LIE ALGEBRAS

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1. Algebras

By scalar ring, we mean an associative, unital, commutative ring. By default we do not assume rings/algebras to be associative. In most of these lectures, the scalar ring will be assumed to be a field, often of characteristic zero and/or algebraically closed.

We fix a scalar ring R. Let A be an R-algebra (that is, an R-module endowed with an R-bilinear product $A \times A \to A$, often denoted $(a,b) \mapsto ab$). Homomorphisms between R-algebras are R-module homomorphisms that are also multiplicative homomorphisms. Subalgebras are R-submodules stable under the product. A 2-sided ideal (or 2-sided R-ideal) I is an R-submodule such that $x \in I, y \in A$ implies that both xy and $yx \in I$. The quotient A/I then canonically inherits a product structure. If S is a scalar R-algebra (that is, an R-algebra that is a scalar ring), then $S \otimes_R A$ is naturally an S-algebra; this is called "extension of scalars".

An R-derivation (or derivation, if R can be omitted) of A is a R-module endomorphism f of A satisfying $f(ab) = f(a)b + af(b)$. For $x, y \in A$, we write $L_x(y) = R_y(x) = xy$; thus L_x, R_x are R-module endomorphisms of A.

The algebra A is said to be

(1) associative if $L_{xy} = L_x L_y$ for all $x, y \in A$, or equivalently if $L_x R_y = R_y L_x$ for all $x, y \in A$. As a formula, this means that $x(yz) = (xy)z$ for all $x, y, z \in A$;
(2) (left) Leibniz if $L_{xy} = L_x L_y - L_y L_x$ for all $x, y \in A$, or equivalently if L_x is a derivation for every x. As a formula, this reads as the "Leibniz-Loday identity" $(xy)z - x(yz) + y(xz) = 0$ for all $x, y, z \in A$.
(3) alternating if $xx = 0$ for all $x \in A$;
(4) skew-symmetric if $xy + yx = 0$ for all $x, y \in A$; (thus alternating implies skew-symmetric and the converse holds if 2 is invertible in A)
(5) Lie if it is both alternating and Leibniz. (For an alternating algebra, the Leibniz-Loday identity can be rewritten as $\mathrm{Jac}(x, y, z) = 0$, where $\mathrm{Jac}(x, y, z) = x(yz) + y(zx) + z(xy)$; this is known as Jacobi identity. Note that in an alternating algebra, the trilinear map Jac is alternating, i.e., vanishes whenever two variables are equal.)

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All these conditions are stable under taking subalgebras and quotients. They are also stable under taking extensions of scalars[1] (for all multilinear conditions this is straightforward; for the alternating condition this easily follows, first using that alternating implies skew-symmetric).

If A is associative, the product defined as commutator bracket $[a, b] = ab - ba$ is Lie. For this reason, it is custom to denote the product in a Lie algebra with brackets (rather than with a dot or no symbol).

For a Lie (or more generally skew-symmetric) algebra, as in the commutative case, we just talk of "ideals" rather than 2-sided ideals.

If A is Leibniz (e.g., Lie) and if B is an R-subalgebra, the 2-sided normalizer $N_A(B)$ is defined as the set of x such that $xB \cup Bx \subset B$ (for Lie algebra, we omit "2-sided"). Then B is a 2-sided ideal of its 2-sided normalizer $N_A(B)$ (exercise).

Let V be an R-module. Let $\mathfrak{gl}_R(V)$ be the set of R-module endomorphisms of V. This is an associative R-algebra, and hence is a Lie R-algebra for the corresponding commutator bracket. Many important Lie algebras are naturally constructed as subalgebras of the latter.

Let A be an R-algebra. Then the set of R-derivations of A is a Lie subalgebra of $\mathfrak{gl}_R(A)$ (exercise: check it), denoted $\mathrm{Der}_R(A)$.

Let $\mathfrak{h}, \mathfrak{n}$ be Lie R-algebras and $j : \mathfrak{h} \to \mathrm{Der}_R(\mathfrak{n})$ an R-algebra homomorphism. The semidirect product $\mathfrak{n} \rtimes_j \mathfrak{h}$ is defined as follows: as an R-module, this is the direct sum $\mathfrak{n} \oplus \mathfrak{h}$. The product is defined as

$$[(n_1, h_1), (n_2, h_2)] = ([n_1, n_2] + j(h_1)n_2 - j(h_2)n_1, [h_1, h_2]).$$

Given a Lie R-algebra $\mathfrak{g}$ and R-submodules $\mathfrak{n}, \mathfrak{h}$, the Lie algebra $\mathfrak{g}$ decomposes as semidirect product $\mathfrak{n} \rtimes \mathfrak{h}$ if and only if $\mathfrak{n}$ is an ideal and $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as R-module. Here j maps $h \in \mathfrak{h}$ to the restriction $(L_h)|_{\mathfrak{n}}$.

If I, J are R-submodules of A, we denote by IJ the R-submodule generated by $\{xy : (x, y) \in I \times J\}$.

Given an algebra A, one defines $A^1 = A$, and, by induction, $A^i = \sum_{j+k=i} A^j A^k$. Here BC is the submodule generated by bc for $(b, c) \in B \times C$. Then A^i is a 2-sided ideal, and $A = A^1 \supset A^2 \supset A^3 \supset \dots$. The sequence $(A^i)_{i \geq 1}$ is called the *lower central series*. The algebra A is said to be *nilpotent* if $A^{i+1} = \{0\}$ for some $i \geq 0$; in this case, it is called i-step nilpotent; the nilpotency class of A is the smallest i for which this holds.

In the case of a Leibniz algebra $\mathfrak{g}$ (and hence of a Lie algebra), one has $\mathfrak{g}^i = \mathfrak{g}\mathfrak{g}^{i-1}$ for all $i \geq 2$ (exercise), which simplifies the definition of the lower central series.

Exercice: for $n \geq 0$, consider the alternating algebra whose underlying R-module has a basis $(x, y_1, \dots, y_n)$, and brackets $[x, y_i] = y_{i+1}$, $1 \leq i < n$. (others being zero, except those following by skew-symmetry). Check that it is a Lie

[1]For fields of prime characteristic p, check that the condition $(\forall x, x^p = x)$, does *not* pass to extensions of scalars.

algebra, whose nilpotency class is equal to n as soon as $R \neq \{0\}$. (It is known as standard filiform Lie algebra of class n over R.)

Given a Leibniz algebra, one defines $\mathfrak{g}^{(0)} = \mathfrak{g}$ and $\mathfrak{g}^{(i+1)} = \mathfrak{g}^{(i)}\mathfrak{g}^{(i)}$. These are 2-sided ideals (exercise), forming the *derived series*. The Leibniz algebra $\mathfrak{g}$ is said to be *solvable* if $\mathfrak{g}^{(i)} = \{0\}$ for some i; it is then called i-step solvable; the smallest such i is called derived length (or solvability length) of $\mathfrak{g}$. We have $\mathfrak{g}^{(i)} \subset \mathfrak{g}^{2^i}$ (with equality for $i = 0, 1$); in particular, nilpotent implies solvable. When the product is zero (that is, $\mathfrak{g}$ is 1-step nilpotent, that is, 1-step solvable), $\mathfrak{g}$ is called abelian. We say that $\mathfrak{g}$ is perfect if $\mathfrak{g} = \mathfrak{g}^{(1)} (= \mathfrak{g}^2)$.

Exercise: If $\mathfrak{g}$ is solvable, its only perfect subalgebra is $\{0\}$. Conversely, assuming that $\mathfrak{g}$ is finite-dimensional over a field, show that if the only perfect subalgebra is $\{0\}$, then $\mathfrak{g}$ is solvable.

Exercise: 1) Let I, J be solvable ideals in a Lie algebra; show that $I + J$ is a solvable ideal. 2) Find a Lie algebra with two abelian ideals I, J such that $I + J$ is not abelian. 3) (harder) Let I, J be 2-sided ideals in an algebra A. Show that $(I + J)^{k+\ell-1} \subset I^k + J^\ell$ for all $k, \ell \geq 1$. Deduce that if I, J are nilpotent, then so is $I + J$.

A representation of a Lie R-algebra $\mathfrak{g}$ in an R-module V is an R-algebra homomorphism $\rho(\mathfrak{g}) \to \mathfrak{gl}_R(V)$, the latter being endowed with its commutator bracket. Endowed with such a homomorphism, V is called a $\mathfrak{g}$-module. By definition $\mathfrak{g}$-submodules are R-submodules that are stable under $\rho(\mathfrak{g})$. The quotient by a $\mathfrak{g}$-submodule is naturally a module as well.

A $\mathfrak{g}$-module V is said to be simple if it is nonzero and its only submodules are $\{0\}$ and V; in this case the representation is said to be irreducible.

The mapping $\mathfrak{g} \to \mathrm{Der}_R(\mathfrak{g})$, mapping x to L_x, is a representation of $\mathfrak{g}$ in inself, called adjoint representation. Its kernel is the center $\mathfrak{z}(\mathfrak{g}) = \{x \in \mathfrak{g} : xy = yx, \forall y \in \mathfrak{g}\}$. Submodules of the adjoint representations are precisely ideals of $\mathfrak{g}$.

An algebra A is said to be simple if its product is not zero[2], and its only 2-sided ideals are $\{0\}$ and A.

2. Small dimension

Now we assume that the ground scalar ring R is a field, now denoted K.

One can tackle the task of classifying Lie algebras of each given dimension, possibly restricting to some subclass, or with restrictions on K.

In each given dimension, abelian Lie algebras form one isomorphy class. In dimension 0,1, these are the only ones. In dimension 2, there is only one other, which we denote $\mathfrak{b}$: it can be described with the basis (x, s) with the bracket being given by $[s, x] = x$ (by this we mean the other bracket follow from Lie

[2]Thus a 1-dimensional space over a field, endowed with the zero bracket, is not considered as a simple Lie algebra. This is just a convention. It plays a role analogous to cyclic groups of prime order in finite group theory, which, in contrast, are considered as simple groups.

algebra axioms: $[s,s]=[x,x]=0$, $[x,s]=-x$). We will often use, when $0\neq 2$ in K, the basis (h,x) of $\mathfrak{b}$ with $h=2s$; thus $[h,x]=2x$.

Exercise: show that indeed every 2-dimensional non-abelian Lie K-algebra is isomorphic to $\mathfrak{b}$.

Note that $\mathfrak{b}$ is solvable with derived length 2, and not nilpotent.

Exercise: 1) Show that every 3-dimensional Lie algebra is either solvable or simple.

2) Show that every solvable 3-dimensional Lie algebra possesses an abelian ideal (first find a nonzero abelian ideal and discuss according to its dimension). Deduce that it is isomorphic to a semidirect product $\mathfrak{g}_M = K^2 \rtimes_M K$, where $M \in \mathfrak{gl}_2(K)$, and the notation meaning that the homomorphism $K \to \mathrm{Der}(K^2) = \mathfrak{gl}_2(K)$ is given by $t \mapsto tA$; two such Lie algebras $\mathfrak{g}_{M_1}, \mathfrak{g}_{M_2}$ are isomorphic if and only if KM_1 and KM_2 are conjugate by some element of $\mathrm{GL}_2(K)$. Deduce a list when $K=\mathbf{C}$, and when $K=\mathbf{R}$.

3) Let $\mathfrak{g}$ be a 3-dimensional Lie algebra in which L_x is nilpotent for every x. Show that $\mathfrak{g}$ is nilpotent.

4) Assume that K is algebraically closed. Let $\mathfrak{g}$ be simple and 3-dimensional. Deduce that there exist a 2-dimensional subalgebra, namely s,x such that $[s,x]=x$ and $x\neq 0$.

5) Assuming in addition that K is not of characteristic 2, deduce that there exists $y\neq 0$ such that $[s,y]=-y$, with (s,x,y) a basis, and in turn deduce that y can be chosen so that $[x,y]=s$.

Exercise: Fix a field K. 1) Let Alg_n be the space of bilinear laws on K^3, and $\mathrm{Alt}_n \subset \mathrm{Alg}_n$ its subspace of alternating bilinear laws. Check that these are subspaces of the space of all maps $K^2 \to K$, of dimension n^3 and $n^2(n-1)/2$ respectively.

2) Let Lie_n be the subset of Alt_n consisting of the Lie algebra laws, i.e., those skew-linear maps $B : K^2 \to K$ satisfying $\mathrm{J}(B)=0$, where $\mathrm{J}(B)$ is the alternating trilinear form $(x,y,z)\mapsto B(x,B(y,z))+B(y,B(z,x))+B(z,B(x,y))$. Check that the inclusion $\mathrm{Lie}_n \subset \mathrm{Alt}_n$ is an equality for $n\le 2$ and is a proper inclusion for all $n\ge 3$.

3) Show that Alt_n is not stable under addition, for all $n\ge 3$.

4) Let u_n be the largest dimension of a subspace contained in Lie_n (note that it depends a priori on K, which is fixed). Show that $\liminf u_n/n^3>0$. (Hint: for $n=2m$ even, write $K^n = V_1\oplus V_2$ with $\dim(V_1)=\dim(V_2)=m$, and describe the set of Lie algebra laws B satisfying $B(V_1,V_1)\subset V_2$ and $B(K^n,V_2)=\{0\}$.)

Note that the set of isomorphism classes of n-dimensional Lie algebras can be identified to the quotient $\mathrm{GL}_n(K)\backslash\mathrm{Lie}_n$. Informally speaking[3], this shows that Lie_n has cubic dimension (i.e., bounded above and below by cubic polynomials), and so is the quotient, since the dimension of $\mathrm{GL}_n(K)$ is quadratic.

[3]This can be restated more rigorously using the language of algebraic geometry.

3. Representations of $\mathfrak{sl}_2$

We fix a ground field K.

Let V be vector space over K, and f a linear endomorphism. If $t \in K$, define $V_t = V_t(f) = \bigcup_k \mathrm{Ker}(f-t)^k$: this is the characteristic subspace of f associated to t. The V_t generate their direct sum. By definition, f is K-trigonalizable if and only if $\bigoplus_{t\in K} V_t = V$; this holds when V is finite-dimensional and K is algebraically closed; this is called the characteristic decomposition of V with respect to f.

Let us pass to the 2-dimensional Lie algebra $\mathfrak{b}$, with its basis (h,x), with $[h,x] = 2x$; we assume here that $2 \neq 0$ in K (this is somewhat an artificial restriction here, but it will be convenient in the sequel).

Proposition 3.1. *Consider a $\mathfrak{b}$-module (V,ρ) (that is, given by a homomorphism of Lie K-algebras $\rho : \mathfrak{b} \to \mathfrak{gl}(V)$). Write $H = \rho(h)$, $X = \rho(x)$. We have*

$$X\big(\mathrm{Ker}((H-t)^k)\big) \subset \mathrm{Ker}((H-(t+2))^k).$$

In particular, we have

$$XV_t \subset V_{t+2}, \quad \forall t \in K.$$

Proof. The relation $[h,x] = 2x$ implies $HX - XH = 2X$, which can be rewritten as $(H-2)X = XH$, and thus for every $t \in K$ we have $(H-2-t)X = X(H-t)$. By an immediate induction, we deduce $(H-2-t)^kX = X(H-t)^k$ for all $k \geq 0$. The formula follows. □

This already has useful consequences:

Corollary 3.2. *Let (V,ρ) be a finite-dimensional $\mathfrak{b}$-module; if K has characteristic $p > 0$, assume in addition that $p \neq 2$ and $\dim(V) < p$. Then $\rho(x)$ is nilpotent.*

Proof. Write $d = \dim(V)$ and $X = \rho(x)$. For every $t \in K$, the $d+1$ elements $t, t+2, \dots, t+2d$ are pairwise distinct in K, and therefore there exists one of them, say $t+2k$, such that $V_{t'} = \{0\}$. Since $X^{t+2k}V_t \subset V_{t+2k}$, we deduce that $X^{t+2k}V_t = \{0\}$; so $X^{t+2d}V_t = \{0\}$. Hence, assuming $V = \bigoplus V_t$, we have $X^{t+2d} = 0$.

This concludes when K is algebraically closed; the general case follows by considering $V \otimes_K L$ as a representation of $\mathfrak{b} \otimes_K L$ for an algebraically closed extension L of K. □

Now let us pass to $\mathfrak{sl}_2(K)$ with $2 \neq 0$ in K. We choose a basis (h,x,y) with $[h,x] = 2x$, $[h,y] = -2y$, and $[x,y] = h$. Originally, this corresponds to the matrices

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

but this matrix interpretation plays no role here. Given a $\mathfrak{sl}_2(K)$-module (V,ρ), we always denote $H = \rho(h)$, $X = \rho(x)$, $Y = \rho(y)$.

Lemma 3.3. *For every $\mathfrak{sl}_2(K)$-module, we have $[X, Y^n] = nY^{n-1}(H - n + 1)$ for all $n \geq 0$.*

The proof is an exercise by induction.

Given an $\mathfrak{sl}_2(K)$-module (V, ρ), also denote $V_t = V_t(h)$ for $t \in K$. By Proposition (3.1), $XV_t \subset V_{t+2}$ for all t. Also, since $(-h, y)$ can play the same role as (h, x) and $V_t(-H) = V_{-t}$, we have $YV_t \subset V_{t-2}$ for all t.

Proposition 3.4. *For every finite-dimensional $\mathfrak{sl}_2(K)$-module (V, ρ), with K of characteristic zero (or, when K has characteristic $p > 2$, of dimension $< p$), we have $V_t = \{0\}$ for every $t \notin \mathbf{Z}1_K$, and H is K-trigonalizable, with $V = \bigoplus_{i \in \mathbf{Z}1_K} V_i$.*

Proof. By contradiction, suppose that $V_t \neq \{0\}$ with $t \notin \mathbf{Z}1_K$. Consider $W = \bigoplus_{n \in \mathbf{Z}} V_{t+2n}$; this is a nonzero submodule; passing to W we can suppose that $V = W$; hence, since $0 \notin t + \mathbf{Z}1_K$, we have $H - k$ invertible on V, for all $k \in \mathbf{Z}$. By Corollary 3.2, Y is nilpotent.

