\Phi(m,n)$ when m is fixed).

4. General varieties

The purpose of this section is to prove Theorem 1.1 in its general formulation (for an arbitrary absolutely irreducible variety). Since the group of birational transformations of an absolutely irreducible variety can be canonically identified with that of an open affine subset, we can, in the sequel, stick to affine varities.

If X is an affine variety over the field K, we define a *specification* of X over a finite field F as an affine variety X'' over F satisfying the following condition. Denoting by B, B'' be the K-algebra of functions of X and the F-algebra of functions on X'', there exists a finitely generated subdomain A of K, a finitely generated A-subalgebra B' of B, a surjective homomorphism $A \to F$, so that $B' \otimes_A F \simeq B''$ as A-algebras, and the natural K-algebra homomorphism $B' \otimes_A K \to B$ is an isomorphism. Note that $\dim(X'') \le \dim(X)$.

Proposition 4.1. *Let X be an affine d-dimensional absolutely irreducible variety over a field K. Then the group $\mathrm{Bir}_K(X)$ is approximable (in the sense of Definition 2.1) by the family of groups $\{\mathrm{Bir}_F(X')\}$, where F ranges over finite fields and X' ranges over d-dimensional specifications of X over F that are absolutely irreducible over F.*

Proof. Let B be the K-algebra of functions on X and L be its field of fractions, so that $\mathrm{Bir}_K(X) = \mathrm{Aut}_K(L)$.

Suppose that a finite symmetric subset W with 1 is given in $\mathrm{Aut}_K(L)$. It consists of a finite family (v_i) of pairwise distinct elements of $\mathrm{Aut}_K(L)$. There exists $f \in B - \{0\}$ such that $v_i(B) \subset B[f^{-1}]$ for all i. Denote by $u_i : B \to B[f^{-1}]$ the K-algebra homomorphism which is the restriction of v_i.

Fix generators $t_1, \dots, t_m$ of B as a K-algebra, so that $B[f^{-1}]$ is generated by $t_1, \dots, t_m, f^{-1}$ as a K-algebra. For each (i,j), we can write $u_i(t_j)$ as a certain polynomial with coefficients in K and $m+1$ indeterminates, evaluated at $(t_1, \dots, t_m, f^{-1})$. Let C_1 be the (finite) subset of K consisting of the coefficients of these polynomials (i,j varying). Also, under the mapping $X_j \mapsto t_j$, the K-algebra B is the quotient of $K[X_1, \dots, X_m]$ by some ideal; we can consider a certain finite set of polynomials with coefficients in K generating this ideal. Let C_2 be the finite subset of K consisting of the coefficients of those polynomials. Also, f can be written as a polynomial in $t_1, \dots, t_m$; let $C_3 \subset K$ consist of the coefficients of this polynomial. Let A_0 be the subring of K generated by $C_1 \cup C_2 \cup C_3$.

Let B_0' be the A_0-subalgebra of B generated by the t_j. By generic flatness [SGA, Lem. 6.7], there exists $s \in A_0 - \{0\}$ such that $B' = B_0'[s^{-1}]$ is flat over $A = A_0[s^{-1}]$. Since A contains coefficients of the polynomials defining B, we have, in a natural way, $B = B' \otimes_A K$. Moreover, $f \in B'$ and the homomorphisms u_i actually map B' to $B'[f^{-1}]$; if u_i' denotes the corresponding restriction map $B' \to B'[f^{-1}]$, then $u_i' \otimes_A K = u_i$ (here we view $- \otimes_A K$ as a functor). In particular, since the u_i are pairwise distinct by definition, the u_i' are pairwise distinct as well. This means that for all $i \neq i'$ there exists an element $x_{ii'} \in B'$ such that $u_i(x_{ii'}) \neq u_{i'}(x_{ii'})$. Let $x \in B' - \{0\}$ be the product of all $u_i(x_{ii'}) - u_{i'}(x_{ii'})$, where $\{i, i'\}$ ranges over pairs of distinct indices. Also, fix k large enough so that the element $g = f^k \prod_i u_i(f) \in B'[f^{-1}] - \{0\}$ belongs to $B' - \{0\}$.

