Finitely presentable, non-hopfian groups with Kazhdan’s Property (T) and infinite outer automorphism group

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Abstract. We give simple examples of Kazhdan groups with infinite outer automorphism groups. This answers a question of Paulin, independently answered by Ollivier and Wise by completely different methods. As arithmetic lattices in (non-semisimple) Lie groups, our examples are in addition finitely presented.

We also use results of Abels about compact presentability of $p$-adic groups to exhibit a finitely presented non-Hopfian Kazhdan group. This answers a question of Ollivier and Wise.

1. Introduction

Recall that a locally compact group is said to have Property (T) if every weakly continuous unitary representation with almost invariant vectors has nonzero invariant vectors.

It was asked by Paulin in [HV, p.134] (1989) whether there exists a group with Kazhdan’s Property (T) and with infinite outer automorphism group. This question remained unanswered until 2004; in particular, it is Question 18 in [Wo].

This question was motivated by the two following special cases. The first is the case of lattices in semisimple groups over local fields, which have long been considered as prototypical examples of groups with Property (T). If $\Gamma$ is such a lattice, Mostow’s rigidity Theorem and the fact that semisimple groups have finite outer automorphism group imply that Out($\Gamma$) is finite. Secondly, a new source of groups with Property (T) appeared when Zuk [Zu] proved that certain models of random groups have Property (T). But they are also hyperbolic, and Paulin proved [Pa] that a hyperbolic group with Property (T) has a finite outer automorphism group.

However, it turns out that various arithmetic lattices in appropriate non-semisimple groups provide examples. For instance, consider the additive group $\text{Mat}_{m\times n}(\mathbb{Z})$ of $m \times n$ matrices over $\mathbb{Z}$, endowed with the action of $\text{GL}_n(\mathbb{Z})$ by left multiplication.

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1 A representation $\pi : G \to \mathcal{U}(\mathcal{H})$ almost has invariant vectors if for every $\varepsilon > 0$ and every finite subset $F \subseteq G$, there exists a unit vector $\xi \in \mathcal{H}$ such that $\|\pi(g)\xi - \xi\| < \varepsilon$ for every $g \in F$.
Proposition 1.1. For every \( n \geq 3, m \geq 1 \), \( \text{SL}_n(\mathbb{Z}) \ltimes \text{Mat}_{m \times n}(\mathbb{Z}) \) is a finitely presented linear group, has Property (T), is non-coHopfian\(^2\), and its outer automorphism group contains a copy of \( \text{PGL}_m(\mathbb{Z}) \), hence is infinite if \( m \geq 2 \).

We later learned that Ollivier and Wise [OW] had independently found examples of a very different nature. They embed any countable group \( G \) in \( \text{Out}(\Gamma) \), where \( \Gamma \) has Property (T), is a subgroup of a torsion-free hyperbolic group, satisfying a certain “graphical” small cancelation condition (see also [BS]). In contrast to our examples, theirs are not, a priori, finitely presented; on the other hand, our examples are certainly not subgroups of hyperbolic groups since they all contain a copy of \( \mathbb{Z}^2 \).

They also construct in [OW] a non-coHopfian group with Property (T) that embeds in a hyperbolic group. Proposition 1.1 actually answers two questions in their paper: namely, whether there exists a finitely presented group with Property (T) and without the coHopfian Property (resp. with infinite outer automorphism group).

Remark 1.2. Another example of a non-coHopfian group with Property (T) is \( \text{PGL}_n(\mathbb{F}_p[X]) \) when \( n \geq 3 \). This group is finitely presentable if \( n \geq 4 \) [RS] (but not for \( n = 3 \) [Be]). In contrast with the previous examples, the Frobenius morphism \( \text{Fr} \) induces an isomorphism onto a subgroup of infinite index, and the intersection \( \bigcap_{k \geq 0} \text{Im}(\text{Fr}^k) \) is reduced to \( \{1\} \).

