QUOTIENTS OF THE MAGMATIC OPERAD: LATTICE STRUCTURES AND CONVERGENT REWRITE SYSTEMS

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ABSTRACT. We study quotients of the magmatic operad, that is the free nonsymmetric operad over one binary generator. In the linear setting, we show that the set of these quotients admits a lattice structure and we show an analog of the Grassmann formula for the dimensions of these operads. In the nonlinear setting, we define comb associative operads, that are operads indexed by nonnegative integers generalizing the associative operad. We show that the set of comb associative operads admits a lattice structure, isomorphic to the lattice of nonnegative integers equipped with the division order. Driven by computer experimentations, we provide a finite convergent presentation for the comb associative operad in correspondence with 3. Finally, we study quotients of the magmatic operad by one cubic relation by expressing their Hilbert series and providing combinatorial realizations.

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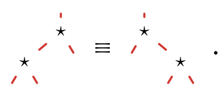
Introduction

Associative algebras are spaces endowed with a binary product $\star$ satisfying among others the associativity law

$$(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)\,. \tag{0.0.1}$$

It is well-known that the associative algebras are representations of the associative (nonsymmetric) operad **As**. This operad can be seen as the quotient of the magmatic operad **Mag** (the free operad of binary trees on the binary generator $\star$) by the operad congruence $\equiv$ satisfying



(0.0.2)

These two binary trees are the syntax trees of the expressions appearing in the above associativity law.

In a more combinatorial context and regardless of the theory of operads, the Tamari order is a partial order on the set of the binary trees having a fixed number of internal nodes γ. This order is generated by the covering relation consisting in rewriting a tree $\mathfrak{t}$ into a tree $\mathfrak{t}'$ by replacing a subtree of $\mathfrak{t}$ of the form of the left member of (0.0.2) into a tree of the form of the right member of (0.0.2). This transformation is known in a computer science context as the right rotation operation [Knu98] and intervenes in algorithms involving binary search trees [AVL62]. The partial order hence generated by the right rotation operation is known as the Tamari order [Tam62] and has a lot of combinatorial and algebraic properties (see for instance [HT72, Cha06]).

A first connection between the associative operad and the Tamari order is based upon the fact that the orientation of (0.0.2) from left to right provides a convergent orientation (a terminating and confluent rewrite relation) of the congruence $\equiv$. The normal forms of the rewrite relation induced by the rewrite rule obtained by orienting (0.0.2) from left to right are right comb binary trees and are hence in one-to-one correspondence with the elements of **As**. Following the ideas developed by Anick for associative algebras [Ani86], the description of an operad by mean of normal forms provides homological informations for this operad. One of the fundamental homological properties for operads is Koszulness [GK94], generalizing Koszul associative algebras [Pri70]: the convergent orientation of (0.0.2) proves that **As** is a Koszul operad [DK10, LV12].

This work is intended to study and collect the possible links between the Tamari order and some quotients of the operad **Mag**. In the long run, the goal is to study quotients $\mathbf{Mag}/_{\equiv}$ of **Mag** where $\equiv$ is an operad congruence generated by equivalence classes of trees of a fixed degree (that is, a fixed number of internal nodes). In particular, we would like to know if the fact that $\equiv$ is generated by equivalence classes of trees forming intervals of the Tamari order implies algebraic properties for $\mathbf{Mag}/_{\equiv}$ (like the description of orientations of its space of relations, of nice bases, and of Hilbert series).

To explore this vast research area, we select to pursue in this paper the following directions. First, we consider the very general set of the quotients of **Mag** seen as an operad in the category of vector spaces. We show that this set of operads forms a lattice, wherein its partial order relation is defined from the existence of operad morphisms (Theorem 2.1.3). We also provide a Grassmann formula (see for instance [Lan02] analog relating the Hilbert series of the operads of the lattice together with their lower-bound and upper-bound (Theorem 2.2.1). Besides, we focus on a special kind of quotients of **Mag**, denoted by $\mathbf{CAs}^{(\gamma)}$, defined by equating the left and right comb binary trees of a fixed degree $\gamma \geqslant 1$. Observe that since $\mathbf{CAs}^{(2)} = \mathbf{As}$, the operads $\mathbf{CAs}^{(\gamma)}$ can be seen as generalizations of **As**. These operads are called comb associative operads. For instance, $\mathbf{CAs}^{(3)}$ is the operad describing the category of the algebras equipped with a binary product $\star$ subjected to the relation

$$((x_1 \star x_2) \star x_3) \star x_4 = x_1 \star (x_2 \star (x_3 \star x_4)) . \tag{0.0.3}$$

We first provide general results about the operads $\mathbf{CAs}^{(\gamma)}$. In particular, we show that the set of these operads forms a lattice which embeds as a poset in the lattice of the quotients of **Mag** aforementioned (Theorems 3.2.8 and 3.2.9). We focus in particular on the study of $\mathbf{CAs}^{(3)}$. Observe that the congruence defining this operad is generated by an equivalence class of trees which is not an interval for the Tamari order. As preliminary computer experiments show, $\mathbf{CAs}^{(3)}$ has oscillating first dimensions (see (3.3.23)), what is rather unusual among all known operads. We provide a convergent orientation of the space of relations of $\mathbf{CAs}^{(3)}$ (Theorem 3.3.1), a description of a basis of the operad in terms of normal forms, and prove that its Hilbert series is rational. For all these, we use rewrite systems on trees [BN98] and the Buchberger algorithm for operads [DK10]. We expose some experimental results obtained with the help of the computer for some operads $\mathbf{CAs}^{(\gamma)}$ with $\gamma \geqslant 4$. All our computer programs are made from scratch in Caml and Python. Finally, we continue the investigation of the quotients of **Mag** by regarding the quotients of **Mag** obtained by equating two trees of degree 3. This leads to ten quotient operads of **Mag**. We provide for some of these combinatorial realizations, mostly in terms of integer compositions.

This text is presented as follows. Section 1 contains preliminaries about operads, binary trees, the magmatic operad, and rewrite systems on binary trees. We also prove and recall some important lemmas about rewrite systems on trees used thereafter. In Section 2, we study the set of all the quotients of **Mag** seen as an operad in the category of vector spaces and its lattice structure. Section 3 is the heart of this article and is devoted to the study of the comb associative operads $\mathbf{CAs}^{(\gamma)}$. Finally, Section 4 presents our results about the quotients of **Mag** obtained by equating two trees of degree 3.

Some of the results presented here were announced in [CCG18].

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General notations and conventions. For any integers a and c, $[a, c]$ denotes the set $\{b \in \mathbb{N} : a \leqslant b \leqslant c\}$ and $[n]$, the set $[1, n]$. The cardinality of a finite set S is denoted by $\# S$. In all this work, $\mathbb{K}$ is a field of characteristic zero.

1. The magmatic operad, quotients, and rewrite relations

We set in this preliminary section our notations about operads. We also provide a definition for the magmatic operad and introduce tools to handle with some of its quotients involving rewrite systems on binary trees.

1.1. Nonsymmetric operads.

A *nonsymmetric operad in the category of sets* (or a *nonsymmetric operad* for short) is a graded set $\mathcal{O} = \bigsqcup_{n \geqslant 1} \mathcal{O}(n)$ together with maps

$$\circ_i : \mathcal{O}(n) \times \mathcal{O}(m) \to \mathcal{O}(n + m - 1), \qquad 1 \leqslant i \leqslant n, 1 \leqslant m, \tag{1.1.1}$$

called *partial compositions*, and a distinguished element $\mathbb{1} \in \mathcal{O}(1)$, the *unit* of $\mathcal{O}$. This data has to satisfy, for any $x \in \mathcal{O}(n)$, $y \in \mathcal{O}(m)$, and $z \in \mathcal{O}$, the three relations

$$(x \circ_i y) \circ_{i+j-1} z = x \circ_i (y \circ_j z), \qquad 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, \tag{1.1.2a}$$

$$(x \circ_i y) \circ_{j+m-1} z = (x \circ_j z) \circ_i y, \qquad 1 \leqslant i < j \leqslant n, \tag{1.1.2b}$$

$$\mathbb{1} \circ_1 x = x = x \circ_i \mathbb{1}, \qquad 1 \leqslant i \leqslant n. \tag{1.1.2c}$$

Since we consider in this work only nonsymmetric operads, we shall call these simply *operads*.

Let us provide some elementary definitions and notations about operads. If x is an element of $\mathcal{O}$ such that $x \in \mathcal{O}(n)$ for an $n \geqslant 1$, the *arity* $|x|$ of x is n. The *complete composition maps* of $\mathcal{O}$ are the map

$$\circ : \mathcal{O}(n) \times \mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_n) \to \mathcal{O}(m_1 + \cdots + m_n) \tag{1.1.3}$$

defined, for any $x \in \mathcal{O}(n)$ and $y_1, \ldots, y_n \in \mathcal{O}$, by

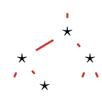
$$x \circ [y_1, \ldots, y_n] := (\cdots ((x \circ_n y_n) \circ_{n-1} y_{n-1}) \cdots) \circ_1 y_1. \tag{1.1.4}$$

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are two operads, a map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is an *operad morphism* if it respects the arities, sends the unit of $\mathcal{O}_1$ to the unit of $\mathcal{O}_2$, and commutes with partial composition maps. A map $\phi : \mathcal{O}_1 \to \mathcal{O}_2$ is an *operad antimorphism* if it respects the arities, sends the unit of $\mathcal{O}_1$ to the unit of $\mathcal{O}_2$, and $\phi(x \circ_i y) = \phi(x) \circ_{|x|-i+1} \phi(y)$ for all $x, y \in \mathcal{O}_1$ and $i \in [|x|]$. We say that $\mathcal{O}_2$ is a *suboperad* of $\mathcal{O}_1$ if $\mathcal{O}_2$ is a subset of $\mathcal{O}_1$ containing the unit of $\mathcal{O}_1$ and the partial composition maps of $\mathcal{O}_2$ are the ones of $\mathcal{O}_1$ restricted on $\mathcal{O}_2$. For any subset $\mathfrak{G}$ of $\mathcal{O}$, the *operad generated* by $\mathfrak{G}$ is the smallest suboperad $\mathcal{O}^{\mathfrak{G}}$ of $\mathcal{O}$ containing $\mathfrak{G}$. When $\mathcal{O} = \mathcal{O}^{\mathfrak{G}}$, we say that $\mathfrak{G}$ is a *generating set* of $\mathcal{O}$. An *operad congruence* is an equivalence relation $\equiv$ on $\mathcal{O}$ such that $\equiv$ respects the arities, and for any $x, x', y, y' \in \mathcal{O}$ such that $x \equiv x'$ and $y \equiv y'$, $x \circ_i y$ is $\equiv$-equivalent to $x' \circ_i y'$ for any valid integer i. Given an operad congruence $\equiv$, the *quotient operad* $\mathcal{O}/_{\equiv}$ of $\mathcal{O}$ by $\equiv$ is the operad of all $\equiv$-equivalence classes endowed with the partial composition maps defined in the obvious way. In the case where all the sets $\mathcal{O}(n)$, $n \geqslant 1$, are finite, the *Hilbert series* $\mathcal{H}_{\mathcal{O}}(t)$ of $\mathcal{O}$ is the series defined by

$$\mathcal{H}_{\mathcal{O}}(t) := \sum_{n \geqslant 1} \# \mathcal{O}(n) \; t^n. \tag{1.1.5}$$

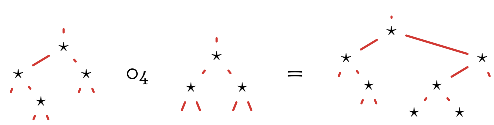
We have provided here definitions about operads in the category of sets. Nevertheless, operads can be defined in the category of $\mathbb{K}$-vector spaces. We call them *linear operads* and we study a class of such operads in Section 2. All the above definitions extend for linear operads, mainly by substituting Cartesian products $\times$ of sets with tensor products $\otimes$ of spaces, maps with linear maps, operad congruences with operad ideals, and cardinalities of sets with space dimensions (for instance in (1.1.5)). If $\mathcal{O}$ is an operad in the category of sets, we denote by $\mathbb{K}\langle\mathcal{O}\rangle$ the corresponding linear operad defined on the linear span of $\mathcal{O}$, where the partial composition maps of $\mathcal{O}$ are extended by linearity on $\mathbb{K}\langle\mathcal{O}\rangle$. Conversely, when $\mathcal{O}$ is a linear operad admitting a basis B such that its unit $\mathbb{1}$ belongs to B and all partial composition maps are internal in B, $\mathcal{O} = \mathbb{K}\langle B\rangle$ and this operad can be studied as a set-theoretic operad B.

1.2. **Binary trees and the magmatic operad.** A *binary tree* is either the *leaf* | or a pair $(\mathfrak{t}_1, \mathfrak{t}_2)$ of binary trees. We use the standard terminology about binary trees (such as *root*, *internal node*, *left child*, *right child*, *etc.*) in this work. Let us recall the main notions. The *arity* $|\mathfrak{t}|$ (resp. *degree* $\deg(\mathfrak{t})$) of a binary tree $\mathfrak{t}$ is its number of leaves (resp. internal nodes). A binary tree $\mathfrak{t}$ is *quadratic* (resp. *cubic*) if $\deg(\mathfrak{t}) = 2$ (resp. $\deg(\mathfrak{t}) = 3$). We shall draw binary trees the root to the top. For instance,


(1.2.1)

is the graphical representation of the binary tree $((|, (|, |)), (|, |))$.

The *magmatic operad* **Mag** is the graded set of all the binary trees where $\mathbf{Mag}(n)$, $n \geqslant 1$, is the set of all the binary trees of arity n. The partial composition maps of **Mag** are grafting of trees: given two binary trees $\mathfrak{t}$ and $\mathfrak{s}$, $\mathfrak{t} \circ_i \mathfrak{s}$ is the binary tree obtained by grafting the root of $\mathfrak{s}$ onto the ith leaf (numbered from left to right) of $\mathfrak{t}$. For instance,


(1.2.2)

is a partial composition in **Mag**. This leads, by definition, to the following complete composition maps of **Mag**. Given $\mathfrak{t} \in \mathbf{Mag}(n)$ and $\mathfrak{s}_1, \dots, \mathfrak{s}_n \in \mathbf{Mag}$, $\mathfrak{t} \circ [\mathfrak{s}_1, \dots, \mathfrak{s}_n]$ is the binary tree obtained by grafting simultaneously the roots of all the $\mathfrak{s}_i$ onto the ith leaves of $\mathfrak{t}$. The unit of **Mag** is the leaf. The number of binary trees of arity $n \geqslant 1$ is the $(n-1)$st Catalan number $\mathrm{cat}(n-1)$ and hence, the Hilbert series of **Mag** is

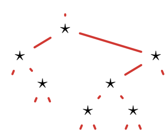
$$\mathcal{H}_{\mathbf{Mag}}(t) = \sum_{n \geqslant 1} \mathrm{cat}(n-1) t^n = \sum_{n \geqslant 1} \binom{2n-1}{n-1} \frac{1}{n} t^n. \tag{1.2.3}$$

The operad **Mag** can be seen as the free operad generated by one binary element $\star$. It satisfies the following universality property. Let $\mathfrak{G} := \mathfrak{G}(2) := \{\star'\}$ be the graded set containing exactly one element $\star'$ of arity 2. For any operad $\mathcal{O}$ and any map $f : \mathfrak{G}(2) \to \mathcal{O}(2)$, there exists a unique operad morphism $\phi : \mathbf{Mag} \to \mathcal{O}$ such that $f = \phi \circ \mathrm{c}$, where c is the map sending $\star'$ to the unique binary tree of degree 1 (and then, arity 2). In other terms, the diagram

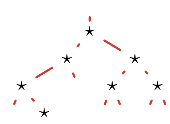
$$\begin{tikzcd} \mathfrak{G} \arrow[r, "f"] \arrow[d, tail, "\mathrm{c}"'] & \mathcal{O} \\ \mathbf{Mag} \arrow[ur, dashed, "\phi"'] & \end{tikzcd} \tag{1.2.4}$$

commutes.

