A spectral gap property for subgroups of finite covolume in Lie groups

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Dedicated to the memory of Andrzej Hulanicki

Abstract

Let G be a real Lie group and H a lattice or, more generally, a closed subgroup of finite covolume in G. We show that the unitary representation $\lambda_{G/H}$ of G on $L^2(G/H)$ has a spectral gap, that is, the restriction of $\lambda_{G/H}$ to the orthogonal of the constants in $L^2(G/H)$ does not have almost invariant vectors. This answers a question of G. Margulis. We give an application to the spectral geometry of locally symmetric Riemannian spaces of infinite volume.

1 Introduction

Let G a locally compact group. Recall that a unitary representation $(\pi, \mathcal{H})$ of G almost has invariant vectors if, for every compact subset Q of G and every $\varepsilon > 0$, there exists a unit vector $\xi \in \mathcal{H}$ such that $\sup_{x \in Q} \|\pi(x)\xi - \xi\| < \varepsilon$. If this holds, we also say that the trivial representation 1_G is weakly contained in π and write $1_G \prec \pi$.

Let H be a closed subgroup of G for which there exists a G-invariant regular Borel measure μ on G/H (see [BHV, Appendix B] for a criterion of the existence of such a measure). Denote by $\lambda_{G/H}$ the unitary representation of G given by left translation on the Hilbert space $L^2(G/H, \mu)$ of the square integrable measurable functions on the homogeneous space G/H. If μ is finite, we say that H has finite covolume in G. In this case, the space $\mathbb{C}1_{G/H}$ of the constant functions on G/H is contained in $L^2(G/H, \mu)$ and is G-invariant as well as its orthogonal complement

$$L^2_0(G/H, \mu) = \left\{ \xi \in L^2(G/H, \mu) \, : \, \int_{G/H} \xi(x) d\mu(x) = 0 \right\}.$$

In case μ is infinite, we set $L^2_0(G/H,\mu) = L^2(G/H,\mu)$.

Denote by $\lambda^0_{G/H}$ the restriction of $\lambda_{G/H}$ to $L^2_0(G/H,\mu)$ (in case μ is infinite, $\lambda^0_{G/H} = \lambda_{G/H}$). We say that $\lambda_{G/H}$ (or $L^2(G/H,\mu)$) has a *spectral gap* if $\lambda^0_{G/H}$ has no almost invariant vectors. In the terminology of [Marg91, Chapter III. (1.8)], H is called weakly cocompact.

By a Lie group we mean a locally compact group G whose connected component of the identity G^0 is open in G and is a real Lie group. We prove the following result which has been conjectured in [Marg91, Chapter III. Remark 1.12].

Theorem 1 *Let G be a Lie group and H a closed subgroup with finite covolume in G. Then the unitary representation $\lambda_{G/H}$ on $L^2(G/H)$ has a spectral gap.*

It is a standard fact that $L^2(G/H)$ has a spectral gap when H is cocompact in G (see [Marg91, Chapter III, Corollary 1.10]). When G is a semisimple Lie group, Theorem 1 is an easy consequence of Lemma 3 in [Bekk98]. Our proof is by reduction to these two cases. The crucial tool for this reduction is Proposition (1.11) from Chapter III in [Marg91] (see Proposition 5 below). From Theorem 1 and again from this proposition, we obtain the following corollary.

Corollary 2 *Let G be a second countable Lie group, H a closed subgroup with finite covolume in G and σ a unitary representation of H. Let $\pi = \mathrm{Ind}_H^G\sigma$ be the representation of G induced from σ. If 1_H is not weakly contained in σ, then 1_G is not weakly contained in π.*

Observe that, by continuity of induction, the converse result is also true: if $1_H \prec \sigma$, then $1_G \prec \pi$.

From the previous corollary we deduce a spectral gap result for some subgroups of G with infinite covolume.

