Appendix A. The Cremona group is not an amalgam

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Let $\mathbf{k}$ be a field. The *Cremona group* $\mathsf{Bir}(\mathbb{P}^d_{\mathbf{k}})$ of $\mathbf{k}$ in dimension d is defined as the group of birational transformations of the d-dimensional $\mathbf{k}$-affine space. It can also be described as the group of $\mathbf{k}$-automorphisms of the field of rational functions $\mathbf{k}(t_1, \dots, t_d)$. We endow it with the discrete topology.

Let us say that a group has *Property* $(\mathrm{F}\mathbf{R})_\infty$ if it satisfies the following

(1) For every isometric action on a complete real tree, every element has a fixed point.

Here we prove the following result.

Theorem A.1. *If* $\mathbf{k}$ *is an algebraically closed field, then* $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ *has Property* $(\mathrm{F}\mathbf{R})_\infty$.

Corollary A.2. *The Cremona group does not decompose as a nontrivial amalgam.*

Recall that a *real tree* can be defined in the following equivalent ways (see [1])

- A geodesic metric space which is 0-hyperbolic in the sense of Gromov;
- A uniquely geodesic metric space for which $[ac] \subset [ab] \cup [bc]$ for all a, b, c;
- A geodesic metric space with no subspace homeomorphic to the circle.

In a real tree, a *ray* is a geodesic embedding of the half-line. An *end* is an equivalence class of rays modulo being at bounded distance. For a group of isometries of a real tree, to *stably fix* an end means to pointwise stabilize a ray modulo eventual coincidence (it means it fixes the end as well as the corresponding Busemann function).

For a group Γ, Property $(\mathrm{F}\mathbf{R})_\infty$ has the following equivalent characterizations:

(2) For every isometric action of Γ on a complete real tree, every finitely generated subgroup has a fixed point.

(3) Every isometric action of Γ on a complete real tree either has a fixed point, or stably fixes a point at infinity (in the sense above).

The equivalence between these three properties is justified in Lemma A.9. Similarly, we can define the weaker *Property* $(\mathrm{FA})_\infty$, replacing complete real trees by ordinary trees (and allowing fixed points to be middle of edges), and the three corresponding equivalent properties are equivalent [3] to the following fourth: the group is not a nontrivial amalgam and has no homomorphism onto the group of integers. In particular, Corollary A.2 follows from Theorem A.1.

Remark A.3. a.– Note that the statement for actions on real trees (rather than trees) is strictly stronger. Indeed, unless $\mathbf{k}$ is algebraic over a finite field, the group $\mathsf{PGL}_2(\mathbf{k}) = \mathsf{Bir}(\mathbb{P}^1_{\mathbf{k}})$ does act isometrically on a real tree with a hyperbolic element (this uses the existence of a nontrivial real-valued valuation on $\mathbf{k}$), but does not have such an action on a discrete tree (see Proposition A.8).

A.3. b.– Note that $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ always has an action on a discrete tree with no fixed point (i.e. no fixed point on the 1-skeleton) when $\mathbf{k}$ is algebraically closed, and more generally whenever $\mathbf{k}$ is an infinitely generated field: write, with the help of a transcendence basis, $\mathbf{k}$ as the union of an increasing sequence of proper subfields $\mathbf{k} = \bigcup \mathbf{k}_n$, then $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ is the increasing union of its proper subgroups $\mathsf{Bir}\left(\mathbb{P}^2_{\mathbf{k}_n}\right)$, and thus acts on the disjoint union of the coset spaces $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})/\mathsf{Bir}\left(\mathbb{P}^2_{\mathbf{k}_n}\right)$, which is in a natural way the vertex set of a tree on which $\mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$ acts with no fixed point (this is a classical construction of Serre [3, Chap I, §6.1]).

A.3. c.– Theorem A.1 could be stated, with a similar proof, for actions on Λ-trees when Λ is an arbitrary ordered abelian group (see [1] for an introduction to Λ-metric spaces and Λ-trees).

In the following, $\mathcal{T}$ is a complete real tree; all actions on $\mathcal{T}$ are assumed to be isometric. We begin by a few lemmas.