We now provide some useful tools about binary trees. Given a binary tree $\mathfrak{t}$, we denote by $\mathrm{p}(\mathfrak{t})$ the *prefix word* of $\mathfrak{t}$, that is the word on $\{0, 2\}$ obtained by a left to right depth-first traversal of $\mathfrak{t}$ and by writing 0 (resp. 2) when a leaf (resp. an internal node) is encountered. For instance,

 $\overset{\mathrm{p}}{\longmapsto}$ 220200222002000. (1.2.5)

The set of all the words on $\{0, 2\}$ is endowed with the lexicographic order $\leqslant$ induced by $0 < 2$. By extension, this defines a total order on each set $\mathbf{Mag}(n)$, $n \geqslant 1$. Indeed, we set $\mathfrak{t} \leqslant \mathfrak{t}'$ if $\mathfrak{t}$ and $\mathfrak{t}'$ have the same arity and $\mathrm{p}(\mathfrak{t}) \leqslant \mathrm{p}(\mathfrak{t}')$. Let also the *left rank* of $\mathfrak{t}$ as the number $\mathrm{lr}(\mathfrak{t})$ of internal nodes in the left branch beginning at the root of $\mathfrak{t}$. For instance,

 $\overset{\mathrm{lr}}{\longmapsto}$ 3. (1.2.6)

Equivalently, $\mathrm{lr}(\mathfrak{t})$ is the length of the prefix of $\mathrm{p}(\mathfrak{t})$ containing only the letter 2. A binary tree $\mathfrak{s}$ is a *subtree* of $\mathfrak{t}$ if it possible to stack $\mathfrak{s}$ onto $\mathfrak{t}$ by possibly superimposing leaves of $\mathfrak{s}$ onto internal nodes of $\mathfrak{t}$. More formally, by using the operad **Mag** and its composition maps, this is equivalent to the fact that $\mathfrak{t}$ expresses as

$$\mathfrak{t} = \mathfrak{r} \circ_i (\mathfrak{s} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_n]) \tag{1.2.7}$$

where $\mathfrak{r}$ and $\mathfrak{r}_1$, ..., $\mathfrak{r}_n$ are binary trees, $i \in [|\mathfrak{r}|]$, and n is the arity of $\mathfrak{s}$. When, on the contrary, $\mathfrak{s}$ is not a subtree of $\mathfrak{t}$, we say that $\mathfrak{t}$ *avoids* $\mathfrak{s}$.

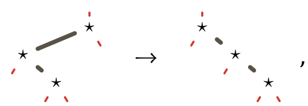
1.3. **Rewrite systems on binary trees.** We present here notions about rewrite systems on binary trees. General notations and notions appear in [BN98].

A *rewrite rule* is an ordered pair $(\mathfrak{s}, \mathfrak{s}')$ of binary trees such that $|\mathfrak{s}| = |\mathfrak{s}'|$. A set S of rewrite rules is a binary relation on **Mag** and it shall be denoted by $\to$. We denote by $\mathfrak{s} \to \mathfrak{s}'$ the fact that $(\mathfrak{s}, \mathfrak{s}') \in \to$. In the sequel, to define a set of rewrite rules $\to$, we shall simply list all the pairs $\mathfrak{s} \to \mathfrak{s}'$ contained in $\to$. The *degree* $\deg(\to)$ of $\to$ is the maximal degree of the binary trees in relation through $\to$. Note that $\deg(\to)$ can be not defined when $\to$ is infinite.

If $\to$ is a set of rewrite rules, we denote by $\Rightarrow$ the *rewrite relation induced* by $\to$. Formally we have

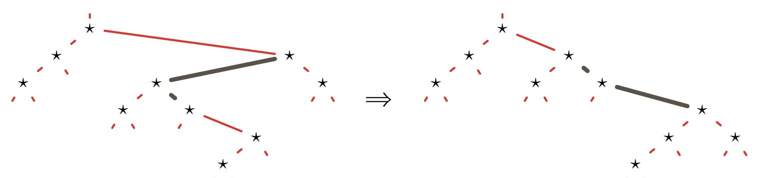
$$\mathfrak{t} \circ_i (\mathfrak{s} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_n]) \Rightarrow \mathfrak{t} \circ_i \left(\mathfrak{s}' \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_n]\right), \tag{1.3.1}$$

if $\mathfrak{s} \to \mathfrak{s}'$ where $n = |\mathfrak{s}|$, and $\mathfrak{t}$, $\mathfrak{r}_1$, …, $\mathfrak{r}_n$ are binary trees. In other words, one has $\mathfrak{t} \Rightarrow \mathfrak{t}'$ if it is possible to obtain $\mathfrak{t}'$ from $\mathfrak{t}$ by replacing a subtree $\mathfrak{s}$ of $\mathfrak{t}$ by $\mathfrak{s}'$ whenever $\mathfrak{s} \to \mathfrak{s}'$. For instance, if $\to$ is the set of rewrite rules containing the single rewrite rule



(1.3.2)

one has



(1.3.3)

The right member of (1.3.3) is obtained by replacing, in the tree of left member of (1.3.3), a subtree equal to the left member of (1.3.2) starting at the right child of its root by the right member of (1.3.2).

Let $\to$ be a set of rewrite rules and $\Rightarrow$ be the rewrite relation induced by $\to$. Since $\Rightarrow$ is in particular a binary relation on **Mag**, the classical notations about closures apply here: we denote by $\overset{+}{\Rightarrow}$ (resp. $\overset{*}{\Rightarrow}$, $\overset{*}{\Leftrightarrow}$) the transitive (resp. reflexive and transitive, and reflexive, symmetric, and transitive) closure of $\Rightarrow$.

When $\mathfrak{t}_0$, $\mathfrak{t}_1$, …, $\mathfrak{t}_k$ are binary trees such that

$$\mathfrak{t}_0 \Rightarrow \mathfrak{t}_1 \Rightarrow \cdots \Rightarrow \mathfrak{t}_k, \tag{1.3.4}$$

we say that $\mathfrak{t}_0$ is *rewritable* by $\Rightarrow$ into $\mathfrak{t}_k$ in k *steps*. When there is no infinite chain

$$\mathfrak{t}_0 \Rightarrow \mathfrak{t}_1 \Rightarrow \mathfrak{t}_2 \Rightarrow \cdots \tag{1.3.5}$$

we say that $\Rightarrow$ is *terminating*. To establish the termination of a rewrite relation, we will use the following criterion.

Lemma 1.3.1. *Let $\to$ be a set of rewrite rules on* **Mag**. *If for any $\mathfrak{t}, \mathfrak{t}' \in$* **Mag** *such that $\mathfrak{t} \to \mathfrak{t}'$ one has $\mathfrak{t} > \mathfrak{t}'$, then the rewrite relation induced by $\to$ is terminating.*

Proof. Observe first that for any binary trees $\mathfrak{t}$ and $\mathfrak{s}$, the prefix word of $\mathfrak{t} \circ_i \mathfrak{s}$ is obtained by replacing the ith 0 of $\mathrm{p}(\mathfrak{t})$ by $\mathrm{p}(\mathfrak{s})$. For this reason, and due to the definition (1.1.4) of $\circ$, for any binary trees $\mathfrak{s}$ and $\mathfrak{r}_1, \dots, \mathfrak{r}_n$ where n is the arity of $\mathfrak{s}$, the prefix word of $\mathfrak{s} \circ [\mathfrak{r}_1, \dots, \mathfrak{r}_n]$ is obtained by replacing from right to left each 0 of $\mathrm{p}(\mathfrak{s})$ by the prefix words of each $\mathfrak{r}_i$. This, together with the definition (1.3.1) of the rewrite relation $\Rightarrow$ induced by $\to$ and the hypothesis of the statement of the lemma, implies that if $\mathfrak{t}$ and $\mathfrak{t}'$ are two binary trees such that $\mathfrak{t} \Rightarrow \mathfrak{t}'$, $\mathrm{p}(\mathfrak{t}) > \mathrm{p}(\mathfrak{t}')$. This means that $\mathfrak{t} > \mathfrak{t}'$ and leads to the fact that any chain $\mathfrak{t}_0 \Rightarrow \mathfrak{t}_1 \Rightarrow \mathfrak{t}_2 \Rightarrow \cdots$ is finite since $\mathfrak{t}_0 > \mathfrak{t}_1 > \mathfrak{t}_2 > \cdots$ and there is a finite number of binary trees of a fixed arity. Therefore, $\Rightarrow$ is terminating. □

A *normal form* for $\Rightarrow$ is a binary tree $\mathfrak{t}$ such that for all binary trees $\mathfrak{t}'$, $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{t}'$ implies $\mathfrak{t}' = \mathfrak{t}$. In other words, a normal form for $\Rightarrow$ is a tree which is not rewritable by $\Rightarrow$. A normal form for $\Rightarrow$ of a binary tree $\mathfrak{t}$ is a normal form $\bar{\mathfrak{t}}$ for $\Rightarrow$ such that $\mathfrak{t} \overset{*}{\Rightarrow} \bar{\mathfrak{t}}$. When no confusion is possible, we simply say normal form instead of normal form for $\Rightarrow$. The set of all the normal forms is denoted by $\mathfrak{N}_{\Rightarrow}$. The trees of $\mathfrak{N}_{\Rightarrow}$ admit the following description, useful for enumerative prospects.

Lemma 1.3.2. *Let $\to$ be a set of rewrite rules on* **Mag** *and $\Rightarrow$ be the rewrite relation induced by $\to$. Then, $\mathfrak{N}_{\Rightarrow}$ is the set of all the binary trees that avoid all the trees appearing as left members of $\to$.*

Proof. Assume first that $\mathfrak{t}$ is a binary tree avoiding all the trees appearing as left members of $\to$. Then, due to the definition (1.3.1) of $\Rightarrow$, $\mathfrak{t}$ is not rewritable by $\Rightarrow$. Hence, $\mathfrak{t}$ is a normal form for $\Rightarrow$. Conversely, assume that $\mathfrak{t} \in \mathfrak{N}_{\Rightarrow}$. In this case, by definition of a normal form, $\mathfrak{t}$ is not rewritable by $\Rightarrow$, so that $\mathfrak{t}$ does not admit any occurrence of a tree appearing as a left member of $\to$. □

When for all binary trees $\mathfrak{t}$, $\mathfrak{s}_1$, and $\mathfrak{s}_2$ such that $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{s}_1$ and $\mathfrak{t} \overset{*}{\Rightarrow} \mathfrak{s}_2$, there exists a binary tree $\mathfrak{t}'$ such that $\mathfrak{s}_1 \overset{*}{\Rightarrow} \mathfrak{t}'$ and $\mathfrak{s}_2 \overset{*}{\Rightarrow} \mathfrak{t}'$, we say that $\Rightarrow$ is *confluent*. Besides, a tree $\mathfrak{t}$ is a *branching tree* for $\Rightarrow$ if there exists two different trees $\mathfrak{s}_1$ and $\mathfrak{s}_2$ satisfying $\mathfrak{t} \Rightarrow \mathfrak{s}_1$ and $\mathfrak{t} \Rightarrow \mathfrak{s}_2$. In this case, the pair $\{\mathfrak{s}_1, \mathfrak{s}_2\}$ is a *branching pair* for $\mathfrak{t}$. Moreover, the branching pair $\{\mathfrak{s}_1, \mathfrak{s}_2\}$ is *joinable* if there exists a binary tree $\mathfrak{t}'$ such that $\mathfrak{s}_1 \overset{*}{\Rightarrow} \mathfrak{t}'$ and $\mathfrak{s}_2 \overset{*}{\Rightarrow} \mathfrak{t}'$. The diamond lemma [New42] is based upon the inspection of the branching pairs of a terminating rewrite relation $\Rightarrow$ in order to prove its confluence.

Lemma 1.3.3. *Let $\to$ be a set of rewrite rules on* **Mag** *and $\Rightarrow$ be the rewrite relation induced by $\to$. Then, if $\Rightarrow$ is terminating and all its branching pairs are joinable, $\Rightarrow$ is confluent.*

When $\Rightarrow$ is terminating and confluent, $\Rightarrow$ is said *convergent*. We shall use the following result to prove that a terminating rewrite relation is convergent.

Lemma 1.3.4. *Let $\to$ be a set of rewrite rules on* **Mag** *having a degree* $\deg(\to)$. *Then, if the rewrite relation $\Rightarrow$ induced by $\to$ is terminating and all its branching pairs made of trees of degrees at most* $2\deg(\to) - 1$ *are joinable, $\Rightarrow$ is convergent.*

Proof. The statement of the lemma is the specialization on rewrite relations on **Mag** of a more general result about rewrite relations appearing in [Gir16, Lemma 1.2.1.]. □

Let us now go back on operads. Let $\equiv$ be an operad congruence of **Mag**. If $\mathfrak{t}$ is a binary tree, we denote by $[\mathfrak{t}]_\equiv$ the $\equiv$-equivalence class of $\mathfrak{t}$. By definition, $[\mathfrak{t}]_\equiv$ is an element of the quotient operad

$$\mathcal{O} := \mathbf{Mag}/_\equiv. \tag{1.3.6}$$

A set of rewrite rules $\to$ on **Mag** is an *orientation* of $\equiv$ if $\overset{*}{\Leftrightarrow}$ and $\equiv$ are equal as binary relations, where $\Rightarrow$ is the rewrite relation induced by $\to$. Moreover, $\to$ is a *convergent* (resp. *terminating*, *confluent*) orientation of $\equiv$ if $\Rightarrow$ is convergent (resp. terminating, confluent). In this text, we call *presentation* of a quotient operad $\mathcal{O}$ of the form (1.3.6) the data of a generating set for the operad congruence $\equiv$. Observe that any orientation of $\equiv$ is a presentation of $\mathcal{O}$, so that the above nomenclature (*convergent*, *terminating*, and *confluent*) still holds for presentations. A presentation is said to be *finite* if it is a finite set.

When $\to$ is a convergent orientation of $\equiv$, the set $\mathfrak{N}_\Rightarrow$ of all normal forms for $\Rightarrow$ is called a *Poincaré-Birkhoff-Witt basis* [Hof10, DK10] (or a *PBW basis* for short) of the quotient operad $\mathcal{O}$. This forms a one-to-one correspondence between the sets $\mathfrak{N}_\Rightarrow(n)$ and $\mathcal{O}(n)$, $n \geqslant 1$. In other words, a PBW basis offers a way to assign with each $\equiv$-equivalence class $[\mathfrak{t}]_\equiv$ a representative $\mathfrak{t}' \in [\mathfrak{t}]_\equiv \cap \mathfrak{N}_\Rightarrow$. A *combinatorial realization* of an operad $\mathcal{O}$ of the form (1.3.6) is an operad $\mathcal{C}$ isomorphic to $\mathcal{O}$ which admits an explicit description of its elements and an explicit description of its partial composition maps. The knowledge of a PBW basis $\mathcal{C} := \mathfrak{N}_\Rightarrow$ of $\mathcal{O}$ provides a combinatorial realization $\mathcal{C}$ of $\mathcal{O}$. Indeed, the partial composition $\mathfrak{t}' \circ_i \mathfrak{s}'$ of two binary trees $\mathfrak{t}'$ and $\mathfrak{s}'$ of $\mathcal{C}$ is the tree obtained by grafting the root of $\mathfrak{s}'$ onto the ith leaf of $\mathfrak{t}'$ and by rewriting by $\Rightarrow$ this tree as much as possible in order to obtain a normal form. This process is well-defined since, by hypothesis, $\Rightarrow$ is convergent.