Let $k \geq 1$ be minimal such that $Y^k = 0$. The relation $0 = [X, Y^k] = kY^{k-1}(H - k + 1)$ holds in W; since $H - k + 1$ is invertible, we deduce that $Y^{k-1} = 0$ and get a contradiction.

When K is algebraically closed, the result follows. In general, we consider an algebraically closed extension L of K, and denote $V_L = V \otimes_K L$, so H extends to an operator H_L on V_L. By the algebraically closed case, the only eigenvalues of H_L are in $\mathbf{Z}1_K$, and hence, we have, for some k, the equality $\prod_{|i| \leq k}(H_L - i)^k = 0$. Hence, by restriction, $\prod_{|i| \leq k}(H - i)^k = 0$. This means that $V = \sum_{|i| \leq k} V_i$. $\square$

For $n \in \mathbf{N}(= \{0, 1, 2, \dots\})$, write $J_n = \{n, n-2, \dots, -n\} \subset \mathbf{Z}$; this subset has $n + 1$ elements. Define V as a vector space over K with basis $(e_n)_{n \in \mathbf{Z}}$, and $\mathbb{V}[n]$ the $(n+1)$-dimensional subspace with basis $(e_i)_{i \in J_n}$. Define

$$He_i = ie_i, \quad X_n e_i = \frac{n - i}{2} e_{i+2}, \quad Y_n e_i = \frac{n + i}{2} e_{i-2}.$$

These define linear endomorphisms of V. By a straightforward computation, we have $[H, X_n] = 2X_n$, $[H, Y_n] = -2Y_n$, and $[X_n, Y_n] = H$. Therefore, $\rho_n : (h, x, y) \mapsto (H, X_n, Y_n)$ defines a representation of the Lie algebra $\mathfrak{sl}_2(K)$ on V. Observe that $\mathbb{V}[n]$ is a submodule for ρ_n; we now systematically endow $\mathbb{V}[n]$ with the structure of $\mathfrak{sl}_2(K)$-module defined by ρ_n.

Exercise Let A be the polynomial K-algebra $K[x, y]$, and write $A[n]$ for its $(n+1)$-dimensional subspace of homogeneous polynomials of degree n. Define the linear endomorphisms of A: $X = L_y \partial_x$ and $Y = L_x \partial_y$. Check that these are derivations of A, preserving $A[n]$ for all n. Check that the Lie subalgebra of $\mathrm{Der}(A)$ generated by X, Y is isomorphic to $\mathfrak{sl}_2(K)$ (for some isomorphism mapping X to X and Y to Y), and that the representation of $\mathfrak{sl}_2(K)$ on $A[n]$ is isomorphic to $\mathbb{V}[n]$.

Lemma 3.5. *Fix $n \in \mathbf{N}$. Suppose that K has characteristic zero, or, if K has characteristic $p > 0$, that $n < p$. Then $\mathbb{V}[n]$ is a simple $\mathfrak{sl}_2(K)$-module.*

Proof. It is nonzero. The characteristic assumption implies the nonvanishing of the coefficients $\frac{n\pm i}{2}$ except $\frac{n+i}{2}$ for $i=-n$ and $\frac{n-i}{2}$ for $i=n$. Thus X_n maps Ke_i onto Ke_{i+2} for all $i \in J_n \smallsetminus \{n\}$, and Y_n maps Ke_i onto Ke_{i-2} for all $i \in J_n \smallsetminus \{-n\}$. In particular, the kernel of the nilpotent endomorphism X_n is reduced to Ke_n. Let W be a nonzero submodule, and $v \in W \smallsetminus \{0\}$. Let $k \geq 0$ be maximal such that $w := X_n^k v \neq 0$. Then $w \in \mathrm{Ker}(X_n) = Ke_n$. So $e_n \in W$. Applying Y_n repeatedly, we deduce that $e_i \in W$ for all $i \in J_n$. So $W = \mathbb{V}[n]$. $\square$

Lemma 3.6. *Let V be a $\mathfrak{sl}_2(K)$-module. Let v be an eigenvector for H, in the kernel of X. Write $w_i = Y^i v$. Then the family $(w_i)_{i\geq 0}$ linearly generates a submodule W of V.*

If K has characteristic zero, or characteristic p and $\dim(V) < p$, then its nonzero elements form a basis of W, and the kernel of X on W is 1-dimensional, reduced to Kv. In particular, if V is a simple $\mathfrak{sl}_2(K)$-module, then the kernel of both X and Y is 1-dimensional, $\dim(V_n) \leq 1$ for all n, and H is diagonalizable.

Proof. Say that $Hv = tv$. The relation $(H+2-i)Y = Y(H-i)$ implies that $Hw_i = (t-2i)w_i$ for all i. The formula $[X, Y^i] = iY^{i-1}(H-i+1)$ of Lemma 3.3, applied to v, yields $Xw_i = i(t-i+1)w_{i-1}$ for all $i \geq 0$. Since $Yw_i = w_{i+1}$, $Xw_0 = 0$, we deduce that the given family generates a $\mathfrak{sl}_2(K)$-submodule.

In characteristic zero, the elements w_i belong to distinct eigenspaces, and hence its nonzero elements form a free family. In characteristic p, $d = \dim(W) < p$ is finite, and the freeness of the family of nonzero elements in $(w_0, \dots, w_d)$ implies that $w_i = 0$ for some $i \leq d$, hence $w_d = 0$, and hence $(w_0, \dots, w_{d-1})$ is a basis for W. Then $d(t-d+1) = 0$ in K, so $t = d-1$ in K, and $i(t-i+1) \neq 0$ in K for $1 \leq i \leq d-1$, so the kernel of X on W is reduced to Kv.

If in addition V is simple, then by the previous paragraph, there is a nonzero submodule on which the kernel of X has dimension 1, and hence by simplicity, this is V and the kernel of X has dimension 1. Changing the roles of (H, X, Y) and $(-H, Y, X)$, we deduce that the kernel of Y also has dimension 1. Moreover, since $(w_0, \dots, w_{d-1})$ is a basis and belong to distinct subspaces V_n, we deduce that $\dim(V_n) \leq 1$ for all n; in particular H is diagonalizable. $\square$

Theorem 3.7. *Every simple $\mathfrak{sl}_2(K)$-module V (of dimension $< p$ when K has positive characteristic p) is isomorphic to $\mathbb{V}[n]$ for $n = \dim(V) - 1$.*

Proof. There exists $n \in \mathbf{Z}$ such that $V_n \neq \{0\}$ and $V_{n+1} = \{0\}$; if K has characteristic $p > 0$, we choose $n \in \{0, \dots, p-1\}$. Choose $v_n \in V_n \smallsetminus \{0\}$. Define $v_{n-2i} = Y^i v_n \in V_{n-2i}$; let $k \geq 0$ be the maximal i such that $v_{n-2i} \neq 0$. By Lemma 3.6, $(v_n, v_{n-2}, \dots, v_{n-2k})$ is a basis of a submodule, and therefore of V by simplicity.

In particular, we see that all non-zero V_i are 1-dimensional. The dimension of V is therefore equal to $k+1$, and the trace of H is therefore equal to $\sum_{i=0}^{k}(n-2i) = (k+1)n - k(k+1) = (k+1)(n-k)$ (in K). Since the trace of $H = [X, Y]$ is zero (in K), in characteristic zero, we deduce that $k = n$. In positive characteristic p, we deduce that p divides $(k+1)(n-k)$ in $\mathbf{Z}$. We get the same conclusion $k=n$ as follows: since $0<k+1=\dim(V)<p$ (in $\mathbf{Z}$), we see that p divides $n-k$; then we see that $n-k$ belong to $\{-p+1,\dots,p-1\}$, and hence we conclude that $k=n$. Accordingly, $(v_i)_{i\in J_n}$ is a basis of V.

For $i\in J_n$, we have $Hv_i=iv_i$. Write $Xv_i=a_iv_{i+2}$ and $Yv_i=b_iv_{i-2}$, with $a_n=b_{-n}=0$; set $b_{n+2}=a_{n-2}=0$ for convenience. Then the relation $[X,Y]=H$ yields the relation (in K)

$$a_{i-2}b_i-a_ib_{i+2}=i,\quad i\in J_n;$$

note that this holds for any basis $(v_i)_{i\in J_n}$ with $v_i\in V_i$, not only the specific one constructed above. Writing $c_i=a_ib_{i+2}$, it yields $c_{i-2}-c_i=i$, which implies, by a simple backwards 2-step induction,

$$a_ib_{i+2}=c_i=\left(\frac{n-i}{2}\right)\left(\frac{n+i+2}{2}\right),\quad \forall i\in J_n. \tag{3.1}$$

By Lemma 3.6, $\mathrm{Ker}(Y)$ is reduced to Kv_{-n}. Thus $Yv_i\neq 0$ for all $i\in J_n\smallsetminus\{-n\}$. Similarly, $Xv_i\neq 0$ for all $i\in J_n\smallsetminus\{n\}$. We now, after choosing $e_n=v_n$ iteratively define, for $i\in J_n$ (with J_n indexed in decreasing order), the element e_{i-2} by $Ye_i=\frac{n+i}{2}e_{i-2}$ for all $i\in J_n\smallsetminus\{-n\}$. Write $Xe_i=a_i'e_{i+2}$, for $i\in J_n\smallsetminus\{n\}$. Then (3.1) holds with b_{i+2} replaced with $\frac{n+i+2}{2}$ and a_i by a_i', which yields $a_i'=\frac{n-i}{2}$ for all $i\in J_n$. Thus V is isomorphic to $\mathbb{V}[n]$. □

Corollary 3.8. *Let $\mathfrak{g}$ be a Lie algebra and V a finite-dimensional, faithful $\mathfrak{g}$-module (V,ρ) (faithful means that the ρ is injective). Suppose that $\mathfrak{g}$ contains a subalgebra isomorphic to $\mathfrak{sl}_2(K)$. Suppose that K has characteristic 0. Then the form $B_\rho:\mathfrak{g}\times\mathfrak{g}\to K$, $(g_1,g_2)\mapsto\mathrm{Trace}(\rho(x)\rho(y))$ is nonzero.*

Proof. Restricting, we can suppose that $\mathfrak{g}=\mathfrak{sl}_2(K)$.

For $n\ge 0$, we have $t_n:=B_{\mathbb{V}[n]}(h,h)=\sum_{i\in J_n}i^2$. So $t_n\in\mathbf{N}$ for all n, and $t_n>0$ whenever $n>0$ (it equals $\frac{n(n+1)(n+2)}{3}$ but we do not need this here). There exist submodules $0\subset V^1\subset V^2\subset\cdots\subset V^k=V$ such that V^i/V^{i-1} is irreducible for all i, say isomorphic to $\mathbb{V}[n_i]$. So $B_V(h,h)=\sum_i t_{n_i}\ge 0$, and is positive as soon as $n_i>0$ for some i. The remaining case is when all n_i are zero. In this case, all V_{n_i} are 1-dimensional, and this yields a homomorphism of $\mathfrak{sl}_2(K)$ into the Lie algebra of strictly upper triangular matrices; the latter is nilpotent and its only perfect subalgebra is $\{0\}$. Thus the representation is zero and cannot be faithful, a contradiction. □

Corollary 3.9. *Let (V,ρ) be a finite-dimensional $\mathfrak{sl}_2(K)$-module (of dimension $<p$ in case of positive characteristic p). Then V is irreducible if and only if* $\mathrm{ad}(h)$ *has only simple eigenvalues, and any two distinct eigenvalues have their difference in* $2\mathbf{Z}$.

Proof. For each n, $\mathbb{V}[n]$ has these properties. Conversely, if V is not irreducible, then it has a submodule W such that, for some m,n, $\mathbb{V}[m]$ is isomorphic to a submodule of W and $\mathbb{V}[n]$ is isomorphic to a submodule of V/W. If both m and n are even (resp. odd), then it follows that 0 (resp. 1) is a double eigenvalue of $\mathrm{ad}(h)$. Otherwise, both 0 and 1 are eigenvalues of $\mathrm{ad}(h)$. □

Proposition 3.10. *Let $\mathfrak{g}$ be a Lie algebra. Let $\mathcal{C}$ be class of (isomorphism classes of) finite-dimensional $\mathfrak{g}$-modules. Let $\hat{\mathcal{C}}$ be the class of finite-dimensional $\mathfrak{g}$-modules all of whose irreducible subquotients belong to $\mathcal{C}$. Equivalent statements:*

(1) *any extension of $\mathfrak{g}$-modules, $0 \to U \to V \xrightarrow{p} W \to 0$ with $U, W \in \mathcal{C}$, splits: there exists a $\mathfrak{g}$-module homomorphism $i : W \to V$ such that $p \circ i = \mathrm{id}_W$.*
(2) *any finite-dimensional $\mathfrak{g}$-module in $\hat{\mathcal{C}}$ is sum of its irreducible submodules;*
(3) *any finite-dimensional $\mathfrak{g}$-module in $\hat{\mathcal{C}}$ is a direct sum of irreducible submodules.*

Proof. The implication (2)⇒(3) holds for each given finite-dimensional $\mathfrak{g}$-module V: consider a submodule W of maximal dimension that splits as a direct sum of irreducible submodules; the assumption implies, if $V \neq W$, that there is an irreducible submodule W' not contained in W; then W and W' generate their direct sum and we contradict the maximality of W.

Assuming (3), in the setting of (1), there exists an irreducible submodule P not contained in U. By simplicity of U, its intersection with U is zero, and by simplicity of W, its image is all of W. So $V = U \oplus P$. Then p restricts to a bijection $P \to W$, whose inverse yields the desired splitting.

Suppose (1) and let us prove (2). Let V be a counterexample of minimal dimension. Clearly, $V \neq \{0\}$, and hence contains a simple submodule U. By induction, V/U is a sum of simple submodules V_i/U. Then by (1), we can write $V_i = U \oplus P_i$ with P_i a submodule. Hence V is sum of U and all P_i. □

Theorem 3.11. *Let K be a field of characteristic zero. Every finite-dimensional representation of $\mathfrak{sl}_2(K)$ is a direct sum of irreducible representations.*

Proof. By Proposition 3.10, we have to prove that for module V and submodule U such that both U and V/U are irreducible, the corresponding exact sequence splits, that is, there exists a submodule W such that $V = U \oplus W$. Equivalently, we have to prove that the set of submodules of V is not reduced to $\{\{0\}, U, V\}$.

By Theorem 3.7, we can suppose that U is isomorphic to $\mathbb{V}[n]$ and W to $\mathbb{V}[m]$, with $n, m \neq 0$.

We conclude in two ways according to whether $m = n$. First suppose that $m \neq n$. Then since $\dim(V_{m+2}) = \dim(V_m) - 1$, there exists a nonzero element v in $\mathrm{Ker}(X) \cap V_m$. Then by Lemma 3.6, there is $k \geq 0$ and a submodule W of V such that $W = \bigoplus_{i=0}^{k} W_{m-2i}$ with $\dim(W_i) \leq 1$ for all i, $\dim(W_m) = 1$ and $\dim(W_{m+2}) = 0$. None of $\{0\}$, U and V satisfies these conditions and we are done.

Now suppose that $m = n$. So, by Theorem 3.7, U and V/U are isomorphic $\mathfrak{sl}_2(K)$-modules. Choosing an isomorphism $V/U \to U$, composed with the obvious maps $V \to V/U \to U \subset V$, yields a nonzero endomorphism t of the

$\mathfrak{sl}_2(K)$-module V, such that $t^2 = 0$. Write $A = K[t]/(t^2)$ (as a vector space, it has the basis $(1, t)$). Thus, X, Y, Z commute with t, and hence are A-module endomorphisms of V. We have $\mathrm{Im}(t) = \mathrm{Ker}(t) = U$, and in particular, each V_i, for $i \in J_n$, is a free A-module of rank 1, generated by any element of $V_i \smallsetminus U_i$.

Choose $v_n \in V_n \smallsetminus U_n$. Define $v_{n-2j} = Y^j v_n$, for $i \leq n$. Working in V/U, by Lemma 3.6, we obtain that v_i is nonzero in V/U for each $i \in J_n$, that is, $v_i \in V_i \smallsetminus U_i$. Therefore $(v_i)_{i \in J_n}$ is a basis of the free A-module V of rank $n+1$.

Also $Hv_n = (n + \lambda t)v_n$ for some $\lambda \in K$. Using that $[H, Y] = -2Y$, we deduce that $Hv_i = (i + \lambda t)v_i$ for all i. We now use that the trace of $H = [X, Y]$, viewed as matrix over A, vanishes. This trace is $\sum_{i \in J_n} (i + \lambda t) = \lambda(n+1)t$. Hence $\lambda = 0$. Thus $Hv = iv$ for all $i \in V_i$ and all $i \in J_n$: H is diagonalizable (as K-linear endomorphism). In particular, v_n is an eigenvector of H; we deduce (Lemma 3.6) that the K-linear subspace spanned by v_i is a $\mathfrak{sl}_2(K)$-submodule of V, and it is a direct summand of U. □

4. Invariant forms

Let A be an R-algebra and M an R-module. An R-bilinear map $f : A \times A \to M$ is said to be invariant if $f(ab, c) = f(a, bc)$ for all a, b, c.

If $\mathfrak{g}$ is a Lie algebra over a field K and ρ a finite-dimensional representation, then the form $B_\rho : (x, y) \mapsto \mathrm{Trace}(\rho(x)\rho(y))$ is a symmetric invariant bilinear form. It is called trace form associated to ρ. When $\rho = \mathrm{ad}$ is the adjoint representation (defined by $\mathrm{ad}(x)(y) = [x, y]$, this is called the Killing form of the finite-dimensional Lie algebra $\mathfrak{g}$.

A Lie algebra $\mathfrak{g}$ over a field K is said to be semisimple if it is finite-dimensional and its Killing form is non-degenerate.