There is a natural map $\phi : \mathrm{Spec}(B') \to \mathrm{Spec}(A)$ consisting in taking the intersection with A. This map is continuous for the Zariski topology. Consider the open subset of $\mathrm{Spec}(B')$ consisting of those primes not containing gx; this is an open subset of $\mathrm{Spec}(B')$ containing $\{0\}$. Since B' is A-flat, the map ϕ is open [SGA, Th. 6.6]. Therefore there exists $a \in A - \{0\}$ such that every prime of A not containing a is of the form $\mathfrak{P} \cap A$ for some prime $\mathfrak{P}$ of B' not containing gx.

Now since B' is A-flat and absolutely integral, by [EGA, 12.1.1] there exists $a' \in A - \{0\}$ such that for every prime $\mathcal{Q}$ of A not containing a', the quotient ring $B' \otimes_A (A/\mathcal{Q}) = B'/\mathcal{Q}B'$ is an absolutely integral $(A/\mathcal{Q})$-algebra.

It follows that if $\mathfrak{m}$ is a maximal ideal of A not containing aa', then $B'/\mathfrak{m}B'$ is an absolutely integral $(A/\mathfrak{m})$-algebra and $\mathfrak{m}B'$ does not contain gx. Let us fix such a maximal ideal $\mathfrak{m} \subset A$ (it exists because in a finitely generated domain, the intersection of maximal ideals is trivial, see for instance [Eis, Th. 4.19]). Since u_i' is a A-algebra homomorphism, it sends $\mathfrak{m}B'$ to $\mathfrak{m}B'[f^{-1}]$, and therefore induces a $(A/\mathfrak{m})$-algebra homomorphism $u_i'' : B'/\mathfrak{m}B' \to B'[f^{-1}]/\mathfrak{m}B'[f^{-1}]$. Since $x \neq 0$ in $B'/\mathfrak{m}B'$, the u_i'' are pairwise distinct.

We need to check that $\dim(B'/\mathfrak{m}B') \leq d$. First, by [Eis, Th. 13.8], $\dim(B') \leq \dim(A) + d$. Now since B' is A-flat, by [Eis, Th. 10.10] we have $\dim(B'/\mathfrak{m}B') \leq \dim(B') - \dim(A_{\mathfrak{m}})$. Since A is a finitely generated domain, and $\mathfrak{m}$ is a maximal ideal, we have $\dim(A_{\mathfrak{m}}) = \dim(A)$ (see Lemma 4.3), and from the two inequalities above we deduce $\dim(B'/\mathfrak{m}B') \leq d$. (Actually both inequalities are equalities (same references): for the first one, [Eis, Th. 13.8] uses the fact that A is universally catenary, which follows in turn from the fact that $\mathbf{Z}$ is universally catenary, which is part of [Eis, Cor. 18.10].)

To conclude it is enough to prove the following claim

Claim 4.2. *The homomorphisms u_i'' uniquely extend to pairwise distinct $(A/\mathfrak{m})$-automorphisms v_i'' of the field of fractions of $B'/\mathfrak{m}B'$ and whenever $v_i v_j = v_k$ we have $v_i'' v_j'' = v_k''$.*

To check the claim, begin with the following general remark. If R is a domain, s a nonzero element, and we have two homomorphisms $\alpha, \beta : R \to R[s^{-1}]$, such that $\alpha(s)$ is nonzero, then α uniquely extends to a homomorphism $R[s^{-1}] \to R[(s\alpha(s))^{-1}]$ and we can define the composite map $\alpha\beta : R \to R[(s\alpha(s))^{-1}]$.

Since $g \neq 0$ in B, this can be applied to the K-algebra homomorphisms $(v_i =)u_i : B \to B[f^{-1}]$, which are given by $t_\ell \mapsto U_{\ell i}(t_1, \dots, t_m)/f^d$, where $U_{\ell i} \in A[X_1, \dots, X_m]$. We thus have, for all ℓ

$$\begin{aligned} v_i(v_j(t_\ell)) =& v_i(U_{\ell j}(t_1, \dots, t_m)/f^d) \\ =& U_{\ell j}(u_i(t_1), \dots, u_i(t_m))/u_i(f)^d \\ =& U_{\ell j}(U_{1i}(t_1, \dots, t_m)/f^d, \dots U_{mi}(t_1, \dots, t_m)/f^d)/u_i(f)^d. \end{aligned}$$