When $\to$ is a terminating but not convergent orientation of $\equiv$, we shall use a variant of the *Buchberger semi-algorithm* for operads [DK10, Section 3.7] to compute a set of rewrite rules $\to'$ such that, as binary relations $\to \subseteq \to'$, and $\to'$ is a convergent orientation of $\equiv$. This semi-algorithm takes as input a finite set of rewrite rules $\to$ and outputs the set of rewrite rules $\to'$ satisfying the property stated above. Here is, step by step, a description of its execution:

(*1*) Set $\to' := \to$ and let $\mathfrak{P}$ be the set of branching trees for $\Rightarrow$.

(*2*) If $\mathfrak{P}$ is empty, the execution stops and the output is $\to'$.

(*3*) Otherwise, let $\mathfrak{t}$ be a branching tree for $\Rightarrow'$. Remove $\mathfrak{t}$ from $\mathfrak{P}$.

(*4*) Let $\{\mathfrak{s}_1, \mathfrak{s}_2\}$ be a branching pair for $\mathfrak{t}$.

(*5*) Let $\bar{\mathfrak{s}_1}$ and $\bar{\mathfrak{s}_2}$ be normal forms of $\mathfrak{s}_1$ and $\mathfrak{s}_2$, respectively.

(*6*) If $\bar{\mathfrak{s}_1}$ is different from $\bar{\mathfrak{s}_2}$, add to $\to'$ the rewrite rule $\max_\preccurlyeq \{\bar{\mathfrak{s}_1}, \bar{\mathfrak{s}_2}\} \to' \min_\preccurlyeq \{\bar{\mathfrak{s}_1}, \bar{\mathfrak{s}_2}\}$.

(*7*) Add to $\mathfrak{P}$ all new branching trees of degrees at most $2\deg(\to') - 1$ created by the rewrite rule created in Step *(6)*.

(*8*) Go to Step *(2)*.

The set of rewrite rules $\to'$ outputted by this semi-algorithm is a *completion* of $\to$. By Lemma 1.3.4, $\Rightarrow'$ is confluent. Notice that, for certain inputs $\to$, this semi-algorithm never stops. Notice also that the computed completion depends on the total order $\preccurlyeq$ on the binary trees of a same arity, the choices at Steps *(3)* and *(4)* as well as the computed normal forms at Step *(5)*.

2. Quotients of the linear magmatic operad

In this section, we equip the set of quotients of the linear magmatic operad with a lattice structure. We also show a Grassmann formula analog for this lattice.

2.1. Lattice structure.

The *linear magmatic operad*, written $\mathbb{K}\langle\mathbf{Mag}\rangle$, is the free linear operad over one binary generator. By definition, for each arity n, $\mathbb{K}\langle\mathbf{Mag}\rangle(n)$ is the vector space with basis $\mathbf{Mag}(n)$ and the compositions maps of $\mathbb{K}\langle\mathbf{Mag}\rangle$ are the extensions by linearity the ones of $\mathbf{Mag}$.

We denote by $\mathcal{I}(\mathbb{K}\langle\mathbf{Mag}\rangle)$ the set of operad ideals of $\mathbb{K}\langle\mathbf{Mag}\rangle$ and we set

$$\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle) := \left\{\mathbb{K}\langle\mathbf{Mag}\rangle/_I : I \in \mathcal{I}(\mathbb{K}\langle\mathbf{Mag}\rangle)\right\}, \tag{2.1.1}$$

as the set of all quotients of the linear magmatic operad. Given $\mathbb{K}\langle\mathbf{Mag}\rangle/_I \in \mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$ and $x \in \mathbb{K}\langle\mathbf{Mag}\rangle$, we denote by $[x]_I$ the I-equivalence class of x. Observe that $\mathbb{K}\langle\mathbf{Mag}\rangle/_I$ is generated as an operad by $[\star]_I$ (where, recall, $\star$ is the binary generator of $\mathbf{Mag}$ and thus also of $\mathbb{K}\langle\mathbf{Mag}\rangle$). Moreover, given two elements $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$, we denote by $\mathrm{Hom}(\mathcal{O}_1, \mathcal{O}_2)$ the set of linear operad morphisms from $\mathcal{O}_1$ to $\mathcal{O}_2$.

Proposition 2.1.1. *For any $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$, the set $\mathrm{Hom}(\mathcal{O}_1, \mathcal{O}_2)$ admits a vector space structure. Moreover, its dimension is equal to 0 or 1 and every nonzero morphism is surjective.*

Proof. Let $I_1, I_2 \in \mathcal{I}(\mathbb{K}\langle\mathbf{Mag}\rangle)$ such that $\mathcal{O}_1 = \mathbb{K}\langle\mathbf{Mag}\rangle/_{I_1}$ and $\mathcal{O}_2 = \mathbb{K}\langle\mathbf{Mag}\rangle/_{I_2}$. Since $\mathcal{O}_1$ is generated by the binary element $[\star]_{I_1}$, a morphism $\varphi : \mathcal{O}_1 \to \mathcal{O}_2$ is uniquely determined by $\varphi([\star]_{I_1})$. Moreover, $\varphi([\star]_{I_1})$ has arity 2 in $\mathcal{O}_2$. Hence, $\varphi([\star]_{I_1})$ belongs to the line spanned by the binary generator of $\mathcal{O}_2$, that is there exists a scalar $\lambda \in \mathbb{K}$ such that $\varphi([\star]_{I_1}) = \lambda[\star]_{I_2}$. If there exists such a λ different from zero, then for every nonzero scalar μ, we have a well-defined operad morphism $\psi : \mathcal{O}_1 \to \mathcal{O}_2$ satisfying $\psi([\star]_{I_1}) = \mu[\star]_{I_2} = \left(\mu\lambda^{-1}\right)\varphi([\star]_{I_1})$. Hence, $\mathrm{Hom}(\mathcal{O}_1, \mathcal{O}_2)$ is either reduced to the zero morphism or it is in one-to-one correspondence with $\mathbb{K}$, which proves that $\mathrm{Hom}(\mathcal{O}_1, \mathcal{O}_2)$ is a vector space of dimension at most 1. Moreover, if φ is different from 0, that is there is a nonzero scalar such that $\varphi([\star]_{I_1}) = \lambda[\star]_{I_2}$, we have $\varphi\left(\lambda^{-1}[\star]_{I_1}\right) = [\star]_{I_2}$, so that φ is surjective. $\square$

We introduce the binary relation $\preceq_{\mathrm{i}}$ on $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$ as follows: we have $\mathcal{O}_2 \preceq_{\mathrm{i}} \mathcal{O}_1$ if the dimension of $\mathrm{Hom}(\mathcal{O}_1, \mathcal{O}_2)$ is equal to 1.

Proposition 2.1.2. *Let $\mathcal{O}_1 = \mathbb{K}\langle\mathbf{Mag}\rangle/_{I_1}$ and $\mathcal{O}_2 = \mathbb{K}\langle\mathbf{Mag}\rangle/_{I_2}$ be two operads of $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$. We have $\mathcal{O}_2 \preceq_{\mathrm{i}} \mathcal{O}_1$ if and only if $I_1 \subseteq I_2$.*

Proof. Since, by Proposition 2.1.1, $\mathrm{Hom}(\mathcal{O}_1, \mathcal{O}_2)$ is a vector space of dimension at most 1, it contains a nonzero morphism if and only if the morphism $\bar{\varphi} : \mathcal{O}_1 \to \mathcal{O}_2$ satisfying $\bar{\varphi}([\star]_{I_1}) = [\star]_{I_2}$ is well-defined, which means that $\mathcal{O}_2 \preceq_{\mathrm{i}} \mathcal{O}_1$ is equivalent to this condition. Moreover, by the universal property of the quotient, $\bar{\varphi}$ is well-defined if and only if I_1 is included in the kernel of the morphism $\varphi : \mathbb{K}\langle\mathbf{Mag}\rangle \to \mathcal{O}_2$ defined by $\varphi(\star) = [\star]_{I_2}$. This kernel is equal to I_2, so that $\bar{\varphi}$ is well-defined if and only if I_1 is included in I_2, which concludes the proof. □

Recall that a *lattice* is a tuple $(E, \leqslant, \wedge, \vee)$ where $\leqslant$ is a partial order relation such that any two elements e and e' of E admit a lower-bound $e \wedge e'$ and an upper-bound $e \vee e'$. In particular, $(\mathcal{I}(\mathbb{K}\langle\mathbf{Mag}\rangle), \subseteq, \cap, +)$ is a lattice, where $\cap$ and $+$ are the intersection and the sum of operad ideals, respectively.

Given two operads $\mathcal{O}_1 = \mathbb{K}\langle\mathbf{Mag}\rangle /_{I_1}$ and $\mathcal{O}_2 = \mathbb{K}\langle\mathbf{Mag}\rangle /_{I_2}$ of $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$, let us define

$$\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2 := \mathbb{K}\langle\mathbf{Mag}\rangle /_{I_1 + I_2} \tag{2.1.2}$$

and

$$\mathcal{O}_1 \vee_{\mathrm{i}} \mathcal{O}_2 := \mathbb{K}\langle\mathbf{Mag}\rangle /_{I_1 \cap I_2}. \tag{2.1.3}$$

Explicitly, for every positive integer n, $(\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2)(n)$ (resp. $(\mathcal{O}_1 \vee_{\mathrm{i}} \mathcal{O}_2)(n)$) is the quotient vector space $\mathbb{K}\langle\mathbf{Mag}\rangle(n)/_{I_1(n)+I_2(n)}$ (resp. $\mathbb{K}\langle\mathbf{Mag}\rangle(n)/_{I_1(n)\cap I_2(n)}$).

Theorem 2.1.3. *The tuple* $(\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle), \preceq_{\mathrm{i}}, \wedge_{\mathrm{i}}, \vee_{\mathrm{i}})$ *is a lattice.*

Proof. First, we observe that the map $\mathcal{O} : \mathcal{I}(\mathbb{K}\langle\mathbf{Mag}\rangle) \to \mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$ defined by $\mathcal{O}(I) := \mathbb{K}\langle\mathbf{Mag}\rangle /_I$ is a bijection: it is surjective by definition of $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$ and it is injective since $\mathcal{O}(I_1) = \mathcal{O}(I_2)$ implies that the kernel of the natural projection $\mathbb{K}\langle\mathbf{Mag}\rangle \to \mathcal{O}(I_1) = \mathcal{O}(I_2)$ is equal to both I_1 and I_2. Moreover, from Proposition 2.1.2, $\mathcal{O}_2 \preceq_{\mathrm{i}} \mathcal{O}_1$ is equivalent to $I_1 \subseteq I_2$, so that $\preceq_{\mathrm{i}}$ is a partial order relation on $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$ and $\mathcal{O}$ is a decreasing bijection. The tuple $(\mathcal{I}(\mathbb{K}\langle\mathbf{Mag}\rangle), \subseteq, \cap, +)$ being a lattice, the decreasing bijection $\mathcal{O}$ induces lattice operations on $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$, precisely $\wedge_{\mathrm{i}}$ and $\vee_{\mathrm{i}}$ by definition. □

The union of generating sets of two operad ideals I_1 and I_2 is a generating set of $I_1 + I_2$, so that the union of generating relations for the two operads $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$ forms a generating set for the relations of $\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2$. However, the authors do not know how to compute a generating set of the intersection of ideals (it is not the intersection of the generating relations), so that we do not know any general method to compute generating relations for $\mathcal{O}_1 \vee_{\mathrm{i}} \mathcal{O}_2$.

2.2. Hilbert series and Grassmann formula.

The statement of the Grassmann formula analog for $(\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle), \preceq_{\mathrm{i}}, \wedge_{\mathrm{i}}, \vee_{\mathrm{i}})$ is the following.

Theorem 2.2.1. *Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be two operads of* $\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle)$*. We have*

$$\mathcal{H}_{\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2}(t) + \mathcal{H}_{\mathcal{O}_1 \vee_{\mathrm{i}} \mathcal{O}_2}(t) = \mathcal{H}_{\mathcal{O}_1}(t) + \mathcal{H}_{\mathcal{O}_2}(t). \tag{2.2.1}$$

Proof. Let $I_1, I_2 \in \mathcal{I}(\mathbb{K}\langle \mathbf{Mag} \rangle)$ be such that $\mathcal{O}_1 = \mathbb{K}\langle \mathbf{Mag} \rangle /_{I_1}$ and $\mathcal{O}_2 = \mathbb{K}\langle \mathbf{Mag} \rangle /_{I_2}$. For every positive integer n, we have

$$\dim\left(\left(\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2\right)(n)\right) + \dim\left(\left(\mathcal{O}_1 \vee_{\mathrm{i}} \mathcal{O}_2\right)(n)\right) = \dim\left(\mathcal{O}_1(n)\right) + \dim\left(\mathcal{O}_2(n)\right). \tag{2.2.2}$$

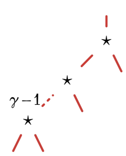
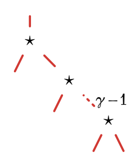
Indeed,

$$\begin{aligned}
&\dim(\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2(n)) + \dim\left(\mathcal{O}_1 \vee_{\mathrm{i}} \mathcal{O}_2(n)\right) \\
&\quad = \dim\left(\mathbb{K}\langle \mathbf{Mag} \rangle (n)/_{(I_1+I_2)(n)}\right) + \dim\left(\mathbb{K}\langle \mathbf{Mag} \rangle (n)/_{(I_1 \cap I_2)(n)}\right) \\
&\quad = \dim\left(\mathbb{K}\langle \mathbf{Mag} \rangle (n)\right) - \dim\left(I_1(n) + I_2(n)\right) + \dim\left(\mathbb{K}\langle \mathbf{Mag} \rangle (n)\right) - \dim\left(I_1(n) \cap I_2(n)\right) \\
&\quad = \dim\left(\mathbb{K}\langle \mathbf{Mag} \rangle (n)\right) - \dim\left(I_1(n)\right) + \dim\left(\mathbb{K}\langle \mathbf{Mag} \rangle (n)\right) - \dim\left(I_2(n)\right) \\
&\quad = \dim\left(\mathbb{K}\langle \mathbf{Mag} \rangle (n)/_{I_1(n)}\right) + \dim\left(\mathbb{K}\langle \mathbf{Mag} \rangle (n)/_{I_2(n)}\right) \\
&\quad = \dim\left(\mathcal{O}_1(n)\right) + \dim\left(\mathcal{O}_2(n)\right).
\end{aligned} \tag{2.2.3}$$

The third equality is due to the Grassmann formula [Lan02] applied to the subspaces $I_1(n)$ and $I_2(n)$ of $\mathbb{K}\langle \mathbf{Mag} \rangle (n)$.

From (2.2.2), and for every positive integer n, the terms of degree n in the left and right members of (2.2.1) are equal, which proves Theorem 2.2.1. □

We terminate this section with an example illustrating the lattice constructions on $\mathcal{Q}(\mathbb{K}\langle \mathbf{Mag} \rangle)$. For that, we introduce various operads, which requires the following notations, also used in Section 3. Let, for any integer $\gamma \geqslant 1$, the binary trees $\mathfrak{c}^{(\gamma)}_{\diagup}$ and $\mathfrak{c}^{(\gamma)}_{\diagdown}$ be respectively the left and the right combs of degree γ. These trees are depicted as

$$\mathfrak{c}^{(\gamma)}_{\diagup} =$$  and $$\mathfrak{c}^{(\gamma)}_{\diagdown} =$$  , (2.2.4)

where the values on the dotted edges denote the number of internal nodes they contain.