Ollivier and Wise also constructed in [OW] the first examples of non-Hopfian groups with Property (T). They asked whether a finitely presented example exists. Although linear finitely generated groups are residually finite, hence Hopfian, we use them to positively answer their question.

Theorem 1.3. There exists a \( S \)-arithmetic lattice \( \Gamma \), and a central subgroup \( Z \subset \Gamma \), such that \( \Gamma \) and \( \Gamma/Z \) are finitely presented, have Property (T), and \( \Gamma/Z \) is non-Hopfian.

The group \( \Gamma \) has a simple description as a matrix group from which Property (T) and the non-Hopfian property for \( \Gamma/Z \) are easily checked (Proposition 2.7). Section 3 is devoted to prove finite presentability of \( \Gamma \). We use here a general criterion for finite presentability of \( S \)-arithmetic groups, due to Abels [A2]. It involves the computation of the first and second cohomology group of a suitable Lie algebra.

2. Proofs of all results except finite presentability of \( \Gamma \)

We need some facts about Property (T).

Lemma 2.1 (see [HV, Chap. 3, Théorème 4]). Let \( G \) be a locally compact group, and \( \Gamma \) a lattice in \( G \). Then \( G \) has Property (T) if and only if \( \Gamma \) has Property (T). \( \square \)

The next lemma is an immediate consequence of the classification of semisimple algebraic groups over local fields with Property (T) (see [Ma, Chap. III, Théorème 5.6]) and S. P. Wang’s results on the non-semisimple case [Wa, Theorem 2.10].

Lemma 2.2. Let \( K \) be a local field, \( G \) a connected linear algebraic group defined over \( K \). Suppose that \( G \) is perfect, and, for every simple quotient \( S \) of \( G \), either

\(^2A\) group is coHopfian (resp. Hopfian) if it is isomorphic to no proper subgroup (resp. quotient) of itself.
Proof of Proposition 1.1. The group $\text{SL}_n(K)$ defined over $k$ has Property (T) by Lemma 2.2 as an injective endomorphism of $\Gamma$ commutes with the left multiplication of $\text{SL}_n(K)$ has Property (T) by Lemma 2.2 as an injective endomorphism of $\Gamma$. Then $G(K)$ has Property (T).

As a semidirect product of two finitely presented groups, it is finitely presented. For every $F$ or $k$ has $S$ is isogeneous to either $\text{Sp}(n,1)$ or $\text{Sp}(1,-20)$. If $\text{char}(K) > 0$, suppose in addition that $G$ has a Levi decomposition defined over $K$. Then $G(K)$ has Property (T).

We now turn to the proof of Theorem 1.3. The following lemma is immediate, and already used in [Ha, Th. 4(iii)] and [A1].

Lemma 2.3. Let $\Gamma$ be a group, $Z$ a central subgroup. Let $\alpha$ be an automorphism of $\Gamma$ such that $\alpha(Z)$ is a proper subgroup of $Z$. Then $\alpha$ induces a surjective, non-injective endomorphism of $\Gamma/Z$, whose kernel is $\alpha^{-1}(Z)/Z$.

Definition 2.4. Fix $n_1, n_2, n_3, n_4 \in \mathbb{N} - \{0\}$ with $n_2, n_3 \geq 3$. We set $\Gamma = G(\mathbb{Z}[1/p])$, where $p$ is any prime, and $G$ is algebraic the group defined as matrices by blocks of size $n_1, n_2, n_3, n_4$:

$$\begin{pmatrix} I_{n_1} & (*)&12 & (*)&13 & (*)&14 \\
0 & (*)&22 & (*)&23 & (*)&24 \\
0 & 0 & (*)&33 & (*)&34 \\
0 & 0 & 0 & I_{n_4} \end{pmatrix},$$

where $(*)$ denote any matrices and $(**)_ii$ denote matrices in $\text{SL}_{n_i}$, $i = 2, 3$.