Proposition 4.1. *Let A be a finite-dimensional K-algebra. Suppose that it admits a non-degenerate symmetric invariant bilinear form f, and possesses no ideal J such that $J^2 = \{0\}$. Then A decomposes as a finite direct product $\prod_{i=1}^n A_i$ of simple K-algebras (orthogonal to each other for f, and each being non-degenerate for f). The A_i are precisely the minimal nonzero 2-sided ideals of A.*

Proof. Let I be a minimal nonzero ideal. Let J be the orthogonal of I. The invariance of f implies that J is a 2-sided ideal (exercise: check it). For $x \in I$, $y \in J$ and $z \in A$, we have $f(xy, z) = f(x, yz) = 0$ since $x \in I$ and $yz \in J$. Using non-degeneracy, we deduce that $xy = 0$. That is, $IJ = 0$; similarly $JI = 0$.

So $I \cap J$ is also a 2-sided ideal. Hence it is equal to either $\{0\}$ or I. If $I \cap J = I$, that is, $I \subset J$, the property $IJ = 0$ implies $I^2 = \{0\}$, which is excluded by the assumptions. Hence, $I \cap J = \{0\}$. Since $\dim(I) = \mathrm{codim}(J)$, we deduce that $I \oplus J = A$ (linearly); since $IJ = JI = 0$, this is a product decomposition.

Since f is non-degenerate in restriction to both I, J, we can pursue by induction until we have a decomposition $A = \prod_{i=1}^n A_i$ in which $A_i \neq \{0\}$ and the only 2-sided ideals of A contained in A_i are A_i and $\{0\}$ (and the A_i, being non-degenerate for f and orthogonal to each other for f). Since 2-sided ideals of A_i are also 2-sided ideals in A and since $A_i^2 \neq 0$ by assumption, this implies that A_i is simple.

Let I be a nonzero 2-sided ideal. So there exists i such that the projection of I on A_i is nonzero, say contains some element x. The set of $x' \in A_i$ such that $x'A_i = A_i x' = 0$ is a 2-sided ideal squaring to zero, an hence we deduce that either xA_i or $A_i x$ is nonzero. Hence either IA_i or $A_i I$ is nonzero. Since A_i is simple, we deduce that $A_i \subset I$. □

Exercice: in the above setting, show that A possesses exactly 2^n ideals. In addition, show that each 2-sided ideal of A, viewed as $\mathbf{Z}$-algebra, is a 2-sided ideal (i.e., is a K-subspace). (Beware that in general, in non-unital K-algebras, there might be ideals as $\mathbf{Z}$-algebra that are not K-subspaces.)

Proposition 4.2. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then every nilpotent ideal is contained in the kernel of the Killing form.*

Proof. For $x \in I$, $\mathrm{ad}(x)$ maps $\mathfrak{g}$ into I, maps I into I^2, etc. For $y \in \mathfrak{g}$, $\mathrm{ad}(x)$ maps each of $\mathfrak{g}$, I, I^2, etc, into itself. Therefore $\mathrm{ad}(x)\mathrm{ad}(y)$ maps $\mathfrak{g}$ into I, maps I into I^2, etc, and thus is nilpotent and has trace zero. This means that x and y are orthogonal for the Killing form. Since this holds for every $y \in \mathfrak{g}$, we deduce that x belongs to the kernel of the Killing form. □

Combining the previous two propositions, we deduce:

Corollary 4.3. *(K arbitrary field) Every semisimple Lie K-algebra is a finite direct product of simple Lie K-algebras.*

Proposition 4.4. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field K. Then every derivation D of $\mathfrak{g}$ is inner, i.e., of the form $\mathrm{ad}(x)$ for some x.*

Proof. Let D be a derivation, and define $\mathfrak{h}$ as the semidirect product $\mathfrak{g} \rtimes_D K$. Let $B_{\mathfrak{g}}$, resp. $B_{\mathfrak{h}}$ be the Killing forms. Let I be the orthogonal of the ideal $\mathfrak{g}$ in $\mathfrak{h}$; as the orthogonal of an ideal, it is an ideal, and since $\mathfrak{g}$ has codimension 1, I has dimension ≥ 1. Since $\mathfrak{g}$ is an ideal, we have $(B_{\mathfrak{h}})|_{\mathfrak{g}\times\mathfrak{g}} = B_{\mathfrak{g}}$. Hence $I \cap \mathfrak{g} = \{0\}$. Hence I is 1-dimensional, and hence $\mathfrak{h} = \mathfrak{g} \times I$, so I is central. Write, in $\mathfrak{h}$, $D = g + z$ with $g \in \mathfrak{g}$ and $z \in I$. Since I is central, we have $\mathrm{ad}(D) = \mathrm{ad}(g)$, which, in restriction to $\mathfrak{g}$, means that D equals the inner derivation $\mathrm{ad}(g)$. □

Exercise: 1) Let $\mathfrak{g}$ be a Lie algebra with center reduced to $\{0\}$ and such that every derivation of $\mathfrak{g}$ is inner. Show that for every Lie algebra $\mathfrak{h}$ containing $\mathfrak{g}$ as an ideal, $\mathfrak{h}$ is direct product of $\mathfrak{g}$ and its centralizer $\{x \in \mathfrak{h} : [x, \mathfrak{g}] = \{0\}\}$.

(Note: By the previous two propositions 4.4, this assumption is satisfied by semisimple Lie algebras.) 2) Show that 2-dimensional nonabelian Lie algebras satisfy this assumption.

Exercise: say that a Lie algebra $\mathfrak{g}$ is radical-free if it admits no nonzero solvable ideal. Let $\mathfrak{g}$ be a Lie algebra and D a derivation. Show that the semidirect product $\mathfrak{g} \rtimes_D K$ is radical-free if and only if D is not an inner derivation.

Remark: if $\mathfrak{g}$ is a finite-dimensional Lie algebra with trivial radical and D is a non-inner derivation, then the corresponding semidirect product $\mathfrak{g} \rtimes_D K$ is a radical-free Lie algebra, but is not perfect. We see later that this cannot occur in characteristic zero (Corollary 6.4).

5. Representation of solvable and nilpotent Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over a field K. For $\alpha \in \mathrm{Hom}(\mathfrak{g}, K)$, write $\mathbb{V}[\alpha]$ as K endowed with the representation $\rho(g)v = \alpha(g)v$: this is an irreducible representation. It is straightforward that any representation in the 1-dimensional space K has this form.

Lemma 5.1. *Let K be an algebraically closed field. Let $\mathfrak{g}$ be a Lie algebra with a faithful finite-dimensional $\mathfrak{g}$-module (V, ρ). If K has positive characteristic p, suppose in addition that* $\dim(V) < p$. *Then* $[\mathfrak{g}, \mathfrak{g}]$ *contains no 1-dimensional ideal of* $\mathfrak{g}$.

Proof. Otherwise, let $\mathfrak{a}$ be such an ideal. We discuss according to whether $\mathfrak{a}$ is central and in both cases, we reach a contradiction.

If $\mathfrak{a}$ is not central, then for some x we have $[x, y] = y$. By Corollary 3.2 (where we use the dimension restriction in positive characteristic), $\rho(y)$ is nilpotent. Let E be the kernel of $\rho(y)$; the relation $\rho(x)\rho(y) - \rho(y)\rho(x) = \beta(x)\rho(y)$ implies that $\rho(x)E \subset E$ for every $x \in \mathfrak{g}$. So E is a submodule, nonzero since $\rho(y)$ is nilpotent. By irreducibility, $V = E$. So $\rho(y) = 0$, contradicting the faithfulness of the representation.

Now suppose that $\mathfrak{a}$ is central in $\mathfrak{g}$. Let t be an eigenvalue of $\rho(y)$. Since y is central, $\mathrm{Ker}(\rho(y) - t)$ is a $\mathfrak{g}$-submodule of V, and hence equals V. So the trace of $\rho(y)$ is equal to $t \dim(V)$. Since $x \mapsto \mathrm{Trace}\rho(x)$ is a homomorphism, it vanishes on y, and hence $t \dim(V) = 0$ in K. Since $0 < \dim(V) < p$, we deduce that $t = 0$. ☐

Theorem 5.2. *Let K be an algebraically closed field of characteristic zero. Let $\mathfrak{g}$ be a solvable Lie algebra and (V, ρ) a finite-dimensional $\mathfrak{g}$-module.*

1. *If V is irreducible then V is isomorphic, as a $\mathfrak{g}$-module, to $\mathbb{V}[\alpha]$ for some α.*
2. *For any function $\alpha : \mathfrak{g} \to K$, denote $V_\alpha = \bigcap_{g\in\mathfrak{g}} \bigcup_{n\geq 0} \mathrm{Ker}(\rho(g) - \alpha(g))^n$. Then $V_\alpha = \{0\}$ whenever $\alpha \notin \mathrm{Hom}(\mathfrak{g}, K)$ (*Hom *denoting Lie K-algebra homomorphisms); the V_α generate their direct sum $\bigoplus_{\alpha\in\mathrm{Hom}(\mathfrak{g},K)} V_\alpha \subset V$.*
3. *If $\mathfrak{g}$ is nilpotent, $V = \bigoplus_{\alpha\in\mathrm{Hom}(\mathfrak{g},K)} V_\alpha$. Moreover, V_α is a $\mathfrak{g}$-submodule for every α (regardless that K is algebraically closed);*

Proof. Let I be the kernel of the representation ρ; it has finite codimension k in $\mathfrak{g}$.

If $\mathfrak{g}$ is abelian, then it is standard linear algebra that $V = \bigoplus V_\alpha$, that all V_α are submodules, and that the common eigenspace $E_\alpha = \bigcap_{g\in\mathfrak{g}} \mathrm{Ker}(\rho(g) - \alpha(g))$

is nonzero as soon as $V_\alpha \neq \{0\}$. The condition $E_\alpha \neq \{0\}$ clearly implies that $\alpha \in \mathrm{Hom}(\mathfrak{g}, K)$; thus $V = \bigoplus_{\alpha \in \mathrm{Hom}(\mathfrak{g},K)} V_\alpha$. If V is irreducible, since $\bigoplus E_\alpha$ is a nonzero submodule, it is equal to V, and irreducibility implies that $V = E_\alpha$ for some α. Since the action is scalar, $V = E_\alpha$ is 1-dimensional, and we deduce that V is isomorphic to $\mathbb{V}[\alpha]$ as $\mathfrak{g}$-module. This proves all the assertions when $\mathfrak{g}/I$ is abelian, and in particular when $k \leq 1$.

The result predicts that, when V is irreducible, $\mathfrak{g}/I$ is abelian. So, consider, by a contradiction, a counterexample with $k \geq 2$ minimal. Write $\mathfrak{h} = \mathfrak{g}/I$. Let $\mathfrak{a}$ be a nonzero abelian ideal contained in the derived subalgebra $[\mathfrak{h}, \mathfrak{h}]$; we can suppose that it has minimal dimension. So $\mathfrak{a}$ is an irreducible $\mathfrak{h}$-module for the adjoint representation, and actually an irreducible $(\mathfrak{h}/\mathfrak{a})$-module. Therefore, by minimality of k, we have $\dim(\mathfrak{a}) = 1$. Using Lemma 5.1 now yields a contradiction.

Let us prove the second assertion. Using that irreducible representations are 1-dimensional, there exist $\mathfrak{g}$-submodules $\{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = V$ such that V_i/V_{i-1} is 1-dimensional, and isomorphic to $\mathbb{V}[\alpha_i]$ for some $\alpha_i \in \mathrm{Hom}(\mathfrak{g}, K)$. Suppose that $V_\alpha \neq \{0\}$, and let i be minimal such that $V_\alpha \subset V_i$. Then $i \geq 1$, and the projection of V_α on V_i/V_{i-1} is nonzero. Since, for every $g \in \mathfrak{g}$, $\rho(g) - \alpha(g)$ is nilpotent on V_α and $\rho(g) - \alpha_i(g)$ is nilpotent on V_i/V_{i-1}, we deduce that $\alpha_i(g) = \alpha(g)$. So $\alpha = \alpha_i$, and thus $\alpha \in \mathrm{Hom}(\mathfrak{g}, K)$.

Next, suppose by contradiction that the sum is not direct: so there exists α and a finite subset I of $\mathrm{Hom}(\mathfrak{g}, K) \smallsetminus \{\alpha\}$, such that $V_\alpha \cap \sum_{\beta \in I} V_\beta \neq \{0\}$. Since K is infinite, there exists $g \in \mathfrak{g}$ such that $\alpha(g) \neq \beta(g)$ for every $\beta \in I$. Then, if V'_t is the characteristic subspace of $\rho(g)$ with respect to $t \in K$, we have $V_\beta \subset V'_{\beta(g)}$ for all g; since $V'_{\alpha(g)} \cap \sum_{\beta \in I} V'_{\beta(g)} = \{0\}$, we deduce that $V_\alpha \cap \sum_{\beta \in I} V_\beta = \{0\}$, a contradiction.

Now suppose that $\mathfrak{g}$ is nilpotent and let us prove the last assertion; consider a counterexample of minimal dimension d; then $d \geq 2$ since the case of dimension 1 is noticed above.

Let $W \subset V$ be a simple submodule, thus of dimension 1 by the above, and isomorphic to $\mathbb{V}[\beta]$ for some $\beta \in \mathrm{Hom}(\mathfrak{g}, K)$. Then $(V/W) = \bigoplus_{\alpha \in \mathrm{Hom}(\mathfrak{g},K)} (V/W)_\alpha$, and $(V/W)_\alpha$ is a submodule of V/W.

Fix α such that $(V/W)_\alpha$ is nonzero. First case: $(V/W)_\alpha \neq V/W$ (i.e., V/W has at least two weights). Let $U(\alpha)$ be the inverse image of $(V/W)_\alpha$ in V; this is a $\mathfrak{g}$-submodule. Then by induction $U(\alpha)$ is sum of common characteristic subspaces, and applying this to all α, we deduce that V is generated by common characteristic subspaces, proving the decomposition. Moreover, $V_\alpha \subset U(\alpha)$, and is therefore a $\mathfrak{g}$-submodule, again by induction. So V is not a counterexample. Thus, we have shown that for a counterexample of minimal dimension, for every irreducible submodule $W \subset V$, we have $(V/W)_\alpha = V/W$ for some α.

For every $v \in V$ and $g \in \mathfrak{g}$, we have $(\rho(g) - \alpha(g))^{\dim(V)-1} v \in W$. If $\alpha = \beta$, we deduce that $(\rho(g) - \alpha(g))^{\dim(V)} v = 0$, and hence $V = V_\alpha$ and is a submodule, so we have a contradiction.

It remains to consider the case when $\alpha \neq \beta$. Consider an irreducible $\mathfrak{g}$-submodule of V/W, thus 1-dimensional, and let T be its inverse image in V. Thus T is 2-dimensional. Choosing a basis (e_1, e_2) of T with $e_1 \in W$, the representation in T can be written as

$$g \mapsto \begin{pmatrix} \beta(g) & u(g) \\ 0 & \alpha(g) \end{pmatrix}.$$

Its image cannot be the whole algebra of upper triangular matrices, because the latter is not nilpotent. Therefore, its image has dimension ≤ 2. Since any nilpotent Lie algebra of dimension ≤ 2 is abelian, we deduce that the image is abelian. Therefore, the abelian case applies, and we deduce that $T = T_\alpha \oplus T_\beta$. We use the uniqueness of the weight of the quotient V/T_α. Since the image of T_β in this quotient is nonzero, we have $V/T_\alpha = (V/T_\alpha)_\beta$. Modding out by T, we deduce that $V/T = (V/T)_\alpha$. But as a quotient of V/W, we have $V/T = (V/T)_\beta$. Therefore, if $V \neq T$, we deduce $\alpha = \beta$, a contradiction. So $V = T = V_\alpha \oplus V_\beta$, which is not a counterexample and again we have a final contradiction. □

Exercise: exhibit one case, with $\mathfrak{g}$ solvable (and K algebraically closed), for which $V \neq \bigoplus_\alpha V_\alpha$.

Corollary 5.3. *Let K be a field of characteristic zero. For every finite-dimensional solvable Lie algebra $\mathfrak{g}$, the derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. First suppose that K is algebraically closed. Consider the adjoint representation. Its kernel is the center $\mathfrak{z}$ of $\mathfrak{g}$; write $\mathfrak{z}' = \mathfrak{z} \cap [\mathfrak{g}, \mathfrak{g}]$. Since irreducible representations have dimension 1 by the theorem, one can embed $\mathfrak{g}/\mathfrak{z}$ into the Lie algebra of upper triangular matrices of size $\dim(\mathfrak{g})$. Its derived subalgebra is nilpotent. This shows that $[\mathfrak{g}, \mathfrak{g}]/\mathfrak{z}'$ is nilpotent. Since $\mathfrak{z}'$ is central, this implies that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

When K is arbitrary, fix an algebraically closed extension; then $\mathfrak{g} \otimes_K L$ satisfies the property, which passes to its Lie K-subalgebra $\mathfrak{g}$. □

Corollary 5.4. *Let K be a field of characteristic zero. Let $\mathfrak{g}$ be a solvable Lie algebra and (V, ρ) a finite-dimensional $\mathfrak{g}$-module.*

(1) *The V_α generate their direct sum;*
(2) *if $\mathfrak{g}$ is nilpotent, then the V_α are $\mathfrak{g}$-submodules;*
(3) *if $\rho(g)$ is nilpotent for every $g \in \mathfrak{g}$, then $V = V_0$. If moreover V is irreducible, then it is 1-dimensional and isomorphic to $\mathbb{V}[0]$.*

Proof. Let L be an algebraically closed extension of K. Define $\mathfrak{g}^L : \mathfrak{g} \otimes_K L$ and $V^L = V \otimes_K L$. We have $V_\alpha = V^L_\alpha \cap V$. Hence these generate their direct sum, and, when $\mathfrak{g}$ is nilpotent, are submodules.