We first recall that the *linear associative operad* is

$$\mathbb{K}\langle \mathbf{As} \rangle := \mathbb{K}\langle \mathbf{Mag} \rangle /_{I_{\mathbb{K}\langle \mathbf{As} \rangle}}, \tag{2.2.5}$$

where $I_{\mathbb{K}\langle \mathbf{As} \rangle}$ is the ideal spanned $\mathfrak{c}^{(2)}_{\diagup} - \mathfrak{c}^{(2)}_{\diagdown}$, and that its Hilbert series is

$$\mathcal{H}_{\mathbb{K}\langle \mathbf{As} \rangle}(t) = \sum_{n \geqslant 1} t^n, \tag{2.2.6}$$

We define the *anti-associative operad* by

$$\mathbf{AAs} := \mathbb{K}\langle \mathbf{Mag} \rangle /_{I_{\mathbf{AAs}}}, \tag{2.2.7}$$

where $I_{\mathbf{AAs}}$ is the ideal spanned by $\mathfrak{c}^{(2)}_{\diagup} + \mathfrak{c}^{(2)}_{\diagdown}$. Using the Buchberger algorithm for operads [DK10, Section 3.7], we check that the set of rewrite rules

$$\left\{\mathfrak{c}^{(2)}_{\diagup} \to -\mathfrak{c}^{(2)}_{\diagdown},\ \mathfrak{c}^{(3)}_{\diagdown} \to 0\right\} \tag{2.2.8}$$

is a convergent presentation of **AAs**. We point out that this statement is false if the characteristic of $\mathbb{K}$ is equal to 2. Moreover, using the convergent presentation (2.2.8), we have

$$\mathcal{H}_{\mathbf{AAs}}(t) = t + t^2 + t^3. \tag{2.2.9}$$

Let us consider the *2-nilpotent operad* [Zin12] defined by

$$2\mathbf{Nil} := \mathbb{K}\langle\mathbf{Mag}\rangle /_{I_{2\mathbf{Nil}}}, \tag{2.2.10}$$

where $I_{2\mathbf{Nil}}$ is the ideal spanned by the two trees $\mathfrak{c}_{\diagup}^{(2)}$ and $\mathfrak{c}_{\diagdown}^{(2)}$. We have

$$\mathcal{H}_{2\mathbf{Nil}}(t) = t + t^2. \tag{2.2.11}$$

We introduce, for every integer $\gamma \geqslant 2$, the (nonlinear) *γ-right comb operad* $\mathbf{RC}^{(\gamma)}$ as follows. For every arity n, we let

$$\mathbf{RC}^{(\gamma)}(n) := \begin{cases} \mathbf{Mag}(n) & \text{if } n \leqslant \gamma, \\ \mathfrak{c}_{\diagdown}^{(n-1)} & \text{otherwise}, \end{cases} \tag{2.2.12}$$

and the partial composition $\mathfrak{t}_1 \circ_i \mathfrak{t}_2$ is the partial composition of $\mathfrak{t}_1$ and $\mathfrak{t}_2$ in **Mag** if the integer $n := |\mathfrak{t}_1| + |\mathfrak{t}_2| - 1$ is smaller than or equal to γ, and $\mathfrak{c}_{\diagdown}^{(n)}$ otherwise. Moreover, by definition of the γ-right comb operad, we have

$$\mathcal{H}_{\mathbf{RC}^{(\gamma)}}(t) = \sum_{1 \leqslant n \leqslant \gamma} \operatorname{cat}(n-1) t^n + \sum_{n \geqslant \gamma+1} t^n. \tag{2.2.13}$$

Lemma 2.2.2. *We have an isomorphism*

$$\mathbf{RC}^{(\gamma)} \simeq \mathbf{Mag}/_{\equiv_{(\gamma)}}, \tag{2.2.14}$$

where $\equiv_{(\gamma)}$ is the smallest operad congruence satisfying $\mathfrak{t} \equiv_{(\gamma)} \mathfrak{c}_{\diagdown}^{(\gamma)}$, where $\mathfrak{t}$ runs over all the binary trees of arity $\gamma + 1$. In other words, $\mathbf{RC}^{(\gamma)}$ is a combinatorial realization of $\mathbf{Mag}/_{\equiv_{(\gamma)}}$.

Proof. Let $\to$ be the set of rewrite rules $\mathfrak{t} \to \mathfrak{c}_{\diagdown}^{(\gamma)}$, where $\mathfrak{t}$ runs over all the binary trees of arity $\gamma + 1$ different from $\mathfrak{c}_{\diagdown}^{(\gamma)}$. The unique normal form of arity $n \geqslant \gamma + 1$ for the rewrite relation $\Rightarrow$ induced by $\to$ is $\mathfrak{c}_{\diagdown}^{(n-1)}$, so that $\to$ is a convergent presentation of $\mathbf{Mag}/_{\equiv_{(\gamma)}}$. Moreover, the normal forms for $\Rightarrow$ of arity $n \leqslant \gamma$ are all the trees of arity n and, by using the convergent presentation $\to$, the compositions of $\mathbf{Mag}/_{\equiv_{(\gamma)}}$ satisfy (2.2.12). Hence, $\to$ is also a convergent presentation of $\mathbf{RC}^{(\gamma)}$ which proves the statement of the lemma. □

Now, we define the *linear γ-right comb operad* $\mathbb{K}\langle\mathbf{RC}^{(\gamma)}\rangle$ as the linear operad spanned by $\mathbf{RC}^{(\gamma)}$. In particular, its Hilbert series is given in (2.2.13), and Lemma 2.2.2 implies that we have $\mathbb{K}\langle\mathbf{RC}^{(\gamma)}\rangle = \mathbb{K}\langle\mathbf{As}\rangle /_{I_{\mathbf{RC}^{(\gamma)}}}$ where $I_{\mathbf{RC}^{(\gamma)}}$ is the ideal spanned by the elements $\mathfrak{t} - \mathfrak{c}_{\diagdown}^{(\gamma)}$, with $\mathfrak{t}$ a binary tree of arity $\gamma + 1$.

The lower-bound and the upper-bound of $\mathbb{K}\langle\mathbf{As}\rangle$ and **AAs** in the lattice $(\mathcal{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle), \preceq_{\mathrm{i}}, \wedge_{\mathrm{i}}, \vee_{\mathrm{i}})$ are described by the following.

Theorem 2.2.3. *We have*

$$\mathbb{K}\langle \mathbf{As}\rangle \wedge_{\mathrm{i}} \mathbf{AAs} = 2\mathbf{Nil} \tag{2.2.15}$$

and

$$\mathbb{K}\langle \mathbf{As}\rangle \vee_{\mathrm{i}} \mathbf{AAs} = \mathbb{K}\langle \mathbf{RC}^{(3)}\rangle. \tag{2.2.16}$$

Proof. The ideal of relations of $\mathbb{K}\langle \mathbf{As}\rangle \wedge_{\mathrm{i}} \mathbf{AAs}$ is equal to $I_{\mathbb{K}\langle \mathbf{As}\rangle} + I_{\mathbf{AAs}}$, so that it is spanned by the two elements $\mathfrak{c}^{(2)}_{\diagup} - \mathfrak{c}^{(2)}_{\diagdown}$ and $\mathfrak{c}^{(2)}_{\diagup} + \mathfrak{c}^{(2)}_{\diagdown}$. By linear transformations applied to these generators, $I_{\mathbb{K}\langle \mathbf{As}\rangle} + I_{\mathbf{AAs}}$ is spanned by $\mathfrak{c}^{(2)}_{\diagup}$ and $\mathfrak{c}^{(2)}_{\diagdown}$, that is, it is equal to $I_{2\mathbf{Nil}}$, which proves (2.2.15).

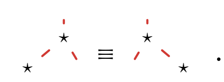
Let us now denote by $\pi : \mathbb{K}\langle \mathbf{Mag}\rangle \to \mathbb{K}\langle \mathbf{As}\rangle \vee_{\mathrm{i}} \mathbf{AAs}$ the natural projection. Let $\mathfrak{t}$ be a tree of arity 4 and let us define $\alpha_{\mathfrak{t}} := \mathfrak{t} - \mathfrak{c}^{(3)}_{\diagdown}$. The elements $\alpha_{\mathfrak{t}}$ belong to $I_{\mathbb{K}\langle \mathbf{As}\rangle}$ and to $I_{\mathbf{AAs}}$ since both $[\mathfrak{t}]_{I_{\mathbf{AAs}}}$ and $[\mathfrak{c}^{(3)}_{\diagdown}]_{I_{\mathbf{AAs}}}$ are equal to $[0]_{I_{\mathbf{AAs}}}$. The last statement is shown using the convergent presentation (2.2.8) of $\mathbf{AAs}$. Hence, the ideal generated by the elements $\alpha_{\mathfrak{t}}$, that is the ideal of relations of $\mathbb{K}\langle \mathbf{RC}^{(3)}\rangle$, is included in $I_{\mathbb{K}\langle \mathbf{As}\rangle} \cap I_{\mathbf{AAs}} = \ker(\pi)$, so that π induces a surjective morphism $\bar{\pi} : \mathbb{K}\langle \mathbf{RC}^{(3)}\rangle \to \mathbb{K}\langle \mathbf{As}\rangle \vee_{\mathrm{i}} \mathbf{AAs}$. We conclude by using Hilbert series: $\mathcal{H}_{\mathbb{K}\langle \mathbf{As}\rangle \vee_{\mathrm{i}} \mathbf{AAs}}(t)$ is computed by using the Grassmann formula analog with Formulas (2.2.6), (2.2.9), and (2.2.11), and it turns out to be equal to $\mathcal{H}_{\mathbb{K}\langle \mathbf{RC}^{(3)}\rangle}(t)$ which is given in (2.2.13). Hence, $\bar{\pi}$ is an isomorphism, which proves (2.2.16). □

3. Generalizations of the associative operad

In this section, we define comb associative operads and we show that the set of such operads admits a lattice structure, isomorphic to the lattice of division for nonnegative integers. We relate this lattice to the one of the linear magmatic quotients considered in the previous section. We also provide a finite convergent presentation of the comb associative operad corresponding to 3.

3.1. Comb associative operads.

Recall first that the *associative operad* $\mathbf{As}$ is the quotient of $\mathbf{Mag}$ by the smallest operad congruence $\equiv$ satisfying



(3.1.1)

We propose here a generalization of $\equiv$ in order to define generalizations of $\mathbf{As}$.

As in Section 2, the left and the right combs of degree γ are denoted by $\mathfrak{c}^{(\gamma)}_{\diagup}$ and $\mathfrak{c}^{(\gamma)}_{\diagdown}$, respectively. In the sequel , we shall employ the drawing convention introduced after (2.2.4): the values on dotted edges in a binary tree denote the number of internal nodes they contain. Moreover, we also employ the convention stipulating that dotted edges with no value have any number of internal nodes. Let us now define for any $\gamma \geqslant 1$ the *γ-comb associative operad* $\mathbf{CAs}^{(\gamma)}$ as the quotient operad $\mathbf{Mag}/_{\equiv^{(\gamma)}}$ where $\equiv^{(\gamma)}$ is the smallest operad congruence of $\mathbf{Mag}$ satisfying

$$\mathfrak{c}^{(\gamma)}_{\diagup} \equiv^{(\gamma)} \mathfrak{c}^{(\gamma)}_{\diagdown}. \tag{3.1.2}$$

Notice that $\equiv^{(1)}$ is trivial so that $\mathbf{CAs}^{(1)} = \mathbf{Mag}$, and that $\equiv^{(2)}$ is the operad congruence defining $\mathbf{As}$ so that $\mathbf{CAs}^{(2)} = \mathbf{As}$. Let also

$$\mathbf{CAs} := \left\{ \mathbf{CAs}^{(\gamma)} : \gamma \geqslant 1 \right\} \tag{3.1.3}$$

be the set of all the γ-comb associative operads.

3.2. **Lattice of comb associative operads.** In order to introduce a lattice structure on $\mathbf{CAs}$, we begin by studying operad morphisms between its elements by mean of intermediate lemmas.

Lemma 3.2.1. *For all positive integers γ and n such that $\gamma \geqslant 2$ and $n \leqslant \gamma + 1$,*

$$\#\mathbf{CAs}^{(\gamma)}(n) = \mathrm{cat}(n-1) - \delta_{n,\gamma+1}, \tag{3.2.1}$$

where $\delta_{x,y}$ is the Kronecker delta.

Proof. Since the equivalence relation $\equiv^{(\gamma)}$ is trivial on the binary trees of degrees $d < \gamma$, and since a binary tree of degree d has arity $n := d + 1$, one has $\#\mathbf{CAs}^{(\gamma)}(n) = \#\mathbf{Mag}(n) = \mathrm{cat}(n-1)$ with $n \leqslant \gamma$. Besides, by definition of $\equiv^{(\gamma)}$, all the $\equiv^{(\gamma)}$-equivalence classes of binary trees of degree γ are trivial, except one due to the fact that $\mathfrak{c}^{(\gamma)}_{\diagup} \neq \mathfrak{c}^{(\gamma)}_{\diagdown}$ and $\mathfrak{c}^{(\gamma)}_{\diagup} \equiv^{(\gamma)} \mathfrak{c}^{(\gamma)}_{\diagdown}$. Therefore, since a binary tree of degree γ has arity $n := \gamma + 1$, $\#\mathbf{CAs}^{(\gamma)}(n) = \#\mathbf{Mag}(\gamma + 1) - 1 = \mathrm{cat}(\gamma + 1 - 1) - 1 = \mathrm{cat}(n-1) - 1$ as stated. □

Lemma 3.2.2. *Let γ and γ' be two positive integers. If there exists an operad morphism $\varphi : \mathbf{CAs}^{(\gamma')} \to \mathbf{CAs}^{(\gamma)}$, then it is surjective and satisfies $\varphi\left([\mathfrak{t}]_{\equiv^{(\gamma')}}\right) = [\mathfrak{t}]_{\equiv^{(\gamma)}}$ for any binary tree $\mathfrak{t}$.*

Proof. The operad $\mathbf{CAs}^{(\gamma')}$ is generated by one binary generator $[\star]_{\equiv^{(\gamma')}}$, which is the image of the binary generator $\star$ of $\mathbf{Mag}$ in $\mathbf{CAs}^{(\gamma')}$. Hence, φ is entirely determined by the image $\varphi\left([\star]_{\equiv^{(\gamma')}}\right)$. Moreover, $\varphi([\star]_{\equiv^{(\gamma')}})$ has to be of arity 2 in $\mathbf{CAs}^{(\gamma)}$, so that we necessarily have $\varphi\left([\star]_{\equiv^{(\gamma')}}\right) = [\star]_{\equiv^{(\gamma)}}$. Hence, if φ exists, it is the unique operad morphism from $\mathbf{CAs}^{(\gamma')}$ to $\mathbf{CAs}^{(\gamma)}$ determined by the image of $[\star]_{\equiv^{(\gamma')}}$. In this case, $[\star]_{\equiv^{(\gamma)}}$ being in the image of φ, the latter is surjective. Finally, it follows that φ sends $[\mathfrak{t}]_{\equiv^{(\gamma')}}$ to $[\mathfrak{t}]_{\equiv^{(\gamma)}}$ by induction on the degree of the binary tree $\mathfrak{t}$. □

Lemma 3.2.3. *Let γ and γ' be two positive integers and $\varphi : \mathbf{CAs}^{(\gamma')} \to \mathbf{CAs}^{(\gamma)}$ be an operad morphism. Then, φ is injective if and only if $\gamma = \gamma'$.*

Proof. Assume that φ is injective. By Lemma 3.2.2, φ is also surjective, so that φ is an isomorphism. If $\gamma \neq \gamma'$, by Lemma 3.2.1, there is a positive integer n such that $\#\mathbf{CAs}^{(\gamma)}(n) \neq \#\mathbf{CAs}^{(\gamma')}(n)$. This is contradictory with the fact that $\mathbf{CAs}^{(\gamma)}$ and $\mathbf{CAs}^{(\gamma')}$ are isomorphic. Hence, $\gamma = \gamma'$.