The centre of $G$ consists of matrices of the form

$$\begin{pmatrix} I_{n_1} & 0 & 0 & (*)&14 \\
0 & I_{n_2} & 0 & 0 \\
0 & 0 & I_{n_3} & 0 \\
0 & 0 & 0 & I_{n_4} \end{pmatrix}.$$  Define $Z$ as the centre of $G(\mathbb{Z})$.

Remark 2.5. This group is related to an example of Abels: in [A1] he considers the same group, but with blocks $1 \times 1$, and $\text{GL}_1$ instead of $\text{SL}_1$ in the diagonal. Taking the points over $\mathbb{Z}[1/p]$, and taking the quotient by a cyclic subgroup if the centre, this provided the first example of a finitely presentable non-Hopfian solvable group.

Remark 2.6. If we do not care about finite presentability, we can take $n_3 = 0$ (i.e. 3 blocks suffice), as in P. Hall’s original solvable example [Ha, Th. 4(iii)].
We begin by easy observations. Identify $GL_n$ to the upper left diagonal block. It acts by conjugation on $G$ as follows:
\[
\begin{pmatrix}
  u & 0 & 0 & 0 \\
  0 & I & 0 & 0 \\
  0 & 0 & I & 0 \\
  0 & 0 & 0 & I
\end{pmatrix}
\cdot
\begin{pmatrix}
  I & A_{12} & A_{13} & A_{14} \\
  0 & B_2 & A_{23} & A_{24} \\
  0 & 0 & B_3 & A_{34} \\
  0 & 0 & 0 & I
\end{pmatrix}
= \begin{pmatrix}
  I & uA_{12} & uA_{13} & uA_{14} \\
  0 & B_2 & A_{23} & A_{24} \\
  0 & 0 & B_3 & A_{34} \\
  0 & 0 & 0 & I
\end{pmatrix}.
\]

This gives an action of $GL_n$ on $G$, and also on its center, and this latter action is faithful. In particular, for every commutative ring $R$, $GL_n(R)$ embeds in $Out(G(R))$.

From now on, we suppose that $R = \mathbb{Z}[1/p]$, and $u = pI_n$. The automorphism of $\Gamma = G(\mathbb{Z}[1/p])$ induced by $u$ maps $Z$ to its proper subgroup $Z^p$. In view of Lemma 2.3, this implies that $\Gamma/Z$ is non-Hopfian.

**Proposition 2.7.** The groups $\Gamma$ and $\Gamma/Z$ are finitely generated, have Property (T), and $\Gamma/Z$ is non-Hopfian.

**Proof.** We have just proved that $\Gamma/Z$ is non-Hopfian. By the Borel-Harish-Chandra Theorem [BHC], $\Gamma$ is a lattice in $G(\mathbb{R}) \times G(\mathbb{Q}_p)$. This group has Property (T) as a consequence of Lemma 2.2. By Lemma 2.1, $\Gamma$ also has Property (T). Finite generation is a consequence of Property (T) [HV, Lemme 10]. Since Property (T) is (trivially) inherited by quotients, $\Gamma/Z$ also has Property (T). $\square$

**Remark 2.8.** This group has a surjective endomorphism with nontrivial finite kernel. We have no analogous example with infinite kernel. Such examples might be constructed if we could prove that some groups over rings of dimension $\geq 2$ such as $SL_n(\mathbb{Z}[X])$ or $SL_n(\mathbb{F}_p[X,Y])$ have Property (T), but this is an open problem [Sh]. The non-Hopfian Kazhdan group of Ollivier and Wise [OW] is torsion-free, so the kernel is infinite in their case.

**Remark 2.9.** It is easy to check that $GL_{n_1}(\mathbb{Z}) \times GL_{n_4}(\mathbb{Z})$ embeds in $Out(\Gamma)$ and $Out(\Gamma/Z)$. In particular, if $\max(n_1, n_4) \geq 2$, then these outer automorphism groups are infinite.

We finish this section by observing that $Z$ is a finitely generated subgroup of the centre of $\Gamma$, so that finite presentability of $\Gamma/Z$ immediately follows from that of $\Gamma$.