For all ℓ, j can write the formal identity

$$U_{\ell j}(X_1/Y, \dots, X_m/Y)Y^\delta = V_{\ell j}(T_1, \dots, T_m, Y)$$

for some $V_{\ell j} \in B[X_1, \dots, X_m, Y]$ and some positive integer δ. Thus $v_iv_j = v_k$ (or equivalently $u_iu_j = u_k$) means that for all ℓ we have the equality in L, for all ℓ

$$U_{\ell j}(U_{1i}(t_1, \dots, t_m)/f^d, \dots U_{mi}(t_1, \dots, t_m)/f^d)/u_i(f)^d = U_{\ell k}(t_1, \dots, t_m)/f^d,$$

that is

$$V_{\ell j}(U_{1i}(t_1, \dots, t_m), \dots U_{mi}(t_1, \dots, t_m)) = U_{\ell k}(t_1, \dots, t_m)u_i(f)^d f^{d\delta - d},$$

which actually holds in $B' \subset L$. This equality still holds modulo the ideal $\mathfrak{m}B'$. Since $g \neq 0$ in $B'/\mathfrak{m}B'$ (i.e., f and $u_j(f)$ are nonzero elements of the domain $B'/\mathfrak{m}B'$), this equality exactly means that $u_i''u_j'' = u_k''$ in the sense above.

Since in particular for every i there exists ι such that v_iv_ι and $v_\iota v_i$ are the identity, $u_i''u_\iota''$ and $u_\iota''u_i''$ are the identity; in particular u_i'' extends to an automorphism v_i'' of the fraction field of $B'/\mathfrak{m}B'$. Since the u_i'' are pairwise distinct, so are the v_i''. Moreover, whenever $u_iu_j = u_k$, we have $u_i''u_j'' = u_k''$ which in turn implies $v_i''v_j'' = v_k''$. So the claim is proved, and hence Proposition 4.1 as well. ☐

We used the following standard lemma.

Lemma 4.3. *Let A be a finitely generated domain. Then for any maximal ideal $\mathfrak{m}$, we have* $\dim(A) = \dim(A_\mathfrak{m})$.

Proof. If the characteristic is positive, A is a finitely generated algebra over the field on p elements, and [Eis, Cor. 13.4] (based on Noether normalization) applies, giving $\dim(A) = \dim(A_\mathfrak{m}) + \dim(A/\mathfrak{m}) = \dim(A_\mathfrak{m})$.

If the characteristic is zero, we use the fact that $\mathbf{Z}$ is universally catenary [Eis, Cor. 18.10], to apply [Eis, Th. 13.8], which yields $\dim(A_\mathfrak{m}) = \dim(\mathbf{Z}_{\mathfrak{m}\cap\mathbf{Z}}) + \dim(A \otimes_\mathbf{Z} \mathbf{Q})$. Since $\mathfrak{m}$ has finite index, $\mathfrak{m} \cap \mathbf{Z} = p\mathbf{Z}$ for some prime p and $\dim(\mathbf{Z}_{\mathfrak{m}\cap\mathbf{Z}}) = 1$. So $\dim(A_\mathfrak{m}) = 1 + \dim(A \otimes_\mathbf{Z} \mathbf{Q})$. Since this value does not depend on $\mathfrak{m}$, we deduce that $\dim(A_\mathfrak{m}) = \dim(A)$. ☐

Proposition 4.4. *For every absolutely irreducible affine variety X over a finite field F, the group $\mathrm{Bir}_F(X)$ is sofic. Actually, its sofic profile is $\preceq n^d$, where $d = \dim(X)$.*

The proof is similar to the one of Proposition 2.3 and left to the reader. The only additional feature is the fact, which follows from the Lang-Weil estimates (making use of the assumption that X is absolutely irreducible), that for some constant $c > 0$ and every finite extension F' of F with q elements, the number of points in $X(F')$ is $\geq cq^d$.

