FINITELY PRESENTABLE, NON-HOPFIAN GROUPS WITH KAZHDAN'S PROPERTY (T) AND INFINITE OUTER AUTOMORPHISM GROUP

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Abstract. We give simple examples of Kazhdan groups with infinite outer automorphism groups. This answers a question of Paulin, independently answered by Ollivier and Wise by completely different methods. As arithmetic lattices in (non-semisimple) Lie groups, our examples are in addition finitely presented.

We also use results of Abels about compact presentability of p-adic groups to exhibit a finitely presented non-Hopfian Kazhdan group. This answers a question of Ollivier and Wise.

1. Introduction

Recall that a locally compact group is said to have Property (T) if every weakly continuous unitary representation with almost invariant vectors[1] has nonzero invariant vectors.

It was asked by Paulin in [HV, p.134] (1989) whether there exists a group with Kazhdan's Property (T) and with infinite outer automorphism group. This question remained unanswered until 2004; in particular, it is Question 18 in [Wo].

This question was motivated by the two following special cases. The first is the case of lattices in *semisimple* groups over local fields, which have long been considered as prototypical examples of groups with Property (T). If Γ is such a lattice, Mostow's rigidity Theorem and the fact that semisimple groups have finite outer automorphism group imply that $\mathrm{Out}(\Gamma)$ is finite. Secondly, a new source of groups with Property (T) appeared when Zuk [Zu] proved that certain models of random groups have Property (T). But they are also hyperbolic, and Paulin proved [Pa] that a hyperbolic group with Property (T) has finite outer automorphism group.

However, it turns out that various arithmetic lattices in appropriate *non-semisimple* groups provide examples. For instance, consider the additive group $\mathrm{Mat}_{mn}(\mathbf{Z})$ of $m \times n$ matrices over $\mathbf{Z}$, endowed with the action of $\mathrm{GL}_n(\mathbf{Z})$ by left multiplication.

Proposition 1.1. *For every $n \geq 3$, $m \geq 1$, $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{mn}(\mathbf{Z})$ is a finitely presented linear group, has Property (T), is non-coHopfian*[2]*, and its outer automorphism group contains a copy of $\mathrm{PGL}_m(\mathbf{Z})$, hence is infinite if $m \geq 2$.*

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[1] A representation $\pi : G \to \mathcal{U}(\mathcal{H})$ almost has invariant vectors if for every $\varepsilon > 0$ and every finite subset $F \subseteq G$, there exists a unit vector $\xi \in \mathcal{H}$ such that $\|\pi(g)\xi - \xi\| < \varepsilon$ for every $g \in F$.

[2] A group is coHopfian (resp. Hopfian) if it is isomorphic to no proper subgroup (resp. quotient) of itself.

We later learned that Ollivier and Wise [OW] had independently found examples of a very different nature. They embed any countable group G in $\mathrm{Out}(\Gamma)$, where Γ has Property (T), is a subgroup of a torsion-free hyperbolic group, satisfying a certain "graphical" small cancellation condition (see also [BS]). In contrast to our examples, theirs are not, a priori, finitely presented; on the other hand, our examples are certainly not subgroups of hyperbolic groups since they all contain a copy of $\mathbf{Z}^2$.

They also construct in [OW] a non-coHopfian group with Property (T) that embeds in a hyperbolic group. Proposition 1.1 actually answers two questions in their paper: namely, whether there exists a finitely presented group with Property (T) and without the coHopfian Property (resp. with infinite outer automorphism group).

Remark 1.2. Another example of non-coHopfian group with Property (T) is $\mathrm{PGL}_n(\mathbf{F}_p[X])$ when $n \geq 3$. This group is finitely presentable if $n \geq 4$ [RS] (but not for $n = 3$ [Be]). In contrast with the previous examples, the Frobenius morphism Fr induces an isomorphism onto a subgroup of *infinite* index, and the intersection $\bigcap_{k\geq 0} \mathrm{Im}(\mathrm{Fr}^k)$ is reduced to $\{1\}$.