Conversely, if $\gamma = \gamma'$, the only operad morphism from $\mathbf{CAs}^{(\gamma)}$ to itself sends the generator $[\star]_{\equiv^{(\gamma)}}$ to itself. This maps extends as an operad morphism into the identity morphism which is of course injective. □

Lemma 3.2.4. *Let γ and γ' be two positive integers. There exists an operad morphism $\varphi : \mathbf{CAs}^{(\gamma')} \to \mathbf{CAs}^{(\gamma)}$ if and only if $\mathfrak{c}^{(\gamma')}_{\diagup} \equiv^{(\gamma)} \mathfrak{c}^{(\gamma')}_{\diagdown}$.*

Proof. Assume that $\varphi : \mathbf{CAs}^{(\gamma')} \to \mathbf{CAs}^{(\gamma)}$ is an operad morphism. Since $\mathfrak{c}_{\diagup}^{(\gamma')} \equiv^{(\gamma')} \mathfrak{c}_{\diagdown}^{(\gamma')}$, we have

$$\varphi\left(\left[\mathfrak{c}_{\diagup}^{(\gamma')}\right]_{\equiv^{(\gamma')}}\right) = \varphi\left(\left[\mathfrak{c}_{\diagdown}^{(\gamma')}\right]_{\equiv^{(\gamma')}}\right). \tag{3.2.2}$$

Now, by using Lemma 3.2.2, we obtain from (3.2.2) the relation

$$\left[\mathfrak{c}_{\diagup}^{(\gamma')}\right]_{\equiv^{(\gamma)}} = \left[\mathfrak{c}_{\diagdown}^{(\gamma')}\right]_{\equiv^{(\gamma)}}, \tag{3.2.3}$$

saying that $\mathfrak{c}_{\diagup}^{(\gamma')} \equiv^{(\gamma)} \mathfrak{c}_{\diagdown}^{(\gamma')}$ as excepted.

Conversely, when $\mathfrak{c}_{\diagup}^{(\gamma')} \equiv^{(\gamma)} \mathfrak{c}_{\diagdown}^{(\gamma')}$, let $\varphi : \mathbf{CAs}^{(\gamma')}(2) \to \mathbf{CAs}^{(\gamma)}(2)$ be the map defined by $\varphi\left([\star]_{\equiv^{(\gamma')}}\right) := [\star]_{\equiv^{(\gamma)}}$. Now, since $\equiv^{(\gamma)}$ is coarser than $\equiv^{(\gamma')}$, φ extends (in a unique way) into an operad morphism, whence the statement of the lemma. □

We define the binary relation $\preceq_{\mathrm{d}}$ on $\mathbf{CAs}$ as follows: we have $\mathbf{CAs}^{(\gamma)} \preceq_{\mathrm{d}} \mathbf{CAs}^{(\gamma')}$ if and only if there exists a morphism $\varphi : \mathbf{CAs}^{(\gamma')} \to \mathbf{CAs}^{(\gamma)}$.

Proposition 3.2.5. *The binary relation $\preceq_{\mathrm{d}}$ is a partial order relation on* $\mathbf{CAs}$*.*

Proof. The binary relation $\preceq_{\mathrm{d}}$ is reflexive since there exists the identity morphism on $\mathbf{CAs}^{(\gamma)}$ for every positive integer γ. It is transitive since the composite of operad morphisms is an operad morphism. Finally, let us assume that there exist tow morphisms $\varphi : \mathbf{CAs}^{(\gamma')} \to \mathbf{CAs}^{(\gamma)}$ and $\psi : \mathbf{CAs}^{(\gamma)} \to \mathbf{CAs}^{(\gamma')}$. In particular, $\psi \circ \varphi$ and $\varphi \circ \psi$ are endomorphisms of $\mathbf{CAs}^{(\gamma')}$ and $\mathbf{CAs}^{(\gamma)}$, respectively. From Lemma 3.2.2, these two morphisms are identity morphisms, so that φ and ψ are injective. From Lemma 3.2.3, γ and γ' are equal, which proves that $\preceq_{\mathrm{d}}$ is anti-symmetric. Hence, $\preceq_{\mathrm{d}}$ is a partial order. □

In order to show that $(\mathbf{CAs}, \preceq_{\mathrm{d}})$ extends into a lattice, we relate $(\mathbf{CAs}, \preceq_{\mathrm{d}})$ with the lattice of integers $(\mathbb{N}, |, \gcd, \mathrm{lcm})$, where $|$ denotes the division relation, gcd denotes the greatest common divisor, and lcm the least common multiple operators, respectively.

Recall that $\mathrm{lr}(\mathfrak{t})$ denotes the left rank of a binary tree $\mathfrak{t}$, as defined in Section 1. Besides, to simplify the notation, we shall write $\bar{a}$ instead of $a - 1$ for any integer a.

Lemma 3.2.6. *Let $\gamma \geqslant 2$ be an integer and let $\mathfrak{t}$ and $\mathfrak{t}'$ be two binary trees. If $\mathfrak{t} \equiv^{(\gamma)} \mathfrak{t}'$, then*

$$\mathrm{lr}(\mathfrak{t}) \pmod{\bar{\gamma}} = \mathrm{lr}\left(\mathfrak{t}'\right) \pmod{\bar{\gamma}}. \tag{3.2.4}$$

Proof. Consider here the rewrite rule $\to$ on $\mathbf{Mag}$ satisfying $\mathfrak{c}_{\diagup}^{(\gamma)} \to \mathfrak{c}_{\diagdown}^{(\gamma)}$. Let us show that the rewrite relation $\Rightarrow$ induced by $\to$ is such that $\mathfrak{t} \Rightarrow \mathfrak{t}'$ implies (3.2.4). Any binary tree $\mathfrak{t}$ decomposes as $\mathfrak{t} = \mathfrak{c}_{\diagup}^{(\mathrm{lr}(\mathfrak{t}))} \circ \left[|, \mathfrak{t}_1, \dots, \mathfrak{t}_{\mathrm{lr}(\mathfrak{t})}\right]$ where the $\mathfrak{t}_i$ are binary trees. Now, if $\mathfrak{t} \Rightarrow \mathfrak{t}'$, then one among the following two cases occurs.

(*i*) The rewrite step is applied into one of the trees $\mathfrak{t}_i$, that is there exists $\mathfrak{t}'_i$ such that $\mathfrak{t}' = \mathfrak{c}_{\diagup}^{(\mathrm{lr}(\mathfrak{t}))} \circ \left[|, \mathfrak{t}_1, \dots, \mathfrak{t}'_i, \dots, \mathfrak{t}_{\mathrm{lr}(\mathfrak{t})}\right]$ so that $\mathrm{lr}\left(\mathfrak{t}'\right) = \mathrm{lr}\left(\mathfrak{t}\right)$.

(*ii*) The rewrite step is applied into the left branch beginning at the root of $\mathfrak{t}$, that is there exists i such that $\mathfrak{t}' = \mathfrak{c}_{\diagup}^{(\mathrm{lr}(\mathfrak{t}) - \bar{\gamma})} \circ \left[|, \mathfrak{t}_1, \dots, \mathfrak{c}_{\diagdown}^{(\bar{\gamma})} \circ \left[\mathfrak{t}_i, \dots, \mathfrak{t}_{i+\bar{\gamma}}\right], \dots, \mathfrak{t}_{\mathrm{lr}(\mathfrak{t})}\right]$ so that $\mathrm{lr}\left(\mathfrak{t}'\right) = \mathrm{lr}\left(\mathfrak{t}\right) - \bar{\gamma} = \mathrm{lr}\left(\mathfrak{t}\right) \pmod{\bar{\gamma}}$.

This implies (3.2.4). Finally, since $\equiv^{(\gamma)}$ is the reflexive, symmetric, and transitive closure of $\Rightarrow$, the statement of the lemma follows. $\square$

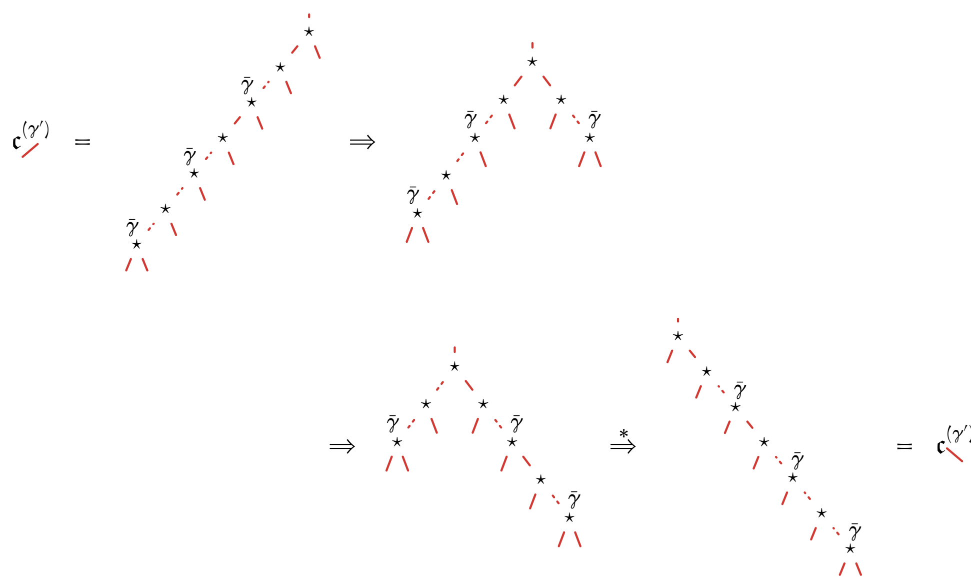
Proposition 3.2.7. *Let γ and γ' be two positive integers such that $\gamma \geqslant 2$. Then, there exists a morphism $\varphi : \mathbf{CAs}^{(\gamma')} \to \mathbf{CAs}^{(\gamma)}$ if and only if $\bar{\gamma} \mid \bar{\gamma}'$.*

Proof. From Lemma 3.2.4, it is enough to show that $\mathfrak{c}^{(\gamma')}_{\diagup} \equiv^{(\gamma)} \mathfrak{c}^{(\gamma')}_{\diagdown}$ if and only if $\bar{\gamma} \mid \bar{\gamma}'$. If $\mathfrak{c}^{(\gamma')}_{\diagup} \equiv^{(\gamma)} \mathfrak{c}^{(\gamma')}_{\diagdown}$, as a consequence of the existence of a surjective morphism φ from $\mathbf{CAs}^{(\gamma')}$ to $\mathbf{CAs}^{(\gamma)}$ and Lemmas 3.2.1 and 3.2.2, one has $\gamma' = 1$ or $\gamma \leqslant \gamma'$. Since

$$\mathrm{lr}\left(\mathfrak{c}^{(\gamma')}_{\diagup}\right) - \mathrm{lr}\left(\mathfrak{c}^{(\gamma')}_{\diagdown}\right) = \gamma' - 1 = \bar{\gamma}', \tag{3.2.5}$$

by using Lemma 3.2.6, we deduce that $\bar{\gamma}'$ is divisible by $\bar{\gamma}$, which shows the direct implication.

Conversely, if $\bar{\gamma} \mid \bar{\gamma}'$, the rewrite rule $\to$ on **Mag** satisfying $\mathfrak{c}^{(\gamma)}_{\diagup} \to \mathfrak{c}^{(\gamma)}_{\diagdown}$ induces the sequence



(3.2.6)

of rewrite steps, where dotted edges denotes left or right comb trees of degree $\gamma - 1$. Hence, since $\equiv^{(\gamma)}$ is the reflexive, symmetric, and transitive closure of $\Rightarrow$, we have $\mathfrak{c}^{(\gamma')}_{\diagup} \equiv^{(\gamma)} \mathfrak{c}^{(\gamma')}_{\diagdown}$. $\square$

Proposition 3.2.7 implies the following result.

Theorem 3.2.8. *The tuple $(\mathbf{CAs}, \preceq_{\mathrm{d}}, \wedge_{\mathrm{d}}, \vee_{\mathrm{d}})$ is a lattice, where $\wedge_{\mathrm{d}}$ and $\vee_{\mathrm{d}}$ are defined, for all positive integers γ and γ', by*

$$\mathbf{CAs}^{(\gamma)} \wedge_{\mathrm{d}} \mathbf{CAs}^{(\gamma')} := \mathbf{CAs}^{\left(\gcd\left(\bar{\gamma}, \bar{\gamma}'\right)\right)} \tag{3.2.7}$$

and

$$\mathbf{CAs}^{(\gamma)} \vee_{\mathrm{d}} \mathbf{CAs}^{(\gamma')} := \mathbf{CAs}^{\left(\mathrm{lcm}\left(\bar{\gamma}, \bar{\gamma}'\right)\right)}. \tag{3.2.8}$$

The lattice $(\mathbb{N}, |, \mathrm{gcd}, \mathrm{lcm})$ admits 1 as minimum element and 0 as maximum element since any nonnegative integer is divisible by 1 and divides 0. These properties translate as follows for $(\mathbf{CAs}, \preceq_{\mathrm{d}}, \wedge_{\mathrm{d}}, \vee_{\mathrm{d}})$: the minimum element is $\mathbf{As} = \mathbf{CAs}^{(2)}$ and the maximum element is $\mathbf{Mag} = \mathbf{CAs}^{(1)}$. Algebraically, this says that any comb associative operad projects onto $\mathbf{As}$ and is a quotient of $\mathbf{Mag}$.

We end this section by relating the lattice $(\mathbb{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle), \preceq_{\mathrm{i}}, \wedge_{\mathrm{i}}, \vee_{\mathrm{i}})$ introduced in Section 2 with the lattice $(\mathbf{CAs}, \preceq_{\mathrm{d}}, \wedge_{\mathrm{d}}, \vee_{\mathrm{d}})$. As explained in Section 1, a set-theoretic operad can be embedded into a linear operad, so that the operads $\mathbf{CAs}^{(\gamma)}$ can be embedded into quotient operads $\mathbb{K}\langle\mathbf{CAs}^{(\gamma)}\rangle$ of $\mathbb{K}\langle\mathbf{Mag}\rangle$. Formally, the operad $\mathbb{K}\langle\mathbf{CAs}^{(\gamma)}\rangle$ is equal to $\mathbb{K}\langle\mathbf{Mag}\rangle/_{I_\gamma}$, where I_γ is the operad ideal of $\mathbb{K}\langle\mathbf{Mag}\rangle$ generated by $\mathfrak{c}^{(\gamma)}_{/} - \mathfrak{c}^{(\gamma)}_{\backslash}$. We obtain a new lattice $(\mathbb{K}\langle\mathbf{CAs}\rangle, \preceq_{\mathrm{d}}, \wedge_{\mathrm{d}}, \vee_{\mathrm{d}})$, where $\mathbb{K}\langle\mathbf{CAs}\rangle$ is the set of all operads $\mathbb{K}\langle\mathbf{CAs}^{(\gamma)}\rangle$. In this linear framework, the condition $\mathbb{K}\langle\mathbf{CAs}^{(\gamma)}\rangle \preceq_{\mathrm{d}} \mathbb{K}\langle\mathbf{CAs}^{(\gamma')}\rangle$ means that the dimension of the space $\mathrm{Hom}\left(\mathbb{K}\langle\mathbf{CAs}^{(\gamma')}\rangle, \mathbb{K}\langle\mathbf{CAs}^{(\gamma)}\rangle\right)$ is equal to 1. Hence, $(\mathbb{K}\langle\mathbf{CAs}\rangle, \preceq_{\mathrm{d}}, \wedge_{\mathrm{d}}, \vee_{\mathrm{d}})$ is related to $(\mathbb{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle), \preceq_{\mathrm{i}}, \wedge_{\mathrm{i}}, \vee_{\mathrm{i}})$ by the following theorem.