3. **Finite presentability of $\Gamma$**

We recall that a Hausdorff topological group $H$ is compactly presented if there exists a compact generating subset $C$ of $H$ such that the abstract group $H$ is the quotient of the group freely generated by $C$ by relations of bounded length. See Abels [A2, §1.1] for more about compact presentability.

Kneser [Kn] has proved that for every linear algebraic $\mathbb{Q}_p$-group, the $S$-arithmetic lattice $G(\mathbb{Z}[1/p])$ is finitely presented if and only if $G(\mathbb{Q}_p)$ is compactly presented. A characterization of the linear algebraic $\mathbb{Q}_p$-groups $G$ such that $G(\mathbb{Q}_p)$ is compactly presented is given in [A2]. This criterion requires the study of a solvable cocompact subgroup of $G(\mathbb{Q}_p)$, which seems tedious to carry out in our specific example.

Let us describe another sufficient criterion for compact presentability, also given in [A2], which is applicable to our example. Let $U$ be the unipotent radical in $G$, and let $S$ denote a Levi factor defined over $\mathbb{Q}_p$, so that $G = S \ltimes U$. Let $u$ be the
Lie algebra of $U$, and $D$ be a maximal $\mathbb{Q}_p$-split torus in $S$. We recall that the first homology group of $u$ is defined as the abelianization

$$H_1(u) = u/[u,u],$$

and the second homology group of $u$ is defined as $\text{Ker}(d_2)/\text{Im}(d_3)$, where the maps

$$u \wedge u \wedge u \xrightarrow{d_3} u \wedge u \xrightarrow{d_2} u$$

are defined by:

$$d_2(x_1 \wedge x_2) = -[x_1,x_2] \quad \text{and} \quad d_3(x_1 \wedge x_2 \wedge x_3) = x_3 \wedge [x_1,x_2] + x_2 \wedge [x_3,x_1] + x_1 \wedge [x_2,x_3].$$

We can now state the result by Abels that we use (see [A2, Theorem 6.4.3 and Remark 6.4.5]).

**Theorem 3.1 (Abels).**

Let $G$ be a connected linear algebraic group over $\mathbb{Q}_p$. Suppose that $G$ is unipotent-by-semisimple, i.e. $G = US$, where $U$ is the unipotent radical and $S$ is a semisimple Levi factor. We furthermore assume that $S$ is split semisimple without any simple factors of rank one. Then $G(\mathbb{Q}_p)$ is compactly presented if and only if the following two conditions are satisfied

1. $H_1(u)^S = \{0\}$;
2. $H_2(u)^S = \{0\}$.

**On the proof.** This relies on [A2, Theorem 6.4.3 and Remark 6.4.5]. A few comments are necessary:

- Condition (1a) of [A2, Remark 6.4.5] involves the orthogonal $\Phi^\perp$ of the subspace generated by roots. It states that $\Phi^\perp$ does not contain dominant weights $\omega_1, \omega_2$ of the $S$-module $H_1(u)$ with $0 \in [\omega_1, \omega_2]$. Since $S$ is semisimple, $\Phi^\perp = \{0\}$ and (1a) just means that 0 is not a dominant weight, which is exactly Condition (i) above. (Note that (i) is actually a necessary and sufficient condition for $G(\mathbb{Q}_p)$ to be compactly generated, see [A2, Theorem 6.4.4].)

- As noticed in [A2, p. 132], Condition (1b) of [A2, Remark 6.4.5] is superfluous when $S$ has no factors of rank $\leq 1$.

- (ii) is a restatement of Condition 2 of [A2, Theorem 6.4.3].

