On Haagerup and Kazhdan Properties

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Yves de Cornulier

de nationalité française
titulaire d'un DEA de Mathématiques, Université Paris VII

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Peter Buser, directeur
Alain Valette, codirecteur
Éva Bayer Fluckiger, présidente du jury
Laurent Bartholdi, rapporteur interne
Étienne Ghys, rapporteur externe
Pierre de la Harpe, rapporteur externe

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Zusammenfassung

Im ersten Kapitel, charakterisieren wir p-adischen algebrischen Gruppen mit Haagerup Eigenschaft. Wir charakterisieren auch zusammenhängenden Lie Gruppen mit Haagerup Eigenschaft **als diskreten Gruppen gesehen**, und wir geben ein Beispiel einer endlich definierbar Gruppe die hat nicht Haagerup Eigenschaft, aber hat keine unendlich Untergruppe mit relativen Eigenschaft (T). Dieses Beispiel begründet die Einführung, im zweiten Kapitel, der relativen Eigenschaft (T) für irgendeine Teilmenge einer lokalkompakte Gruppe. In einer zusammenhängenden Lie Gruppe, wir charakterisieren die Teilmengen mit relativen Eigenschaft (T). Wir führen "Auflösungenëin um diese Ergebnisse auf ihren [diskrete Untergruppen] auszudehnen.

Im dritten Kapitel charakterisieren wir zusammenhängenden Lie Gruppen die ein endlich erzeugbar Untergruppe mit Eigenschaft (T) haben. Im vierten Kapitel geben wir ein Beispiel einer endlich definierbar Gruppe mit der Hopf Eigenschaft und unendlich äußeren automorphismus Gruppe. Im fünften Kapitel charakterisieren wir lokal nilpotenten Gruppen, alle dessen unitären Darstellungen null reduzierte 1-kohomologie haben. Im sechsten Kapitel zeigen wir dass, wenn F eine endliche vollkommene Gruppe ist und X eine Menge ist, denn das direkte Produkt F^X **stark geschrankt** ist. Das meint dass sie keine [isometric] Wirkung auf einen metrischen Raum besitzt. Endlich im siebten Kapitel, stellen wir einige kurzen Bemerkungen zusammen, davon eine Liste von geöffneten Fragen.

Abstract

In the first chapter, we characterize p-adic linear algebraic groups with the Haagerup Property. We also characterize connected Lie groups having the Haagerup Property **viewed as discrete groups**, and we provide an example of a finitely presented group not having the Haagerup Property, but having no infinite subgroup with relative Property (T). This example motivates the introduction in the second chapter of the relative Property (T) for an arbitrary subset of a locally compact group. In a connected Lie group, we characterize the subsets with relative Property (T). We introduce a notion of "resolutions" so as to extend the latter results to lattices in connected Lie groups.

In the third chapter, we characterize connected Lie groups having a dense finitely generated subgroup with Property (T). In the fourth chapter, we provide an example of a finitely presented group having Property (T), non-Hopfian, and with infinite outer automorphism group. In the fifth chapter, we characterize locally nilpotent groups for which all unitary representations have vanishing reduced 1-cohomology. In the sixth chapter, we show that if F is a finite perfect group, and X is any set, then the unrestricted direct product F^X is **strongly bounded**. This means that it has no isometric action on any metric space with unbounded orbits. Finally in the seventh chapter, we collect several short notes, including a list of open questions.

Introduction

La thèse que voici a été réalisée sous la direction conjointe de Peter Buser (École Polytechnique Fédérale de Lausanne) et Alain Valette (Université de Neuchâtel), entre octobre 2003 et décembre 2005.

Bref historique du sujet. En introduisant la propriété (T) en 1967 [Kaz67], Kazhdan a apporté une méthode totalement novatrice pour montrer que des groupes sont de type fini, reposant sur leurs représentations unitaires. Elle s'applique à de nombreux réseaux dans des groupes de Lie. En dehors de l'engendrement fini, la propriété (T) a des conséquences tout-à-fait non triviales, comme le fait que tout sous-groupe d'indice fini a un abélianisé fini. Elle joue ainsi un rôle essentiel dans un résultat de Margulis, disant que, pour les réseaux dans certains groupes de Lie, tout quotient strict est fini (voir [Mar91]). La propriété (T) relative, introduite à l'origine comme outil technique servant par exemple à démontrer que $\mathrm{SL}_3(\mathbf{R})$ a la propriété (T), est aujourd'hui considérée comme intéressante en elle-même ; elle peut en effet se voir comme une version faible de la propriété (T) et possède des conséquences similaires.

La propriété de Haagerup, introduite dans [AkWa81], apparaît notamment parce qu'elle est satisfaite par les groupes les plus “simples”, à savoir les groupes virtuellement résolubles (et plus généralement moyennables), les groupes libres et plus généralement les groupes kleiniens, les groupes de Coxeter, les groupes de Thompson. Pour beaucoup de gens, son principal intérêt réside dans le fait que, pour un groupe donné, elle implique la conjecture de Baum-Connes. En ce qui me concerne, je m'y suis surtout intéressé en tant que négation forte de la propriété (T) relative (et vice-versa).

Propriété de Haagerup et propriété (T) relative. On renvoie au chapitre 0 pour les définitions de la propriété (T) relative et de la propriété de Haagerup. Rappelons simplement qu'un groupe localement compact, σ-compact G a la propriété de Haagerup s'il agit proprement par isométries sur un espace de Hilbert [CCJJV01]. D'autre part, si H est un sous-groupe de G, on dit que la paire (G, H) a la propriété (T) relative [HaVa89] si, pour toute action de G par isométries sur un espace de Hilbert, H fixe au moins un point ; si (G, G) a la propriété (T) relative on dit simplement que G a la propriété (T).

J'ai été amené à étudier les problèmes suivants :

(1) Généraliser la classification des groupes de Lie connexes ayant la propriété de Haagerup, aux groupes algébriques p-adiques.

(2) Trouver un groupe n'ayant pas la propriété de Haagerup, mais n'ayant aucun

sous-groupe avec la propriété (T) relative (ce problème est posé dans [CCJJV01, Chapitre 7]).

(2bis) Soit Γ un réseau irréductible dans $SO(4,1) \times SO(3,2)$. Est-ce que Γ (qui n'a pas la propriété de Haagerup) répond à (2) ?

Le problème (1) est résolu dans le Chapitre 1.

Théorème. (Voir Theorem 1.23.)

Soit G ou bien un groupe de Lie connexe, ou bien $G = \mathbf{G}(\mathbf{K})$, où $\mathbf{G}$ est un groupe algébrique linéaire sur un corps local $\mathbf{K}$ de caractéristique zéro. Les assertions suivantes sont équivalentes.

(i) G a la propriété de Haagerup.

(ii) Pour tout sous-groupe fermé non compact H de G, la paire (G, H) n'a pas la propriété (T) relative.

(iii) On a une décomposition $G = MS$, pour des sous-groupes fermés M et S, où $[M, S] = 1$, M est moyennable, et S est semi-simple, ayant tous ses facteurs simples de rang un, et de plus, si $\mathbf{K} = \mathbf{R}$ ou dans le cas des groupes de Lie, S n'a pas de facteur localement isomorphe à $\mathrm{Sp}(n, 1)$ pour un $n \geq 2$, ou au groupe exceptionnel de rang un $F_{4(-20)}$.

La démarche est la suivante : soit G un groupe algébrique sur $\mathbf{Q}_p$, R son radical moyennable, et S un facteur de Levi semi-simple sans facteurs compacts. Supposons que $[S, R] \neq 1$. Alors G possède un sous-groupe fermé isomorphe, à un revêtement d'ordre 2 près, au produit semi-direct de SL_2 par un sous-groupe normal N, qui est soit une représentation irréductible non triviale, soit un groupe de Heisenberg, sur lequel SL_2 agit irréductiblement modulo le centre. Dans les deux cas, on vérifie que $(\mathrm{SL}_2 \ltimes N, N)$ a la propriété (T) relative, ce qui est une obstruction à la propriété de Haagerup.

Un résultat récent [GHW05, §5, Theorem 4] montre que $\mathrm{SL}_2(\mathbf{C})$, vu comme groupe **discret**, a la propriété de Haagerup. Il me semblait que les méthodes développées pour résoudre (1) pouvaient permettre de résoudre le problème suivant :

(3) Déterminer exactement les groupes de Lie connexes qui, vus comme groupes discrets, ont la propriété de Haagerup.

La réponse est que les exemples donnés dans [GHW05] sont essentiellement les seuls.

Théorème. (Voir Theorem 1.28.) **Soit G un groupe de Lie connexe, et $\mathfrak{g}$ son algèbre de Lie. Alors G, vu comme groupe discret, a la propriété de Haagerup si et seulement si $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C}$ est isomorphe à une algèbre de Lie de la forme $\mathfrak{sl}_2(\mathbf{C})^n \times \mathfrak{r}$ avec $\mathfrak{r}$ résoluble.**

De manière inattendue, cette étude a permis de résoudre le problème (2). En effet, le groupe $SO_3(\mathbf{R}) \ltimes \mathbf{R}^3$, vu comme groupe discret, n'a pas la propriété de Haagerup alors qu'on peut vérifier par ailleurs qu'il n'a aucun sous-groupe avec la propriété (T) relative, comme conséquence du résultat de [GHW05] mentionné plus haut. Il restait à piocher un sous-groupe de type fini convenable.

Théorème. (Voir Remark 1.34.) **Le groupe de type fini $SO_3(\mathbf{Z}[1/5]) \ltimes \mathbf{Z}[1/5]^3$ n'a pas la propriété de Haagerup, mais n'a aucun sous-groupe infini avec la propriété (T) relative.**

Le fait d'avoir résolu (2) signifiait que l'existence d'un sous-groupe avec la propriété (T) relative n'était plus la seule obstruction connue à la propriété de Haagerup. Dans le cas de $\mathrm{SO}_3(\mathbf{Z}[1/5]) \ltimes \mathbf{Z}[1/5]^3$, l'obstruction a l'inconvénient de ne pas être intrinsèque : elle consiste en effet à plonger le groupe comme réseau dans un groupe n'ayant pas la propriété de Haagerup. Il est alors peu à peu devenu clair dans mon esprit qu'il était naturel de formuler la définition de propriété (T) relative pour une paire (G, X) où G est toujours un groupe, mais X est non plus un sous-groupe mais un sous-ensemble quelconque. J'en viens alors au chapitre 2, qui est le point central de ma thèse. Dans la première partie, j'étends le résultat consistant à démontrer l'équivalence entre diverses définitions à la propriété (T) relative par rapport à un sous-ensemble. En fait, le plus souvent, les arguments donnés dans [Jol05] pour la propriété (T) relative par rapport à un sous-groupe s'étendent sans difficulté. Puis, je montre que, dans un groupe localement compact, la propriété relative est "contenue" dans les sous-groupes à engendrement compact : cela étend un résultat de Kazhdan, à savoir que les groupes ayant la propriété (T) sont compactement engendrés ; cependant, si on reprend l'argument de Kazhdan, on obtient seulement que si (G, X) a la propriété (T) relative, alors X est contenu dans un sous-groupe ouvert Ω à engendrement compact de G. L'autre moitié consiste à trouver un sous-groupe à engendrement compact H, contenant Ω, tel que (H, X) a la propriété (T) relative. Enfin, dans le cas d'un groupe G localement compact à engendrement compact, je définis une métrique (H-métrique ; H comme Hilbert), essentiellement canonique, telle qu'une partie $X \subset G$ est bornée si et seulement si (G, X) a la propriété (T) relative.

Parallèlement, j'avais cherché à caractériser les groupes de Lie connexes G tel que $(G, \mathrm{rad}(G))$ a la propriété (T) relative : les cas bien compris jusque là étaient à radical abélien (typiquement, $\mathrm{SL}_2(\mathbf{R}) \ltimes \mathbf{R}^2$). Un lemme sur les extensions centrales, inspiré par [CCJJV01, Chapitre 4], permettait de "remonter" dans le radical, et finalement d'obtenir la caractérisation voulue[1]. Une fois cela montré, je me suis aperçu que cela donnait une réponse à une question plus générale, à savoir : caractériser, dans G, les parties X telles que (G, X) a la propriété (T) relative. La caractérisation a la forme suivante : on définit un sous-groupe caractéristique fermé $R_T(G)$ de G (le "T-radical").

Théorème. (Voir Corollary 2.51.) **Soit X une partie du groupe de Lie connexe G. Alors (G, X) a la propriété (T) relative si et seulement si l'image de X dans G/R_T est relativement compacte.**

Le résultat vaut également (Corollary 2.47) pour un groupe algébrique p-adique ; cela fait l'objet de la deuxième partie du chapitre 2. Si G a la propriété de Haagerup, R_T doit être compact et on peut alors montrer sans difficulté qu'il est en fait trivial. On réobtient ainsi une nouvelle fois la réponse à (1) (bien que l'essentiel du travail dans le chapitre 1 reste largement indépendant). Le fait, nouveau, qu'un groupe de Lie connexe n'ayant pas la propriété de Haagerup possède un sous-groupe **normal** non trivial avec la propriété (T) relative (à savoir R_T) est utilisé par Florian Martin dans son étude de la cohomologie réduite des groupes de Lie connexes [Mrt05].

[1] **On trouve, ceci dit, un raisonnement similaire dans [Sha99t].**

Passons à la troisième partie du chapitre 2. Son point de départ est l'étude de la question (2bis), que j'avais attaquée avant de répondre à (2). Donnons-nous donc Γ comme dans la question (2bis) et Λ un sous-groupe ; soit p la projection vers $\mathrm{SO}(4,1)$; par irréductibilité, elle est d'image dense. Une observation immédiate est que si (Γ,Λ) a la propriété (T) relative, alors $p(\Lambda)$ est relativement compact. Cependant, un argument simple montre que l'ensemble des éléments elliptiques (autrement dit : la réunion des sous-groupes compacts) de $\mathrm{SO}(4,1)$ est d'intérieur non vide ; on obtient ainsi l'existence de sous-groupes cycliques infinis $\Lambda \subset \Gamma$ tels que $p(\Lambda)$ est relativement compact[2], ce qui soulève naturellement la question de savoir si, réciproquement, cela implique que (Γ,Λ) a la propriété (T) relative. C'est là qu'intervient un résultat de Margulis [Mar91, Chapitre 3, §6], qui implique cette réciproque. Ainsi, la réponse à (2bis) est négative.

Théorème. Voir Theorem 2.82 et Proposition 2.84. **Dans** $\mathrm{SO}(4,1)$ **ou** $\mathrm{SU}(2,1)$**, il existe des sous-groupes** $\Lambda \subset \Gamma$ **de type fini, tel que** (Γ,Λ) **a la propriété (T) relative et** Λ **infini, mais on ne peut pas choisir** Λ **distingué dans** Γ**.**

Cela répond au problème suivant, posé notamment par Popa :

(4) Existe-t-il des paires (G,H) avec la propriété (T) relative, H sous-groupe de G, qui ne proviennent pas d'un exemple[3] où H est normal dans G ?

En effet, on peut vérifier que si Γ est un sous-groupe de $\mathrm{SO}(4,1)$, il n'a aucun sous-groupe normal infini avec la propriété (T) relative. Pour faire le lien avec la H-métrique mentionnée plus haut, on peut montrer que la H-métrique de Γ est équivalente à la métrique provenant de la racine carrée de la métrique des mots de $\mathrm{SO}(4,1)$ via p, et qu'ainsi, pour tout $X \subset \Gamma$, (Γ, X) a la propriété (T) relative si et seulement si $p(X)$ est relativement compact.

Le but de la troisième partie du chapitre 2 est de donner une généralisation conceptuelle de ce type de phénomène. On systématise ainsi des idées déjà présentes chez Lubotzky-Zimmer [LuZi89], et Margulis [Mar91]. Remarquons que, ci-dessus, ce qui intervenait était le groupe Γ, et la projection p vers $\mathrm{SO}(4,1)$. Le cadre général est celui-ci : un groupe localement compact G, et un morphisme p d'image dense vers un autre groupe localement compact Q. J'appelle un tel morphisme une **résolution**[4] si toute représentation de G proche de la représentation triviale contient une sous-représentation non nulle factorisant par Q via p. Lorsque p est le quotient par un sous-groupe normal N, dire que $G \to Q = G/N$ est une résolution revient à dire que (G,N) a la propriété (T) relative. Le concept est nouveau quand, par exemple, p n'est pas surjectif, par exemple dans le cas de la projection $\Gamma \to \mathrm{SO}(4,1)$, étudié plus haut. Le théorème dans [Mar91, Chapitre III, §6] s'interprète sur cet exemple comme ceci : la projection $\mathrm{SO}(3,2) \times \mathrm{SO}(4,1) \to \mathrm{SO}(4,1)$ est une résolution, donc cela est hérité par le réseau ; la projection de Γ sur $\mathrm{SO}(4,1)$ est aussi une résolution. Dans le

[2] **En prenant des exemples explicites de** Γ**, on obtient des exemples avec** Λ **libre non abélien.**

[3] **Par "paires qui proviennent d'un exemple où** H **est normal dans** G**", on entend des exemples tels que** $(\mathbf{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z} \times \{0\})$ **ou** $((\mathbf{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2) * \mathbf{Z}, \mathbf{Z}^2)$**, qui se ramènent immédiatement au cas bien connu de la paire** $(\mathbf{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$**.**

[4] **Cette terminologie, peut-être maladroite, m'est inspirée aussi bien par la notion de résolution d'une singularité, utilisée en géométrie, que par la notion de résolution en algèbre homologique ; l'idée ici est qu'on remplace** G **par un objet plus simple** Q**, mais qui contient la même information, au moins lorsqu'on s'intéresse aux représentations unitaires proches de la représentation triviale.**

cas de $\mathrm{SO}_3(\mathbf{Z}[1/5]) \ltimes \mathbf{Z}[1/5]^3$ mentionné plus haut, l'inclusion dans $\mathrm{SO}_3(\mathbf{Z}[1/5]) \ltimes \mathbf{R}^3$ est une résolution, ce qui permet de montrer l'existence de parties infinies avec la propriété relative (alors qu'on a évoqué plus haut le fait qu'il n'y a aucun **sous-groupe** avec la propriété (T) relative).

La troisième partie du chapitre 2 consiste également à étudier systématiquement les résolutions et notamment à montrer plusieurs caractérisations connues dans le cas de la propriété relative à un sous-groupe normal ; contrairement au cas de la propriété (T) relative par rapport à un sous-ensemble étudié dans la première partie du chapitre 2, les preuves demandent cette fois-ci des arguments nouveaux. On éclaircit également un point technique délicat : ce qu'on a appelé "résolution" plus haut est appelé "prérésolution" dans le chapitre 2, où dans la définition de résolution, on demande une hypothèse supplémentaire, à savoir que la sous-représentation non nulle factorisant par Q ait presque des vecteurs invariants, vue comme représentation de Q. Cette hypothèse joue un rôle important en pratique. Cependant, dans le cas σ-compact, on montre finalement (§2.3.8), mais de façon très indirecte, que cette hypothèse est en fait superflue, c'est-à-dire que les notions de résolutions et prérésolutions coïncident.

Sous-groupes denses avec la propriété (T) dans les groupes de Lie connexes. Soit G un groupe localement compact. L'existence, dans G, d'un sous-groupe dense de type fini Γ qui, vu comme groupe discret, a la propriété (T), a des applications remarquables. En particulier, quand $G = \mathrm{SO}(n)$ pour $n \geq 5$, cela a permis à Margulis et Sullivan [Mar80, Sul81] de démontrer que, sur les parties Lebesgue-mesurables de la sphère S^{n-1}, la seule moyenne G-invariante est la mesure de Lebesgue.

Dans le chapitre 3, je classifie les groupes de Lie connexes ayant un sous-groupe de type fini dense avec la propriété (T). Cela étend le cas, dû à Margulis, d'un groupe de Lie connexe compact.

Théorème. (Voir Theorem 3.3) **Soit G un groupe de Lie connexe. Alors G a un sous-groupe de type fini dense avec la propriété (T) si et seulement si il vérifie les conditions suivantes :**

- **G a la propriété (T),**
- **le cercle R/Z n'est pas un quotient de G,**
- **le groupe SO_3 n'est pas un quotient de G.**

La condition que G ait la propriété (T) peut aussi, à partir d'un résultat de Wang [Wan82], se caractériser en termes de quotients, voir Theorem 3.1.

Dans le fait que les conditions du théorème sont nécessaires, le seul point non trivial est le fait que SO_3 n'est pas un quotient de G. Ce résultat est dû à Zimmer [Zim84]. La réciproque, consistant à montrer que les conditions du théorèmes sont suffisantes, est prouvée en six étapes.

Dans la première étape, on suppose que G est un groupe algébrique réel défini sur $\mathbf{Q}$: on utilise alors un argument standard pour projeter de façon dense un réseau sur G.

Dans la seconde étape, on se ramène au cas où G a une algèbre de Lie parfaite ; puis on montre, dans la troisième étape, que cela implique que la sous-algèbre de Lie obtenue en enlevant les facteurs simples compacts est encore parfaite.

Dans la quatrième étape, on prouve un lemme montrant qu'une algèbre de Lie sur **R** avec quelques propriétés convenables est quotient d'une autre vérifiant les mêmes propriétés et qui est en plus définie sur **Q**. Le principal ingrédient est un résultat de Witte [Wit05] disant que les représentations réelles des groupes de Lie semi-simples ont toujours des formes sur **Q**.

On utilise cela ainsi que la première étape dans la cinquième étape pour prouver le théorème dans le cas où G est algébrique réel.

Dans la sixième étape, on prouve le cas général ; on doit en fait travailler avec une extension d'un groupe algébrique réel par un centre discret infini.

Groupes de Kazhdan avec un groupe d'automorphismes extérieurs infini. Dans l'Astérisque de de la Harpe et Valette en 1989 [HaVa89], il y a une série de questions alors ouvertes, dont celle-ci, due à Frédéric Paulin :

(6) Existe-t-il un groupe ayant la propriété (T) et ayant un groupe d'automorphismes extérieurs infini ?

Une réponse positive à cette question est donnée dans le chapitre 4.

Théorème. (Voir Proposition 4.1.) **Le groupe (linéaire, de présentation finie) $\mathrm{SL}_3(\mathbf{Z}) \ltimes (\mathbf{Z}^3 \oplus \mathbf{Z}^3)$ a la propriété (T) et a un groupe d'automorphismes extérieurs infini.**

Il se trouve que la question (6) a été résolue indépendamment et presque simultanément par Y. Ollivier et D. Wise [OlWi05], par une manière beaucoup plus élaborée. Si leurs exemples sont beaucoup moins élémentaires (par exemple, ils ne sont pas, a priori, de présentation finie ni résiduellement finis), leur méthode, consistant en une "machine de Rips à noyau de Kazhdan", a des conséquences pas du tout triviales telles que : tout groupe avec la propriété (T) est quotient d'un groupe hyperbolique sans torsion avec la propriété (T) (voir le paragraphe 7.3). Dans leur papier, ils construisent un groupe de type fini, non hopfien, avec la propriété (T). Ils demandent alors :

(7) Existe-t-il un groupe de présentation finie ayant la propriété (T) et non hopfien ?

Cette question est résolue dans le chapitre 4.

Théorème. (Voir Theorem 4.3.) **Il existe un groupe non hopfien de présentation finie, quotient d'un groupe linéaire par un sous-groupe cyclique, qui a la propriété (T).**

Ce groupe est obtenu en modifiant la construction par Abels (1979) [Abe79], du premier groupe résoluble de présentation finie non hopfien, ce qui répondait alors à une vieille question de P. Hall. La preuve est basée sur des travaux postérieurs d'Abels [Abe87], donnant un critère pour qu'un groupe p-arithmétique soit de présentation finie, et consistant essentiellement à calculer et étudier les premiers groupes d'homologie du radical unipotent d'une algèbre de Lie convenable.

Annulation de la 1-cohomologie réduite. Un résultat de Shalom dit que, pour un groupe de type fini, la propriété (T), i.e. l'annulation de la 1-cohomologie de toutes les représentations unitaires, est équivalente à l'annulation de la 1-cohomologie **réduite** de toutes les représentations unitaires. Appelons cela la propriété ($\overline{\text{FH}}$). Le résultat de Shalom ne s'étend pas à des groupes dénombrables quelconques : en effet, un groupe qui est une limite directe de groupes avec la propriété (T), par exemple un groupe localement fini, a la propriété ($\overline{\text{FH}}$). Une question de F. Martin et A. Valette est de savoir si la réciproque est vraie :

(8) Soit G un groupe dénombrable avec la propriété ($\overline{\text{FH}}$). Est-ce que G est limite directe de groupes avec la propriété (T) ?

F. Martin a prouvé que la réponse est oui dans le cas où G est hypercentral (c'est-à-dire que tout quotient non trivial de G a un centre non trivial). Le cas qui semblait alors naturel à étudier est celui des groupes localement nilpotents. Je prouve le résultat suivant dans le chapitre 5.

Théorème. (Voir Theorem 5.1.) **Un groupe localement nilpotent a la propriété ($\overline{\text{FH}}$) si et seulement si son abélianisé est un groupe de torsion.**

Or il existe des exemples connus de groupes infinis localement nilpotents, sans torsion et parfaits, ce qui répond ainsi négativement à (8).

La preuve du théorème est basée sur la "propriété ($\overline{\text{FH}}$) relative". La dernière partie du chapitre 5 consiste à étendre le résultat au cas des groupes localement compacts, qui pose quelques difficultés techniques supplémentaires.

Groupes non dénombrables avec la propriété (FH), et groupes fortement bornés. On a la question suivante, posée dans [wor01] ainsi que dans une version préliminaire de [BHV05] :

(5) Est-ce que l'équivalence entre la propriété (T) et la propriété (FH) (propriété du point fixe pour des actions isométriques sur des Hilberts), connue dans le cas σ-compact, s'étend à des groupes localement compacts quelconques ? (On sait déjà que (T)$\Rightarrow$(FH), dû à Delorme (voir [BHV05]), est valable pour tous groupes topologiques.)

On sait qu'un groupe discret avec la propriété (T) est forcément de type fini, si bien que la question se réduit, dans le cas discret, à : existe-t-il un groupe non dénombrable avec la propriété (FH) ?

C'était le point de départ pour le chapitre 6. La question (5) est ainsi résolue par le résultat suivant.

Théorème. (Voir Proposition 6.1.) **Soit G un groupe dénombrable. Alors G se plonge dans un groupe de cardinalité $\aleph_1$ ayant la propriété (FH). En particulier, il existe des groupes discrets ayant la propriété (FH) mais pas la propriété (T).**

J'obtiens d'autres exemples à partir de l'observation qu'une certaine propriété, introduite récemment par G. Bergman [Ber05] et étudiée depuis par plusieurs auteurs, est équivalente à la propriété suivante : un groupe est **fortement borné** si toute action sur un espace métrique est à orbites bornées. En particulier, cela implique la propriété (FH) ; or Bergman a prouvé que le groupe des permutations d'un ensemble infini a cette propriété, ce qui a été généralisé depuis à divers groupes d'automorphismes. Pour ma part, je montre le résultat suivant.

Théorème. (Voir Theorem 6.13 et Theorem 6.15.) **Sont fortement bornés :**

– **les groupes ω_1-existentiellement clos, et**
– **le produit S^X, S étant un groupe fini parfait et X un ensemble infini quelconque.**

En particulier, S^X donne un exemple de groupe infini moyennable (car localement fini) ayant la propriété (FH).

Je rassemble par la suite des papiers plus courts et des notes isolées.

• Une note s'intéressant au groupe $G(K)$ des K-points d'un groupe algébrique linéaire G : sous certaines hypothèses, on donne des conditions nécessaires et suffisantes pour que $G(K)$ ait la propriété de Haagerup (comme groupe discret), resp. pour que $G(K)$ possède un sous-groupe infini avec la propriété (T).

• Une note contenant la généralisation suivante de [GHW05, §5, Theorem 4], à savoir que si A est un anneau commutatif réduit, alors $\mathrm{SL}_2(A)$ a la propriété de Haagerup. Comme application, les groupes résiduellement libres ont la propriété de Haagerup.

• Une note contenant l'observation faite plus haut du fait qu'en combinant les résultats d'Ollivier-Wise et ceux de Shalom, tout groupe ayant la propriété (T) est quotient d'un groupe hyperbolique sans torsion avec la propriété (T), ce qui répond à une question d'Ollivier et Wise.

• Une note faisant le lien entre un résultat de Vaserstein [Vas88] et un résultat de Shalom [Sha99p], avec pour exemple de conséquence le résultat suivant, répondant notamment à une question posée dans [BHV05] : le groupe (non localement compact) des lacets dans $\mathrm{SL}_n(\mathbf{R})$ a la propriété (T) si $n \geq 3$.

• Une note contenant un lemme, inspiré par des travaux de Shalom [Sha99p], permettant de prouver la propriété (T) relative par rapport à un sous-groupe normal abélien. Il s'applique directement à $(\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathbf{Z}^n, \mathbf{Z}^n)$, c'est-à-dire sans utiliser le fait que $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathbf{Z}^n$ est un réseau dans $\mathrm{SL}_n(\mathbf{R}) \ltimes \mathbf{R}^n$.

• Une courte note qui est un lemme concernant les fonctions conditionnellement de type négatif mesurables sur les groupes localement compacts : je montre qu'une telle fonction, sous la condition qu'elle soit bornée sur tout compact, est presque partout égale à une fonction conditionnellement de type négatif **continue**. C'est basé sur un résultat analogue dans le cas des fonctions de type positif [dLGl65].

• Pour finir, une liste de question ouvertes, certaines originales, concernant la propriété (T) et la propriété de Haagerup.

Contents

Chapter 0

Preliminaries

0.1 Representations

Let $\mathcal{H}$ be a Hilbert space over $\mathbf{C}$ (resp. $\mathbf{R}$). Denote by $\mathcal{U}(\mathcal{H})$ (resp. $\mathcal{O}(\mathcal{H})$) the group of unitary (resp. orthogonal) automorphisms of $\mathcal{H}$. There are several natural topologies on this group; we consider here the strong operator topology, defined as the weakest topology making the function $T \mapsto T(v)$ (with values in $\mathcal{H}$) continuous for all $v \in \mathcal{H}$.

Let G be a topological group. A unitary (resp. orthogonal) representation is a continuous morphism $G \to \mathcal{U}(\mathcal{H})$ (resp. $G \to \mathcal{O}(\mathcal{H})$). An equivalent datum is an action by unitary (resp. orthogonal) automorphisms of G on $\mathcal{H}$, given by a map $G \times \mathcal{H} \to \mathcal{H}$ which is supposed to be separately continuous.

Similarly, we can define groups of affine automorphisms $\mathcal{AU}(\mathcal{H}) \simeq \mathcal{U}(\mathcal{H}) \ltimes \mathcal{H}$ (resp. $\mathcal{AO}(\mathcal{H}) \simeq \mathcal{O}(\mathcal{H}) \ltimes \mathcal{H}$) by taking the group generated by unitary (resp. orthogonal) automorphisms and translations. The strong affine operator topology is similarly defined, and affine unitary (resp. affine orthogonal) representations of topological groups can be analogously defined.

Remark 0.1. The kernel of a representation (of any kind as above) of a topological group is closed, and hence always contains $\overline{\{1\}}$. Therefore, it is harmless in general to work, in this context, with Hausdorff topological groups.

Lemma 0.2 (Mazur-Ulam). **If $\mathcal{H}$ is a real Hilbert space, then every isometry of $\mathcal{H}$ is a□ne, i.e. belongs to $\mathcal{AO}(\mathcal{H})$.**

Definition 0.3. A topological group G is **a-T-menable** if it is Hausdorff and there exists a continuous isometric action α of G on a Hilbert space $\mathcal{H}$ which is metrically proper, i.e. such that, for every $v \in \mathcal{H}$ (equivalently, for some $v \in \mathcal{H}$) and every $r \in \mathbf{R}_+$, the set $\{g \in G : \|\alpha(g)v - v\| \leq r\}$ is compact.

Note that an a-T-menable group is necessarily locally compact and σ-compact: indeed, $U_n = \{g \in G : \|\alpha(g)v - v\| \leq n\}$ is a compact neighbourhood of 1 in G, and $\bigcup_n U_n = G$.

Lemma 0.4. **Let G be a group acting by isometries on a Hilbert space[1]. The following are equivalent.**

(i) **G fixes a point in $\mathcal{H}$.**

(ii) **There is a bounded G-orbit in $\mathcal{H}$.**

(iii) **All G-orbits in $\mathcal{H}$ are bounded.**

Note that (i)⇒(ii)⇔(iii) holds for every isometric action on a nonempty metric space. (ii)⇒(i) is a consequence of the "centre Lemma": every nonempty bounded subset X of a Hilbert space is contained in a unique ball of minimal radius. If X is a G-orbit, then the ball, hence its centre, must be invariant under G.

Definition 0.5. Let G be a topological group, and H a subgroup. The pair (G, H) has **relative Property (FH)** if, for every continuous isometric action of G on a Hilbert space $\mathcal{H}$, one of the following equivalent conditions is satisfied:

- H fixes a point in $\mathcal{H}$.
- There is a bounded H-orbit in $\mathcal{H}$.
- All H-orbits in $\mathcal{H}$ are bounded.

The following result is immediate from the definitions.

Lemma 0.6. **Let G be an a-T-menable topological group, and H a subgroup. If (G, H) has relative Property (FH), then H is relatively compact in G (i.e. has compact closure).**

Definition 0.7. Let π be an orthogonal or unitary representation of a topological group G on a Hilbert space $\mathcal{H}$. It is said to have **almost invariant vectors**[2] if for every compact[3] subset $K \subset G$ and every $\varepsilon > 0$, there exists a (K, ε)-invariant vector in $\mathcal{H}$, i.e. $\xi \in \mathcal{H}$ such that $\|v\| = 1$ and $\sup_{g\in K} \|\pi(g)\xi - \xi\| \leq \varepsilon$.

Note that the zero representation does **not** have almost invariant vectors.

Definition 0.8. Let π be an orthogonal or unitary representation of a Hausdorff topological group G on a Hilbert space $\mathcal{H}$. It is said to be C_0 if for every $\xi, \eta \in \mathcal{H}$, $\langle\pi(g)\xi, \eta\rangle \to 0$ when $g \to \infty$; that is, for every $\xi, \eta \in \mathcal{H}$ and $\varepsilon > 0$, there exists a compact subset $K \subset G$ such that $\sup_{g\in G-K} |\langle\pi(g)\xi, \eta\rangle| \leq \varepsilon$.

---

[1] **We do not specify whether the Hilbert space is real or complex: if it is complex, it can also be viewed as a real Hilbert space.**

[2] **Although it is sometimes used in a general setting, this definition is relevant especially in the case when G is locally compact. Indeed, in some groups such as $\mathcal{AO}(\mathcal{H})$, there is a natural notion of bounded subsets, which is weaker than that of relatively compact subsets, and leads to an alternative definition of "having almost invariant vectors".**

[3] **In this definition, we do not assume that compact subsets are Hausdorff. Note however that, in a topological group G, a subset is compact if and only if its image in the Hausdorff group $G/\{1\}$ is compact.**

Note that if a Hausdorff topological group G has a nonzero C_0-representation in a Hilbert space $\mathcal{H}$, then G is locally compact. Indeed, fix $\xi \in \mathcal{H}$ of norm 1. For some compact subset K, $\sup_{g \in G-K} |\langle \pi(g)\xi, \xi \rangle| \leq 1/2$. But if G is not locally compact, 1 must belong to the closure $\overline{G-K}$, and, since $\langle \pi(g)\xi, \xi \rangle$ is continuous, we obtain $\|\xi\|^2 \leq 1/2$, a contradiction.

Conversely, if G is locally compact, then it has at least one faithful C_0-representation, namely the left regular representation λ_G on $L^2(G)$.

Note also that if G is compact, then every representation of G is C_0. Conversely, if the trivial one-dimensional representation of G is C_0, then G is compact.

Definition 0.9. A topological group G has the **Haagerup Property** if it is Hausdorff and there exists an orthogonal C_0-representation of G with almost invariant vectors.

Note that a topological group with the Haagerup Property is locally compact. The following characterization is due to Akemann and Walter [AkWa81].

Theorem 0.10. **A topological group is a-T-menable if and only if it is σ-compact and has the Haagerup Property.**

In the case of non-σ-compact locally compact groups, the following result holds: such a group G has the Haagerup Property if and only if all its open compactly generated (equivalently: all its open σ-compact) subgroups have the Haagerup Property (see Proposition 0.25).

Let us turn to relative Property (T). We present it here in the standard context; we give slightly more general (and natural) statements in Chapter 2.

Definition 0.11. Let G be a topological group and H a subgroup. The pair (G, H) has relative Property (T) if, for every orthogonal representation of G with almost invariant vectors, there exist H-invariant vectors.

If (G, G) has relative Property (T), we simply say that G has Property (T).

Observe that if $H_1 \subset H_2$ and if (G, H_2) has relative Property (T), then $(G, \overline{H_1})$ also has relative Property (T). If H is relatively compact, it follows from Lemma 0.4 that (G, H) has relative Property (T). If G has the Haagerup Property, then the converse holds.

Theorem 0.12 (Delorme, Guichardet, Jolissaint). **Let G be a topological group, H a subgroup. If (G, H) has relative Property (T), then it has relative Property (FH). If G is locally compact, σ-compact, then the converse is true.**

The converse is not true in general, even when $G = H$ is a discrete group (see Chapter 6).

0.2 Functions on groups

Let X be any set. Denote by $\mathbf{R}^{(X)}$ the real vector space with basis X, and $\mathbf{R}_0^{(X)}$ its hyperplane defined as the kernel of the linear form $(u_x)_{x \in X} \mapsto \sum_{x \in X} u_x$.

A (real-valued) kernel is a function $\Phi : X \times X \to \mathbf{R}$. It must be viewed as a matrix with coefficients indexed by X. Such a kernel defines a bilinear form B_Φ of $\mathbf{R}^{(X)}$, defined on the basis by $B(e_x, e_y) = K(x, y)$.

Definition 0.13. The kernel Φ is called **positive definite**[4] if the corresponding bilinear form is symmetric and non-negative. That is, $K(x,y) = K(y,x)$ for all x, y, and, for all n, all $x_1, \ldots, x_n \in X$, $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$, $\sum_{i,j=1}^n \lambda_i \lambda_j K(x_i, x_j) \geq 0$.

Definition 0.14. The kernel Ψ is called **conditionally negative definite** if it vanishes on the diagonal and if the corresponding bilinear form is symmetric, and non-positive in restriction to $\mathbf{R}_0^{(X)}$. That is, $K(x,x) = 0$, $K(x,y) = K(y,x)$ for all x, y, and, for all n, all $x_1, \ldots, x_n \in X$, $\lambda_1, \ldots, \lambda_n \in \mathbf{R}$ such that $\sum_{i=1}^n \lambda_i = 0$, $\sum_{i,j=1}^n \lambda_i \lambda_j K(x_i, x_j) \leq 0$.

There are similar definitions for complex-valued functions: a complex-valued kernel is **positive definite** if the corresponding semi-linear form on $\mathbf{C}^{(X)}$ is hermitian and non-negative, and is **conditionally negative definite** if it is zero on the diagonal and if the corresponding semi-linear form is hermitian, and non-positive in restriction to $\mathbf{C}_0^{(X)}$. Observe that this coincides with the above definitions in the case of a complex-valued kernel which actually takes real values.

A conditionally negative definite kernel on G takes non-negative values. In the complex-valued case, it takes values of non-negative real part.

Definition 0.15. Let G be a group. A real or complex valued function f on G is definite positive (resp. conditionally negative definite) if the kernel $(g,h) \mapsto f(g^{-1}h)$ is so.

If φ is a positive definite function on a group G, then $\varphi(1) \in \mathbf{R}_+$ and, for all $g \in G$, $|\varphi(g)| \leq \varphi(1)$; it is called normalized if $\varphi(1) = 1$.

0.3 Correspondence

The two following lemmas are straightforward.

Lemma 0.16. **If π is an orthogonal (resp. unitary) representation of a topological group G on a real (resp. complex) Hilbert space $\mathcal{H}$, then, for every $\xi \in \mathcal{H}$, the coefficient $g \mapsto \langle \pi(g)\xi, \xi \rangle$ is a real-valued (resp. complex-valued) positive definite function on G.**

Lemma 0.17. **If α is an affine isometric action of a topological group G on a real Hilbert space $\mathcal{H}$, then, for every $v \in \mathcal{H}$, the coefficient $g \mapsto \|\alpha(g)v - v\|^2$ is a real-valued continuous conditionally negative definite function on G.**

In both cases, it can be shown that such functions arise this way:

Proposition 0.18 (GNS Construction). **Let φ be a continuous, real-valued (resp. complex-valued) positive definite function on a topological group G. Then there exists an orthogonal (resp. unitary) representation π on a real (resp. complex) Hilbert space $\mathcal{H}$, and $\xi \in \mathcal{H}$ such that $\varphi(g) = \langle \pi(g)\xi, \xi \rangle$ for all $g \in G$. If φ is C_0 (i.e. G is Hausdorff and $\lim_{g\to\infty} \varphi(g) = 0$), then π can be chosen C_0.**

[4] Observe that this widely used terminology is awkward since the corresponding bilinear form is not assumed to be definite (i.e. non-degenerate).

The latter assertion is a consequence of [Dix69, Proposition 13.4.10].

Proposition 0.19 (affine GNS Construction). **Let ψ be a continuous, real-valued (resp. complex-valued) conditionally negative definite function on a topological group G. Then there exists an a□ne isometric action α on a Hilbert space $\mathcal{H}$, and $v \in \mathcal{H}$ such that $\psi(g) = \|\alpha(g)v - v\|^2$ for all $g \in G$. Moreover, we can choose $\mathcal{H}$ with a complex structure making the linear part of the a□ne action unitary.**

Both propositions hold in a more general and natural setting, in the context of kernels on arbitrary sets. There is, in addition, a form of uniqueness which we do not state here.

The following results follows from Proposition 0.19.

Proposition 0.20. **If G is a topological group and H a subgroup, then the following are equivalent:**

- **(G, H) has relative Property (FH).**
- **For every a□ne isometric action of G on a complex Hilbert space, H has a fixed point.**
- **Every continuous conditionally negative definite function on G is bounded in restriction to H.**

Proposition 0.21. **If G is a topological group, then the following are equivalent:**

- **G is a-T-menable.**
- **There exists an a□ne isometric action on a complex Hilbert space which is proper.**
- **There exists a proper conditionally negative definite function on G.**

In both cases, we see that working with affine isometric action on real or complex Hilbert spaces does not change the definition of relative Property (FH) and a-T-menability.

Proposition 0.22. **Let G be a Hausdor□ topological group. The following are equivalent.**

(i) **G has the Haagerup Property.**

(ii) **There exists a unitary C_0-representation of G with almost invariant vectors.**

(iii) **There exist a net (φ_i) of continuous, real-valued normalized C_0 positive definite functions on G which tends to 1 uniformly on compact subsets.**

(iv) **There exist a net (φ_i) of continuous, complex-valued normalized C_0 positive definite functions on G which tends to 1 uniformly on compact subsets.**

Proof: (iii)⇒(iv) is trivial. (iv)⇒(iii): use the fact that if φ is a complex-valued positive definite function on G, then $|\varphi|^2$ is a real-valued one.

(i)⇒(iii): let π be an orthogonal representation of G with almost invariant vectors. For every $\varepsilon > 0$, K compact, pick a (K, ε)-invariant vector, and let $\varphi_{\varepsilon,K}$ be the corresponding coefficient. Then $\varphi_{\varepsilon,K} \to 1$ when $\varepsilon \to 0$ and K becomes big.

(iii)⇒(i): let (φ_i) be a net of continuous, positive definite normalized C_0 functions on G, tending to 1 uniformly on compact subsets. By the GNS Construction (Proposition 0.18), we can associate to φ a real Hilbert space $\mathcal{H}_i$, a vector ξ_i such that $\varphi_i(g) = \langle \pi(g)\xi_i, \xi_i \rangle$ for all $g \in G$, and we can ask π_i to be C_0. Note that $\|\xi_i\| = 1$ since φ_i is normalized. Then G acts on the direct Hilbertian sum $\bigoplus_{i \in I} \mathcal{H}_i$ through $\bigoplus_{i \in I} \pi_i$. This representation almost has invariant vectors, and is C_0.

The equivalence between (ii) and (iv) is similar to that between (i) and (iii), proving all equivalences. Note that (ii)⇒(i) is immediate, and its converse can also be proved by a complexification argument. □

Proposition 0.22 shows that the Haagerup Property can be indifferently defined for unitary or orthogonal representations.

The same holds for relative Property (T), using a complexification argument. However, in full generality, only one implication is known.

Proposition 0.23. **Let G be a topological group, and H a subgroup. Consider the following properties.**

- **(i)** (G, H) **has relative Property (T).**
- **(ii)** **For every net (φ_i) of continuous, normalized positive definite functions on G converging to 1 uniformly on compact subsets, the convergence is uniform on H.**

Then (ii)⇒(i), and the converse is true when G is locally compact, σ-compact.

Actually, the known proofs of (i)⇒(ii) pass through relative Property (FH), and that is why G locally compact, σ-compact is needed.

In Chapter 2, we will use (ii) rather than (i) as definition of relative Property (T).

0.4 Stability Properties

Lemma 0.24. **Let G be a locally compact group with the Haagerup Property, and let H be a closed subgroup. Then H has the Haagerup Property.**

This immediately follows from the definition. Note that the corresponding statement for a-T-menable groups is also trivial.