Ollivier and Wise also constructed in [OW] the first examples of non-Hopfian groups with Property (T). They asked whether a finitely presented example exists. Although linear finitely generated groups are residually finite, hence Hopfian, we use them to answer positively their question.

Theorem 1.3. *There exists a S-arithmetic lattice Γ, and a central subgroup $Z \subset \Gamma$, such that Γ and Γ/Z are finitely presented, have Property (T), and Γ/Z is non-Hopfian.*

The group Γ has a simple description as a matrix group from which Property (T) and the non-Hopfian property for Γ/Z are easily checked (Proposition 2.7). Section 3 is devoted to prove finite presentability of Γ. We use here a general criterion for finite presentability of S-arithmetic groups, due to Abels [A2]. It involves the computation of the first and second cohomology group of a suitable Lie algebra.

2. Proofs of all results except finite presentability of Γ

We need some facts about Property (T).

Lemma 2.1 (see [HV, Chap. 3, Théorème 4])**.** *Let G be a locally compact group, and Γ a lattice in G. Then G has Property (T) if and only if Γ has Property (T).* $\square$

The next lemma is an immediate consequence of the classification of semisimple algebraic groups over local fields with Property (T) (see [Ma, Chap. III, Theorem 5.6]) and S. P. Wang's results on the non-semisimple case [Wa, Theorem 2.10].

Lemma 2.2. *Let $\mathbf{K}$ be a local field, G a connected linear algebraic group defined over $\mathbf{K}$. Suppose that G is perfect, and, for every simple quotient S of G, either S has $\mathbf{K}$-rank ≥ 2, or $\mathbf{K} = \mathbf{R}$ and S is isogeneous to either $\mathrm{Sp}(n, 1)$ $(n \geq 2)$ or $\mathrm{F}_{4(-20)}$. If $\mathrm{char}(\mathbf{K}) > 0$, suppose in addition that G has a Levi decomposition defined over $\mathbf{K}$. Then $G(\mathbf{K})$ has Property (T).* $\square$

Proof of Proposition 1.1. The group $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{mn}(\mathbf{Z})$ is linear in dimension $n+m$. As a semidirect product of two finitely presented groups, it is finitely presented.

For every $k \geq 2$, it is isomorphic to its proper subgroup $\mathrm{SL}_n(\mathbf{Z}) \ltimes k\mathrm{Mat}_{mn}(\mathbf{Z})$ of finite index k^{mn}.

The group $\mathrm{GL}_m(\mathbf{Z})$ acts on $\mathrm{Mat}_{mn}(\mathbf{Z})$ by right multiplication. Since this action commutes with the left multiplication of $\mathrm{SL}_n(\mathbf{Z})$, $\mathrm{GL}_m(\mathbf{Z})$ acts on the semidirect product $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{mn}(\mathbf{Z})$ by automorphisms, and, by an immediate verification, this gives an embedding of $\mathrm{GL}_m(\mathbf{Z})$ if n is odd or $\mathrm{PGL}_m(\mathbf{Z})$ if n is even into $\mathrm{Out}(\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{mn}(\mathbf{Z}))$ (it can be shown that this is an isomorphism if n is odd; if n is even, the image has index two). In particular, if $m \geq 2$, then $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{mn}(\mathbf{Z})$ has infinite outer automorphism group.

On the other hand, in view of Lemma 2.1, it has Property (T) (actually for all $m \geq 0$): indeed, $\mathrm{SL}_n(\mathbf{Z}) \ltimes \mathrm{Mat}_{mn}(\mathbf{Z})$ is a lattice in $\mathrm{SL}_n(\mathbf{R}) \ltimes \mathrm{Mat}_{mn}(\mathbf{R})$, which has Property (T) by Lemma 2.2 as $n \geq 3$. □

We now turn to the proof of Theorem 1.3. The following lemma is immediate, and already used in [Ha] and [A1].