Now suppose that $\rho(g)$ is nilpotent for every $g \in \mathfrak{g}$. We have $V^L = \bigoplus_{\alpha \in \mathrm{Hom}(\mathfrak{g}^L, L)} (V^L)_\alpha$. Fix α such that $(V^L)_\alpha \neq \{0\}$. For every $g \in \mathfrak{g}$, $\alpha(g)$ is an eigenvalue of $\rho(g)$, but the latter is nilpotent (this is inherited from V to V^L). Hence $\alpha(g) = 0$ for all $g \in \mathfrak{g}$. Note that $\mathfrak{g}$ M-linearly spans $\mathfrak{g}^L$, and hence $\alpha = 0$. So $V^L =$

$(V^L)_0$. So $V_0 = (V^L)_0 \cap V = V$. Let W be an irreducible L-subspace of V^L. Then W is 1-dimensional, and hence elements of W belong to the intersection $\bigcap_{g\in\mathfrak{g}} \mathrm{Ker}(\rho(g)_{V^L})$. Since all $\rho(g)$ are matrices over K, this intersection is also nonzero at the level of V. Hence $\bigcap_{g\in\mathfrak{g}} \mathrm{Ker}(\rho(g)_{V^L}) \neq \{0\}$. If V is irreducible, we deduce that it is 1-dimensional and with null action, i.e., isomorphic to $\mathbb{V}[0]$. □

Corollary 5.5 (Engel's theorem)**.** *Let K be a field of characteristic zero. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra such that* $\mathrm{ad}(x)$ *is nilpotent for every $x \in \mathfrak{g}$. Then $\mathfrak{g}$ is nilpotent.*

Proof. We first assert that $\mathfrak{g}$ is solvable. Let $\mathfrak{g}$ be a counterexample of minimal dimension. Let $\mathfrak{h}$ be a maximal solvable subalgebra of $\mathfrak{g}$, so $\mathfrak{h} \neq \mathfrak{g}$. Under the adjoint action, we view $\mathfrak{g}$ as an $\mathfrak{h}$-module. Then $\mathfrak{g}/\mathfrak{h}$ contains an irreducible $\mathfrak{h}$-submodule $\mathfrak{m}/\mathfrak{h}$. By Corollary 5.4(3), $\mathfrak{m}/\mathfrak{h}$ has dimension 1 and has a null action, which implies $[\mathfrak{h},\mathfrak{m}] \subset \mathfrak{h}$. In particular, $\mathfrak{m}$ is contained in the normalizer $\mathfrak{n}$ of $\mathfrak{h}$. As $\mathfrak{m}/\mathfrak{h}$ is a 1-dimensional subalgebra, and hence $\mathfrak{m}$ is a subalgebra with $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}$. So $\mathfrak{m}$ is solvable, and this contradicts the maximality of $\mathfrak{h}$.

So $\mathfrak{g}$ is solvable. Let $\{0\} = \mathfrak{g}^{[0]} \subset \mathfrak{g}^{[1]} \subset \dots \mathfrak{g}^{[k]}$ be submodules (under the adjoint representation) such that each successive quotient is irreducible. By Corollary 5.4(3), each $\mathfrak{g}^{[i]}/\mathfrak{g}^{[i-1]}$ is 1-dimensional, with null action. Choosing a compatible basis, we can therefore express all $\mathrm{ad}(x)$, $x \in \mathfrak{g}$, as strictly upper triangular matrices. So $\mathfrak{g}$ is nilpotent. □

Remark: Engel's theorem holds over arbitrary fields; see [J, Chap. II.2].

Lemma 5.6. *Let R be a scalar ring. Let A be an algebra (with product denoted by $[\cdot,\cdot]$) and D an R-linear derivation of A. Then for all $t, u \in R$, $x, y \in A$, and $n \in \mathbf{N}$ we have*

$$(D-t-u)^n[x,y] = \sum_{k=0}^{n} \binom{n}{k} [(D-t)^k x, (D-u)^{n-k} y].$$

Proof. By induction on n; the case $n = 0$ is clear. Suppose that $n \geq 1$ and that the formula is proved for $n-1$. Then

$$\begin{aligned}(D-t-u)^n(xy) =&(D-t-u)(D-t-u)^{n-1}(xy)\\ =&(D-t-u)\sum_{k=0}^{n-1}\binom{n-1}{k}[(D-t)^k x, (D-u)^{n-1-k} y].\end{aligned}$$

Use that $(D-t-u)[(D-t)^k x, (D-u)^{n-1-k} y]$ is equal to

$$[(D-t)^{k+1} x, (D-u)^{n-1-k} y] + [(D-t)^k x, (D-u)^{n-k} y],$$

and then (as in the proof of the classical binomial expansion) change the variable $k+1$ to k in the left-hand term, use the formula $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$ to obtain the desired formula. (Exercise: fill in details.) □

Theorem 5.7. *Let K be a field of characteristic zero. Let $\mathfrak{h}$ be a nilpotent Lie K-algebra. Let A be a finite-dimensional K-algebra (with product denoted as $[\cdot,\cdot]$), and $\rho:\mathfrak{g}\to\mathrm{Der}(A)$ a K-algebra homomorphism. Let $A\supset\bigoplus_{\alpha\in\mathrm{Hom}(\mathfrak{g},K)}A_\alpha$ be the characteristic decomposition of A. Then this is an algebra grading: $[A_\alpha,A_\beta]\subset A_{\alpha+\beta}$ for all $\alpha,\beta\in\mathrm{Hom}(\mathfrak{h},K)$.*

Proof. Write $d=\dim(A)$. Fix $\alpha,\beta\in\mathrm{Hom}(\mathfrak{h},K)$. For $x\in A_\alpha$, $y\in A_\beta$ and $g\in\mathfrak{h}$, we have $x\in\mathrm{Ker}(\rho(g)-\alpha(g))^d$ and $y\in\mathrm{Ker}(\rho(g)-\beta(g))^d$. Then the formula of Lemma 5.6, for $t=\alpha(g)$ and $u=\beta(h)$, shows that $[x,y]$ belongs to the kernel of $(\rho(g)-\alpha(g)-\beta(g))^{2d}$. Since this holds for all $g\in\mathfrak{h}$, we deduce that $[x,y]\in A_{\alpha+\beta}$. $\square$

6. Cartan subalgebras

Let $\mathfrak{g}$ be a Lie algebra. A Cartan subalgebra is a nilpotent subalgebra, equal to its normalizer.

Assume that we work over an infinite ground field K. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. For $x\in\mathfrak{g}$, write $\mathfrak{g}_0(x)=\mathrm{Ker}(\mathrm{ad}(x)^{\dim(\mathfrak{g})}$, the characteristic subspace of $\mathrm{ad}(x)$ with respect to the eigenvalue zero ("null-characteristic subspace of $\mathrm{ad}(x)$"). We say that $x\in\mathfrak{g}$ is a regular element if $\dim\mathfrak{g}_0(x)$ is minimal, that is, equals $\min_{y\in\mathfrak{g}}\dim(\mathfrak{g}_0(y))$. The existence of regular elements is obvious.

Exercise: show that in the space of matrices, having a centralizer of dimension $\geq k$ is a Zariski-closed condition (i.e., can be defined as zero set of a certain set of polynomials). Deduce that the set of regular elements is a (nonempty) Zariski-open subset of $\mathfrak{g}$.

Theorem 6.1. *Let K be an infinite field. Let $\mathfrak{g}$ be a finite-dimensional Lie K-algebra. Then for every regular element $x\in\mathfrak{g}$, the null-characteristic subspace $\mathfrak{g}_0(x)$ of $\mathrm{ad}(x)$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. First suppose that K is algebraically closed. Fix $x\in\mathfrak{g}$. Write $\mathfrak{g}=\bigoplus_{t\in K}\mathfrak{g}_t$, the characteristic decomposition with respect to $\mathrm{ad}(x)$. So $\mathfrak{g}_0=\mathfrak{g}_0(x)$. Write $\mathfrak{g}_*=\bigoplus_{t\neq0}\mathfrak{g}_t$. By Theorem 5.7 (with $(Kx,\mathfrak{g})$ playing the role of (g,A)), $(\mathfrak{g}_t)_{t\in K}$ is a grading of $\mathfrak{g}$, and in particular, $\mathfrak{g}_0$ is a subalgebra and $[\mathfrak{g}_0,\mathfrak{g}_*]\subset\mathfrak{g}_*$.

We first check (for arbitrary $x\in\mathfrak{g}$) that $\mathfrak{g}_0$ is equal to its own normalizer $\mathfrak{n}$. Since $x\in\mathfrak{g}_0$, $\mathfrak{n}$ is $\mathrm{ad}(x)$ invariant, and hence $\mathfrak{n}$ is a graded subspace of $\mathfrak{g}$. Thus, if, by contradiction, $\mathfrak{n}\neq\mathfrak{g}_0$, there exists $t\neq0$ such that $\mathfrak{n}_t\neq\{0\}$. Since $\mathfrak{n}_t$ is $\mathrm{ad}(x)$-invariant, there exists an eigenvector, so there exists $y\in\mathfrak{n}_t\smallsetminus\{0\}$ such that $[x,y]=ty$. So $\mathrm{ad}(y)(t^{-1}x)=-y$; this contradicts the assumption that y normalizes $\mathfrak{g}_0$.

Now we check that $\mathfrak{g}_0$ is nilpotent. By Corollary 5.5 (Engel's Theorem) it is enough to show that $\mathrm{ad}(y)$ is nilpotent for every $y\in\mathfrak{g}_0$. Assume otherwise. For $y\in\mathfrak{g}_0$, write $N_y=\mathrm{ad}(y)|_{\mathfrak{g}_0}$ and $T_y=\mathrm{ad}(y)|_{\mathfrak{g}_*}$. Then N_x is nilpotent, T_x

is invertible. We have to show that N_y is nilpotent. Assume, by contradiction, otherwise.

The set of $s \in K$ such that $T_x + sT_y = T_{x+sy}$ is invertible is the complement of a finite subset F_*. Also, the set F_0 of $s \in K$ such that $N_x + sN_y$ is nilpotent is finite: otherwise $N_x + sN_y$ is nilpotent for all s, namely $(N_x + N_y)^d = 0$ for all s, with $d = \dim(\mathfrak{g})$; expanding and taking the term of degree d yields $N_y^d = 0$, a contradiction. So, for $s \notin F_0 \cup F_*$ and $z = x + sy$, we have N_z not nilpotent and T_z is invertible. Thus, $\dim \mathfrak{g}_0(z) < \dim \mathfrak{g}_0(x)$, contradicting that x is regular. $\square$

We henceforth assume that K has characteristic zero.

We say that $\mathfrak{h}$ is a split (or K-split) Cartan subalgebra if the adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ can be made upper triangular in some basis. We say that $\mathfrak{g}$ is a split (or K-split) Lie algebra if it admits a split Cartan subalgebra (beware that this does not always mean that all Cartan subalgebras are split). If $\mathfrak{h}$ is a split Cartan subalgebra, we have $\mathfrak{g} = \bigoplus_{\alpha \in \mathrm{Hom}(\mathfrak{h},K)} \mathfrak{g}_\alpha$. We call this a Cartan grading of $\mathfrak{g}$.

Let $\mathfrak{g}$ be endowed with a Cartan grading.

Lemma 6.2. *For all $\alpha, \beta \in \mathrm{Hom}(\mathfrak{g}_0, K)$ such that $\mathfrak{g}_\beta \neq \{0\}$, we have $\beta|_{[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]} \in \mathbf{Q}(\alpha|_{[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]})$.*

Proof. Write $\mathfrak{g}_{\beta+\mathbf{Z}\alpha} = \bigoplus_{n\in\mathbf{Z}} \mathfrak{g}_{\beta+n\alpha}$. This is a $\mathfrak{g}_{0+\mathbf{Z}\alpha}$-submodule of $\mathfrak{g}$. For $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, write $z = [x, y]$; both $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$ preserve $\mathfrak{g}_{\beta+\mathbf{Z}\alpha}$, and hence their commutator $\mathrm{ad}(z)$, restricted to $\mathfrak{g}^{[\beta,\alpha]}$, has trace zero. Computing this trace componentwise, we obtain the equality

$$0 = \sum_{n\in\mathbf{Z}} (\dim \mathfrak{g}_{\beta+n\alpha})(\beta + n\alpha)(z),$$

which can be rewritten as

$$\dim(\mathfrak{g}_{\beta+\mathbf{Z}\alpha})\beta(z) = -\left(\sum_{n\in\mathbf{Z}} n \dim \mathfrak{g}_{\beta+n\alpha}\right)\alpha(z);$$

by linearity the latter equality holds for all $z \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Since the characteristic is zero and $\mathfrak{g}_\beta \neq \{0\}$, this dimension is nonzero in K, and we deduce that in restriction to $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ we have $\beta = q\alpha$ with $q = \frac{\sum_{n\in\mathbf{Z}} n \dim(\mathfrak{g}_{\beta-n\alpha})}{\dim(\mathfrak{g}_{\beta+\mathbf{Z}\alpha})}$. $\square$

Proposition 6.3. *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K of characteristic zero. Then the kernel of the Killing form is a solvable ideal.*

Proof. If I is the kernel of the Killing form and L is an extension of K, the kernel of the Killing form of the Lie L-algebra $\mathfrak{g} \otimes_K L$ is equal to $I \otimes_K L$. Hence, we can suppose that K is algebraically closed, and we fix a Cartan grading on $\mathfrak{g}$.

We start proving the following claim: if $\mathfrak{g}$ is a nonzero perfect finite-dimensional Lie algebra over K, then its Killing form B is not zero.

Assume by contradiction that $B = 0$. Since $\mathfrak{g}$ is perfect, we have $\mathfrak{g}_0 = \mathfrak{g}_0 \cap [\mathfrak{g}, \mathfrak{g}] = \sum_\alpha [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Fix $z \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. Then

$$B(z, z) = \mathrm{Trace}(\mathrm{ad}(z)^2) = \sum_\beta \dim(\mathfrak{g}_\beta)\beta(z)^2.$$

By Lemma 6.2, we have $\beta(z) = q_{\alpha,\beta}\alpha(z)$ for some $q_{\alpha,\beta} \in \mathbf{Q}$. So

$$0 = B(z, z) = \sum_\beta \dim(\mathfrak{g}_\beta)^2 q_{\alpha,\beta}^2 \alpha(z)^2.$$

Hence, for every β such that $\mathfrak{g}_\beta \neq \{0\}$, every α and every $z \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ we have $\beta(z) = q_{\alpha,\beta}\alpha(z) = 0$. Since $\mathfrak{g}_0 = \sum_\alpha [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$, we deduce that $\beta(z) = 0$ for all $z \in \mathfrak{g}_0$, i.e., $\mathfrak{g}_\beta \neq \{0\}$ implies $\beta = 0$. This means that $\mathfrak{g} = \mathfrak{g}_0$ is nilpotent, a contradiction with $\mathfrak{g}$ being perfect and nonzero.

The claim being proved, let us prove the proposition. Let I be the kernel of the Killing form, $(I^{(n)})$ its derived series, and $J = \bigcap_n I^{(n)}$. Then J is a perfect ideal. If by contradiction $J \neq \{0\}$, then by the previous fact, its Killing form is nonzero. It is straightforward that the Killing form of an ideal is the restriction of the Killing form of the larger algebra, and hence the Killing form of $\mathfrak{g}$ does not vanish on $J \times J$. This contradicts the assumption that J is contained in the kernel of the Killing form. Hence $J = \{0\}$, which means that I is solvable. $\square$

In characteristic zero, we therefore have a converse to Proposition 4.2.

Corollary 6.4. *Let K be a field of characteristic zero, and $\mathfrak{g}$ a finite-dimensional Lie K-algebra. Equivalent properties:*

(1) $\mathfrak{g}$ is semisimple (i.e., has a non-degenerate Killing form);
(2) $\mathfrak{g}$ has no nonzero abelian ideal (or the same with "abelian" replaced with "solvable", or "nilpotent")
(3) $\mathfrak{g}$ is isomorphic to a finite direct product of simple K-algebras.

Proof. If we have a nonzero solvable ideal, its derived series consists of ideals (exercise) and hence its last nonzero term is a nonzero abelian ideal. So the 3 differents readings of (2) are equivalent (with K arbitrary).

For an arbitrary field, (1)⇒(3) is the contents of Proposition 4.1 respectively (with K arbitrary). Also, (3)⇒(1) is immediate: if we have a nonzero solvable ideal, its projection to some simple factor is a nonzero solvable ideal, and hence the simple factor is solvable, which is not possible.

Finally, (2)⇒(1) is the implication making this a corollary: suppose that $\mathfrak{g}$ has no nonzero solvable ideal. By the proposition, the kernel of the Killing form is solvable, and hence is zero; hence the Killing form is non-degenerate. $\square$

Given an algebra $\mathfrak{g}$ graded in an abelian group Λ and B a bilinear form on $\mathfrak{g}$, we say that B is concentrated in degree zero if $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = \{0\}$ for all $\alpha, \beta \in \Lambda$ such that $\alpha + \beta \neq 0$.

Proposition 6.5. *Let $\mathfrak{g}$ be a finite-dimensional Lie K-algebra.*

(1) *if $\mathfrak{g}$ is endowed with a grading in a torsion-free abelian group, the Killing form of $\mathfrak{g}$ is concentrated in degree 0;*
(2) *if $\mathfrak{g}$ is endowed with a Cartan grading, then every invariant symmetric bilinear form B is concentrated in degree 0.*

Proof. For $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$ with $\alpha + \beta \neq 0$, $\mathrm{ad}(x)\mathrm{ad}(y)$ shifts the degree by $\alpha+\beta$, and hence is nilpotent, since the grading abelian group is torsion-free. This establishes the first assertion.