Proposition 0.25. **Let G be a locally compact group. Then G has the Haagerup Property if and only if every open, compactly generated subgroup of G has the Haagerup Property.**

Proof: The proof is the same as in [CCJJV01, Proposition 6.1.1], except that we do not suppose σ-compactness. We use the characterization in terms of positive definite functions. Let K be a compact subset of G, and let $\varepsilon > 0$. Then K is contained in an open, compactly generated subgroup H of G. Since H has the Haagerup Property, there exists a real-valued C_0 positive definite function φ on H which is $\geq 1 - \varepsilon$ on K. The function $\tilde{\varphi}$ on G which coincides with φ on H and is zero outside is positive definite and C_0 on G, and $\geq 1 - \varepsilon$ on K. This ends the proof. □

Proposition 0.26 ([CCJJV01, Proposition 6.1.5]). **Let G be a locally compact group, and let H be a closed subgroup which is co-Følner in G, i.e. there exists a G-invariant mean on G/H. If H has the Haagerup Property, then so does G.**

This applies when H has finite covolume. Another application is the case when H is normal in G. This gives:

Corollary 0.27. **Let $1 \to N \to G \to Q \to 1$ be an extension of locally compact groups. Suppose that N has the Haagerup Property and that Q is amenable. Then G also has the Haagerup Property.**

Chapter 1

Kazhdan and Haagerup Properties in algebraic groups over local fields

1.1 Lie algebras and minimal $\mathfrak{s}$-subalgebras

In the sequel, all Lie algebras are finite-dimensional over a field of characteristic zero, denoted by K, or $\mathbf{K}$ when it is a local field. If $\mathfrak{g}$ is a Lie algebra, denote by $\mathrm{rad}(\mathfrak{g})$ its radical and $Z(\mathfrak{g})$ its centre, $D\mathfrak{g}$ its derived subalgebra, and $\mathrm{Der}(\mathfrak{g})$ the Lie algebra of all derivations of $\mathfrak{g}$. If $\mathfrak{h}_1, \mathfrak{h}_2$ are Lie subalgebras of $\mathfrak{g}$, $[\mathfrak{h}_1, \mathfrak{h}_2]$ denotes the Lie subalgebra generated by the brackets $[h_1, h_2]$, $(h_1, h_2) \in \mathfrak{h}_1 \times \mathfrak{h}_2$.

Let $\mathfrak{g}$ be a Lie algebra with radical $\mathrm{rad}(\mathfrak{g}) = \mathfrak{r}$ and semisimple Levi factor $\mathfrak{s}$ (so that $\mathfrak{g} \simeq \mathfrak{s} \ltimes \mathfrak{r}$). We focus here on aspects of $\mathfrak{g}$ related to the action of $\mathfrak{s}$. This suggests the following definitions.

If $\mathfrak{s}$ is a Lie algebra, we define a **Lie $\mathfrak{s}$-algebra** to be a Lie algebra $\mathfrak{n}$ endowed with a morphism $i : \mathfrak{s} \to \mathrm{Der}(\mathfrak{n})$, defining a **completely reducible** linear action of $\mathfrak{s}$ on $\mathfrak{n}$. (This latter technical condition is empty if $\mathfrak{s}$ is semisimple.)

A Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ naturally embeds in the semidirect product $\mathfrak{s} \ltimes \mathfrak{n}$, so that we write $i(s)(n) = [s, n]$ for $s \in \mathfrak{s}$, $n \in \mathfrak{n}$.

By the **trivial irreducible module** of $\mathfrak{s}$ we mean a one-dimensional vector space endowed with a trivial action of $\mathfrak{s}$. We say that a module (over a Lie algebra or over a group) is **full** if it is completely reducible and does not contain the trivial irreducible module.

Definition 1.1. Let $\mathfrak{s}$ be a Lie algebra. We say that a Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ is **minimal** if $[\mathfrak{s}, \mathfrak{n}] \neq 0$, and for every $\mathfrak{s}$-subalgebra $\mathfrak{n}'$ of $\mathfrak{n}$, either $\mathfrak{n}' = \mathfrak{n}$ or $[\mathfrak{s}, \mathfrak{n}'] = 0$.

Note that, clearly, a Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ satisfying $[\mathfrak{s}, \mathfrak{n}] \neq 0$ contains a minimal $\mathfrak{s}$-subalgebra. We establish the following characterization of minimal $\mathfrak{s}$-algebras:

Theorem 1.2. **Let $\mathfrak{s}$ be a Lie algebra. A solvable Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ is minimal if and only if it satisfies the following conditions 1), 2), 3), and 4):**

1) $\mathfrak{n}$ **is 2-nilpotent (that is,** $[\mathfrak{n}, D\mathfrak{n}] = 0$**).**

2) $[\mathfrak{s}, \mathfrak{n}] = \mathfrak{n}$**.**

3) $[\mathfrak{s}, D\mathfrak{n}] = 0$**.**

4) $\mathfrak{n}/D\mathfrak{n}$ **is irreducible as a $\mathfrak{s}$-module.**

Definition 1.3. We call a solvable Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ **almost minimal** if it satisfies conditions 1), 2), and 3) of Theorem 1.2.

This definition has the advantage to be invariant under field extensions. Note that an almost minimal solvable Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ automatically satisfies the following Condition 4'): $\mathfrak{n}/D\mathfrak{n}$ is a full $\mathfrak{s}$-module.

Proposition 1.4. **Let $\mathfrak{n}$ be a solvable Lie $\mathfrak{s}$-algebra.**

1) The Lie $\mathfrak{s}$-subalgebra $[\mathfrak{s}, \mathfrak{n}]$ is an ideal in $\mathfrak{n}$ (and also in $\mathfrak{s} \ltimes \mathfrak{n}$), and $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{n}]] = [\mathfrak{s}, \mathfrak{n}]$.

2) If, moreover, $[\mathfrak{s}, D\mathfrak{n}] = 0$, then $[\mathfrak{s}, \mathfrak{n}]$ is an almost minimal Lie algebra (see Definition 1.3).

Proof: 1) Let $\mathfrak{v}$ be the subspace generated by the brackets $[s, n]$, $(s, n) \in \mathfrak{s} \times \mathfrak{n}$. Since the action of $\mathfrak{s}$ is completely reducible (see the definition of Lie $\mathfrak{s}$-algebra), it is immediate that $[\mathfrak{s}, \mathfrak{n}]$ and $[\mathfrak{s}, [\mathfrak{s}, \mathfrak{n}]]$ both coincide with the Lie subalgebra generated by $\mathfrak{v}$. Then, using Jacobi identity,

$$[\mathfrak{n}, [\mathfrak{s}, \mathfrak{n}]] \;=\; [\mathfrak{n}, [\mathfrak{s}, [\mathfrak{s}, \mathfrak{n}]]] \;\subseteq\; [\mathfrak{s}, [\mathfrak{n}, [\mathfrak{s}, \mathfrak{n}]]] + [[\mathfrak{s}, \mathfrak{n}], [\mathfrak{s}, \mathfrak{n}]] \;\subseteq\; [\mathfrak{s}, [\mathfrak{n}, \mathfrak{n}]] + [\mathfrak{s}, \mathfrak{n}] \;\subseteq\; [\mathfrak{s}, \mathfrak{n}].$$

2) Let $\mathfrak{z}$ be the linear subspace generated by the commutators $[v, w]$, $v, w \in \mathfrak{v}$. By Jacobi identity,

$$[\mathfrak{v}, \mathfrak{z}] \;=\; [[\mathfrak{s}, \mathfrak{v}], \mathfrak{z}] \;\subseteq\; [[\mathfrak{s}, \mathfrak{z}], \mathfrak{v}] + [\mathfrak{s}, [\mathfrak{v}, \mathfrak{z}]] \;\subseteq\; [[\mathfrak{s}, D\mathfrak{n}], \mathfrak{v}] + [\mathfrak{s}, D\mathfrak{n}] = 0.$$

Thus, the subspace $\mathfrak{n}' = \mathfrak{v} \oplus \mathfrak{z}$ is a 2-nilpotent Lie $\mathfrak{s}$-subalgebra of $\mathfrak{n}$. The Lie subalgebra $[\mathfrak{s}, \mathfrak{n}']$ contains $\mathfrak{v}$, hence also contains $\mathfrak{z}$, so $[\mathfrak{s}, \mathfrak{n}']$ is equal to $\mathfrak{n}'$. Thus Conditions 1) and 2) of Definition 1.3 are satisfied, while Condition 3) follows immediately from the hypothesis $[\mathfrak{s}, D\mathfrak{n}] = 0$. □

Proof of Theorem 1.2. Suppose that the four conditions are satisfied. Condition 4 implies $\mathfrak{n} \neq 0$. Then Condition 2 implies $[\mathfrak{s}, \mathfrak{n}] = \mathfrak{n} \neq 0$. Let $\mathfrak{n}' \subseteq \mathfrak{n}$ be a $\mathfrak{s}$-subalgebra. Then, by irreducibility (Condition 4), either $D\mathfrak{n} + \mathfrak{n}' = D\mathfrak{n}$ or $D\mathfrak{n} + \mathfrak{n}' = \mathfrak{n}$. In the first case, $\mathfrak{n}'$ centralizes $\mathfrak{s}$. In the second case, $\mathfrak{n} = [\mathfrak{s}, \mathfrak{n}] = [\mathfrak{s}, \mathfrak{n}' + D\mathfrak{n}] = [\mathfrak{s}, \mathfrak{n}'] \subseteq \mathfrak{n}'$, using Conditions 1 and 2, and the fact that $\mathfrak{n}'$ is a $\mathfrak{s}$-subalgebra.

Conversely, suppose that $\mathfrak{n}$ is minimal. Since $\mathfrak{n}$ is solvable, $D\mathfrak{n}$ is a proper $\mathfrak{s}$-subalgebra, so that, by minimality, $[\mathfrak{s}, D\mathfrak{n}] = 0$. By Proposition 1.4, $[\mathfrak{s}, \mathfrak{n}]$ is a nonzero almost minimal Lie $\mathfrak{s}$-subalgebra of $\mathfrak{n}$, hence satisfies 1), 2), 3). The minimality implies that 4) is also satisfied. □

The classification of (almost) minimal solvable Lie $\mathfrak{s}$-algebras can be deduced from the classification of irreducible $\mathfrak{s}$-modules. Let $\mathfrak{v}$ be a full $\mathfrak{s}$-module (equivalently, an abelian Lie $\mathfrak{s}$-algebra satisfying $[\mathfrak{s}, \mathfrak{v}] = \mathfrak{v}$). Recall that a bilinear form φ on $\mathfrak{v}$ is called $\mathfrak{s}$-invariant if it satisfies $\varphi([s, v], w) + \varphi(v, [s, w]) = 0$ for all $s \in \mathfrak{s}$, $v, w \in \mathfrak{v}$. Let $\mathrm{Bil}_{\mathfrak{s}}(\mathfrak{v})$ (resp. $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})$) denote the space of all $\mathfrak{s}$-invariant bilinear (resp. alternating bilinear) forms on $\mathfrak{v}$. Denote by $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$ the linear dual of $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})$.

Definition 1.5. We define the Lie $\mathfrak{s}$-algebra $\mathfrak{h}(\mathfrak{v})$ as follows: as a vector space, $\mathfrak{h}(\mathfrak{v}) = \mathfrak{v} \oplus \mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$; it is endowed with the following bracket:

$$[(x,z),(x',z')] = (0, e_{x,x'}) \qquad x,x' \in \mathfrak{v} \;\; z,z' \in \mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$$

where $e_{x,x'} \in \mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$ is defined by $e_{x,x'}(\varphi) = \varphi(x,x')$.

This is a 2-nilpotent Lie $\mathfrak{s}$-algebra under the action $[s,(x,z)] = ([s,x],0)$, which is almost minimal. Other almost minimal Lie $\mathfrak{s}$-algebras can be obtained by taking the quotient by a linear subspace of the centre. The following theorem states that this is the only way to construct almost minimal solvable Lie $\mathfrak{s}$-algebras.

Theorem 1.6. **If $\mathfrak{n}$ is an almost minimal solvable Lie $\mathfrak{s}$-algebra, then it is isomorphic (as a $\mathfrak{s}$-algebra) to $\mathfrak{h}(\mathfrak{v})/Z$, for some full $\mathfrak{s}$-module $\mathfrak{v}$ and some subspace Z of $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$. It is minimal if and only if $\mathfrak{v}$ is irreducible.**

Moreover, the almost minimal $\mathfrak{s}$-algebras $\mathfrak{h}(\mathfrak{v})/Z$ and $\mathfrak{h}(\mathfrak{v})/Z'$ are isomorphic if and only if Z' and Z are in the same orbit for the natural action of $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v})$ on the Grassmannian of $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$.

Remark 1.7. If $\mathfrak{s}$ is semisimple, $\mathfrak{s} \ltimes \mathfrak{h}(\mathfrak{v})$ is the universal central extension of the perfect Lie algebra $\mathfrak{s} \ltimes \mathfrak{v}$.

Proof of Theorem 1.6. Let $\mathfrak{n}$ be an almost minimal solvable Lie $\mathfrak{s}$-algebra. Let $\mathfrak{v}$ be the subspace generated by the brackets $[s,n]$, $(s,n) \in \mathfrak{s} \times \mathfrak{n}$. Since $\mathfrak{n}$ is almost minimal, $\mathfrak{v}$ is a complementary subspace of $D\mathfrak{n}$, and is a full $\mathfrak{s}$-module. If $u \in D\mathfrak{n}^*$, consider the alternating bilinear form ϕ_u on $\mathfrak{v}$ defined by $\phi_u(x,y) = u([x,y])$. This defines a mapping $D\mathfrak{n}^* \to \mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})$ which is immediately seen to be injective. By duality, this defines a surjective linear map $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^* \to D\mathfrak{n}$, whose kernel we denote by Z. It is immediate from the definition of $\mathfrak{h}(\mathfrak{v})$ that this map extends to a surjective morphism of Lie $\mathfrak{s}$-algebras $\mathfrak{h}(\mathfrak{v}) \to \mathfrak{n}$ with kernel Z. This proves that $\mathfrak{n}$ is isomorphic to $\mathfrak{h}(\mathfrak{v})/Z$.

The second assertion is immediate.

The third assertion follows from the proof of the first one, where we made no choice. Namely, take an isomorphism $\psi : \mathfrak{h}(\mathfrak{v})/Z \to \mathfrak{h}(\mathfrak{v})/Z'$. It gives by restriction an $\mathfrak{s}$-automorphism φ of $\mathfrak{v}$, which induces a unique automorphism $\tilde{\varphi}$ of $\mathfrak{h}(\mathfrak{v})$. Let p and p' denote the natural projections in the following diagram of Lie $\mathfrak{s}$-algebras:

$$\begin{array}{ccc} \mathfrak{h}(\mathfrak{v}) & \xrightarrow{p} & \mathfrak{h}(\mathfrak{v})/Z \\ {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\psi} \\ \mathfrak{h}(\mathfrak{v}) & \xrightarrow[p']{} & \mathfrak{h}(\mathfrak{v})/Z' \end{array}$$

This diagram is commutative: indeed, $p' \circ \tilde{\varphi}$ and $\psi \circ p$ coincide in restriction to $\mathfrak{v}$, and $\mathfrak{v}$ generates $\mathfrak{h}(\mathfrak{v})$ as a Lie algebra. This implies $Z = \mathrm{Ker}(\psi \circ p) = \mathrm{Ker}(p' \circ \tilde{\varphi}) = \tilde{\varphi}^{-1}(Z')$. □

Assume that $\mathfrak{s} = \mathfrak{sl}_2$. This case is essential in view of Theorem 1.6, and there is a simple description for it. Up to isomorphism, there exists exactly one irreducible

$\mathfrak{s}$-module $\mathfrak{v}_n$ of dimension n for every $n \geq 1$. Since $\mathfrak{v}_n$ is absolutely irreducible for all n, by Schur's Lemma, $\mathrm{Bil}_{\mathfrak{s}}(\mathfrak{v}_n)$ is at most one dimensional for all n. In fact, it is one-dimensional. Indeed, take the usual basis (H, X, Y) of $\mathfrak{sl}_2$ satisfying $[H, X] = 2X$, $[H, Y] = -2Y$, $[X, Y] = H$, and take the basis $(e_0, \dots, e_{n-1})$ of $\mathfrak{v}_n$ so that $H.e_i = (n-1-2i)e_i$, $X.e_i = (n-i)e_{i-1}$, and $Y.e_i = (i+1)e_{i+1}$, with the convention $e_{-1} = e_n = 0$. Then $Bil_{\mathfrak{s}}(\mathfrak{v}_n)$ is generated by the form φ_n defined by

$$\varphi_n(e_i, e_{n-1-i}) = (-1)^i \binom{i}{n-1}; \quad \varphi(e_i, e_j) = 0 \text{ if } i + j \neq n-1.$$

For odd n, φ_n is symmetric so that $Alt_{\mathfrak{s}}(\mathfrak{v}_n) = 0$; for even n, φ_n is symplectic and generates $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v}_n)$. For even n, denote by $\mathfrak{h}_{n+1}$ the one-dimensional central extension $\mathfrak{h}(\mathfrak{v}_n)$, well-known as the $(n+1)$-dimensional Heisenberg Lie algebra.

Thus, when $\mathfrak{s} = \mathfrak{sl}_2(K)$, Theorem 1.6 reduces as:

Proposition 1.8. **Up to isomorphism, the minimal solvable Lie $\mathfrak{sl}_2$-algebras are $\mathfrak{v}_n$ and $\mathfrak{h}_{2n-1}$ ($n \geq 2$).**

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical and $\mathfrak{s}$ a semisimple factor. Write $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$ by separating K-anisotropic and K-isotropic factors[1]. The ideal $\mathfrak{s}_c \ltimes \mathfrak{r}$ does not depend on the choice of $\mathfrak{s}$, and is sometimes called the **amenable radical** of $\mathfrak{g}$.

Definition 1.9. We call $\mathfrak{g}$ M-decomposed if $[\mathfrak{s}_{nc}, \mathfrak{r}] = 0$. Equivalently, $\mathfrak{g}$ is M-decomposed if the amenable radical is a direct factor of $\mathfrak{g}$.

The interest in studying $\mathfrak{sl}_2$-algebras lies in the following proposition.

Proposition 1.10. **Let $\mathfrak{g}$ be a Lie algebra, and keep notation as above. Suppose that $\mathfrak{g}$ is not M-decomposed. Then there exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$.**

This result is essentially due to [CDSW05], where it is not explicitly stated, but it is actually proved in the proof of Proposition 8.2 there (under the assumption $K = \mathbf{R}$, but their argument generalizes to any field of characteristic zero). This was a starting point for the present chapter.

Proof of Proposition 1.10. Since $\mathfrak{s}_{nc}$ is semisimple and K-isotropic, it is generated by its subalgebras K-isomorphic to $\mathfrak{sl}_2$. Since $[\mathfrak{s}_{nc}, \mathfrak{r}] \neq 0$, this implies that there exists some subalgebra $\mathfrak{s}'$ of $\mathfrak{s}_{nc}$ which is K-isomorphic to $\mathfrak{sl}_2$ and such that $[\mathfrak{s}', \mathfrak{r}] \neq 0$. Then the result is clear from Proposition 1.8. Notice that the proof gives the following slight refinement: $\mathfrak{h}$ can be chosen so that $\mathrm{rad}(\mathfrak{h}) \subseteq \mathrm{rad}(\mathfrak{g})$. □

When $K = \mathbf{R}$, another important case of Theorem 1.6 is $\mathfrak{s} = \mathfrak{so}_3$. Since the complexification of $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2(\mathbf{C})$, the irreducible complex $\mathfrak{s}$-modules make up a family $(\mathfrak{d}_n^{\mathbf{C}})$ $(n \geq 1)$; $\dim_{\mathbf{C}}(\mathfrak{d}_n^{\mathbf{C}}) = n$, which are the symmetric powers of the natural action of $\mathfrak{su}_2 = \mathfrak{so}_3$ on $\mathbf{C}^2$.

[1] c **and** nc **respectively stand for "non-compact" and "compact"; this is related to the fact that if S is a simple algebraic group defined over the local field K, then its Lie algebra is K-isotropic if and only if $S(\mathrm{K})$ is not compact.**

If $n = 2m+1$ is odd, then this is the complexification of a real $\mathfrak{so}_3$-module $\mathfrak{d}^{\mathbf{R}}_{2m+1}$ (of dimension n). If $n = 2m$ is even, $\mathfrak{d}^{\mathbf{C}}_n$ is irreducible as a $4m$-dimensional real $\mathfrak{so}_3$-module, we call it $\mathfrak{u}_{4m}$.

These two families $(\mathfrak{d}^{\mathbf{R}}_{2n+1})$ and $(\mathfrak{u}_{4n})$ make up all irreducible real $\mathfrak{so}_3$-modules.

Proposition 1.11. **The irreducible real $\mathfrak{so}_3$-modules make up two families: a family $(\mathfrak{d}^{\mathbf{R}}_{2n+1})$ of $(2n+1)$-dimensional modules $(n \geq 0)$, absolutely irreducible, and a family $(\mathfrak{u}_{4n})$ of $4n$-dimensional modules $(n \geq 1)$, not absolutely irreducible, preserving a quaternionic structure.** □

Since $(\mathfrak{d}^{\mathbf{R}}_{2n+1})$ is absolutely irreducible, the space of invariant bilinear forms on $(\mathfrak{d}^{\mathbf{R}}_{2n+1})$ is generated by a scalar product, so that $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{d}^{\mathbf{R}}_{2n+1}) = 0$

On the other hand, $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is three-dimensional, and is spanned by the imaginary parts of an invariant quaternionic hermitian form.

In order to classify the minimal solvable $\mathfrak{so}_3$-algebras, we must determine the orbits of the natural action of $\mathrm{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ on $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$. It is a standard fact that $\mathrm{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is isomorphic to the group of nonzero quaternions, that $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ naturally identifies with the set of imaginary quaternions, and that the action of $\mathrm{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ on $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is given by conjugation of quaternions. This implies that it acts transitively on each component of the Grassmannian.

For $i = 0, 1, 2, 3$, let Z_i be a fixed $(3-i)$-dimensional linear subspace of $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$. Denote by $\mathfrak{hu}^i_{4n}$ the minimal Lie $\mathfrak{so}_3$-algebra $\mathfrak{h}(\mathfrak{u}_{4n})/Z_i$; of course, $\mathfrak{hu}^0_{4n} = \mathfrak{u}_{4n}$ and $\mathfrak{hu}^3_{4n} = \mathfrak{h}(\mathfrak{u}_{4n})$.

Proposition 1.12. **Up to isomorphism, the minimal solvable Lie $\mathfrak{so}_3(\mathbf{R})$-algebras are $\mathfrak{d}^{\mathbf{R}}_{2n+1}$ $(n \geq 1)$ and $\mathfrak{hu}^i_{4n}$ $(n \geq 1$, $i = 0, 1, 2, 3)$.** □

There is a statement analogous to Proposition 1.10.

Proposition 1.13. **Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{R}$. Suppose that $[\mathfrak{s}_c, \mathfrak{r}] \neq \{0\}$. Then $\mathfrak{g}$ has a Lie subalgebra which is isomorphic to either $\mathfrak{so}_3 \ltimes \mathfrak{d}^{\mathbf{R}}_{2n+1}$ or $\mathfrak{so}_3 \ltimes \mathfrak{hu}^i_{4n}$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$.** □

Combining Propositions 1.10 and 1.13, we obtain the following result.

Proposition 1.14. **Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{R}$. Suppose that $[\mathfrak{s}, \mathfrak{r}] \neq \{0\}$. Then $\mathfrak{g}$ has a Lie subalgebra which is isomorphic one of the following:**

- $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ **for some** $n \geq 2$,
- $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ **for some** $n \geq 2$,
- $\mathfrak{so}_3 \ltimes \mathfrak{d}^{\mathbf{R}}_{2n+1}$ **for some** $n \geq 1$, **or**
- $\mathfrak{so}_3 \ltimes \mathfrak{hu}^i_{4n}$ **for some** $i = 0, 1, 2, 3$ **and some** $n \geq 1$. □

1.2 Corresponding results for algebraic groups and connected Lie groups

1.2.1 Minimal algebraic subgroups

We now give the corresponding statements and results for algebraic groups.

Let S be a reductive K-group. A K-S-group means a linear K-group endowed with a K-action of S by automorphisms.

Recall that the Lie algebra functor gives an equivalence of categories between the category of unipotent K-groups and the category of nilpotent Lie K-algebras. If S is semisimple and simply connected with Lie algebra $\mathfrak{s}$, it induces an equivalence of categories between the category of unipotent K-S-groups and the category of nilpotent Lie S-algebras over K. If S is not simply connected (in particular, if S is not semisimple), this is no longer an essentially surjective functor, but it remains fully faithful.

A minimal (resp. almost minimal) solvable S-group N is defined similarly as in the case of Lie algebras; since it satisfies $[S,N]=N$, it is automatically unipotent. Moreover, N is a minimal (resp. almost minimal) solvable K-S-group if and only if its Lie algebra $\mathfrak{n}$ is a minimal (resp. almost minimal) solvable Lie $\mathfrak{s}$-algebra. Proposition 1.4 and Theorem 1.2 also immediately carry over into the context of algebraic groups.

If S is reductive and V is a K-S-module, we define the unipotent K-S-group $H(V)$ as follows: as a variety, $H(V)=V\oplus\mathrm{Alt}_S(V)^*$; it is endowed with the following group law:

$$(x,z)(x',z')=(x+x',z+z'+e_{x,x'}) \qquad x,x'\in V \quad z,z'\in\mathrm{Alt}_S(V)^* \tag{1.2.1}$$

where $e_{x,x'}\in\mathrm{Alt}_S(V)^*$ is defined by $e_{x,x'}(\varphi)=\varphi(x,x')$. This is a K-S-group under the action $s.(x,z)=(s.x,z)$. It is clear that its Lie algebra is isomorphic as a Lie K-S-algebra to $\mathfrak{h}(\mathfrak{v})$, where $V=\mathfrak{v}$ is viewed as a $\mathfrak{s}$-module. Here is the analog of Theorem 1.6.

Theorem 1.15. **If N is an almost minimal solvable K-S-group, then it is isomorphic (as a K-S-group) to $H(V)/Z$, for some full K-S-module V and some K-subspace Z of $\mathrm{Alt}_S(V)^*$. It is minimal if and only if V is irreducible.**

Moreover, the almost minimal K-S-groups $H(V)/Z$ and $H(V)/Z'$ are isomorphic if and only if Z' and Z are in the same orbit for the natural action of $\mathrm{Aut}_S(V)$ on the Grassmannian of $\mathrm{Alt}_S(V)^*$.

1.2.2 The example SL_2

The simply connected K-group with Lie algebra $\mathfrak{sl}_2$ is SL_2. Denote by V_n and H_{2n-1} the SL_2-groups corresponding to $\mathfrak{v}_n$ and $\mathfrak{h}_{2n-1}$. These are the solvable minimal SL_2-groups over K. The only non-simply connected K-group with Lie algebra $\mathfrak{sl}_2$ is the adjoint group PGL_2; thus the minimal solvable PGL_2-groups over K are V_{2n-1} for $n\geq 2$.

Remark 1.16. It is convenient, in algebraic groups, to deal with the unipotent radical rather than with the radical. It is straightforward to see that a reductive subgroup S of a linear algebraic group centralizes the radical if and only if it centralizes the unipotent radical. Indeed, suppose $[S, R_u] = 1$. We always have $[S, R/R_u] = 1$ since R/R_u is central in G^0/R_u and S is connected (G^0 denoting the unit component of G). Since S is reductive, this implies that S acts trivially on R.

Let G be a linear algebraic group over the field K of characteristic zero, R its radical, S a Levi factor, decomposed as $S_{nc}S_c$ by separating K-isotropic and K-anisotropic factors.

Proposition 1.17. **Suppose that $[S_{nc}, R] \neq 1$. Then G has a K-subgroup which is K-isomorphic to either $\mathrm{SL}_2 \ltimes V_n$, $\mathrm{PGL}_2 \ltimes V_{2n-1}$, or $\mathrm{SL}_2 \ltimes H_{2n-1}$ for some $n \geq 2$.**

Let us mention the translation into the context of connected Lie groups, which is immediate from the Lie algebraic version.

Proposition 1.18. **Let G be a real Lie group. Suppose that $[S_{nc}, R] \neq 1$. Then there exists a Lie subgroup H of G which is locally isomorphic to $\mathrm{SL}_2(\mathbf{R}) \ltimes V_n(\mathbf{R})$ or $\mathrm{SL}_2(\mathbf{R}) \ltimes H_{2n-1}(\mathbf{R})$ for some $n \geq 2$.**

Remark 1.19. 1) An analogous result holds with complex Lie groups.

2) The Lie subgroup H is not necessarily closed; this is due to the fact that $\mathrm{SL}_2(\mathbf{R})$ and $H_{2n-1}(\mathbf{R})$ have noncompact centre. For instance, take an element z of the centre of H that generates an infinite discrete subgroup, and take the image of H in the quotient of $H \times \mathbf{R}/\mathbf{Z}$ by (z, α), where α is irrational.

3) It can easily be shown that, if the Lie group G is linear, then the subgroup H is necessarily closed. In a few words, this is because the derived subgroup of the radical is unipotent, hence simply connected, and the centre of the semisimple part is finite.

1.2.3 The example SO_3

We go on with the notation introduced before Proposition 1.11. In the context of algebraic $\mathbf{R}$-groups as in the context of connected Lie groups, the simply connected group corresponding to $\mathfrak{so}_3(\mathbf{R})$ is $\mathrm{SU}(2)$. The only non-simply connected corresponding group is $\mathrm{SO}_3(\mathbf{R})$.

The irreducible $\mathrm{SU}(2)$-modules corresponding to $\mathfrak{d}_{2m+1}^{\mathbf{R}}$ and $\mathfrak{u}_{4n}$ are denoted by $D_{2n+1}^{\mathbf{R}}$ and U_{4n}. Among those, only $D_{2n+1}^{\mathbf{R}}$ provide $\mathrm{SO}_3(\mathbf{R})$-modules.

Denote by HU_{4n}^i the unipotent $\mathbf{R}$-group corresponding to $\mathfrak{hu}_{4n}^i$, $i = 0, 1, 2, 3$.

Remark 1.20. It can be shown that the maximal unipotent subgroups of $\mathrm{Sp}(n, 1)$ are isomorphic to HU_{4n}^3.

Proposition 1.21. **Up to isomorphism, the minimal solvable Lie $\mathrm{SO}_3(\mathbf{R})$-algebras are $D_{2n+1}^{\mathbf{R}}$ for $n \geq 1$; the other minimal solvable Lie $\mathrm{SU}(2)$-algebras are HU_{4n}^i, for $n \geq 1$, $i = 0, 1, 2, 3$.** □

Proposition 1.22. **Let G be a linear algebraic $\mathbf{R}$-group. Suppose that $[S_c, R] \neq 1$. Then G has a $\mathbf{R}$-subgroup which is $\mathbf{R}$-isomorphic to either $\mathrm{SU}(2) \ltimes D^{\mathbf{R}}_{2n+1}$, $\mathrm{SO}_3(\mathbf{R}) \ltimes D^{\mathbf{R}}_{2n+1}$, or $\mathrm{SU}(2) \ltimes HU^i_{4n}$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$.**

Let G be a real Lie group. Suppose that $[S_c, R] \neq 1$. Then G has a Lie subgroup which is locally isomorphic to either $\mathrm{SU}(2) \ltimes D^{\mathbf{R}}_{2n+1}$ or $\mathrm{SU}(2) \ltimes HU^i_{4n}$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$. □

1.3 The Haagerup Property

We provide corresponding statements for Proposition 1.10 in the realm of algebraic groups and connected Lie groups. As a consequence, using Proposition 1.17, we get the following theorem, which was the initial motivation for the results above.

Theorem 1.23. **Let G be either a connected Lie group, or $G = \mathbf{G}(\mathbf{K})$, where $\mathbf{G}$ is a linear algebraic group over the local field $\mathbf{K}$ of characteristic zero. Let $\mathfrak{g}$ be its Lie algebra. The following are equivalent.**

- **(i)** **G has Haagerup's property.**
- **(ii)** **For every noncompact closed subgroup H of G, (G, H) does not have relative Property (T).**
- **(iii)** **The following conditions are satisfied:**
  - **$\mathfrak{g}$ is M-decomposed.**
  - **All simple factors of $\mathfrak{g}$ have $\mathbf{K}$-rank ≤ 1.**
  - **(in the case of Lie groups or when $\mathbf{K} = \mathbf{R}$) No simple factor of $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}(n, 1)$ ($n \geq 2$) or $\mathfrak{f}_{4(-20)}$.**
- **(iv)** **$\mathfrak{g}$ contains no isomorphic copy of any one of the following Lie algebras**
  - **$\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$,**
  - **(in the case of Lie groups or when $\mathbf{K} = \mathbf{R}$) $\mathfrak{sp}(2, 1)$.**

Another proof of the equivalence between (i), (ii) and (iii) will be given in Chapter 2, Corollary 2.49. In the case of connected Lie groups, it was already proved, in a different way, in [CCJJV01, Chap. 4].

Remark 1.24. The notion of M-decomposed (real) Lie algebras also appears in other contexts: heat kernel on Lie groups [Var96], Rapid Decay Property [CPS05], weak amenability [CDSW05].

Let us proceed to the proof of Theorem 1.23. We need some preliminary results.

Proposition 1.25. **Let $\mathbf{K}$ be a local field of characteristic zero and $n \geq 1$. Then the pairs $(\mathrm{SL}_2(\mathbf{K}) \ltimes V_n(\mathbf{K}), V_n(\mathbf{K}))$, $(\mathrm{PGL}_2(\mathbf{K}) \ltimes V_n(\mathbf{K}), V_n(\mathbf{K}))$, $(\mathrm{SL}_2(\mathbf{K}) \ltimes H_n(\mathbf{K}), H_n(\mathbf{K}))$, (S**

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Proof: This follows from Theorem 2.46 in Chapter 2. However, to make the chapters independent, we include a sketch of proof.

The first (and the fourth) case is well-known; it follows, for instance, from Furstenberg's theory [Fur76] of invariant probabilities on projective spaces, which implies that $\mathrm{SL}_2(\mathbf{K})$ does not preserve any probability on $V_n(\mathbf{K})$ (more precisely, on its Pontryagin dual) other than the Dirac measure at zero. See, for instance, the proof of [HaVa89, Chap. 2, Proposition 2]. The second case is an immediate consequence of the first. For the third (resp. fifth) case, we invoke [CCJJV01, Proposition 4.1.4], with $S = \mathrm{SL}_2(\mathbf{K})$, $N = H_n(\mathbf{K})$, even if the hypotheses are slightly different (unless $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$): the only modification is that, since here $[N, S]$ is not necessarily connected, we must show that its image in the unitary group U_n is connected so as to justify Lie's Theorem. Otherwise, it would have a nontrivial finite quotient. This is a contradiction, since $[N, S]$ is generated by divisible elements; this is clear, since, as the group of $\mathbf{K}$-points of an unipotent group, it has a well-defined logarithm. □

Corollary 1.26. **Let G be either a connected Lie group, or $G = \mathbf{G}(\mathbf{K})$, where $\mathbf{G}$ is a linear algebraic group over the local field $\mathbf{K}$ of characteristic zero. Suppose that the Lie algebra $\mathfrak{g}$ of G contains a subalgebra $\mathfrak{h}$ isomorphic to either $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$. Then G has a noncompact closed subgroup with relative Property (T). In particular, G does not have Haagerup's property.**

Proof: Let us begin by the case of algebraic groups. By [Bor91, Chap. II, Corollary 7.9], since $\mathfrak{h}$ is perfect, it is the Lie algebra of a closed $\mathbf{K}$-subgroup H of G. Since H must be $\mathbf{K}$-isomorphic to either $\mathrm{SL}_2 \ltimes V_m$, $\mathrm{PGL}_2 \ltimes V_{2m-1}$, or $\mathrm{SL}_2 \ltimes H_{2m-1}$ for some $m \geq 2$, Proposition 1.25 implies that $G(\mathbf{K})$ has a noncompact closed subgroup with relative Property (T).

In the case of Lie groups, we obtain a Lie subgroup which is the image of an immersion i of $\widetilde{\mathrm{SL}_2}(\mathbf{R}) \ltimes N$, where N is either $V_n(\mathbf{R})$ or $H_{2n-1}(\mathbf{R})$, for some $n \geq 2$, into G. By Proposition 1.25, $(G, \overline{i(N)})$ has Property (T). We claim that $\overline{i(N)}$ is not compact. Suppose the contrary. Then it is solvable and connected, hence it is a torus. It is normal in the closure H of $i(G)$. Since the automorphism group of a torus is totally disconnected, the action by conjugation of H on $\overline{i(N)}$ is trivial; that is, $i(N)$ is central in H. This is a contradiction. □

Proof of Theorem 1.23. As already noticed in Chapter 0, (i)⇒(ii) is immediate from the definition of relative Property (T). We are going to prove (ii)⇒(iv)⇒(iii)⇒(i).

For the implication (iii)⇒(i), in the algebraic case, G is isomorphic, up to a finite kernel, to $S_{nc}(\mathbf{K}) \times \mathrm{Mr}(\mathbf{K})$, where Mr denotes the amenable radical of $\mathbf{G}$. The group $\mathrm{Mr}(\mathbf{K})$ is amenable, hence has Haagerup's property. The group $S_{nc}(\mathbf{K})$ also has Haagerup's property: if $\mathbf{K}$ is Archimedean, it maps, with finite kernel, onto a product of groups isomorphic to $\mathrm{PSO}_0(n, 1)$ or $\mathrm{PSU}(n, 1)$ ($n \geq 2$), and these groups have Haagerup's property, by a result of Faraut and Harzallah, see [BHV05, Chap. 2]. If $\mathbf{K}$ is non-Archimedean, then $S_{nc}(\mathbf{K})$ acts properly on a product of trees (one for each simple factor) [BoTi], and this also implies that it has Haagerup's property [BHV05, Chap. 2].

The same argument also works for connected Lie groups when the semisimple part has finite centre; in particular, this is fulfilled for linear Lie groups and their finite coverings. The case when the semisimple part has infinite centre is considerably more involved, see [CCJJV01, Chap. 4].

(ii)⇒(iv) Suppose that (iv) is not satisfied. If $\mathfrak{g}$ contains a copy of $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$, then, by Corollary 1.26, G does not satisfy (ii). If $\mathbf{K} = \mathbf{R}$, we consider G as a Lie group with finitely many components. By a standard argument, since $\mathrm{Sp}(2,1)$ is simply connected with finite centre (of order 2), an embedding of $\mathfrak{sp}(2,1)$ into $\mathfrak{g}$ corresponds to a closed embedding of $\mathrm{Sp}(2,1)$ or $\mathrm{PSp}(2,1)$ into G. Since $\mathrm{Sp}(2,1)$ has Property (T) [BHV05, Chap. 3], this contradicts (ii).

(iv)⇒(iii) If $\mathfrak{g}$ is not M-decomposed, then, by Proposition 1.10, it contains a copy of $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$.

If $\mathfrak{g}$ has a simple factor $\mathfrak{s}$, then $\mathfrak{s}$ embeds in $\mathfrak{g}$ through a Levi factor. If $\mathfrak{s}$ has $\mathbf{K}$-rank ≥ 2, then it contains a subalgebra isomorphic to either $\mathfrak{sl}_3$ or $\mathfrak{sp}_4$ [Mar91, Chap I, (1.6.2)], and such a subalgebra contains a subalgebra isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{v}_2$ (resp. $\mathfrak{sl}_2 \ltimes \mathfrak{v}_3$) [BHV05, 1.4 and 1.5].

Finally, if $\mathbf{K} = \mathbf{R}$ and $\mathfrak{s}$ is isomorphic to either $\mathfrak{sp}(n,1)$ for some $n \geq 2$ or $\mathfrak{f}_{4(-20)}$, then it contains a copy of $\mathfrak{sp}(2,1)$. □

Remark 1.27. Conversely, $\mathfrak{sp}(n,1)$ does not contain any subalgebra isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for any $n \geq 2$; this can be shown using results of [CDSW05] about weak amenability.

We now use Proposition 1.12 to prove (ii)⇒(i) in the following result (while the reverse implication is essentially due to [GHW05, §5, Theorem 4]).

Theorem 1.28. **Let G be a connected Lie group. Then the following are equivalent:**
(i) G is locally isomorphic to $\mathrm{SO}_3(\mathbf{R})^a \times \mathrm{SL}_2(\mathbf{R})^b \times \mathrm{SL}_2(\mathbf{C})^c \times R$, for a solvable Lie group R and integers a, b, c.
(ii) G has Haagerup's property (when endowed with the discrete topology).

Remark 1.29. Assertion (i) of Theorem 1.28 is equivalent to: (ii') **The complexification $\mathfrak{g}_\mathbf{C}$ of $\mathfrak{g}$ is M-decomposed, and its semisimple part is isomorphic to $\mathfrak{sl}_2(\mathbf{C})^n$ for some n.**

For instance, $\mathrm{SO}_3(\mathbf{R}) \ltimes \mathbf{R}^3$ has a countable subgroup which does not have Haagerup's property. An explicit example is given by $\mathrm{SO}_3(\mathbf{Z}[1/p]) \ltimes \mathbf{Z}[1/p]^3$. It can also be shown that this group has no infinite subgroup with relative Property (T). This answers an open question in [CCJJV01, Section 7.1]. This group is not finitely presented (this is a consequence of [Abe87, Theorem 2.6.4]); we give a similar example in Remark 1.34 which is, in addition, finitely presented.

We need some preliminary observations in view of the proof of Theorem 1.28. Let us exhibit some subgroups in the groups provided by Propositions 1.17 and 1.22.

Observation 1.30. Let G denote $\mathrm{SL}_2 \ltimes V_n$, $\mathrm{PGL}_2 \ltimes V_{2n-1}$, or $\mathrm{SL}_2 \ltimes H_{2n-1}$ for some $n \geq 2$, and R its radical. Then, for every field K of characteristic zero, $G(K)$ contains $G(\mathbf{Z})$ as a subgroup. On the other hand, the pair $(G(\mathbf{Z}), R(\mathbf{Z}))$ has Property (T), this is because $G(\mathbf{Z})$ is a lattice in $G(\mathbf{R})$.

Observation 1.31. Now, let G denote $\mathrm{SU}(2) \ltimes D^{\mathbf{R}}_{2n+1}$, $\mathrm{SO}_3(\mathbf{R}) \ltimes D^{\mathbf{R}}_{2n+1}$, or $\mathrm{SU}(2) \ltimes HU^i_{4n}$ for some $i = 0, 1, 2, 3$. These groups are all defined over $\mathbf{Q}$: this is obvious at least for all but $\mathrm{SU}(2) \ltimes HU^i_{4n}$ for $i = 1, 2$; for these two, this is because the subspace Z_i can be chosen rational in the definition of HU^i_{4n}.

Let R be the radical of G and S a Levi factor defined over $\mathbf{Q}$. Let F be a number field of degree three over $\mathbf{Q}$, not totally real. Let $\mathcal{O}$ be its ring of integers. Then $G(\mathcal{O})$ embeds diagonally as an irreducible lattice in $G(\mathbf{R}) \times G(\mathbf{C})$. Its projection Γ in $G(\mathbf{R})$ does not have Haagerup's property, since otherwise $G(\mathbf{C})$ would also have Haagerup's property by Proposition 0.26, and this is excluded since it does not satisfy $[S_{nc}, R] = 1$, see Proposition 1.26 (the anisotropic Levi factor becomes isotropic after complexification).

Proposition 1.32. **Let G be a real Lie group, R its radical, S a semisimple factor. Suppose that $[S, R] \neq 1$. Then G has a countable subgroup without Haagerup's property.**

Proof: First case: $[S_{nc}, R] \neq 1$. Then, by Proposition 1.18, G has a Lie subgroup H isomorphic to a quotient of $\widetilde{H} = \widetilde{\mathrm{SL}_2(\mathbf{R})} \ltimes R(\mathbf{R})$ by a discrete central subgroup, where $R = V_n$ or H_{2n-1}, for some $n \geq 2$. Denote by $\widetilde{H}(\mathbf{Z})$ the inverse image of $\mathrm{SL}_2(\mathbf{Z}) \ltimes R(\mathbf{Z})$ in $\widetilde{H}$. By the observation above, $(\widetilde{H}(\mathbf{Z}), R(\mathbf{Z}))$ has Property (T), so that its image in H, which we denote by $H(\mathbf{Z})$, satisfies $(H(\mathbf{Z}), R_G(\mathbf{Z}))$ has Property (T), where $R_G(\mathbf{Z})$ means the image of $R(\mathbf{Z})$ in G. Observe that $R_G(\mathbf{Z})$ is infinite: if $R = V_n$, this is $V_n(\mathbf{Z})$; if $R = H_{2n-1}$, this is a quotient of $H_{2n-1}(\mathbf{Z})$ by some central subgroup. Accordingly, $H(\mathbf{Z})$ does not have Haagerup's property.

Second case: $[S_c, R] \neq 1$. By Proposition 1.22, G has a Lie subgroup H isomorphic to a central quotient of $\mathrm{SU}(2)(\mathbf{R}) \ltimes R$, where $R = D^{\mathbf{R}}_{2n+1}$ or HU^i_{4n}, for some $n \geq 1$ and $i = 0, 1, 2, 3$.

First suppose that the radical of H is simply connected. Then, by Observation 1.31, H has a subgroup without the Haagerup property.

Now, let us deal with the case when $H = \widetilde{H}/Z$, where Z is a discrete central subgroup. Then $\widetilde{H}$ has a subgroup Γ as above which does not have Haagerup's property. Let W denote the centre of $\widetilde{H}$. The kernel of the projection of Γ to H is given by $\Gamma \cap Z$. We use the following trick: we apply an automorphism α of $\widetilde{H}$ such that $\alpha(\Gamma) \cap Z$ is finite. It follows that the image of $\alpha(\Gamma)$ in H does not have Haagerup's property.