Theorem 3.2.9. *The inclusion $\iota : (\mathbb{K}\langle\mathbf{CAs}\rangle, \preceq_{\mathrm{d}}) \to (\mathbb{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle), \preceq_{\mathrm{i}})$ is nondecreasing. In particular, for all positive integers γ and γ', we have*

$$\mathbb{K}\langle\mathbf{CAs}^{\left(\mathrm{gcd}\left(\bar{\gamma}, \bar{\gamma}'\right)\right)}\rangle \preceq_{\mathrm{i}} \mathbb{K}\langle\mathbf{CAs}^{(\gamma)}\rangle \wedge_{\mathrm{i}} \mathbb{K}\langle\mathbf{CAs}^{(\gamma')}\rangle \tag{3.2.9}$$

and

$$\mathbb{K}\langle\mathbf{CAs}^{(\gamma)}\rangle \vee_{\mathrm{i}} \mathbb{K}\langle\mathbf{CAs}^{(\gamma')}\rangle \preceq_{\mathrm{i}} \mathbb{K}\langle\mathbf{CAs}^{\left(\mathrm{lcm}\left(\bar{\gamma}, \bar{\gamma}'\right)\right)}\rangle. \tag{3.2.10}$$

Note that $(\mathbb{K}\langle\mathbf{CAs}\rangle, \preceq_{\mathrm{d}}, \wedge_{\mathrm{d}}, \vee_{\mathrm{d}})$ does not embed as a sublattice of $(\mathbb{Q}(\mathbb{K}\langle\mathbf{Mag}\rangle), \preceq_{\mathrm{i}}, \wedge_{\mathrm{i}}, \vee_{\mathrm{i}})$, that is ι is not a lattice morphism. Consider for instance $\gamma = 3$ and $\gamma' = 4$, so that

$$\mathbb{K}\langle\mathbf{CAs}^{(3)}\rangle \wedge_{\mathrm{d}} \mathbb{K}\langle\mathbf{CAs}^{(4)}\rangle = \mathbb{K}\langle\mathbf{CAs}^{(2)}\rangle = \mathbb{K}\langle\mathbf{As}\rangle, \tag{3.2.11}$$

whereas

$$\mathbb{K}\langle\mathbf{CAs}^{(3)}\rangle \wedge_{\mathrm{i}} \mathbb{K}\langle\mathbf{CAs}^{(4)}\rangle = \mathbb{K}\langle\mathbf{Mag}\rangle/_{I} \tag{3.2.12}$$

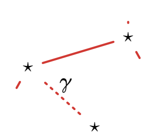
where I is the ideal of $\mathbb{K}\langle\mathbf{Mag}\rangle$ generated by $\mathfrak{c}^{(3)}_{/} - \mathfrak{c}^{(3)}_{\backslash}$ and $\mathfrak{c}^{(4)}_{/} - \mathfrak{c}^{(4)}_{\backslash}$.

3.3. **Completion of comb associative operads.** We are now looking for finite convergent presentations of comb associative operads. By definition, the operad $\mathbf{CAs}^{(\gamma)}$ is the quotient of $\mathbf{Mag}$ by the operad congruence spanned by the rewrite rule

$$\mathfrak{c}^{(\gamma)}_{/} \to \mathfrak{c}^{(\gamma)}_{\backslash}. \tag{3.3.1}$$

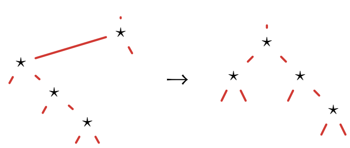
This rewrite rule is compatible with the lexicographic order on prefix words presented at the beginning of Section 1 in the sense that the prefix word of the left member of (3.3.1) is lexicographically greater than the prefix word of the right one.

However, the rewrite relation $\Rightarrow$ induced by $\to$ is not confluent for $\gamma \geqslant 3$. Indeed, we have

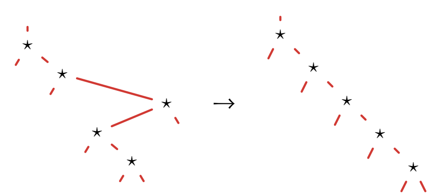
$\mathfrak{c}^{(\gamma+1)}_{/} \Rightarrow$  and $\mathfrak{c}^{(\gamma+1)}_{/} \Rightarrow$  , (3.3.2) and the two right members of (3.3.2) form a branching pair which is not joinable (since these two trees are normal forms of $\Rrightarrow$).

In order to transform the rewrite relation induced by (3.3.1) into a convergent one, we apply the Buchberger algorithm for operads [DK10, Section 3.7] with respect to the lexicographic order on prefix words. We first focus on the special case $\gamma = 3$.

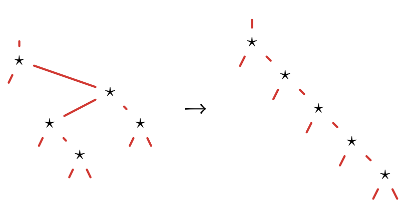
3.3.1. *The 3-comb associative operad.* The Buchberger algorithm applied on binary trees of degrees 4 to 7 provides the new rewrite rules



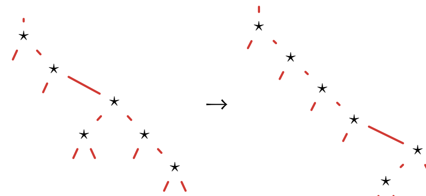
(3.3.3),



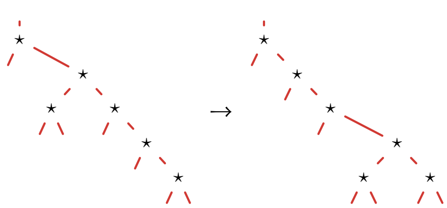
, (3.3.4)



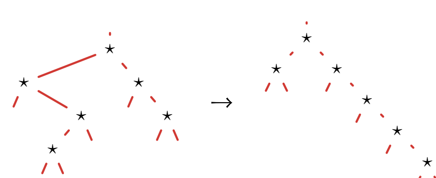
(3.3.5),



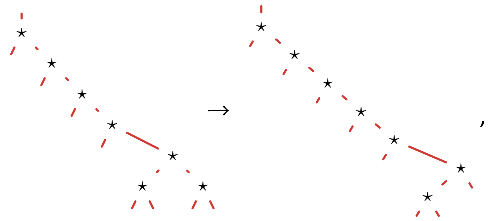
, (3.3.6)



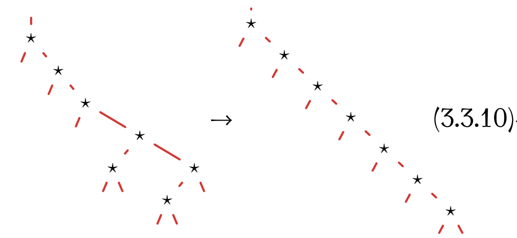
, (3.3.7)



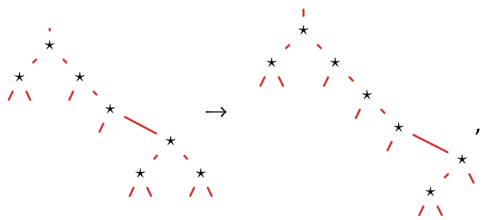
(3.3.8),



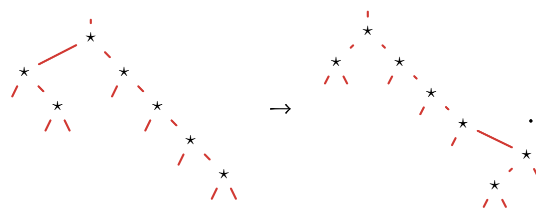
(3.3.9)



(3.3.10)



(3.3.11)



(3.3.12)

Using prefix words, these new relations write as

$$22020200 \to 220020200, \quad (3.3.13)$$

$$20202202000 \to 20202020200, \quad (3.3.14)$$

$$20220200200 \to 20202020200, \quad (3.3.15)$$

$$2020220020200 \to 2020202022000, \quad (3.3.16)$$

$$2022002020200 \to 2020202200200, \quad (3.3.17)$$

$$2202200020200 \to 2200202020200, \quad (3.3.18)$$

$$202020202200200 \to 202020202022000, \quad (3.3.19)$$

$$202020220022000 \to 202020202020200, \quad (3.3.20)$$

$$220020202200200 \to 220020202022000, \quad (3.3.21)$$

$$220200202020200 \to 220020202022000. \quad (3.3.22)$$

Theorem 3.3.1. *The set $\to$ of rewrite rules containing (3.3.1), and (3.3.3)—(3.3.12) is a finite convergent presentation of* $\mathbf{CAs}^{(3)}$.

Proof. Let us show that the rewrite relation $\Rightarrow$ induced by $\to$ is convergent. First, for every relation $\mathfrak{t} \to \mathfrak{t}'$, we have $\mathfrak{t} > \mathfrak{t}'$. Therefore, by Lemma 1.3.1, $\Rightarrow$ is terminating. Moreover, the greatest degree of a tree appearing in $\to$ is 7 so that, from Lemma 1.3.4, to show that $\Rightarrow$ is convergent, it is enough to prove that each tree of degree at most 13 admits exactly one normal form. Equivalently, this amounts to show that the number of normal forms of trees of arity $n \leqslant 14$ is equal to $\#\mathbf{CAs}^{(3)}(n)$. By computer exploration, we get the same sequence

$$1, 1, 2, 4, 8, 14, 20, 19, 16, 14, 14, 15, 16, 17 \tag{3.3.23}$$

for $\#\mathbf{CAs}^{(3)}(n)$ and for the numbers of normal forms of arity n, when $1 \leqslant n \leqslant 14$, which proves the statement of the theorem. $\square$

The rewrite rule $\to$ has, arity by arity, the cardinalities

$$0, 0, 0, 1, 1, 2, 3, 4, 0, 0, \dots . \tag{3.3.24}$$

We also obtain from Theorem 3.3.1 the following consequences.

Proposition 3.3.2. *The set of the trees avoiding as subtrees the ones appearing as left members of $\to$ is a PBW basis of $\mathbf{CAs}^{(3)}$.*

Proof. By definition of PBW bases and Theorem 3.3.1, the set $\mathfrak{N}_{\Rightarrow}$ is a PBW basis of $\mathbf{CAs}^{(3)}$ where $\Rightarrow$ is the rewrite relation induced by $\to$. Now, by Lemma 1.3.2, $\mathfrak{N}_{\Rightarrow}$ can be described as the set of the trees avoiding the left members of $\to$, whence the statement. □

Proposition 3.3.3. *The Hilbert series of $\mathbf{CAs}^{(3)}$ is*

$$\mathcal{H}_{\mathbf{CAs}^{(3)}}(t) = \frac{t}{(1-t)^2}\left(1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11}\right). \tag{3.3.25}$$

Proof. From Proposition 3.3.2, for any $n \geqslant 1$, the dimension of $\mathbf{CAs}^{(3)}(n)$ is the number of trees that avoid as subtrees the left members of $\to$. Now, by using a result of [Gir18] (see also [Row10, KP15]) providing a system of equations for the generating series of the trees avoiding some sets of subtrees, we obtain Expression (3.3.25) for the considered family. □

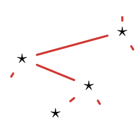
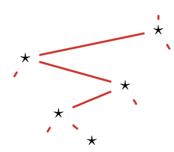
For $n \leqslant 10$, the dimensions of $\mathbf{CAs}^{(3)}(n)$ are provided by Sequence (3.3.23) and for all $n \geqslant 11$, the Taylor expansion of (3.3.25) shows that

$$\#\mathbf{CAs}^{(3)}(n) = n + 3. \tag{3.3.26}$$

Let us describe the elements of the PBW basis of $\mathbf{CAs}^{(3)}$ for arity $n \geqslant 11$. By Proposition 3.3.2, these elements are the normal forms of the rewrite relation induced by $\to$. Let for any $d \geqslant 0$, the binary tree $\mathfrak{z}_d$ defined recursively by

$$\mathfrak{z}_d := \begin{cases} | & \text{if } d = 0, \\ \mathfrak{z}_{d-1} \circ_{\left\lfloor \frac{d-1}{2} \right\rfloor + 1} \star & \text{otherwise.} \end{cases} \tag{3.3.27}$$

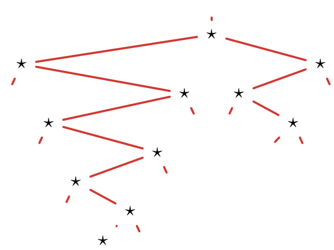
For instance,

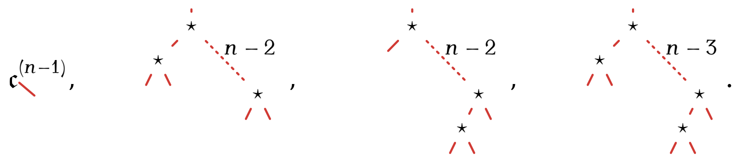
$\mathfrak{z}_4 =$  and $\mathfrak{z}_5 =$  . (3.3.28)

The normal forms split into two families. The first one is the set of the $n - 1$ trees of the form

$$\mathfrak{z}_d \circ_{d+1} \mathfrak{z}_{n-1-d}, \tag{3.3.29}$$

for any $d \in [n-1]$. For example, for $n = 12$,

$\mathfrak{z}_8 \circ_9 \mathfrak{z}_3 =$  (3.3.30) is a tree of this first family. The second family contains the following four trees



(3.3.31)

Observe that each of these $n+3$ trees avoids the left members of the convergent presentation given in Theorem 3.3.1. Moreover, we have the following.

Proposition 3.3.4. *For any two trees* $\mathfrak{t}$ *and* $\mathfrak{t}'$, *if one of them is of the form* (3.3.31) *and* $i \in [|\mathfrak{t}|]$, *then the normal form for the rewrite relation induced by* $\to$ *of* $\mathfrak{t} \circ_i \mathfrak{t}'$ *is a tree of the form* (3.3.31).

Proof. None of the trees of the form (3.3.29) contain the left or the right member of the rewrite rule (3.3.1) generating the congruence relation $\equiv^{(3)}$. Hence, these trees are alone in their equivalence classes. Moreover, the trees of the form (3.3.31) contain the right member of the rewrite rule (3.3.1) generating the congruence relation $\equiv^{(3)}$, so that the composition of such a tree with another tree is not alone in its equivalence class, and thus it does not belong to the family (3.3.29). □

Proposition 3.3.4 says that the family of trees (3.3.31) is absorbing for the partial composition.

Computer explorations allow us to conjecture the multiplication table of the exhibited PBW basis of $\mathbf{CAs}^{(3)}$. However, we do note have a simple description of this table. For instance, the partial composition $\mathfrak{t} \circ_i \mathfrak{t}'$ where $\mathfrak{t}$ is a tree of the form (3.3.29) and $\mathfrak{t}'$ is a tree of the form (3.3.31) can be fully described by 36 cases depending on the values of various parameters associated with $\mathfrak{t}$, $\mathfrak{t}'$, and i.

3.3.2. *Higher comb associative operads.* We run the same algorithm for $\mathbf{CAs}^{(\gamma)}$ when $\gamma \in [9]$. Table 1 shows the number of rewrite rules needed to obtain complete orientations.

From these computer explorations, we conjecture that the algorithm we use does not provide a finite convergent presentation of $\mathbf{CAs}^{(\gamma)}$, when $\gamma \geqslant 4$. We point out that for $\mathbf{CAs}^{(4)}$ new rewrite rules still appear in arity 42. Moreover, the total number of rewrite rules at this arity is 3149. Our program was too slow to compute further the completions of $\mathbf{CAs}^{(5)}$, $\mathbf{CAs}^{(6)}$, $\mathbf{CAs}^{(7)}$, and $\mathbf{CAs}^{(8)}$.