For the second assertion, assume the contrary: consider $x_0 \in \mathfrak{g}_\alpha$, $y_0 \in \mathfrak{g}_\beta$, such that $B(x_0, y_0) \neq 0$. Fix any $h \in \mathfrak{h}$ such that $(\alpha + \beta)(h) \neq 0$. Define $x_n = (\mathrm{ad}(h) - \alpha(h)I)^n x_0$ and $y_n = (\mathrm{ad}(h) - \beta(h)I)^n y_0$. Then x_n and y_n are zero for n large enough. So, there exists $n, m \geq 0$ such that $B(x_n, y_m) \neq 0$ and $B(x_{n+1}, y_m) = B(x_n, y_{m+1}) = 0$. Since $x_{n+1} = [h, x_n] - \alpha(h)x_n$ the relation $B(x_{n+1}, y_m)$ reads as $B([h, x_n], y_n) = \alpha(h)B(x_n, y_n)$. Similarly, $B(x_n, [h, y_n]) = \beta(h)B(x_n, y_n)$. The invariance of B implies that these two numbers are opposite, so $(\alpha + \beta)(h)B(x_n, y_n) = 0$; since this is nonzero, we have a contradiction. □

Proposition 6.6. *Let $\mathfrak{g}$ be a finite-dimensional Lie K-algebra with a Cartan grading $(\mathfrak{g}_\alpha)$. Let $\Phi = \{\alpha \in \mathrm{Hom}(\mathfrak{g}_0, K) : \mathfrak{g}_\alpha \neq \{0\}\}$ be the set of roots. Then $K = \bigcap_{\alpha\in\Phi} \mathrm{Ker}(\alpha)$ is contained in the kernel of the Killing form.*

Proof. For $x, y \in \mathfrak{g}_0$, and B the Killing form, we have

$$B(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)) = \sum_\alpha \dim(\mathfrak{g}_\alpha)\alpha(x)\alpha(y).$$

In particular, if $x \in K$, we have $B(x, y) = 0$. Since we also have $B(\mathfrak{g}_0, \mathfrak{g}_\alpha) = \{0\}$ for $\alpha \neq 0$ (by Proposition 6.5), we deduce that x belongs to the kernel of the Killing form. □

Let $\mathfrak{g}$ be endowed with a Cartan grading. Let B be an invariant, non-degenerate symmetric bilinear form on $\mathfrak{g}$. By Proposition 6.5 it is non-degenerate on $\mathfrak{g}_0$, and hence for every $\alpha \in \mathfrak{g}_0$ there exists a unique $h_\alpha'^B \in \mathfrak{g}_0$ such that $B(h_\alpha'^B, \cdot) = \alpha$ on $\mathfrak{g}_0$.[4] When B is the Killing form $\langle\cdot,\cdot\rangle$ (thus assumed non-degenerate, i.e., $\mathfrak{g}$ is split semisimple, we write it as h'_α (the prime is there because it will be convenient in the sequence to renormalize it and define $h_\alpha = \frac{2}{\langle h'_\alpha, h'_\alpha\rangle} h'_\alpha$; at the moment we do not even know that the denominator does not vanish).

Proposition 6.7. *Let $\mathfrak{g}$ be as above. Then $h_\alpha^B \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ for all α such that $\mathfrak{g}_\alpha \neq \{0\}$. More precisely, for every 1-dimensional $\mathfrak{g}_0$-submodule Kx of $\mathfrak{g}_\alpha$, we have $[\mathfrak{g}_{-\alpha}, Kx] = Kh_\alpha^B$.*

Proof. Let x be a common $\mathfrak{g}_0$-eigenvector in $\mathfrak{g}_\alpha$. For any $y \in \mathfrak{g}_{-\alpha}$ and $z \in \mathfrak{g}_0$, we have

$$B(z, [x, y]) = B([z, x], y) = B(\alpha(z)x, y) = B(z, h_\alpha^B)B(x, y) = B(z, B(x, y)h_\alpha^B)$$

[4]In the lectures, this definition as well as the next proposition will be stated in the case of semisimple Lie algebras and Killing form, at the beginning of the next chapter.

hence (using non-degeneracy on $\mathfrak{g}_0$) for all $y \in \mathfrak{g}_{-\alpha}$ we have $[x,y] = \langle x,y\rangle h_\alpha^B$. Choosing y such that $\langle x,y\rangle = 1$ (using that $B(\cdot,\cdot)$ yields a duality between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$), we obtain the result. □

7. Structure of semisimple Lie algebras

We now consider a semisimple Lie algebra $\mathfrak{g}$ over a field of characteristic zero endowed with a Cartan grading, and its Killing form $\langle\cdot,\cdot\rangle$. Call $\Phi = \{\alpha \in \mathrm{Hom}(\mathfrak{g}_0, K) : \mathfrak{g}_\alpha \neq \{0\}\}$ the set of roots, and $\Phi^* = \Phi \smallsetminus \{0\}$. The dimension of $\mathfrak{g}_0$ is called the rank[5] of $\mathfrak{g}$ (it does not depend on the choice of Cartan grading, since by definition all Cartan subalgebras have the same dimension).

By either assertion of Proposition 6.7, the Killing form is concentrated in degree zero, so $\langle\mathfrak{g}_\alpha, \mathfrak{g}_\beta\rangle = \{0\}$ whenever $\alpha+\beta \neq 0$, and the Killing form induces a duality between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ for all α. Recall that h'_α is the element of $\mathfrak{g}_0$ characterized by the property $\langle h'_\alpha, h\rangle = \alpha(h)$ for all $h \in \mathfrak{g}_0$ (so $h'_\alpha = -h'_{-\alpha}$, and is nonzero for $\alpha \neq 0$).

Proposition 7.1. *The Cartan subalgebra $\mathfrak{g}_0$ is abelian and linearly generated by the h'_α when α ranges over Φ^*.*

Proof. By Proposition 6.6, $\bigcap_{\alpha\in\Phi^*}\mathrm{Ker}(\alpha)$ is contained in the kernel of the Killing form, which is $\{0\}$. Since this intersection contains $[\mathfrak{g}_0,\mathfrak{g}_0]$, it follows that $\mathfrak{g}_0$ is abelian. This intersection is also the orthogonal of the subspace spanned by the h_α, and hence this subspace is all of $\mathfrak{g}_0$. □

Proposition 7.2. *We have $\langle h'_\alpha, h'_\alpha\rangle \neq 0$ for every $\alpha \in \Phi^*$ (so the "coroot" $h_\alpha = \frac{2}{\langle h'_\alpha, h'_\alpha\rangle} h'_\alpha$ is well-defined).*

Proof. By Proposition 6.7, $h'_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. By Lemma 6.2, we can write, for every $\beta \in \Phi$, $\beta(h'_\alpha) = q_\beta\alpha(h'_\alpha)$ for every β, for some rational q_β. If by contradiction $\langle h'_\alpha, h'_\alpha\rangle = 0$, then this number being equal to $\alpha(h'_\alpha)$, we deduce $\beta(h'_\alpha) = 0$ for all $\beta \in \Phi$. Since $\bigcap_{\beta\in\Phi}\mathrm{Ker}(\beta) = \{0\}$ by Proposition 6.6, we deduce $h'_\alpha = 0$, and hence $\alpha = 0$. □

Proposition 7.3. *For every nonzero root $\alpha \in \Phi^*$, we have $\dim(\mathfrak{g}_\alpha) = 1$, while $\dim(\mathfrak{g}_{n\alpha}) = 0$ for every $n \in \mathbf{N}_{\geq 2}$.*

Proof. By Proposition 6.7, there exists $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $h_\alpha = [x,y]$, and such that Ky is a 1-dimensional $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-\alpha}$; thus $[Kx, Ky] = [\mathfrak{g}_\alpha, Ky] = Kh_\alpha$. Write $M = \left(\bigoplus_{n\geq 1}\mathfrak{g}_{n\alpha}\right) \oplus Kh_\alpha \oplus Ky$.

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[5]When $\mathfrak{g}$ is not semisimple, we can still call $\dim(\mathfrak{g}_0)$ the Cartan rank of $\mathfrak{g}$, although it does maybe not deserve to be called "rank". We can call effective rank of $\mathfrak{g}$ the dimension of $\mathfrak{g}_0/\bigcap_{\alpha\in\Phi}\mathrm{Ker}(\alpha)$, that is, the dimension of the linear span of Φ in $\mathfrak{g}_0^*$; it is at most equal to the Cartan rank, with equality for semisimple Lie algebras (and not only them). For nilpotent Lie algebras, the Cartan rank equals the dimension while the effective rank is zero.

Hence M is stable under both $\mathrm{ad}(x)$ and $\mathrm{ad}(y)$, and hence their commutator $\mathrm{ad}(h_\alpha)$ has trace zero on M. This trace can be computed as

$$\left(-1+\sum_{n\geq 1} n\dim(\mathfrak{g}_{n\alpha})\right)\langle h_\alpha, h_\alpha\rangle.$$

Since K has characteristic zero and given that $\langle h_\alpha, h_\alpha\rangle \neq 0$ (Proposition 7.2), this can be zero only if $\dim(\mathfrak{g}_\alpha)=1$ and $\dim(\mathfrak{g}_{n\alpha})=0$ for all $n\geq 2$. □

Corollary 7.4. *For each $\alpha \in \Phi^*$ and $\beta \in \Phi$ and $x \in \mathfrak{g}_\beta$, we have $[h_\alpha, x] = \frac{\langle h'_\alpha, h'_\beta\rangle}{\langle h'_\alpha, h'_\alpha\rangle}x$; in particular for $x \in \mathfrak{g}_\alpha$ we have $[h_\alpha, x]=2x$.* □

Corollary 7.5. *For every nonzero root $\alpha \in \Phi^*$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = Kh_\alpha$. In particular, $\mathfrak{s}_\alpha = \mathfrak{g}_{-\alpha}\oplus Kh_\alpha \oplus \mathfrak{g}_\alpha$ is isomorphic to $\mathfrak{sl}_2(K)$, with an isomorphism mapping h_α to h.*

Proof. The first assertion follows from Propositions 6.7 and 7.3. For the second, choose nonzero elements $x \in \mathfrak{g}_\alpha$ and $y \in g_{-\alpha}$; by the first assertion, $[x,y]$ is a nonzero scalar multiple of h_α, so we can renormalize y to assume that it equals h_α. By Corollary 7.4, we have $[h_\alpha, x]=2x$ and $[h_\alpha, y] = -[h_{-\alpha}, y] = -2y$. □

Theorem 7.6. *(Only in this theorem, $\mathfrak{g}$ is not assumed semisimple; K still has characteristic zero.) Let a finite-dimensional Lie algebra $\mathfrak{g}$ be endowed with a Cartan grading. Then $\mathfrak{g}$ is semisimple if and only if it satisfies the three following conditions:*

(1) $\bigcap_{\alpha\in\Phi^*}\mathrm{Ker}(\alpha)=\{0\}$
(2) $\forall\alpha\in\Phi^*$, $\dim(\mathfrak{g}_\alpha)=1$;
(3) $\forall\alpha\in\Phi^*$, $\alpha|_{[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]}\neq 0$.

Proof. That $\mathfrak{g}$ semisimple satisfies these conditions has already been checked. Suppose they are satisfied. Let $\mathfrak{n}$ be the kernel of the Killing form. Being normalized by $\mathfrak{g}_0$, $\mathfrak{n}$ is a graded ideal. It follows from the last two assumptions that for $\alpha \in \Phi^*$, $\mathfrak{g}_\alpha$ is contained in copy of $\mathfrak{sl}_2(K)$. Hence $\mathfrak{n}\subset\mathfrak{g}_0$; in particular, $[\mathfrak{n}, \mathfrak{g}_\alpha]\subset \mathfrak{g}_\alpha$; since $\mathfrak{n}$ is an ideal, it follows that $\mathfrak{n}$ centralizes $\mathfrak{g}_\alpha$ for every α, and hence $\mathfrak{n}\subset\bigcap_{\alpha\in\Phi^*}\mathrm{Ker}(\alpha)$. By the first assumption, we deduce $\mathfrak{n}=\{0\}$. □

Given $\alpha,\beta\in\Phi^*$, define $m_{\alpha,\beta}=m, n_{\alpha,\beta}=n$ by the requirement that $k\alpha+\beta$ is a root for all $k\in\{-m,\dots,n\}$, and not for $k=-m-1, n+1$, and define $\mathfrak{v}_{\alpha,\beta}=\bigoplus_{k=-m}^{n}\mathfrak{g}_{k\alpha+\beta}$. This is a $\mathfrak{s}_\alpha$-submodule of $\mathfrak{g}$.

By Proposition 7.3, $k\alpha+\beta\neq 0$ for every $k\in\mathbf{Z}$, so $\dim(\mathfrak{g}_{k\alpha+\beta})=1$ for all integer k in $[-m,n]$, again by Proposition 7.3.

Proposition 7.7. *Let α,β be nonzero roots such that $\alpha+\beta$ is also a nonzero root. Then $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta}$.*

Proof. Choose a nonzero $x\in\mathfrak{g}_\alpha$. Since $\alpha+\beta$ is a root, we have $n=n_{\alpha,\beta}\geq 1$. The eigenvalues of $\mathrm{ad}(h_\alpha)|_{\mathfrak{v}_{\alpha,\beta}}$, with multiplicity, are $(k\alpha+\beta)(h_\alpha)$ for $-m_{\alpha,\beta}\leq$

$k \le n_{\alpha,\beta}$. Since $k\alpha(h_\alpha) = 2k$, we deduce that these eigenvalues are distinct and their difference lie in $2\mathbf{Z}$. Therefore, by Corollary 3.9, $\mathfrak{v}_{\alpha,\beta}$ is an irreducible $\mathfrak{s}_\alpha$-module; in particular, the kernel of $\mathrm{ad}(x)|_{\mathfrak{v}_{\alpha,\beta}}$ is 1-dimensional and reduced to $\mathfrak{g}_{n\alpha+\beta}$. Hence $\mathrm{ad}(x)$ is injective on $\mathfrak{g}_\beta$, and thus $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. □

Proposition 7.8. *Given $\alpha, \beta \in \Phi^*$, with $\alpha \neq \pm\beta$; write $m = m_{\alpha,\beta}$ and $n = n_{\alpha,\beta}$. Then*
$$2\frac{\langle h'_\beta, h'_\alpha\rangle}{\langle h'_\alpha, h'_\alpha\rangle} = m - n.$$

Proof. For $h \in \mathfrak{g}_0$, the trace of $\mathrm{ad}(h)$ on $\mathfrak{v}_{\alpha,\beta}$ is
$$\langle h, h'_\alpha\rangle \sum_{k=-m}^{n} k + \langle h, h'_\beta\rangle \sum_{k=-m}^{n} 1 = (m+n+1)\left(\frac{n-m}{2}\langle h, h'_\alpha\rangle + \langle h, h'_\beta\rangle\right)$$

Since $\mathfrak{v}_{\alpha,\beta}$ is a $\mathfrak{s}_\alpha$-submodule, this trace vanishes for $h = h'_\alpha$. The formula follows. □

This first allows to improve the second part of Proposition 7.3.

Proposition 7.9. *If $\alpha \in \Phi^*$, $t \in K$ and $t\alpha \in \Phi$, then $t \in \{-1, 0, 1\}$.*

Proof. Write $\beta = t\alpha$ and suppose $t \notin \{-1, 0, 1\}$. We apply Proposition 7.8; the existence of m, n follows from the assumption on $\mathfrak{g}$. The formula reads as $2t = m - n$, so $t \in \frac{1}{2}\mathbf{Z}$. Switching the role of α and β, we also deduce $1/t \in \frac{1}{2}\mathbf{Z}$. So $t \in \{\pm 1/2, \pm 2\}$. The case $t = \pm 2$ is excluded by Proposition 7.3, and so is the $t = \pm 1/2$ by switching α and β. □

Proposition 7.10. *For all nonzero roots α, β, we have $\langle h'_\alpha, h'_\beta\rangle \in \mathbf{Q}$ (and hence $\langle h_\alpha, h_\beta\rangle \in \mathbf{Q}$ as well).*

Proof. By Proposition 7.8, it is enough to show that $\langle h'_\alpha, h'_\alpha\rangle \in \mathbf{Q}$. This is by definition equal to
$$\mathrm{Tr}(\mathrm{ad}(h'_\alpha)^2) = \sum_\beta \langle h'_\alpha, h'_\beta\rangle^2.$$

Hence
$$\frac{1}{\langle h'_\alpha, h'_\alpha\rangle} = \sum_\beta \frac{\langle h'_\alpha, h'_\beta\rangle^2}{\langle h'_\alpha, h'_\alpha\rangle^2} \in \mathbf{Q}.$$ □

Lemma 7.11. *Let V be a finite-dimensional vector space over K (here, an arbitrary field) with a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$, and I a subset of V, linear spanning V, such that $\langle x, y\rangle \in F$ for all $x, y \in V$ and some subfield F of V. Then, for any basis J of V as a K-linear space, I is contained in the F-linear span of J.*

Proof. First suppose that J is an orthogonal basis. Then for every $x \in I$, we have $x = \sum_{y \in J} \frac{\langle x,y \rangle}{\langle y,y \rangle} y$ and the result holds. In general, write $J = (e_1, \dots, e_k)$; then we can orthogonalize, namely find, for all i, e'_i with $e'_i - e_i$ in the F-linear span of $\{e_j : j < i\}$, with $J' = \{e'_i : 1 \le i \le k\}$ an orthonormal basis as well, and it still holds that $\langle x, y \rangle \in F$ for all $x \in I \cup J'$. So every $x \in I$ belongs to the F-linear span of J', which coincides with the F-linear span of J. □

For any subfield F of K, write $\mathfrak{g}_0^F$ the K-linear span of Φ (where Φ is identified to $\{h'_\alpha : \alpha \in \Phi\}$). By Lemma 7.11, we have $\dim_F(\mathfrak{g}_0^F) = \dim_K(\mathfrak{g}_0)$.

In a field F, say that $t \in F$ is positive, written $t > 0$, if t is a sum of a nonempty finite number of nonzero squares. A real field is a field in which 0 is not positive. We say that a bilinear form B, on a vector space over a real field is definite positive if $B(x, x) > 0$ for all $x \neq 0$.

Proposition 7.12. *For every subfield F of K which is a real field, the Killing form is positive-definite on the F-linear span of Φ. More generally, for every subfield F of K, field extension F' of F such that F is a real field, the extension of the Killing form to $\mathfrak{g}_0^{F'} = \mathfrak{g}_0^F \otimes_F F'$ is definite-positive.*

Proof. Fix a basis J of $\mathfrak{g}_0$ contained in Φ, and consider an element of $\mathfrak{g}_0^F$, which can be, by Lemma 7.11, in the form $v = \sum_{y \in J} t_y y$ with $t_y \in F$. We have

$$\begin{aligned}\langle v, v \rangle &= \sum_{(y,z) \in J^2} t_y t_z \langle y, z \rangle = \sum_{(y,z) \in J^2} t_y t_z \mathrm{Tr}(\mathrm{ad}(y)\mathrm{ad}(z)) \\ &= \sum_{(y,z) \in J^2} t_y t_z \sum_\alpha \alpha(y)\alpha(z) = \sum_\alpha \left(\sum_{y \in J} t_y \alpha(y) \right)^2 = \sum_\alpha \alpha(v)^2;\end{aligned}$$

If $v \neq 0$, there exists α such that $\alpha(v) \neq 0$ and hence, since $\beta(v) = \sum_y \langle \beta, y \rangle \in F$ for all β, we deduce that $\langle v, v \rangle > 0$, and the Killing form is therefore positive-definite.