Lemma 2.3. *Let Γ be a group, Z a central subgroup. Let α be an automorphism of Γ such that $\alpha(Z)$ is a proper subgroup of Z. Then α induces a surjective, non-injective endomorphism of Γ/Z, whose kernel is $\alpha^{-1}(Z)/Z$.* □

Definition 2.4. Fix $n_1, n_2, n_3, n_4 \in \mathbf{N} - \{0\}$ with $n_2, n_3 \geq 3$. We set $\Gamma = G(\mathbf{Z}[1/p])$, where p is any prime, and G is algebraic the group defined as matrices by blocks of size n_1, n_2, n_3, n_4:

$$\begin{pmatrix} I_{n_1} & (*)_{12} & (*)_{13} & (*)_{14} \\ 0 & (**)_{22} & (*)_{23} & (*)_{24} \\ 0 & 0 & (**)_{33} & (*)_{34} \\ 0 & 0 & 0 & I_{n_4} \end{pmatrix},$$

where $(*)$ denote any matrices and $(**)_{ii}$ denote matrices in SL_{n_i}, $i = 2, 3$.

The centre of G consists of matrices of the form $\begin{pmatrix} I_{n_1} & 0 & 0 & (*)_{14} \\ 0 & I_{n_2} & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & I_{n_4} \end{pmatrix}$. Define Z as the centre of $G(\mathbf{Z})$.

Remark 2.5. This group is related to an example of Abels: in [A1] he considers the same group, but with blocks 1×1, and GL_1 instead of SL_1 in the diagonal. Taking the points over $\mathbf{Z}[1/p]$, and taking the quotient by a cyclic subgroup if the centre, this provided the first example of finitely presentable non-Hopfian solvable group.

Remark 2.6. If we do not care about finite presentability, we can take $n_3 = 0$ (i.e. 3 blocks suffice).

We begin by easy observations. Identify GL_{n_1} to the upper left diagonal block. It acts by *conjugation* on G as follows:

$$\begin{pmatrix} u & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \cdot \begin{pmatrix} I & A_{12} & A_{13} & A_{14} \\ 0 & B_2 & A_{23} & A_{24} \\ 0 & 0 & B_3 & A_{34} \\ 0 & 0 & 0 & I \end{pmatrix} = \begin{pmatrix} I & uA_{12} & uA_{13} & uA_{14} \\ 0 & B_2 & A_{23} & A_{24} \\ 0 & 0 & B_3 & A_{34} \\ 0 & 0 & 0 & I \end{pmatrix}.$$

This gives an action of GL_{n_1} on G, and also on its centre, and this latter action is faithful. In particular, for every commutative ring R, $\mathrm{GL}_{n_1}(R)$ embeds in $\mathrm{Out}(G(R))$.

From now on, we suppose that $R = \mathbf{Z}[1/p]$, and $u = pI_{n_1}$. The automorphism of $\Gamma = G(\mathbf{Z}[1/p])$ induced by u maps Z to its proper subgroup Z^p. In view of Lemma 2.3, this implies that Γ/Z is non-Hopfian.

Proposition 2.7. *The groups Γ and Γ/Z are finitely generated, have Property (T), and Γ/Z is non-Hopfian.*

Proof. We have just proved that Γ/Z is non-Hopfian. By the Borel-Harish-Chandra Theorem [BHC], Γ is a lattice in $G(\mathbf{R}) \times G(\mathbf{Q}_p)$. This group has Property (T) as a consequence of Lemma 2.2. By Lemma 2.1, Γ also has Property (T). Finite generation is a consequence of Property (T) [HV, Lemme 10]. Since Property (T) is (trivially) inherited by quotients, Γ/Z also has Property (T). $\square$

Remark 2.8. This group has a surjective endomorphism with nontrivial finite kernel. We have no analogous example with infinite kernel. Such examples might be constructed if we could prove that some groups over rings of dimension ≥ 2 such as $\mathrm{SL}_n(\mathbf{Z}[X])$ or $\mathrm{SL}_n(\mathbf{F}_p[X,Y])$ have Property (T), but this is an open problem [Sh]. The non-Hopfian Kazhdan group of Ollivier and Wise [OW] is torsion-free, so the kernel is infinite in their case.