This allows to suppose that $\Gamma \cap Z$ is finite, so that the image of Γ in H does not have Haagerup's property. Let us construct such an automorphism.

Observe that the representations of $\mathrm{SU}(2)$ can be extended to the direct product $\mathbf{R}^* \times \mathrm{SU}(2)$ by making $\mathbf{R}^*$ act by scalar multiplication. This action lifts to an action of $\mathbf{R}^* \times \mathrm{SU}(2)$ on HU^i_{4n}, where the scalar a acts on the derived subgroup of HU^i_{4n} by multiplication by a^2.

Now, working in the unit component of the centre W of $\widetilde{H}$, which we treat as a vector space, we can take a so that $a^2 \cdot (\Gamma \cap W)$ avoids $Z \cap W$ (a clearly exists, since Γ and Z are countable). $\square$

If G is a topological group, denote by G_d the group G endowed with the discrete topology.

Proof of Theorem 1.28. We remind that we must prove, for a connected Lie group G, the equivalence between

(i) G is locally isomorphic to $\mathrm{SO}_3(\mathbf{R})^a \times \mathrm{SL}_2(\mathbf{R})^b \times \mathrm{SL}_2(\mathbf{C})^c \times R$, with R solvable and integers a, b, c, and

(ii) G_d has Haagerup's property.

The implication (i)$\Rightarrow$(ii) is, essentially, a deep and recent result of Guentner, Higson, and Weinberger [GHW05, §5, Theorem 4], which implies that $(\mathrm{PSL}_2(\mathbf{C}))_d$ has Haagerup's property. Let G be as in (i), and S its semisimple factor. Then G/S is solvable, so that, by Proposition 0.26, we can reduce to the case when $G = S$. Now, let Z be the centre of the semisimple group G, and embed G_d in $(G/Z)_d \times G$, where G_d means G endowed with the discrete topology. This is a discrete embedding. Since G has Haagerup's property, this reduces the problem to the case when G has trivial centre. So, we are reduced to the cases of $\mathrm{SO}_3(\mathbf{R})$, $\mathrm{PSL}_2(\mathbf{R})$, and $\mathrm{PSL}_2(\mathbf{C})$. The two first groups are contained in the third, so that the result follows from the Guentner-Higson-Weinberger Theorem.

Conversely, suppose that G does not satisfy (i).

If $[S, R] \neq 1$, then, by Proposition 1.32, G_d does not have Haagerup's property. Otherwise, observe that the simple factors allowed in (i) are exactly those of geometric rank one (viewing $\mathrm{SL}_2(\mathbf{C})$ as a complex Lie group). Hence, S has a factor W which is not of geometric rank one. Then the result is provided by Lemma 1.33 below. □

Lemma 1.33. **Let S be a simple Lie group which is not locally isomorphic to $\mathrm{SO}_3(\mathbf{R})$, $\mathrm{SL}_2(\mathbf{R})$ or $\mathrm{SL}_2(\mathbf{C})$. Then S_d does not have Haagerup's property.**

Proof: Let Z be the centre of S, so that $S/Z \simeq G(\mathbf{R})$ for some $\mathbf{R}$-algebraic group G. By assumption, $G(\mathbf{C})$ has factors of higher rank, hence does not have Haagerup's property. Let F be a number field of degree three over $\mathbf{Q}$, not totally real. Let $\mathcal{O}$ be its ring of integers. Then $G(\mathcal{O})$ embeds diagonally as an irreducible lattice in $G(\mathbf{R}) \times G(\mathbf{C})$, and is isomorphic to its projection in $G(\mathbf{R})$. Let Γ be the inverse image in $S \times G(\mathbf{C})$ of $G(\mathcal{O})$. Then Γ is a lattice in $S \times G(\mathbf{C})$. Hence, by [CCJJV01, Proposition 6.1.5], Γ does not have Haagerup's property. Note that the projection Γ' of Γ into S has finite kernel, contained in the centre of $G(\mathbf{C})$. So Γ' neither has Haagerup's property, and is a subgroup of S. □

Remark 1.34. Theorem 1.28 is no longer true if we replace the statement "G_d has Haagerup's property" by "G_d has no infinite subgroup with relative Property (T)". Indeed, let $G = K \ltimes V$, where K is locally isomorphic to $\mathrm{SO}_3(\mathbf{R})^n$ and V is a vector space on which K acts nontrivially. Suppose that (G_d, H) has Property (T) for some subgroup H. Then $(G_d/V, H/(H \cap V))$ has Property (T). In view of the Guentner-Higson-Weinberger Theorem (see the proof of Theorem 1.28), $H/(H \cap V)$ is finite. On the other hand, since G has Haagerup's property, $H \cap V$ must be relatively compact, and this implies that $H \cap V = 1$. Thus, H is finite.

Motivated by this example, it is easy to exhibit finitely generated groups without the Haagerup Property and do not have infinite subgroups with relative Property (T). For instance, let $n \geq 3$, and q be the quadratic form $\sqrt{2}\, x_0^2 + x_1^2 + x_2^2 + \cdots + x_{n-1}^2$. Let $G(R) = \mathrm{SO}(q)(R) \ltimes R^n$ and write, for any commutative $\mathbf{Q}(\sqrt{2})$-algebra R,

$H(R) = \mathrm{SO}(q)(R)$. Then $\Gamma = G(\mathbf{Z}[\sqrt{2}])$ is such an example. The fact that Γ has no infinite subgroup Λ with relative Property (T) can be seen without making use of the Guentner-Higson-Weinberger Theorem: first observe that $H(\mathbf{Z}[\sqrt{2}])$ is a cocompact lattice in $\mathrm{SO}(n-1,1)$, hence has Haagerup's property. So the projection of Λ in $H(\mathbf{Z}[\sqrt{2}])$ is finite. So, passing to a finite index subgroup if necessary, we can suppose that Λ is contained in the subgroup $\mathbf{Z}[\sqrt{2}]^n$ of $\Gamma = \mathrm{SO}(q)(\mathbf{Z}[\sqrt{2}]) \ltimes \mathbf{Z}[\sqrt{2}]^n$. But then the closure L of Λ in the subgroup $\mathbf{R}^n$ of the amenable group $G(\mathbf{R}) = \mathrm{SO}(q)(\mathbf{R}) \ltimes \mathbf{R}^n$ is not compact, and $(G(\mathbf{R}), L)$ has Property (T). This is a contradiction.

On the other hand, Γ does not have Haagerup's property, since it is a lattice in $G(\mathbf{R}) \ltimes G^\sigma(\mathbf{R})$ (use Proposition 0.26), where σ is the nontrivial automorphism of $\mathbf{Q}(\sqrt{2})$, and $G^\sigma(\mathbf{R}) \simeq \mathrm{SO}(n-1,1) \ltimes \mathbf{R}^n$ does not have Haagerup's property, by Corollary 1.26. Note that Γ, as a cocompact lattice in a connected Lie group, is finitely presented.

We derive some other results with the help of Theorem 1.6.

Proposition 1.35. **There exists a continuous family $(\mathfrak{g}_t)$ of pairwise non-isomorphic real (or complex) Lie algebras satisfying the following properties:**
(i) $\mathfrak{g}_t$ is perfect, and
(ii) the simply connected Lie group corresponding to $\mathfrak{g}_t$ has Property (T).

Note that Proposition 1.35 with only (i) may be of independent interest; we do not know if it had already been observed. On the other hand, it is well-known that there exist continuously many pairwise non-isomorphic complex n-dimensional nilpotent Lie algebras if $n \geq 7$.

Proposition 1.36. **There exists a continuous family (G_t) of pairwise non-isomorphic connected Lie groups with Property (T), and with isomorphic Lie algebras.**

Proof of Proposition 1.35. We must construct a continuous family of connected Lie groups with Property (T) and with perfect and pairwise non-isomorphic Lie algebras.

Consider $\mathfrak{s} = \mathfrak{sp}_{2n}(\mathbf{R})$ $(n \geq 2)$. Let $\mathfrak{v}_i$, $i = 1,2,3,4$ be four nontrivial absolutely irreducible, $\mathfrak{s}$-modules which are pairwise non-isomorphic and all preserve a symplectic form[2]. Then $\mathfrak{v} = \bigoplus_{i=1}^4 \mathfrak{v}_i$ is a full $\mathfrak{s}$-module and $\mathrm{Aut}_\mathfrak{s}(\mathfrak{v}) = \prod_{i=1}^4 \mathrm{Aut}_\mathfrak{s}(\mathfrak{v}_i) \simeq (\mathbf{R}^*)^4$. In particular, $\mathrm{Alt}_\mathfrak{s}(\mathfrak{v})^* \simeq \mathbf{R}^4$ and $\mathrm{Aut}_\mathfrak{s}(\mathfrak{v})$ acts diagonally on it. The action on the 2-Grassmannian, which is 4-dimensional, is trivial on the scalars, so that its orbits are at most 3-dimensional. So there exists a continuous family (P_t) of 2-planes in $\mathrm{Alt}_\mathfrak{s}(\mathfrak{v})^*$ which are in pairwise distinct orbits for the action of $\mathrm{Aut}_\mathfrak{s}(\mathfrak{v})$. By Theorem 1.15, the Lie $\mathfrak{s}$-algebras $\mathfrak{h}(\mathfrak{v})/P_t$ are pairwise non-isomorphic, and so the Lie algebras $\mathfrak{g}_t = \mathfrak{s} \ltimes \mathfrak{h}(\mathfrak{v})/P_t$ are pairwise non-isomorphic. The Lie algebras $\mathfrak{g}_t$ are perfect, and the corresponding Lie groups G_t have Property (T): this immediately follows from Wang's classification [Wan82, Theorem 1.9]. □

[2] **There exist infinitely many such modules, which can be obtained by taking large irreducible components of the odd tensor powers of the standard $2n$-dimensional $\mathfrak{s}$-module.**

Remark 1.37. These examples have 2-nilpotent radical. This is, in a certain sense, optimal, since there exist only countably many isomorphism classes of Lie algebras over $\mathbf{R}$ with abelian radical, and only a finite number for each dimension.

Proof of Proposition 1.36. We must construct a continuous family (H_t) of locally isomorphic, pairwise non-isomorphic connected Lie groups with Property (T). The proof is actually similar to that of Proposition 1.35. Use the same construction, but, instead of taking the quotient G_t by P_t, take the quotient H_t by a lattice Γ_t of P_t. If we take the quotient of H_t by its biggest compact normal subgroup P_t/Γ_t, we obtain G_t. By the proof of Proposition 1.35, the groups G_t are pairwise non-isomorphic. Accordingly, the groups H_t are pairwise non-isomorphic. □

Chapter 2

Relative Kazhdan Property

Throughout this chapter, by **morphism** between topological groups we mean continuous group homomorphisms. If X is a Hausdorff topological space, $Y \subset X$ is **relatively compact** if its closure in X is compact.

2.1 Relative Property (T)

2.1.1 Property (T) relative to subsets

Definition 2.1. Let G be a locally compact group, and X any subset. We say that (G, X) has relative Property (T) if, for every net (φ_i) of continuous normalized positive definite functions which converges to 1 uniformly on compact subsets, the convergence is uniform on X.

We say that (G, X) has relative Property (FH) if every continuous conditionally definite negative function on G is bounded on X.

If φ is a positive definite function on G, then so is $|\varphi|^2$. Thus, the definition of relative Property (T) remains unchanged if we only consider real-valued positive definite functions or even non-negative real-valued positive definite functions.

The definition of relative Property (FH) extends that given in Chapter 0 in the case when X is a subgroup. In the case of relative Property (T), the definitions do not coincide; however we show below that they coincide for locally compact, σ-compact groups. For general locally compact groups, we only know that the definition given here implies that given in Chapter 0. In this chapter, we only deal with the definition given here, which seems more tractable.

Question 2.2 ([AkWa81]). For a σ-compact locally compact group G, are the following equivalent (the implication (1)⇒(2) being trivial):

(1) G is a-T-menable.

(2) For every $X \subset G$, (G, X) has relative Property (FH) if and only if $\overline{X}$ is compact.

Remark 2.3. If G is locally compact but not σ-compact, (1) and (2) of Question 2.2 are not equivalent: (1) is always false, while a characterization of (2) is less clear. For instance, if G is abelian, then (2) is fulfilled, but it is shown in Chapter 6 that if F is a non-nilpotent finite group, then $F^{\mathbf{N}}$, viewed as a discrete group, does not satisfy (2); moreover, if F is perfect, then $F^{\mathbf{N}}$ has Property (FH). On the other hand, being locally finite, these groups are amenable, hence Haagerup.

It is maybe worth comparing Question 2.2 to the following result, essentially due to [GHW05]:

Proposition 2.4. **The locally compact, σ-compact group G is a-T-menable if and only if there exists a family $(\psi_n)_{n\in\mathbf{N}}$ of continuous conditionally negative definite functions on G, such that, for every sequence (M_n) of positive numbers, $\{g \in G,\ \forall n,\ \psi_n(g) \le M_n\}$ is compact.**

Proof: The direct implication is trivial (take any proper function ψ, and $\psi_n = \psi$ for all n). Conversely, suppose the existence of a family (ψ_n) satisfying the condition. Let (K_n) be an increasing sequence of compact subsets of G whose interiors cover G. There exists a sequence (ε_n) such that $\varepsilon_n\psi_n \le 2^{-n}$ on K_n. Set $\psi = \sum_n \varepsilon_n\psi_n$; since the series is convergent uniformly on compact subsets, ψ is well-defined and continuous. Then, for every $M < \infty$, the set $\{\psi \le M\}$ is contained in $\{g,\ \forall n,\ \psi_n \le M/\varepsilon_n\}$, which is, by assumption, compact. □

Definition 2.5 (Akemann, Walter [AkWa76]). A locally compact group has the weak dual Riemann-Lebesgue Property if, for every $\varepsilon, \eta > 0$ and every compact subset K of G, there exists a compact subset Ω of G such that, for every $x \in G - \Omega$, there exists a normalized, real-valued, positive definite[1] function φ on G such that $\varphi(x) \le \eta$ and $\|1 - \varphi\|_\infty^K \le \varepsilon$.

We use the following lemma several times in the sequel.

Lemma 2.6. **Fix $0 < \varepsilon < 1$. Let G be a locally compact group and X a subset. Then (G, X) has relative Property (T) if and only if, for every net (φ_i) of normalized, real-valued continuous positive definite functions which converges to 1 uniformly on compact subsets, eventually $|\varphi_i| > \varepsilon$ on X.**

Proof: The forward implication is trivial. Suppose that (G, X) does not have relative Property (T). Then there exists a net (φ_i) of normalized, real-valued continuous positive definite functions which converges to 1 uniformly on compact subsets, such that $\alpha = \sup_i \inf_{g\in X} \varphi_i(g) < 1$. Then, for some $n \in \mathbf{N}$, $\alpha^n < \varepsilon$. Hence, (φ_i^n) is a net of normalized, continuous positive definite functions which converges to 1 uniformly on compact subsets, but, for no i, $|\varphi_i^n| > \varepsilon$ on X. □

Proposition 2.7. **Let G be a locally compact group. Then G has the weak dual Riemann-Lebesgue Property if and only if every subset X of G such that (G, X) has relative Property (T) is relatively compact.**

Proof: Suppose that G has the weak dual Riemann-Lebesgue Property. Let X be a non-relatively compact subset of G.

[1] **In [AkWa76], the assumption is Re$(\varphi(x)) \le \eta$, since they deal with complex-valued functions, but, since, for every complex-valued function φ, if φ is positive definite, so is $|\varphi|^2$, the definition here is equivalent to theirs.**

For every $i = (K_i, \varepsilon_i, \eta_i)$, with K_i a compact subset of G, and $\varepsilon_i, \eta_i > 0$, there exists a compact subset Ω_i of G, such that, for every $x \notin \Omega_i$, there exists a normalized positive definite function φ on G such that $\varphi(x) \leq \eta_i$ and $\|1 - \varphi\|_\infty^{K_i} \leq \varepsilon_i$. Choose $x_i \in X - \Omega_i$, choose φ_i so that $\varphi_i(x_i) \leq \eta_i$ and $\|1 - \varphi_i\|_\infty^{K_i} \leq \varepsilon_i$. When $i \to \infty$ (meaning that K_i becomes big and ε_i, η_i become small), φ_i tends to 1 uniformly on compact subsets, but the convergence is not uniform on X since $\varphi_i(x_i) < \eta_i$.

Conversely, suppose that G does not have the weak dual Riemann-Lebesgue Property. There exist a compact K_0 and $\varepsilon_0, \eta_0 > 0$ such that, if we write $V_0 = \{\varphi$ positive definite such that $\|1-\varphi\|_\infty^{K_0} \leq \varepsilon_0\}$, then $X = \{x \in G,\ \forall \varphi \in V_0,\ \varphi(x) \geq \eta_0\}$ is not relatively compact. But V_0 is a neighbourhood of 1 for the topology of uniform convergence on compact subsets. Hence, using Lemma 2.6, (G, X) has relative Property (T). □

The following result of [AkWa81] will also be a consequence of Proposition 2.7 and Theorem 2.16.

Theorem 2.8 (Akemann, Walter). **Let G be a locally compact group. If G satisfies (2) of Question 2.2, then G has the weak dual Riemann-Lebesgue Property, and the converse is true if G is σ-compact.**

Question 2.9 ([AkWa81]). For locally compact groups, is the Haagerup Property equivalent to the (a priori weaker) weak dual Riemann-Lebesgue Property?

Remark 2.10. It follows from Theorem 2.8 that Questions 2.2 and 2.9 are equivalent for locally compact, σ-compact groups.

Remark 2.11. If (P) is either the Haagerup Property or the weak dual Riemann-Lebesgue Property, then Property (P) is inherited by closed subgroups, and a locally compact group has Property (P) if and only if all its open, compactly generated subgroup have Property (P). For the Haagerup Property, this is proved in [CCJJV01, Proposition 6.1.1]. For the weak dual Riemann-Lebesgue Property, this follows from Proposition 2.7 and Theorem 2.30.

In particular, Question 2.9 reduces to the compactly generated case.

Definition 2.12. We say that G satisfies the TH alternative if it is either Haagerup, or has a subset X with noncompact closure, such that (G, X) has relative Property (T).

Question 2.9 becomes: does every locally compact group satisfy the TH alternative?

Remark 2.13. Here is an obstruction to the Haagerup Property for a locally compact, compactly generated group G, which does not formally imply the existence of a non-relatively compact subset with relative Property (T). Let w belong to the Stone-Cech boundary $\beta G \smallsetminus G$ of G. Let us say that (G, w) has relative Property (T) if, for every conditionally negative definite function ψ on G, its canonical extension $\tilde{\psi} : \beta G \to \mathbf{R}_+ \cup \{\infty\}$ satisfies $\tilde{\psi}(w) < \infty$.

It is clear that relative Property (T) for (G, w) prevents G from being Haagerup. On the other hand, there is no reason why this should imply the existence of a non-relatively compact subset with relative Property (T).

2.1.2 Various equivalences

Definition 2.14. Let G be a locally compact group and $X \subset G$. Given a unitary representation π of G and $\varepsilon \geq 0$, a (X, ε)-invariant vector for π is a nonzero vector in the representation such that $\|\pi(g)\xi - \xi\| \leq \varepsilon\|\xi\|$ for every $g \in X$.

Definition 2.15. Let G be a locally compact group, X, W subsets, $\varepsilon, \eta > 0$. We say that (W, η) is a ε-Kazhdan pair for (G, X) if, for unitary representation π of G which has a (W, η)-invariant vector, then π has a (X, ε)-invariant vector. Given G, X, W, ε, if such $\eta > 0$ exists, we say that W is a ε-Kazhdan subset for (G, X).

The following result generalizes a result due to Jolissaint [Jol05] when X is a subgroup. The notation $1 \prec \pi$ means that the unitary representation π almost has invariant vectors.

Theorem 2.16. **Let G be a locally compact group, and $X \subset G$. Consider the following properties.**

- **(1)** (G, X) **has relative Property (T).**
- **(2)** **For every $\varepsilon > 0$, there exists a compact ε-Kazhdan subset for (G, X).**
- **(2')** **For some $\varepsilon < \sqrt{2}$, there exists a compact ε-Kazhdan subset for (G, X).**
- **(3)** **For every $\varepsilon > 0$ and every unitary representation π of G such that $1 \prec \pi$, π has a (X, ε)-invariant vector.**
- **(3')** **For some $\varepsilon < \sqrt{2}$ and every unitary representation π of G such that $1 \prec \pi$, π has a (X, ε)-invariant vector.**
- **(4)** (G, X) **has relative Property (FH).**

Then the following implications hold:

$$\begin{array}{ccccccc} (1) & \Longrightarrow & (2) & \Longrightarrow & (3) & & \\ & & \Downarrow & & \Downarrow & & \\ & & (2') & \Longrightarrow & (3') & \Longrightarrow & (4). \end{array}$$

Moreover, if G is σ-compact, then (4)⇒(1), so that they are all equivalent.

Proof: (1)⇒(2) Suppose the contrary. There exists $\varepsilon > 0$ such that, for every (η, K), $\eta > 0$ and $K \subset G$ compact, there exists a unitary representation $\pi_{\eta,K}$ of G which has a (K, η)-invariant unit vector $\xi_{\eta,K}$, but has no (X, ε)-invariant vector. Denote by $\varphi_{\eta,K}$ the corresponding coefficient. Then, when $\eta \to 0$ and K becomes big, $\varphi_{\eta,K}$ converges to 1, uniformly on compact subsets. By relative Property (T), the convergence is uniform on X. It follows that, for some K and some η, $\pi_{\eta,K}$ has a ε-invariant vector, a contradiction.

(2)⇒(2'), (2)⇒(3), (2')⇒(3'), and (3)⇒(3') are immediate.

(3')⇒(4) Let ψ be a conditionally negative definite function on G, and, for $t > 0$, let $(\pi_t, \mathcal{H}_t)$ be the cyclic representation of G associated with the function of positive type $e^{-t\psi}$. Set $\rho_t = \pi_t \otimes \overline{\pi_t}$. Since $\pi_t \to 1_G$ when $t \to 0$, so does ρ_t.

Suppose that ψ is not bounded on X: $\psi(x_n) \to \infty$ for some sequence (x_n) in X. Then we claim that for every $t > 0$ and every $\xi \in \mathcal{H}_t \otimes \overline{\mathcal{H}_t}$, we have $\langle \rho_t(x_n)\xi, \xi\rangle \to 0$ when $n \to \infty$. Equivalently, for every $\xi \in \mathcal{H}_t \otimes \overline{\mathcal{H}_t}$ of norm one, $\|\rho_t(x_n)\xi - \xi\| \to \sqrt{2}$. This is actually established in the proof of [Jol05, Lemma 2.1] (where the assumption that $X = H$ is a subgroup is not used for this statement).

By Lebesgue's dominated convergence Theorem, it follows that if ρ denotes the representation $\bigoplus_{t>0} \rho_t$, then $\langle \rho(x_n)\xi, \xi\rangle \to 0$ for every ξ. In particular, for every $\varepsilon < \sqrt{2}$, ρ has no (X, ε)-invariant vector. Since $1 \prec \rho$, this contradicts (3').

(4)⇒(1) The proof is a direct adaptation of that of the analogous implication in [AkWa81, Theorem 3]. We suppose that G is σ-compact and that (G, X) has relative Property (FH). Let (φ_i) be a net of nonnegative real-valued positive definite normalized functions on G which converges to 1 uniformly on compact subsets. Suppose by contradiction that the convergence is not uniform on X. Then there exists $\varepsilon > 0$ such that we can extract a sequence φ_n, and pick a sequence (x_n) of elements of X, such that $1 - \varphi_n(x_n) \geq \varepsilon$ for all n.

Let (K_n) be an increasing sequence of compact subsets of G whose interiors cover G. By an immediate induction, we can extract (k_n) so that $\sup_{g \in K_n}(1 - \varphi_{k_n}(g)) \leq 4^{-n}$. Now set $\psi(g) = \sum_n 2^n(1 - \varphi_{k_n}(g))$. Since the series converges uniformly on compact sets, ψ is well-defined, and continuous. Then, by [BHV05, Proposition C.2.3], ψ is a conditionally negative definite function on G. Moreover, $\psi(x_{k_n}) \geq 2^n(1 - \varphi_{k_n}(x_{k_n})) \geq 2^n\varepsilon$, so that ψ is not bounded on X, a contradiction. □

Remark 2.17. There is a direct proof of (1)⇒(4). Suppose that (G, X) has relative Property (T). Let ψ be conditionally negative definite function on G. By Schönberg's Theorem, $e^{-t\psi}$ is positive definite for all $t > 0$, and tends to 1 when $t \to 0$, uniformly on compact subsets. By relative Property (T), the convergence is uniform on X. This easily implies that ψ is bounded on X.

Remark 2.18. When $X = H$ is a subgroup, we retrieve a result of [Jol05]. Note that, in this case, by a well-known application of the "Lemma of the centre" [BHV05, Lemma 2.2.7], Condition (2') of Theorem 2.16 can be chosen with $\varepsilon = 0$, i.e. becomes: for every unitary representation of G such that $1 \prec \pi$, there exists a nonzero vector fixed by H.

Remark 2.19. When G is not σ-compact, whether the implication (3')⇒(1) holds is not known, except when X is a normal subgroup [Jol05]. On the other hand, even if $X = G$, (4)⇒(3') does not hold for general locally compact groups, even discrete (see Chapter 6).

2.1.3 Relative Property (T) can be read on irreducible representations.

The following lemma, due to Choquet (unpublished), is proved in [Dix69, B.14 p. 355].

Lemma 2.20. **Let K be a compact, convex subset of a locally convex space E. Let x be an extremal point of K. Let $\mathcal{W}$ be the set of all open half-spaces of E which contain x. Then $\{W \cap K,\ W \in \mathcal{W}\}$ is a neighbourhood basis of x in K.**

Denote $\mathcal{P}(G)$ (resp. $\mathcal{P}_1(G)$) (resp. $\mathcal{P}_{\leq 1}(G)$) the set of all (complex-valued) positive definite function φ on G (resp. such that $\varphi(1) = 1$) (resp. such that $\varphi(1) \leq 1$).

Recall that $\varphi \in \mathcal{P}(G)$ is **pure** if it satisfies one of the two equivalent conditions: (i) φ is associated to an irreducible representation; (ii) φ belongs to an extremal axis of the convex cone $\mathcal{P}(G)$.

Theorem 2.21. **Let G be a locally compact group and X a subset. The following are equivalent:**

(i) (G, X) has relative Property (T).

(ii) For every net of continuous, normalized pure **positive definite functions on G which converges to 1, the convergence is uniform on X.**

Proof: (i)$\Rightarrow$(ii) is trivial; suppose that G satisfies (ii).

In the space $L^\infty(G) = L^1(G)^*$, endowed with the weak* topology, let $\mathcal{W}$ be the set of all open half-spaces of $L^\infty(G)$ which contain the constant function 1. Finally set $K = \mathcal{P}_{\leq 1}(G)$.

We first recall Raikov's Theorem [Dix69, Théorème 13.5.2]: on $\mathcal{P}_1(G)$, the weak* topology coincides with the topology of uniform convergence on compact subsets.

Let $\mathcal{L}$ be the set of all continuous linear forms u on $L^\infty(G)$ such that $u(1) = 1$ and $u|_K \leq 1$. Since K is convex and compact for the weak*-topology, by Lemma 2.20, $\{\{u > 1 - \varepsilon\} \cap K,\ u \in \mathcal{L}, \varepsilon > 0\}$ is a basis of open neighbourhoods of 1 in $\mathcal{P}_{\leq 1}(G)$.

Hence, by (ii), and using Raikov's Theorem, for every $1 > \varepsilon > 0$, there exists $u \in \mathcal{L}$ and $\eta > 0$ such that, for every pure $\varphi \in \mathcal{P}_1(G)$, $u(\varphi) > 1-\eta$ implies $\varphi \geq 1-\varepsilon$ on X.

Let $\varphi = \sum \lambda_i \varphi_i$ be a convex combination of continuous, normalized positive definite functions φ_i associated to irreducible representations. Suppose that $u(\varphi) > 1 - \eta\varepsilon$.

Decompose φ as $\sum \lambda_j \varphi_j + \sum \lambda_k \varphi_k$, where $u(\varphi_j) > 1 - \eta$ and $u(\varphi_k) \leq 1 - \eta$.

Then

$$1-\eta\varepsilon \ < \ u(\varphi) \ = \ \sum \lambda_j u(\varphi_j)+\sum \lambda_k u(\varphi_k) \ \leq \ \sum \lambda_j+\sum \lambda_k(1-\eta) \ = \ 1-\eta\sum \lambda_k,$$

so that $\sum \lambda_k \leq \varepsilon$. Hence, on X, we have

$$\varphi \ = \ \sum \lambda_j \varphi_j + \sum \lambda_k \varphi_k \ \geq \ \sum \lambda_j (1-\varepsilon) - \sum \lambda_k \ \geq \ \sum \lambda_j - 2\varepsilon \ \geq \ 1 - 3\varepsilon.$$

Set $K_{u,\varepsilon\eta} = \{\varphi \in K,\ u(\varphi) > 1 - \varepsilon\eta\}$, and $K_{cp} = \{\varphi \in K,\ \varphi$ is a convex combination of continuous, normalized pure positive definite functions on $G\}$. By [BHV05, Theorem C.5.5], K_{cp} is weak* dense in $\mathcal{P}_1(G)$. Since $K_{u,\varepsilon\eta}$ is open in K, this implies that $K_{cp} \cap K_{u,\varepsilon\eta}$ is weak*-dense in $\mathcal{P}_1(G) \cap K_{u,\varepsilon\eta}$. By Raikov's Theorem, it is also dense for the topology of uniform convergence on compact subsets. Hence, since for all $\varphi \in K_{cp} \cap K_{u,\varepsilon\eta}$, $\varphi \geq 1 - 3\varepsilon$ on X, the same holds for all $\varphi \in \mathcal{P}_1(G) \cap K_{u,\varepsilon\eta}$. $\square$

Theorem 2.22. **Let G be a locally compact, σ-compact group. The following are equivalent.**

(1) (G,X) has relative Property (T).

(2) For every $\varepsilon > 0$, there exists a neighbourhood V of 1_G in $\hat{G}$ such that every $\pi \in V$ has a (X,ε)-invariant vector.

(2') For some $\varepsilon < \sqrt{2}$, there exists a neighbourhood V of 1_G in $\hat{G}$ such that every $\pi \in V$ has a (X,ε)-invariant vector.

Proof: (2)⇒(2') is trivial.

(2')⇒(1). By a result of Kakutani and Kodaira [Com84, Theorem 3.7], there exists a compact normal subgroup K of G such that G/K is second countable. So we can suppose that G is second countable. Let $1 \prec \pi$. Arguing as in [DeKi68, proof of Lemme 1], π contains a nonzero subrepresentation which is entirely supported by V. We conclude by Lemma 2.24 below that π has a ε'-invariant vector, where $\varepsilon < \varepsilon' < \sqrt{2}$. This proves that (3') of Theorem 2.16 is satisfied.

(1)⇒(2). Let π_i be a net of irreducible unitary representations which converges to the trivial representation, and fix $\varepsilon > 0$. By [BHV05, Proposition F.2.4], there exists a net of normalized positive definite functions (φ_i), such that φ_i is associated to π_i for all i, and such that φ_i tends to 1 uniformly on compact subsets. By relative Property (T), the convergence is uniform on X, and it follows that eventually π_i has a (X,ε)-invariant vector. □

Remark 2.23. The special case when X is a subgroup is claimed without proof in [HaVa89, Chap. 1, 18.].

Let G be a second countable, locally compact group, and $X \subset G$. Let (Z,μ) be measured space $0 < \mu(Z)$, with μ σ-finite. Let $(H_z)_{z\in Z}$ be a measurable field of Hilbert spaces [Dix69, A 69], and denote by Γ the space of measurable vector fields. Let (π_z) be a field of unitary representations, meaning that $z \mapsto \pi_z(g)x(z)$ is measurable, for every $x \in \Gamma$, $g \in G$. Recall that, by definition, there exists a sequence (x_n) in Γ such that, for every $z \in Z$, the family $(x_n(z))$ is total. Set $\pi = \int^{\oplus} \pi_z d\mu(z)$.

Lemma 2.24. **Fix $\varepsilon > 0$. Suppose that, for every z, π_z has a (X,ε)-invariant vector. Then π has a (X,ε')-invariant vector for every $\varepsilon' > \varepsilon$.**

Proof: Fix $0 < \eta < 1$. First note that, upon replacing the family (x_n) by the family of all its rational combinations, we can suppose that, for every $z \in Z$ and every $v \in H_z$ of norm one, there exists n such that $\|v - x_n(z)\| \leq \eta$. In particular, if v is (X,ε)-invariant, then, for all $x \in X$, $\|\pi_z(x)x_n(z) - x_n(z)\| \leq \varepsilon + 2\eta$ and $\|x_n(z)\| \geq \|1-\eta\|$, so that $x_n(z)$ is $(X,(\varepsilon+2\eta)/(1-\eta))$-invariant.

Now define, for all $n \geq 0$:

$$A_n = \{z \in Z,\ 1-\eta \leq \|x_n(z)\| \leq 1+\eta,\ x_n(z) \text{ is } (X,(\varepsilon+2\eta)/(1-\eta))\text{-invariant}\}.$$

We have $\bigcup A_n = Z$ by the remark above. Using that X is separable, it is immediate that A_n is measurable for every n. Accordingly, there exists n_0 such that

$\mu(A_{n_0}) > 0$. Using that μ is σ-finite, there exists a measurable subset $B \subset A_{n_0}$ such that $0 < \mu(B) < \infty$. Define ξ as the field

$$z \mapsto \begin{cases} x_{n_0}(z), & z \in B; \\ 0, & \text{otherwise.} \end{cases}$$

Then it is clearly measurable, and

$$\|\xi\|^2 = \int_B \|x_{n_0}(z)\|^2 d\mu(z) \geq (1-\eta)^2 \mu(B),$$

and, for every $g \in X$,

$$\|\pi(g)\xi - \xi\|^2 = \int_B \|\pi_z(g)x_{n_0}(z) - x_{n_0}(z)\|^2 d\mu(z) \leq ((\varepsilon + 2\eta)^2(1+\eta)^2/(1-\eta)^2)\mu(B).$$

It follows that $\xi \neq 0$ and is $(X, (\varepsilon+2\eta)(1+\eta)/(1-\eta)^2)$-invariant. Finally, for every $\varepsilon' > \varepsilon$, we can choose η sufficiently small so that $(\varepsilon + 2\eta)(1+\eta)/(1-\eta)^2 \leq \varepsilon'$. □

2.1.4 Some stability results

We note for reference the following immediate but useful result:

Proposition 2.25. **Let G be locally compact and $X_1, \dots, X_n$ be subsets. Denote by $X_1 \dots X_n$ the pointwise product $\{x_1 \dots x_n,\ (x_1, \dots, x_n) \in X_1 \times \cdots \times X_n\}$. Suppose that, for every i, (G, X_i) has relative Property (T) (resp. (FH)).**
Then $(G, X_1 \dots X_n)$ has relative Property (T) (resp. (FH)).

Proof: It suffices to prove the case when $n = 2$, since then the result follows by induction. For the case of Property (FH), this follows from the inequality, for all conditionally negative definite functions ψ: $\psi(gh)^{1/2} \leq \psi(g)^{1/2} + \psi(h)^{1/2}$. For the case of Property (T), a similar inequality holds since, if φ is normalized positive definite, then $1 - |\varphi|^2$ is conditionally negative definite. □

Example 2.26. 1) If $G_1, \dots, G_n$ are locally compact groups, and $X_i \subset G_i$, and if (G_i, X_i) has relative Property (T) (resp. (FH)) for every i, then $(\prod G_i, \prod X_i)$ also has relative Property (T) (resp. (FH)).

2) Fix $n \geq 3$, let A a topologically finitely generated locally compact commutative ring, and set $G = \mathrm{SL}_n(A)$. Denote by $V_{n,m}$ the elements in G which are products of $\leq m$ elementary matrices. Then it follows from [Sha99p, Corollary 3.5] that $(G, V_{n,m})$ has relative Property (T) for all m. It is not known whether, for such A, there exists m such that $V_{n,m} = E_n(A)$, the subgroup generated by elementary matrices; this seems to be an open question whenever A has Krull dimension ≥ 2, e.g. $A = \mathbf{Z}[X]$, or $A = \mathbf{F}_p[X, Y]$.

The following proposition is trivial.

Proposition 2.27. **Let G be a locally compact group, H a closed subgroup, and $Y \subset X \subset H$. If (H, X) has relative Property (T) (resp. (FH)), then so does (G, Y).** □

Proposition 2.28 (Stability by extensions). **Let G be a locally compact group, N a closed normal subgroup, and $X \subset G$. Denote by $p : G \to G/N$ the projection.**

If (G, N) and $(G/N, p(X))$ have relative Property (T) (resp. (FH)), then so does (G, X).

Proof: The assertion about relative Property (FH) is immediate; that about relative Property (T) is straightforward, using [BHV05, Lemma B.1.1]: for every compact subset K of G/N, there exists a compact subset $\tilde{K}$ of G such that $p(\tilde{K}) = K$. □

2.1.5 Relative Property (T) and compact generation

It is well-known that a locally compact group with Property (T) is compactly generated. We generalize this result. The following lemma is such a generalization, but we are going to use it to prove something stronger.

Lemma 2.29. **Let G be a locally compact group, and $X \subset G$ such that (G, X) has relative Property (T). Then X is contained in an open, compactly generated subgroup of G.**

Proof: For every open, compactly generated subgroup Ω of G, let λ_Ω be the quasi-regular representation of G on $\ell^2(G/\Omega)$. Let $\delta_\Omega \in \ell^2(G/\Omega)$ be the Dirac function on G/Ω. Let φ_Ω be the corresponding coefficient. Then φ_Ω tends to 1, uniformly on compact subsets, when Ω becomes big. Hence, since the convergence is uniform on X, there exist Ω such that $|1 - \varphi_\Omega| < 1$ on X. This means that, for all $g \in X$, $0 < \langle \pi_\Omega(g)\delta_\Omega, \delta_\Omega\rangle = \langle \delta_{g\Omega}, \delta_\Omega\rangle \in \{0, 1\}$, so that $\delta_\Omega = \delta_{g\Omega}$ for all $g \in X$, that is, $g \in \Omega$. Hence, $X \subset \Omega$. □

The following theorem shows that, in a certain sense, all the information about relative Property (T) lies within compactly generated subgroups.

Theorem 2.30. **Let G be a locally compact group, and $X \subset G$ a subset. Then (G, X) has relative Property (T) if and only if there exists an open, compactly generated subgroup H such that $X \subset H$ and (H, X) has relative Property (T).**

Proof: By Lemma 2.29, there exists $\Omega \supset X$ an open, compactly generated subgroup. Let (K_i) be an increasing net of open, relatively compact subsets, covering G, and denote by H_i the subgroup generated by K_i. We can suppose that $\Omega \subset H_i$ for all i.

Suppose by contradiction that, for every i, (H_i, X) does not have Property (T). Then, using Lemma 2.6, for all i and all n, there exists a normalized, continuous positive definite function $\varphi_{i,n}$ on H_i, such that $\varphi_{i,n} \geq 1 - 2^{-n}$ on K_i and $\inf_X \varphi_{i,n} \leq 1/2$. Since H_i is open in G, we can extend $\varphi_{i,n}$ to all of G, by sending the complement of H_i to 0. It is clear that the net $(\varphi_{i,n})$ tends to 1 uniformly on compact subsets of G, but $\inf_X \varphi_{i,n} \leq 1/2$. This contradicts that (G, X) has Property (T). □

2.1.6 H-metric

First recall that a length function on a group G is a function $L : G \to \mathbf{R}_+$ satisfying the subadditivity condition $L(gh) \leq L(g) + L(h)$ for all g, h, and such that $L(1) = 0$

and $L(g) = L(g^{-1})$ for all $g \in G$. A length function defines a (maybe non-separated) left-invariant metric on G by $d(g,h) = L(g^{-1}h)$.

Observe that if L_1, L_2 are two length functions, then so is $L = \max(L_1, L_2)$. Indeed, we can suppose $L_1(gh) \geq L_2(gh)$. Then $L(gh) = L_1(gh) \leq L_1(g) + L_1(h) \leq L(g) + L(h)$.

Also observe that a pointwise limit of length functions is a length function. If follows that the upper bound of a family of length functions, provided that it is everywhere finite, is a length function.

Now let G be a locally compact, compactly generated group, and K a relatively compact, open generating subset.

Define Ψ_K as the upper bound of all (continuous, real-valued) conditionally negative definite functions ψ such that $\psi \leq 1$ on K.

Recall that if ψ is a real-valued conditionally negative definite function, then $\psi^{1/2}$ is a length function. It follows that $\Psi_K^{1/2}$ is a length function. It is easily checked that it defines a separated metric on G, whose closed balls are closed (for the initial topology). We call it the H-metric.

It is easy to observe that if K and L are two open, relatively compact generating subsets, then there exist constants $A, A' > 0$ such that $A\Psi_K \leq \Psi_L \leq A'\Psi_K$. Accordingly, the identity map defines a bi-Lipschitz map between these two metrics, and the choice of K is not essential at all.

Proposition 2.31. **Let G be a locally compact, compactly generated group, and X a subset. Then (G, X) has relative Property (T) if and only if X is bounded for the H-metric.**

Proof: First recall that, since G is σ-compact, relative Property (T) and relative Property (FH) are equivalent by Theorem 2.16.

If X is bounded for the H-metric, and ψ is a (continuous, real-valued) conditionally negative definite function on X, then, for some constant $\alpha > 0$, $\alpha\psi \leq 1$ on K. So $\psi \leq \alpha^{-1}\Psi_K$ which is bounded on X, and thus (G, X) has relative Property (FH).

Conversely, suppose that X is not bounded for the H-metric. Then there exist a sequence of (continuous, real-valued) conditionally negative definite functions ψ_n, bounded by 1 on K, and a sequence x_n of X such that $\psi_n(x_n) \geq 4^n$. Set $\psi = \sum 2^{-n}\psi_n$. Since the convergence is uniform on compact subsets, ψ is a well-defined continuous conditionally negative definite function on G, and $\psi(x_n) \geq 2^n$, so that ψ is not bounded on X, and (G, X) does not have relative Property (FH). □

Corollary 2.32. **Let G be a locally compact, compactly generated group.**

1) G has Property (T) is and only if it is bounded for the H-metric.

2) G has the weak dual Riemann-Lebesgue Property if and only if it is proper for the H-metric (that is, the balls for the H-metric are compact for the initial topology).

It is maybe interesting comparing the H-metric with the word metric (relative to any compact generating set). A result in this direction was communicated to us by V. Lafforgue: if G does not have Property (T), if L_K denotes the word length with respect to the compact generating set K, and $H_K = \Psi_K^{1/2}$ is the length in the H-metric, then, for every $0 < C < 1/2$, there exists a sequence (x_n) in G such that, for all n, $L_K(x_n) \leq n$ and $H_K(x_n) \geq C\sqrt{n}$.

2.2 Relative Property (T) in Lie groups and p-adic algebraic groups

2.2.1 Preliminaries

Given a locally compact group G, we can naturally raise the problem of determining for which subsets X the pair (G, X) has relative Property (T).

Here is a favourable case, where the problem is completely solved.

Lemma 2.33. **Let G be a locally compact group, and N a normal subgroup such that (G, N) has relative Property (T) and G/N is Haagerup. Let X be any subset of G. Then (G, X) has relative Property (T) if and only if the image of X in G/N is relatively compact.**

Proof: The condition is clearly necessary, since relative Property (T) is inherited by images. Conversely, if the image of X in G/N is relatively compact, there exists a compact subset K of G such that X is contained in $KN = \{kn,\ (k, n) \in K \times N\}$.

Let ψ be a continuous, conditionally negative definite function on G. Then ψ is bounded on N and on K, hence on KN, hence on X. This proves that (G, X) has relative Property (FH). In view of Theorem 2.16, this is sufficient if G is σ-compact. Actually, we can reduce to this case: indeed, by Theorem 2.30, there exists an open, compactly generated subgroup H of G, which contains N and can be supposed to contain K, such that (H, N) has relative Property (T). □

Recall the key result, due to Shalom [Sha99t, Theorem 5.5] (see also [BHV05, Section 1.4]).

Proposition 2.34. **Let G be a locally compact group and N a closed normal abelian subgroup. Assume that the only mean on the Borel subsets of the Pontryagin dual $\widehat{N} = \mathrm{Hom}(N, \mathbf{R}/\mathbf{Z})$, invariant under the action of G by conjugation, is the Dirac measure at zero. Then the pair (G, N) has relative Property (T).**

This result allows to prove relative Property (T) for certain normal abelian subgroups. Since we also deal with nilpotent subgroups, we use the following proposition, which generalizes [CCJJV01, Proposition 4.1.4].

Proposition 2.35. **Let G be a locally compact, σ-compact group, N a closed subgroup, and let Z be a closed, central subgroup of G contained in $\overline{[N, N]}$. Suppose that every morphism of N into a compact Lie group has an abelian image.**

Suppose that the pair $(G/Z, N/Z)$ has Property (T). Then (G, N) has Property (T).

Proof: It suffices to show that (G, Z) has relative Property (T). Indeed, since the pairs (G, Z) and $(G/Z, N/Z)$ have relative Property (T), it then follows by Proposition 2.28 that (G, N) has relative Property (T).