We now return to our particular example $G$ from Definition 2.4; it is unipotent-by-semisimple and the semisimple Levi factor $S = \text{SL}_{n_2} \times \text{SL}_{n_3}$ is split with no factor of rank one, so it fulfills the assumptions of Theorem 3.1. So its compact presentability is equivalent to Conditions (i) and (ii) of this theorem. Keep the previous notation $S$, $D$, $U$, $u$, so that $S$ (resp. $D$) denoting in our case the diagonal by blocks (resp. diagonal) matrices in $G$, and $U$ denotes the matrices in $G$ all of whose diagonal blocks are the identity. The set of indices of the matrix is partitioned

\[\begin{array}{c}
3\text{In the published version, the corresponding theorem is a misquotation of Abels’ theorem. This version is corrected. The numbering of the lemmas is not affected. The reference to the erratum is published as Proc. Amer. Math. Soc. 139 (2011), 383–384. It essentially includes }
\end{array}\]

The verifications in the published version were enough to be applicable to the correct version of Theorem 3.1, so no change in the computations has been done.
as $I = I_1 \sqcup I_2 \sqcup I_3 \sqcup I_4$, with $|I_j| = n_j$ as in Definition 2.4. It follows that, for every field $K$,

$$u(K) = \left\{ T \in \text{End}(K^I), \forall j, T(K^{I_j}) \subset \bigoplus_{i < j} K^{I_i} \right\}.$$ 

Throughout, we use the following notation: a letter such as $i_k$ (or $j_k$, etc.) implicitly means $i_k \in I_k$. Define, in an obvious way, subgroups $U_{ij}$, $i < j$, of $U$, and their Lie algebras $u_{ij}$.

We begin by checking Condition (i) of Theorem 3.1. This follows from the following lemma.

**Lemma 3.2.** For any two weights of the action of $D$ on $H_1(u)$, $0$ is not on the segment joining them. In particular, $0$ is not a weight of the action of $D$ on $H_1(u)$.

**Proof.** Recall that $H_1(u) = u/[u, u]$. So it suffices to look at the action on the supplement $D$-subspace $u_{12} \oplus u_{23} \oplus u_{34}$ of $[u, u]$. Identifying $S$ with $\text{SL}_{n_2} \times \text{SL}_{n_3}$, we denote $(A, B)$ an element of $D \subset S$. We also denote by $e_{pq}$ the matrix whose coefficient $(p, q)$ equals one and all others are zero.

$$(A, B) \cdot e_{i_1j_1} = a_{i_1}^{-1}e_{i_1j_1}, \quad (A, B) \cdot e_{j_2k_3} = a_{j_2}b_{k_3}^{-1}e_{j_2k_3}, \quad (A, B) \cdot e_{k_3l_4} = b_{k_3}e_{k_3l_4}.$$ 

Since $S = \text{SL}_{n_2} \times \text{SL}_{n_3}$, the weights for the adjoint action on $u_{12} \oplus u_{23} \oplus u_{34}$ live in $M/P$, where $M$ is the free $\mathbb{Z}$-module of rank $n_2 + n_3$ with basis $(u_{1}, \ldots, u_{n_2}, v_{1}, \ldots, v_{n_3})$, and $P$ is the plane generated by $\sum_j u_{j_2}$ and $\sum_k v_{k_3}$. Thus, the weights are (modulo $P$) $-u_{j_2}, u_{j_2} - v_{k_3}$, $1 \leq j_2 \leq n_2, 1 \leq k_3 \leq n_3$.

Using that $n_2, n_3 \geq 3$, it is clear that no nontrivial positive combination of two weights (viewed as elements of $\mathbb{Z}^{n_2 + n_3}$) lies in $P$. 

We must now check Condition (ii) of Theorem 3.1, and therefore compute $H_2(u)$ as a $D$-module.