Remark 2.9. It is easy to check that $\mathrm{GL}_{n_1}(\mathbf{Z}) \times \mathrm{GL}_{n_4}(\mathbf{Z})$ embeds in $\mathrm{Out}(\Gamma)$ and $\mathrm{Out}(\Gamma/Z)$. In particular, if $\max(n_1, n_2) \geq 2$, then these outer automorphism groups are infinite.

We finish this section by observing that Z is a finitely generated subgroup of the centre of Γ, so that finite presentability of Γ/Z immediately follows from that of Γ.

3. Finite presentability of Γ

We recall that a Hausdorff topological group H is *compactly presented* if there exists a compact generating subset C of H such that the abstract group H is the quotient of the group freely generated by C by relations of bounded length. See [A2, §1.1] for more about compact presentability.

Kneser [Kn] has proved that for every linear algebraic $\mathbf{Q}_p$-group, the S-arithmetic lattice $G(\mathbf{Z}[1/p])$ is finitely presented if and only if $G(\mathbf{Q}_p)$ is compactly presented. A characterization of the linear algebraic $\mathbf{Q}_p$-groups G such that $G(\mathbf{Q}_p)$ compactly presented is given in [A2]. This criterion requires the study of a solvable cocompact subgroup of $G(\mathbf{Q}_p)$, which seems hard to carry out in our specific example.

Let us describe another sufficient criterion for compact presentability, also given in [A2], which is applicable to our example. Let U be the unipotent radical in G, and let S denote a Levi factor defined over $\mathbf{Q}_p$, so that $G = S \ltimes U$. Let $\mathfrak{u}$ be the Lie algebra of U, and D be a maximal $\mathbf{Q}_p$-split torus in S. We recall that the first homology group of $\mathfrak{u}$ is defined as the abelianization

$$H_1(\mathfrak{u}) = \mathfrak{u}/[\mathfrak{u}, \mathfrak{u}],$$

and the second homology group of $\mathfrak{u}$ is defined as $\mathrm{Ker}(d_2)/\mathrm{Im}(d_3)$, where the maps

$$\mathfrak{u} \wedge \mathfrak{u} \wedge \mathfrak{u} \stackrel{d_3}{\to} \mathfrak{u} \wedge \mathfrak{u} \stackrel{d_2}{\to} \mathfrak{u}$$

are defined by:

$$d_2(x_1 \wedge x_2) = -[x_1, x_2] \text{ and } d_3(x_1 \wedge x_2 \wedge x_3) = x_3 \wedge [x_1, x_2] + x_2 \wedge [x_3, x_1] + x_1 \wedge [x_2, x_3].$$

We can now state the result by Abels that we use (see [A2, Theorem 6.4.3 and Remark 6.4.5]).

Theorem 3.1. *Let G be a connected linear algebraic group over $\mathbf{Q}_p$. Suppose that the following assumptions are fulfilled:*

(i) *G is $\mathbf{Q}_p$-split.*
(ii) *G has no simple quotient of $\mathbf{Q}_p$-rank one.*
(iii) *0 does not lie on the segment joining two dominant weights for the adjoint representation of S on $H_1(\mathfrak{u})$.*
(iv) *0 is not a dominant weight for an irreducible subrepresentation of the adjoint representation of S on $H_2(\mathfrak{u})$.*