The same proof applies to the generalized statement. □

When $K = \mathbf{C}$, we usually consider $F = \mathbf{R}$ in this statement; when K is arbitrary, one usually consider $F = \mathbf{Q}$ and $F' = \mathbf{R}$, and we view Φ as a subset of the Euclidean space $\mathfrak{g}_0^{\mathbf{R}}$, although the latter is no longer considered as a subset of $\mathfrak{g}$.

For nonzero elements $\alpha, \beta \in \mathfrak{g}_0^{\mathbf{R}}$, define $\theta_{\alpha,\beta} = \arccos\left(\frac{\langle \alpha, \beta \rangle}{\|\alpha\| \|\beta\|}\right) \in [0, \pi]$, the angle between α and β.

Proposition 7.13. *For any $\alpha, \beta \in \Phi^*$ with $\|\beta\| \ge \|\alpha\|$, $\langle \alpha, \beta \rangle \le 0$, we have one of the following:*

(a) $\theta_{\alpha,\beta} = \pi/2$ (that is, $\langle \alpha, \beta \rangle = 0$)
(b) $\theta_{\alpha,\beta} = \pi/3$, $\|\beta\| = \|\alpha\|$; $\alpha + \beta \in \Phi$
(c) $\theta_{\alpha,\beta} = \pi/4$, $\|\beta\| = \sqrt{2}\|\alpha\|$; $\alpha + \beta, 2\alpha + \beta \in \Phi$

(d) $\theta_{\alpha,\beta} = \pi/6$, $\|\beta\| = \sqrt{3}\|\alpha\|$; $\alpha+\beta, 2\alpha+\beta, 3\alpha+\beta, 3\alpha+2\beta \in \Phi$;
(e) $\theta_{\alpha,\beta} = 0$; $\alpha = -\beta$.
In particular, if they are not orthogonal, we have $\|\beta\|^2 = t_{\alpha,\beta}\|\alpha\|^2$ *with* $t_{\alpha,\beta} \in \{1,2,3\}$.

Proof. Write $c = -\cos(\theta_{\alpha,\beta}) \in [0,1]$. The cases $c = 0$ and $c = 1$ yield respectively (a) and (e); in the latter case, we use Proposition 7.9 to deduce $\alpha = -\beta$. Otherwise, $0 < c < 1$.

We first Proposition 7.9, which implies that $\hat{\theta}_{\alpha,\beta} > 0$ (equivalently, $c < 1$), and, to start with, the consequence of Proposition 7.8, namely that $s := 2\frac{-\langle\alpha,\beta\rangle}{\langle\beta,\beta\rangle} \in \mathbf{Z}$ and $t := 2\frac{-\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbf{Z}$. Indeed, this can be rewritten as $s = 2c\frac{\|\alpha\|}{\|\beta\|} \in \mathbf{Z}$ and $t = 2c\frac{\|\beta\|}{\|\alpha\|} \in \mathbf{Z}$. Multiplying, this yields $st = 4c^2 \in \mathbf{Z}$; since $0 < c < 1$ is a cosine, we deduce $st = 4c^2 \in \{1,2,3\}$. Since $\|\beta\| \geq \|\alpha\|$, we deduce that $t \geq s$, and hence $s = 1$. So $\|\beta\|/\|\alpha\| = 2c = \sqrt{4c^2} \in \{1, \sqrt{2}, \sqrt{3}\}$, so the corresponding value of c is $1/2$, $1/\sqrt{2}$, $\sqrt{3}/2$ respectively, corresponding to the given values of the angles given in Cases (b), (c), (d) respectively.

Now use more precisely Proposition 7.8, which says that $t = n - m$, where $k\alpha + \beta \in \Phi$ for all $k \in \{1,\dots,n\}$, and $m \geq 0$. So $n \geq t$. In the above items, the value of t is respectively $0,1,2,3$, and the additional assertion follows in each case, except, in the last case, $3\alpha+2\beta$. But denoting $\beta' = -\beta-3\alpha'$ and $\alpha' = -\beta-2\alpha'$, we have $\|\beta'\| = \sqrt{3}\|\alpha'\|$, $\langle\alpha',\beta'\rangle < 0$, and hence applying the result to this pair, we deduce that $3\alpha'+\beta' = 3\alpha+2\beta$ belongs to Φ. ∎

8. Root systems

Definition 8.1. Let E be a Euclidean space. A root system is a finite subset Φ of E satisfying $0 \in \Phi$, $\Phi = -\Phi$, and satisfying the conclusion of Proposition 7.13.

For $F \subset E$, we write $F^* = F \smallsetminus \{0\}$. We call a subset F of E irreducible if it is not reduced to $\{0\}$, and cannot be written as $F_1 \cup F_2$ with $\langle F_1, F_2\rangle = \{0\}$, $F_1^* \neq F^* \neq F_2^*$.

We write this section separately, because it is pure Euclidean geometry.

Definition 8.2. Let E be a Euclidean space and F a subset of E, with $F = -F$. We say that $B \subset F$ is a fundamental basis of F if

- B is linearly independent;
- we have $F^* = (\Sigma B \cap F) \cup -(\Sigma B \cap F)$, where ΣB is the subsemigroup generated by B (the set of nonempty sums of elements of B)
- $\langle x, y\rangle \leq 0$ for all $x, y \in B$.

Lemma 8.3. *Let E be a Euclidean space. Let F be root system in E, or more generally a finite subset of E such that*

- *$F = -F$;*
- *for all $x, y \in F$ such that $\langle x, y\rangle < 0$, we have $x + y \in F$.*

Let ℓ be a linear form on E, such that ℓ does not vanish on $F \smallsetminus \{0\}$. Write $F_+[\ell] = \{x \in F : \ell(x) > 0\}$ and $F_+^1[\ell] = F_+[\ell] \smallsetminus (F_+[\ell] + F_+[\ell])$. Then $F_1^+[\ell]$ is a fundamental basis of F and $F_+[\ell] = F \cap \Sigma(F_1^+[\ell])$. (In particular, $F_+[\ell]$ and $F_+^1[\ell]$ determine each other.) Moreover, every fundamental basis of F has this form (for some ℓ).

Proof. Enumerating elements of F_+ as $x_1, \dots, x_k$ with $\ell(x_1) \le \ell(x_2) \le \dots$, by an immediate induction, we see that x_i belongs to the $\mathbf{Z}$-span of F_+^1. Since $F = -F$ and ℓ does not vanish on $F \smallsetminus \{0\}$, the complement of $F \smallsetminus \{0\}$ is $-F \smallsetminus \{0\}$.

Suppose by contradiction that $x, y \in F_+^1$ with $\langle x, y\rangle > 0$. Then $\langle -x, y\rangle < 0$, and $-x, y \in F$, so $-x + y \in F$, and hence $x - y \in F$ as well. Up to switch x and y, we can suppose that $\ell(x - y) > 0$, so $x = (x - y) + y$ does not belong to F_+^1, a contradiction.

Consider, by contradiction, a nontrivial combination between elements of F_+^1. Gathering coefficients of the same sign, write it as $w_U = w_V$, where $w_U = \sum_{x \in U} t_x x$, with U, V disjoint subsets of F_+^1, U nonempty, and $t_x > 0$ for all $x \in U \cup V$. In particular, $\ell(w_U) > 0$, and hence $w_U \neq 0$. So $0 < \langle w_U, w_U\rangle = \langle w_U, w_V\rangle \le 0$, a contradiction. Hence F_+^1 is a free family.

The inclusion $\Sigma(F_+^1) \cap F \subset F_+$ is clear. Conversely, by construction if $x \in F_+ \smallsetminus \Sigma(F_+^1)$ with $\ell(x)$ minimal, then $x \notin F_+^1$, so we can write $x = y + z$ with $y, z \in F_+$ so $\ell(y), \ell(z) < \ell(x)$ and thus by minimality, one has $y, z \in \Sigma(F_+^1)$. Hence x also belongs to $\Sigma(F_+^1)$, a contradiction.

Finally, let B be a fundamental basis; choose ℓ with $\ell = 1$ on B. Then the corresponding F_+^1 (determined by ℓ) contains B, and since it is a basis of the span of F as well as B, we deduce that $F_+^1 = B$. □

Definition 8.4. A spread system in E is a subset B of $E \smallsetminus \{0\}$ such that the angle between any two distinct elements of B belongs to $\{\pi/2, 2\pi/3, 3\pi/4, 5\pi/6\}$. It is called a normed spread system if, in addition, it satisfies the norm compatibility of root systems: if $\alpha, \beta \in B$ are distinct and non-orthogonal, and $\|\beta\| \ge \|\alpha\|$ and the angle between them is $2\pi/3$, resp. $3\pi/4$, resp. $5\pi/6$, then $\|\beta\| = \|\alpha\|$, resp. $\|\beta\| = \sqrt{2}\|\alpha\|$, resp. $\|\beta\| = \sqrt{3}\|\alpha\|$.

By Lemma 8.3, for very root system Φ (and choice of linear form ℓ not vanishing on Φ^*), the subset Φ_+^1, called set of fundamental roots (relative to ℓ), is a linearly independent normed spread system.

Definition 8.5. The (non-oriented) Dynkin diagram of a spread system P is the graph whose set of vertices is P, with an edge between any two non-orthogonal roots:

- labeled by 3, or denoted as a simple edge, if the angle is $2\pi/3$;
- labeled by 4, or denoted as a double edge, if the angle is $3\pi/4$;
- labeled by 6, or denoted as a triple edge, if the angle is $5\pi/6$.

Given a normed spread system, its (oriented) Dynkin diagram consists in endowing each edge labeled by 4 or 6, using an arrow from the largest to the smallest vector (to memorize the convention, think of the arrow as a $>$ sign!).

Definition 8.6. We call (non-oriented) Dynkin diagram a finite set endowed with a function from the set of 2-element subsets to $\{2, 3, 4, 6\}$, and represent it as a graph according to the previous rules; an orientation on a Dynkin diagram means a choice of orientation on edges labeled 4 or 6. We call a Dynkin diagram realizable (resp. freely realizable, resp. non-freely realizable) if it the Dynkin diagram of some spread system (resp. of some linearly independent spread system, resp. of some linearly dependent spread system). Given an orientation, we call it strongly realizable if it is the oriented Dynkin diagram of some normed spread system.

Given a spread system, if we replace each each element with a scalar multiple, we obtain another spread system, with the same Dynkin diagram. We call this "rescaling". The following shows that up to rescaling, a spread system is entirely determined by its Dynkin diagram:

Proposition 8.7. *Let $F \subset E$, $F' \subset E'$ be spread systems, and $u : F \to F'$ a bijection inducing an isomorphism of Dynkin diagrams. Then there is a unique isometry $\bar{u}$ from the $\mathbf{R}$-span of F onto the $\mathbf{R}$-span of F', such that $\bar{u}(v)$ is positively collinear to $u(v)$ for all $v \in F$. In particular, F cannot be both freely and non-freely realizable.*

Proof. Uniqueness is clear, since one necessarily has $\|\bar{u}(v)\| = \frac{\|v\|}{\|u(v)\|}u(v)$ for all $v \in F$.

We can suppose that F spans E and F' spans E'. Choose a bijection $u : F \to F'$ defining a graph isomorphism. We can rescale elements of F' to ensure that $\|u(v)\| = \|v\|$ for all $v \in F'$.

Consider the space $\mathbf{R}^F$, with basis $(e_v)_{v \in F}$. Define the symmetric bilinear form defined by $B(e_v, e_{v'}) = \langle v, v' \rangle$, extended by bilinearity. Let ψ be the unique linear map mapping $e_v \in \mathbf{R}^F$ to $v \in E$. Then $B(x, x') = \langle \psi(x), \psi(x') \rangle$ for all $x, x' \in \mathbf{R}^F$. Hence $\mathrm{Ker}(\psi)$ equals the kernel K of B, and ψ induces an isometric isomorphism from $\mathbf{R}^F/K$ to E.

Also, define ψ' as the unique linear map mapping e_v to $u(v)$. Note that u is a graph isomorphism means that $\theta_{u(v),u(v')}) = \theta_{v,v'}$ for all $v, v' \in V$. Since

$$\langle u(v), u(v') \rangle = \|u(v)\|.\|u(v')\| \cos(\theta_{u(v),u(v')}) = \|v\|.\|v'\| \cos(\theta_{v,v'}) = \langle v, v' \rangle$$

for all $v, v' \in V$, the same reasoning applies to ψ', which induces an isometric isomorphism from $\mathbf{R}^F/K$ to E. Composing, we deduce a isometric isomorphism from E to E' extending u. $\square$

(See pictures of Dynkin diagrams in separate file)

Proposition 8.8. *All Dynkin diagrams A_n, B_n C_n, D_n, $E_{6,7,8}$, F_4, G_2 are strongly freely realizable. All (non-oriented) Dynkin diagrams $\tilde{A}_n$, $\tilde{B}_n$, $\tilde{C}_n$, $\tilde{D}_n$, $\tilde{E}_{6,7,8}$, $\tilde{F}_4$, $\tilde{G}_2$ are non-freely realizable.*

Proof. The proof consists in giving a list of vectors and can mechanically be checked by a computation of scalar products.

Use $(e_1 \dots, e_k)$ as a basis of $\mathbf{R}^k$; write $\mathbf{R}_0^{k+1}$ for the (k-dimensional) hyperplane of vectors with sum 0. Then A_n ($n \ge 1$) can be freely realized by the vectors $(e_1-e_2, e_2-e_3, \dots, e_n-e_{n+1})$, which form a basis of $\mathbf{R}_0^{n+1}$. Adding $\xi^A = e_{n+1}-e_1$, we non-freely realize $\tilde{A}_n$ for $n \ge 2$.

Now start from the vectors $e_1 - e_2, \dots, e_{n-1} - e_n$ (realizing A_{n-1}). Then

- adding $-e_1$, resp. $-2e_1$ to the list, we strongly freely realize B_n, resp. C_n;
- adding $-e_1 - e_2$, we strongly freely realize D_n ($n \ge 4$);
- adding $-e_1$ and e_n, we non-freely realize $\tilde{C}_n$ ($n \ge 2$);
- adding $-e_1$ and $e_{n-1} + e_n$, we non-freely realize $\tilde{B}_n$ ($n \ge 3$);
- adding $-e_1 - e_2$ and $e_{n-1} + e_n$, we non-freely realize $\tilde{D}_n$ ($n \ge 4$).

It remains to deal with exceptional Dynkin diagrams, namely of type EFG:

- In $\mathbf{R}_0^3$, consider the vectors $(1, -1, 0)$ and $(-1, 2, 1)$; it freely realizes G_2; adding the vector $(1, 0, -1)$ non-freely realizes $\tilde{G}_2$.
- In $\mathbf{R}^4$, consider the vectors $(1, -1, 0, 0)$, $(0, 1, -1, 0)$, $(0, 0, 1, 0)$, $-\frac{1}{2}(1, 1, 1, 1)$. It freely realizes F_4. Adding the vector $(0, 0, 0, 1)$ non-freely realizes $\tilde{F}_4$.
- In $H_8 = \mathbf{R}^8$, consider the vectors

  $$\alpha_1 = -\frac{1}{2}(1, 1, \dots, 1); \quad \alpha_2 = e_2 + e_3; \quad \alpha_i = -e_{i-1} + e_i, \ i \in \{3, \dots, 8\}.$$

  Let H_7 be the hyperplane of equation $t_1 = t_8$, and H_6 the hyperplane of H_6, of equation $t_1 = t_7 = t_8$. Then for $i = 6, 7, 8$, $(\alpha_1, \dots, \alpha_i)$ is a basis of H_i, and freely realizes E_i. Moreover, define

  $$\xi_8 = e_1 - e_8, \quad \xi_7 = e_1 + e_8 \in H_7, \quad \xi_6 = (1, 1, -1, -1, -1, -1, 1, 1) \in H_6.$$

  (Thus ξ_8 is orthogonal to α_j for $j \neq 8$, ξ_7 is orthogonal to α_j for $j \neq 1, 8$; ξ_6 is orthogonal to α_j for $j \neq 3, 7, 8$; $\theta_{\xi_8,\alpha_8} = \theta_{\xi_7,\alpha_1} = \theta_{\xi_6,\alpha_3} = 2\pi/3$.) Then, for each of $i = 6, 7, 8$, $(\alpha_1, \dots, \alpha_i, \xi_i)$ non-freely realizes $\tilde{E}_i$ (in H_i). $\square$

Theorem 8.9. *An oriented connected Dynkin diagram is strongly freely realizable if and only each of its components is so.*

Consider a nonempty oriented connected Dynkin diagram X. Equivalences:

(1) the oriented Dynkin diagram X is isomorphic to one among A_n ($n \ge 1$), B_n ($n \ge 2$), C_n ($n \ge 3$), D_n ($n \ge 4$), $E_{6,7,8}$, F_4, G_2;

(2) the non-oriented Dynkin diagram X is isomorphic to one among A_n ($n \ge 1$), $B_n = C_n$ ($n \ge 2$), D_n ($n \ge 4$), $E_{6,7,8}$, F_4, G_2;

(3) the oriented Dynkin diagram X is strongly freely realizable;

(4) *the (non-oriented) Dynkin diagram X is freely realizable. An oriented Dynkin diagram is strongly freely realizable if and only if all its components are strongly freely realizable.*

Proof. The first assertion is straightforward.

The implications (3)⇒(4) and (1)⇔(2) are trivial (noting that C_2 is isomorphic to B_2 as oriented graph). The implication (1)⇒(3) is part of the contents of Proposition 8.8.

It remains to prove (4)⇒(2). Let n be the number of vertices. If $n = 1, 2$ there is nothing to prove (since A_2, $B_2 = C_2$, G_2 are allowed).