We use an argument similar to the proof of [Wan82, Lemma 1.6]. To show that (G, Z) has relative Property (T), we use the characterization by nets of irreducible representations (see Theorem 2.22). Let π_i be a net of irreducible representations of G converging to the trivial representation: we must show that eventually π_i factors through Z. Let $\overline{\pi}_i$ be the contragredient representation of π_i. Then $\pi_i \otimes \overline{\pi_i}$ converges to the trivial representation. By irreducibility, π_i is scalar in restriction to Z, hence $\pi_i \otimes \overline{\pi_i}$ is trivial on Z, so factors through G/Z. Since $(G/Z, N/Z)$ has Property (T), the restriction to N of $\pi_i \otimes \overline{\pi_i}$ eventually contains the trivial representation. By a standard argument [BHV05, Appendix 1], this means that $\pi_i|_N$ eventually contains a finite-dimensional representation ρ_i.

Remark that $\overline{\rho_i(N)}$ is a compact Lie group; so it is, by assumption, abelian. This means that $[N, N]$ acts trivially; hence Z does so as well: ρ_i is trivial on Z. Hence, for large i, π_i has nonzero Z-invariants vectors; by irreducibility, π_i is trivial on Z. Accordingly (G, Z) has Property (T). □

We shall use the following well-known result of Furstenberg [Fur76].

Theorem 2.36 (Furstenberg). **Let $\mathbf{K}$ be a local field, V a finite dimensional K-vector space. Let $G \subset \mathrm{PGL}(V)$ be a Zariski connected (but not necessarily Zariski closed) subgroup, whose closure is not compact. Suppose that G preserves a probability measure μ on the projective space $P(V)$. Then there exists a proper projective G-invariant subspace $W \subsetneq P(V)$ such that $\mu(W) = 1$.**

Remark 2.37. Observe that a subgroup of $\mathrm{PGL}(V)$ preserves an invariant mean on $P(V)$ if and only if it preserves a probability: indeed, a mean gives rise to a normalized positive linear form on $L^\infty(P(V))$, and restricts to a normalized positive linear form on $C(P(V))$, defining a probability.

We say that a topological group G is discompact[2] if there is no nontrivial morphism of G to a compact group.

Remark 2.38. If G is a discompact locally compact group, then it has trivial abelianization. Indeed, it follows that its abelianization is also discompact, so has trivial Pontryagin dual, so it trivial by Pontryagin duality.

Corollary 2.39. **Let G be a discompact locally compact group. Let V be a finite-dimensional vector space over K, and let $G \to \mathrm{GL}(V)$ be any continuous representation. Then G preserves a probability on $P(V)$ if and only if G has a nonzero fixed point on V.**

Proof: If $G(\mathbf{K})$ fixes a point $x \in V - \{0\}$, then it fixes the Dirac measure at the point $\mathbf{K}x$ of $P(V)$. Conversely, suppose that G preserves a probability on $P(V)$. Let W be a G-stable projective subspace, minimal among those such that $\mu(W) \neq 0$. Then, by Theorem 2.36, the image of G in $\mathrm{PGL}(W)$ is relatively compact, hence trivial since G is discompact. Accordingly, G has a fixed point in $P(V)$, namely every element of W. Thus G fixed a line Kx in V. The action of G on Kx defines a morphism $G \to \mathbf{K}^*$; in view of Remark 2.38, G acts trivially on Kx, hence fixes x. □

Example 2.40. (1) Let G be a simply connected, simple group over $\mathbf{K}$, of positive $\mathbf{K}$-rank. Then $G(\mathbf{K})$ is discompact. Indeed, $G(\mathbf{K})$ is generated by elements whose

[2] **This is often called “minimally almost periodic”, but we prefer the terminology “discompact”, introduced in [Sha99t].**

conjugacy classes contain 1 in their closure: this follows from the following observations: $G(\mathbf{K})$ is simple [Mar91, Chap I, Theorem 1.5.6 and Theorem 2.3.1(a)], and there exists a subgroup of G isomorphic to either $\mathrm{SL}_2(\mathbf{K})$ or $\mathrm{PSL}_2(\mathbf{K})$ [Mar91, Chap I, Proposition 1.6.3]. Accordingly, every morphism of $G(\mathbf{K})$ into a compact group has trivial image.

(2) Let G be a connected, noncompact, simple Lie group. Then G is discompact. Indeed, such a group is generated by connected subgroups locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$, hence is generated by elements whose conjugacy class contains 1.

The following proposition is essentially due to M. Burger [Bur91, Proposition 7] (see also [Val94, Proposition 2.3]).

Proposition 2.41. **Let V be a finite-dimensional space over a local field $\mathbf{K}$. Let G be any topological group, and $\rho : G \to \mathrm{GL}(V)$ a continuous representation. Then $(G \ltimes V, V)$ has relative Property (T) if and only if G preserves no probability on $P(V^*)$. In particular, if G is discompact, then $(G \ltimes V, V)$ has relative Property (T) if and only if G fixes no point in V^*.**

Proof: Suppose that $(G \ltimes V, V)$ does not have Property (T). We can suppose that G is endowed with the discrete topology, so that $G \ltimes V$ is locally compact. By Proposition 2.34, G preserves a mean on $V^* - \{0\}$ (recall that there is a $\mathrm{GL}(V)$-equivariant identification between the linear dual V^* and the Pontryagin dual $\hat{V}$). So G preserves a mean on $P(V^*)$. Since $P(V^*)$ is compact, this implies that G also preserves a probability on $P(V^*)$ (see Remark 2.37).

Conversely, suppose, by contradiction that G preserves a probability on $P(V^*)$ and $(G \ltimes V, V)$ has relative Property (T). By Theorem 2.36, the finite index subgroup G_0 of G (its unit component in the inverse image of the Zariski topology from $\mathrm{GL}(V)$) preserves a nonzero subspace $W \subset V^*$, such that the image of the morphism $G_0 \to \mathrm{PGL}(W)$ has compact closure. Since W is a subspace of V^*, W^* is a quotient of V. By Corollary 4.1(2) in [Jol05], $(G_0 \ltimes V, V)$ has relative Property (T), and so has $(G_0 \ltimes W^*, W^*)$. This implies that $(\overline{\rho(G_0)} \ltimes W^*, W^*)$ also has relative Property (T). But $\overline{\rho(G_0)} \ltimes W^*$ is amenable, so that W^* is compact, and this is a contradiction.

The second assertion follows from Corollary 2.39. □

Remark 2.42. It is worth noting that, in Proposition 2.41, and in view of Corollary 2.39, relative Property (T) for $(G \ltimes V, V)$ only depends on the closure (for the ordinary topology) of the image of G in $\mathrm{PGL}(V)$.

2.2.2 Relative Property (T) in algebraic groups over local fields of characteristic zero

We denote by $\mathbf{K}$ a local field of characteristic zero. Here is the main lemma of this subsection.

Lemma 2.43. **Let G be a linear algebraic $\mathbf{K}$-group, which decomposes as $S \ltimes R$, where S is semisimple and $\mathbf{K}$-isotropic, and R is unipotent.**

Suppose that $[S, R] = R$. Then $(G(\mathbf{K}), R(\mathbf{K}))$ has relative Property (T).

Proof: Upon replacing S by its universal cover, we can suppose that S is simply connected. We then argue by induction on the dimension of R. If the dimension is zero, there is nothing to prove; suppose $R \neq 1$. Let Z be the last nonzero term of its descending central series.

First case: Z is central in G. The hypothesis $[S, R] = R$ implies that R is not abelian. Hence $Z \subset [R, R]$, so that $Z(\mathbf{K}) \subset [R, R](\mathbf{K})$. By [BoTi, Lemma 13.2] $[R, R](\mathbf{K}) = [R(\mathbf{K}), R(\mathbf{K})]$, so that $Z(\mathbf{K}) \subset [R(\mathbf{K}), R(\mathbf{K})]$. We must check that the hypotheses of Proposition 2.35 are fulfilled. Let W be a compact Lie group, and $R(\mathbf{K}) \to W$ a morphism with dense image: we must show that W is abelian. Since $R(\mathbf{K})$ is solvable, the connected component W_0 is abelian. Moreover, $R(\mathbf{K})$ is divisible, so W/W_0 is also divisible; this implies $W = W_0$. Accordingly, by Proposition 2.35, since $(G(\mathbf{K})/Z(\mathbf{K}), R(\mathbf{K})/Z(\mathbf{K}))$ has relative Property (T) by induction hypothesis, it follows that $(G(\mathbf{K}), R(\mathbf{K}))$ has relative Property (T).

Second case: Z is not central in G. Set $N = [S, Z]$. Then $[S, N] = N$. By Proposition 2.41 and in view of Example 2.40(1), $(G(\mathbf{K}), N(\mathbf{K}))$ has relative Property (T). By the induction assumption, $((G/N)(\mathbf{K}), (R_u/N)(\mathbf{K}))$, which coincides with $(G(\mathbf{K})/N(\mathbf{K}), R_u(\mathbf{K})/N(\mathbf{K}))$, has relative Property (T). Hence $(G(\mathbf{K}), R_u(\mathbf{K}))$ has relative Property (T). □

Let G be a linear algebraic group over $\mathbf{K}$. We denote by R_u its unipotent radical, and L a reductive Levi factor (so that $G_0 = L \ltimes R_u$). We decompose L as an almost product $L_m L_{nm}$, where L_m (resp. L_{nm}) includes the centre of L, and the simple factors of rank zero (resp. includes the simple factors of positive rank)[3].

Let R be the radical of G, S a Levi factor, and decompose it as $S_c S_{nc}$, where S_c (resp. S_{nc} is the sum of all factors of rank 0 (resp. of positive rank)).

If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{h}_1$, $\mathfrak{h}_2$ are two subspaces, we denote by $[\mathfrak{h}_1, \mathfrak{h}_2]$ (resp. $[\mathfrak{h}_1, \mathfrak{h}_2]_v$) the Lie algebra (resp. the subspace) generated by the $[h_1, h_2]$, $(h_1, h_2) \in \mathfrak{h}_1 \times \mathfrak{h}_2$.

Lemma 2.44. **For every Levi factors L, S of respectively R_u and R, $[L_{nm}, R_u] = [S_{nc}, R]$, and this is a K-characteristic subgroup of G.**

Proof: We can work within the Lie algebra. We first justify that $[\mathfrak{l}_{nm}, \mathfrak{r}_u]$ is an ideal: indeed,

$$[\mathfrak{l}, [\mathfrak{l}_{nm}, \mathfrak{r}_u]] \subset [[\mathfrak{l}, \mathfrak{l}_{nm}], \mathfrak{r}_u] + [\mathfrak{l}_{nm}, [\mathfrak{l}, \mathfrak{r}_u]] \subset [\mathfrak{l}_{nm}, \mathfrak{r}_u]$$

since $[\mathfrak{l}, \mathfrak{l}_{nm}] \subset \mathfrak{l}_{nm}$ and $[\mathfrak{l}, \mathfrak{r}_u] \subset \mathfrak{r}_u$. On the other hand,

$$\begin{aligned} [\mathfrak{r}_u, [\mathfrak{l}_{nm}, \mathfrak{r}_u]] \; &= \; [\mathfrak{r}_u, [\mathfrak{l}_{nm}, [\mathfrak{l}_{nm}, \mathfrak{r}_u]]] \\ \subset \; [\mathfrak{l}_{nm}, [\mathfrak{r}_u, [\mathfrak{l}_{nm}, \mathfrak{r}_u]]] \; &+ \; [[\mathfrak{l}_{nm}, \mathfrak{r}_u], [\mathfrak{l}_{nm}, \mathfrak{r}_u]] \; \subset \; [\mathfrak{l}_{nm}, \mathfrak{r}_u]. \end{aligned}$$

It follows that $[L_{nm}, R_u]$ is a normal subgroup of G. By [BoSe64, (5.1)], the $\mathbf{K}$-conjugacy class of L does not depend of the choice of L. So the same thing holds for L_{nm} (which is $\mathbf{K}$-characteristic in L). Accordingly, $[L_{nm}, R_u]$ is a $\mathbf{K}$-characteristic subgroup of G.

[3] $(n)m$ **stands for (non-)amenable.**

Now, since S_{nc} is a reductive $\mathbf{K}$-subgroup of G, again using [BoSe64, (5.1)], upon $\mathbf{K}$-conjugating, we can suppose that $S_{nc} \subset L$, so that finally $S_{nc} = L_{nm}$, and $R = L_r \ltimes R_u$, where L_r is the unit component of centre of L. Since $[L_{nc}, L_r] = 1$, we obtain $[S_{nc}, R] = [L_{nc}, R] = [L_{nc}, R_u]$. □

Let S_{nh} be the sum of all simple factors H of S_{nc} such that $H(\mathbf{K})$ is not Haagerup (equivalently: has Property (T)): these are factors of rank ≥ 2, and also, when $\mathbf{K} = \mathbf{R}$, factors locally isomorphic to $\mathrm{Sp}(n,1)$ or $F_{4(-20)}$.

Definition 2.45. Define R_T as the $\mathbf{K}$-subgroup $S_{nh}[S_{nc}, R]$ of G.

Theorem 2.46. ***R_T is a $\mathbf{K}$-characteristic subgroup of G, $G(\mathbf{K})/R_T(\mathbf{K})$ is Haagerup, and $(G(\mathbf{K}), R_T(\mathbf{K}))$ has relative Property (T).***

Proof: It follows from Lemma 2.44 that $[S_{nc}, R]$ is unipotent. Consider the $\mathbf{K}$-subgroup $W = S_{nc}[S_{nc}, R]$ of G. Applying Lemma 2.43 to W, we obtain that $(G(\mathbf{K}), [S_{nc}, R](\mathbf{K}))$ has relative Property (T). Since $(G(\mathbf{K}), S_{nh}(\mathbf{K}))$ also has relative Property (T) and $[S_{nc}, R](\mathbf{K})$ is a normal subgroup, we obtain that $(G(\mathbf{K}), R_T(\mathbf{K}))$ has relative Property (T) by Proposition 2.25.

To show that R_T is a $\mathbf{K}$-characteristic subgroup, we can work modulo the subgroup $[S_{nc}, R]$ which is $\mathbf{K}$-characteristic by Lemma 2.44. But, in $G/[S_{nc}, R]$, S_{nc} is a direct factor and can be characterized as the biggest normal subgroup which is connected, semisimple, and isotropic; and S_{nh} is $\mathbf{K}$-characteristic in S_{nc}. It follows that R_T is $\mathbf{K}$-characteristic.

Finally, $H = G/R_T$ is almost the direct product of a semisimple group H_s such that $H_s(\mathbf{K})$ is Haagerup, and its amenable radical H_m, such that $H_m(\mathbf{K})$ is amenable, hence Haagerup. So $H(\mathbf{K})$ is Haagerup, and contains $G(\mathbf{K})/R_T(\mathbf{K})$ as a closed subgroup. □

So we are in position to apply Lemma 2.33.

Corollary 2.47. ***Let X be a subset of $G(\mathbf{K})$. Then $(G(\mathbf{K}), X)$ has relative Property (T) if and only if the image of X is $G(\mathbf{K})/R_T(\mathbf{K})$ is relatively compact.*** □

We retrieve a result of Wang (his statement is slightly different but equivalent to this one).

Corollary 2.48 (Wang). ***$G(\mathbf{K})$ has Property (T) if and only if $S_{nh}[S_{nc}, R_u](\mathbf{K})$ is cocompact in $G(\mathbf{K})$.*** □

Corollary 2.49. ***$G(\mathbf{K})$ is Haagerup if and only if $S_{nh} = [S_{nc}, R] = 1$.*** □

This was proved differently in Chapter 1, Theorem 1.23.

2.2.3 Relative Property (T) in Lie groups

Let G be a Lie group (connected, even if it is straightforward to generalize to a Lie group with finitely many components), R its radical, S a Levi factor (not necessarily closed), decomposed as $S_c S_{nc}$ by separating compact and noncompact factors. Let S_{nh} be the sum of all simple factors of S_{nc} which have Property (T).

Set $R_T = \overline{S_{nh}[S_{nc}, R]}$.

Theorem 2.50. R_T **is a characteristic subgroup of** G**,** G/R_T **is Haagerup, and** (G, R_T) **has relative Property (T).**

Proof: The first statement can be proved in the same lines as in the algebraic case.

It is immediate that G/R_T is locally isomorphic to a direct product $M \times S$ where M is amenable and S is semisimple with all simple factors locally isomorphic to SO$(n,1)$ or SU$(n,1)$. By [CCJJV01, Chap. 4], G/R_T is Haagerup.

Finally, let us show that (G, R_T) has relative Property (T). First, note that we can reduce to the case when G is simply connected. Indeed, let $p : \tilde{G} \to G$ be the universal covering. Then $p(\tilde{H}) = H$, for $H = R, S_{nc}, S_{nh}$, where $\tilde{H}$ is the analytic subgroup of $\tilde{G}$ which lies over H. If the simply connected case is done, then $(\tilde{G}, \widetilde{S_{nh}}[\widetilde{S_{nc}}, \tilde{R}])$ has relative Property (T). It follows that $(G, p(\widetilde{S_{nh}}[\widetilde{S_{nc}}, \tilde{R}]))$ also has relative Property (T), and the closure of $p(\widetilde{S_{nh}}[\widetilde{S_{nc}}, \tilde{R}])$ is equal to R_T.

Now suppose that G is simply connected. Then the subgroup $S_{nc}[S_{nc}, R]$ is closed and isomorphic to $S_{nc} \ltimes [S_{nc}, R]$. Arguing as in the proof of Lemma 2.43 (using Example 2.40(2) instead of (1)), $(S_{nc} \ltimes [S_{nc}, R], [S_{nc}, R])$ has relative Property (T). Hence $(G, [S_{nc}, R])$ also has relative Property (T), and, as in the proof of Theorem 2.46, it implies that $(G, S_{nh}[S_{nc}, R])$ has relative Property (T). □

So we are again in position to apply Lemma 2.33.

Corollary 2.51. **Let** X **be a subset of** G**. Then** (G, X) **has relative Property (T) if and only if the image of** X **in** G/R_T **is relatively compact.**

We also retrieve a result of Wang in the case of connected Lie groups.

Corollary 2.52 (Wang). **The connected Lie group** G **has Property (T) if and only if** $\overline{S_{nh}[S_{nc}, R_u]}$ **is cocompact in** G**.** □

Corollary 2.53 ([CCJJV01], chap. 4). **The connected Lie group** G **is Haagerup if and only if**[4] $S_{nh} = [S_{nc}, R] = 1$**.**

Proof of Corollary 2.53. The hypothesis implies that S_{nh} and $W = [S_{nc}, R]$ are both relatively compact. So $S_{nh} = 1$. Now, since $[S_{nc}, [S_{nc}, R]] = [S_{nc}, R]$, we have $[S_{nc}, \overline{W}] = \overline{W}$. But, since W is a compact, connected, and solvable Lie group, it is a torus; since S_{nc} is connected, its action on $\overline{W}$ is necessarily trivial, so that $W \subset [S_{nc}, \overline{W}] = 1$. □

Remark 2.54. If G is a connected Lie group without the Haagerup Property, the existence of a noncompact closed subgroup with relative Property (T) was proved in [CCJJV01], and later established by another method (Theorem 1.23), where the result was generalized to linear algebraic groups over local fields of characteristic zero. However, in both cases, the subgroup constructed is not necessarily normal, while R_T is.

[4] **Note that we used, in the proof of Theorem 2.50, one implication from [CCJJV01, Chap. 4], namely,** $S_{nh} = [S_{nc}, R] = 1$ **implies** G **Haagerup. This result is easy when** S_{nc} **has finite centre, but, otherwise, is much more involved. Accordingly, only the reverse implication can be considered as a corollary of the present work.**

Remark 2.55. In this remark, given a locally compact group G, we say that a closed, normal subgroup N is a T-radical if G/N is Haagerup and (G, N) has relative Property (T).

It is natural to ask about the uniqueness of T-radicals when they exist. Observe that if N, N' are T-radicals, then the image of N in G/N' is relatively compact, and vice versa. In particular, if G is discrete, then all T-radicals are commensurable.

This is no longer the case if G is not discrete, for instance, set $G = SL(2, \mathbf{Z}) \ltimes \mathbf{R}^2$. Then the subgroups $a\mathbf{Z}^2$, for $a \neq 0$, are all T-radicals, although two of them may have trivial intersection. We thus see that G does not necessarily have a minimal T-radical.

Let G be a finitely generated solvable group with infinite locally finite centre. Then, every finite subgroup of the centre is a T-radical, but G has no infinite T-radical, so has no maximal T-radical. An example of such a group G is the group of matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & u^n & c \\ 0 & 0 & 1 \end{pmatrix}$, for $a, b, c \in \mathbf{F}_p[u, u^{-1}]$, $n \in \mathbf{Z}$, and p a fixed prime.

However, if G is a connected Lie group, it can be shown that G has a minimal and a maximal T-radical. The minimal one is R_T, as defined above: indeed, if H is a quotient of G with the Haagerup Property, then S_{nh} and $[S_{nc}, R]$ are necessarily contained in the kernel. The maximal one is found by taking the preimage of the maximal normal compact subgroup of G/R_T; it can immediately be generalized to any connected locally compact group.

Remark 2.56. Following Shalom [Sha99t], if G is a topological group and H is a subgroup, we say that (G, H) has **strong** relative Property (T) if there exists a Kazhdan pair (K, ε) for the pair (G, H) with K finite (and $\varepsilon > 0$). More precisely, this means that every unitary representation with a (K, ε)-invariant vector has a H-invariant vector. In this context, it is natural to equip $\hat{G}$ with the topology inherited from $\widehat{G_d}$, the unitary dual of G_d, where G_d denotes G with the discrete topology. As for the case of relative Property (T), it can be checked that (G, H) has strong relative Property (T) if and only if, for every net π_i in $\widehat{G}$ which converges to 1 in $\widehat{G_d}$, eventually π_i has a H-invariant vector. Then it is straightforward from the proof that Proposition 2.35 remains true for strong relative Property (T). On the other hand, Proposition 2.34 is actually true with strong Property (T) [Sha99t, Theorem 5.5]. It then follows from the proofs that, if G is a connected Lie group, then (G, R_T) has strong relative Property (T), and similarly for algebraic groups over local fields of characteristic zero.

2.3 Framework for irreducible lattices: resolutions

In this section, we make a systematic study of ideas relying on work of Lubotzky and Zimmer [LuZi89], and later apparent in [Mar91, Chap. III, 6.] and [BeLo97].

Given a locally compact group G, when can we say that we have a good quantification of Kazhdan's Property (T)? Lemma 2.33 provides a satisfactory answer whenever G has a normal subgroup N such that G/N is Haagerup and (G, N) has relative Property (T). We have seen in Section 2.2 that this is satisfied in a large class of groups. However, this is not inherited by lattices. A typical example is the case of an irreducible lattice Γ in a product of noncompact simple Lie groups $G\times H$, where G has Property (T) and H is Haagerup. In such an example, although $\Gamma\cap G=\{1\}$, G can be thought as a "ghost" normal sugroup of Γ, and is the "kernel" of the projection $\Gamma\to H$. Relative Property (T) for the pair $(G\times H, G)$ can be restated by saying that the projection $G\times H\to H$ is a "resolution". By a theorem essentially due Margulis, this notion is inherited by lattices, so that, in this case, the projection $\Gamma\to H$ is a resolution.

Before giving rigorous definitions, we need some elementary preliminaries.

2.3.1 Q-points

We recall that an action by isometries α of a topological group G on a metric space X is continuous if the function $g\mapsto\alpha(g)x$ is continuous for every $x\in X$. All the functions and actions here are supposed continuous.

Let $f:G\to Q$ be a morphism between topological groups, with dense image. Recall that, for any Hausdorff topological space X, a function $u:G\to X$ factors through Q if and only if, for every net (g_i) in G such that $f(g_i)$ converges in Q, $u(g_i)$ converges in X; note that the factorization $Q\to X$ is unique.

Definition 2.57. Let $f:G\to Q$ be a morphism between topological groups, with dense image. Let α be an action of G by isometries on a metric space X. We call $x\in X$ a Q-point if the orbital map $g\mapsto\alpha(g)x$ factors through Q.

Proposition 2.58. **Suppose that X is a complete metric space. The set X^Q of Q-points in X is closed, G-invariant, and the action α^Q of G on X^Q factors through Q. In particular, $\mathrm{Ker}(f)\subset\mathrm{Ker}(\alpha^Q)$.**

Proof: The two latter statements are immediate; let us show that X^Q is closed. Let (y_n) be a sequence in X^Q, converging to a point $y\in X$. Write $\alpha(g)y_n=w_n(f(g))$, where w_n is a continuous function: $Q\to X$. If $m,n\in\mathbf{N}$, $d(w_m(f(g)),w_n(f(g)))=d(\alpha(g)y_m,\alpha(g)y_n)=d(y_m,y_n)$. It follows that $\sup_{q\in f(G)}d(w_m(q),w_n(q))\to 0$ when $m,n\to\infty$. On the other hand, since $f(G)$ is dense in Q, $\sup_{q\in f(G)}d(w_m(q),w_n(q))=\sup_{q\in Q}d(w_m(q),w_n(q))$. Accordingly, (w_n) is a Cauchy sequence for the topology of uniform convergence on Q. Since X is complete, this implies that (w_n) converges to a continuous function $w:Q\to X$. Clearly, for all $g\in G$, $\alpha(g)y=w(f(g))$, so that $y\in X^Q$. The last statement is immediate. □

Proposition 2.59. **Suppose that X is complete. Let $x\in X$. Equivalences:**

1) $x\in X^Q$**.**

2) The mapping $g\mapsto d(x,\alpha(g)x)$ **factors through** Q**.**

3) For every net (g_i) **in** G **such that** $f(g_i)\to 1$**,** $d(x,\alpha(g_i)x)\to 0$**.**

Proof: 1)⇒2)⇒3) is immediate.

Suppose 3). Let (g_i) be a net in G such that $f(g_i)$ converges in Q. Then $(f(g_i^{-1}g_j))$ converges to 1 when $i,j\to\infty$, so that

$$d(\alpha(g_i)x,\alpha(g_j)x)=d(y,\alpha(g_i^{-1}g_j)x)\to 0,$$

i.e. $(\alpha(g_i)x)$ is Cauchy. Hence, it converges since X is complete. This means that $g \mapsto \alpha(g)x$ factors through Q, i.e. $x \in X^Q$. □

Proposition 2.60. **1) Suppose that X is a complete CAT(0) space. Then X^Q is a closed, totally geodesic subspace.**

2) If $X = \mathscr{H}$ is the Hilbert space of a unitary representation π of G, then $\mathscr{H}^Q$ is a closed subspace, defining a subrepresentation π^Q of π (we refer to elements in $\mathscr{H}^Q$ as Q-vectors rather that Q-points). For every $\xi \in \mathscr{H}$, $\xi \in \mathscr{H}^Q$ if and only if the corresponding coe□cient $g \mapsto \langle \xi, \pi(g)\xi \rangle$ factors through Q.

3) If $X = \mathscr{H}$ is an a□ne Hilbert space, then $\mathscr{H}^Q$ is a closed a□ne subspace (possibly empty). For every $v \in \mathscr{H}$, $v \in \mathscr{H}^Q$ if and only if the corresponding conditionally negative definite function $g \mapsto \|v - g \cdot v\|^2$ factors through Q.

Proof: 1) The first statement is immediate since, for all $\lambda \in \mathbf{R}$, the function $(c, c') \mapsto (1-\lambda)c + \lambda c'$ is continuous (actually 1-Lipschitz) on its domain of definition.

2) If $\mathscr{H}$ is the Hilbert space of a unitary representation, then $\mathscr{H}^Q$ is immediately seen to be a linear subspace, and is closed by Proposition 2.58. Note that this also can be derived as a particular case of 1). The nontrivial part of the last statement in 2) follows from Proposition 2.59.

3) is similar. □

Lemma 2.61. **Let $G \to Q$ be a morphism with dense image, and (π_i) is a family of unitary representations of G. Then $(\bigoplus \pi_i)^Q = \bigoplus \pi_i^Q$.**

Proof: The inclusion $\bigoplus \pi_i^Q \subset (\bigoplus \pi_i)^Q$ is trivial. Let p_i denote the natural projections, set $\pi = \bigoplus \pi_i$, and write $\mu_\xi(g) = \pi(g)\xi$. Then, if ξ is a Q-vector, i.e. if μ_ξ factors through Q, then $p_i \circ \mu_\xi$ also factors through Q. But $p_i \circ \mu_\xi = \mu_{p_i(\xi)}$, so that $p_i(\xi)$ is a Q-vector. □

2.3.2 Resolutions

Convention 2.62. If $G \to Q$ is a morphism with dense image, and π is a representation of G factoring through a representation $\tilde{\pi}$ of Q, we write $1_Q \prec \pi$ rather than $1_Q \prec \tilde{\pi}$ or $1 \prec \tilde{\pi}$ to say that $\tilde{\pi}$ almost has invariant vectors (note that $1_Q \prec \pi$ implies $1_G \prec \pi$, but the converse is not true in general). Similarly, if (π_i) is a net of representations of G factoring through representations $\tilde{\pi}_i$ of Q, when we write $\pi_i \to 1_Q$, we mean for the Fell topology on representations of Q.

Definition 2.63 (Resolutions). Let G be a locally compact group, and $f : G \to Q$ a morphism to another locally compact group Q, such that $f(G)$ is dense in Q.

We say that f is a **resolution** (of G) if, for every unitary representation π of G which almost has invariant vectors, then $1_Q \prec \pi^Q$, meaning that π^Q, viewed as a representation of Q, almost has invariant vectors (in particular, $\pi^Q \neq 0$).

We call f a Haagerup resolution if Q is Haagerup.

The definition of resolution generalizes the notion of relative Property (T) of a normal subgroup, since $G \to G/N$ is a resolution if and only if (G, N) has relative Property (T).

Remark 2.64. It is natural to ask if, in Definition 2.63, the condition $1_Q \prec \pi^Q$ can be weakened into $\pi^Q \neq 0$. In §2.3.8, we show, in a very indirect way, that, under the assumption that G and Q are σ-compact, this is indeed the case. However, it is much more convenient to work with this **a priori** stronger definition given here.

Proposition 2.65. **Let $f : G \to Q$ be a morphism between locally compact groups, with dense image. Equivalences:**

(1) f is a resolution.

(2) For every net (π_i) of unitary representations of G which converges to 1_G, $\pi_i^Q \to 1_Q$.

Proof: (2)⇒(1) is trivial. Suppose (1). Let $\pi_i \to 1_G$. Then, for every subnet (π_j), $1_G \prec \bigoplus_j \pi_j$. By (1), $1_Q \prec (\bigoplus_j \pi_j)^Q$, which equals $\bigoplus_j \pi_j^Q$ by Lemma 2.61. Hence $\pi_i^Q \to 1_Q$. □

Corollary 2.66. **Let $G \to Q$ be a resolution. Then for every net (π_i) of irreducible representations which converges to 1_G, eventually π_i factors through a representation $\tilde{\pi}_i$ of Q, and $\tilde{\pi}_i \to 1_Q$.** □

The converse of Corollary 2.66 is more involved, and is proved (Theorem 2.96) under the mild hypothesis that G is σ-compact.

Thus, a resolution allows to convey properties about the neighbourhood of 1_Q in $\hat{Q}$ into properties about the neighbourhood of 1_G in $\hat{G}$. For instance, this is illustrated by Property (τ) (see Section 2.3.5).

The following proposition generalizes the fact that relative Property (T) is inherited by extensions (Proposition 2.28), and is one of our main motivations for having introduced resolutions.

Proposition 2.67. **Let $f : G \to Q$ be a resolution, and $X \subset G$. Then (G, X) has relative Property (T) if and only if $(Q, f(X))$ does.**

Proof: The condition is trivially sufficient. Suppose that $(Q, f(X))$ has relative Property (T). Fix $\varepsilon > 0$, and let π be a unitary representation of G such that $1_G \prec \pi$. Since $G \to Q$ is a resolution, $1_Q \prec \pi^Q$. Hence, by Property (T), π^Q has a $(f(X), \varepsilon)$-invariant vector; this is a (X, ε)-invariant vector for π. □

Recall that a morphism between locally compact spaces is proper if the inverse image of any compact subset is compact. It is easy to check that a morphism $G \to H$ between locally compact groups is proper if and only if its kernel K is compact, its image Q is closed in H, and the induced map $G/K \to Q$ is an isomorphism of topological groups.

Corollary 2.68. **Let $f : G \to Q$ be a Haagerup resolution. Then for every $X \subset G$, (G, X) has relative Property (T) if and only if $\overline{f(X)}$ is compact.**

Accordingly, either f is a proper morphism, so that G is also Haagerup, or there exists a noncompact closed subset $X \subset G$ such that (G, X) has relative Property (T). In particular, G satisfies the TH alternative. □

The following theorem generalizes (case when $Q = \{1\}$) compact generation of locally compact groups with Property (T) [Kaz67]; in this more specific direction, it generalizes Proposition 2.8 of [LuZi89].

Theorem 2.69. **Let $f : G \to Q$ be a resolution. Then G is compactly generated if and only if Q is.**

We need the following lemma, which is also used later.

Lemma 2.70. **Let $f : G \to Q$ be a morphism with dense image between topological groups, and let Ω be an open neighbourhood of 1 of Q. Then, for all n, $f^{-1}(\Omega^n) \subset f^{-1}(\Omega)^{n+1}$.**

Proof: Let x belong to $f^{-1}(\Omega^n)$. Write $f(x) = u_1 \dots u_n$ with $u_i \in \Omega$. Let $(\varepsilon_1, \dots, \varepsilon_n) \in \Omega^n$. Set $v_i = u_i\varepsilon_i$ and $v_0 = u_1 \dots u_n(v_1 \dots v_n)^{-1}$, so that $f(x) = v_0 v_1 \dots v_n$. If all ε_i are chosen sufficiently close to 1, then $v_0 \in \Omega$ and $u_i\varepsilon_i \in \Omega$ for all i; by density of $f(G)$, we can also impose that $u_i\varepsilon_i \in f(G)$ for all i. We fix $\varepsilon_1, \dots, \varepsilon_n$ so that all these conditions are satisfied. Since $v_0 = f(x)(v_1 \dots v_n)^{-1}$, v_0 also belongs to $f(G)$. For all i, write $v_i = f(x_i)$, so that $x = kx_0x_1 \dots x_n$ with $k \in \mathrm{Ker}(f)$. Set $y_0 = kx_0$. Then $x = y_0x_1 \dots x_n \in f^{-1}(\Omega)^{n+1}$. □

Proof of Theorem 2.69. If G is compactly generated, so is Q (only supposing that the morphism has dense image). Indeed, if K is a compact generating set of G, then $f(K)$ generates a dense subgroup of Q. It follows that, if K' is a compact subset of Q containing $f(K)$ in its interior, then K' generates Q.

Conversely, suppose that Q is compactly generated, and let Ω be an open, relatively compact generating set. For every open, compactly generated subgroup H of G, let φ_H be its characteristic function. Then, when H becomes big, φ_H converges to 1, uniformly on compact subsets of G. Since $(G, f^{-1}(\Omega))$ has relative Property (T) by Proposition 2.67, it follows that $f^{-1}(\Omega)$ is contained in a compactly generated subgroup H of G. By Lemma 2.70, $f^{-1}(\Omega^n) \subset f^{-1}(\Omega)^{n+1} \subset H$. Since $Q = \bigcup \Omega^n$, it follows that $G = H$. □

2.3.3 Lattices and resolutions

The following theorem generalizes the fact that Property (T) is inherited by lattices.

Theorem 2.71. **Let G be a locally compact group, N a closed, normal subgroup. Suppose that (G, N) has relative Property (T) (equivalently, the projection $f : G \to G/N$ is a resolution).**

Let H be a closed subgroup of finite covolume in G, and write $Q = \overline{f(H)}$. Then $f : H \to Q$ is a resolution.

Theorem 2.71 is a slight strengthening of [Mar91, Chap. III, (6.3) Theorem]. To prove it, we need the following lemma, all of whose arguments are borrowed from [BeLo97].

Lemma 2.72. **Let G, N, H, Q be as in Theorem 2.71.**

For every representation π of H which factors through Q, if $1_H \prec \pi$, then $1_Q \prec \pi$.

Proof: By [Jol05, Corollary 4.1(2)], $(f^{-1}(Q), N)$ has relative Property (T); hence, upon replacing G by $f^{-1}(Q)$, we can suppose that $Q = G/N$.

Denote by $\tilde{\pi}$ the factorization of π through Q, and by $\hat{\pi}$ the (intermediate) factorization of ρ through G, so that $\pi = \hat{\pi}|_H$.

By continuity of induction and since H has finite covolume, $1_G \prec \mathrm{Ind}_H^G \pi$. On the other hand, $\mathrm{Ind}_H^G \pi = \mathrm{Ind}_H^G \hat{\pi}|_H = \hat{\pi} \otimes L^2(G/H) = \hat{\pi} \oplus (\hat{\pi} \otimes L_0^2(G/H))$.

We claim that $1_G \not\prec \hat{\pi} \otimes L_0^2(G/H)$. It follows that $1_G \prec \hat{\pi}$. Since every compact subset of G/N is the image of a compact subset of G [BHV05, Lemma B.1.1], it follows that $1_{G/N} \prec \tilde{\pi}$.

It remains to prove the claim. Since $\hat{\pi}|_N$ is a trivial representation, $\hat{\pi}|_N \otimes L_0^2(G/H)|_N$ is a multiple of $L_0^2(G/H)|_N$. But, by [BeLo97, Lemma 2], $L_0^2(G/H)$ does not contain any nonzero N-invariant vector. Accordingly, neither does $\hat{\pi} \otimes L_0^2(G/H)$. Hence, by relative Property (T), $1_G \not\prec \hat{\pi} \otimes L_0^2(G/H)$. □

Proof of Theorem 2.71. Let π be a unitary representation of H, and suppose that $1_H \prec \pi$. Using [Mar91, Chap. III, (6.3) Theorem] twice[5], $\pi^Q \neq 0$, and its orthogonal in π does not almost contain invariant vectors. It follows that $1_H \prec \pi^Q$. By Lemma 2.72, $1_Q \prec \pi^Q$. □

Remark 2.73. The conclusion of Lemma 2.72 is false if we drop the assumption that (G, N) has relative Property (T), as the following example shows.

Set $G = \mathbf{Z} \times \mathbf{R}/\mathbf{Z}$, $N = \mathbf{Z} \times \{0\}$, and H the cyclic subgroup generated by $(1, \alpha)$, where $\alpha \in (\mathbf{R} - \mathbf{Q})/\mathbf{Z}$.

The projection $p : H \to \mathbf{R}/\mathbf{Z}$ has dense image. Hence, the Pontryagin dual morphism: $\hat{p} : \mathbf{Z} \simeq \widehat{\mathbf{R}/\mathbf{Z}} \to H^* \simeq \mathbf{R}/\mathbf{Z}$ also has dense image. Take a sequence (χ_n) of pairwise distinct nontrivial characters of $\mathbf{R}/\mathbf{Z}$ such that $\hat{p}(\chi_n)$ tends to 0. Then the direct sum $\pi = \bigoplus \chi_n$ does not weakly contain the trivial representation (otherwise, since $\mathbf{R}/\mathbf{Z}$ has Property (T), it would contain the trivial representation), but $\pi \circ p|_H$ weakly contains the trivial representation 1_H.

We can now combine the results of Section 2.2 with Theorem 2.71.

Let G be a finite direct product of Lie groups and algebraic groups over local fields of characteristic zero: $G = L \times \prod_{i=1}^n H_i(\mathbf{K}_i)$. Write $R_T(G) = R_T(L) \times \prod_{i=1}^n R_T(H_i)(\mathbf{K}_i)$, where R_T is defined in Sections 2.2.2 and 2.2.3. Observe that, by Theorems 2.50 and 2.46, (G, R_T) has relative Property (T) and G/R_T is Haagerup. Denote by $f : G \to G/R_T(G)$ the quotient morphism.

Corollary 2.74. **Let G be a finite product of Lie groups and (rational points of) algebraic groups over local fields of characteristic zero. Let Γ be a closed subgroup of finite covolume in G. Then there exists a Haagerup resolution for Γ, given by $f : \Gamma \to \overline{f(\Gamma)}$.** □

[5] **The assumption in [Mar91] is that N has Property (T), but it is clear from the proof that relative Property (T) for (G, N) is su□cient.**

2.3.4 The factorization Theorem

Theorem 2.75. **Let G be a locally compact group, $f : G \to Q$ a resolution, $u : G \to H$ a morphism to a locally compact group, with dense image, where H is Haagerup. Then there exists a compact, normal subgroup K of H, and a factorization $Q \to H/K$ making the following diagram commutative**

$$\begin{array}{ccc} G & \xrightarrow{f} & Q \\ {\scriptstyle u}\downarrow & & \vdots \\ H & \xrightarrow{p} & H/K. \end{array}$$

Proof: H has a C_0-representation π with almost invariant vectors. So $\pi \circ u$ almost has invariant vectors, so, upon passing to a subrepresentation, we can suppose that $\pi \circ u$ factors through a representation $\tilde{\pi}$ of Q. We fix a normalized coefficient φ of $\tilde{\pi} \circ f$.

Let (g_i) be a net in G such that $f(g_i) \to 1$. Then $\varphi(g_i) \to 1$. This implies that $u(g_i)$ is bounded in H, since φ is a C_0 function. Let K be the set of all limits of $u(g_i)$ for such nets (g_i). Then K is a compact, normal subgroup of H (it is normal thanks to the density of $u(G)$ in H).

Let p be the projection: $H \to H/K$. We claim that $p \circ u$ factors through Q. Indeed, if (g_i) is a net in G such that $(f(g_i))$ is Cauchy in Q, then $p \circ u$ is also Cauchy in H/K. This implies that $p \circ u$ factors through Q. □

2.3.5 Applications to Property (τ) and related properties

Recall that a representation of a group is said to be **finite** if its kernel has finite index.

Definition 2.76. We recall that a topological group G has Property (τ) (resp. (τ_{FD})) if for every net (π_i) of finite (resp. finite dimensional) irreducible unitary representations of G which converges to 1_G, eventually $\pi_i = 1_G$.

We say that a topological group G has Property (FH$_{FD}$) if every isometric action of G on a finite-dimensional Hilbert space has a fixed point. Equivalently, every finite-dimensional unitary representation has vanishing 1-cohomology.

We say that a topological group G has Property (FH$_F$) if every finite unitary representation has vanishing 1-cohomology.

The topological group G has Property (FAb$^{\mathbf{R}}$) (resp. (FAb)) if for every closed subgroup of finite index H of G, $\mathrm{Hom}(H, \mathbf{R}) = 0$ (resp. $\mathrm{Hom}(H, \mathbf{Z}) = 0$).

It turns out that Properties (FH$_F$) and (FAb$^{\mathbf{R}}$) are equivalent. This is shown in [LuZu05][6] using induction of unitary representations and Shapiro's Lemma. Alternatively, this can be shown using induction of affine representations.

Note also that (FAb$^{\mathbf{R}}$) implies (FAb), and they are clearly equivalent for finitely generated groups; while $\mathbf{R}$ or $\mathbf{Q}$ satisfy (FAb) but not (FAb$^{\mathbf{R}}$).

[6] **The group there is assumed to be finitely generated but this has no importance.**

Property (τ) clearly implies (FAb) [LuZu05]; it is observed there that the first Grigorchuk group does not have Property (τ), but has Property (FH_{FD}) since all its linear representations are finite. However, no finitely presented group is known to satisfy (FAb) but not Property (τ).

We provide below an example of a finitely generated group which has Property (τ) (hence (FH_F)) but not (FH_{FD}). We do not know if this is the first known example.

We begin by a general result.

Proposition 2.77. **Let G, Q be locally compact, and $G \to Q$ be a resolution. Let (P) be one of the Properties: (T), (τ), (τ_{FD}), (FAb), ($\mathrm{FAb}^{\mathbf{R}}$), ($\mathrm{FH}_F$), ($\mathrm{FH}_{FD}$). Then G has Property (P) if and only if Q does.**

Proof: In all cases, Property (P) for G clearly implies Property (P) for Q.

Let us show the converse. For (T), (τ), and (τ_{FD}) this follows directly from Proposition 2.65.

Suppose that G does not have Property (FAb). Let $N \subset G$ be a closed normal subgroup of finite index such that $\mathrm{Hom}(N, \mathbf{Z}) \neq 0$. Let M be the kernel of a morphism of N onto $\mathbf{Z}$, and set $K = \bigcap_{g \in G/N} gMg^{-1}$. Then K is the kernel of the natural diagonal morphism $N \to \prod_{g \in G/N} N/gMg^{-1} \simeq \mathbf{Z}^{G/N}$. It follows that N/K is a nontrivial free abelian group of finite rank, and K is normal in G, so that $H = G/K$ is infinite, finitely generated, virtually abelian. Since H is Haagerup, by Theorem 2.75, Q maps onto the quotient of H by a finite subgroup F. Since H/F is also infinite, finitely generated, virtually abelian, Q does not have Property (FAb).

The case of Property ($\mathrm{FAb}^{\mathbf{R}}$) can be proved similarly; since ($\mathrm{FAb}^{\mathbf{R}}$) is equivalent to ($\mathrm{FH}_F$) which is treated below, we omit the details.

Suppose that G does not have Property (FH_{FD}). Let G act isometrically on a Euclidean space E with unbounded orbits, defining a morphism $\alpha : G \to \mathrm{Isom}(E)$. Set $H = \overline{\alpha(G)}$. Since $\mathrm{Isom}(E)$ is Haagerup, so is H. By Theorem 2.75, H has a compact normal subgroup K such that Q has a morphism with dense image into H/K. Observe that the set of K-fixed points provides an action of H/K on a non-empty affine subspace of E, with unbounded orbits. So Q does not have Property (FH_{FD}).