Then $G(\mathbf{Q}_p)$ is compactly presented. □

We now return to our particular example of G, observe that it is clearly $\mathbf{Q}_p$-split, and that its simple quotients are SL_{n_2} and SL_{n_3}, which have rank greater than one. Keep the previous notations S, D, U, $\mathfrak{u}$, so that S (resp. D) denoting in our case the diagonal by blocks (resp. diagonal) matrices in G, and U denotes the matrices in G all of whose diagonal blocks are the identity. The set of indices of the matrix is partitioned as $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$, with $|I_j| = n_j$ as in Definition 2.4. It follows that, for every field K,

$$\mathfrak{u}(K) = \left\{ T \in \mathrm{End}(K^I),\ \forall j,\ T(K^{I_j}) \subset \bigoplus_{i<j} K^{I_i} \right\}.$$

Throughout, we use the following notation: a letter such as i_k (or j_k, etc.) implicitly means $i_k \in I_k$. Define, in an obvious way, subgroups U_{ij}, $i<j$, of U, and their Lie algebras $\mathfrak{u}_{ij}$.

We begin by checking Condition (iii) of Theorem 3.1.

Lemma 3.2. *For any two weights of the action of D on $H_1(\mathfrak{u})$, 0 is not on the segment joining them.*

Proof. Recall that $H_1(\mathfrak{u}) = \mathfrak{u}/[\mathfrak{u},\mathfrak{u}]$. So it suffices to look at the action on the supplement D-subspace $\mathfrak{u}_{12} \oplus \mathfrak{u}_{23} \oplus \mathfrak{u}_{34}$ of $[\mathfrak{u},\mathfrak{u}]$. Identifying S with $\mathrm{SL}_{n_2} \times \mathrm{SL}_{n_3}$, we denote (A,B) an element of $D \subset S$. We also denote by e_{pq} the matrix whose coefficient (p,q) equals one and all others are zero.

$$(A,B)\cdot e_{i_1j_2} = a_{j_2}^{-1} e_{i_1j_2}, \quad (A,B)\cdot e_{j_2k_3} = a_{j_2} b_{k_3}^{-1} e_{j_2k_3}, \quad (A,B)\cdot e_{k_3\ell_4} = b_{k_3} e_{k_3\ell_4}.$$

Since $S = \mathrm{SL}_{n_2} \times \mathrm{SL}_{n_3}$, the weights for the adjoint action on $\mathfrak{u}_{12} \oplus \mathfrak{u}_{23} \oplus \mathfrak{u}_{34}$ live in M/P, where M is the free $\mathbf{Z}$-module of rank n_2+n_3 with basis $(u_1,\dots,u_{n_2},v_1,\dots,v_{n_3})$, and P is the plane generated by $\sum_{j_2} u_{j_2}$ and $\sum_{k_3} v_{k_3}$. Thus, the weights are (modulo P) $-u_{j_2}$, $u_{j_2}-v_{k_3}$, v_{k_3} ($1 \le j_2 \le n_2$, $1 \le k_3 \le n_3$).

Using that $n_2, n_3 \ge 3$, it is clear that no nontrivial positive combination of two weights (viewed as elements of $\mathbf{Z}^{n_2+n_3}$) lies in P. □

We must now check Condition (iv) of Theorem 3.1, and therefore compute $H_2(\mathfrak{u})$ as a D-module.

Lemma 3.3. $\mathrm{Ker}(d_2)$ *is generated by*

(1) $\mathfrak{u}_{12} \wedge \mathfrak{u}_{12}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{34} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{13}$, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{34}$.

(2) $\mathfrak{u}_{14} \wedge \mathfrak{u}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$.
(3) $e_{i_1j_2} \wedge e_{k_2\ell_3}$ $(j_2 \neq k_2)$, $e_{i_2j_3} \wedge e_{k_3\ell_4}$ $(j_3 \neq \ell_3)$.
(4) $e_{i_1j_2} \wedge e_{k_2\ell_4}$ $(j_2 \neq k_2)$, $e_{i_1j_3} \wedge e_{k_3\ell_4}$ $(j_3 \neq k_3)$.
(5) *Elements of the form* $\sum_{j_2} \alpha_{j_2}(e_{i_1j_2} \wedge e_{j_2k_3})$ *if* $\sum_{j_2} \alpha_{j_2} = 0$, *and* $\sum_{j_3} \alpha_{j_3}(e_{i_2j_3} \wedge e_{j_3k_4})$ *if* $\sum_{j_3} \alpha_{j_3} = 0$.
(6) *Elements of the form* $\sum_{j_2} \alpha_{j_2}(e_{i_1j_2} \wedge e_{j_2k_4}) + \sum_{j_3} \beta_{j_3}(e_{i_1j_3} \wedge e_{j_3k_4})$ *if* $\sum_{j_2} \alpha_{j_2} + \sum_{j_3} \beta_{j_3} = 0$.