Recall that a subgroup H of a topological group G is called *co-amenable* in G if there is a G-invariant mean on the space $C^b(G/H)$ of bounded continuous functions on G/H. When G is locally compact, this is equivalent to $1_G \prec \lambda_{G/H}$; this property has been extensively studied by Eymard [Eyma72] for which he refers to as the amenability of the homogeneous space G/H. Observe that a normal subgroup H in G is co-amenable in G if and only if the quotient group G/H is amenable.

Corollary 3 *Let G be a second countable Lie group and H a closed subgroup with finite covolume in G. Let L be a closed subgroup of H. Assume that L is not co-amenable in H. Then $\lambda_{G/L}$ has a spectral gap.*

Corollary 3 is a direct consequence of Corollary 2, since the representation $\lambda_{G/L}$ on $L^2(H/L)$ is equivalent to the induced representation $\mathrm{Ind}_H^G \lambda_{H/L}$.

Here is a reformulation of the previous corollary. Let G be a Lie group and H a closed subgroup with finite covolume in G. If a subgroup L of H is co-amenable in G, then L is co-amenable in H. Observe that the converse (if L is co-amenable in H, then L is co-amenable in G) is true for any topological group G and any closed subgroup H which is co-amenable in G (see [Eyma72, p.16]).

Using methods from [Leuz03] (see also [Broo86]), we obtain the following consequence for the spectral geometry of infinite coverings of locally symmetric Riemannian spaces of finite volume. Recall that a lattice in the locally compact group G is a discrete subgroup of G with finite covolume.

Corollary 4 *Let G be a semisimple Lie group with finite centre and maximal compact subgroup K and let Γ be a torsion-free lattice G. Let $\widetilde{V}$ be a covering of the locally symmetric space $V = K\backslash G/\Gamma$. Assume that the fundamental group of $\widetilde{V}$ is not co-amenable in Γ.*

(i) We have $h(\widetilde{V}) > 0$ for the Cheeger constant $h(\widetilde{V})$ of $\widetilde{V}$.

(ii) We have $\lambda_0(\widetilde{V}) > 0$, where $\lambda_0(\widetilde{V})$ is the bottom of the L^2 -spectrum of the Laplace– Beltrami operator on $\widetilde{V}$.

There is in general no uniform bound for $h(\widetilde{V})$ or $\lambda_0(\widetilde{V})$ for all coverings $\widetilde{V}$. However, it was shown in [Leuz03] that, when G has Kazhdan's Property (T), such a bound exists for *every* locally symmetric space $V = K\backslash G/\Gamma$. Observe also that if, in the previous corollary, the fundamental group of $\widetilde{V}$ is co-amenable in Γ and has infinite covolume, then $h(\widetilde{V}) = \lambda_0(\widetilde{V}) = 0$, as shown in [Broo81].

2 Proofs of Theorem 1 and Corollary 4

The following result of Margulis (Proposition (1.11) in Chapter III of [Marg91]) wil be crucial.

Proposition 5 *([Marg91]) Let G be a second countable locally compact group, H a closed subgroup of G, and σ a unitary representation of H. Assume that $\lambda_{G/H}$ has a spectral gap and that 1_H is not weakly contained in σ. Then 1_G is not weakly contained in $\mathrm{Ind}_H^G\sigma$.*

In order to reduce the proof of Theorem 1 to the semisimple case, we will use several times the following proposition.

Proposition 6 *Let G a separable locally compact group and H_1 and H_2 be closed subgroups of G with $H_1 \subset H_2$ and such that G/H_2 and H_2/H_1 have G-invariant and H_2-invariant regular Borel measures, respectively. Assume that the H_2-representation λ_{H_2/H_1} on $L^2(H_2/H_1)$ and that the G-representation λ_{G/H_2} on $L^2(G/H_2)$ have both spectral gaps. Then the G-representation λ_{G/H_1} on $L^2(G/H_1)$ has a spectral gap.*