The case of Property (FH_F) can be treated similarly, noting that α maps a subgroup of finite index to translations, and this is preserved after restricting to the action on an affine subspace. □

Remark 2.78. The case of Property (FH_{FD}) in Proposition 2.77 contains as a particular case Theorem B in [BeLo97], without making use of the Vershik-Karpushev Theorem, while the proof given in [BeLo97] does.

Proposition 2.77 justifies why we do not have restricted the definitions of Property (τ), etc., to discrete groups (as is usually done), since in many cases, when we have a resolution $G \to Q$, Q is non-discrete. For instance, all these properties are easy or trivial to characterize for connected Lie groups.

Proposition 2.79. **Let G be a connected locally compact group. Then**

1) G has Property (τ).

2) G **has Property (**FH_F**) if and only if** $\mathrm{Hom}(G, \mathbf{R}) = 0$**.**
3) G **has Property (**τ_{FD}**) if and only if** $\mathrm{Hom}(G, \mathbf{R}) = 0$**.**
4) G **has Property (**FH_{FD}**) if and only if every amenable quotient of** G **is compact.**

Proof: Since G has no proper closed finite index subgroup, 1) and 2) are immediate.

3) The condition is clearly necessary. Conversely, suppose that $\mathrm{Hom}(G, \mathbf{R}) = 0$. Let W be the intersection of all kernels of finite-dimensional unitary representations of G. Clearly, it suffices to show that G/W has Property (τ_{FD}). By [Dix69, Théorème 16.4.6], $G/W \simeq \mathbf{R}^n \times K$ for some compact group K. The assumption then implies $n = 0$, so that G/W is compact, so that G has Property (τ_{FD}).

4) Suppose that G does not have Property (FH_{FD}). Let K be a compact normal subgroup of G such that G/K is a Lie group [MoZi55]. Then there exists an unbounded isometric affine action of G on some Euclidean space. Upon restricting to the orbit of a K-fixed point, we can suppose that K is contained in the kernel N of this action. Necessarily, the Lie group G/N is not compact, and amenable since it embeds in the amenable Lie group $O(n) \ltimes \mathbf{R}^n$.

Conversely, suppose G has a noncompact amenable quotient H. Since H does not have Property (T), by a result of Shalom [Sha00] (see [BHV05, Section 3.2]), there exists an irreducible unitary representation π of H with non-vanishing 1-reduced cohomology. By [Mrt05, Theorem 3.1], π is finite-dimensional[7]. □

Proposition 2.80. **Fix** $n \geq 5$**, set** $\Gamma = \mathrm{SO}_n(\mathbf{Z}[2^{1/3}]) \ltimes \mathbf{Z}[2^{1/3}]^n$**. Then** Γ **is a finitely presentable group, has Property** (τ_{FD}) **(hence Property** (τ)**, hence Property (**FH_F**)), but not (**FH_{FD}**).**

Proof: Note that $\mathrm{SO}_n(\mathbf{C})$ and $\mathrm{SO}_n(\mathbf{C}) \ltimes \mathbf{C}^n$ have Property (T). Since Γ is an irreducible lattice in the Lie group $(\mathrm{SO}_n(\mathbf{R}) \ltimes \mathbf{R}^n) \times (\mathrm{SO}_n(\mathbf{C}) \ltimes \mathbf{C}^n)$, it is finitely presentable, and, moreover, by Theorem 2.71, $\Gamma \to \mathrm{SO}_n(\mathbf{R}) \ltimes \mathbf{R}^n$ is a resolution.

By Proposition 2.79, $\mathrm{SO}_n(\mathbf{R}) \ltimes \mathbf{R}^n$ has Property (τ_{FD}). Thus Γ also has Property (τ_{FD}) by Proposition 2.77. On the other hand, the embedding of Γ in $\mathrm{SO}_n(\mathbf{R}) \ltimes \mathbf{R}^n$ provides an isometric action of Γ with unbounded orbits on the n-dimensional Euclidean space. □

Remark 2.81. It is asked in [LuZi89] whether there exists a finitely generated group with Property (τ) but not (τ_{FD}). Obvious non-finitely generated examples are $\mathbf{Q}$ and $\mathbf{R}$. It may be tempting to find a finitely generated group Γ with a resolution $\Gamma \to \mathbf{R}$, but unfortunately no such Γ exists. Indeed, since Γ is discrete and $\mathrm{Hom}(\Gamma, \mathbf{R}) \neq 0$, there exists a discrete, nontrivial, torsion-free abelian quotient Λ of Γ. By Theorem 2.75, there exists a factorization: $\mathbf{R} \to \Lambda$, necessarily surjective. This is a contradiction since $\mathbf{R}$ is connected.

2.3.6 Subgroups of simple Lie groups

Let G be a connected simple Lie group, with Lie algebra $\mathfrak{g}$. We are interested in subgroups $\Gamma \subset G$, viewed as discrete groups.

[7] **It is possible to prove 4) more directly, but we have used Shalom's and Martin's results to make short.**

1) If $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbf{R}) \simeq \mathfrak{so}(2,1) \simeq \mathfrak{su}(1,1)$, $\mathfrak{sl}_2(\mathbf{C}) \simeq \mathfrak{so}(3,1)$, or $\mathfrak{so}_3(\mathbf{R})$, then every $\Gamma \subset G$ is Haagerup [GHW05, §5, Theorem 4].

2) If G has Property (T) and $\mathfrak{g} \not\simeq \mathfrak{so}_3(\mathbf{R})$, then there exists $\Gamma \subset G$ with Property (T): if G is noncompact, take any lattice. If G is compact, this is due to Margulis [Mar91, chap. III, Proposition 5.7]. We recall the simple argument: writing $G = H(\mathbf{R})$ with H absolutely simple, defined over $\mathbf{Q}$, $H(\mathbf{Z}[2^{1/3}])$ is an irreducible lattice in $G \times H(\mathbf{C})$ [BoHC62]. By the assumption on G, H has $\mathbf{C}$-rank ≥ 2, so that $H(\mathbf{C})$ has Property (T). It follows that the projection of $H(\mathbf{Z}[2^{1/3}])$ on G is a dense subgroup with Property (T).

3) If $\mathfrak{g} \simeq \mathfrak{so}(n,1)$ with $n \geq 5$ or $\mathfrak{g} \simeq \mathfrak{su}(n,1)$ with $n \geq 3$, then then there exists $\Gamma \subset G$ with Property (T): it suffices to observe that G contains a subgroup locally isomorphic to $\mathrm{SO}(n)$ ($n \geq 5$) or $\mathrm{SU}(n)$ ($n \geq 3$), and such a subgroup contains an infinite subgroup with Property (T) by 2).

4) There are only two remaining cases: $\mathfrak{g} \simeq \mathfrak{so}(4,1)$ and $\mathfrak{g} \simeq \mathfrak{su}(2,1)$. We are going to show that the behaviour there is different from that in preceding examples.

The only result already known is that if $\mathfrak{g} \simeq \mathfrak{so}(4,1)$ or $\mathfrak{g} \simeq \mathfrak{su}(2,1)$, then no infinite $\Gamma \subset G$ can have Property (T); this follows from 1) since such Γ would be contained in a maximal compact subgroup. This result is generalized in the following theorem.

Theorem 2.82. **Let G be a connected Lie group, locally isomorphic to either $\mathrm{SO}(4,1)$ or $\mathrm{SU}(2,1)$.**

Let $\Gamma \subset G$, and view Γ as a discrete group.

1) If $\Lambda \subset \Gamma$ is a normal subgroup such that (Γ, Λ) has relative Property (T), then Λ is a finite subgroup of G.

2) If Γ is not dense, and if $\Lambda \subset \Gamma$ is a subgroup such that (Γ, Λ) has relative Property (T), then Λ is a finite subgroup of G.

3) If Γ is dense, and $X \subset \Gamma$ is a normal subset (i.e. invariant under conjugation) such that (Γ, X) has relative Property (T), then X is a finite subset of the centre of G.

Suppose that G is locally isomorphic to $\mathrm{SU}(2,1)$. Then we have stronger statements:

4) If Γ is not dense, then Γ is Haagerup.

5) If $X \subset \Gamma$ is a normal subset and (Γ, X) has relative Property (T), then X is a finite subset of G.

Proof: Fix a subset $X \subset \Gamma$ such that (Γ, X) has relative Property (T). We make a series of observations.

a) First note that G is Haagerup, a fact due to [FaHa74] (and [CCJJV01, Chap. 4] in the case of $\widetilde{\mathrm{SU}}(2,1)$). By relative Property (T), $\overline{X}$ must compact. Denote by $\mathfrak{h}$ the Lie algebra of $\overline{\Gamma}$.

b) Suppose, in this paragraph b), that Γ is dense in G, i.e. $\mathfrak{h} = \mathfrak{g}$; and suppose that X is a normal subset. Then $\overline{X}$ is a compact, normal subset in G. Let Z be the centre of G, and fix $h \in X$. Then the conjugacy class of h in G/Z is relatively compact. Let M be the symmetric space associated to G/Z, and fix $y \in M$. Then the function $g \mapsto d(ghg^{-1}y, y)$ is bounded, so that h has bounded displacement length. Since M is $\mathrm{CAT}(-1)$ and geodesically complete, this implies that h acts as the identity, i.e. $h \in Z$. Accordingly, $X \subset Z$, so that X is discrete. Since it is relatively compact, it is finite. This proves 3).

c) Suppose that Γ is Zariski dense modulo Z, but not dense. Then Γ is discrete. Indeed, Γ is contained in the stabilizer W of $\mathfrak{h}$ for the adjoint action. Since W is Zariski closed modulo Z, this implies that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, so that, since $\mathfrak{h} \neq \mathfrak{g}$ and $\mathfrak{g}$ is simple, $\mathfrak{h} = \{0\}$, i.e. Γ is discrete. Since G is Haagerup, this implies that Γ is Haagerup.

d) Now suppose that $Z = 1$ and Γ is not Zariski dense. Let N be the Zariski closure of Γ. Let R_u be the unipotent radical of N, and $L = CS$ a Levi factor, with abelian part C, and semisimple part S. The possibilities for simple factors in S are rather restricted. The complexification $G_{\mathbf{C}}$ is isomorphic to either $\mathrm{PSO}_5(\mathbf{C})$ or $\mathrm{PSL}_3(\mathbf{C})$. In both cases, by a dimension argument, the only possible simple subgroups of $G_{\mathbf{C}}$ are, up to isogeny, $\mathrm{SL}_2(\mathbf{C})$, and maybe $\mathrm{SL}_3(\mathbf{C})$ in $\mathrm{PSO}_5(\mathbf{C})$; however, $\mathfrak{sl}_3(\mathbf{C})$ does not embed in $\mathfrak{so}_5(\mathbf{C})$ as we see, for instance, by looking at their root systems. So the only possible factors in S are, up to isogeny, $\mathrm{SL}_2(\mathbf{C})$, $\mathrm{SL}_2(\mathbf{R})$, and $\mathrm{SO}_3(\mathbf{R})$. By [GHW05, §5, Theorem 4], the image of Γ in N/R_u is Haagerup, so that the image of X in H/R_u is finite.

e) We keep the assumptions of d), and suppose moreover that $X = \Lambda$ is a subgroup. Since Λ is relatively compact, and R_u is unipotent, $\Lambda \cap R_u = \{1\}$. Since we proved in d) that the image of Λ in N/R_u is finite, this implies that Λ is finite.

Now let us drop the assumption $Z = 1$. Then the image of Λ modulo Z is finite, so that, by the case $Z = 1$, Λ is virtually contained in Z. This implies that Λ is discrete, hence finite since it is also relatively compact.

In view of c), d), and e), 2) is now proved; observe that 1) is an immediate consequence of 2) and 3).

f) Now suppose that $\mathfrak{g} \simeq \mathfrak{su}(2,1)$, and let us prove 4). Observe that 5) is an immediate consequence of 3) and 4).

We first suppose that $Z = 1$, and that Γ is not Zariski dense. So we continue with the notation of d). Write $S = S_c S_{nc}$ by separating compact and noncompact simple factors.

Suppose that $S_c \neq 1$. This is a compact subgroup, so, upon conjugating, we can suppose that it is contained in the maximal subgroup $\mathrm{PS}(\mathrm{U}(2) \times \mathrm{U}(1))$. The Lie algebra of S_c is identified with $\mathfrak{su}(2)$.

Claim 2.83. The only proper subalgebra of $\mathfrak{su}(2,1)$ properly containing $\mathfrak{su}(2)$ is $\mathfrak{s}(\mathfrak{u}(2) \times \mathfrak{u}(1))$.

Proof of the claim: Let $\mathfrak{k}$ be such a subalgebra. If $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{u}(2) \times \mathfrak{u}(1))$, then $\mathfrak{k} = \mathfrak{s}(\mathfrak{u}(2) \times \mathfrak{u}(1))$ by a dimension argument.

Otherwise, we claim that the action of $\mathfrak{k}$ on $\mathbf{C}^3$ is irreducible. Let us consider the decomposition $\mathbf{C}^3 = \mathbf{C}^2 \oplus \mathbf{C}$. Let A be the $\mathbf{C}$-subalgebra of $\mathrm{M}_3(\mathbf{C})$ generated by $\mathfrak{k}$. Since the action of $\mathfrak{su}(2)$ on $\mathbf{C}^2$ is irreducible, A contains $\mathrm{M}_2(\mathbf{C}) \times \mathrm{M}_1(\mathbf{C})$. In particular, the only possible stable subspaces are $\mathbf{C}^2$ and $\mathbf{C}$. Now observe that since they are orthogonal to each other, if one is stable by $\mathfrak{k}$, then so is the other. So, if $\mathfrak{k}$ does not act irreducibly, it preserves these subspaces; this means that $\mathfrak{k} \subset \mathfrak{s}(\mathfrak{u}(2) \times \mathfrak{u}(1))$. □

By the claim, N is virtually isomorphic to a connected Lie group locally isomorphic to either $\mathrm{SO}_3(\mathbf{R})$ or $\mathrm{SO}_3(\mathbf{R}) \times \mathbf{R}$. So, by [GHW05, §5, Theorem 4], Γ is Haagerup.

Otherwise, $S_c = 1$. Since G is Haagerup, by [CCJJV01, Chap. 4] (or Corollary 2.53), $[S_{nc}, R_u] = \{1\}$ so that S_{nc} is, up to a finite kernel, a direct factor of N. Since we have proved in d) that the only possible simple factor appearing in S_{nc} are locally isomorphic to $\mathrm{SL}_2(\mathbf{R})$ or $\mathrm{SL}_2(\mathbf{C})$,[8] in view of [GHW05, §5, Theorem 4], this implies that Γ has a subgroup of finite index which is Haagerup, so that Γ is Haagerup.

Finally let us drop the hypothesis $Z = 1$. Let N be the preimage in G of the Zariski closure of Γ in G/Z. There are two possible cases:

• N has finitely many connected components. Then, by Theorem 1.28 (which relies on similar arguments), every subgroup of N is Haagerup for the discrete topology.

• N has infinitely many connected components. Then N is almost the direct product of Z and N/Z, so that, by the case $Z = 1$, every subgroup of N is Haagerup for the discrete topology. □

The following proposition shows that the statements in Theorem 2.82 are, in a certain sense, optimal: in 1), the assumption that Λ be a normal subgroup cannot be dropped, etc.

Proposition 2.84. **Let G be a connected Lie group, locally isomorphic to either $\mathrm{SO}(4,1)$ or $\mathrm{SU}(2,1)$.**

1) G has finitely presented subgroups $\Gamma \supset \Lambda$, such that Λ is infinite and (Γ, Λ) has relative Property (T).

2) If G is locally isomorphic to $\mathrm{SO}(4,1)$, then G has a finitely presented subgroup Γ and an infinite normal subset $X \subset \Gamma$ such that (Γ, X) has relative Property (T).

Proof: 1) First suppose that $G = \mathrm{SU}(2,1)$, and write $G = H(\mathbf{R})$, where $H(R)$ is defined, for every commutative ring R as the set of matrices (A, B) (rather denoted $A+iB$) which satisfy the relation $({}^tA - i\,{}^tB)J(A+iB) = J$, where[9] J is the diagonal matrix $\mathrm{diag}(1, 1, -1)$. Let K be defined as the upper-left 2×2 block in H, so that $K(\mathbf{R}) \simeq \mathrm{SU}(2)$. Observe that $H(\mathbf{C}) \simeq \mathrm{SL}_3(\mathbf{C})$ and $K(\mathbf{C}) \simeq \mathrm{SL}_2(\mathbf{C})$.

Then $\Gamma = H(\mathbf{Z}[2^{1/3}])$ embeds as a lattice in $H(\mathbf{R}) \times H(\mathbf{C})$. By Theorem 2.71, the projection p of Γ into $H(\mathbf{R}) \simeq \mathrm{SU}(2,1)$ is a resolution. Set $\Lambda = K(\mathbf{Z}[2^{1/3}])$. Then Λ is a lattice in $K(\mathbf{R}) \times K(\mathbf{C})$, so embeds as a cocompact lattice in $K(\mathbf{C}) \simeq \mathrm{SL}_2(\mathbf{C})$. On the other hand, since $p(\Lambda)$ is relatively compact (it is dense in $\mathrm{SU}(2)$), by Proposition 2.67, (Γ, Λ) has relative Property (T). Note that, as lattices in connected Lie groups, they are finitely presentable.

Let us now suppose that G is locally isomorphic to $\mathrm{SU}(2,1)$, and let Z be its centre. Let Γ, Λ be as above, and let Γ_0, Λ_0 be their projection in $G/Z \times \mathrm{SL}_3(\mathbf{C})$. Finally, let Γ_1, Λ_1 be their preimage in $G \times \mathrm{SL}_3(\mathbf{C})$. If Z is finite, then it is immediate that (Γ_1, Λ_1) has relative Property (T), and that they are finitely presented. So we suppose that $G = \widetilde{\mathrm{SU}(2,1)}$. Let q be the projection $G \times \mathrm{SL}_3(\mathbf{C}) \to \mathrm{SU}(2,1) \times \mathrm{SL}_2(\mathbf{C})$, and observe that $\Gamma_1 = q^{-1}(\Gamma)$ and $\Lambda_1 = q^{-1}(\Lambda)$.

[8] **Actually, it is easily checked that $\mathfrak{sl}_2(\mathbf{C})$ does not embed in $\mathfrak{su}(2,1)$.**

[9] **This relation must be understood as a relation where i is a formal variable satisfying $i^2 = -1$. In other words, this means ${}^tAJA + {}^tBJB = J$ and ${}^tAJB - {}^tBJA = 0$.**

Since $K(\mathbf{R}) \times K(\mathbf{C}) \simeq \mathrm{SU}(2) \times \mathrm{SL}_2(\mathbf{C})$ is simply connected, $W = q^{-1}(K(\mathbf{R}) \times K(\mathbf{C}))$ is isomorphic to $K(\mathbf{R}) \times K(\mathbf{C}) \times \mathbf{Z}$, and contains Λ_1 as a lattice. So we can define Λ_2 as the projection of Λ_1 into the unit component W_0, which is isomorphic to Λ, hence finitely presentable. Since the projection of Λ_2 on $G = K(\mathbf{R})$ is relatively compact, by Proposition 2.67, (Γ_1, Λ_2) has relative Property (T).

A similar example can be constructed in $\mathrm{SO}(4,1)$, projecting an irreducible lattice from $\mathrm{SO}(4,1) \times \mathrm{SO}_5(\mathbf{C})$. Since $\mathrm{SO}(4,1)$ has finite fundamental group, we do not have to care with some of the complications of the previous example.

2) Observe that $\mathrm{SO}(4,1)$ has a subgroup isomorphic to $\mathrm{SO}_3(\mathbf{R}) \ltimes \mathbf{R}^3$. Indeed, if we write $\mathrm{SO}(4,1)$ as $\{A,\ {}^tAJA = J\}$, where $J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, then it contains the following subgroup which is isomorphic to $\mathrm{SO}_3(\mathbf{R}) \ltimes \mathbf{R}^3$:

$$P = \left\{ \begin{pmatrix} 1 & -{}^tvA & -{}^tvv/2 \\ 0 & A & v \\ 0 & 0 & 1 \end{pmatrix},\ A \in \mathrm{SO}(3),\ v \in \mathbf{R}^3 \right\}.$$

Now consider the subgroup $\Gamma = \mathrm{SO}_3(\mathbf{Z}[2^{1/3}]) \ltimes \mathbf{Z}[2^{1/3}]^3$. Then Γ embeds as a lattice in $(\mathrm{SO}_3(\mathbf{C}) \ltimes \mathbf{C}^3) \times (\mathrm{SO}_3(\mathbf{R}) \ltimes \mathbf{R}^3)$. By Theorem 2.50, $((\mathrm{SO}_3(\mathbf{C}) \ltimes \mathbf{C}^3) \times (\mathrm{SO}_3(\mathbf{R}) \ltimes \mathbf{R}^3), \mathbf{C}^3)$ has relative Property (T). Therefore, by Theorem 2.71, the inclusion morphism $\Gamma \to \mathrm{SO}_3(\mathbf{Z}[2^{1/3}]) \ltimes \mathbf{R}^3$ is a resolution. Let B be the Euclidean unit ball in $\mathbf{R}^3$. Then, by Proposition 2.67, $(\Gamma, \mathbf{Z}[2^{1/3}]^3 \cap B)$ has relative Property (T). Finally observe that $X = \mathbf{Z}[2^{1/3}]^3 \cap B$ is a normal subset in Γ.

Now observe that Γ is contained in P, hence is contained in the unit component $\mathrm{SO}_0(4,1)$. The only other connected Lie group with Lie algebra $\mathfrak{so}(4,1)$ is its universal covering (of degree 2); taking the preimage of Γ and X, we obtain the required pair with relative Property (T). □

Remark 2.85. Examples similar to $\Gamma = \mathrm{SO}_3(\mathbf{Z}[2^{1/3}]) \ltimes \mathbf{Z}[2^{1/3}]^3$ (see the proof of Proposition 2.84) were already introduced in Chapter 1 (Remark 1.34), where it was observed that they provide the first known examples of groups without the Haagerup Property, but have no infinite subgroup with relative Property (T). We have made here more concrete the negation of the Haagerup Property by exhibiting an infinite subset with relative Property (T).

2.3.7 Affine resolutions

Although they are probably known to the specialists, we found no reference for the following lemmas.

Lemma 2.86. **Let M be a CAT(0) metric space. Let X be a nonempty bounded subset, and let $B'(c,r)$ be the closed ball of minimal radius containing X [BrHa99, Chap. II, Corollary 2.8(1)]. Suppose that X is contained in another ball $B'(c',r')$. Then**

$$d(c,c')^2 \leq r'^2 - r^2.$$

Proof: Suppose the contrary, so that $d^2 = \|c - c'\|^2 > r'^2 - r^2 \geq 0$. For $t \in [0,1]$, set $p_t = (1-t)c + tc'$, which is a well-defined point on the geodesic segment $[cc']$.

By [Bro89, (**) p.153], for every $z \in M$ and every $t \in [0,1]$,

$$d(z, p_t)^2 \leq (1-t)d(z,c)^2 + td(z,c')^2 - t(1-t)d^2. \tag{2.3.1}$$

It follows that if $z \in B'(c,r) \cap B'(c',r')$, then $d(z,p_t)^2 \leq (1-t)r^2 + tr'^2 - t(1-t)d^2$; denote by $u(t)$ this expression. By an immediate calculation, $u(t)$ is minimal for $t = t_0 = (d^2 + r^2 - r'^2)/(2d^2)$, which belongs to $]0,1]$ by assumption. Since $u(0) = r^2$, it follows that $u(t_0) < r^2$. Since this is true for all $z \in B'(c,r) \cap B'(c',r')$, this implies that X is contained in a closed ball of radius $u(t_0)^{1/2} < r$, contradiction. □

Lemma 2.87. **Let M be a complete CAT(0) space. Let K be a nonempty closed convex bounded subset, and $B'(c,r)$ the ball of minimal radius containing K. Then $c \in K$.**

Proof: Suppose that $c \notin K$, and let p be its projection on K [BrHa99, Chap. II, Proposition 2.4]. Fix $x \in K$. Then for every $p' \in [px]$, $d(p,c) \leq d(p',c)$. Hence, by (2.3.1), for all $t \in [0,1]$, $d(p,c)^2 \leq (1-t)d(p,c)^2 + td(x,c)^2 - t(1-t)d(p,x)^2$. Taking the limit, after dividing by t, when $t \to 0$, gives $d(x,p)^2 \leq d(x,c)^2 - d(p,c)^2$, so that $d(x,p)^2 \leq r^2 - d(p,c)^2$. In other words, $K \subset B'(p, (r^2 - d(p,c)^2)^{1/2})$. This contradicts the minimality of r. □

Lemma 2.88. **Let M be a complete CAT(0) space. Let (F_n) be a decreasing sequence of nonempty closed convex bounded subsets. Then $\bigcap F_n \neq \emptyset$.**

Proof: Let $B'(c_n, r_n)$ be the ball of minimal radius containing F_n. Observe that $c_n \in F_n$ by Lemma 2.87. Moreover, (r_n) is non-increasing, hence converges.

On the other hand, if $m \leq n$, then $F_n \subset F_m$. Applying Lemma 2.86, we get $d(c_n, c_m)^2 \leq r_n^2 - r_m^2$. Therefore, (c_n) is Cauchy, hence has a limit c, which belongs to $\bigcap F_n$. □

Theorem 2.89. **Let $f : G \to Q$ be a morphism with dense image between locally compact groups. Let G act by isometries on a complete CAT(0) space M.**

Suppose that there exists a neighbourhood Ω of 1 in Q, such that, for some $w \in M$, $f^{-1}(\Omega)w$ is bounded. Then M^Q is nonempty.

Proof: Let (Ω_n) be a sequence of compact symmetric neighbourhoods of 1 in Q, contained in Ω, such that $\Omega_{n+1} \cdot \Omega_{n+1} \subset \Omega_n$ for all n. Set $V_n = f^{-1}(\Omega_n)$.

By the assumption on Ω, $V_n \cdot w$ is bounded for all n. Let $B'(c_n, r_n)$ be the minimal ball containing $V_n \cdot w$. Note that the sequence (c_n) is bounded since $d(c_n, w) \leq r_n \leq r_0$ for all n.

Then, for all $g \in V_{n+1}$, $g^{-1}V_{n+1} \cdot w \subset V_n \cdot w \subset B'(c_n, r_n)$, so that $V_{n+1} \cdot w \subset B'(gc_n, r_n)$. By Lemma 2.86, we have

$$d(c_{n+1}, gc_n) \leq \sqrt{(r_n - r_{n+1})(r_n + r_{n+1})} \leq \sqrt{2r_0(r_n - r_{n+1})}.$$

Specializing this inequality to $g = 1$, we obtain

$$d(c_{n+1}, c_n) \leq \sqrt{2r_0(r_n - r_{n+1})},$$

and combining the two previous inequalities, we get, for all $g \in V_{n+1}$,

$$d(c_n, gc_n) \leq d(c_{n+1}, c_n) + d(c_{n+1}, gc_n) \leq 2\sqrt{2r_0(r_n - r_{n+1})}.$$

Set $u(n) = \sup\{2\sqrt{2r_0(r_m - r_{m+1})},\ m \geq n\}$. Since (r_n) is non-increasing and nonnegative, $r_n - r_{n+1} \to 0$, so that $u(n) \to 0$.

Note that, for all g, the function $x \mapsto d(x, gx)$ is continuous and convex on M [BrHa99, Chap. II, Proposition 6.2]. It follows that $F_n = \{v \in \mathscr{H},\ \forall g \in V_{n+1},\ d(v, gv) \leq u(n)\}$ is closed and convex. Set $K_n = F_n \cap B'(w, r_0)$. Then (K_n) is a decreasing sequence of closed, convex, bounded subsets of M, nonempty since $c_n \in K_n$. By Lemma 2.88, $\bigcap K_n$ is nonempty; pick a point y in the intersection. We claim that $y \in X^Q$: to see this, let us appeal to Proposition 2.59. Let g_i be a net in G such that $f(g_i) \to 1$.

Set $n_i = \sup\{n, g_i \in \Omega_{n+1}\} \in \mathbf{N} \cup \{\infty\}$. Then $n_i \to \infty$ since all Ω_n are neighbourhoods of 1 in Q, and $d(y, g_i y) \leq u(n_i)$ for all i (where we set $u(\infty) = 0$). It follows that $d(y, g_i y) \to 0$. By Proposition 2.59, $y \in X^Q$. □

Definition 2.90. Let $f : G \to Q$ be a morphism with dense image between locally compact groups. We call it an affine resolution if, for every isometric action of G on an affine Hilbert space, there exists an affine G-invariant subspace such that the action of G on this subspace factors through Q.

Theorem 2.91. **Let G, Q be locally compact groups, $f : G \to Q$ be a morphism with dense image. The following implications (1)⇒(2)⇒(3)⇔(4) hold. Moreover, if G is σ-compact, then (4)⇒(1), so that they are all equivalent.**

(1) $G \to Q$ is a resolution.

(**(2)**

action of G, contradicting $b \notin B^1(G,\pi)$. Hence the linear part of α^Q is a nonzero subrepresentation of π, so that π^Q is nonzero. □

Let π be a unitary representation of G on a Hilbert space $\mathscr{H}$, such that $1_G \prec \pi$. We must show that $1_Q \prec \pi^Q$. Again, since the case when $1 \leq \pi$ is trivial, we suppose that $1 \nleq \pi$. Let ρ be the orthogonal of π^Q. By the claim, $1_G \nprec \rho$. It follows that $1_G \prec \pi^Q$, so that we can suppose that $\pi = \pi^Q$, i.e. π factors through a representation $\tilde{\pi}$ of Q.

Claim 2.93. The natural continuous morphism $\hat{f} : Z^1(Q,\tilde{\pi}) \to Z^1(G,\pi)$ is bijective.

Proof: It is clearly injective. Let $b \in Z^1(G,\pi)$. Since f is an affine resolution, one can write $b(g) = b'(g) + \pi(g)v - v$ ($\forall g \in G$), where $b' \in Z^1(G,\pi)$ factors through Q and $v \in \mathscr{H}$. Since π also factors through Q, this implies that so does b, meaning that b belongs to $\mathrm{Im}(\hat{f})$. □

Since G and Q are σ-compact, $Z^1(G,\pi)$ and $Z^1(Q,\tilde{\pi})$ are Fréchet spaces. Since $\hat{f} : Z^1(Q,\tilde{\pi}) \to Z^1(G,\pi)$ is bijective, by the open mapping Theorem, it is an isomorphism. Note that it maps $B^1(Q,\tilde{\pi})$ bijectively onto $B^1(G,\pi)$, and that $B^1(G,\pi)$ is not closed in $Z^1(G,\pi)$, as we used in the proof of the first claim. It follows that $B^1(Q,\tilde{\pi})$ is not closed in $Z^1(Q,\tilde{\pi})$. Using the converse in Guichardet's result [BHV05, Theorem 2.13.2], $1_Q \prec \tilde{\pi}$. □

As a corollary of Theorems 2.89 and 2.91, we get:

Corollary 2.94. Let G, Q be locally compact groups, and let $f : G \to Q$ be a resolution. Let G act on a complete CAT(0) space M, and suppose that there exists a G-equivariant proper embedding i of M in a Hilbert space. Then $M^Q \neq \emptyset$.

Proof: Let Ω be a compact neighbourhood of 1 in Q, and set $V = f^{-1}(\Omega)$. Fix $x \in M$, and set $\psi(g) = \|i(gx)\|^2$. Then ψ is conditionally negative definite on G. By Proposition 2.67, ψ is bounded on V. This implies that the hypothesis of Theorem 2.89 is fulfilled. □

There are many metric spaces for which there automatically exists such an equivariant embedding; namely, those metric spaces M which have a $\mathrm{Isom}(M)$-equivariant embedding in a Hilbert space. Thus the hypotheses of Corollary are satisfied, for instance when

- M is a Hilbert space,
- M is a tree, or a complete $\mathbf{R}$-tree [HaVa89, Chap. 6, Proposition 11].
- M is a real or complex hyperbolic space (maybe infinite-dimensional) [FaHa74],
- M is a finite-dimensional CAT(0) cube complex.

For instance:

Corollary 2.95. Let G, Q be locally compact groups, and let $f : G \to Q$ be a resolution. Then G has Property (FA) if and only if Q does.

Proof: If G has Property (FA), so does Q. Let us show the converse. By a result of Alperin and Watatani (see [HaVa89, Chap. 6]), every tree equivariantly embeds in a Hilbert space (more precisely, the distance is a conditionally negative definite kernel). It follows that, for every action of G on a tree, there exists a nonempty

G-invariant subtree on which the action factors through Q. The result immediately follows. □

Theorem 2.91 allows us to prove the converse of Corollary 2.66.

Theorem 2.96. **Let $f : G \to Q$ be a morphism between locally compact groups, with dense image, and suppose G σ-compact. Then f is a resolution if and only if, for every net (π_i) of irreducible representation of G converging to 1_G, eventually π_i factors through a representation $\tilde{\pi}_i$ of Q, and $\tilde{\pi}_i \to 1_Q$.**

Proof: The condition is clearly necessary. Suppose that it is satisfied. Let us show that (2) of Theorem 2.91 is satisfied, and let us use Theorem 2.22. Fix $\varepsilon > 0$, let $X \subset G$ be such that $\overline{f(X)}$ is compact, and let (π_i) be a net of irreducible representation of G converging to 1_G. Then eventually π_i factors through a representation $\tilde{\pi}_i$ of Q, and $\tilde{\pi}_i \to 1_Q$. Since $\overline{f(X)}$ is compact, this implies that, eventually, $\tilde{\pi}_i$ has a $(\overline{f(X)}, \varepsilon)$-invariant vector, so that π_i has a (X, ε)-invariant vector. □

2.3.8 Preresolutions

Definition 2.97. Let $f : G \to Q$ be a morphism between locally compact, with dense image.

We say that f is a **preresolution** if, for every unitary representation π of G, $1_G \prec \pi$ implies $\pi^Q \neq 0$.

We say that f satisfies Condition (K) if, for every unitary representation π of Q, if $1_G \prec \pi \circ f$, then $1_Q \prec \pi$.

Note that Condition (K) is satisfied whenever $G \to Q$ is the quotient by a normal subgroup. On the other hand, if $\mathbf{Z} \to \mathbf{R}/\mathbf{Z}$ is any dense embedding, it is easy to check that it does not satisfy Condition (K) (see Remark 2.73).

By definition, f is a resolution if and only if it is a preresolution and satisfies Condition (K). It is natural to ask whether Condition (K) is actually redundant in this definition.

With the help of Theorem 2.91, we show that the answer is positive when G is σ-compact (see Corollary 2.104). We first need some preliminary results.

Proposition 2.98. **If $f : G \to Q$ is a morphism between locally compact groups, with dense image. Equivalences:**

(i) $G \to Q$ is a preresolution.

(ii) There exists a "Kazhdan pair" (K, ε), K compact in G, and $\varepsilon > 0$, such that every representation of G with a (K, ε)-invariant vector has a nonzero subrepresentation which factors through Q.

(iii) For every set $\mathcal{R}$ of unitary representations of G which do not contain a nonzero subrepresentation which factors through Q, 1_G is isolated in $\mathcal{R} \cup \{1_G\}$.

Proof: (iii)⇒(ii). If (ii) is not satisfied, choose, for every (K, ε), K compact, $\varepsilon > 0$, a representation $\pi_{K,\varepsilon}$ which does not contain a nonzero subrepresentation factoring through Q. Then 1_G is not isolated in $1_G \cup \{\pi_{K,\varepsilon},$ varying $(K, \varepsilon)\}$; this contradicts (iii).

(ii)⇒(i) is trivial.

(i)⇒(iii). If (iii) is not satisfied, let $\pi_i \to 1_G$, where π_i does not contain a nonzero subrepresentation factoring through Q. Then $1_G \prec \bigoplus \pi_i$, so that $\bigoplus \pi_i$ contains a subrepresentation ρ which factors through Q. By Lemma 2.61, so does π_i for some i. This is a contradiction. □

If $f : G \to Q$ is a morphism between topological groups, we say that $X \subset G$ is Q-compact (resp. Q-relatively compact) if $f(X)$ (resp. $\overline{f(X)}$) is compact.

Lemma 2.99. **Let π be a unitary representation of G. Let ξ, ξ' be any vectors, and φ, φ' the corresponding positive definite functions. Then**

$$\|\varphi - \varphi'\|_\infty \leq \|\xi - \xi'\|(\|\xi\| + \|\xi'\|), \quad \text{and}$$
$$\|\pi(g)\xi - \xi\|^2 = 2(\|\xi\|^2 - \mathrm{Re}(\varphi(g))).$$

Proof: For all $g \in G$,

$$\begin{aligned}\varphi(g) - \varphi'(g) &= \langle \pi(g)\xi, \xi\rangle - \langle \pi(g)\xi', \xi'\rangle \\ &= \langle \pi(g)\xi, \xi - \xi'\rangle + \langle \pi(g)(\xi - \xi'), \xi'\rangle,\end{aligned}$$

hence $|\varphi(g) - \varphi'(g)| \leq \|\xi - \xi'\|(\|\xi\| + \|\xi'\|)$ by the Cauchy-Schwarz inequality.

The second assertion is obtained by an immediate calculation. □

Proposition 2.100. **Let (K, ε) a Kazhdan pair for a preresolution $G \to Q$, with K compact and $\varepsilon > 0$. Let $(\pi, \mathscr{H})$ be a unitary representation of G, and let P be the orthogonal projection onto $\mathscr{H}^Q$. Take $\delta \in [0, 1]$. Then, for every $(K, \delta\varepsilon)$-invariant vector $\xi \in \mathscr{H}$, we have**

$$\|\xi - P\xi\| \leq \delta\|\xi\|, \text{ and}$$
$$\|\xi - \frac{1}{\|P\xi\|}P\xi\| \leq 2\delta\|\xi\| \text{ (provided } \delta < 1 \text{ and } \xi \neq 0).$$

For every positive definite function φ on G, if Re$(\varphi) \geq 1 - \delta^2\varepsilon^2/2$ on K, then there exists a positive definite function φ' on G, factoring through Q, such that $\varphi(1) = \varphi'(1)$, and $|\varphi - \varphi'| \leq 4\delta\varphi(1)$ on G.

Proof: Write $\xi = \xi_1 + \xi_2$, with $\xi_1 = P\xi$. For every $g \in K$, $\|\xi - \pi(g)\xi\| \leq \delta\varepsilon\|\xi\|$. Since the representation $(\mathscr{H}^Q)^\perp$ has no subrepresentation which factors through Q, there exists $g \in K$ such that $\varepsilon\|\xi_2\| \leq \|\xi_2 - \pi(g)\xi_2\|$. Hence

$$\varepsilon^2\|\xi_2\|^2 \leq \|\xi_2 - \pi(g)\xi_2\|^2 = \|\xi - \pi(g)\xi\|^2 - \|\xi_1 - \pi(g)\xi_1\|^2 \leq \|\xi - \pi(g)\xi\|^2 \leq \delta^2\varepsilon^2\|\xi\|^2.$$

Since $\xi_2 = \xi - P\xi$, this gives $\|\xi - P\xi\| \leq \delta\|\xi\|$. The second inequality follows easily, and the statement on positive definite functions follows by Lemma 2.99 and a GNS construction. □

As a corollary, we get:

Corollary 2.101. **Let $G \to Q$ be a preresolution. For every net (φ_i) of positive definite normalized functions converging to 1 uniformly on compact subsets, there exists another net (φ'_i) positive definite normalized functions on G, factoring through Q, such that $\|\varphi - \varphi'_i\|_\infty \to 0$.** □

Lemma 2.102. **Let G be a locally compact group, and $f : G \to Q$ a resolution. Let $\psi : G \to \mathbf{R}_+$ be a conditionally negative definite function on G. Then, there exists a symmetric, open neighbourhood A of 1 in Q such that ψ is bounded on $f^{-1}(A)$.**

Proof: By Schönberg's Theorem, for every $t > 0$, $e^{-t\psi}$ is definite positive; moreover, it tends to 1, uniformly on compact subsets, when $t \to 0$. By Corollary 2.101, there exists, for every $t > 0$, a normalized positive definite function φ_t such that $\|e^{-t\psi} - \varphi_t\|_\infty \to 0$. For some fixed $t > 0$, $\|e^{-t\psi} - \varphi_t\| \leq 1/4$. Since φ_t factors through a continuous function on Q, there exists a symmetric, open neighbourhood A of 1 in Q such that $|1 - \varphi_t| \leq 1/4$ on $f^{-1}(A)$. Hence $1 - e^{-t\psi} \leq 1/2$ on $f^{-1}(A)$, so that $\psi \leq \log(2)/t$ on $f^{-1}(A)$. □

Theorem 2.103. **Let G be a locally compact group, $f : G \to Q$ be a preresolution. Then f satisfies Condition (3) of Theorem 2.91, namely, (G, B) has relative Property (FH) for every Q-compact subset B of G.**

Proof: Let ψ be a conditionally negative definite function on G. Let K be a compact subset of Q, and let us show that ψ is bounded on $f^{-1}(K)$. Let A be as in Lemma 2.102, so that ψ is bounded on $B = f^{-1}(A)$.

Let Ω be the subgroup of Q generated by A; it is open. Then K is contained in the union of finitely many cosets $q_1\Omega, \ldots, q_n\Omega$. Since Ω is open and $f(G)$ dense, we can choose $q_i \in f(G)$, say, $q_i = f(g_i)$.

Set $L = \bigcup_{i=1}^n q_i^{-1}(K \cap q_i\Omega)$, so that $K \subset \bigcup_{i=1}^n q_i L$.

Since L is a compact subset of Ω, L is contained in A^n for some n. So, by Lemma 2.70, $f^{-1}(L)$ is contained in B^{n+1}. Since ψ is bounded on B, it is also bounded on B^{n+1}, hence on $f^{-1}(L)$.

It follows that ψ is bounded on $\bigcup_{i=1}^n q_i f^{-1}(L) = f^{-1}(\bigcup_{i=1}^n g_i L)$, which contains $f^{-1}(K)$. □

Corollary 2.104. **Let G, Q be locally compact, σ-compact groups, and $f : G \to Q$ a morphism with dense image. Then f is a resolution if and only if it is a preresolution.**

Proof: This follows from Theorems 2.91 and 2.103. □

Chapter 3

Dense subgroups with Property (T) in Lie groups

In this chapter, we characterize connected Lie groups which have a dense finitely generated subgroup Γ with Property (T) (when viewed as a discrete group). The existence of such a dense subgroup is a strengthening of Property (T); this has been used by Margulis and Sullivan [Mar80, Sul81] to solve the Ruziewicz Problem in dimension $n \geq 4$, namely that the Lebesgue measure is the only mean on the measurable subsets of S^n, invariant under SO_{n+1}.

We begin by recalling a result which characterizes connected Lie groups with Property (T). This is due to S. P. Wang [Wan82], but we give a slightly different formulation.

Theorem 3.1. **Let G be a connected Lie group. Then G has Property (T) if and only if**

(i) Every amenable[1] quotient of G is compact, and

(ii) No simple quotient of G is locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$ for some $n \geq 2$.

In Theorem 3.1, Condition (i) is shown in Chapter 2 (Proposition 2.79) to be equivalent to: every isometric action of G on a Euclidean space has a fixed point. Condition (ii) can be interpreted as: every isometric action on a real or complex finite-dimensional hyperbolic space has a fixed point. Thus Theorem 3.1 has the following geometric reformulation.

Theorem 3.2. **Let G be a connected Lie group. Then G has Property (T) if and only if every isometric action of G on a Euclidean, real hyperbolic or complex hyperbolic space has a fixed point.**

Here is the main result of the chapter.

Theorem 3.3. **Let G be a connected Lie group. Then G has a dense, finitely generated subgroup with Property (T) if and only if G has Property (T), and**

(iii) R/Z is not a quotient of G (that is, $\overline{[G,G]} = G$).

(iv) $\mathrm{SO}_3(\mathrm{R})$ is not a quotient of G.

[1] **Recall that a connected Lie group is amenable if and only if its radical is cocompact.**

Remark 3.4. It is easy to check that, for a connected Lie group with Property (T), ((iii) and (iv)) is equivalent to: $\mathrm{Hom}(G, \mathrm{PSL}_2(\mathbf{C})) = \{1\}$, which means, geometrically, that every isometric action on the three-dimensional real hyperbolic space is identically trivial.

The only nontrivial point as regards the necessary condition in Theorem 3.3 is due to Zimmer [Zim84], who shows that $\mathrm{SO}_3(\mathbf{R})$ has no infinite finitely generated subgroup with Property (T). The sufficient condition, constructing a dense subgroup with Property (T), was proved by Margulis [Mar91, chap. III, Proposition 5.7] for G compact.

We state some related results after the proof.

Notation: the Lie algebra of a Lie group or an algebraic group is denoted by the corresponding Gothic letter.

Proof of Theorem 3.1. If the connected Lie group G has Property (T), then Conditions (i) and (ii) are satisfied, since non-compact amenable groups, and connected Lie groups locally isomorphic to $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ for some $n \geq 2$ do not have Property (T) (see [HaVa89, §6.d]).

We deduce the converse from S. P. Wang's classification [Wan82]: denote by R the radical, and S_{nc} the sum of all noncompact simple factors in a Levi factor. If G does not have Property (T), then either (1) S_{nc} does not have Property (T), or (2) $W = \overline{S_{nc}[S_{nc}, R]} \cap R$ is not cocompact in R. In the case (1), (ii) is not satisfied. On the other hand, it is easily seen that W is a normal subgroup of G. So, in case (2), taking the quotient, we can suppose that $W = 1$. So G is locally isomorphic to $S_{nc} \times R_m$, where R_m denotes the amenable radical RS_c, and $\overline{S_{nc}} \cap R = 1$. This implies that G is actually the direct product of R and S_{nc}. So either R or S_{nc} does not have Property (T), giving either the negation of (i) or (ii). □

Proof of Theorem 3.3. If G has a finitely generated dense subgroup Γ with Property (T), then G has Property (T) (indeed, Property (T) is inherited by morphism with dense image, as follows immediately from the definition); (iii) is also clearly satisfied (because Γ has finite abelianization), and also (iv) by [Zim84] (see also [HaVa89, Chap. 6, 26]). We must show that, conversely, these conditions are sufficient.