**Lemma 3.3.** $\text{Ker}(d_2) \subset u \wedge u$ is linearly spanned by

1. $u_{12} \wedge u_{12}$, $u_{23} \wedge u_{23}$, $u_{34} \wedge u_{34}$, $u_{13} \wedge u_{24}$, $u_{12} \wedge u_{13}$, $u_{24} \wedge u_{34}$, $u_{12} \wedge u_{34}$.
2. $u_{14} \wedge u$, $u_{13} \wedge u_{13}$, $u_{24} \wedge u_{24}$, $u_{13} \wedge u_{24}$.
3. $e_{i_1j_2} \wedge e_{k_3l_4}$ $(j_2 \neq k_2)$, $e_{i_2j_3} \wedge e_{k_3l_4}$ $(j_3 \neq k_3)$.
4. $e_{i_1j_2} \wedge e_{k_3l_4}$ $(j_2 \neq k_2)$, $e_{i_1j_2} \wedge e_{k_3l_4}$ $(j_3 \neq k_3)$.
5. Elements of the form $\sum_j \alpha_j (e_{i_1j_2} \wedge e_{j_2k_3})$ if $\sum_j \alpha_j = 0$, and

$\sum_{j_3} \alpha_{j_3} (e_{i_2j_3} \wedge e_{j_3k_4})$ if $\sum_{j_3} \alpha_{j_3} = 0$.
6. Elements of the form $\sum_j \alpha_j (e_{i_1j_2} \wedge e_{j_2k_3}) + \sum_{j_3} \beta_{j_3} (e_{i_1j_3} \wedge e_{j_3k_4})$ if

$\sum_j \alpha_j + \sum_{j_3} \beta_{j_3} = 0$.

**Proof.** First observe that $\text{Ker}(d_2)$ contains $u_{ij} \wedge u_{kl}$ when $[u_{ij}, u_{kl}] = 0$. This corresponds to (1) and (2). The remaining cases are $u_{12} \wedge u_{23}$, $u_{23} \wedge u_{34}$, $u_{12} \wedge u_{24}$, $u_{13} \wedge u_{34}$.

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4: Compared to the published version, only the second sentence in the statement of Lemma 3.2 has been added, and the proof has not been modified, although the statement is stronger than what is actually needed for the corrected version given here of Theorem 3.1.
On the one hand, \( \text{Ker}(d_2) \) also contains \( e_{i_1 j_2} \wedge e_{k_4 \ell_3} \) if \( j_2 \neq k_2 \), etc.; this corresponds to elements in (3), (4). On the other hand, \( d_2(e_{i_1 j_2} \wedge e_{j_2 k_3}) = -e_{i_1 k_3} \), \( d_2(e_{i_2 j_3} \wedge e_{j_3 k_4}) = -e_{i_2 k_4} \), \( d_2(e_{i_1 j_2} \wedge e_{j_2 k_4}) = -e_{i_1 k_4} \), \( d_2(e_{i_1 j_3} \wedge e_{j_3 k_4}) = -e_{i_1 k_4} \). The lemma follows.

\[ \square \]

**Definition 3.4.** Denote by \( b \) (resp. \( \mathfrak{h} \)) the subspace spanned by elements in (2), (4), and (6) (resp. in (1), (3), and (5)) of Lemma 3.3.

**Proposition 3.5.** \( \text{Im}(d_3) = b \), and \( \text{Ker}(d_2) = b \oplus \mathfrak{h} \) as D-module. In particular, \( H_2(u) \) is isomorphic to \( \mathfrak{h} \) as a D-module.

**Proof.** We first prove, in a series of facts, that \( \text{Im}(d_3) \supset b \).

**Fact.** \( u_{14} \wedge u \) is contained in \( \text{Im}(d_3) \).

**Proof.** If \( z \in u_{14} \), then \( d_3(x \wedge y \wedge z) = z \wedge [x, y] \). This already shows that \( u_{14} \wedge (u_{13} \oplus u_{24} \oplus u_{14}) \) is contained in \( \text{Im}(d_3) \), since \( [u, u] = u_{13} \oplus u_{24} \oplus u_{14} \).