Proof Recall that, for any closed subgroup H of G, the representation $\lambda_{G/H}$ is equivalent to the representation $\mathrm{Ind}_H^G 1_H$ induced by the identity representation 1_H of H. Hence, we have, by transitivity of induction,

$$\lambda_{G/H_1} = \mathrm{Ind}_{H_1}^G 1_{H_1} = \mathrm{Ind}_{H_2}^G(\mathrm{Ind}_{H_1}^{H_2} 1_{H_1}) = \mathrm{Ind}_{H_2}^G \lambda_{H_2/H_1}.$$

We have to consider three cases:

• *First case:* H_1 has finite covolume in G, that is, H_1 has finite covolume in H_2 and H_2 has finite covolume in G. Then

$$\lambda^0_{G/H_1} = \lambda^0_{G/H_2} \oplus \mathrm{Ind}_{H_2}^G \lambda^0_{H_2/H_1}.$$

By assumption, $\lambda^0_{H_2/H_1}$ and λ^0_{G/H_2} do not weakly contain 1_{H_2} and 1_G, respectively. It follows from Proposition 5 that $\mathrm{Ind}_{H_2}^G \lambda^0_{H_2/H_1}$ does not weakly contain 1_G. Hence, λ^0_{G/H_1} does not weakly contain 1_G.

• *Second case:* H_1 has finite covolume in H_2 and H_2 has infinite covolume in G. Then

$$\lambda_{G/H_1} = \lambda_{G/H_2} \oplus \mathrm{Ind}_{H_2}^G \lambda^0_{H_2/H_1}.$$

By assumption, $\lambda^0_{H_2/H_1}$ and λ_{G/H_2} do not weakly contain 1_{H_2} and 1_G. As above, using Proposition 5, we see that λ_{G/H_1} does not weakly contain 1_G.

• *Third case:* H_1 has infinite covolume in H_2. By assumption, λ_{H_2/H_1} does not weakly contain 1_{H_2}. By Proposition 5 again, it follows that $\lambda_{G/H_1} = \mathrm{Ind}_{H_2}^G \lambda_{H_2/H_1}$ does not weakly contain 1_G. ■

For the reduction of the proof of Theorem 1 to the case where G is second countable, we will need the following lemma.

Lemma 7 *Let G be a locally compact group and H a closed subgroup with finite covolume. The homogeneous space G/H is σ-compact.*

Proof Let μ be the G-invariant regular probability measure on the Borel subsets of G/H. Choose an increasing sequence of compact subsets K_n of G/H with $\lim_n \mu(K_n) = 1$. The set $K = \bigcup_n K_n$ has μ-measure 1 and is therefore dense in G/H. Let U be a compact neighbourhood of e in G. Then $UK = G/H$ and $UK = \bigcup_n UK_n$ is σ-compact.■

Proof of Theorem 1

Through several steps the proof will be reduced to the case where H is a lattice in G and where G is a connected semisimple Lie group with trivial centre and without compact factors.

• *First step:* we can assume that G is σ-compact and hence second-countable. Indeed, let $p : G \to G/H$ be the canonical projection. Since every compact subset of G/H is the image under p of some compact subset of G (see [BHV, Lemma B.1.1]), it follows from Lemma 7 that there exists a σ-compact subset K of G such that $p(K) = G/H$. Let L be the subgroup of G generated by $K \cup U$ for a neighbourhood U of e in G. Then L is a σ-compact open subgroup of G. We show that $L \cap H$ has a finite covolume in L and that $\lambda_{G/H}$ has a spectral gap if $\lambda_{L/L\cap H}$ has a spectral gap.

Since LH is open in G, the homogeneous space $L/L \cap H$ can be identified as L-space with LH/H. Therefore $L \cap H$ has finite covolume in L. On other hand, the restriction of $\lambda_{G/H}$ to L is equivalent to the L-representation $\lambda_{L/L\cap H}$, since $LH/H = p(L) = G/H$. Hence, if $\lambda_{L/L\cap H}$ has a spectral gap, then $\lambda_{G/H}$ has a spectral gap.

• *Second step:* we can assume that G is connected. Indeed, let G^0 be the connected component of the identity of G. We show that $G^0 \cap H$ has a finite covolume in G^0 and that $\lambda_{G/H}$ has a spectral gap if $\lambda_{G^0/G^0\cap H}$ has a spectral gap.