If $n = 3$, the proposed possibilities are A_3 and $B_3 = C_3$, and we have to discard all others. Write $\hat{\theta} = \min(\theta, \pi - \theta) \in [0, \pi/2]$. Denote the three vectors as α, β, γ, with $\hat{\theta}_{\alpha,\beta} \le \hat{\theta}_{\alpha,\gamma} \le \hat{\theta}_{\beta,\gamma}$. Since they are non-coplanar, we have the strict triangle inequality $\theta_{-\alpha,\beta} + \theta_{-\alpha,\gamma} > \theta_{\beta,\gamma}$. First, this yields $\theta_{-\alpha,\beta} + \theta_{-\alpha,\gamma} > \pi/2$. This discards the possibilities that $(\hat{\theta}_{\alpha,\beta}, \hat{\theta}_{\alpha,\gamma})$ is $(\pi/6, \pi/6)$, $(\pi/6, \pi/4)$, $(\pi/6, \pi/3)$, or $(\pi/4, \pi/4)$. In addition, this discards the possibilities that $(\hat{\theta}_{\alpha,\beta}, \hat{\theta}_{\alpha,\gamma}, \hat{\theta}_{\beta,\gamma})$ is among $(\pi/4, \pi/3, \pi/3)$ or $(\pi/3, \pi/3, \pi/3)$. The only remaining possibilities are $(\pi/4, \pi/3, \pi/2)$, $(\pi/3, \pi/3, \pi/2)$ (which correspond to $B_3 = C_3$ and A_3 respectively), or $(\pi/k, \pi/2, \pi/2)$ for $k = 2, 3, 4, 6$, which corresponds to a non-connected Dynkin diagram.

If $n \ge 4$, first we claim that the diagram has no edge labeled by 6, and no 2 consecutive edges labeled by 4. Indeed, otherwise, it has an induced connected subgraph on 3 vertices with (at least) one edge labeled by 6 or two edges labeled by 4, and this has been excluded.

Next, we claim that the diagram has no loop. Loops with only edges labeled by 3 are excluded, since the graph $\tilde{A}_n$ is non-freely realizable for $n \ge 2$. If the loop has length ≥ 5 and at least two edges labeled by 4, then it has an induced subgraph isomorphic to $\tilde{C}_n$ for some $n \ge 2$ and hence is non-freely realizable. If has a single edge labeled by 4 and others by 3 and $n \ge 6$, then it has an induced subgraph isomorphic to $\tilde{F}_4$, and this is non-freely realizable. The remaining cases are: a loop of size 4 with 1 or 2 non-consecutive edges labeled by 4, or a loop of size 5 with a single edge labeled by 4. Choosing the vectors to be of radius $\sqrt{2}$, if realizable, the corresponding matrix of scalar products is the following:

$$\begin{pmatrix} 2 & -1 & 0 & -\sqrt{2} \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -\sqrt{2} & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & -\sqrt{2} \\ -1 & 2 & -\sqrt{2} & 0 \\ 0 & -\sqrt{2} & 2 & -1 \\ -\sqrt{2} & 0 & -1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 0 & 0 & -\sqrt{2} \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -\sqrt{2} & 0 & 0 & -1 & 2 \end{pmatrix}.$$

The determinant of each of these matrices ($-1 - 2\sqrt{2}$, -7, and $-2 - 2\sqrt{2}$ respectively) is negative, so all these cases are discarded (since any Gram matrix has a non-negative determinant), and we conclude that the diagram has no loop.

Next, we claim that there is at most one edge labeled by 4: otherwise, there is an induced subgraph isomorphic to $\tilde{C}_n$ for some $n \geq 2$, which is non-freely realizable, and this is excluded.

Next, we claim that if there is an edge labeled by 4, then the diagram is filiform. Indeed, otherwise, there is an induced subgraph isomorphic to $\tilde{B}_n$ for some $n \geq 3$, which is non-freely realizable, and this is excluded.

Finally, if there is an edge labeled by 4, we claim that the diagram is of type $B_n = C_n$ ($n \geq 2$) or F_4: otherwise, since we know that it is filiform with a single edge labeled by 4, it contains an induced subgraph isomorphic to $\tilde{F}_4$, and this excluded.

It remains to consider the case of a loop-free diagram with only edges labeled by 3. If there are two branching vertices, or a vertex of degree ≥ 4, then there is an induced subgraph isomorphic to $\tilde{D}_n$ for some $n \geq 4$, which is non-freely realizable, and this is excluded.

Finally, either we have a graph of type A_n, or we have a graph with a single trivalent branching points, and branches of size (not counting the branching point) $1 \leq k_1 \leq k_2 \leq k_3$ (so $n = 1 + k_1 + k_2 + k_3$). Call this $X(k_1, k_2, k_3)$.

If $k_1 \geq 2$, then we have an induced subgraph isomorphic to $X(2,2,2) = \tilde{E}_6$. So $k_1 = 1$. If $k_2 \geq 3$, then we have an induced subgraph isomorphic to $\tilde{E}_7$. So $k_2 \leq 2$.

If $k_2 = 1$, then the graph has type $X(1,1,k_3) = D_{k_3+3}$. Otherwise, $k_2 = 2$. If $k_3 \geq 5$, then we have an induced subgraph isomorphic to $X(1,2,5) = \tilde{E}_8$. The remaining possibilities are $X(1,2,k)$ for $k = 2,3,4$, which correspond to E_6, E_7, and E_8. $\square$

Let Φ be a root system (with ℓ). For $x = \sum_{\alpha \in \Phi_+} t_\alpha \alpha$ in the span of Φ, define the subset $\{\alpha \in \Phi_+ : t_\alpha \neq 0\}$, called the support of x. Write $|x| = \sum_\alpha |t_\alpha|$.

Lemma 8.10. *For every positive root β there exists a fundamental root α in the support of β such that $\langle \alpha, \beta \rangle > 0$ and $\beta - \alpha \in \Phi_+ \cup \{0\}$.*

Proof. Write $\beta = \sum_{\alpha \in F} n_\alpha \alpha$ with F a nonempty subset of Φ_+, and n_α a positive integer for all $\alpha \in F$. Then

$$0 < \|\beta\|^2 = \sum_{\alpha \in F} n_\alpha \langle \beta, \alpha \rangle.$$

So there exists $\alpha \in F$ such that $\langle \beta, \alpha \rangle > 0$. Hence $\beta - \alpha \in \Phi$. Since $\beta - \alpha = \sum_{\gamma \in F \smallsetminus \{\alpha\}} n_\gamma \gamma + (n_\alpha - 1)\alpha$, we deduce that $\beta - \alpha \in \Phi_+ \cup \{0\}$. $\square$

Let B, F be subsets of a Euclidean space. Define, for $k = 1, 2, 3$

$$u_k(B, F) = \{k\alpha + \beta : \alpha \in B, \beta \in F : \langle \alpha, \beta \rangle < 0, \|\beta\|^2 \geq k\|\alpha\|^2\}.$$

Define inductively

$$B^{(n)} = \emptyset, n \le 0,\ B^{(1)} = B, \quad B^{(n)} = \bigcup_{k=1}^{3} u_k(B, B^{(n-k)}),\ (n \ge 2), \quad B^{(\infty)} = \bigcup_{n \ge 1} B^{(n)}.$$

This yields an effective procedure to construct elements of a root system starting with a fundamental basis.

Proposition 8.11. *Let Φ be a root system with a fundamental basis B and $\Phi_+ = \Sigma B \cap \Phi$ the corresponding set of positive roots. Then $\Phi_+ = B^{(\infty)}$ (and $\Phi = B^{(\infty)} \cup \{0\} \cup (-B^{(\infty)})$). In particular, if Φ, Ψ are root systems with a common fundamental basis, then they are equal.*

Proof. Choose a linear form ℓ to be equal to 1 on B. We prove, by induction on $n \ge 1$, that $\Phi \cap \ell^{-1}(\{n\}) = B^{(n)}$. This is clear for $n = 1$, and for arbitrary n the inclusion $\supset$ follows from the root system axioms.

For $n \ge 2$, let γ belong to $\Phi \cap \ell^{-1}(\{n\})$. By Lemma 8.10, there exists a fundamental root α in the support of γ such that $\langle \alpha, \gamma \rangle > 0$; set $\beta = \gamma - \alpha \in \Phi$. By induction, we have $\beta \in B^{(n-1)}$.

First suppose that $\|\alpha\| \ge \|\gamma\|$, and write $\|\gamma\|^2 = t\|\alpha\|^2$ with $t \le 1$; we have $\langle \alpha, \gamma \rangle = \langle \alpha, \alpha \rangle / 2$. Then $\langle \beta, \alpha \rangle = -\|\alpha\|^2/2 < 0$. Hence $\gamma = \alpha + \beta$ belongs to $B^{(n)}$.

Otherwise, $\|\gamma\|^2 = k\|\alpha\|^2$ for $k \in \{2, 3\}$ (so $\langle \alpha, \gamma \rangle = \langle \gamma, \gamma \rangle / 2$). In this case, $\delta = \gamma - k\alpha$ belongs to $\Phi \cap \ell^{-1}(\{n - k\})$.

We need to check that $k < n$; if $k = 1$ this holds; if $k = 2, 3$, since $\gamma - i\alpha$ is a nonzero root for $0 \le i \le k$, we have $\ell(\gamma - i\alpha) \ne 0$, and it follows (by an intermediate value argument) that $\ell(\gamma - i\alpha) \ge 1$ for all such i, so for $i = k$ this yields $n - k = \ell(\delta) \ge 1$.

Hence, by induction, δ belongs to $B^{(n-k)}$. Then $\langle \delta, \alpha \rangle = -\|\gamma\|^2/2 < 0$ and $\|\delta\|^2 = \|\gamma\|^2 = k\|\alpha\|^2$, so $\gamma = \delta + k\alpha$ belongs to $B^{(n)}$. ☐

Remark 8.12. If B is a normed spread system, then it is immediate (by induction) from the definition that $\{\|v\| : v \in B^{(\infty)}\} = \{\|v\| : v \in B\}$: the construction does not provide new norms.

It follows that if B has no edge labeled by 6 (so we can renormalize on each component so that the norms are in $\{1, \sqrt{2}\}$, then the induction simplifies to $B^{(n+1)} = u_1(B, B^{(n-1)}) \cup u_2(B, B^{(n-2)})$. In turn, if B has only simple edge (no edge labeled by 4 or 6), then the induction simplifies to $B^{(n+1)} = u_1(B, B^{(n-1)})$.

(Note that $\|\beta\| \in \{\|\alpha\|, \|\delta\|\}$. Thus, by induction, $\|\beta\| = \|\alpha'\|$ for some $\alpha' \in B$. In particular, if the Dynkin diagram only has edges labeled 3, then all nonzero roots have the same norm and in particular, the only possible angle (up to $\theta \mapsto \pi - \theta$) between non-collinear non-orthogonal roots is $\pi/3$; in particular it is enough to consider $k = 1$ in the above algorithm. Similarly, if the Dynkin diagram only has edges labeled by 3 and 4, the only possible angles are $\pi/3$ and $\pi/4$; in particular, it is enough to consider only $k = 1, 2$ in the above algorithm.)

Lemma 8.13. *Let Φ be a subset of a Euclidean space, all of whose norms lie in $\{\sqrt{2}, 2\}$. Then Φ is a root system if and only if it satisfies the following conditions:*

(1) $0 \in \Phi$, $\Phi = -\Phi$*;*
(2) for all $\alpha, \beta \in \Phi$, *one has* $\langle \alpha, \beta \rangle \in \mathbf{Z}$*;*
(3) for all $\alpha, \beta \in \Phi$ *such that* $\max(\|\alpha\|, \|\beta\|) = 2$, *one has* $\langle \alpha, \beta \rangle \in 2\mathbf{Z}$*;*
(4) for all $\alpha, \beta \in \Phi$ *such that* $\langle \alpha, \beta \rangle < 0$, *one has* $\alpha + \beta \in \Phi$*;*
(5) for all $\alpha, \beta \in \Phi$ *such that* $\langle \alpha, \beta \rangle < 0$ *and* $\|\beta\| > \|\alpha\|$, *one has* $2\alpha + \beta \in \Phi$.

Proof. A root system with norms in $\{\sqrt{2}, 2\}$ indeed satisfies these conditions. Conversely, if satisfied, for any α, β with $\|\beta\| \geq \|\alpha\|$ and $\langle \alpha, \beta \rangle < 0$, $\alpha \neq -\beta$, one has $(\|\alpha\|, \|\beta\|, \langle \alpha, \beta \rangle)$ is one of $(\sqrt{2}, \sqrt{2}, -1)$, $(2, 2, -2)$, or $(\sqrt{2}, 2, -2)$; this gives one of the allowed configurations (excluding the angle $5\pi/6$), $\alpha + \beta$ belongs to Φ, and in the third case, $2\alpha + \beta$ belongs to Φ.

Finally, Φ is a subset of the 2-ball whose distinct elements have distance $\geq \sqrt{2}$, hence is finite. □

Proposition 8.14. *Every normed spread system B is contained in a root system Φ; we can choose Φ to be generated by B, in the sense that no root system properly contained in Φ contains B.*

Proof. We can suppose that the Dynkin diagram has a single component.

First suppose that the Dynkin diagram has type G_2. Denote the fundamental roots by α and β with $\|\beta\|^2 = 3\|\alpha\|^2$. The the algorithm successively outputs $\alpha + \beta$, $2\alpha + \beta$, $3\alpha + \beta$, $3\alpha + 2\beta$. That this defines a root system is checked by hand.

Next, we suppose that the Dynkin diagram has no edge labeled by 6. Let us first show that B (arbitrary normed spread system) is contained in a root system.

Define $B^{[1]} = B \cup (-B) \cup \{0\}$ and inductively

$$B^{[n]} = B^{[n-1]} \cup \bigcup_{p+q=n} \{\alpha + \beta : \alpha \in B^{[p]}, \beta \in B^{[q]}, \langle \alpha, \beta \rangle < 0\}$$

$$\cup \bigcup_{2p+q=n} \{2\alpha + \beta : \alpha \in B^{[p]}, \beta \in B^{[q]}, \langle \alpha, \beta \rangle < 0, \|\beta\| = \sqrt{2}\|\alpha\|\}.$$

Let us prove that $\Phi = \bigcup_n B^{[n]}$ is a root system.

Up to renormalization, we can suppose that the set of norms of elements of B is contained in $\{\sqrt{2}, 2\}$. It follows that the set of norms of elements of Φ is also contained in $\{\sqrt{2}, 2\}$; we call elements of norm $\sqrt{2}$ small and elements of norm 2 large.

Let Λ be the $\mathbf{Z}$-submodule of the ambient Euclidean space, generated by B. Denote by B' the subset $\{2\alpha : \alpha \in B : \|\alpha\| = \sqrt{2}\} \cup \{\alpha \in B : \|\alpha\| = 2\}$. Let Λ' be the $\mathbf{Z}$-submodule generated by B' (it has finite index in Λ).

Clearly, Φ is contained in Λ. Since the scalar product is integral on $B \times B$, it is therefore integral on $\Lambda \times \Lambda$, and hence on $\Phi \times \Phi$. Moreover, since all large roots produced are obtained as sum of two large roots, or sum of a large root and twice a small root, the set of large roots is contained in Λ'. Since the scalar product takes even values on $B \times B'$, we deduce that the scalar product takes even values on $\Lambda \times \Lambda'$, and hence the scalar product of any root with a large root is even.

This yields the integrality conditions in Lemma 8.13. The stability condition is obvious from the inductive definition of $B^{[n]}$.

It is clear that B generates Φ in the specified sense. □

Proposition 8.15. *Let B be a free normed spread system, and suppose that B is contained in a root system Φ, and generates Φ. Then B is a fundamental basis of Φ.*

Proof. We can suppose that B is irreducible, so Φ is also irreducible. We can suppose that Φ spans the ambient Euclidean space.

In type A_n, we write $B = \{e_i - e_{i+1} : 1 \le i \le n\} \subset \mathbf{R}_0^{n+1}$ and define $\Phi_+ = \{e_i - e_j : 1 \le i < j \le n+1\}$. Clearly Φ_+ is contained in Φ, and $\Phi_+ \cup (-\Phi_+) \cup \{0\}$ is a root system, so equals Φ. Since every element of Φ_+ belongs to ΣB, we deduce that B is a fundamental basis. A similar explicit argument holds in types D_n, B_n and C_n.

The exceptional cases can be checked by long computations, but also with little computation as follows: one first observes that in the root system of type B_n (resp. C_n), the set of large (resp. small) roots is a root system of type D_n, and the set of small roots has type nA_1 (all non-collinear roots are orthogonal). Here it is understood that $D_3 = A_3$ and $D_2 = 2A_1$.

- the spread system of type G_2 does not embed isometrically into any other irreducible root system (which have no $\pi/6$ angle);
- the spread system of type F_4 does not embed isometrically into any other irreducible root system: indeed, it is the only one (with G_2) with the property of having both small and long non-orthogonal non-collinear roots.
- the spread system E_6 does not embed into D_n for any n, by a specific elementary verification. As a consequence, E_i does not embed into any irreducible root system of type ABCD.

This being done, suppose that B has exceptional type (EFG). Since B cannot be embedded isometrically into a root system of classical type, we deduce that Φ has exceptional type, and since there is at most one such type by dimension, we deduce that Φ has the same type as B. This means, B' being a fundamental basis of B, that the type of B and B' are the same. Since B and B' have the same norms (which is the set of norms achieved by Φ), we deduce that there is an isometry f mapping B onto B'; by Lemma 8.7, it extends to a self-isometry of E. Since Φ is the root system generated by both B and B', we have $f(\Phi) = \Phi$. So $B = f^{-1}(B')$ is a fundamental basis. □

Lemma 8.16. *Let V be a real vector space and W a finite (or countable) union of affine subspaces of codimension ≥ 2. Then any two points x, x' in $V \smallsetminus W$ can* *be joined by a path $[x,y]\cup[y,x']$ consisting of two segments. In particular, it is connected for any topological vector space structure on V.*

Let X be a finite union of affine subspaces of codimension ≥ 1. Then for $x,x'\notin X$, such a path meets X only at finitely many points (and we can choose y to be outside X).