Proof. First observe that $\mathrm{Ker}(d_2)$ contains $\mathfrak{u}_{ij} \wedge \mathfrak{u}_{kl}$ when $[\mathfrak{u}_{ij}, \mathfrak{u}_{kl}] = 0$. This corresponds to (1) and (2). The remaining cases are $\mathfrak{u}_{12} \wedge \mathfrak{u}_{23}$, $\mathfrak{u}_{23} \wedge \mathfrak{u}_{34}$, $\mathfrak{u}_{12} \wedge \mathfrak{u}_{24}$, $\mathfrak{u}_{13} \wedge \mathfrak{u}_{34}$.

On the one hand, $\mathrm{Ker}(d_2)$ also contains $e_{i_1j_2} \wedge e_{k_2\ell_3}$ if $j_2 \neq k_2$, etc.; this corresponds to elements in (3), (4). On the other hand, $d_2(e_{i_1j_2} \wedge e_{j_2k_3}) = -e_{i_1k_3}$, $d_2(e_{i_2j_3} \wedge e_{j_3k_4}) = -e_{i_2k_4}$, $d_2(e_{i_1j_2} \wedge e_{j_2k_4}) = -e_{i_1k_4}$, $d_2(e_{i_1j_3} \wedge e_{j_3k_4}) = -e_{i_1k_4}$. The lemma follows. □

Definition 3.4. Denote by $\mathfrak{b}$ (resp. $\mathfrak{h}$) the subspace generated by elements in (2), (4), and (6) (resp. in (1), (3), and (5)) of Lemma 3.3.

Proposition 3.5. $\mathrm{Im}(d_3) = \mathfrak{b}$, *and* $\mathrm{Ker}(d_2) = \mathfrak{b} \oplus \mathfrak{h}$ *as D-module. In particular, $H_2(\mathfrak{u})$ is isomorphic to $\mathfrak{h}$ as a D-module.*

Proof. We first prove, in a series of facts, that $\mathrm{Im}(d_3) \supset \mathfrak{b}$.

Fact. $\mathfrak{u}_{14} \wedge \mathfrak{u}$ is contained in $\mathrm{Im}(d_3)$.

Proof. If $z \in \mathfrak{u}_{14}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. This already shows that $\mathfrak{u}_{14} \wedge (\mathfrak{u}_{13} \oplus \mathfrak{u}_{24} \oplus \mathfrak{u}_{14})$ is contained in $\mathrm{Im}(d_3)$, since $[\mathfrak{u}, \mathfrak{u}] = \mathfrak{u}_{13} \oplus \mathfrak{u}_{24} \oplus \mathfrak{u}_{14}$.

Now, if $(x, y, z) \in \mathfrak{u}_{24} \times \mathfrak{u}_{12} \times \mathfrak{u}_{34}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. Since $[\mathfrak{u}_{24}, \mathfrak{u}_{12}] = \mathfrak{u}_{14}$, this implies that $\mathfrak{u}_{14} \wedge \mathfrak{u}_{34} \subset \mathrm{Im}(d_3)$. Similarly, $\mathfrak{u}_{14} \wedge \mathfrak{u}_{12} \subset \mathrm{Im}(d_3)$.