The subgroup G^0H is open in G and has finite covolume in G as it contains H. It follows that G^0H has finite index in G since G/G^0H is discrete. Hence λ_{G/G^0H} has a spectral gap.

On the other hand, since G^0H is closed in G, the homogeneous space $G^0/G^0 \cap H$ can be identified as a G^0-space with G^0H/H. Therefore $G^0 \cap H$ has finite covolume in G^0. The restriction of $\lambda_{G^0H/H}$ to G^0 is equivalent to the G^0-representation $\lambda_{G^0/G^0\cap H}$.

Suppose now that $\lambda_{G^0/G^0\cap H}$ has a spectral gap. Then the G^0H-representation $\lambda_{G^0H/H}$ has a spectral spectral, since $L^2_0(G^0H/H) \cong L^2_0(G^0/G^0\cap H)$ as G_0-representations. An application of Proposition 6 with $H_1 = H$ and $H_2 = G^0H$ shows that $\lambda_{G/H}$ has a spectral gap. Hence, we can assume that G is connected.

• *Third step:* we can assume that H is a lattice in G. Indeed, let H^0 be the connected component of the identity of H and let $N_G(H^0)$ be the normalizer of H^0 in G. Observe that $N_G(H^0)$ contains H. By [Wang76, Theorem 3.8], $N_G(H^0)$ is cocompact in G. Hence, $\lambda_{G/N_G(H^0)}$ has a spectral gap. It follows from the previous proposition that $\lambda_{G/H}$ has a spectral gap if $\lambda_{N_G(H^0)/H}$ has a spectral gap.

On the other hand, since H^0 is a normal subgroup of H, we have

$$L^2_0(N_G(H^0)/H) \cong L^2_0((N_G(H^0)/H^0)/(H/H^0)),$$

as $N_G(H^0)$-representations. Hence, $\lambda_{N_G(H^0)/H}$ has a spectral gap if and only if $\lambda_{\overline{N}/\overline{H}}$ has a spectral gap, where $\overline{N} = N_G(H^0)/H^0$ and $\overline{H} = H/H^0$.

The second step applied to the Lie group $\overline{N}/\overline{H}$ shows that $\lambda_{\overline{N}/\overline{H}}$ has a spectral gap if $\lambda_{\overline{N}^0/(\overline{N}^0\cap\overline{H})}$ has a spectral gap. Observe that $\overline{N}^0 \cap \overline{H}$ is a lattice in the connected Lie group $\overline{N}^0$, since $\overline{H}$ is discrete and since H has finite covolume in $N_G(H^0)$.

This shows that we can assume that H is a lattice in the connected Lie group G.

• *Fourth step:* we can assume that G is a connected semisimple Lie group with no compact factors. Indeed, let $G = SR$ be a Levi decomposition of G, with R the solvable radical of G and S a semisimple subgroup. Let C be the maximal compact normal subgroup of S. It is proved in [Wang70, Theorem B, p.21] that HCR is closed in G and that HCR/H is compact. Hence, by the previous proposition, $\lambda_{G/H}$ has a spectral gap if $\lambda_{G/HCR}$ has a spectral gap.

The quotient $\overline{G} = G/CR$ is a connected semisimple Lie group with no compact factors. Moreover, $\overline{H} = HCR/CR$ is a lattice in $\overline{G}$ since $HCR/CR \cong H/H\cap CR$ is discrete and since HCR has finite covolume in G. Observe that $\lambda_{G/HCR}$ is equivalent to $\lambda_{\overline{G}/\overline{H}}$ as G-representation.

• *Fifth step:* we can assume that G has trivial centre. Indeed, let Z be the centre of G. It is known that ZH is discrete (and hence closed) in G (see

[Ragh72, Chapter V, Corollary 5.17]). Hence, ZH/H is finite and $\lambda_{ZH/H}$ has a spectal gap.