Lemma A.4. *Let $x_0, \dots, x_k$ be points in a real tree $\mathcal{T}$ and $s \geq 0$. Assume that $d(x_i, x_j) = s|i-j|$ holds for all i, j such that $|i-j| \leq 2$. Then it holds for all i, j.*

Proof. This is an induction; for $k \leq 2$ there is nothing to prove. Suppose $k \geq 3$ and the result known up to $k-1$, so that the formula holds except maybe for $\{i, j\} = \{0, k\}$. Join x_i to x_{i+1} by segments. By the induction, the $k-1$ first segments, and the $k-1$ last segments, concatenate to geodesic segments. But the first and the last of these k segments are also disjoint, otherwise picking the "smallest" point in the last segment that also belongs to the first one, we find an injective loop, contradicting that $\mathcal{T}$ is a real tree. Therefore the k segments concatenate to a geodesic segment and $d(x_0, x_k) = sk$. □

Lemma A.5. *If $\mathbf{k}$ is any field and $d \geq 3$, then $\Gamma = \mathsf{SL}_d(\mathbf{k})$ has Property $(\mathrm{F}\mathbf{R})_\infty$. In particular, if $\mathbf{k}$ is algebraically closed, then $\mathsf{PGL}_d(\mathbf{k})$ has Property $(\mathrm{F}\mathbf{R})_\infty$.*

Proof. Let Γ act on $\mathcal{T}$. Let F be a finite subset of Γ. Every element of F can be written as a product of elementary matrices. Let A be the (finitely generated) subring of $\mathbf{k}$ generated by all entries of those matrices. Then $F \subset \mathsf{EL}_d(A)$, the subgroup of $\mathsf{SL}_d(A)$ generated by elementary matrices. By the Shalom-Vaserstein Theorem (see [2]), $\mathsf{EL}_d(A)$ has Kazhdan's Property (T) and in particular has a fixed point in $\mathcal{T}$, so F has a fixed point in $\mathcal{T}$. (There certainly exists a more elementary proof, but this one also shows that for every isometric action of $\mathsf{SL}_d(\mathbf{k})$ on a Hilbert space, every finitely generated subgroup fixes a point.) □

Fix the following notation: $G = \mathsf{Bir}(\mathbb{P}^2_{\mathbf{k}})$; $H = \mathsf{PGL}_3(\mathbf{k}) = \mathsf{Aut}(\mathbb{P}^2_{\mathbf{k}}) \subset G$; σ is the Cremona involution, acting in affine coordinates by $\sigma(x, y) = (x^{-1}, y^{-1})$. The Max Noether Theorem is that $G = \langle H, \sigma \rangle$. Let C be the standard Cartan subgroup of H, that is, the semidirect product of the diagonal matrices by the

Weyl group (of order 6). Let $\mu \in H$ be the involution given in affine coordinates by $\mu(x,y) = (1-x, 1-y)$.

Lemma A.6. *We have $\langle C, \mu \rangle = H$.*

Proof. We only give a sketch, the details being left to the reader. In GL_3, μ can be written as the matrix $\begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Multiply μ by its conjugate by a suitable diagonal matrix to obtain an elementary matrix; conjugating by elements of C provide all elementary matrices and thus we obtain all matrices with determinant one; since C also contains diagonal matrices, we are done. □

Lemma A.7. *Let G act on $\mathcal{T}$ so that H has no fixed point and has a (unique) stably fixed end. Then G stably fixes this unique end.*

Proof. Let ω be the unique end stably fixed by H (recall that if it is represented by a ray (x_t), this means that for every $h \in H$ there exists $t_0 = t_0(h)$ such that h fixes x_t for all $t \geq t_0$). Then $\sigma H \sigma^{-1}$ stably fixes $\sigma\omega$. In particular, since $\sigma C \sigma^{-1} = C$, the end $\sigma\omega$ is also stably fixed by C. If $\sigma\omega = \omega$, then ω is stably fixed by σ and then by the Max Noether Theorem, ω is stably fixed by G. Otherwise, let D be the line joining ω and $\sigma\omega \neq \omega$. Since both ends of D are stably fixed by C, the line D is pointwise fixed by C. Also, μ stably fixes the end ω and therefore for some t, x_t is fixed by μ and therefore, by Lemma A.6, is fixed by all of H, contradicting the assumption. □

Proof of Theorem A.1. Note that $\mu \in H$ and $\mu\sigma$ has order three. It follows that $\sigma = (\mu\sigma)\mu(\mu\sigma)^{-1}$. Using the Max Noether Theorem, it follows that $H_1 = H$ and $H_2 = \sigma H \sigma^{-1}$ generate G.