Proof. Let $(W_i)_{i\in I}$ be the given subspaces. For $x\in V$, let $H_i(x)$ be the affine subspace spanned by $W_i\cup\{x\}$; it has codimension ≥ 1, and $H(x)=\bigcup_{i\in I}H_i(x)$.

If $x\notin W_i$ and $y\notin H_i(x)$, then the line (xy) has empty intersection with W_i. So if $x\notin W$ and $y\notin H(x)$, then $[x,y]\cap W$ is empty. Thus, for any x,x', choose $y\notin H(x)\cup H(x')$, and then $([x,y]\cup[y,x'])\cup W$ is empty.

Let H be a hyperplane constituting X. Both segments $[x,y]$ and $[y,x']$ are not contained in H, and hence each intersect H at most twice. □

Proposition 8.17. *Let Φ be a root system spanning E. For a nonzero root α, let r_α be the orthogonal reflection mapping α to $-\alpha$. Then $r_\alpha(\Phi)=\Phi$. The subgroup W generated by $\{r_\alpha:\alpha\in W\}$ is finite.*

Proof. For any $\beta\in\Phi$, we have to check that $r_\alpha(\beta)\in\Phi$. This is clear if $\beta\in\mathbf{R}\alpha$, since β is mapped to $-\beta$, or when β is orthogonal to α, since it is mapped to itself. Otherwise, changing β to $-\beta$, we can suppose that $\langle\alpha,\beta\rangle<0$. If $\|\alpha\|\geq\|\beta\|$, then r_α maps β to $\beta+\alpha\in\Phi$. Otherwise we have $\|\alpha\|<\|\beta\|$, the angle between α and β is $3\pi/4$ or $5\pi/6$, and r_α maps β to $\beta+2\alpha$ or $\beta+3\alpha$ respectively, and this belongs to Φ in the respective case.

The kernel of the W-action on Φ is trivial, since Φ spans E. Since Φ is finite, we deduce the finiteness assertion. □

Definition 8.18. W is called the Weyl group of the root system Φ.

Proposition 8.19. *W acts transitively on the set of fundamental bases of Φ. In particular, the fundamental bases of Φ have isomorphic (as labeled graphs) Dynkin diagrams.*

Proof. Let X (resp. W) be the set of elements of E orthogonal at least one nonzero root, resp to at least two non-collinear roots. So X is a finite union of hyperplanes, and $W\subset X$ is a finite union of subspaces of codimension 2.

For $\xi\in E$, write $\Phi_+(\xi)=\{\alpha\in\Phi^*:\langle\alpha,\xi\rangle\geq 0$. We have $\Phi_+(\xi)\cup(-\Phi_+(\xi))=\Phi^*$, and $\Phi_+(\xi)\cap(-\Phi_+(\xi))=\{\alpha\in\Phi^*:\langle\alpha,\xi\rangle=0$. In particular, this intersection is empty if $\xi\notin X$; if $\xi\in X\smallsetminus W$, then this intersection is reduced to $\{\alpha,-\alpha\}$ for some $\alpha\in\Phi^*$; denote $\alpha=\alpha_\xi$ (we will always consider the pair $\pm\alpha_\xi$, so the choice does not matter).

By a straightforward verification, we have $r_\alpha(\Phi_+(\xi))=\Phi_+(r_\alpha(\xi))$ for all $\alpha\in\Phi^*$ and $\xi\in E$. When ξ ranges over its connected component $C(\xi)$ in $E\smallsetminus X$, the subset $\Phi_+(\xi)$ does not vary.

Now consider ξ_0 and ξ_1 in $E\smallsetminus X$, and using Lemma 8.16, join them by a continuous and piecewise affine path $(\xi_t)_{t\in[0,1]}$ valued in $E\smallsetminus W$, such that $S=\{t:\xi_t\in X\}\subset\,]0,1[$ is finite and (ξ_t) is smooth at each $t\in S$.

For t ranging over each given component of $[0,1] \smallsetminus S$, $\Phi_+(\xi_t)$ does not vary. For $s \in S$, let α_s be the unique (up to sign) root that is orthogonal to s, and H_s the orthogonal of α_s. There exists a neighborhood V of s, $v \in E \smallsetminus \{0\}$ such that $\xi_t = (t-s)v + \xi_s$ for all $t \in V$; decompose v as $v' + v''$ with $v' \in \mathbf{R}\alpha_s$ (nonzero) and $v' \in H_s$. Write $\xi'_t = (t-s)v' + \xi_s$. For t close enough to s, ξ_t and ξ'_t are in the same component of $E \smallsetminus X$. Moreover, $r_{\alpha_s}(\xi'_{s+t}) = \xi'_{s-t}$ for all t. Hence, for $t > 0$ small enough, we have $r_{\alpha_s}(\Phi_+(\xi_{s+t})) = \Phi_+(\xi_{s-t})$. Iterating and denoting $S = \{s_1 < s_2 < \cdots < s_k\}$ we deduce that $\Phi_+(\xi_1) = r_{\alpha_{s_k}} \dots r_{\alpha_{s_1}} \Phi_+(\xi_0)$. □

9. Back to semisimple Lie algebras

Proposition 9.1. *The root system $\Phi \in E$ determines the Lie algebra up to isomorphism. More precisely, consider the graded vector space $\mathfrak{g} = \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha$, with $\mathfrak{g}_0 = E$ and each $\mathfrak{g}_\alpha = K$ for each nonzero α. Say that a Lie bracket $[\cdot,\cdot]$ on $\mathfrak{g}$ is compatible if it preserves the grading, and $[h,x] = \langle \alpha, h\rangle x$ for every $h \in \mathfrak{g}_0$, every $\alpha \in \Phi$, and every $x \in \mathfrak{g}_\alpha$.*

Then any two compatible Lie brackets are conjugated by some linear automorphism of $\mathfrak{g}$ preserving the grading and acting as the identity on $\mathfrak{g}_0$. Moreover, if there at least one such Lie bracket, then there is one such that the structure constant are rational (as soon as the scalar product between elements of Φ are all rational).

Proof. We now choose ℓ not vanishing on any difference of two roots. We thus view Φ as a totally ordered set, with $\alpha < \beta$ if $\ell(\alpha) < \ell(\beta)$.

Let P be the set of pairs (α, δ) of elements of Φ, such that $\alpha + \delta$ is a nonzero root. Let P^+ be its subset of pairs such that $\alpha, \delta > 0$. Let $P^+_{\min}$ be the subset of P^+ of such pairs satisfying: $\alpha < \delta$ and for every $(\beta,\gamma) \in P$ such that $\beta+\gamma = \alpha+\delta$, we have $\alpha \le \beta$.

Recall that for $\alpha \in \Phi^*$, h_α is the unique scalar multiple of h'_α such that $\alpha(h_\alpha) = 2$. For each $\alpha \in \Phi_+$, choose a nonzero element e_α in $\mathfrak{g}_\alpha$, and choose $e_{-\alpha} \in \mathfrak{g}_{-\alpha}$ determined by the condition $[e_\alpha, e_{-\alpha}] = h_\alpha$.

For $(\alpha,\beta) \in P$, we write $[e_\alpha, e_\beta] = u_{\alpha,\beta} e_{\alpha+\beta}$ (we also write $u_{\alpha,\beta} = u^{(\alpha+\beta)}_{\alpha,\beta}$ sometimes to emphasize $\alpha+\beta$). By Proposition 7.7, we have $u_{\alpha,\beta} \neq 0$. Say that the basis is normalized if $u_{\alpha,\delta} = 1$ for every $(\alpha,\delta) \in P^+_{\min}$.

We can always choose the basis to be normalized. Indeed, the map $(\alpha,\delta) \mapsto \alpha + \beta$ is injective on $P^+_{\min}$. Therefore, for each $(\alpha,\delta) \in P^+_{\min}$, we can replace $e_{\pm(\alpha+\delta)}$ with $u^{\pm 1}_{\alpha,\delta} e_{\pm(\alpha+\delta)}$. This change of basis preserves the previous properties and replaces $u_{\alpha,\delta}$ with 1.

We now prove that the structure constants $u_{\beta,\gamma}$ are uniquely determined, when the basis is normalized. Let us start with a preliminary computation, which will be used several times.

Consider $(\alpha,\delta),(\beta,\gamma)\in P$ such that $\varepsilon:=\alpha+\delta=\beta+\gamma$. If $\beta-\alpha$ is a root (which is automatic if $\langle\alpha,\beta\rangle>0$), we have

$$\begin{aligned} u_{\beta,\gamma}e_\varepsilon=[e_\beta,e_\gamma]=&\frac{1}{u_{\alpha,\beta-\alpha}}[[e_\alpha,e_{\beta-\alpha}],e_\gamma]\\ =&\frac{1}{u_{\alpha,\beta-\alpha}}\left([e_\alpha,[e_{\beta-\alpha},e_\gamma]]+[[e_\alpha,e_\gamma],e_{\beta-\alpha}]\right)\\ =&\frac{u_{\beta-\alpha,\gamma}}{u_{\alpha,\beta-\alpha}}[e_\alpha,e_\delta]+\frac{1}{u_{\alpha,\beta-\alpha}}[[e_\alpha,e_\gamma],e_{\beta-\alpha}]; \end{aligned}\tag{9.1}$$

if in addition $\alpha+\gamma$ is a not a root, we deduce

$$u^{(\varepsilon)}_{\beta,\gamma}=\frac{u^{(\delta)}_{\beta-\alpha,\gamma}}{u^{(\beta)}_{\alpha,\beta-\alpha}}u^{(\varepsilon)}_{\alpha,\delta}.\tag{9.2}$$

We now first show that for every non-fundamental root $\varepsilon>0$, writing $\varepsilon=\alpha+\delta$ where $(\alpha,\delta)\in P^+_{\min}$, the structure constants $u_{\beta,\gamma}$ for $\beta+\gamma=\varepsilon$, $\beta,\gamma>0$, uniquely determined by $u_{\alpha,\delta}(=1)$, and are rational. We can suppose that $\alpha,\beta,\gamma,\delta$ are pairwise distinct. We argue by induction on $\ell(\varepsilon)$.

Let us first do it entirely when E is linearly generated by $\alpha,\beta,\gamma,\delta$. Since $\delta=\beta+\gamma-\alpha$, E has at most dimension 3. Then Φ is irreducible of dimension 2 or 3, since otherwise one of $\alpha,\beta,\gamma,\delta$ would be orthogonal to all three others and cannot sum to a root any of them. By the classification, the type is A_2, $B_2=C_2$, G_2, A_3, B_3, or C_3.

We have to determine all ways to describe ε as a sum of two positive roots; if there is a single way (up to ordering), there is nothing to do. So we classify, in each type, the ways a positive root can be written in at least two ways as a sum of two positive roots.

1. in type A_2, B_2, there is none. In type G_2 with fundamental roots a,b and other positive roots $a+b,2a+b,3a+b,3a+2b$, we have the possibility $(3a+b)=b+(3a+b)=(a+b)+(2a+b)$.
2. in types A_3,B_3,C_3, write the fundamental roots as a,b,c with a orthogonal to c; so the positive roots of length 2 are $a+b$ and $b+c$ we have the possibility $a+b+c=a+(b+c)=(a+b)+c$;
3. in type B_3 (with a the smaller root), the positive roots of length ≥ 3 are $2a+b$, $a+b+c$, $2a+b+c$, $2a+2b+c$; we have the possibility $2a+b+c=a+(a+b+c)=(2a+b)+c$, as well as $2a+2b+c=b+(2a+b+c)=(a+b)+(a+b+c)=(b+c)+(2a+b)$.
4. in type C_3 (with a the largest root), the positive roots of length ≥ 3 are $a+2b$, $a+b+c$, $a+2b+c$, $a+2b+2c$; we have the possibility $a+2b+c=c+(a+2b)=b+(a+b+c)=(a+b)+(b+c)$, as well as $a+2b+2c=c+(a+2b+c)=(b+c)+(a+b+c)$.

These are all possibilities. In all cases but one exception, such an equality is written as $\alpha+\delta=\beta+\gamma$, such that $\beta-\alpha$ is a root and $\alpha+\gamma$ is not a root. So Formula 9.2 applies: $u^{(\varepsilon)}_{\beta,\gamma} = (u^{(\delta)}_{\beta-\alpha,\gamma}/u^{(\beta)}_{\alpha,\beta-\alpha})u^{(\varepsilon)}_{\alpha,\delta}$. This is relation of proportionality between $u_{\alpha,\delta}$ and $u_{\beta,\gamma}$, in terms of (rational) structure constants that were already imposed by induction.

The only case where this does not apply is in case C_3, the double equality $c + (a + 2b) = b + (a + b + c) = (a + b) + (b + c)$. In this case, choosing $(\alpha, \beta, \gamma, \delta) = (b, a + b, b + c, a + b + c)$ satisfies these conditions, so we have a given relation of proportionality between $u_{b,a+b+c}$ and $u_{a+b,b+c}$. Next, choosing $(\alpha, \beta, \gamma, \delta) = (c, b + c, a + b, a + 2b)$, we still have $\beta - \alpha$ a root, so we have the above formula, but in this case $\alpha + \gamma$ is indeed a root, and the relation (9.1) yields a linear combination with only nonzero (given, rational) coefficients between $u_{c,a+2b}$, $u_{a+b,b+c}$, and $u_{b,a+b+c}$. Combining these two combinations yields a relation of proportionality between each two of $u_{c,a+2b}$, $u_{a+b,b+c}$, and $u_{b,a+b+c}$ (with rational, already determined coefficients).

Hence $u_{\beta,\gamma}$ is well-determined, and rational, whenever $\beta, \gamma > 0$.

Now we have to check that the values of $u_{\alpha,\delta}$ for all pairs $(\alpha, \delta) \in P^+$, determine all other values, and generate the same subfield. It is enough to consider the case of irreducible roots systems in dimension 2.

We leave the case of G_2 to the reader (since G_2 does not embed into a larger connected Dynkin diagram, the proof remains complete in higher dimension). So we are in type A_2 or B_2.

We say that a nonzero root is large if it has maximal norm, and small if it has minimal norm (so in type A_2, all nonzero roots are both small and large).

It is convenient to renormalize: define $E_\alpha = e_\alpha$ for $\alpha > 0$ and $E_\alpha = \frac{\langle\alpha,\alpha\rangle}{2} e_\alpha$ for $\alpha < 0$: thus $[E_\alpha, E_{-\alpha}] = h'_\alpha$. Write $[E_\alpha, E_\beta] = v_{\alpha,\beta} = E_{\alpha+\beta}$ for $(\alpha, \beta) \in P$.

For α, β, γ nonzero such that $\alpha+\beta+\gamma = 0$, the Jacobi identity on $(E_\alpha, E_\beta, E_\gamma)$ yields $v_{\beta,\gamma}h'_\alpha + v_{\gamma,\alpha}h'_\beta + v_{\alpha,\beta}h'_\gamma = 0$, which yields

$$v_{\beta,\gamma} = v_{\gamma,\alpha} = v_{\alpha,\beta} \tag{9.3}$$

For α, β nonzero roots such that $\alpha - \beta$ is not a root and $\alpha + \beta$ is a nonzero root, the Jacobi identity on $(E_\alpha, E_{-\alpha}, E_\beta)$ yields the equality $v_{\alpha,\beta}v_{\alpha+\beta,-\alpha} = \langle\alpha, \beta\rangle$. Using that (9.3) gives $v_{\alpha+\beta,-\alpha} = v_{-\alpha,-\beta}$, we deduce $v_{\alpha,\beta}v_{-\alpha,-\beta} = \langle\alpha, \beta\rangle$.

Now let α_1, α_2 be arbitrary nonzero roots with $\alpha_1 \neq \pm\alpha_2$, and set $\alpha_3 = -\alpha_1 - \alpha_2$. Suppose (this is is automatic when we are not in type G_2) that there is i (i is counted modulo 3) such that $\alpha_i - \alpha_{i+1}$ is not a root. By (9.3) $v_{\alpha_j,\alpha_{j+1}}$ does not depend on j (j counted modulo 3), and $v_{-\alpha_j,-\alpha_{j+1}}$ does not depend on j. Writing $k = \langle\alpha_i, \alpha_{i+1}\rangle$, we have $v_{\alpha_i,\alpha_{i+1}}v_{-\alpha_i,-\alpha_{i+1}} = k$, and hence $v_{\alpha_j,\alpha_{j+1}}v_{-\alpha_j,-\alpha_{j+1}} = k$. Since at least one of the six pairs (α_j, α_{j+1}), $(-\alpha_j, -\alpha_{j+1})$ belongs to P^+, we see that v_{α_1,α_2} is determined by the restriction of v to P^+ (and is rational as soon as v is rational on P^+).

(In type G_2, it remains to consider α_1, α_2 small roots with negative scalar product.) □

Theorem 9.2. *Given a root system Φ, there exists a compatible Lie bracket (in the sense of the previous proposition). That is, there is a semisimple K-split Lie algebra whose root system is isometric (after rescaling on irreducible factors) to Φ.*

This is a long proof!

One approach is case-by-case, by computing roots in $\mathfrak{sl}_{n+1}$ (type A_n), $\mathfrak{so}_{2n}$ (type D_n), $\mathfrak{so}_{2n+1}$ (type B_n), and $\mathfrak{sp}(2n)$ (type C_n).

Then one has to provide tables in each of the exceptional types. More precisely, one has to specify the coefficient $u_{\alpha,\beta}$ for each pair (α, β) of nonzero roots such that $\alpha + \beta$ is a root. In the largest case, E_8, there are 240 roots and therefore about 30000 unordered pairs of distinct roots. The Jacobi relation has to be checked on each (unordered) triple of roots, and there are about 2000000 such triples. This is accessible with a good software; however using the symmetries of the root system (and a choice of entries with good symmetries), restricting to triples whose sum is also a root, drastically reduces these numbers (I'm not sure to which extent).

Another approach is uniform, and consists in defining the Lie algebra associated to Φ by a presentation by generators and relations. See Chapter 7 in Carter's book [**?**].

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