Now, if \( (x, y, z) \in u_{24} \times u_{13} \times u_{14} \), then \( d_3(x \wedge y \wedge z) = z \wedge [x, y] \). Since \([u_{24}, u_{12}] = u_{14} \), this implies that \( u_{14} \wedge u_{34} \subseteq \text{Im}(d_3) \). Similarly, \( u_{14} \wedge u_{12} \subseteq \text{Im}(d_3) \).

Finally we must prove that \( u_{14} \wedge u_{23} \subseteq \text{Im}(d_3) \). This follows from the formula \( e_{i_1 j_4} \wedge e_{k_2 \ell_3} = d_3(e_{i_1 m_2} \wedge e_{k_2 \ell_3} \wedge e_{m_2 j_4}) \), where \( m_2 \neq k_2 \) (so that we use that \( |f_2| \geq 2 \)).

\[ \square \]

**Fact.** \( u_{13} \wedge u_{13} \) and, similarly, \( u_{24} \wedge u_{24} \), are contained in \( \text{Im}(d_3) \).

**Proof.** If \( (x, y, z) \in u_{12} \times u_{23} \times u_{13} \), then \( d_3(x \wedge y \wedge z) = z \wedge [x, y] \). Since \([u_{12}, u_{23}] = u_{13} \), this implies that \( u_{13} \wedge u_{13} \subseteq \text{Im}(d_3) \).

\[ \square \]

**Fact.** \( u_{13} \wedge u_{24} \) is contained in \( \text{Im}(d_3) \).

**Proof.** \( d_3(e_{i_1 k_2} \wedge e_{k_3 \ell_3} \wedge e_{k_2 j_4}) = e_{k_2 j_4} \wedge e_{i_1 \ell_3} \wedge e_{k_3 \ell_3} \). Since we already know that \( e_{i_1 j_4} \wedge e_{k_2 \ell_3} \in \text{Im}(d_3) \), this implies \( e_{k_2 j_4} \wedge e_{i_1 \ell_3} \in \text{Im}(d_3) \).

\[ \square \]

**Fact.** The elements in (4) are in \( \text{Im}(d_3) \).

**Proof.** \( d_3(e_{i_1 j_2} \wedge e_{j_2 k_3} \wedge e_{\ell_3 m_4}) = -e_{i_1 k_3} \wedge e_{\ell_3 m_4} \) if \( k_3 \neq \ell_3 \). The other case is similar.

\[ \square \]

**Fact.** The elements in (6) are in \( \text{Im}(d_3) \).

**Proof.** \( d_3(e_{i_1 j_2} \wedge e_{j_2 k_3} \wedge e_{k_3 \ell_4}) = -e_{i_1 k_3} \wedge e_{k_3 \ell_4} + e_{i_1 j_2} \wedge e_{j_2 \ell_4} \). Such elements linearly span all elements as in (6).

\[ \square \]

Conversely, we must check \( \text{Im}(d_3) \supset b \). By straightforward verifications:

- \( d_3(u_{14} \wedge u \wedge u) \subseteq u_{14} \wedge u \).
- \( d_3(u_{13} \wedge u_{23} \wedge u_{24}) = 0 \).
- \( d_3(u_{12} \wedge u_{24} \wedge u_{23}) \), \( d_3(u_{13} \wedge u_{24} \wedge u_{34}) \), \( d_3(u_{12} \wedge u_{13} \wedge u_{24}) \), \( d_3(u_{12} \wedge u_{23} \wedge u_{34}) \) are all contained in \( u_{14} \wedge u \).
- \( d_3(u_{12} \wedge u_{13} \wedge u_{24}) \), \( d_3(u_{12} \wedge u_{23} \wedge u_{24}) \), and similarly \( d_3(u_{23} \wedge u_{24} \wedge u_{34}) \) are contained in \( u_{14} \wedge u_{23} + u_{13} \wedge u_{24} \).
- The only remaining case is that of \( u_{12} \wedge u_{23} \wedge u_{34} \): \( d_3(e_{i_1 j_2} \wedge e_{j_2 k_3} \wedge e_{k_3 \ell_4}) = \delta_{k_3 \ell_4} e_{i_1 j_2} \wedge e_{j_2 k_3} - \delta_{j_2 \ell_4} e_{i_1 k_3} \wedge e_{k_3 \ell_4} \), which lies in (4) or in (6).
Finally \( \text{Im}(d_3) = \mathfrak{b} \).