However, the completion algorithm we use depends on the chosen order on the trees. In order to find out if the completion algorithm leads to a finite convergent presentation using a different order, we run the following backtracking algorithm. For every branching pair $\{\mathfrak{t}_1, \mathfrak{t}_2\}$ which is not joinable, we recursively try to find a completion of $\to$ by adding either the rule $\mathfrak{t}_1 \to \mathfrak{t}_2$ or $\mathfrak{t}_2 \to \mathfrak{t}_1$. If at any moment the rewrite relation $\Rightarrow$ induced by $\to$ loops (that is $\Rightarrow$ is not antisymmetric), we simply reject it. We do not find any finite presentation for $\mathbf{CAs}^{(4)}$, $\mathbf{CAs}^{(5)}$, and $\mathbf{CAs}^{(6)}$ until arity 12. We conjecture that there is no

γ	Cardinalities of completions of $\mathbf{CAs}^{(\gamma)}$																									
1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	1	2	3	4	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	0	3	4	5	18	22	11	12	15	19	25	36	44	52	68	79	93	105	106	109	107
5	0	0	0	0	0	1	1	0	0	4	5	8	18	31	36	48	73	111	172	272	455	783					
6	0	0	0	0	0	0	1	1	0	0	0	5	6	11	23	30	48	73	117	204	348	589	1004				
7	0	0	0	0	0	0	0	1	1	0	0	0	0	6	7	16	24	32	49	88	150	261	475	854			
8	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	7	8	21	29	34	53	93	172	311	565		
9	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	8	9	28	30	36	57	101	185	348	648

TABLE 1. The sequences of the cardinalities, arity by arity, of the rewrite rules being completions of orientations of $\equiv^{(\gamma)}$.

finite convergent presentation of $\mathbf{CAs}^{(\gamma)}$ when $\gamma \geqslant 4$ and when the left and the right members of the rewrite rules are trees belonging to **Mag**.

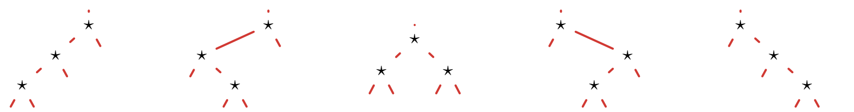
Thanks to the partial completions presented in Table 1, we can compute the following first dimensions of $\mathbf{CAs}^{(\gamma)}$. Table 2 shows the first dimensions of the operads $\mathbf{CAs}^{(\gamma)}$ for $\gamma \in [9]$.

γ	Dimensions of $\mathbf{CAs}^{(\gamma)}$																
1	1	1	2	5	14	42	132	429	1430	4862	16796	58786	208012	742900	2674440	9694845	35357670
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	2	4	8	14	20	19	16	14	14	15	16	17	18	19	20
4	1	1	2	5	13	35	96	264	724	1973	5335	14390	38872	105141	284929	774254	2111088
5	1	1	2	5	14	41	124	384	1210	3861	12440	40392	131997	433782	1432696	4752857	15829261
6	1	1	2	5	14	42	131	420	1375	4576	15431	52598	180895	626862	2186504	7670138	27041833
7	1	1	2	5	14	42	132	428	1420	4796	16432	56966	199444	704140	2503914	8959699	32236657
8	1	1	2	5	14	42	132	429	1429	4851	16718	58331	205632	731272	2620176	9449688	34276116
9	1	1	2	5	14	42	132	429	1430	4861	16784	58695	207452	739840	2658936	9620232	35011566

TABLE 2. The sequences, arity by arity, of the dimensions of $\mathbf{CAs}^{(\gamma)}$.

4. Equating two cubic trees

In this section, we explore all the quotients of **Mag** obtained by equating two trees of degree 3. We denote by $\mathfrak{a}_i$ the ith cubic tree for the lexicographic order, that is



$\mathfrak{a}_1$ $\mathfrak{a}_2$ $\mathfrak{a}_3$ $\mathfrak{a}_4$ $\mathfrak{a}_5$

We denote by $\mathbf{Mag}^{\{i,j\}}$ the quotient operad $\mathbf{Mag}/_{\equiv}$, where $\equiv$ is the operad congruence generated by $\mathfrak{a}_i \equiv \mathfrak{a}_j$. We have already studied the operad $\mathbf{Mag}^{\{1,5\}} = \mathbf{CAs}^{(3)}$ in Section 3.3.1.

4.1. **Anti-isomorphic classes of quotients.** Some of the quotients $\mathbf{Mag}^{\{i,j\}}$ are anti-isomorphic one of the other. Indeed, the map $\phi : \mathbf{Mag} \to \mathbf{Mag}$ sending any binary tree $\mathfrak{t}$ to the binary tree obtained by exchanging recursively the left and right subtrees of $\mathfrak{t}$ is an anti-isomorphism of $\mathbf{Mag}$. For this reason, the $\binom{5}{2} = 10$ quotients $\mathbf{Mag}^{\{i,j\}}$ of $\mathbf{Mag}$ fit into the six equivalence classes

$$\left\{\mathbf{Mag}^{\{1,2\}}, \mathbf{Mag}^{\{4,5\}}\right\}, \left\{\mathbf{Mag}^{\{1,3\}}, \mathbf{Mag}^{\{3,5\}}\right\}, \left\{\mathbf{Mag}^{\{1,4\}}, \mathbf{Mag}^{\{2,5\}}\right\}, \\ \left\{\mathbf{CAs}^{(3)}\right\}, \left\{\mathbf{Mag}^{\{2,3\}}, \mathbf{Mag}^{\{3,4\}}\right\}, \left\{\mathbf{Mag}^{\{2,4\}}\right\} \tag{4.1.1}$$

of anti-isomorphic operads.

Given an operad $\mathcal{O}$ with partial compositions $\circ_i$, we consider the partial compositions $\bar{\circ}_i$ defined by $x \,\bar{\circ}_i\, y := x \circ_{|x|-i+1} y$ for all $x, y \in \mathcal{O}$ and $i \in [|x|]$. The reader can easily check the assertions of the following lemma.

Lemma 4.1.1. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two anti-isomorphic operads and let ϕ be an anti-isomorphism between $\mathcal{O}_1$ and $\mathcal{O}_2$. Then,*

(i) $\mathcal{H}_{\mathcal{O}_1}(t) = \mathcal{H}_{\mathcal{O}_2}(t)$*;*

(ii) if $\to^{(1)}$ is a convergent presentation of $\mathcal{O}_1$, then the set of rewrite rules $\to^{(2)}$ satisfying $\phi(x) \to^{(2)} \phi(y)$ for any $x, y \in \mathcal{O}_1$ whenever $x \to^{(1)} y$, is a convergent presentation of $\mathcal{O}_2$;

(iii) If $(\mathcal{O}, \circ_i)$ is a combinatorial realization of $\mathcal{O}_1$, then $(\mathcal{O}, \bar{\circ}_i)$ is a combinatorial realization of $\mathcal{O}_2$.

4.2. **Quotients on integer compositions.** Four among the six equivalence classes of the quotients $\mathbf{Mag}^{\{i,j\}}$ of $\mathbf{Mag}$ can be realized in terms of operads on integer compositions. Let us review these.

4.2.1. *Operads $\mathbf{Mag}^{\{1,2\}}$ and $\mathbf{Mag}^{\{4,5\}}$.* The reader can check, using the Buchberger algorithm for operads, that the rewrite rule $\mathfrak{a}_2 \to \mathfrak{a}_1$ is a convergent presentation of $\mathbf{Mag}^{\{1,2\}}$. The operads $\mathbf{Mag}^{\{1,2\}}$ and $\mathbf{Mag}^{\{4,5\}}$ are anti-isomorphic, so that by Lemma 4.1.1, the rewrite rule $\mathfrak{a}_4 \to \mathfrak{a}_5$ is a convergent presentation of $\mathbf{Mag}^{\{4,5\}}$. In a similar fashion as Proposition 3.3.3, we compute the following result thanks to [Gir18].

Theorem 4.2.1. *The Hilbert series of $\mathbf{Mag}^{\{1,2\}}$ and $\mathbf{Mag}^{\{4,5\}}$ are*

$$\mathcal{H}_{\mathbf{Mag}^{\{1,2\}}}(t) = \mathcal{H}_{\mathbf{Mag}^{\{4,5\}}}(t) = t\frac{1-t}{1-2t}. \tag{4.2.1}$$

A Taylor expansion of series (4.2.1) shows the following.

Proposition 4.2.2. *For all $n \geqslant 2$,*

$$\#\mathbf{Mag}^{\{1,2\}}(n) = \#\mathbf{Mag}^{\{4,5\}}(n) = 2^{n-2}. \tag{4.2.2}$$

Many graduate sets of combinatorial objects are enumerated by powers of 2. We choose to present a combinatorial realization of $\mathbf{Mag}^{\{1,2\}}$ based on integer compositions. Recall that an *integer composition* is a finite sequence of positive integers. If $\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)$ is an integer composition, we denote by $s_{i,j}(\boldsymbol{\lambda})$ the number $1 + \sum_{i \leqslant k \leqslant j} \boldsymbol{\lambda}_k$. The *arity* of $\boldsymbol{\lambda}$ is $s_{1,p}(\boldsymbol{\lambda})$. Observe that the empty integer composition ϵ is the unique object of arity 1. The graded set of all integer compositions is denoted by $\mathcal{C}$.

Given an integer $i \geqslant 1$, we define the binary operation $\circ_i^{(1,2)} : \mathcal{C}(n) \times \mathcal{C}(m) \to \mathcal{C}(n+m-1)$ for any integer compositions $\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)$ and $\boldsymbol{\mu} := (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q)$ of respective arities $n \geqslant i$ and $m \geqslant 1$ by

$$\boldsymbol{\lambda} \circ_i^{(1,2)} \boldsymbol{\mu} := \begin{cases} (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q) & \text{if } i = n, \\ (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \boldsymbol{\lambda}_{k+1} + m - 1, \boldsymbol{\lambda}_{k+2}, \dots, \boldsymbol{\lambda}_p) & \text{otherwise,} \end{cases} \tag{4.2.3}$$

where $k \geqslant 0$ is such that $s_{1,k}(\boldsymbol{\lambda}) \leqslant i < s_{1,k+1}(\boldsymbol{\lambda})$.

Proposition 4.2.3. *The operad $\left(\mathcal{C}, \circ_i^{(1,2)}\right)$ is a combinatorial realization of $\mathbf{Mag}^{\{1,2\}}$.*

Proof. We have to show that the operads $\left(\mathcal{C}, \circ_i^{(1,2)}\right)$ and $\mathbf{Mag}^{\{1,2\}}$ are isomorphic. The set of normal forms of arity n for the rewrite relation $\Rightarrow$ induced by the rule $\mathfrak{a}_2 \to \mathfrak{a}_1$ is

$$\left\{ \mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)} : (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p) \in \mathcal{C}(n) \right\} \tag{4.2.4}$$

where

$$\mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)} := \mathfrak{c}_{\diagdown}^{(p)} \circ \left[\mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_1 - 1)}, \dots, \mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_p - 1)}, | \right]. \tag{4.2.5}$$

Thus, the map $\phi : \mathbf{Mag}^{\{1,2\}}(n) \to \mathcal{C}(n)$ defined by

$$\phi \left(\mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)} \right) := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p) \tag{4.2.6}$$

is a bijection. Let us show that ϕ is an operad morphism. Let $\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)$ and $\boldsymbol{\mu} := (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q)$ be two integer compositions of respective arities n and m, and let

$$\mathfrak{t} := \mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)} \in \mathbf{Mag}^{\{1,2\}}(n) \quad \text{and} \quad \mathfrak{t}' := \mathfrak{c}_{\diagup}^{(\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q)} \in \mathbf{Mag}^{\{1,2\}}(m). \tag{4.2.7}$$

The tree $\mathfrak{t} \circ_n \mathfrak{t}'$ is equal to $\mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q)}$, so that $\phi(\mathfrak{t} \circ_n \mathfrak{t}') = \phi(\mathfrak{t}) \circ_n^{(1,2)} \phi(\mathfrak{t}')$. Let $i \in [n-1]$ and k be such that $s_{1,k}(\boldsymbol{\lambda}) \leqslant i < s_{1,k+1}(\boldsymbol{\lambda})$, so that $\mathfrak{t} \circ_i \mathfrak{t}'$ is equal to

$$\mathfrak{c}_{\diagdown}^{(p)} \circ \left[\mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_1 - 1)}, \dots, \mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_k - 1)}, \mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_{k+1} - 1)} \circ_{i+1-s_{1,k}(\boldsymbol{\lambda})} \mathfrak{t}', \mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_{k+2} - 1)}, \dots, \mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_p - 1)}, | \right]. \tag{4.2.8}$$

The tree (4.2.8) rewrites by $\Rightarrow$ into

$$\mathfrak{c}_{\diagdown}^{(p)} \circ \left[\mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_1 - 1)}, \dots, \left(\mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_{k+1} - 1)} \circ_{i+1-s_{1,k}(\boldsymbol{\lambda})} \mathfrak{c}_{\diagup}^{(\boldsymbol{\mu}_1 - 1)} \right) \circ_{i+1-s_{1,k}(\boldsymbol{\lambda})+\boldsymbol{\mu}_1} \mathfrak{c}_{\diagup}^{(\boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_q)}, \dots, \mathfrak{c}_{\diagup}^{(\boldsymbol{\lambda}_p - 1)}, | \right] \tag{4.2.9}$$

which rewrite itself by $\Rrightarrow$ into

$$\mathfrak{c}^{(p)} \circ \left[\mathfrak{c}^{(\boldsymbol{\lambda}_1-1)}, \dots, \mathfrak{c}^{(\boldsymbol{\lambda}_k-1)}, \mathfrak{c}^{(\boldsymbol{\lambda}_{k+1}-1+\boldsymbol{\mu}_1)} \circ_{i+1-s_{1,k}(\boldsymbol{\lambda})+\boldsymbol{\mu}_1} \mathfrak{c}^{(\boldsymbol{\mu}_2,\dots,\boldsymbol{\mu}_q)}, \mathfrak{c}^{(\boldsymbol{\lambda}_{k+2}-1)}, \dots, \mathfrak{c}^{(\boldsymbol{\lambda}_p-1)}, \mathsf{I}\right] \tag{4.2.10}$$

in $\boldsymbol{\mu}_1 - 1$ steps. By iterating $q-1$ times the rewrite steps passing from (4.2.8) to (4.2.10), we get

$$\mathfrak{t} \circ_i \mathfrak{t}' \Rrightarrow \mathfrak{c}^{(\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_k,\boldsymbol{\lambda}_{k+1}+m-1,\boldsymbol{\lambda}_{k+2},\dots,\boldsymbol{\lambda}_p)}, \tag{4.2.11}$$

so that $\phi(\mathfrak{t} \circ_i \mathfrak{t}') = \phi(\mathfrak{t}) \circ_i^{(1,2)} \phi(\mathfrak{t}')$. Therefore ϕ is an operad morphism. $\square$

From Lemma 4.1.1, we deduce that $\left(\mathcal{C}, \bar{\circ}_i^{(1,2)}\right)$ is a combinatorial realization of $\mathbf{Mag}^{\{4,5\}}$.

4.2.2. *Operads* $\mathbf{Mag}^{\{1,3\}}$ *and* $\mathbf{Mag}^{\{3,5\}}$. The reader can check that the rewrite rule $\mathfrak{a}_3 \to \mathfrak{a}_1$ is a convergent presentation of $\mathbf{Mag}^{\{1,3\}}$. By Lemma 4.1.1, the rewrite rule $\mathfrak{a}_3 \to \mathfrak{a}_5$ is a convergent presentation of $\mathbf{Mag}^{\{3,5\}}$. Thanks to [Gir18], the Hilbert series of $\mathbf{Mag}^{\{1,3\}}$ and $\mathbf{Mag}^{\{3,5\}}$ are equals to (4.2.1). Thus, for all $n \geqslant 2$, $\#\mathbf{Mag}^{\{1,3\}}(n)$ and $\#\mathbf{Mag}^{\{3,5\}}(n)$ are equal to (4.2.2).