From Propositions 4.1 and 4.4 we deduce

Corollary 4.5. *For every absolutely irreducible affine variety X over a field K, the group $\mathrm{Bir}_K(X)$ is sofic. Actually, its sofic profile is $\preceq n^d$, where $d = \dim(X)$.*

5. A nonlinear subgroup of the Cremona group

We provide in this section an example of a finitely generated subgroup of $\mathrm{Cr}_2(\mathbf{C})$ that is not linear over any field. It is 3-solvable and actually lies in the Jonquières subgroup, that is, the group of birational transformations preserving the partition of $\mathbf{C}^2$ by horizontal lines.

If $f \in K(X)$ and $g \in K(X)^\times$, define $\alpha_f, \mu_g \in \mathrm{Cr}_2(K)$ by

$$\alpha_f(x, y) = (x, y + f(x)); \quad \mu_g(x, y) = (x, yg(x)).$$

We have

$$\alpha_{f+f'} = \alpha_f \alpha_{f'}; \quad \mu_{gg'} = \mu_g \mu_{g'}; \quad \mu_g \alpha_f \mu_g^{-1} = \alpha_{fg}.$$

Also for $t \in K$, define $s_t \in \mathrm{Cr}_2(K)$ by $(x, y) = (x + t, y)$, so that

$$s_t \alpha_{f(X)} s_t^{-1} = \alpha_{f(X-t)}; \quad s_t \mu_{g(X)} s_t^{-1} = \mu_{g(X-t)}.$$

Consider the subgroup Γ_n of $\mathrm{Cr}_2(K)$ generated by s_1 and α_{X^n} ($n \geq 0$).

Lemma 5.1. *The group Γ_n is nilpotent of class at most $n + 1$; moreover if K has characteristic zero the nilpotency length of Γ_n is exactly $n + 1$, and Γ_n is torsion-free.*

Proof. Consider the largest group R_n, generated by s_1 and by the abelian subgroup A_n consisting of all α_P, where P ranges over polynomials of degree at most n. Then A_n is normalized by s_1 and $[s_1, A_n] \subset A_{n-1}$ for all $n \geq 1$, while $A_0 = \{1\}$. Therefore R_n is nilpotent of class at most $n+1$, and therefore so is Γ_n. Conversely, the n-iterated group commutator $[s_1, [s_1, \cdots, [s_1, \alpha_{X^n}] \cdots]]$ is equal to $\alpha_{\Delta^n X^n}$, where Δ is the discrete differential operator $\Delta P(X) = -P(X) + P(X - 1)$. So if K has characteristic zero (or $p > n$) then $\Delta^n X^n \neq 0$ and Γ_n is not n-nilpotent. In this case it is also clear that R_n is torsion-free. $\square$

Now assume that K has characteristic zero and consider the group $G \subset \mathrm{Cr}_2(\mathbf{Q}) \subset \mathrm{Cr}_2(K)$ generated by $\{s_1, \alpha_1, \mu_X\}$.

Proposition 5.2. *The group $G \subset \mathrm{Cr}_2(\mathbf{Q})$ is solvable of length three; it is not linear over field.*

Proof. From the conjugation relations above it is clear that the subgroup generated by s_1, all α_f and μ_g, is solvable of length at most three. If we restrict to those g of the form $\prod_{n\in\mathbf{Z}}(X-n)^{k_n}$ (where (k_n) is finitely supported), we obtain a subgroup containing Γ, that is clearly torsion-free.

Since $\mu_X^n\alpha_1\mu_X^{-n} = \alpha_{X^n}$, we see that G contains Γ_n for all n, which is nilpotent of length exactly $n+1$. Therefore it has no linear representation over any field.

[Sketch of proof of the latter (well-known) result: in characteristic $p > 0$, any torsion-free nilpotent subgroup is abelian, so this discards this case. Otherwise in characteristic zero, since any finite index subgroup of a torsion-free nilpotent group of nilpotency length $n+1$ still has nilpotency length $n+1$, the existence of a linear representation of G into $\mathrm{GL}_d(\mathbf{C})$ implies the existence of a Lie subalgebra of $\mathfrak{gl}_d(\mathbf{C})$ of nilpotency length $n+1$ for all n; this necessarily implies $n+1 \le d^2$, and since n is unbounded this is a contradiction.]