It follows from Lemma 3.3 that \( \text{Ker}(d_2) = \mathfrak{h} \oplus \mathfrak{b} \). Since \( \mathfrak{b} = \text{Im}(d_3) \), this is a \( D \)-submodule. Let us check that \( \mathfrak{h} \) is also a \( D \)-submodule; the computation will be used in the sequel.

The action of \( S \) on \( u \) by conjugation is given by:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & A & 0 & 0 \\
0 & 0 & B & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
X_{12} & X_{13} & X_{14} \\
0 & X_{23} & X_{24} \\
0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
X_{12}A^{-1} & X_{13}B^{-1} & X_{14} \\
0 & 0 & AX_{23}B^{-1} \\
0 & 0 & 0 & BX_{24} \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

We must look at the action of \( D \) on the elements in (1), (3), and (5). We fix \( (A, B) \in D \subset S \cong \text{SL}_{n_2} \times \text{SL}_{n_3} \), and we write \( A = \sum_j a_j e_{j_2j_3} \) and \( B = \sum_k b_k e_{k_3k_3} \).

- (1):

\[
(A, B) \cdot e_{i_1j_2} \wedge e_{k_1\ell_2} = e_{i_1j_2}A^{-1} \wedge e_{k_1\ell_2}A^{-1} = a_j^{-1} e_{i_1j_2} \wedge e_{k_1\ell_2}.
\]

The action on other elements in (1) has a similar form.

- (3) \((j_2 \neq k_2)\):

\[
(A, B) \cdot e_{i_1j_2} \wedge e_{k_2\ell_3} = e_{i_1j_2}A^{-1} \wedge Ae_{k_2\ell_3}B^{-1} = a_j^{-1} a_{k_2} b_{\ell_3} e_{i_1j_2} \wedge e_{k_2\ell_3}.
\]

The action on the other elements in (3) has a similar form.

- (5) \((\sum_j \alpha_{j_2} = 0)\):

\[
(A, B) \cdot \sum_j \alpha_{j_2} (e_{i_1j_2} \wedge e_{j_2k_3}) = \sum_j \alpha_{j_2} (e_{i_1j_2}A^{-1} \wedge Ae_{j_2k_3}B^{-1})
\]

\[
= \sum_j \alpha_{j_2} a_j^{-1} (e_{i_1j_2} \wedge a_j e_{j_2k_3}^{-1} e_{j_2k_3}) = b_{k_3}^{-1} \left( \sum_j \alpha_{j_2} (e_{i_1j_2} \wedge e_{j_2k_3}) \right).
\]

The other case in (5) has a similar form. \( \square \)

**Lemma 3.6.** \( \theta \) is not a weight for the action of \( D \) on \( \text{H}_2(u) \).

**Proof.** As described in the proof of Lemma 3.2, we think of weights as elements of \( M/P \). Hence, we describe weights as elements of \( M = \mathbb{Z}^{n_2+n_3} \) rather than \( M/P \), and must check that no weight lies in \( P \).

- (1) In (3.1), the weight is \(-u_{j_2} - u_{\ell_2}\), hence does not belong to \( P \) since \( n_2 \geq 3 \).

  The other verifications are similar.

- (3) In (3.2), the weight is \(-u_{j_2} + u_{k_2} - v_{\ell_3}\), hence does not belong to \( P \). The other verification for (3) is similar.

- (5) In (3.3), the weight is \(-v_{k_3}\), hence does not belong to \( P \). The other verification is similar. \( \square \)

Finally, Lemmas 3.2 and 3.6 imply that the conditions (i) and (ii) of Theorem 3.1 are satisfied, so that \( \Gamma \) is finitely presented. \( \square \)

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