Like in Section 4.2.1, we choose a combinatorial realization based on integer compositions. Given an integer $i \geqslant 1$, we define the binary operation $\circ_i^{(1,3)} : \mathcal{C}(n) \times \mathcal{C}(m) \to \mathcal{C}(n+m-1)$ for any integer compositions $\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)$ and $\boldsymbol{\mu} := (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q)$ of respective arities $n \geqslant i$ and $m \geqslant 1$ by

$$\boldsymbol{\lambda} \circ_i^{(1,3)} \boldsymbol{\mu} := \begin{cases} (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{i-1}, \boldsymbol{\mu}_1 + s_{i,p}(\boldsymbol{\lambda}), \boldsymbol{\mu}_2, \dots, \boldsymbol{\mu}_q) & \text{if } i \leqslant p+1, \\ (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{k-1}, \boldsymbol{\lambda}_k + m - 1, \boldsymbol{\lambda}_{k+1}, \dots, \boldsymbol{\lambda}_p) & \text{otherwise,} \end{cases} \tag{4.2.12}$$

where $k \geqslant 0$ is such that $k + 1 + s_{k+1,p}(\boldsymbol{\lambda}) \leqslant i < k + s_{k,p}(\boldsymbol{\lambda})$.

Proposition 4.2.4. *The operad* $\left(\mathcal{C}, \circ_i^{(1,3)}\right)$ *is a combinatorial realization of* $\mathbf{Mag}^{\{1,3\}}$.

Proof. The proof is similar to the one of Proposition 4.2.3, thus we just give an outline of it. We have to show that the operads $\left(\mathcal{C}, \circ_i^{(1,3)}\right)$ and $\mathbf{Mag}^{\{1,3\}}$ are isomorphic. The set of normal forms of arity n for the rewrite rule $\Rrightarrow$ induced by $\mathfrak{a}_3 \to \mathfrak{a}_1$ is

$$\left\{\mathfrak{c}^{(\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_p)} : (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p) \in \mathcal{C}(n)\right\} \tag{4.2.13}$$

where

$$\mathfrak{c}^{(\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_p)} := \mathfrak{c}^{(\boldsymbol{\lambda}_1)} \circ_2 \left(\mathfrak{c}^{(\boldsymbol{\lambda}_2)} \circ_2 \left(\cdots \left(\mathfrak{c}^{(\boldsymbol{\lambda}_{p-1})} \circ_2 \mathfrak{c}^{(\boldsymbol{\lambda}_p)}\right) \cdots\right)\right). \tag{4.2.14}$$

Thus, it is possible to show that the map $\phi : \mathbf{Mag}^{\{1,3\}}(n) \to \mathcal{C}(n)$ defined by

$$\phi\left(\mathfrak{c}^{(\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_p)}\right) := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p) \tag{4.2.15}$$

is an operad isomorphism from $\mathbf{Mag}^{\{1,3\}}$ to $\left(\mathcal{C}, \circ_i^{(1,3)}\right)$. $\square$

From Lemma 4.1.1, we deduce that $\left(\mathcal{C}, \bar{\circ}_i^{(1,3)}\right)$ is a combinatorial realization of $\mathbf{Mag}^{\{3,5\}}$.

4.2.3. *Operads* $\mathbf{Mag}^{\{1,4\}}$ *and* $\mathbf{Mag}^{\{2,5\}}$. The reader can check that the rewrite rule $\mathfrak{a}_4 \to \mathfrak{a}_1$ is a convergent presentation of $\mathbf{Mag}^{\{1,4\}}$. By Lemma 4.1.1, the rewrite rule $\mathfrak{a}_2 \to \mathfrak{a}_5$ is a convergent presentation of $\mathbf{Mag}^{\{2,5\}}$. Thanks to [Gir18], the Hilbert series of $\mathbf{Mag}^{\{1,4\}}$ and $\mathbf{Mag}^{\{2,5\}}$ are equals to (4.2.1). Thus, for $n \geqslant 2$, $\#\mathbf{Mag}^{\{1,4\}}(n)$ and $\#\mathbf{Mag}^{\{2,5\}}(n)$ are equal to (4.2.2).

Like in Section 4.2.1, we choose a combinatorial realization based on integer compositions. Given an integer $i \geqslant 1$, we define the binary operation $\circ_i^{(2,5)} : \mathcal{C}(n) \times \mathcal{C}(m) \to \mathcal{C}(n+m-1)$ for any integer compositions $\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)$ and $\boldsymbol{\mu} := (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q)$ of respective arities $n \geqslant i$ and $m \geqslant 1$ by

$$\boldsymbol{\lambda} \circ_i^{(2,5)} \boldsymbol{\mu} := \begin{cases} (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_{q-1}, \boldsymbol{\mu}_q + \boldsymbol{\lambda}_{k+1}, \dots, \boldsymbol{\lambda}_p) & \text{if } i = s_{1,k}(\boldsymbol{\lambda}), \\ (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, i - s_{1,k}(\boldsymbol{\lambda}), \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q, s_{1,k+1}(\boldsymbol{\lambda}) - i, \boldsymbol{\lambda}_{k+2}, \dots, \boldsymbol{\lambda}_p) & \text{otherwise,} \end{cases} \tag{4.2.16}$$

where $k \geqslant 0$ is such that $s_{1,k}(\boldsymbol{\lambda}) \leqslant i < s_{1,k+1}(\boldsymbol{\lambda})$.

Proposition 4.2.5. *The operad* $\left(\mathcal{C}, \circ_i^{(2,5)}\right)$ *is a combinatorial realization of* $\mathbf{Mag}^{\{2,5\}}$.

Proof. The proof is similar to the one of Proposition 4.2.3, thus we just give an outline of it. We have to show that the operad $\left(\mathcal{C}, \circ_i^{(2,5)}\right)$ and $\mathbf{Mag}^{\{2,5\}}$ are isomorphic. The set of normal forms of arity n for the rewrite rule $\Rightarrow$ induced by $\mathfrak{a}_2 \to \mathfrak{a}_5$ is (4.2.4). Thus, it is possible to show that the map (4.2.6) is an operad isomorphism from $\mathbf{Mag}^{\{2,5\}}$ to $\left(\mathcal{C}, \circ_i^{(2,5)}\right)$. □

From Lemma 4.1.1, we deduce that $\left(\mathcal{C}, \bar{\circ}_i^{(2,5)}\right)$ is a combinatorial realization of $\mathbf{Mag}^{\{1,4\}}$.

4.2.4. *Operad* $\mathbf{Mag}^{\{2,4\}}$. The reader can check that the rewrite rules $\mathfrak{a}_2 \to \mathfrak{a}_4$ and $\mathfrak{a}_4 \to' \mathfrak{a}_2$ are both convergent presentations of $\mathbf{Mag}^{\{2,4\}}$. Thanks to [Gir18], the Hilbert series of $\mathbf{Mag}^{\{2,4\}}$ is equal to (4.2.1). Thus, for $n \geqslant 2$, $\#\mathbf{Mag}^{\{2,4\}}(n)$ is equal to (4.2.2).

Like in Section 4.2.1, we choose a combinatorial realization based on integer compositions. Given an integer $i \geqslant 1$, we define the binary operation $\circ_i^{(2,4)} : \mathcal{C}(n) \times \mathcal{C}(m) \to \mathcal{C}(n+m-1)$ for any integer compositions $\boldsymbol{\lambda} := (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_p)$ and $\boldsymbol{\mu} := (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_q)$ of respective arities $n \geqslant i$ and $m \geqslant 1$ by

$$\boldsymbol{\lambda} \circ_i^{(2,4)} \boldsymbol{\mu} := \begin{cases} (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_{q-1}, \boldsymbol{\mu}_q + \boldsymbol{\lambda}_{k+1}, \boldsymbol{\lambda}_{k+2}, \dots, \boldsymbol{\lambda}_p) & \text{if } i = s_{1,k}(\boldsymbol{\lambda}), \\ (\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k, i - s_{1,k}(\boldsymbol{\lambda}), \boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_{q-1}, \boldsymbol{\mu}_q + s_{1,k+1}(\boldsymbol{\lambda}) - i, \boldsymbol{\lambda}_{k+2}, \dots, \boldsymbol{\lambda}_p) & \text{otherwise,} \end{cases} \tag{4.2.17}$$

where $k \geqslant 0$ is such that $s_{1,k}(\boldsymbol{\lambda}) \leqslant i < s_{1,k+1}(\boldsymbol{\lambda})$.

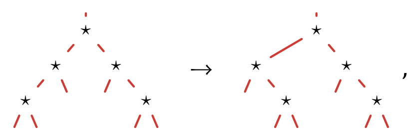
Proposition 4.2.6. *The operads* $\left(\mathcal{C}, \circ_i^{(2,4)}\right)$ *and* $\left(\mathcal{C}, \bar{\circ}_i^{(2,4)}\right)$ *are combinatorial realizations of* $\mathbf{Mag}^{\{2,4\}}$.

Proof. The proof is similar to the one of Proposition 4.2.3, thus we just give an outline of it. We have to show that the operad $\left(\mathcal{C}, \circ_i^{(2,4)}\right)$ and $\mathbf{Mag}^{\{2,4\}}$ are isomorphic. The set of normal forms of arity n for the rewrite rule $\Rightarrow$ induced by $\mathfrak{a}_2 \to \mathfrak{a}_4$ is (4.2.4). Thus, it is possible to show that the map (4.2.6) is an operad isomorphism from $\mathbf{Mag}^{\{2,4\}}$ to $\left(\mathcal{C}, \circ_i^{(2,4)}\right)$. □

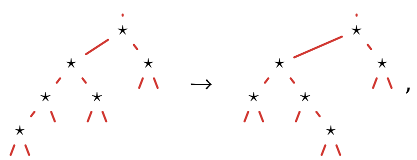
4.2.5. *Non-isomorphism of the operads.* As shown in the previous sections, the operads of the four considered equivalence classes $\left\{\mathbf{Mag}^{\{1,2\}}, \mathbf{Mag}^{\{4,5\}}\right\}$, $\left\{\mathbf{Mag}^{\{1,3\}}, \mathbf{Mag}^{\{3,5\}}\right\}$, $\left\{\mathbf{Mag}^{\{1,4\}}, \mathbf{Mag}^{\{2,5\}}\right\}$, and $\left\{\mathbf{Mag}^{\{2,4\}}\right\}$ have the same Hilbert series. Even if they can be realized on the same set of integer compositions, all these operads are pairwise non-isomorphic (and also non-anti-isomorphic). Indeed, any (anti-)isomorphism between two of these operads necessarily maps the generator of the first to the generator of the second, and since by definition the nontrivial relations between the generators are different from one operad to another, the operads cannot by (anti-)isomorphic. This remark is also valid for the corresponding linear operads.

4.3. **Quotients with complicated presentations.** We do not find finite convergent presentations for the operads $\mathbf{Mag}^{\{2,3\}}$ and $\mathbf{Mag}^{\{3,4\}}$. However, thanks to computer explorations, we conjecture that the rewrite rules

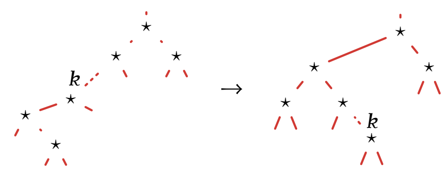
$$\mathfrak{a}_4 \to \mathfrak{a}_3, \tag{4.3.1}$$



(4.3.2)



(4.3.3)



$k \geqslant 1$ (4.3.4)

form a convergent presentation of $\mathbf{Mag}^{\{3,4\}}$. We checked that presentation until arity 40. From this rewrite relation $\to$, we also conjecture that the Hilbert series of $\mathbf{Mag}^{\{2,3\}}$ and, by Lemma 4.1.1, of $\mathbf{Mag}^{\{3,4\}}$ are

$$\mathcal{H}_{\mathbf{Mag}^{\{2,3\}}}(t) = \mathcal{H}_{\mathbf{Mag}^{\{3,4\}}}(t) = \frac{t}{(1-t)^3}\left(1 - 2t + 2t^2 + t^4 - t^6\right). \tag{4.3.5}$$

By Taylor expansion, we have the sequence

$$1, 1, 2, 4, 8, 14, 21, 29, 38, 48 \tag{4.3.6}$$

for the first dimensions of $\mathbf{Mag}^{\{2,3\}}$ and $\mathbf{Mag}^{\{3,4\}}$. For $n \geqslant 5$,

$$\#\mathbf{Mag}^{\{2,3\}}(n) = \#\mathbf{Mag}^{\{3,4\}}(n) = \frac{n(n+1)}{2} - 7. \tag{4.3.7}$$

Conclusion and perspectives

In this paper, we have considered some quotients of the magmatic operad in both the linear and the set-theoretic frameworks. We focused mainly our study on comb associative operads and collected properties by using computer exploration and rewrite systems on trees. There are many ways to extend this work. Here follow some few further research directions.

A first research direction consists in finding convergent presentations for (all or most of) the operads $\mathbf{CAs}^{(\gamma)}$ in order to describe algebraic and combinatorial properties of them (as describing explicit bases, computing Hilbert series, and providing combinatorial realizations). This has been done only in the case $\gamma = 3$. For some other cases, we only have conjectural and experimental data (see Section 3.3.2).

Following ideas existing for word rewriting theory [GGM15], a second axis consists in allowing new generators for the operads $\mathbf{CAs}^{(\gamma)}$ in order to obtain finite convergent presentations when $\gamma \geqslant 4$. Indeed, the Buchberger semi-algorithm works by adding rewrite rules to a set of rewrite rules to obtain a convergent rewrite system. An orthogonal procedure consists rather in adding new generators (new labels for internal nodes in the trees) in order to obtain convergent rewrite systems. More generally, we also would like to use these ideas for other magmatic quotients, such as the operad $\mathbf{Mag}^{\{3,4\}}$ that we did not study entirely in Section 4.

A third axis consists in studying if the completion of presentations of quotients of the magmatic operad maintains links with the lattice structure introduced in Section 2. More precisely, assuming that we have completed the presentations of the quotients $\mathcal{O}_1$ and $\mathcal{O}_2$ of $\mathbf{Mag}$, as well as the one of the lower-bound $\mathcal{O}_1 \wedge_{\mathrm{i}} \mathcal{O}_2$, the question consists in designing an algorithm for computing a completion of a presentation of the upper-bound $\mathcal{O}_1 \vee_{\mathrm{i}} \mathcal{O}_2$. Of course, the same question also makes sense for the lattice of comb associative operads introduced in Section 3.

Let us address now a perspective fitting more in a combinatorial context. As mentioned in the introduction of this article, we suspect that some combinatorial properties of quotients $\mathbf{Mag}/_{\equiv}$ of $\mathbf{Mag}$ derive from properties of the equivalence relation generating the operad congruence $\equiv$. More precisely, we would like to investigate if, when this equivalence relation is a set of Tamari intervals (or is closed by interval, or satisfies some other classical properties coming from poset theory), one harvests a nice description of the Hilbert series and of a combinatorial realization of $\mathbf{Mag}/_{\equiv}$.

A last research axis relies on the study on the 2-magmatic operad $2\mathbf{Mag}$, that is, the free operad generated by two binary elements. The analog of the associative operad in this context is the operad $2\mathbf{As}$ [LR06] defined as the quotient of $2\mathbf{Mag}$ by the congruence saying that the two generators are associative. This operad has a nice combinatorial realization in terms of alternating bicolored Schröder trees. The question consists here in generalizing our main results for the quotients of $2\mathbf{Mag}$ and the generalizations of $2\mathbf{As}$ (that is, the definition of analogs of comb associative operads and the study of their presentations).

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