First step: suppose that $G = H(\mathbf{R})_0$, where H is a linear algebraic group defined over $\mathbf{Q}$ (the subscript 0 means the connected component in the Hausdorff topology). It is well-known that $H(\mathbf{R})_0$ is an open subgroup of finite index in $H(\mathbf{R})$ [BoTi, Corollaire 14.5]. Consider the normal subgroup $W = S_{nc}[S_{nc}, R]$ of H, where S_{nc} denotes the sum of all simple $\mathbf{R}$-isotropic factors in a Levi factor S. Then $W(\mathbf{R})$ is cocompact in $H(\mathbf{R})$ (since $H(\mathbf{R})$ has Property (T)). The hypotheses (iii) and (iv) then imply that H/W is, modulo its finite centre, a product of simple factors of $\mathbf{C}$-rank ≥ 2. This implies that $S[S, R] = H$, and that $(H/R)(\mathbf{C})$ has Property (T). By [Wan82], this implies that $H(\mathbf{C})$ has Property (T).

Now fix a number field of degree 3 over $\mathbf{Q}$, not totally real, and $\mathcal{O}$ its ring of integers: for instance, $\mathcal{O} = \mathbf{Z}[2^{1/3}]$. Then, since H is perfect, by the Borel-Harish-Chandra Theorem [BoHC62], $H(\mathcal{O})$ embeds as an irreducible lattice in $H(\mathbf{R}) \times H(\mathbf{C})$, which has Property (T). So its projection on $G = H(\mathbf{R})$ is a dense subgroup with Property (T). This proves the case of the first step.

Let $\mathfrak{g}$ be the Lie algebra of $\mathfrak{g}$. Set $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$, where $\mathfrak{r}$ is the radical. Let $\mathfrak{s}_{nc}$ be the sum of all factors of positive $\mathbf{R}$-rank of $\mathfrak{s}$, and let $\mathfrak{g}_{nc}$ be the preimage of $\mathfrak{s}_{nc}$ in $\mathfrak{g}$: this is an ideal of $\mathfrak{g}$.

Second step: we can reduce to the case when the Lie algebra $\mathfrak{g}_{nc}$ is perfect.

Set $\mathfrak{h} = \bigcap_{n\geq 0} D^n\mathfrak{g}$, where D means the derived subalgebra. Then $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, generating a normal Lie subgroup H (not necessarily closed) of G. Moreover, $G/\overline{H}$ is solvable, hence trivial by the assumption (iii). This means that H is dense in G. Accordingly, since any dense subgroup of H is dense in G, we can replace G by H and thus suppose that $\mathfrak{g}$ is perfect.

Third step: let us show that if $\mathfrak{g}$ is perfect, and if (i) and (iii) are satisfied, then $\mathfrak{g}_{nc}$ is also perfect (that is, $[\mathfrak{s}_{nc}, \mathfrak{r}] = \mathfrak{r}$).

Consider the adjoint action of G on the quotient $\mathfrak{g}/D(\mathfrak{g}_{nc})$. This defines a morphism $f : G \to \mathrm{GL}(\mathfrak{g}/D(\mathfrak{g}_{nc}))$, such that $\overline{f(G)}$ is amenable. Therefore, the Lie group $f(G)$ is also amenable, hence compact. This implies that $\mathfrak{g}/D(\mathfrak{g}_{nc})$ is a compact Lie algebra [Hel78, Chap. 2, §5], that is, the direct product of an abelian Lie algebra and a semisimple compact Lie algebra. Since $\mathfrak{g}$ is perfect, this implies that $\mathfrak{g}/D(\mathfrak{g}_{nc})$ is semisimple. Since $\mathfrak{g}_{nc}/D(\mathfrak{g}_{nc})$ is an abelian ideal in $\mathfrak{g}/D(\mathfrak{g}_{nc})$, we conclude that $D(\mathfrak{g}_{nc}) = \mathfrak{g}_{nc}$.

Fourth step. We begin by the following standard lemma.

Lemma 3.5. **Let $\mathfrak{g}$ be a Lie algebra, and $\mathfrak{n}$ a nilpotent ideal. Let π denote the projection: $\mathfrak{g} \to \mathfrak{g}/[\mathfrak{n},\mathfrak{n}]$. Let $X \subseteq \mathfrak{g}$ satisfy: $\pi(X)$ generates $\mathfrak{g}/[\mathfrak{n},\mathfrak{n}]$. Then X generates $\mathfrak{g}$.**

Proof: Argue by induction on the length of the descending central series of $\mathfrak{n}$. If $\mathfrak{n}$ is abelian, the result is trivial. Otherwise, let $\mathfrak{z}$ be the least nonzero term of the descending central series of $\mathfrak{n}$. By induction hypothesis, X generates $\mathfrak{g}$ modulo $\mathfrak{z}$. On the other hand, $\mathfrak{z}$ is contained in $[\mathfrak{n},\mathfrak{n}]$, that is, $\mathfrak{z}$ is generated by some elements of the form $[n, n']$, for some $n, n' \in \mathfrak{n}$. Since $\mathfrak{z}$ is central in $\mathfrak{n}$, these elements can be chosen modulo $\mathfrak{z}$, so that they can be taken in the subalgebra generated by X. This implies that $\mathfrak{z}$ is contained in the subalgebra generated by X, so that X generates $\mathfrak{g}$. □

The fourth step consists in proving the following lemma.

Lemma 3.6. **Let $\mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{r}$ be a Lie algebra over $\mathbf{R}$, with $\mathfrak{s}$ semisimple and $\mathfrak{r}$ nilpotent. Then there exists a Lie algebra $\mathfrak{h} = \mathfrak{s} \ltimes \mathfrak{n}$, defined over $\mathbf{Q}$, with $\mathfrak{n}$ nilpotent, and a surjection $p : \mathfrak{h} \to \mathfrak{g}$ which is the identity on the Levi factor, and maps $\mathfrak{n}$ onto $\mathfrak{r}$. If, moreover, $[\mathfrak{s}_{nc}, \mathfrak{r}] = \mathfrak{r}$, we can impose $[\mathfrak{s}_{nc}, \mathfrak{n}] = \mathfrak{n}$.**

Proof: Let $\mathfrak{v}$ be a complementary $\mathfrak{s}$-subspace of $[\mathfrak{r},\mathfrak{r}]$ in $\mathfrak{r}$. By [Wit05], we can fix a $\mathbf{Q}$-form of $\mathfrak{s}$, together with a $\mathbf{Q}$-form of $\mathfrak{v}$, so that the representation of $\mathfrak{s}$ on $\mathfrak{v}$ is defined over $\mathbf{Q}$. By Lemma 3.5, $\mathfrak{r}$ is generated by $\mathfrak{v}$ as a Lie algebra. Let m be such that $\mathfrak{r}$ is m-nilpotent, and let $\mathfrak{n}$ be the free m-nilpotent Lie algebra generated by the vector space $\mathfrak{v}$. The action of $\mathfrak{s}$ on $\mathfrak{v}$, which is defined over $\mathbf{Q}$, extends naturally to an action on $\mathfrak{n}$, also defined over $\mathbf{Q}$. On the other hand, by the universal property of the embedding $\mathfrak{v} \to \mathfrak{n}$, the identity map $\mathfrak{v} \to \mathfrak{v}$ extends to a Lie algebra morphism of $\mathfrak{n}$ onto $\mathfrak{r}$, which is actually a morphism of $\mathfrak{s}$-modules: indeed, let $s \in \mathfrak{s}$. Then s gives a derivation on $\mathfrak{n}$, whose image is a derivation of $\mathfrak{r}$ which coincides, in restriction to $\mathfrak{v}$, with $\mathfrak{ad}(s)$. Since $\mathfrak{r}$ is generated by $\mathfrak{v}$, this implies that they coincide on all of $\mathfrak{r}$. Therefore, the surjection $\mathfrak{n} \to \mathfrak{r}$ extends to a surjection $\mathfrak{s} \ltimes \mathfrak{n} \to \mathfrak{s} \ltimes \mathfrak{r}$.

If $[\mathfrak{s}, \mathfrak{r}] = \mathfrak{r}$, the condition $[\mathfrak{s}_{nc}, \mathfrak{n}] = \mathfrak{n}$ is immediate, since then $[\mathfrak{s}_{nc}, \mathfrak{n}]$ contains $\mathfrak{v}$. □

In view of the second and third steps, the hypotheses are now: $\mathfrak{g}_{nc}$ is perfect, and $\mathfrak{g}$ has no simple factor $\mathfrak{s}$ isomorphic to $\mathfrak{so}(3)$, $\mathfrak{so}(n,1)$ or $\mathfrak{su}(n,1)$.

Fifth step: suppose that $G = H(\mathbf{R})_0$, where H is a connected linear algebraic group defined over $\mathbf{R}$ such that $\mathfrak{h} = \mathfrak{g}$ is perfect. Choose $p : \hat{\mathfrak{h}} \to \mathfrak{h}$ as in Lemma 3.6, and let $\mathfrak{p} \subset \hat{\mathfrak{h}} \times \mathfrak{h}$ be its graph. Using [Bor91, Corollary 7.9] twice, $\hat{\mathfrak{h}}$ is the Lie algebra of a simply connected linear algebraic group $\hat{H}$ defined over $\mathbf{Q}$, and $\mathfrak{p}$ is the Lie algebra of a $\mathbf{R}$-closed subgroup $P \subset \hat{H} \times H$. Since $\mathfrak{p} \cap \mathfrak{h} = \{0\}$, by [Bor91, Corollary 6.12], $P \cap H$ is finite. Since p is onto, the projection of $P(\mathbf{R})$ into $H(\mathbf{R})$ is Zariski dense; thus $W = P \cap H$ is normal in H. Replacing H by H/W if necessary, we assume that $W = \{1\}$. Since the projection $\mathfrak{p} \to \hat{\mathfrak{h}}$ is onto, the projection of $P(\mathbf{R})$ on $\hat{H}(\mathbf{R})$ contains an open subgroup for the Hausdorff topology, but this topology is connected since we have chosen $\hat{H}$ simply connected. Hence P is the graph of a morphism of $\hat{H}$ onto H, still denoted by P.

By the first step, $\hat{H}(\mathbf{R})$ has a finitely generated dense subgroup $\hat{\Gamma}$ with Property (T). It follows that $P(\hat{\Gamma}) \cap G$ is a dense subgroup with Property (T) in G.

Sixth step. We now conclude. We have reduced to the case when $\mathfrak{g}$ is perfect; in this setting, the hypotheses (recalled before Step 5) only depend on $\mathfrak{g}$. So we can reduce to the case when G is simply connected: indeed, if G satisfies the above hypotheses, and if we can prove that its universal cover has a dense subgroup with Property (T), this one projects densely on G. Accordingly, we also suppose that G is simply connected. Since $\mathfrak{g}$ is perfect, there exists a linear algebraic $\mathbf{R}$-group H with Lie algebra $\mathfrak{g}$, so that there exists a discrete, central subgroup Z of G such that G/Z is isomorphic to $H(\mathbf{R})_0$. By the fifth step, $H(\mathbf{R})_0 = G/Z$ has a dense subgroup Γ with Property (T).

Let $\tilde{\Gamma}$ be the preimage of Γ in G. Define Z_n as the kernel of the natural morphism $D^n(\tilde{\Gamma}) \to D^n(\Gamma)$, so that we have, for all n, an exact sequence:

$$1 \to Z_n \to D^n(\tilde{\Gamma}) \to D^n(\Gamma) \to 1.$$

Then (Z_n) is a decreasing sequence of subgroups of Z. Moreover, since Γ has Property (T), for every n, $D^n(\Gamma)$ has finite index in Γ. Accordingly, for each n such that $D^n(\tilde{\Gamma})/D^{n+1}(\tilde{\Gamma})$ is infinite, we have $\mathrm{rk}(Z_{n+1}) < \mathrm{rk}(Z_n)$. This implies the existence of n such that $D^n(\tilde{\Gamma})$ has finite abelianization. Therefore, by Serre's Theorem on central extensions [HaVa89, Théorème 12], $D^n(\tilde{\Gamma})$ has Property (T). We finally claim that $D^n(\tilde{\Gamma})$ is dense in G: this follows from the fact that $\tilde{\Gamma}$ is dense in G and G is topologically perfect. □

Theorem 3.3 can be compared to the following result.

Proposition 3.7. **Let G be a connected Lie group. Then G has an infinite, finitely generated subgroup with Property (T) if and only if G has at least a simple factor which is not locally isomorphic to $\mathrm{SO}(3)$, $\mathrm{SL}_2(\mathbf{R})$, $\mathrm{SL}_2(\mathbf{C})$, $\mathrm{SO}(4,1)$, $\mathrm{SU}(2,1)$.**

Proof: Suppose that G has such a simple factor S; through a Levi factor, S embeds in G as a (non-necessarily closed) subgroup of G. If S has Property (T), then it has a dense (hence infinite) subgroup with Property (T) (this follows from Theorem 3.3, since we have excluded SO(3)). Otherwise, S is locally isomorphic to $\mathrm{SU}(n, 1)$ ($n \geq 3$) or $\mathrm{SO}(n, 1)$ ($n \geq 5$). Then S has a compact subgroup K locally isomorphic to $\mathrm{SU}(n)$ ($n \geq 3$) or $\mathrm{SO}(n)$ ($n \geq 5$). By [Mar91, chap. III, Proposition 5.7] (or Theorem 3.3), K has a dense (hence infinite) finitely generated subgroup with Property (T).

Conversely, if G contains an infinite subgroup Γ with Property (T), then, since Γ is not virtually solvable, the projection of Γ modulo the radical is infinite, so that we are reduced to the case when G is semisimple; we assume this now. Similarly, the projection of Γ modulo the centre is infinite. So now we suppose that G is a connected, centre-free semisimple Lie group, hence a direct product of simple factors. The projection into at least one factor, say, S, must be infinite. It then suffices to show that S cannot be locally isomorphic to one of the five groups quoted in the proposition. Since each of these five groups has the Haagerup Property [HaVa89, §6.d], i.e. acts properly on a Hilbert space, Γ must be contained in a maximal compact subgroup. Thus Γ embeds in $\mathrm{SO}_3(\mathbf{R})$, and this is in contradiction with Zimmer's result already used above [Zim84]. □

Remark 3.8. In contrast, it is proved in Chapter 2 (Proposition 2.84) that, in $\mathrm{SO}_0(4, 1)$ and $\mathrm{SU}(2, 1)$, there exists infinite subgroups $\Lambda \subset \Gamma$, such that (Γ, Λ) has **relative** Property (T). Moreover, Λ cannot be chosen normal, and Γ is necessarily dense.

In some "minimal" cases, an infinite subgroup with Property (T) is necessarily dense or Zariski dense.

Proposition 3.9. **Let G be a simple, connected Lie group, locally isomorphic to $\mathrm{SO}(5)$, $\mathrm{SU}(3)$, $\mathrm{Sp}_4(\mathbf{R})$, or $\mathrm{SL}_3(\mathbf{R})$. Then every infinite, finitely generated subgroup with Property (T) is either dense, or discrete and Zariski dense [2].**

Proof: Projecting modulo the centre, we can suppose that G in center-free and thus is algebraic. Let Γ be an infinite subgroup and H its Zariski closure; suppose by contradiction $H \neq G$. Then H has a simple factor S which is not one of the five groups quoted in Proposition 3.7. Observe that $\dim(S) < \dim(G)$. If G is either SU(3) or $\mathrm{SL}_3(\mathbf{R})$, then this implies $\dim(S) < 8$ and thus S is one of the five groups quoted in Proposition 3.7. If G is either SO(5) or $\mathrm{Sp}_4(\mathbf{R})$, then $\dim(G) = 10$ and we must have $\dim(S) = 8$, otherwise we contradict again Proposition 3.7. But passing to the complexification, we get an embedding of the simple 8-dimensional subalgebra SL_3 into the simple 10-dimensional simple Lie algebra $\mathrm{Sp}_4(\simeq \mathrm{SO}_5)$, and this does not exist (the root system A_2 does not embed in the root system B_2), a contradiction.

Finally Γ is Zariski dense, so that the Lie algebra of its Hausdorff closure is normalized by all of G, hence is either trivial or all of $\mathfrak{g}$, i.e. Γ is either discrete or dense. □

[2] **Note that G is not necessarily algebraic, however this statement makes sense if we define a Zariski dense subset as a subset which is Zariski dense modulo the centre.**

Question 3.10. Does there exist an infinite, discrete subgroup of $\mathrm{SL}_3(\mathbf{R})$, $\mathrm{Sp}_4(\mathbf{R})$, or $\mathrm{Sp}(2,1)$ which has Property (T), but is not a lattice?

Remark 3.11. Following Shalom [Sha99t], a locally compact group has **strong Property (T)** if it has a **finite** Kazhdan subset.

The following implications are immediate: G has a dense finitely generated subgroup with Property (T) $\Rightarrow$ G has strong Property (T) $\Rightarrow$ G has Property (T).

Shalom proves that a connected Lie group G with Property (T) has strong Property (T) if and only if $\mathbf{R}/\mathbf{Z}$ is not a quotient of G, i.e. if G is topologically perfect. The most remarkable result is that $\mathrm{SO}_3(\mathbf{R})$ has strong Property (T); this is actually a reformulation of a deep result of Drinfeld [Dri84].

We would like to generalize Lemma 3.6, and thus Theorem 3.3 to algebraic groups over $\mathbf{Q}_p$, $\mathrm{SO}_3(\mathbf{R})$ being replaced by anisotropic simple groups whose complexification is a product of rank one factors. This would be possible if the following question has a positive answer:

Question 3.12. Let G be a simply connected, semisimple group over $\mathbf{Q}_p$. Let V be a rational representation of G. Does V have a $\mathbf{Q}$-form?

Two hints towards a positive answer to this question are:

- The answer is positive when $\mathbf{Q}_p$ is replaced by $\mathbf{R}$ [Wit05].
- Every simply connected semisimple group over $\mathbf{Q}_p$ has a $\mathbf{Q}$-form. This is a straightforward consequence of [BoHa78, Theorem B].

Remark 3.13. Question 3.12 is equivalent to the following one: if $\mathfrak{g}$ is a perfect Lie algebra over $\mathbf{Q}_p$ with abelian radical, does $\mathfrak{g}$ have a $\mathbf{Q}$-form? Note that the answer is negative if we replace "with abelian radical" by "with 2-nilpotent radical". Indeed, there are $2^{\aleph_0}$ non-isomorphic such Lie algebras; this can be proved in a similar way as Proposition 1.35.

Remark 3.14. If $\mathbf{Q}_p$ is replaced by an arbitrary field $K \supset \mathbf{Q}$, the answer to Question 3.12 is negative in general; actually a semisimple simply connected K-group need not have a $\mathbf{Q}$-form. For instance, if $K = \mathbf{Q}(\sqrt{2})$, and q is the quadratic form $\sqrt{2}x_1^2 + x_2^2 + x_3^2$, then $\mathrm{SO}(q)$ has no $\mathbf{Q}$-form.

Chapter 4

Finitely presentable, non-Hopfian groups with Kazhdan's Property (T) and infinite outer automorphism group

4.1 Introduction

It was asked by Paulin in [HaVa89, p.134] (1989) whether there exists a finitely generated group with Kazhdan's Property (T) and with infinite outer automorphism group. This question remained unanswered until 2004; in particular, it is Question 18 in [wor01].

This question was motivated by the two following special cases. The first is the case of lattices in **semisimple** groups over local fields, which have long been considered as prototypical examples of groups with Property (T). If Γ is such a lattice, Mostow's rigidity Theorem and the fact that semisimple groups have finite outer automorphism group imply that $\mathrm{Out}(\Gamma)$ is finite. Secondly, a new source of groups with Property (T) appeared when Zuk [Zuk96] proved that certain models of random groups have Property (T). But they are also hyperbolic, and Paulin proved [Pau91] that a hyperbolic group with Property (T) has finite outer automorphism group.

However, it turns out that various arithmetic lattices in appropriate **non-semi-simple** groups provide examples. For instance, consider the additive group $\mathrm{M}_{mn}(\mathbf{Z})$ of $m \times n$ matrices over $\mathbf{Z}$, endowed with the action of $\mathrm{GL}_n(\mathbf{Z})$ by left multiplication.

Proposition 4.1. **For every $n \geq 3$, $m \geq 1$, $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{M}_{mn}(\mathbf{Z})$ is a finitely presented linear group, has Property (T), is non-coHopfian[1], and its outer automorphism group contains a copy of $\mathrm{PGL}_m(\mathbf{Z})$, hence is infinite if $m \geq 2$.**

Ollivier and Wise [OlWi05] have independently found examples of a very different nature. They embed any countable group G in $\mathrm{Out}(\Gamma)$, where Γ has Property (T), is

[1] **A group is coHopfian (resp. Hopfian) if it is isomorphic to no proper subgroup (resp. quotient) of itself.**

a subgroup of a torsion-free hyperbolic group, satisfying a certain "graphical" small cancellation condition (see also [BeSz05]). In contrast to our examples, theirs are not, a priori, finitely presented; on the other hand, our examples are certainly not subgroups of hyperbolic groups since they all contain a copy of $\mathbf{Z}^2$.

They also construct in [OlWi05] a non-coHopfian group with Property (T) which embeds in a hyperbolic group. Proposition 4.1 actually answers two questions in their paper: namely, whether there exists a finitely presented group with Property (T) and without the coHopfian Property (resp. with infinite outer automorphism group).

Remark 4.2. Another example of a non-coHopfian group with Property (T) is $\mathrm{PGL}_n(\mathbf{F}_p[X])$ ($n \geq 3$). This group is finitely presentable if $n \geq 4$ [ReSo76]. In contrast with the previous examples, the Frobenius morphism Fr induces an isomorphism onto a subgroup of **infinite** index, and the intersection $\bigcap_{k\geq 0} \mathrm{Im}(\mathrm{Fr}^k)$ is reduced to $\{1\}$.

Ollivier and Wise also constructed in [OlWi05] the first examples of non-Hopfian groups with Property (T). They asked whether a finitely presented example exists. Although linear finitely generated groups are residually finite, hence Hopfian, we use them to answer positively their question.

Theorem 4.3. *There exists a p-arithmetic lattice Γ, and a central subgroup $Z \subset \Gamma$, such that Γ and Γ/Z are finitely presented, have Property (T), and Γ/Z is non-Hopfian.*

The group Γ has a simple description as a matrix group from which Property (T) and the non-Hopfian property for Γ/Z are easily checked (Proposition 4.12). Section 4.3 is devoted to prove finite presentability of Γ. We use here a general criterion for finite presentability of p-arithmetic groups, due to Abels [Abe87]. It involves the computation of the first and second cohomology group of a suitable Lie algebra.

4.2 Proofs of all results except finite presentability of Γ

We need some facts about Property (T). The first is obvious:

Lemma 4.4. *Property (T) is inherited by quotients.* □

Lemma 4.5 (see [HaVa89], Chap. 3, Théorème 4). *Let G be a locally compact group, and Γ a lattice in G. Then G has Property (T) if and only if Γ has Property (T).* □

The next lemma is an immediate consequence of S. P. Wang's classification [Wan82, Theorem 2.10].

Lemma 4.6. *Let K be a local field of characteristic 0, G an algebraic group defined over K, and $\mathfrak{g}$ its Lie algebra. Suppose that $\mathfrak{g}$ is perfect, and, for every simple quotient $\mathfrak{s}$ of $\mathfrak{g}$, either $\mathfrak{s}$ has K-rank ≥ 2, or $\mathrm{K} = \mathrm{R}$, and $\mathfrak{s}$ is isomorphic to either $\mathfrak{sp}(n,1)$ ($n \geq 2$) or $\mathfrak{f}_{4(-20)}$. Then $G(\mathrm{K})$ has Property (T).* □

Proof of Proposition 4.1. The group $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{M}_{mn}(\mathbf{Z})$ is linear in dimension $n+m$. As a semidirect product of two finitely presented groups, it is finitely presented. For every $k \geq 2$, it is isomorphic to its proper subgroup $\mathrm{SL}_n(\mathbf{Z}) \ltimes k\mathrm{M}_{mn}(\mathbf{Z})$ of finite index k^{mn}.

The group $\mathrm{GL}_m(\mathbf{Z})$ acts on $\mathrm{M}_{mn}(\mathbf{Z})$ by right multiplication. Since this action commutes with the left multiplication of $\mathrm{SL}_n(\mathbf{Z})$, $\mathrm{GL}_m(\mathbf{Z})$ acts on the semidirect product $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{M}_{mn}(\mathbf{Z})$ by automorphisms, and, by an immediate verification, this gives an embedding of $\mathrm{GL}_m(\mathbf{Z})$ if n is odd or $\mathrm{PGL}_m(\mathbf{Z})$ if n is even into $\mathrm{Out}(\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{M}_{mn}(\mathbf{Z}))$ (it can be shown that this is an isomorphism if n is odd; if n is even, the image has index two). In particular, if $m \geq 2$, then $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{M}_{mn}(\mathbf{Z})$ has infinite outer automorphism group.

On the other hand, in view of Lemma 4.5, it has Property (T) (actually for all $m \geq 0$): indeed, $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{M}_{mn}(\mathbf{Z})$ is a lattice in $\mathrm{SL}_n(\mathbf{R}) \ltimes \mathrm{M}_{mn}(\mathbf{R})$, which has Property (T) by Lemma 4.6 as $n \geq 3$. □

We now turn to the proof of Theorem 4.3. The following lemma is immediate, and already used in [Hal61] and [Abe79].

Lemma 4.7. **Let Γ be a group, Z a central subgroup. Let α be an automorphism of Γ such that $\alpha(Z)$ is a proper subgroup of Z. Then α induces a surjective, non-injective endomorphism of Γ/Z, whose kernel is $\alpha^{-1}(Z)/Z$.** □

Definition 4.8. Fix $n_1, n_2, n_3, n_4 \in \mathbf{N}^*$ with $n_2, n_3 \geq 3$. We set $\Gamma = G(\mathbf{Z}[1/p])$, where p is any prime, and G is the algebraic group defined as matrices by blocks of size n_1, n_2, n_3, n_4:

$$\begin{pmatrix} I_{n_1} & (*)_{12} & (*)_{13} & (*)_{14} \\ 0 & (**)_{22} & (*)_{23} & (*)_{24} \\ 0 & 0 & (**)_{33} & (*)_{34} \\ 0 & 0 & 0 & I_{n_4} \end{pmatrix},$$

where $(*)$ denote any matrices and $(**)_{ii}$ denote matrices in SL_{n_i}, $i = 2, 3$.

The centre of G consists of matrices of the form $\begin{pmatrix} I_{n_1} & 0 & 0 & (*)_{14} \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4} \end{pmatrix}$. Define Z as the centre of $G(\mathbf{Z})$.

Remark 4.9. This group is related to an analogous example of Abels: in [Abe79] he considers the same group, but with blocks 1×1, and GL_1 instead of SL_1 in the diagonal. Taking the points over $\mathbf{Z}[1/p]$, and taking the quotient by a cyclic subgroup if the centre, this provided the first example of a finitely presentable non-Hopfian solvable group.

Remark 4.10. If we do not care about finite presentability, we can take $n_3 = 0$ (i.e. 3 blocks suffice).

We begin by easy observations.

Lemma 4.11. **If K is any local field, then $G(\mathrm{K})$ has Property (T).**

Proof: This follows from Lemma 4.6 if $\mathbf{K}$ has characteristic zero; actually [Wan82] also covers the case when $\mathbf{K}$ has positive characteristic. □

Identify GL_{n_1} to the upper left diagonal block. It acts by **conjugation** on G as follows:

$$\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & A_{12} & A_{13} & A_{14} \\ 0 & B_2 & A_{23} & A_{24} \\ 0 & 0 & B_3 & A_{34} \\ 0 & 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & uA_{12} & uA_{13} & uA_{14} \\ 0 & B_2 & A_{23} & A_{24} \\ 0 & 0 & B_3 & A_{34} \\ 0 & 0 & 0 & I \end{pmatrix}.$$

This gives an action of GL_{n_1} on G, and also on its centre, and this latter action is faithful. In particular, for every commutative ring R, $\mathrm{GL}_{n_1}(R)$ embeds in $\mathrm{Out}(G(R))$.

From now on, we suppose that $R = \mathbf{Z}[1/p]$, and $u = pI_{n_1}$. The automorphism of $\Gamma = G(\mathbf{Z}[1/p])$ induced by u maps Z to its proper subgroup Z^p. In view of Lemma 4.7, this implies that Γ/Z is non-Hopfian.

Proposition 4.12. **The groups Γ and Γ/Z are finitely generated, have Property (T), and Γ/Z is non-Hopfian.**

Proof: We have just proved that Γ/Z is non-Hopfian. By the Borel-Harish-Chandra Theorem [BoHC62], Γ is a lattice in $G(\mathbf{R}) \times G(\mathbf{Q}_p)$. Thus, Property (T) follows from Lemmas 4.11 and 4.5. Finite generation is a consequence of Property (T) [HaVa89, Lemme 10]. Property (T) for Γ/Z follows from Lemma 4.4. □

Remark 4.13. This group has a surjective endomorphism with nontrivial finite kernel. We have no analogous example with infinite kernel. Such examples might be constructed if we could prove that some groups over rings of dimension ≥ 2 such as $\mathrm{SL}_n(\mathbf{Z}[X])$ or $\mathrm{SL}_n(\mathbf{F}_p[X, Y])$ have Property (T), but this is an open problem [Sha99p]. The non-Hopfian Kazhdan group of Ollivier and Wise [OlWi05] is torsion-free, so the kernel is infinite in their case.

Remark 4.14. It is easy to check that $\mathrm{GL}_{n_1}(\mathbf{Z}) \times \mathrm{GL}_{n_4}(\mathbf{Z})$ embeds in $\mathrm{Out}(\Gamma)$ and $\mathrm{Out}(\Gamma/Z)$. In particular, if $\max(n_1, n_4) \geq 2$, then these outer automorphism groups are infinite.

We finish this section by observing that Z is a finitely generated subgroup of the centre of Γ, so that finite presentability of Γ/Z immediately follows from that of Γ.

4.3 Finite presentability of Γ

We recall that a Hausdorff topological group H is **compactly presented** if there exists a compact generating subset C of H such that the abstract group H is the quotient of the group freely generated by C by relations of bounded length. See [Abe87, §1.1] for more about compact presentability.

Kneser [Kne64] has proved that for every linear algebraic $\mathbf{Q}_p$-group, the S-arithmetic lattice $G(\mathbf{Z}[1/p])$ is finitely presented if and only if $G(\mathbf{Q}_p)$ is compactly presented. A characterization of linear algebraic $\mathbf{Q}_p$-groups G such that $G(\mathbf{Q}_p)$ is compactly presented is given in [Abe87].

Let U be the unipotent radical in G, and let S denote a Levi factor defined over $\mathbf{Q}_p$, so that $G = S \ltimes U$. Let $\mathfrak{u}$ be the Lie algebra of U, and D be a maximal $\mathbf{Q}_p$-split torus in S. We recall that the first homology group of $\mathfrak{u}$ is defined as the abelianization

$$H_1(\mathfrak{u}) = \mathfrak{u}/[\mathfrak{u},\mathfrak{u}],$$

and the second homology group of $\mathfrak{u}$ is defined as $\mathrm{Ker}(d_2)/\mathrm{Im}(d_3)$, where the maps

$$\mathfrak{u}\wedge\mathfrak{u}\wedge\mathfrak{u} \stackrel{d_3}{\to} \mathfrak{u}\wedge\mathfrak{u} \stackrel{d_2}{\to} \mathfrak{u}$$

are defined by:

$$d_2(x_1\wedge x_2) = -[x_1,x_2] \text{ and } d_3(x_1\wedge x_2\wedge x_3) = x_3\wedge[x_1,x_2]+x_2\wedge[x_3,x_1]+x_1\wedge[x_2,x_3].$$

More details about homology of Lie algebras can be found in [Abe87, Chap. V]. We can now state the result we use of Abels (see [Abe87, Theorem 6.4.3 and Remark 6.4.5]).

Theorem 4.15 (Abels). Let G be a connected linear algebraic group over $\mathbf{Q}_p$. Suppose that the following assumptions are fulfilled:

- **(i)** **G is $\mathbf{Q}_p$-split.**
- **(ii)** **G has no simple quotient of $\mathbf{Q}_p$-rank one.**
- **(iii)** **0 does not lie on the segment joining two dominant weights for the adjoint representation of S on $H_1(\mathfrak{u})$.**
- **(iv)** **0 is not a dominant weight for an irreducible subrepresentation of the adjoint representation of S on $H_2(\mathfrak{u})$.**

Then $G(\mathbf{Q}_p)$ is compactly presented. □

We now return to our particular example of G, observe that it is clearly $\mathbf{Q}_p$-split, and that its simple quotients are SL_{n_2} and SL_{n_3} which have rank greater than one. Keep the previous notations D, U, $\mathfrak{u}$, so that D denotes the diagonal matrices in G, and U denotes the matrices in G all of whose diagonal blocks are the identity. Moreover, let $S \simeq \mathrm{SL}_{n_2}\times\mathrm{SL}_{n_3}$ denote the diagonal blocks in G; this is a Levi factor of the unipotent radical U, and D is a maximal torus of S which is split over $\mathbf{Q}_p$.

We introduce some notation: the set of indices of the matrix is partitioned as $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$, with $|I_j| = n_j$ as in Definition 4.8. It follows that, for every field K

$$\mathfrak{u}(K) = \left\{ T \in \mathrm{End}(K^I),\ \forall j,\ T(K^{I_j}) \subset \bigoplus_{i<j} K^{I_i} \right\}.$$

Throughout, we use the following notation: a letter such as i_k (or j_k, etc.) implicitly means $i_k \in I_k$.

Define, in an obvious way, subgroups U_{ij}, $i<j$, of U.

We begin by checking Condition (iii) of Theorem 4.15.

Lemma 4.16. **For any two weights of the action of D on $H_1(\mathfrak{u})$, 0 is not on the segment joining them.**

Proof: Recall that $H_1(\mathfrak{u}) = \mathfrak{u}_{\mathrm{ab}}$. So it suffices to look at the action on the supplement D-subspace $\mathfrak{u}_{12} \oplus \mathfrak{u}_{23} \oplus \mathfrak{u}_{34}$ of $[\mathfrak{u}, \mathfrak{u}]$:

$$(A, B) \cdot e_{i_1 j_2} = a_{j_2}^{-1} e_{i_1 j_2}, \quad (A, B) \cdot e_{j_2 k_3} = a_{j_2} b_{k_3}^{-1} e_{j_2 k_3}, \quad (A, B) \cdot e_{k_3 \ell_4} = b_{k_3} e_{k_3 \ell_4}.$$

Since $S = \mathrm{SL}_{n_2} \times \mathrm{SL}_{n_3}$, the weights live in M/P, where M is the free $\mathbf{Z}$-module of rank $n_2 + n_3$ with basis $(u_1, \dots, u_{n_2}, v_1, \dots, v_{n_3})$, and P is the plane generated by $\sum_{j_2} u_{j_2}$ and $\sum_{k_3} v_{k_3}$. Thus, the weights are $-u_{j_2}$, $u_{j_2} - v_{k_3}$, v_{k_3} ($1 \le j_2 \le n_2$, $1 \le k_3 \le n_3$).

Using that $n_2, n_3 \ge 3$, it is clear that no nontrivial positive combination of two weights (viewed as elements of $\mathbf{Z}^{n_2+n_3}$) lies in P. $\square$

We must now check Condition (iv) of Theorem 4.15, and therefore compute $H_2(\mathfrak{u})$ as a D-module.

Lemma 4.17. $\mathrm{Ker}(d_2)$ **is generated by**

- **(1)** $\mathfrak{u}_{12} \wedge \mathfrak{u}_{12}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{34} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{13}$, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{34}$.
- **(2)** $\mathfrak{u}_{14} \wedge \mathfrak{u}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$.
- **(3)** $e_{i_1 j_2} \wedge e_{k_2 \ell_3}$ ($j_2 \neq k_2$), $e_{i_2 j_3} \wedge e_{k_3 \ell_4}$ ($j_3 \neq k_3$).
- **(4)** $e_{i_1 j_2} \wedge e_{k_2 \ell_4}$ ($j_2 \neq k_2$), $e_{i_1 j_3} \wedge e_{k_3 \ell_4}$ ($j_3 \neq k_3$).
- **(5) Elements of the form** $\sum_{j_2} \alpha_{j_2} (e_{i_1 j_2} \wedge e_{j_2 k_3})$ **if** $\sum_{j_2} \alpha_{j_2} = 0$, **and** $\sum_{j_3} \alpha_{j_3} (e_{i_2 j_3} \wedge e_{j_3 k_4})$ **if** $\sum_{j_3} \alpha_{j_3} = 0$.
- **(6) Elements of the form** $\sum_{j_2} \alpha_{j_2} (e_{i_1 j_2} \wedge e_{j_2 k_4}) + \sum_{j_3} \beta_{j_3} (e_{i_1 j_3} \wedge e_{j_3 k_4})$ **if** $\sum_{j_2} \alpha_{j_2} + \sum_{j_3} \beta_{j_3} = 0$.

Proof: First observe that $\mathrm{Ker}(d_2)$ contains $\mathfrak{u}_{ij} \wedge \mathfrak{u}_{kl}$ when $[\mathfrak{u}_{ij}, \mathfrak{u}_{kl}] = 0$. This corresponds to (1) and (2). The remaining cases are $\mathfrak{u}_{12} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{34}$.

On the one hand, $\mathrm{Ker}(d_2)$ also contains $e_{i_1 j_2} \wedge e_{k_2 \ell_3}$ if $j_2 \neq k_2$, etc.; this corresponds to elements in (3), (4). On the other hand, $d_2(e_{i_1 j_2} \wedge e_{j_2 k_3}) = -e_{i_1 k_3}$, $d_2(e_{i_2 j_3} \wedge e_{j_3 k_4}) = -e_{i_2 k_4}$, $d_2(e_{i_1 j_2} \wedge e_{j_2 k_4}) = -e_{i_1 k_4}$, $d_2(e_{i_1 j_3} \wedge e_{j_3 k_4}) = -e_{i_1 k_4}$. The lemma follows. $\square$

Definition 4.18. Denote by $\mathfrak{b}$ (resp. $\mathfrak{h}$) the subspace generated by elements in (2), (4), and (6) (resp. in (1), (3), and (5)) of Lemma 4.17.

Proposition 4.19. $\mathrm{Im}(d_3) = \mathfrak{b}$, **and** $\mathrm{Ker}(d_2) = \mathfrak{b} \oplus \mathfrak{h}$ **as** D**-module. In particular,** $H_2(\mathfrak{u})$ **is isomorphic to** $\mathfrak{h}$ **as a** D**-module.**

Proof: We first prove, in a series of facts, that $\mathrm{Im}(d_3) \supset \mathfrak{b}$.

Fact 4.20. $\mathfrak{u}_{14} \wedge \mathfrak{u}$ is contained in $\mathrm{Im}(d_3)$.

Proof: If $z \in \mathfrak{u}_{14}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. This already shows that $\mathfrak{u}_{14} \wedge (\mathfrak{u}_{13} \oplus \mathfrak{u}_{24} \oplus \mathfrak{u}_{14})$ is contained in $\mathrm{Im}(d_3)$, since $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{u}_{13} \oplus \mathfrak{u}_{24} \oplus \mathfrak{u}_{14}$.

Now, if $(x, y, z) \in \mathfrak{u}_{24} \times \mathfrak{u}_{12} \times \mathfrak{u}_{34}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. Since $[\mathfrak{u}_{24}, \mathfrak{u}_{12}] = \mathfrak{u}_{14}$, this implies that $\mathfrak{u}_{14} \wedge \mathfrak{u}_{34} \subset \mathrm{Im}(d_3)$. Similarly, $\mathfrak{u}_{14} \wedge \mathfrak{u}_{12} \subset \mathrm{Im}(d_3)$.

Finally we must prove that $\mathfrak{u}_{14} \wedge \mathfrak{u}_{23} \subset \mathrm{Im}(d_3)$. This follows from the formula $e_{i_1 j_4} \wedge e_{k_2 \ell_3} = d_3(e_{i_1 m_2} \wedge e_{k_2 \ell_3} \wedge e_{m_2 j_4})$, where $m_2 \neq k_2$ (so that we use that $|I_2| \ge 2$). $\square$

Fact 4.21. $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$ and, similarly, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$, are contained in $\mathrm{Im}(d_3)$.

Proof: If $(x, y, z) \in \mathfrak{u}_{12} \times \mathfrak{u}_{23} \times \mathfrak{u}_{13}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. Since $[\mathfrak{u}_{12}, \mathfrak{u}_{23}] = \mathfrak{u}_{13}$, this implies that $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13} \subset \mathrm{Im}(d_3)$. □

Fact 4.22. $\mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$ is contained in $\mathrm{Im}(d_3)$.

Proof: $d_3(e_{i_1k_2} \wedge e_{k_2\ell_3} \wedge e_{k_2j_4}) = e_{k_2j_4} \wedge e_{i_1\ell_3} + e_{i_1j_4} \wedge e_{k_2\ell_3}$. Since we already know that $e_{i_1j_4} \wedge e_{k_2\ell_3} \in \mathrm{Im}(d_3)$, this implies $e_{k_2j_4} \wedge e_{i_1\ell_3} \in \mathrm{Im}(d_3)$. □

Fact 4.23. The elements in (4) are in $\mathrm{Im}(d_3)$.

Proof: $d_3(e_{i_1j_2} \wedge e_{j_2k_3} \wedge e_{\ell_3m_4}) = -e_{i_1k_3} \wedge e_{\ell_3m_4}$ if $k_3 \neq \ell_3$. The other case is similar. □

Fact 4.24. The elements in (6) are in $\mathrm{Im}(d_3)$.

Proof: $d_3(e_{i_1j_2} \wedge e_{j_2k_3} \wedge e_{k_3\ell_4}) = -e_{i_1k_3} \wedge e_{k_3\ell_4} + e_{i_1j_2} \wedge e_{j_2\ell_4}$. Such elements generate all elements as in (6). □

Conversely, we must check $\mathrm{Im}(d_3) \subset \mathfrak{b}$. By straightforward verifications:

- $d_3(\mathfrak{u}_{14} \wedge \mathfrak{u} \wedge \mathfrak{u}) \subset \mathfrak{u}_{14} \wedge \mathfrak{u}$.
- $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{24}) = 0$
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{24})$, $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34})$, $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{34})$, $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34})$ are all contained in $\mathfrak{u}_{14} \wedge \mathfrak{u}$.
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{23}) \subset \mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$, and similarly $d_3(\mathfrak{u}_{23} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34}) \subset \mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$.
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{24})$ and similarly $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{34})$ are contained in $\mathfrak{u}_{14} \wedge \mathfrak{u}_{23} + \mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$.
- The only remaining case is that of $\mathfrak{u}_{12} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{34}$: $d_3(e_{i_1j_2} \wedge e_{j'_2k_3} \wedge e_{k'_3\ell_4}) = \delta_{k_3k'_3} e_{i_1j_2} \wedge e_{j'_2\ell_4} - \delta_{j_2j'_2} e_{i_1k_3} \wedge e_{k'_3\ell_4}$, which lies in (4) or in (6).

Finally $\mathrm{Im}(d_3) = \mathfrak{b}$.

It follows from Lemma 4.17 that $\mathrm{Ker}(d_2) = \mathfrak{h} \oplus \mathfrak{b}$. Since $\mathfrak{b} = \mathrm{Im}(d_3)$, this is a D-submodule. Let us check that $\mathfrak{h}$ is also a D-submodule; the computation will be used in the sequel.

The action of S on $\mathfrak{u}$ by **conjugation** is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ 0 & 0 & X_{23} & X_{24} \\ 0 & 0 & 0 & X_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & X_{12}A^{-1} & X_{13}B^{-1} & X_{14} \\ 0 & 0 & AX_{23}B^{-1} & AX_{24} \\ 0 & 0 & 0 & BX_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We must look at the action of D on the elements in (1), (3), and (5). We fix (A, B) diagonal in S, $A = \sum_{j_2} a_{j_2} e_{j_2j_2}$, $B = \sum_{k_3} b_{k_3} e_{k_3k_3}$.

- (1):

$$(A, B) \cdot e_{i_1j_2} \wedge e_{k_1\ell_2} = e_{i_1j_2} A^{-1} \wedge e_{k_1\ell_2} A^{-1} = a_{j_2}^{-1} a_{\ell_2}^{-1} e_{i_1j_2} \wedge e_{k_1\ell_2}. \tag{4.3.1}$$

The action on other elements in (1) has a similar form.

- (3) ($j_2 \neq k_2$):

$$(A, B) \cdot e_{i_1j_2} \wedge e_{k_2\ell_3} = e_{i_1j_2} A^{-1} \wedge A e_{k_2\ell_4} B^{-1} = a_{j_2}^{-1} a_{k_2} b_{\ell_3}^{-1} e_{i_1j_2} \wedge e_{k_2\ell_3}. \tag{4.3.2}$$

The action on the other elements in (3) has a similar form.