By the previous proposition, $\lambda_{G/H}$ has a spectral gap if $\lambda_{G/ZH}$ has a spectral gap. Now, $\overline{G} = G/Z$ is a connected semisimple Lie group with no compact factors and with trivial centre, $\overline{H} = ZH/Z$ is a lattice in $\overline{G}$ and $\lambda_{G/ZH}$ is equivalent to $\lambda_{\overline{G}/\overline{H}}$.

• *Sixth step:* by the previous steps, we can assume that H is a lattice in a connected semisimple Lie group G with no compact factors and with trivial centre. In this case, the claim was proved in Lemma 3 of [Bekk98]. This completes the proof of Theorem 1.■

Proof of Corollary 4

The proof is identical with the proof of Theorems 3 and 4 in [Leuz03]; we give a brief outline of the arguments. Let Λ be the fundamental group of $\widetilde{V}$. First, it suffices to prove Claims (i) and (ii) for G/Γ instead of $K\backslash G/\Gamma$ (see Section 4 in [Leuz03]). So we assume that $\widetilde{V} = G/\Lambda$.

Equip G with a right invariant Riemannian metric and G/Λ with the induced Riemannian metric. Observe that G/Λ has infinite volume, since Λ is of infinite index in Γ. Claim (ii) is a consequence of (i), by Cheeger's inequality $\frac{1}{4}h(G/\Lambda)^2 \leq \lambda_0(G/\Lambda)$. Recall that the Cheeger constant $h(G/\Lambda)$ of G/Λ is the infimum over all numbers $A(\partial\Omega)/V(\Omega)$, where Ω is an open submanifold of G/Λ with compact closure and smooth boundary $\partial\Omega$, and where $V(\Omega)$ and $A(\partial\Omega)$ are the Lebesgue measures of Ω and $\partial\Omega$.

To prove Claim (i), we proceed exactly as in [Leuz03]. By Corollary 3, there exists a compact neighbourhood H of the identity in G and a constant $\varepsilon > 0$ such that

$$(*) \qquad \varepsilon\|\xi\| \leq \sup_{h\in H} \|\lambda_{G/\Lambda}(h)\xi - \xi\|, \qquad \text{for all} \quad \xi \in L^2(G/\Lambda).$$

Let Ω be an open submanifold of G/Λ with compact closure and smooth boundary $\partial\Omega$. By [Leuz03, Proposition 1], we can find an open subset $\widetilde{\Omega}$ of G/Λ compact closure and smooth boundary such that, for all $h \in H$,

$$(**) \qquad V(U_{|h|}(\partial\Omega)) \leq CV(\widetilde{\Omega})\frac{A(\partial\Omega)}{V(\Omega)},$$

where the constant $C > 0$ only depends on H. Here, $|h|$ denotes the distance $d_G(e, g)$ of h to the group unit and, for a subset S of G/Λ, $U_r(S)$ is the tubular neighbourhood

$$U_r(\partial\Omega) = \{x \in G/\Lambda \,:\, d_{G/\Lambda}(x, S) \leq r\}$$

By Inequality $(*)$, applied the characteristic function $\chi_{\widetilde{\Omega}}$ of $\widetilde{\Omega}$, there exists $h \in H$ such that

$$\varepsilon^2 V(\widetilde{\Omega}) \leq \|\lambda_{G/\Lambda}(h)\chi_{\widetilde{\Omega}} - \chi_{\widetilde{\Omega}}\|^2 = V(X),$$

where

$$X = \left\{x \in G/\Lambda \,:\, x \in \widetilde{\Omega}, hx \notin \partial\widetilde{\Omega}\right\} \bigcup \left\{x \in G/\Lambda \,:\, x \notin \widetilde{\Omega}, hx \in \partial\widetilde{\Omega}\right\}.$$

One checks that $X \subset U_{|h|}(\partial\Omega)$. It follows from Inequalities $(*)$ and $(**)$ that $\dfrac{\varepsilon^2}{C} \leq \dfrac{A(\Omega)}{V(\Omega)}$. Hence, $0 < \dfrac{\varepsilon^2}{C} \leq h(G/\Lambda)$. ■

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