Consider an action of G on $\mathcal{T}$. By Lemmas A.5 and A.7, we only have to consider the case when H has a fixed point; in this case, let us show that G has a fixed point. Assume the contrary. Let $\mathcal{T}_i$ be the set of fixed points of H_i ($i = 1, 2$); they are exchanged by σ and since $\langle H_1, H_2 \rangle = G$, we see that the two trees $\mathcal{T}_1$ and $\mathcal{T}_2$ are disjoint. Let $\mathcal{S} = [x_1, x_2]$ be the minimal segment joining the two trees ($x_i \in \mathcal{T}_i$) and $s > 0$ its length. Then $\mathcal{S}$ is pointwise fixed by $C \subset H_1 \cap H_2$ and reversed by σ.

Claim. For all $k \geq 1$, the distance of x_1 with $(\sigma\mu)^k x_1$ is exactly sk.

The claim is clearly a contradiction since $(\sigma\mu)^3 = 1$. To check the claim, let us apply Lemma A.4 to the sequence $((\sigma\mu)^k x_1)$: namely to check that

$$d((\sigma\mu)^k x_1, (\sigma\mu)^\ell x_1) = |k - \ell| s$$

for all k, ℓ it is enough to check it for $|k - \ell| \leq 2$; by translation it is enough to check it for $k = 1, 2$ and $\ell = 0$. For $k = 1$, $d(\sigma\mu x_1, x_1) = d(\sigma x_1, x_1) = d(x_2, x_1) = s$. Since $\langle C, \mu \rangle = H$ by Lemma A.6, the image of $[x_1, x_2]$ by μ is a segment $[x_1, \mu x_2]$ intersecting the segment $[x_1, x_2]$ only at x_1; in particular, $d(x_2, \mu x_2) = 2s$. Hence, under the assumptions of the claim

$$d(\sigma\mu\sigma\mu x_1, x_1) = d(\mu\sigma x_1, \sigma x_1) = d(\mu x_2, x_2) = 2s.$$

This proves the claim for $k = 2$ and the proof is complete. $\square$

For reference we include

Proposition A.8. *If* $\mathbf{k}$ *is algebraically closed, the group* $\mathsf{PGL}_2(\mathbf{k})$ *has Property* $(\mathrm{FA})_\infty$ *but, unless* $\mathbf{k}$ *is an algebraic closure of a finite field, does not satisfy* $(\mathrm{F}\mathbf{R})_\infty$.

Proof. If $\mathbf{k}$ has characteristic zero, the group $\mathsf{PGL}_2(\mathbf{k})$ has the property that the square of every element is divisible (i.e. has nth roots for all $n > 0$). This implies that no element can act hyperbolically on a discrete tree: indeed, in the automorphism group of a tree, the translation length of any element is an integer and the translation length of x^n is n times the translation length of x. If $\mathbf{k}$ has characteristic p the same argument holds: for every x, x^{2p} is divisible.

On the other hand, let I be a transcendence basis of $\mathbf{k}$ and assume it nonempty, and $x_0 \in I$. Set $\mathbf{L} = \mathbf{k}(I - \{i_0\})$, so that $\mathbf{k}$ is an algebraic closure of $\mathbf{L}(x_0)$. The nontrivial discrete valuation of $\mathbf{L}(\!(x_0)\!)$ uniquely extends to a nontrivial, $\mathbf{Q}$-valued valuation on an algebraic closure. It restricts to a non-trivial $\mathbf{Q}$-valued valuation on $\mathbf{k}$.

The remaining case is the case of an algebraic closure of the rational field $\mathbf{Q}$; pick any prime p and restrict the p-valuation from an algebraic closure of $\mathbf{Q}_p$.

Now if F is any field valued in $\mathbf{R}$, then $\mathsf{PGL}_2(F)$ has a natural action on a real tree, on which an element $\mathrm{diag}(a, a^{-1})$, for $|a| > 1$, acts hyperbolically.

(If $\mathbf{k}$ is algebraic over a finite field, then $\mathsf{PGL}_2(\mathbf{k})$ is locally finite and thus satisfies $(\mathrm{FA})_\infty$.) $\square$

Lemma A.9. *The three definitions of* $(\mathrm{F}\mathbf{R})_\infty$ *in the introduction are equivalent.*

Sketch of proof. The implications (3)⇒(2)⇒(1) are clear. (1)⇒(2) is proved for trees in [3, Chap. I §6.5], the argument working for real trees. Now assume (2) and let us prove (3). Fix a point x_0. For every finite subset F of the group, let $\mathcal{S}_F$ be the segment joining x_0 to the set of F-fixed points. Then the union of $\mathcal{S}_F$, when F ranges over finite subsets of the group, is a geodesic emanating from 0. If it is bounded, its other extremity (which exists by completeness) is a fixed point. Otherwise, it defines a stably fixed end. $\square$

References

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