• (5) ($\sum_{j_2} \alpha_{j_2} = 0$)

$$(A,B)\cdot \sum_{j_2} \alpha_{j_2}(e_{i_1 j_2} \wedge e_{j_2 k_3}) \;=\; \sum_{j_2} \alpha_{j_2}(e_{i_1 j_2} A^{-1} \wedge A e_{j_2 k_3} B^{-1})$$

$$= \sum_{j_2} \alpha_{j_2} a_{j_2}^{-1}(e_{i_1 j_2} \wedge a_{j_2} b_{k_3}^{-1} e_{j_2 k_3}) \;=\; b_{k_3}^{-1}\left(\sum_{j_2} \alpha_{j_2}(e_{i_1 j_2} \wedge e_{j_2 k_3})\right). \qquad (4.3.3)$$

The other case in (5) has a similar form. □

Lemma 4.25. **0 is not a weight for the action of D on $H_2(\mathfrak{u})$.**

Proof: First recall that the weight space is M/P, as described in the proof of Lemma 4.16. Hence, we describe weights as elements of $M = \mathbf{Z}^{n_2+n_3}$ rather than M/P, and must check that no weight lies in P.

• (1). In (4.3.1), the weight is $-u_{j_2} - u_{\ell_2}$, hence does not belong to P since $n_2 \geq 3$. The other verifications are similar.

• (3). In (4.3.2), the weight is $-u_{j_2} + u_{k_2} - v_{\ell_3}$, hence does not belong to P. The other verification for (3) is similar.

• (5). In (4.3.3), the weight is $-v_{k_3}$, hence does dot belong to P. The other verification is similar. □

Finally Lemmas 4.16 and 4.25 imply that the conditions of Theorem 4.15 are satisfied, so that Γ is finitely presented.

Chapter 5

Groups with vanishing reduced 1-cohomology

5.1 Introduction

Our object of study is the 1-cohomology of unitary representations of locally compact groups. Let G be a locally compact group, π a unitary representation in a Hilbert space $\mathscr{H}$. The space $Z^1(G,\pi)$ is defined as the space of continuous functions $b : G \to \mathscr{H}$ satisfying the 1-cocycle condition: for all $g, h \in G$, $b(gh) = \pi(g)b(h) + b(g)$. The subspace of $Z^1(G,\pi)$ of 1-coboundaries, namely, 1-cocycles that can be written as $g \mapsto \xi - \pi(g)\xi$ for some $\xi \in \mathscr{H}$, is called $B^1(G,\pi)$. The first cohomology group of π is defined as $H^1(G,\pi) = Z^1(G,\pi)/B^1(G,\pi)$.

Locally compact, σ-compact groups G such that $H^1(G,\pi) = 0$ for every unitary representation form a well-understood class since Delorme and Guichardet have proved that it coincides with the class of locally compact groups with Kazhdan's Property (T) (see [HaVa89]).

However, the 1-cohomology space $H^1(G,\pi)$ has a bad behaviour in some respects, as Guichardet as pointed out [Gui80]. Given a family (π_i) of representations, it may happen that $H^1(G,\pi_i) = 0$ for every i, but $H^1(G, \bigoplus \pi_i) \neq 0$; this phenomenon arises even when $G = \mathbf{Z}$ and π_i is a well-chosen family of one-dimensional representations.

The space $Z^1(G,\pi)$ has a natural topology, that of uniform convergence on compact subsets. The first reduced cohomology group is defined as $\overline{H^1}(G,\pi) = Z^1(G,\pi)/\overline{B^1(G,\pi)}$. In contrast to the non-reduced case, Guichardet [Gui80, Chap. III, 2.4] has proved that the reduced cohomology is well-behaved under orthogonal decompositions, and, more generally, direct integral of unitary representations.

In this paper, we focus on the class of groups with vanishing reduced 1-cohomology, i.e. groups G such that $\overline{H^1}(G,\pi) = 0$ for every unitary representation. The main result in this field is due to Shalom [Sha00]: the class of locally compact, **compactly generated** groups G with vanishing reduced 1-cohomology coincides with the class of locally compact groups with Kazhdan's Property (T). On the other hand, the investigation of non-compactly generated locally compact groups with vanishing 1-cohomology has been launched by F. Martin [Mrt03]. He observes that groups which are direct limits of groups with Property (T) have vanishing reduced 1-cohomology. For a countable discrete **hypercentral**[1] group G, he proves the equivalence between the following properties:

(i) $\overline{H^1}(G,\pi)=0$ for every representation of G.

(ii) G_{ab} is locally finite.

(iii) $H^1(G,\pi)=0$ for every irreducible representation of G.

(iv) G is locally finite.

The main goal in this chapter is to extend the equivalence between (i) and (ii) to all locally compact, locally nilpotent groups. A locally compact group is locally nilpotent if every compact subset is contained in a nilpotent subgroup; discrete hypercentral groups are locally nilpotent[2].

Theorem 5.1. **Let G be a locally compact locally nilpotent group. Equivalences:**
(i) $\overline{H^1}(G,\pi)=0$ **for every unitary representation** π **of** G**.**
(ii) $\mathrm{Hom}(G,\mathbf{R})=\{0\}$**.**

There appears a new phenomenon: while a discrete hypercentral group G such that $\mathrm{Hom}(G,\mathbf{R})=\{0\}$ is locally finite [Rob82], there exist nontrivial torsion-free perfect locally nilpotent groups. Such groups are not direct limits of groups with Property (T) and have vanishing reduced 1-cohomology by Theorem 5.1. This answers negatively 5.1.7 of [Mrt03]. By the way, we exhibit a locally finite group not satisfying (iii), namely, the group of permutations with finite support of a countable set.

The proof of Theorem 5.1 uses, in a crucial way, some relative notions of vanishing of 1-cohomology, introduced in the preliminaries below, which are variants of relative Property (T). In Section 5.3, we prove Theorem 5.1 for discrete groups. The case of non-discrete groups requires some further arguments which are independent of the rest of the paper, and is carried out in Section 5.4.

5.2 Preliminaries

Let G be a locally compact group, and H a closed subgroup. Recall that (G,H) has relative Property (FH) if, for every isometric action of G on an affine Hilbert space, H has a fixed point. Equivalently, for every unitary representation π of G, the natural morphism in 1-cohomology $H^1(G,\pi)\to H^1(H,\pi)$ is zero. (If G is σ-compact, then it is equivalent to relative Property (T), see [Jol05].)

In analogy, we say that (G,H) has relative Property ($\overline{\mathrm{FH}}$) if, for every isometric action α of G on an affine Hilbert space, H almost has fixed points, that is, for every compact $K\subset H$ and every $\varepsilon>0$, there exists a (K,ε)-fixed point for the action, i.e. a point v such that $\sup_{g\in K}\|v-\alpha(g)v\|\le\varepsilon$.

This means that, for every unitary representation π of G, the natural morphism $\overline{H^1}(G,\pi)\to\overline{H^1}(H,\pi|_H)$ is zero. If (G,G) has relative Property ($\overline{\mathrm{FH}}$), we say that G has Property ($\overline{\mathrm{FH}}$).

[1] **A topological group G is hypercentral if $G=N_\alpha(G)$ for sufficiently large α, where $(N_\alpha(G))$ denotes the transfinite ascending central series.**

[2] **This is not true in the non-discrete case: indeed, any residually nilpotent group (such as a congruence subgroup in $\mathrm{SL}_n(\mathbf{Z})$) embeds in a product of finite nilpotent groups, which is a hypercentral compact group.**

Finally, we say that (G, H) has relative Property (FHI) (resp. ($\overline{\text{FHI}}$)) if, for every **irreducible** unitary representation π of G, the natural morphism $H^1(G,\pi) \to H^1(H, \pi|_H)$ (resp. $\overline{H^1}(G,\pi) \to \overline{H^1}(H, \pi|_H)$) is zero.

Proposition 5.2. **Let G be a σ-compact, locally compact group and H a closed subgroup. The following are equivalent:**
(i) (G, H) has relative Property ($\overline{\text{FH}}$).
(ii) (G, H) has relative Property ($\overline{\text{FHI}}$).

Proof: (i) trivially implies (ii), so let us suppose (ii). By [Com84, Theorem 3.7], there exists a compact normal subgroup K of G such that G/K is second countable. Hence, upon replacing G by G/K, we can suppose that G is second countable. Let π be any unitary representation of G, and disintegrate it as $\pi = \int^{\oplus} \pi_i di$. Let $b \in Z^1(G, \pi)$, and disintegrate it as $b = \int^{\oplus} b_i di$, where $b_i \in Z^1(G, \pi_i)$. Then, by (ii), $b_i|_H \in \overline{B^1(H, \pi_i)}$ for every i. By [Gui80, Chap. 3, §2], it follows that $b \in \overline{B^1(H, \pi)}$. Thus (G, H) has relative Property ($\overline{\text{FH}}$). □

Corollary 5.3. **Under the hypotheses of Proposition 5.2, relative Property (FHI) implies relative Property ($\overline{\text{FH}}$).** □

The converse is false, even when $H = G$ (see Remark 5.13 below).

5.3 Reduced 1-cohomology of locally nilpotent groups

We use the following result of Guichardet [Gui72, Corollaire 5].

Proposition 5.4 (Guichardet). **Let G be a nilpotent locally compact group, and π a nontrivial irreducible re0Td [(41(e)-0.29611(s)0.157245(e)-0.294068(n)0.310405(t)-0.298152atv)** (G, π)

Proof: Let α be an affine action of G on a Hilbert space with irreducible linear part. Let v be a H-fixed point. Then $H'v$ is a compact orbit for H'. By the centre Lemma [HaVa89, 3.b], H' fixes a point. □

Lemma 5.8. **Let G' be another subgroup containing H. If the pair (G', H) has relative Property ($\overline{\text{FH}}$), then so does (G, H).** □

Lemma 5.9. **(G, H) has relative Property ($\overline{\text{FH}}$) if and only if for every compactly generated, closed subgroup M of H, (G, M) has Property ($\overline{\text{FH}}$).** □

Let G be a discrete, locally nilpotent group. Denote by $T_a(G)$ the inverse image in G of the torsion subgroup of G_{ab}.

Proposition 5.10. **Let G be a discrete, locally nilpotent group. Then $(G, T_a(G))$ has relative Property ($\overline{\text{FH}}$).**

Proof: Let Γ be a finitely generated subgroup of $T_a(G)$, with generators γ_i, $i = 1, \ldots, n$. For suitable positive integers n_i, γ^{n_i} can be written as a product of commutators. This involves finitely many elements of Γ, so that $\Gamma \subset T_a(H)$ for some finitely generated subgroup $H \subset G$. Therefore, in view of Lemmas 5.8 and 5.9, we can suppose that G is finitely generated.

By Corollary 5.5, $(G, D(G))$ has relative Property (FHI). Since $D(G)$ has finite index in $T_a(G)$, by Lemma 5.7, the pair $(G, T_a(G))$ has relative Property (FHI). By Corollary 5.3, $(G, T_a(G))$ also has Property ($\overline{\text{FH}}$). □

Corollary 5.11. **Let G be a locally nilpotent, discrete group. Equivalences:**

(i) G **has Property** ($\overline{\text{FH}}$)

(ii) G_{ab} **is torsion.**

(iii) $\text{Hom}(G, \mathbf{R}) = \{0\}$**.**

Proof: (ii)⇒(i) If G_{ab} is torsion, then $G = T_a(G)$, so that, by Proposition 5.10, G has Property ($\overline{\text{FH}}$).

(i)⇒(iii) If G has Property ($\overline{\text{FH}}$), then $0 = \overline{H^1}(G, 1_G) = H^1(G, 1_G) = \text{Hom}(G, \mathbf{C}) \simeq \text{Hom}(G, \mathbf{R})^2$.

(iii)⇒(ii) This well-known result immediately follows from the injectivity of $\mathbf{R}$ as a $\mathbf{Z}$-module. □

Remark 5.12. F. Martin [Mrt03, Corollaire 5.4.8] has obtained Corollary 5.11 in the case when G is a discrete countable hypercentral group. In this case the conditions are also equivalent to: G **is locally finite** (compare the next remark).

Remark 5.13. Let G be a σ-compact, locally compact group. Consider the following properties:

(1) Every compact subset of G is contained in an open subgroup with Property (T).

(2) Every compact subset of G is contained in an open subgroup H such that (G, H) has relative Property (T).

(3) G has Property ($\overline{\text{FH}}$).

First note that if G has the Haagerup Property, i.e. acts properly by isometries on a Hilbert space, then (1) and (2) are clearly equivalent to saying that $G = \bigcup G_n$ for some increasing sequence of compact subgroups G_n. Recall [CCJJV01] that amenable groups have the Haagerup Property, and in particular locally nilpotent groups do.

In general, (1)⇒(2)⇒(3), and, by a result of Shalom [Sha00] (see also [BHV05, Chap. 3]), these properties are all equivalent to Property (T) if G is compactly generated. F. Martin [Mrt03, 5.1.7] and A. Valette (oral communication) have asked whether (3)⇒(1) holds in full generality. We deduce from Corollary 5.11 that the answer is negative. For example, let Γ be the group of infinite $\mathbf{Q} \times \mathbf{Q}$ matrices with integer entries, which are upper triangular with 1 on the diagonal, and with finitely many nonzero non-diagonal terms. Then Γ, usually referred as a "McLain group", is a perfect, locally nilpotent, torsion-free group [Rob82, 12.1.9]. By the remark above, since Γ is locally nilpotent but not locally finite, it cannot satisfy (2); however it satisfies (3) by Corollary 5.11.

As regards the implication (2)⇒(1), we have no counterexample. However, we conjecture that there exists a countable group satisfying (2) but not (1). We think that the methods which would lead to such a counterexample might be strictly more interesting that the counterexample itself.

Remark 5.14. Corollary 5.11 gives no information about which locally nilpotent groups have Property (FHI). We do not know any example of, say, a countable locally nilpotent group G such that $\mathrm{Hom}(G, \mathbf{R}) = 0$ and G does not have Property (FHI).

On the other hand, there exists a countable locally finite group without Property (FHI). Indeed, let G be the group of permutations with finite support of $\mathbf{N}$. Note that since G is locally finite, G has Property ($\overline{\text{FH}}$). Let π be its natural representation on $\ell^2(\mathbf{N})$. Then it is easily checked that π is irreducible. On the other hand, $H^1(G, \pi)$ is "big": indeed, to every function $f : \mathbf{N} \to \mathbf{C}$ is associated a formal coboundary $g \mapsto f - \pi(g)f$, which is a coboundary if and only if $f \in \mathbf{C} + \ell^2(\mathbf{N})$.

However, we do not know any group with Property (FHI) which does not satisfy (1) of Remark 5.13.

Remark 5.15. Yehuda Shalom has pointed out to us that the class of amenable groups with Property ($\overline{\text{FH}}$) is stable under quasi-isometries (as defined, without finite generation assumption, in [Sha04]). This follows from Theorems 2.1.7 and 3.2.1 of [Sha04]. The McLain group of Remark 5.13 shows that this class does not coincide with the class of locally finite groups which is also (as an easy consequence of the definition) stable under quasi-isometries.

Remark 5.16. In contrast with Property (FH), Properties ($\overline{\text{FH}}$) and (FHI) are not preserved under extensions. Indeed, let F be any nontrivial finite abelian group, and let Γ be an infinite group with Property (T). Then Γ, as a group with Property (T), and $F^{(\Gamma)}$, as a locally finite abelian group, have both Properties ($\overline{\text{FH}}$) and (FHI). On the other hand, by a result independently obtained by P.-A. Cherix, F. Martin, A. Valette [CMV05] and Neuhauser [Neu05], the wreath product $G = F \wr \Gamma = F^{(\Gamma)} \rtimes \Gamma$, which is finitely generated, does not have Kazhdan's Property (T). By Shalom's result stated above, G does not have Properties ($\overline{\text{FH}}$) and (FHI).

5.4 The locally compact case

The proof of Theorem 5.1, achieved in Corollary 5.11 in the discrete case, is slightly more involved in the locally compact case.

Let G be a locally compact group. We say that $x \in G$ is **elliptic** if the subgroup generated by x is relatively compact in G. We say that x is ab-elliptic if the image of x in G_{ab} is elliptic. We denote by $E_{\mathrm{ab}}(G)$ the set of ab-elliptic elements in G. Note that, clearly, $[G,G] \subset E_{\mathrm{ab}}(G)$.

Lemma 5.17. **For every locally compact group, $E_{\mathrm{ab}}(G)$ is a closed, normal subgroup of G. Moreover $G/E_{\mathrm{ab}}(G)$ is isomorphic to a direct product $\mathbf{R}^n \times \Gamma$, where $n \in \mathbf{N}$ and Γ is a discrete torsion-free abelian group.**

Proof: We can suppose that G is abelian, so that the first assertion is clear.

The second assertion is an immediate and well-known consequence of the structure of locally compact abelian groups. For convenience, we recall the proof. Upon taking the quotient by $E_{\mathrm{ab}}(G)$, we can suppose that G is abelian without elliptic elements. Under these hypotheses, G has an open subgroup V which is a Lie group, hence isomorphic to $\mathbf{R}^n$ for some n. By injectivity of V as a $\mathbf{Z}$-module, V is a direct factor. Since G has no elliptic element, it follows that G/V is torsion-free. □

Lemma 5.18. **Let G be a locally compact, compactly generated group, and (N_i) be an increasing net of closed, normal subgroups. Suppose that, for all i, G/N_i is abelian and has no nontrivial elliptic element. Then, for some $N \subset G$, for large i, $N_i = N$.**

Proof: By assumption, and using Lemma 5.17, for all i, G/N_i is isomorphic to $\mathbf{R}^{n_i} \times \mathbf{Z}^{m_i}$ for suitable integers n_i, m_i, which decrease with i. It follows that there exists (n,m) such that, for large i, $(n_i, m_i) = (n,m)$. Now $\mathbf{R}^n \times \mathbf{Z}^m$ is Hopfian, in the sense that every continuous surjective endomorphism is an isomorphism. The result follows. □

Lemma 5.19. **Let G be a locally compact group. Let $K \subset E_{\mathrm{ab}}(G)$ be a compact subset. Then there exists a closed, compactly generated subgroup H of G such that $K \subset E_{\mathrm{ab}}(H)$.**

Proof: Let G_i be a net of open, compactly generated subgroups containing K and covering G. Set $M_i = E_{\mathrm{ab}}(G_i)$, and $M = \bigcup M_i$.

Fix $j \in I$, and observe that $(G_j \cap M_i)_{i \geq j}$ is a net of normal subgroups of G_j, and, for all $i \geq j$, $G_j/(G_j \cap M_i)$ is abelian without nontrivial elliptic element, since it embeds as an open subgroup in G_i/M_i. By Lemma 5.18, the net $(G_j \cap M_i)$ is eventually constant, hence equal to $G_j \cap M$.

Since this is true for all j, it follows that M is a closed, normal subgroup of G, and $G_j/(G_j \cap M)$ is abelian without elliptic elements for all j. Hence G/M is abelian without elliptic elements, so that the image of K in G/M is trivial, i.e. $K \subset M$. Now fix j. Since $K \subset G_j$, $K \subset M \cap G_j$, which is equal to $M_i \cap G_j$ for some i. Accordingly, $K \subset E_{\mathrm{ab}}(G_i)$. □

We can now generalize Proposition 5.10 to the locally compact case.

Proposition 5.20. **Let G be a locally nilpotent locally compact group. Then $(G, E_{\rm ab}(G))$ has relative Property $(\overline{\rm FH})$.**

Proof: Let Ω be a closed subgroup of $E_{ab}(G)$, generated by a compact subset K. By Lemma 5.19, there exists an open subgroup H containing K such that $K \subset E_{\rm ab}(H)$, so that $\Omega \subset E_{\rm ab}(H)$. Therefore, in view of Lemmas 5.8 and 5.9, we can suppose that G is compactly generated. So we can go on as in the proof of Proposition 5.10. □

Corollary 5.21. **Let G be a locally nilpotent locally compact group. Equivalences:**

(i) G **has Property** $(\overline{\rm FH})$

(ii) $G_{\rm ab}$ **is elliptic.**

(iii) $\mathrm{Hom}(G, \mathbf{R}) = \{0\}$**.**

Proof: (ii)⇒(i)⇒(iii) are proved as in Corollary 5.11.

(iii)⇒(ii) We must show that $G/E_{\rm ab}(G)$ is trivial. By Lemma 5.17, it is isomorphic to $\mathbf{R}^n \times \Gamma$ for some torsion-free abelian group Γ. Now (iii) implies $n = 0$, and, using the injectivity of $\mathbf{R}$ as a $\mathbf{Z}$-module, (iii) also implies $\Gamma = \{1\}$. □

Chapter 6

Uncountable groups with Property (FH) and strongly bounded groups

6.1 Introduction

Let us say that a group is strongly bounded if every isometric action on a metric space has bounded orbits.

We observe that the class of discrete, strongly bounded groups coincides with a class of groups which has recently emerged since a preprint of Bergman [Ber05], sometimes referred to as "groups with uncountable strong cofinality", or "groups with Bergman's Property". This class contains no countably infinite group, but contains symmetric groups over infinite sets [Ber05], various automorphism groups of infinite structures such as 2-transitive chains [DrHo05], full groups of certain equivalence relations [Mil04], oligomorphic permutation groups with ample generics [KeRo05]; see [Ber05] for more references.

In Section 6.2, which is independent of the rest of the chapter, we prove that every countable group embeds in a group with cardinality $\aleph_1$ and Property (FH).

In Section 6.4, we prove that ω_1-existentially closed groups are strongly bounded. This strengthens a result of Sabbagh [Sab75], who proved that they have cofinality $\neq \omega$.

In Section 6.5, we prove that if G is any finite perfect group, and I is any set, then G^I, endowed with the discrete topology, is strongly bounded. This strengthens a result of Koppelberg and Tits [KoTi74], who proved that this group has Serre's Property (FA). This group has finite exponent and is locally finite, hence amenable. In contrast, all previously known infinite strongly bounded groups contain a non-abelian free group.

6.2 Groups with cardinality $\aleph_1$ and Property (FH)

Proposition 6.1. **Let G be a countable group. Then G embeds in a group of cardinality $\aleph_1$ with Property (FH).**

It was asked in [wor01] whether the equivalence between Kazhdan's Property (T) and Property (FH), due to Delorme and Guichardet (see [BHV05, Chap. 2]) holds for more general classes of groups than locally compact σ-compact groups; in particular, whether it holds for general locally compact groups.

The answer is negative, even if we restrict to discrete groups: this follows from the existence of uncountable strongly bounded groups, combined with the fact that Kazhdan's Property (T) implies finite generation [BHV05].

The proof of Proposition 6.1 rests on two ingredients.

Theorem 6.2 (Delzant). **If G is any countable group, then G can be embedded in a group with Property (T).**

Sketch of proof: this is a corollary of the following result, first claimed by Gromov [Gro87, Theorem 5.6.E], and subsequently independently proved by Delzant [Del96] and Olshanskii [Ols95]: if H is any non-elementary word hyperbolic group, then H is SQ-universal, that is, every countable group embeds in a quotient of H. Thus, the result follows from the stability of Property (T) by quotients, and the existence of non-elementary word hyperbolic groups with Property (T); for instance, uniform lattices in $\mathrm{Sp}(n,1)$, $n \geq 2$ (see [HaVa89]). $\square$

Let $\mathcal{C}$ be any class of metric spaces, let G be a group. Say that G has Property (F$\mathcal{C}$) if for every isometric action of G on a space $X \in \mathcal{C}$, all orbits are bounded. For instance, if $\mathcal{C}$ is the class of all Hilbert spaces, then we get Property (FH).

Proposition 6.3. **Let G be a group in which every countable subset is contained in a subgroup with Property (F$\mathcal{C}$). Then G has Property (F$\mathcal{C}$).**

Proof: Let us take an affine isometric action of G on a metric space X in $\mathcal{C}$, and let us show that it has bounded orbits. Otherwise, there exists $x \in X$, and a sequence (g_n) in G such that $d(g_n x, x) \to \infty$. Let H be a subgroup of G with Property (F$\mathcal{C}$) containing all g_n. Since Hx is not bounded, we have a contradiction. $\square$

Proof of Proposition 6.1. We make a standard transfinite induction on ω_1 (as in [Sab75]), using Theorem 6.2. For every countable group Γ, choose a proper embedding of Γ into a group $F(\Gamma)$ with Property (T) (necessarily finitely generated). Fix $G_0 = G$, $G_{\alpha+1} = F(G_\alpha)$ for every $\alpha < \omega_1$, and $G_\lambda = \varinjlim_{\beta<\lambda} G_\lambda$ for every limit ordinal $\lambda \leq \omega_1$. It follows from Proposition 6.3 that G_{ω_1} has Property (FH). Since all embeddings $G_\alpha \to G_{\alpha+1}$ are proper, G_{ω_1} is not countable, hence has cardinality $\aleph_1$. $\square$

6.3 Strongly bounded groups

We recall that a group G is **strongly bounded** if every isometric action of G on a metric space has bounded orbits.

Remark 6.4. Let G be a strongly bounded group. Then every isometric action of G on a nonempty complete CAT(0) space has a fixed point; in particular, G has Property (FH) and Property (FA), which mean, respectively, that every isometric action of G on a Hilbert space (resp. simplicial tree) has a fixed point. This follows from the Bruhat-Tits fixed point lemma, which states that every action of a group on a complete CAT(0) space which has a bounded orbit has a fixed point (see [BrHa99, Chap. II, Corollary 2.8(1)]). This provides many additional examples of uncountable groups with Property (FH).

Definition 6.5. We say that a group G is **Cayley bounded** if, for every generating subset $U \subset G$, there exists some n (depending on U) such that every element of G is a product of n elements of $U \cup U^{-1} \cup \{1\}$. This means every Cayley graph of G is bounded.

A group G is said to have cofinality ω if it can be expressed as the union of an increasing sequence of proper subgroups; otherwise it is said to have cofinality $\neq \omega$.

The combination of these two properties, sometimes referred as "uncountable strong cofinality"[1], has been introduced and is extensively studied in Bergman's preprint [Ber05]; see also [DrGö05, DrHo05]. Note that an uncountable group with cofinality $\neq \omega$ is not necessarily Cayley bounded: the free product of two uncountable groups of cofinality $\neq \omega$, or the direct product of an uncountable group of cofinality $\neq \omega$ with $\mathbf{Z}$, are obvious counterexamples.

The following result can be compared to [Ber05, Lemma 10]:

Proposition 6.6. **A group G is strongly bounded if and only if it is Cayley bounded and has cofinality $\neq \omega$.**

Proof: Suppose that G is not Cayley bounded. Let U be a generating subset such that G the corresponding Cayley graph is not bounded. Since G acts transitively on it, it has an unbounded orbit.

Suppose that G has cofinality ω. Then G acts on a tree with unbounded orbits [Ser77, Chap I, §6.1].

Conversely, suppose that G has has cofinality $\neq \omega$ and is Cayley bounded. Let G act isometrically on a metric space. Let $x \in X$, let $K_n = \{g \in G \mid d(x, gx) < n\}$, and let H_n be the subgroup generated by K_n. Then $G = \bigcup K_n = \bigcup H_n$. Since G has cofinality $\neq \omega$, $H_n = G$ for some n, so that K_n generates G. Since G is Cayley bounded, and since K_n is symmetric, $G \subset (K_n)^m$ for some m. This easily implies that $G \subset K_{nm}$, so that the orbit of x is bounded. □

Remark 6.7. It follows that a countably infinite group Γ is not strongly bounded: indeed, either Γ has a finite generating subset, so that the corresponding Cayley graph is unbounded, or else Γ is not finitely generated, so is an increasing union of a sequence of finitely generated subgroups, so has cofinality ω.

Definition 6.8. If G is a group, and $X \subset G$, define

$$\mathcal{G}(X) = X \cup \{1\} \cup \{x^{-1},\ x \in X\} \cup \{xy \mid x, y \in X\}.$$

The following proposition is immediate from Proposition 6.6 and is essentially contained in Lemma 10 of [Ber05].

[1] **In the literature, it is sometimes referred as "Bergman's Property" or "Strong Bergman Property"; Bergman's Property also sometimes refers to Cayley boundedness without cofinality assumption.**

Proposition 6.9. **The group G is strongly bounded if and only if, for every increasing sequence (X_n) of subsets such that $\bigcup_n X_n = G$ and $\mathcal{G}(X_n) \subset X_{n+1}$ for all n, one has $X_n = G$ for some n.** $\square$

Remark 6.10. The first Cayley bounded groups with uncountable cofinality were constructed by Shelah [She80, Theorem 2.1]. They seem to be the only known to have a uniform bound on the diameter of Cayley graphs. They are torsion-free. These groups are highly non-explicit and their construction, which involves small cancellation theory, rests on the Axiom of Choice.

The first explicit examples, namely, symmetric groups over infinite sets, are due to Bergman [Ber05]. The first explicit torsion-free examples, namely, automorphism groups of double transitive chains, are due to Droste and Holland [DrHo05].

Remark 6.11. It is easy to observe that groups with cofinality $\neq \omega$ also have a geometric characterization; namely, a group G has cofinality $\neq \omega$ if and only if every isometric action of G on an **ultrametric** metric space has bounded orbits.

Remark 6.12. In [BHV05, §2.6], it is proved that an infinite solvable group never has Property (FH). In particular, an infinite solvable group is never strongly bounded. This latter result is improved by Khelif [Khe05] who proves that an infinite solvable group is never Cayley bounded. On the other hand, it is not known whether there exist uncountable solvable groups with cofinality $\neq \omega$.

6.4 ω_1-existentially closed groups

Recall that a group G is ω_1-existentially closed if every countable set of equations and inequations with coefficients in G which has a solution in a group containing G, has a solution in G. Sabbagh [Sab75] proved that every ω_1-existentially closed group has cofinality $\neq \omega$. We give a stronger result:

Theorem 6.13. **Every ω_1-existentially closed group G is strongly bounded.**

Proof: Let G act isometrically on a nonempty metric space X. Fix $x \in X$, and define $\ell(g) = d(gx, x)$ for all $g \in G$. Then ℓ is a length function, i.e. satisfies $\ell(1) = 0$ and $\ell(gh) \leq \ell(g) + \ell(h)$ for all $g, h \in G$. Suppose by contradiction that ℓ is not bounded. For every n, fix $c_n \in G$ such that $\ell(c_n) \geq n^2$. Let C be the group generated by all c_n. By the proof of the HNN embedding Theorem [LySc77, Theorem 3.1], C embeds naturally in the group

$$\Gamma = \langle C, a, b, t\, ;\; c_n = t^{-1}b^{-n}ab^n ta^{-n}b^{-1}a^n \ (n \in \mathbf{N})\rangle,$$

which is generated by a, b, t. Since G is ω_1-existentially closed, there exist $\bar{a}, \bar{b}, \bar{t}$ in G such that the group generated by C, $\bar{a}$, $\bar{b}$, and $\bar{t}$ is naturally isomorphic to Γ. Set $M = \max(\ell(\bar{a}), \ell(\bar{b}), \ell(\bar{t}))$. Then, since ℓ is a length function and c_n can be expressed by a word of length $4n+4$ in a, b, c, we get $\ell(c_n) \leq M(4n+4)$ for all n, contradicting $\ell(c_n) \geq n^2$. $\square$

It is known [Sco51] that every group embeds in a ω_1-existentially closed group. Thus, we obtain:

Corollary 6.14. **Every group embeds in a strongly bounded group.** □

Note that this was already a consequence of the strong boundedness of symmetric groups [Ber05], but provides a better cardinality: if $|G| = \kappa$, we obtain a group of cardinality $\kappa^{\aleph_0}$ rather that 2^κ.

6.5 Powers of finite groups

Theorem 6.15. **Let G be a finite perfect group, and I a set. Then the (unrestricted) product G^I is strongly bounded.**

Remark 6.16. Conversely, if I is infinite and G is not perfect, then G^I maps onto the direct product $\mathbf{Z}/p\mathbf{Z}^I$ for some prime p. Since the latter is an infinite-dimensional vector space over $\mathbf{F}_p$, it maps onto $\mathbf{Z}/p\mathbf{Z}^{(\mathbf{N})}$, which has clearly cofinality ω and is not Cayley bounded, as we see by taking as generating subset the canonical basis of $\mathbf{Z}/p\mathbf{Z}^{(\mathbf{N})}$. Thus, G^I also has cofinality ω and is not Cayley bounded.

Remark 6.17. By Theorem 6.15, every Cayley graph of G^I is bounded. If I is infinite and $G \neq 1$, one can ask whether we can choose a bound which does not depend on the choice of the Cayley graph. The answer is negative: indeed, for all $n \in \mathbf{N}$, observe that the Cayley graph of G^n has diameter exactly n if we choose the union of all factors as generating set. By taking a morphism of G^I onto G^n and taking the preimage of this generating set, we obtain a Cayley graph for G^I whose diameter is exactly n.

Our remaining task is to prove Theorem 6.15. The proof is an adequate modification of the original proof of the (weaker) result of Koppelberg and Tits [KoTi74], which states that G^I has cofinality $\neq \omega$.

If A is a ring with unity, and $X \subset A$, define

$$\mathcal{R}(X) = X \cup \{-1, 0, 1\} \cup \{x + y \mid x, y \in X\} \cup \{xy \mid x, y \in X\}.$$

It is clear that $\bigcup_{n\in\mathbf{N}} \mathcal{R}^n(X)$ is the subring generated by X.

Recall that a Boolean algebra is an associative ring with unity which satisfies $x^2 = x$ for all x. Such a ring has characteristic 2 (since $2 = 2^2 - 2$) and is commutative (since $xy - yx = (x + y)^2 - (x + y)$). The ring $\mathbf{Z}/2\mathbf{Z}$ is a Boolean algebra, and so are all its powers $\mathbf{Z}/2\mathbf{Z}^E = \mathscr{P}(E)$, for any set E.

Proposition 6.18. **Let E be a set, and $(\mathscr{X}_i)_{i\in\mathbf{N}}$ an increasing sequence of subsets of $\mathscr{P}(E)$. Suppose that $\mathcal{R}(\mathscr{X}_i) \subset \mathscr{X}_{i+1}$ for all i. Suppose that $\mathscr{P}(E) = \bigcup_{i\in\mathbf{N}} \mathscr{X}_i$. Then $\mathscr{P}(E) = \mathscr{X}_i$ for some i.**

Remark 6.19. 1) We could have defined, in analogy of Definition 6.5, the notion of strongly bounded ring (although the terminology "uncountable strong cofinality" seems more appropriate in this context). Then Proposition 6.18 can be stated as: if E is infinite, the ring $\mathscr{P}(E) = \mathbf{Z}/2\mathbf{Z}^E$ is strongly bounded. If E is infinite, note that, as an **additive group**, it maps onto $\mathbf{Z}/2\mathbf{Z}^{(\mathbf{N})}$, so has cofinality ω and is not Cayley bounded.

Proof of Proposition 6.18. Suppose the contrary. If $X \subseteq E$, denote by $\mathscr{P}(X)$ the power set of X, and view it as a subset of $\mathscr{P}(E)$. Define $\mathscr{L} = \{X \in \mathscr{P}(E) \mid \forall i,\ \mathscr{P}(X) \nsubseteq \mathscr{X}_i\}$. The assumption is then: $E \in \mathscr{L}$.

Observation: if $X \in \mathscr{L}$ and $X' \subset X$, then either X' or $X - X'$ belongs to $\mathscr{L}$. Indeed, otherwise, some $\mathscr{X}_i$ would contain $\mathscr{P}(X')$ and $\mathscr{P}(X - X')$, and then $\mathscr{X}_{i+1}$ would contain $\mathscr{P}(X)$.

We define inductively a decreasing sequence of subsets $B_i \in \mathscr{L}$, and a non-decreasing sequence of integers (n_i) by:

$$B_0 = E;$$

$$n_i = \inf\{t \mid B_i \in \mathscr{X}_t\};$$

$$B'_{i+1} \subset B_i \quad \text{and} \quad B'_{i+1} \notin \mathscr{X}_{n_i+1};$$

$$B_{i+1} = \begin{cases} B'_{i+1}, & \text{if } B'_{i+1} \in \mathscr{L}, \\ B_i - B'_{i+1}, & \text{otherwise.} \end{cases}$$

Define also $C_i = B_i - B_{i+1}$. The sets C_i are pairwise disjoint.

Fact 6.20. For all i, $B_{i+1} \notin \mathscr{X}_{n_i}$ and $C_i \notin \mathscr{X}_{n_i}$.

Proof: Observe that $\{B_{i+1}, C_i\} = \{B'_{i+1}, B_i - B'_{i+1}\}$. We already know $B'_{i+1} \notin \mathscr{X}_{n_i+1}$, so it suffices to check $B_i - B'_{i+1} \notin \mathscr{X}_{n_i}$. Otherwise, $B'_{i+1} = B_i - (B_i - B'_{i+1}) \in \mathcal{R}(\{B_i, B_i - B'_{i+1}\}) \subset \mathcal{R}(\mathscr{X}_{n_i}) \subset \mathscr{X}_{n_i+1}$; this is a contradiction. □

This fact implies that the sequence (n_i) is strictly increasing. We now use a diagonal argument. Let $(N_j)_{j\in\mathbf{N}}$ be a partition of $\mathbf{N}$ into infinite subsets. Set $D_j = \bigsqcup_{i\in N_j} C_i$ and $m_j = \inf\{t \mid D_j \in \mathscr{X}_t\}$, and let l_j be an element of N_j such that $l_j > \max(m_j, j)$.

Set $X = \bigsqcup_j C_{l_j}$. For all j, $D_j \cap X = C_{l_j} \notin \mathscr{X}_{l_j}$. On the other hand, $D_j \in \mathscr{X}_{m_j} \subset \mathscr{X}_{l_j-1}$ since $l_j \geq m_j + 1$. This implies $X \notin \mathscr{X}_{l_j-1} \supset \mathscr{X}_j$ for all j, contradicting $\mathscr{P}(E) = \bigcup_{i\in\mathbf{N}} \mathscr{X}_i$. □

The following corollary, of independent interest, was suggested to me by Romain Tessera.

Corollary 6.21. **Let A be a finite ring with unity (but not necessarily associative or commutative). Let E be a set, and $(\mathscr{X}_i)_{i\in\mathbf{N}}$ an increasing sequence of subsets of A^E. Suppose that $\mathcal{R}(\mathscr{X}_i) \subset \mathscr{X}_{i+1}$ for all i. Suppose that $A^E = \bigcup_{i\in\mathbf{N}} \mathscr{X}_i$. Then $A^E = \mathscr{X}_i$ for some i.**

Proof: By reindexing, we can suppose that $\mathscr{X}_0$ contains the constants. Write $\mathscr{Y}_i = \{J \subset E \mid 1_J \in \mathscr{X}_{3i}\}$. If $J, K \in \mathscr{Y}_i$, $1_{J\cap K} = 1_J 1_K \in \mathscr{X}_{3i+1} \subset \mathscr{X}_{3i+3}$, so that $J \cap K \in \mathscr{Y}_{i+1}$, and $1_{J\Delta K} = 1_J + 1_K - 2.1_J 1_K \in \mathscr{X}_{3i+3}$, so that $J \,\square\, K \in \mathscr{Y}_{i+1}$. By Proposition 6.18, $\mathscr{Y}_m = \mathscr{P}(E)$ for some m. It is then clear that $A^E = \mathscr{X}_n$ for some n (say, $n = 3m + 1 + \lceil \log_2 |A| \rceil$). □

If A is a Boolean algebra, and $X \subset A$, we define

$$\mathcal{D}(X) = X \cup \{0,1\} \cup \{x + y \mid x, y \in X \text{ such that } xy = 0\} \cup \{xy \mid x, y \in X\}.$$

$$\mathcal{I}_k(X) = \{x_1 x_2 \ldots x_k \mid x_1, \ldots, x_k \in X\}.$$
$$\mathcal{V}_k(X) = \{x_1 + x_2 + \ldots x_k \mid x_1, \ldots, x_k \in X \text{ such that } x_i x_j = 0 \ \ \forall i \neq j\}.$$

The following lemma contains some immediate facts which will be useful in the proof of the main result.

Lemma 6.22. **Let A be a Boolean algebra, and $X \subset A$ a symmetric subset (i.e. closed under $x \mapsto 1 - x$) such that $0 \in X$. Then, for all $n \geq 0$,**
1) $\mathcal{R}^n(X) \subset \mathcal{D}^{2n}(X)$**, and**
2) $\mathcal{D}^n(X) \subset \mathcal{V}_{2^{2^n}}(\mathcal{I}_{2^n}(X))$**.**

Proof: 1) It suffices to prove $\mathcal{R}(X) \subset \mathcal{D}^2(X)$. Then the statement of the lemma follows by induction. Let $u \in \mathcal{R}(X)$. If $u \notin \mathcal{D}(X)$, then $u = x + y$ for some $x, y \in X$. Then $u = (1-x)y + (1-y)x \in \mathcal{D}^2(X)$.
2) Is an immediate induction. □

Definition 6.23 ([KoTi74]). Take $n \in \mathbf{N}$, and let G be a group. Consider the set of functions $G^n \to G$; this is a group under pointwise multiplication. The elements $m(g_1, \ldots, g_n)$ in the subgroup generated by the constants and the canonical projections are called **monomials**. Such a monomial is **homogeneous** if $m(g_1, \ldots, g_n) = 1$ whenever at least one g_i is equal to 1.

Lemma 6.24 ([KoTi74]). **Let G be a finite group which is not nilpotent. Then there exist $a \in G$, $b \in G - \{1\}$, and a homogeneous monomial $f : G^2 \to G$, such that $f(a, b) = b$.**

For the convenience of the reader, we reproduce the proof from [KoTi74].

Lemma 6.25 ([KoTi74]). **Let G be a group, $g \in G$, and g' an element of the subgroup generated by the conjugates of g. Then there exists a homogeneous monomial $f : G \to G$ such that $f(g) = g'$.**

Proof: Write $g' = \prod c_i g^{\alpha_i} c_i^{-1}$. Then $x \mapsto \prod c_i x^{\alpha_i} c_i^{-1}$ is a homogeneous monomial and $f(g) = g'$. □

Lemma 6.26. **Let G be a finitely generated group. Suppose that G is not nilpotent. Then there exists $a \in G$ such that the normal subgroup of G generated by a is not nilpotent.**

Proof: Fix a finite generating subset S of G. For every $s \in S$, denote by N_s the normal subgroup of G generated by s. Since finitely many nilpotent normal subgroups generate a nilpotent subgroup, it immediately follows that if all N_s are nilpotent, then G is nilpotent. □

Proof of Lemma 6.24. Let G be a finite group which is not nilpotent. We must show that there exist $a \in G$, $b \in G - \{1\}$, and a homogeneous monomial $f : G^2 \to G$, such that $f(a, b) = b$.

Take a as in Lemma 6.26, and A the normal subgroup generated by a. Let A_1 be the upper term of the ascending central series of A. We define inductively the sequences $(a_i)_{i \in \mathbf{N}}$ and $(b_i)_{i \in \mathbf{N}}$ such that

$$b_i \in A - A_1, \qquad a_i \in A \qquad \text{and} \qquad b_{i+1} = [a_i, b_i] \in A - A_1.$$

Since G is finite, there exist integers m, m' such that $m < m'$ and $b_m = b_{m'}$. Set $b = b_m$, and for all i, choose, using Lemma 6.25, a homogeneous monomial f_i such that $f_i(a) = a_i$. Then the monomial

$$f: \quad (x, y) \mapsto [f_{m'-1}(x), [f_{m'-2}(x), \dots, [f_m(x), y], \dots]]$$

satisfies $f(a, b) = b$. □

Remark 6.27. If G is a group, and $f(x_1, \dots, x_n)$ is a homogeneous monomial with $n \geq 2$, then $m(g_1, \dots, g_n) = 1$ whenever at least one g_i is central: indeed, we can then write, for all $x_1, \dots, x_n$ with x_i central,

$$m(x_1, \dots, x_i, \dots, x_n) = m'(x_1, \dots, \widehat{x_i}, \dots, x_n) x_i^k.$$

By homogeneity in x_i, $m'(x_1, \dots, \widehat{x_i}, \dots, x_n) = 1$, and we conclude by homogeneity in x_j for any $j \neq i$.

Accordingly, if (C_α) denotes the (transfinite) ascending central series of G, an immediate induction on α shows that if $f(a, b) = b$ for some homogeneous monomial f, $a \in G$ and $b \in C_\alpha$, then $b = 1$. In particular, if G is nilpotent (or even residually nilpotent), then the conclusion of Lemma 6.24 is always false.

Lemma 6.28. **Let G be a finite group, I a set, and $H = G^I$. Suppose that $f(a, b) = b$ for some $a, b \in G$, and some homogeneous monomial f, and let N be the normal subgroup of G generated by b. Let (X_m) be an increasing sequence of subsets of H such that $\mathcal{G}(X_m) \subset X_{m+1}$ (see Definition 6.8), and $\bigcup X_m = H$. Then $N^I \subset X_m$ for m big enough.**

Proof: Suppose the contrary. If $x \in G$ and $J \subset I$, denote by x_J the element of G^I defined by $x_J(i) = x$ if $i \in J$ and $x_J(i) = 1$ if $i \notin J$.

Denote by $\bar{f} = f^I$ the corresponding homogeneous monomial: $H^2 \to H$. Upon extracting, we can suppose that all c_I, $c \in G$, are contained in X_0. In particular, the "constants" which appear in $\bar{f}$ are all contained in X_0.