Finally we must prove that $\mathfrak{u}_{14} \wedge \mathfrak{u}_{23} \subset \mathrm{Im}(d_3)$. This follows from the formula $e_{i_1j_4} \wedge e_{k_2\ell_3} = d_3(e_{i_1m_2} \wedge e_{k_2\ell_3} \wedge e_{m_2j_4})$, where $m_2 \neq k_2$ (so that we use that $|I_2| \geq 2$). □

Fact. $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$ and, similarly, $\mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$, are contained in $\mathrm{Im}(d_3)$.

Proof. If $(x, y, z) \in \mathfrak{u}_{12} \times \mathfrak{u}_{23} \times \mathfrak{u}_{13}$, then $d_3(x \wedge y \wedge z) = z \wedge [x, y]$. Since $[\mathfrak{u}_{12}, \mathfrak{u}_{23}] = \mathfrak{u}_{13}$, this implies that $\mathfrak{u}_{13} \wedge \mathfrak{u}_{13} \subset \mathrm{Im}(d_3)$. □

Fact. $\mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$ is contained in $\mathrm{Im}(d_3)$.

Proof. $d_3(e_{i_1k_2} \wedge e_{k_2\ell_3} \wedge e_{k_2j_4}) = e_{k_2j_4} \wedge e_{i_1\ell_3} + e_{i_1j_4} \wedge e_{k_2\ell_3}$. Since we already know that $e_{i_1j_4} \wedge e_{k_2\ell_3} \in \mathrm{Im}(d_3)$, this implies $e_{k_2j_4} \wedge e_{i_1\ell_3} \in \mathrm{Im}(d_3)$. □

Fact. The elements in (4) are in $\mathrm{Im}(d_3)$.

Proof. $d_3(e_{i_1j_2} \wedge e_{j_2k_3} \wedge e_{\ell_3m_4}) = -e_{i_1k_3} \wedge e_{\ell_3m_4}$ if $k_3 \neq \ell_3$. The other case is similar. □

Fact. The elements in (6) are in $\mathrm{Im}(d_3)$.

Proof. $d_3(e_{i_1j_2} \wedge e_{j_2k_3} \wedge e_{k_3\ell_4}) = -e_{i_1k_3} \wedge e_{k_3\ell_4} + e_{i_1j_2} \wedge e_{j_2\ell_4}$. Such elements generate all elements as in (6). □

Conversely, we must check $\mathrm{Im}(d_3) \subset \mathfrak{b}$. By straightforward verifications:

- $d_3(\mathfrak{u}_{14} \wedge \mathfrak{u} \wedge \mathfrak{u}) \subset \mathfrak{u}_{14} \wedge \mathfrak{u}$.

- $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{24}) = 0$
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{24})$, $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34})$, $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{34})$, $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34})$ are all contained in $\mathfrak{u}_{14} \wedge \mathfrak{u}$.
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{13} \wedge \mathfrak{u}_{23}) \subset \mathfrak{u}_{13} \wedge \mathfrak{u}_{13}$, and similarly $d_3(\mathfrak{u}_{23} \wedge \mathfrak{u}_{24} \wedge \mathfrak{u}_{34}) \subset \mathfrak{u}_{24} \wedge \mathfrak{u}_{24}$.
- $d_3(\mathfrak{u}_{12} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{24})$ and similarly $d_3(\mathfrak{u}_{13} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{34})$ are contained in $\mathfrak{u}_{14} \wedge \mathfrak{u}_{23} + \mathfrak{u}_{13} \wedge \mathfrak{u}_{24}$.
- The only remaining case is that of $\mathfrak{u}_{12} \wedge \mathfrak{u}_{23} \wedge \mathfrak{u}_{34}$: $d_3(e_{i_1 j_2} \wedge e_{j_2' k_3} \wedge e_{k_3' \ell_4}) = \delta_{k_3 k_3'} e_{i_1 j_2} \wedge e_{j_2' \ell_4} - \delta_{j_2 j_2'} e_{i_1 k_3} \wedge e_{k_3' \ell_4}$, which lies in (4) or in (6).