The fact that G is not 2-solvable (=metabelian) can be checked by hand, but also follows from the fact that every torsion-free finitely generated metabelian group is linear over a field of characteristic zero [Re]. $\square$

With little further effort, it actually follows from the same argument that G is not linear over any finite product of fields (and therefore over any reduced commutative ring): indeed at least one of the projections should contain torsion-free nilpotent subgroups of arbitrary large nilpotency length.

Remark 5.3. It is unknown whether there exists a finitely generated subgroup of the group $\mathrm{Aut}(\mathbf{C}^2)$ of *polynomial automorphisms* of $\mathbf{C}^2$, that is not linear in characteristic zero. A construction in the same fashion does not work: indeed let E be the group of elementary automorphisms, namely of the form $(x, y) \mapsto (\alpha x + P(y), \beta y + c)$ for $(\alpha, \beta, c, P) \in \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C} \times \mathbf{C}[X]$. Then, although E is not linear (since by the argument above, it contains all Γ_n), every finitely generated subgroup of E is linear over $\mathbf{C}$.

To see this, write E as a semidirect product $(\mathbf{C}^* \times (\mathbf{C}^* \ltimes \mathbf{C})) \ltimes \mathbf{C}[X]$, where the action on $\mathbf{C}[X]$ is by $(\alpha, \beta, c) \cdot P(X) = \alpha P(\beta X + c)$. In particular, this action stabilizes the subgroup $\mathbf{C}_n[X]$ of polynomials of degree at most n. Therefore any finitely generated subgroup of E is contained in the subgroup $(\mathbf{C}^* \times (\mathbf{C}^* \ltimes \mathbf{C})) \ltimes \mathbf{C}_n[X]$ for some $n \ge 1$. This is a (finite-dimensional) complex Lie group whose center is easily shown to be trivial, so its adjoint representation is a faithful complex linear representation.

A nice observation by Cerveau and Déserti [CD, Lemme 5.2] is that the Cremona group has no faithful linear representation in characteristic zero. Actually, an easy refinement of the same argument provides a stronger result.

Proposition 5.4. *If K is an algebraically closed field, there is no nontrivial finite-dimensional linear representation of $\mathrm{Cr}_2(K)$ over any field.*

(Note that since the Cremona group is not simple by a recent difficult result of Cantat and Lamy [CL], the non-existence of a faithful representation does not formally imply the non-existence of a nontrivial representation.)

Proof of Proposition 5.4. In $\mathrm{Cr}_2(K)$, there is a natural copy of $G = (K^\times)^2 \rtimes \mathbf{Z}$, where $\mathbf{Z}$ acts by the automorphism $\sigma(x, y) = (x, xy)$ of $(K^\times)^2$. Here, it corresponds, in affine coordinates, to the group of transformations of the form

$$(x_1, x_2) \mapsto (\lambda_1 x_1, x_1^n \lambda_2 x_2) \quad \text{for} \quad (\lambda_1, \lambda_2, n) \in (K^\times)^2 \times \mathbf{Z}.$$

Consider an linear representation $\rho : G \to \mathrm{GL}_n(F)$, where F is any field (here G is viewed as a discrete group). If p is a prime which is nonzero in K and if $\omega_p \in K$ is a primitive p-root of unity, set $\alpha_p(x_1, x_2) = (\omega_p x_1, \omega_p x_2)$ and $\beta_p(x_1, x_2) = (x_1, \omega_p x_2)$. Then $\sigma\alpha_p\sigma^{-1}\alpha_p^{-1} = \beta_p$ and commutes with both σ and α_p. An argument of Birkhoff [Bi, Lemma 1] shows that if $\rho(\alpha_p) \neq 1$ then $n > p$ (the short argument given in the proof of [CD, Lemme 5.2] for F of characteristic zero works if it is assumed that p is not the characteristic of F).

Picking p to be greater than n and the characteristics of K and F, this shows that if we have an arbitrary representation $\pi : \mathrm{Cr}_2(K) \to \mathrm{GL}_n(F)$, the restriction of π to $\mathrm{PGL}_3(K)$ is not faithful; since $\mathrm{PGL}_3(K)$ is simple, this implies that π is trivial on $\mathrm{PGL}_3(K)$; since $\mathrm{Cr}_2(K)$ is generated by $\mathrm{PGL}_3(K)$ as a normal subgroup, this yields the conclusion. □

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