Hence we have, for all m, $\bar{f}(X_m, X_m) \subset X_{m+d}$, where d depends only on the length of f. For $J, K \subset I$, we have the following relations:

$$a_I . a_J^{-1} = a_{I-J}, \tag{6.5.1}$$

$$\bar{f}(a_J, b_K) = b_{J \cap K}, \tag{6.5.2}$$

$$\bar{f}(a_J, b_I) = b_J, \tag{6.5.3}$$

$$\text{If } J \cap K = \emptyset, \quad b_J . b_K = b_{J \sqcup K}. \tag{6.5.4}$$

For all m, write $\mathscr{W}_m = \{J \in \mathscr{P}(I) \mid a_J \in X_m\}$, and let $\mathscr{A}_m$ be the Boolean algebra generated by $\mathscr{W}_m$. Then $\bigcup_m \mathscr{A}_m = \mathscr{P}(I)$. By Proposition 6.18, there exists some M such that $\mathscr{A}_M = \mathscr{P}(I)$. Set $\mathscr{X}_n = \mathcal{R}^n(\mathscr{W}_M)$. Then, since $\mathscr{A}_M = \mathscr{P}(I)$, $\bigcup_n \mathscr{X}_n = \mathscr{P}(I)$. Again by Proposition 6.18, there exists some N such that $\mathscr{X}_N = \mathscr{P}(I)$. So, by 1) of Lemma 6.22, we get

$$\mathcal{D}^{2N}(\mathscr{W}_M) = \mathscr{P}(I). \tag{6.5.5}$$

Define, for all m, $\mathscr{Y}_m = \{J \in \mathscr{P}(I) \mid b_J \in X_m\}$. Then from (6.5.3) we get: $\mathscr{W}_m \subset \mathscr{Y}_{m+d}$; from (6.5.2) we get: if $J \in \mathscr{W}_m$ and $K \in \mathscr{Y}_m$, then $J \cap K \in \mathscr{Y}_{m+d}$; and from (6.5.4) we get: if $J, K \in \mathscr{Y}_m$ and $J \cap K = \emptyset$, then $J \sqcup K \in \mathscr{Y}_{m+1}$.

By induction, we deduce $\mathcal{I}_k(\mathscr{W}_m) \subset \mathscr{Y}_{m+kd}$ for all k, and $\mathcal{V}_k(\mathscr{Y}_m) \subset \mathscr{Y}_{m+k}$ for all k. Composing, we obtain $\mathcal{V}_k(\mathcal{I}_l(\mathscr{W}_m)) \subset \mathcal{V}_k(\mathscr{Y}_{m+ld}) \subset \mathscr{Y}_{m+ld+k}$. By 2) of Lemma 6.22, we get $\mathcal{D}^n(\mathscr{W}_m) \subset \mathscr{Y}_{m+2^n d+2^{2^n}}$. Hence, using (6.5.5), we obtain $\mathscr{P}(I) = \mathscr{Y}_D$, where $D = M + 4^N d + 2^{4^N}$.

Let B denote the subgroup generated by b, so that N is the normal subgroup generated by B. Let r be the order of b. Then B^I is contained in X_{D+r}. Moreover, there exists R such that every element of N is the product of R conjugates of elements of B. Then, using that $c_I \in X_0$ for all $c \in G$, N^I is contained in X_{D+r+3R}. □

Theorem 6.29. **Let G be a finite group, and let N the last term of its descending central series (so that $[G, N] = N$). Let I be any set, and set $H = G^I$. Let (X_m) be an increasing sequence of subsets of H such that $\mathcal{G}(X_m) \subset X_{m+1}$ and $\bigcup X_m = H$. Then $N^I \subset X_m$ for m big enough.**

Proof: Let G be a counterexample with $|G|$ minimal. Let W be a normal subgroup of G such that W^I is contained in X_m for large m, and which is maximal for this property. Since G is a counterexample, $N \nsubseteq W$. Hence G/W is not nilpotent, and is another counterexample, so that, by minimality, $W = \{1\}$. Since G is not nilpotent, there exists, by Lemma 6.24, $a \in G$, $b \in G - \{1\}$, and a homogeneous monomial $f : G^2 \to G$, such that $f(a, b) = b$. So, if M is the normal subgroup generated by b, M^I is contained, by Lemma 6.28, in X_i for large i. This contradicts the maximality of W $(= \{1\})$. □

In view of Proposition 6.9, Theorem 6.15 immediately follows from Theorem 6.29.

Question 6.30. Let G be a finite group, and N a subgroup of G which satisfies the conclusion of Theorem 6.29 (I being infinite). Is it true that, conversely, N must be contained in the last term of the descending central series of G? We conjecture that the answer is positive, but the only thing we know is that N must be contained in the derived subgroup of G.

Remark 6.31. We could have introduced a relative definition: if G is a group and $X \subseteq G$ is a subset, we say that (G, X) is strongly bounded if, for every isometric action of G on any metric space M and every $m \in M$, then the "X-orbit"Xm is bounded. Note that G is strongly bounded if only if (G, G) is strongly bounded. Proposition 6.9 generalizes as: (G, X) is strongly bounded if for every sequence (X_n) of subsets of G such that $\bigcup_n X_n = G$ and $\mathcal{G}(X_n) \subset X_{n+1}$ for all n, one has $X_n \supseteq X$ for some n.

Theorem 6.29 is actually stronger than Theorem 6.15: it states that if G is a finite group, if N is the last term of its descending central series, and if I is any set, then the pair (G^I, N^I) is strongly bounded. In particular, it has relative Property (FH): for every isometric action of G^I on an affine Hilbert space, N^I has a fixed point. This shows that a solvable group can have an infinite subgroup with relative Property (FH) (compare Remark 6.12). We do not know if this can happen in a nilpotent group (see also Question 6.30).

Question 6.32. We do not assume the continuum hypothesis. Does there exist a strongly bounded group with cardinality $\aleph_1$?

It seems likely that a variation of the argument in [She80] might provide examples.

Question 6.33. Let (G_n) be a sequence of finite perfect groups. When is the product $\prod_{n\in\mathbf{N}} G_n$ strongly bounded?

It follows from Theorem 6.15 that if the groups G_n have bounded order, then $\prod_{n\in\mathbf{N}} G_n$ is strongly bounded. If all G_n are simple, Saxl, Shelah and Thomas prove [SST96, Theorems 1.7 and 1.9] that $\prod_{n\in\mathbf{N}} G_n$ has cofinality $\neq \omega$ if and only if there does **not** exist a fixed (possibly twisted) Lie type L, a sequence (n_i) and a sequence (q_i) of prime powers tending to infinity, such that $G_{n_i} \simeq L(q_i)$ for all i. Does this still characterize infinite strongly bounded products of non-abelian finite simple groups?

Chapter 7

Short notes

7.1 Complements on subgroups of algebraic groups

Recall that two algebraic K-groups are called geometrically isogeneous (resp. isomorphic) if they are isogeneous (resp. isomorphic) over an algebraic closure $\hat{K}$ of K. For instance, the $\mathbf{R}$-groups SO_3 and PGL_2 are geometrically isomorphic.

The following result is an analog of Theorem 1.28 in the case of algebraic groups.

Theorem 7.1. **Let G be a linear algebraic K-group. Suppose that $\mathrm{char}(K) = 0$, and that G is defined over some number field $F \subset K$. Then $G(K)$ is Haagerup if and only if G is geometrically isogeneous to ${\mathrm{SL}_2}^n \times R$ for some n, and R solvable.**

Proof: Suppose that G is geometrically isogeneous to ${\mathrm{SL}_2}^n \times R$ for some n, and R solvable. Then, since $G(K)$ embeds in $G(\hat{K})$, it suffices to show that $G(\hat{K})$ is Haagerup. This is an easy consequence of Theorem 7.7.

Conversely, suppose that $G(K)$ is Haagerup. Then so is $G(F)$. Let S denote a semisimple Levi factor of the radical in G, defined over F. It is known [BoTi, Théorème 2.14(b)] that S splits over some finite extension L of F. As a classical consequence of the Cebotarev Theorem, L embeds in F_v for some (in fact, infinitely many) non-Archimedean valuation(s) on F.

On the other hand, $G(F)$ embeds as a co-Følner discrete subgroup in $G(\mathbf{A}_F)$ (recall that the only obstructions for being a lattice are characters[1]). This implies that $G(\mathbf{A}_F)$ is Haagerup, so that $G(F_v)$ is Haagerup.

Since G is split over F_v, this implies, by Theorem 1.23, that $[S, R_u] = 1$, and that $S(F_v)$ is Haagerup, so has no factor of rank ≥ 2, so that S is geometrically isogeneous to ${\mathrm{SL}_2}^n$ for some n, and G is geometrically isogeneous to the product of ${\mathrm{SL}_2}^n$ by a solvable linear algebraic group. □

Remark 7.2. It is not clear at all that the assumption that G is defined over some number field can be dropped. For instance, let $n = 5$ or 6 and q be the quadratic form $\sum_{i=1}^{n} t_i x_i^2$ over the field of rational fractions $K = \mathbf{Q}(t_1, \dots, t_n)$. We do not

[1] **More precisely: let H be the intersection of all kernels of rational characters of G. Then $H(F)$ is a lattice, hence is co-Følner in $H(\mathbf{A}_F)$, which is co-abelian, hence co-Følner in $G(\mathbf{A}_F)$; this implies that $H(F)$, hence $G(F)$, is co-Følner in $G(\mathbf{A}_F)$.**

know whether $G = \mathrm{SO}(q)(K)$ is Haagerup. The point is that G has many dense embedding in simple groups of rank one. We can prove that G has no infinite subgroup with Property (T):

Let $H \subset G$ be an infinite subgroup with Property (T). Then H is finitely generated, hence contains a nontorsion element z generating an infinite cyclic subgroup Z. By Proposition 7.3 below, H is conjugated to a subgroup of $\mathrm{GL}(n, \hat{\mathbf{Q}})$. Thus z has its eigenvalues in $\hat{\mathbf{Q}}$. Since one of these eigenvalues is not a root of unity, it has modulus > 1 in some completion of $\mathbf{Q}$. Since z is an orthogonal matrix in $\mathrm{O}(n, \mathbf{R})$, this completion cannot be $\mathbf{R}$; so it is $\mathbf{Q}_p$ for some p. Since $n \leq 6$, we can choose $t_1, \ldots, t_n \in \mathbf{Q}_p$ so that the group $\mathrm{SO}(q)(\mathbf{Q}_p)$ has rank one[2] and hence is Haagerup. The image of Z in $\mathrm{SO}(q)(\mathbf{Q}_p)$ is discrete, because of the eigenvalue of modulus different from one, and is relatively compact, since H has Property (T). This is a contradiction.

Proposition 7.3 (Selberg [Sel60]). **Let $G \subset GL(n, \mathbf{R})$ be an algebraic subgroup, defined over some subfield K. Let Γ be a finitely generated group, and $\Gamma \to G$ be a representation such that $H^1(\Gamma, \mathfrak{g}) = 0$. Then $g\pi(\Gamma)g^{-1} \subset GL(n, \hat{K})$ for some $g \in G$.**

Let us now discuss the existence of an infinite subgroup with Property (T). As in Proposition 3.7, it is immediate to reduce to the case of simple groups. Moreover, changing the ground field, we can deal with absolutely simple groups; it is also harmless to deal with simply connected groups.

Theorem 7.4. **Let G be an absolutely simple, simply connected algebraic group defined over the local field K of characteristic zero. Then $G(\mathrm{K})$ has no infinite subgroup with Property (T) if and only if either**

- **$\mathrm{K} \neq \mathrm{R}$, and G is geometrically isomorphic to SL_2, or**
- **$\mathrm{K} = \mathrm{R}$, and G is isomorphic to the universal covering of $\mathrm{SO}(3)$, $\mathrm{SO}(n,1)$ ($n = 2, 3, 4$), or $\mathrm{SU}(2,1)$.**

Moreover, if $G(\mathrm{K})$ has Property (T), then, when it exists, the infinite subgroup with Property (T) can be chosen dense.

Proof:

- The case when $\mathbf{K} = \mathbf{R}$ is entirely contained in Theorem 3.3 and Proposition 3.7 of Chapter 3, and covers the case when $\mathbf{K} = \mathbf{C}$. □
- Suppose that $\mathbf{K}$ is non-Archimedean, and suppose that G is not geometrically isogeneous to SL_2. Suppose that $\mathbf{K} = K_v$ for some non-Archimedean valuation v on the number field K. Then, by [BoHa78, Theorem B], we can suppose that G is defined over K, and, without changing the isomorphism class of G over $\mathbf{K}$, that G is split over all Archimedean places. Now, $\Gamma = G((\mathcal{O}_K)_v)$ is

[2] **Indeed, there exists an anisotropic quadratic form q_0 in four variables [Ser70, Chapitre IV, §2.3, Corollaire of Théorème 7]. We can choose $q(x) = q_0(x_1, x_2, x_3, x_4) + x_5^2 - x_6^2$. In the corresponding quadratic space, the maximal isotropic subspaces have dimension one: this follows from [Ser70, Chapitre IV, §1.6, Corollaire of Théorème 4].**

a lattice in $G(\mathbf{K}) \times \prod_{s \in S} G(K_s)$, where S denote the set of all Archimedean places of K. Since G is split over all Archimedean places and is simple of geometric rank ≥ 2, the group $\prod_{s \in S} G(K_s)$ has Property (T).

If the $\mathbf{K}$-rank of G is not one, then Γ has Property (T) and projects densely onto $G(\mathbf{K})$.

If the $\mathbf{K}$-rank of G is one, let $H = G(\mathcal{O}_{\mathbf{K}}) \times \prod_{s \in S} G(K_s)$. Then, since H is open in $G(\mathbf{K}) \times \prod_{s \in S} G(K_s)$, the group $\Gamma \cap H$ is a lattice in H, hence has Property (T), and projects densely onto $G(\mathcal{O}_{\mathbf{K}})$.

Still supposing that $\mathbf{K} \neq \mathbf{R}$, let us turn to the converse. Then $G(\mathbf{K})$ embeds in some product of groups SL_2, so the result is immediate from [Zim84]. □

When K is not a local field, we have no general characterization of groups $G(K)$ having an infinite subgroup with Property (T). We already gave, in Remark 7.2, an example of a group G which is geometrically isomorphic to SO_6 but such that $G(K)$ has no infinite subgroup with Property (T), where $K = \mathbf{Q}(t_1, \ldots, t_6)$.

When K is a number field, we can sometimes conclude.

Proposition 7.5. **Let G be an absolutely simple K-group, where K is a number field. Suppose that, for every embedding $K \to \mathbf{R}$, $G(\mathbf{R})$ has Property (T). Then either**

- **G is geometrically isogeneous to SL_2, or**
- **$G(K)$ has an infinite subgroup with Property (T).**

Proof: Suppose that G is not geometrically isogeneous to SL_2; it follows that G has geometric rank ≥ 2. There exists some completion K_v such that G is split over K_v. Then $G((\mathcal{O}_K)_v)$ has Property (T), and is infinite. □

Example 7.6. Let $K = \mathbf{Q}(\sqrt{2})$, $q = \sum_{i=1}^{4} t_i^2 + \sqrt{2} t_5^2 - \sqrt{2} t_6^2$, and $G = \mathrm{SO}(q)$. We are not able to answer whether or not $G(K)$ has an infinite subgroup with Property (T).

7.2 Haagerup Property for subgroups of SL_2 and residually free groups

The purpose of this short note is to point out a straightforward generalization of the following theorem:

Theorem 7.7 (Guentner-Higson-Weinberger [GHW05, §5, Theorem 4]). **Let K be a field, and G be a subgroup of $\mathrm{SL}(2,K)$. Then G has the Haagerup property (as a discrete group).**

Using theorem 7.7, we obtain the following generalization.

Theorem 7.8. **Let R be a reduced (= without nilpotent elements) commutative ring, and G be a subgroup of $\mathrm{SL}(2,R)$. Then G has the Haagerup property (as a discrete subgroup).**

Proof: 1) Suppose R is a finite product of fields. Then it is an immediate consequence of theorem 7.7 (since the Haagerup property is stable under taking finite direct products and (closed) subgroups).

2) General case. We can suppose that G is finitely generated, hence that R is finitely generated as a ring. So R is Noetherian, hence has a finite number of minimal prime ideals $\mathfrak{p}_i$. Since R is reduced, $\bigcap \mathfrak{p}_i = \{0\}$, so that R embeds in $\prod R/\mathfrak{p}_i$, hence in the finite product $\prod K_i$, where $K_i = \mathrm{Frac}(R/\mathfrak{p}_i)$. So case 1 applies. □

Remark 7.9.

- The assumption that R is reduced cannot be dropped. For instance, $\mathrm{SL}(2,\mathbf{Z}[t]/t^2)$ does not have the Haagerup property. Indeed, if H is the kernel of the natural morphism $\mathrm{SL}(2,\mathbf{Z}[t]/t^2) \to \mathrm{SL}(2,\mathbf{Z})$, then H is infinite, while the pair $(\mathrm{SL}(2,\mathbf{Z}[t]/t^2), H)$ has Kazhdan's relative Property (T). This can be seen by embedding it as a lattice in the Lie group $\mathrm{SL}(2,\mathbf{R}[t]/t^2)$, which is isomorphic to $\mathrm{SL}(2,\mathbf{R}) \ltimes \mathfrak{sl}(2,\mathbf{R})$, where the action of $\mathrm{SL}_2(\mathbf{R})$ on the vector space $V = \mathfrak{sl}(2,\mathbf{R})$ is the adjoint action. This is the three-dimensional real irreducible representation of $\mathrm{SL}_2(\mathbf{R})$, so that it is well-known that $(\mathrm{SL}_2(\mathbf{R}) \ltimes V, V)$ has Property (T); see, for instance, Chapter 1 in [BHV05].

- The commutativity assumption cannot be dropped, even in theorem 7.7. Indeed, let $\mathbf{H}$ be the skew-field of Hamilton quaternions. Then $\mathrm{SL}(2,\mathbf{H})$ has infinite subgroups with Kazhdan's property (T): recall that $\mathrm{SL}(2,\mathbf{H}) \simeq \mathrm{SO}(5,1)$, the latter contains $\mathrm{SO}(5)$ as a subgroup, and it is well-known that $\mathrm{SO}(5)$ has infinite subgroups with property (T) (for instance, obtained by projecting an irreducible lattice of $\mathrm{SO}(5) \times \mathrm{SO}(2,3)$).

Here is an application of theorem 7.8. Recall that a group G is called residually free [Bau67] is it satisfies one of the (clearly) equivalent conditions:

- (i) For all $x \in G \backslash \{1\}$, there exist a (nonabelian) free group F and a morphism $f: G \to F$ such that $f(x) \neq 1$.

- (ii) G embeds in a product of free groups.
- (iii) G embeds in a product of free groups of finite rank.

Theorem 7.10. **Let G be a residually free group. Then G has the Haagerup property.**

Proof: It suffices to show that any product $\prod_{i\in I} F_i$ of free groups of finite rank has the Haagerup property. But such a product embeds in $\prod_{i\in I} \mathrm{SL}(2,\mathbf{Z}) = \mathrm{SL}(2,\mathbf{Z}^I)$. So this follows from theorem 7.8. □

Remark 7.11. The Haagerup property is not closed under infinite products (with the discrete topology). If it were, all residually finite groups would have the Haagerup property! For instance, the discrete group $\prod_i \mathrm{SL}(n,\mathbf{Z}/p^i\mathbf{Z})$ does not have the Haagerup property if $n \geq 3$, since it contains the infinite Kazhdan group $\mathrm{SL}(n,\mathbf{Z})$ as a subgroup. On the other hand, we do not know if the class of **torsion-free** groups with the Haagerup property is closed under infinite products.

Remark 7.12. V. Guirardel pointed out to us that, using some nontrivial properties of residually free groups, theorem 7.10 follows directly from theorem 7.7. Indeed, a residually free group can be embedded in $\mathrm{SL}(2,R)$, where R is a **finite** product of fields. The first ingredient is that a residually free group can be embedded in a finite product of fully residually free groups. The second ingredient is that a fully residually free group can be embedded in the ultraproduct ${}^*F_2$, which embeds in $\mathrm{SL}(2,{}^*\mathbf{Q})$, and ${}^*\mathbf{Q}$ is a field. For details and and many other interesting properties of (fully) residually free groups, see [ChGu05].

7.3 A note on quotients of word hyperbolic groups with Property (T)

All groups in this note are discrete and countable. Shalom [Sha00, Theorem 6.7] has proved the following interesting result about Property (T).

Theorem 7.13 (Shalom, 2000). **For every group G with Property (T), there exists a finitely presented group G_0 with Property (T) which maps onto G.**

In other words, this means that, given a finite generating subset for G, only finitely many relations suffice to imply Property (T). This can be interpreted in the topology of marked groups [Cha00] as: Property (T) is an open property. See [Gro03, 3.8] for a generalization to other fixed point properties.

A word hyperbolic group is a finitely generated group whose Cayley graph satisfies a certain condition, introduced by Gromov, meaning that, at large scale, it is negatively curved. We refer to [GhHa90] for a precise definition that we do not need here. We only mention here that word hyperbolic groups are necessarily finitely presented, that word hyperbolicity is a fundamental notion in combinatorial group theory as in geometric topology. Word hyperbolic groups are groups with "many" quotients, and thus can be considered as a generalization of free groups.

It was asked [wor01, Question 16] whether every group with Property (T) is quotient of a group with Property (T) with finiteness conditions stronger than finite presentation. We give an answer here by showing that we can impose word hyperbolicity.

Proposition 7.14. **For every group G with Property (T), there exists a torsion-free word hyperbolic group G_0 with Property (T) which maps onto G.**

Note that Proposition 7.14 contains Theorem 7.13 as a corollary; however it is proved by combining Theorem 7.13 with the following remarkable result of Ollivier and Wise [OlWi05]. Since it involves some technical definitions, we do not quote it in full generality.

Theorem 7.15 (Ollivier and Wise, 2005). **To every finitely presented group Q, we can associate a short exact sequence $1 \to N \to G \to Q \to 1$ such that**

1. **G is torsion-free, word hyperbolic,**
2. **N is 2-generated and has property (T).**

Corollary 7.16. **For every finitely presented group Q with Property (T), there exists a torsion-free word-hyperbolic group G with Property (T) mapping onto Q with finitely generated kernel.**

Proof: Apply Theorem 7.15 to Q, so that G lies in an extension $1 \to N \to G \to Q \to 1$, where N has Property (T) and Q has Property (T). Since Q has Property (T) and since Property (T) is stable under extensions, G also has Property (T). □

Remark 7.17. Corollary 7.16 answers a question at the end of [OlWi05].

Proof of Proposition 7.14: Let G be a group with Property (T). By Theorem 7.13, there exists a finitely presented group Q with Property (T) mapping onto G, and by Corollary 7.16, there exists a torsion-free word hyperbolic group G_0 with Property (T) mapping onto Q, so that G_0 maps onto G. □

Question 7.18. 1) In Theorem 7.15, can G be chosen, in addition, residually finite? In [Wis03], a similar result is proved, G being torsion-free, word hyperbolic, residually finite, and N finitely generated, but never having Property (T).

2) Let G be a word hyperbolic group (maybe torsion-free), and H a quotient of G generated by r elements. Does there exist an intermediate quotient which is both word hyperbolic and generated by r elements? (The analog statement with "word hyperbolic" replaced by "finitely presented" is immediate.) The motivation is that, in Proposition 7.14, we would like to have G_0 generated by no more elements than G. Theorem 7.15 only tells us that if G is r-generated, then G_0 can be chosen $(r+2)$-generated.

Question 7.19. Following [Ser77], a group has Property (FA) if every isometric action on a simplicial tree has a fixed point. Is it true that every group with Property (FA) is a quotient of a finitely presented group with Property (FA)?

7.4 Kazhdan Property for spaces of continuous functions

All the rings here are unitary and commutative. If R is a ring, let $\mathrm{E}(n,R)$ denote the subgroup of $\mathrm{SL}(n,R)$ generated by elementary matrices. If $\mathrm{E}(n,R)$ is normal in $\mathrm{SL}(n,R)$, the quotient is denoted by $SK_{1,n}(R)$.

If R is a topological ring, then $\mathrm{E}(n,R)$ and $\mathrm{SL}(n,R)$ are topological groups for the topology induced by the inclusion in R^{n^2}. We say that a topological ring is **topologically finitely generated** it it has a finitely generated dense subring.

For any topological spaces X, Y, we denote by $\mathcal{C}(X,Y)$ the set of all continuous functions $X \to Y$. If $\mathbf{K}$ denotes $\mathbf{R}$ or $\mathbf{C}$, $\mathcal{C}(X,\mathbf{K})$ is a topological ring for the compact-open topology, which coincides with the topology of uniform convergence on compact subsets.

It is known [Vas86] that, if $n \geq 3$,

$$\mathrm{E}(n,\mathcal{C}(X,\mathbf{K})) = \{u : X \to \mathrm{SL}(n,\mathbf{K}) \text{ homotopically trivial}\}$$

(this is immediate if X is compact, for all $n \geq 2$).

In this note, and in contrast to the remaining of the present work, we deal with non-locally compact groups.

Theorem 7.20. Let $n \geq 3$, and $X \subset \mathrm{R}^d$ be a topological subspace of a Euclidean space. Let $\mathrm{K} = \mathrm{R}$ or C. Endow $\mathcal{C}(X,\mathrm{K})$ with the topology of uniform convergence on compact subsets. Then $\mathrm{E}(n,\mathcal{C}(X,\mathrm{K}))$ has Kazhdan's Property (T).

Corollary 7.21. Let $n \geq 3$, and $X \subset \mathrm{R}^d$ be a compact subset. Endow X with the topology of uniform convergence. Then $\mathrm{SL}(n,\mathcal{C}(X,\mathrm{K}))$ has Kazhdan's Property (T) if and only if the discrete group $SK_{1,n}(\mathcal{C}(X,\mathrm{K})) = \mathrm{SL}(n,\mathcal{C}(X,\mathrm{K}))/\mathrm{E}(n,\mathcal{C}(X,\mathrm{K}))$ does.

Proof: Since X is compact[3], $\mathrm{E}(n,\mathcal{C}(X,\mathbf{K}))$, is open (hence closed) in $\mathrm{SL}(n,\mathcal{C}(X,\mathbf{K}))$. The corollary follows from the trivial fact that Property (T) is stable under quotients and extensions. □

Example 7.22. Fix $k \geq 1$ and $n \geq 3$. Then $SK_{1,n}(\mathcal{C}(S^k,\mathbf{K})) = \pi_k(\mathrm{SL}(n,\mathbf{K}))$, which is an abelian group. It follows that $\mathrm{SL}(n,\mathcal{C}(S^k,\mathbf{K}))$ has Kazhdan's Property (T) if and only if $\pi_k(\mathrm{SL}(n,\mathbf{K}))$ is finite; it is known (see [MiTo91]) that it is infinite if and only if:

- $\mathbf{K} = \mathbf{C}$, k is odd, and $3 \leq k \leq 2n-1$, or
- $\mathbf{K} = \mathbf{R}$, ($k \equiv -1 \pmod 4$ and $3 \leq k \leq 2n-1$) or (n is even and $k = n-1$)

In particular, $\pi_k(\mathrm{SL}(n,\mathbf{K}))$ is finite for $k = 1$, k even, or $k \geq 2n$.

Example 7.23. Let W denote a Cantor set. It is straightforward to show that, for every connected manifold M, all maps $W \to M$ are homotopic. Thus, $SK_{1,n}(\mathcal{C}(W,\mathbf{K}))$ is trivial, and accordingly, for all $n \geq 3$, $\mathrm{SL}(n,\mathcal{C}(W,\mathbf{K}))$ has Kazhdan's Property (T).

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[3] **If X is not compact, we do not know whether E$(n,\mathcal{C}(X,\mathbf{K}))$ is closed in SL$(n,\mathcal{C}(X,\mathbf{K}))$ for the topology of uniform convergence on compact subsets. However, we can restate the corollary as follows: SL$(n,\mathcal{C}(X,\mathbf{K}))$ has Kazhdan's Property (T) if and only if the group $\overline{SK}_{1,n}(\mathcal{C}(X,\mathbf{K})) =$ SL$(n,\mathcal{C}(X,\mathbf{K}))/\overline{\mathrm{E}(n,\mathcal{C}(X,\mathbf{K}))}$ does.**

Theorem 7.20 rests on two results: a K-theoretic result of Vaserstein (Theorem 7.27), and the work of Shalom on Kazhdan's Property (T) (Theorem 7.26).

In [Sha99p], Shalom introduces new methods to establish Kazhdan's Property (T) for the special linear groups over certain rings. This leads, for instance, to the first proof that $\Gamma = \mathrm{SL}(n, \mathbf{Z})$, $n \geq 3$, has Property (T), that does not use the embedding of Γ into $\mathrm{SL}(n, \mathbf{R})$ as a lattice (see also §7.6).

Before stating his main result, let us begin with a definition.

Definition 7.24. If G is a group and $S \subset G$ is a subset, we say that G is boundedly generated by S if there exist $m < \infty$ such that every $g \in G$ is a product of at most m elements in S.

Theorem 7.25 (Shalom, [Sha99p]). **Let $n \geq 3$, and let R be a topologically finitely generated ring. Suppose that $\mathrm{SL}(n, R)$ is boundedly generated by elementary matrices. Then $\mathrm{SL}(n, R)$ has Kazhdan's Property (T).**

As an application, Shalom proves bounded elementary generation for the loop group $\mathrm{SL}(n, \mathcal{C}(S^1, \mathbf{C}))$ ($n \geq 3$), and deduces that it has Property (T). He asks if the same holds for $\mathbf{R}$ instead of $\mathbf{C}$, noting that $\mathrm{SL}(n, \mathcal{C}(S^1, \mathbf{R}))$ is not generated by elementary matrices (since $\pi_1(\mathrm{SL}(n, \mathbf{R})) \neq 1$). This is answered positively by Theorem 7.20; see Example 7.22.

Actually, without modification, the proof of Theorem 7.25 ([Sha99p], see also [BHV05]) gives a stronger statement.

Theorem 7.26. **Let $n \geq 3$, let R be a topologically finitely generated commutative ring, and suppose that $\mathrm{E}(n, R)$ is boundedly generated by elementary matrices. Then $\mathrm{E}(n, R)$ has Kazhdan's Property (T).**

Theorem 7.25 is the particular case of Theorem 7.26 when $\mathrm{E}(n, R) = \mathrm{SL}(n, R)$.

Theorem 7.26 is a strong motivation for studying bounded elementary generation for the group $\mathrm{E}(n, R)$. For instance, this is an open question for $R = \mathbf{F}_p[X, Y]$ or $R = \mathbf{Z}[X]$, for all $n \geq 3$. We now focus on the case when $R = \mathcal{C}(X, \mathbf{K})$, where $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$.

The notion of dimension of a topological space involved here is defined in [Vas71], and it will be sufficient for our purposes to know that $\dim(X)$ is finite for every topological subspace of a Euclidean space.

Theorem 7.27 (Vaserstein, [Vas88]). **Let X be a finite dimensional topological space, and let $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$, and fix $n \geq 3$. Then $\mathrm{E}(n, \mathcal{C}(X, \mathbf{K}))$ is boundedly generated by elementary matrices.**

Now we show how Theorems 7.26 and 7.27 imply Theorem 7.20. Since $X \subset \mathbf{R}^d$ for some d, X is finite dimensional, so that Theorem 7.27 applies: $\mathrm{E}(n, \mathcal{C}(X, \mathbf{K}))$ is boundedly generated by elementary matrices.

It remains to show that Theorem 7.26 applies, that is, $\mathcal{C}(X, \mathbf{K})$ is topologically finitely generated for the compact-open topology (that is, the topology of uniform convergence on compact subsets).

Let $p_1, \ldots, p_d$ be the projections of X on the d coordinates of $\mathbf{R}^d$, and let A be the $\mathbf{K}$-subalgebra of $\mathcal{C}(X, \mathbf{K})$ generated by $p_1, \ldots, p_d$. By the Stone-Weierstrass Theorem, A is dense in $\mathcal{C}(X, \mathbf{K})$ for the topology of uniform convergence on compact subsets. Then the finite family $\{(p_j), \sqrt{2}\}$ (resp. $\{(p_j), \sqrt{2}, i\}$) generates a dense subring in $\mathcal{C}(X, \mathbf{K})$ if $\mathbf{K} = \mathbf{R}$ (resp. if $\mathbf{K} = \mathbf{C}$). □

Remark 7.28.

1. The hypothesis in Theorem 7.20 that X is homeomorphic to a subset of an Euclidean space is close to being necessary in order to apply Theorem 7.26. Indeed, suppose that $\mathcal{C}(X, \mathbf{K})$ is endowed with a topology such that the evaluation functions $f \mapsto f(x)$ are continuous. Besides, suppose that $\mathcal{C}(X, \mathbf{K})$ is topologically finitely generated as a ring by $p_1, \ldots p_d$. Then there exists a continuous injection of X in some Euclidean space, given by $x \mapsto (p_1(x), \ldots, p_d(x))$.

2. If X is metrizable and non-compact, and $\mathcal{C}(X, \mathbf{K})$ is endowed with the uniform convergence topology, then an easy growth argument shows that $\mathcal{C}(X, \mathbf{K})$ is not topologically finitely generated.

It would be interesting to generalize Theorem 7.27 to general semisimple Lie groups without compact factors, and Theorem 7.26 to semisimple groups without compact factors and with Property (T), or at least to higher rank ones. Theorem 7.25 is extended to symplectic groups in [Neu03]. On the other hand, if G is a connected compact simple Lie group, $\mathcal{C}(S^1, G)$ does not have Kazhdan's Property (T) [BHV05, Exercise 4.4.5].

7.5 A lemma about conditionally negative definite functions

If G is a group, we define **unnormalized** conditionally negative definite functions as conditionally negative definite (real-valued) functions, but weakening the hypothesis $\psi(1) = 0$ into $\psi(1) \geq 0$. Note that $\psi - \psi(1)$ is a (normalized) conditionally negative definite function (see [BeFo75] for details, where the abelian assumption is useless).

Proposition 7.29. **Let G be a locally compact group, and ψ a measurable unnormalized conditionally negative definite function, locally bounded. Then there exists a unique** continuous **unnormalized conditionally negative definite function $\bar{\psi}$ of G such that $\psi = \bar{\psi}$ locally almost everywhere.**

Proof: Let ψ a measurable unnormalized conditionally negative definite function. Then $\varphi = e^{-\psi}$ is a measurable positive definite function on G. By a result of de Leeuw and Glicksberg ([dLGl65], see [HeRo70], (32.12) Theorem), φ decomposes as a sum of two positive definite functions: $\varphi = \varphi_c + \varphi_s$, where φ_c is continuous and $\varphi_s = 0$ locally almost everywhere. Since ψ is locally bounded, φ is locally bounded from zero, so that φ_c does not vanish on G. Set $\bar{\psi} = -\log(\varphi_c)$. Then $\bar{\psi}$ is continuous, and $\psi = \bar{\psi}$ locally almost everywhere, so that $\bar{\psi}$ is unnormalized conditionally negative definite. Uniqueness is trivial. □

We need to deal with unnormalized functions in Theorem 7.29: indeed, suppose that G is any locally compact, non-discrete group. The function ψ defined by: $\psi(1) = 0$, $\psi(g) = 1$ elsewhere is (normalized) conditionally negative definite, but $\bar{\psi}$, the constant function 1, is not normalized. However, we have the following corollary.

Corollary 7.30. **Let G be a locally compact group, and ψ a measurable conditionally negative definite function, locally bounded. Then there exists a unique** continuous **conditionally negative definite function $\bar{\psi}$ on G, and a unique constant $a \in \mathbf{R}_+$, such that $\psi = \bar{\psi} + a$ locally almost everywhere.** □

7.6 A criterion for relative Property (T)

Recall the following theorem of Shalom.

Theorem 7.31 ([Sha99t], see also [BHV05], Chapter 1). **Let G be a locally compact group and N a closed normal abelian subgroup. Assume that the only mean on the Borel subsets of the Pontryagin dual $\widehat{N} = \mathrm{Hom}(N, \mathbf{R}/\mathbf{Z})$, invariant under the action of G by conjugation, is the Dirac measure at zero. Then the pair (G, N) has Property (T).**

It allows, for example, to prove Property (T) for $(SL(2,\mathbf{K}) \ltimes \mathbf{K}^2, \mathbf{K}^2)$ for every local field $\mathbf{K}$. On the other hand, it does not apply to $SL(2,\mathbf{Z}) \ltimes \mathbf{Z}^2$, since the action of $SL(2,\mathbf{Z})$ on $\mathbf{Z}^2$ has many finite orbits. We propose a generalization of Theorem 7.31 which does apply to $SL(2,\mathbf{Z}) \ltimes \mathbf{Z}^2$. It is strongly inspired by Shalom's work to prove Property (T) for $(SL(2,\mathbf{Z}) \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$ without using that $SL(2,\mathbf{Z}) \ltimes \mathbf{Z}^2$ is a lattice in $SL(2,\mathbf{R}) \ltimes \mathbf{R}^2$. The main difference between the following work and Shalom's result is that we do not care on explicit Kazhdan constants. This explains why our proof is much shorter that Shalom's one ([Sha99p]; see also [BHV05], Chapter 4).

If H is a locally compact abelian group, recall that the weak topology on H is the coarsest topology that makes all characters $\chi \in \hat{H}$ continuous. It coincides with the usual topology if and only if H is compact.

Theorem 7.32. **Let G be a locally compact group and N a closed, normal, abelian subgroup. Suppose that there exists a weak neighbourhood V of 1 in $\hat{N}$ such that the only G-invariant mean μ on $\hat{N}$ satisfying $\mu(V) = 1$ is the Dirac measure at the unit. Then (G, N) has Property (T).**

Proof: Since V is a weak neighbourhood, there exist $\alpha > 0$ and $g_1, \ldots, g_m$ in N such that V contains $\{\chi \in \hat{N},\ \forall k = 1 \ldots m,\ |1 - \chi(g_k)| < \alpha\}$.

Let $(\pi, \mathscr{H})$ be a unitary representation of G such that $1 \prec \pi$ and such that N has no invariant vector. We are going to exhibit a G-invariant mean μ on $\hat{N} - \{1\}$ such that $\mu(V) = 1$.

Let (K_i, ε_i) be an increasing net with K_i compact and $\alpha > \varepsilon_i > 0$, such that K_i contains all g_k, $\varepsilon_i \to 0$, $\bigcup K_i = G$. Let ξ_i be a (K_i, ε_i)-invariant vector.

Let E be the projection valued measure associated to $\pi|_{\hat{N}}$, so that $\pi(g) = \int_{\hat{N}} \chi(g) dE(\chi)$ for all $g \in N$. For $\xi \in \mathscr{H}$, let μ_ξ be the probability on $\hat{N}$ defined by $\mu_\xi(B) = \langle E(B)\xi, \xi\rangle$.

We have:

$$\|\pi(g)\xi - \xi\|^2 = \int_{\hat{N}} |1 - \chi(g)|^2 d\mu_\xi(\chi) \qquad \forall g \in N,\ \xi \in \mathscr{H}$$

$$\|\pi(g_k)\xi_i - \xi_i\|^2 = \int_{\hat{N}} |1 - \chi(g_k)|^2 d\mu_{\xi_i}(\chi) \leq \varepsilon_i^2$$

Let $A_k = \{\chi \in \hat{N},\ |1 - \chi(g_k)| < \alpha\}$ and B_k its complement in $\hat{N}$.

$$\int_{B_k} |1 - \chi(g_k)|^2 d\mu_{\xi_i}(\chi) \leq \varepsilon_i^2$$

$$\alpha^2 \mu_{\xi_i}(B_k) \leq \varepsilon_i^2$$

$$\mu_{\xi_i}(\bigcup_{k=1}^{m} A_k) \geq 1 - m\varepsilon_i^2/\alpha^2$$

$$\mu_{\xi_i}(V) \geq 1 - m\varepsilon_i^2/\alpha^2$$

Since π as no N-invariant vector, $\mu_{\xi_i}(\{1\}) = 0$ for all i. We can suppose that (μ_{ξ_i}) converges to a mean μ, in the sense that $\mu(B) = \lim_i \mu_{\xi_i}(B)$ for all Borel subsets B. In particular, $\mu(\{1\}) = 0$ and $\mu(V) = 1$.

It is easily checked, using

7.7 Open questions

(1) Let G be a locally compact group, and N a closed, normal subgroup of G. Suppose that G has the Haagerup Property. In general, G/N does not have the Haagerup Property (e.g. when G is a free group). However, does G/N have the Haagerup Property in the following cases:

  (a) N is connected?

  (b) N is amenable?

  (c) More specifically, when N is central in G?

(2) The Haagerup Property is not stable under extensions, as shown by the examples $\mathrm{SL}_2(\mathbf{Z}) \ltimes \mathbf{Z}^2$ and $\mathrm{SL}_2(\mathbf{R}) \ltimes \mathbf{R}^2$. However, does the converse of (1)(c) hold; namely, is it true that the Haagerup Property is stable under central extensions? This is asked in [CCJJV01, Chap. 7]. This is true in the realm of connected Lie groups [CCJJV01, Chap. 4].

(3) ([AkWa81], see §2.1.1) Let G be a locally compact, compactly generated group. Suppose that G does not have the Haagerup Property. Does this implies the existence of a non-compact, closed subset $X \subset G$ such that (G, X) has relative Property (T)?

(4) Does the implication (3) $\Rightarrow$ (1) in Theorem 2.16 hold without assuming the locally compact group G σ-compact (where relative Property (T) is defined as in Definition 2.1, in terms of definite positive functions)? This is known when X is a normal subgroup of G [Jol05].

(5) Let A be a finitely generated ring and fix $n \geq 3$. When does $\mathrm{SL}_n(A)$ have Property (T)? This is known to hold for many rings A of Krull dimension 1; however there is no ring of dimension 2 (e.g. $\mathbf{Z}[X]$ or $\mathbf{F}_p[X,Y]$) for which this is known to hold. In general, it is known that $(\mathrm{SL}_{n-1}(A) \ltimes A^{n-1}, A^{n-1})$ has relative Property (T) [Sha99p].

(6) Let V be an abelian group (say, discrete). Let G be a discrete group acting on V by automorphisms. When does the pair $(G \ltimes V, V)$ have relative Property (T)? This is well understood when V is torsion-free of finite rank. It is widely open in the test-case when V is an infinite dimensional vector space over a prime field K ($\mathbf{F}_p$ or $\mathbf{Q}$).

  No such examples with relative Property (T) are known when $K = \mathbf{Q}$. When $K = \mathbf{F}_p$, some examples are known, for instance when $V = A^n$, with $n \geq 2$ and A an infinite, finitely generated ring of characteristic p, and $G = \mathrm{SL}_n(A)$.

(7) Does there exist an infinite subgroup with Property (T) in one of the following groups:

  (a) $\mathrm{Homeo}(S^1)$

(b) $\mathrm{Homeo}_+(\mathbf{R})$ (i.e. does there exist a non-trivial left-orderable[4] group with Property (T)? [wor01, Question 19])

(c) $\mathrm{Diff}^r_+(\mathbf{R})$ for a given $r > 0$?

By a result of Navas [Nav02], $\mathrm{Diff}^r_+(S^1)$ has no infinite subgroup with Property (T) for $r > 3/2$.

(8) Let G be a bi-orderable group. Does it have the Haagerup Property? Note that, if Γ is a non-abelian free subgroup of $\mathrm{SL}_2(\mathbf{Z})$, then $\Gamma \ltimes \mathbf{Z}^2$ is left-orderable, but does not have the Haagerup Property since $(\Gamma \ltimes \mathbf{Z}^2, \mathbf{Z}^2)$ has relative Property (T). On the other hand, a nontrivial finitely generated bi-orderable group maps onto $\mathbf{Z}$, hence fails to have Property (T).

(9) Let G be a residually torsion-free nilpotent group. Does it have the Haagerup Property? It is easy to check that, in such a group G, there is no infinite subgroup H such that (G, H) has relative Property (T). On the other hand, a residually torsion-free nilpotent group is bi-orderable.

(10) Does the braid group B_n have the Haagerup Property? This is true and easy for $n \leq 3$, but unknown for $n \geq 4$. It is known that B_n has a finite index subgroup which is residually torsion-free nilpotent, namely the pure braid group P_n.

(11) Let $\mathbf{K}$ be a non-Archimedean local field, and let G be a linear algebraic group over $\mathbf{K}$. It is true that, if $G(\mathbf{K})$ has the Haagerup Property, then G is $\mathbf{K}$-isogeneous to the direct product of a semisimple group S with simple factors of $\mathbf{K}$-rank one with a group M such that $M/\mathrm{rad}(M)$ is $\mathbf{K}$-anisotropic (i.e. $M(\mathbf{K})$ is amenable)? This was established for $\mathrm{char}(\mathbf{K}) = 0$ in Chapter 1. Note that, conversely, this condition implies that $G(\mathbf{K})$ has the Haagerup Property.

(12) Let $\mathbf{A}$ be a locally compact, reduced commutative ring. Is it true that $\mathrm{SL}_2(\mathbf{A})$ has the Haagerup Property? This is true for $\mathbf{A}$ discrete [GHW05] (see also Section 7.2).

(13) In the groups $\mathrm{SL}_3(\mathbf{R})$, $\mathrm{Sp}_4(\mathbf{R})$, $\mathrm{Sp}(2, 1)$, does there exist a discrete subgroup of infinite covolume, with Property (T) (see Proposition 3.9)?

[4] **A group is left-orderable (resp. bi-orderable) if it has a total order invariant under left (resp. both left and right) translations.**

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Curriculum Vitae

Données personnelles

Nom de famille : de Cornulier Lucinière
Prénoms : Yves Alain Pierre
Nom d'usage : Yves de Cornulier
Date de naissance : 17 Novembre 1981
Lieu de naissance : Marseille (France)
Nationalité : française
Institut : Institut de Géométrie, Algèbre et Topologie (IGAT), EPFL
Adresse professionnelle : EPFL SB IGAT-GE, MA A2 417 (Bâtiment MA), Station 8, CH-1015 Lausanne
Téléphone professionnel : 0(0 41) 216 93 25 43
E-mail : yves.cornulier@epfl.ch

Études

1998 : Baccalauréat (Nantes, France)
2003 : Diplôme d'Études Approfondies (DEA) de mathématiques, Univ. Paris VII (France)