Finally $\mathrm{Im}(d_3) = \mathfrak{b}$.

It follows from Lemma 3.3 that $\mathrm{Ker}(d_2) = \mathfrak{h} \oplus \mathfrak{b}$. Since $\mathfrak{b} = \mathrm{Im}(d_3)$, this is a D-submodule. Let us check that $\mathfrak{h}$ is also a D-submodule; the computation will be used in the sequel.

The action of S on $\mathfrak{u}$ by *conjugation* is given by:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & B & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & X_{12} & X_{13} & X_{14} \\ 0 & 0 & X_{23} & X_{24} \\ 0 & 0 & 0 & X_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & X_{12}A^{-1} & X_{13}B^{-1} & X_{14} \\ 0 & 0 & AX_{23}B^{-1} & AX_{24} \\ 0 & 0 & 0 & BX_{34} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We must look at the action of D on the elements in (1), (3), and (5). We fix $(A, B) \in D \subset S \simeq \mathrm{SL}_{n_2} \times \mathrm{SL}_{n_3}$, and we write $A = \sum_{j_2} a_{j_2} e_{j_2 j_2}$ and $B = \sum_{k_3} b_{k_3} e_{k_3 k_3}$.

- (1):

$$(A,B) \cdot e_{i_1 j_2} \wedge e_{k_1 \ell_2} = e_{i_1 j_2} A^{-1} \wedge e_{k_1 \ell_2} A^{-1} = a_{j_2}^{-1} a_{\ell_2}^{-1} e_{i_1 j_2} \wedge e_{k_1 \ell_2}. \tag{3.1}$$

  The action on other elements in (1) has a similar form.
- (3) ($j_2 \neq k_2$):

$$(A,B) \cdot e_{i_1 j_2} \wedge e_{k_2 \ell_3} = e_{i_1 j_2} A^{-1} \wedge A e_{k_2 \ell_4} B^{-1} = a_{j_2}^{-1} a_{k_2} b_{\ell_3}^{-1} e_{i_1 j_2} \wedge e_{k_2 \ell_3}. \tag{3.2}$$

  The action on the other elements in (3) has a similar form.
- (5) ($\sum_{j_2} \alpha_{j_2} = 0$)

$$\begin{aligned} (A,B) \cdot \sum_{j_2} \alpha_{j_2} (e_{i_1 j_2} \wedge e_{j_2 k_3}) &= \sum_{j_2} \alpha_{j_2} (e_{i_1 j_2} A^{-1} \wedge A e_{j_2 k_3} B^{-1}) \\ = \sum_{j_2} \alpha_{j_2} a_{j_2}^{-1} (e_{i_1 j_2} \wedge a_{j_2} b_{k_3}^{-1} e_{j_2 k_3}) &= b_{k_3}^{-1} \left(\sum_{j_2} \alpha_{j_2} (e_{i_1 j_2} \wedge e_{j_2 k_3}) \right). \end{aligned} \tag{3.3}$$

  The other case in (5) has a similar form. $\square$

Lemma 3.6. *0 is not a weight for the action of D on $H_2(\mathfrak{u})$.*

Proof. As described in the proof of Lemma 3.2, we think of weights as elements of M/P. Hence, we describe weights as elements of $M = \mathbf{Z}^{n_2+n_3}$ rather than M/P, and must check that no weight lies in P.

1. In (3.1), the weight is $-u_{j_2} - u_{\ell_2}$, hence does not belong to P since $n_2 \geq 3$. The other verifications are similar.
3. In (3.2), the weight is $-u_{j_2} + u_{k_2} - v_{\ell_3}$, hence does not belong to P. The other verification for (3) is similar.

(5) In (3.3), the weight is $-v_{k_3}$, hence does dot belong to P. The other verification is similar. □

Finally, Lemmas 3.2 and 3.6 imply that the conditions of Theorem 3.1 are satisfied, so that Γ is finitely presented. □

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