Generalizations of the associative operad and convergent rewrite systems

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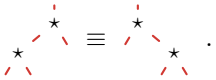
Abstract

The associative operad is the quotient of the magmatic operad by the operad congruence identifying the two binary trees of degree 2. We introduce here a generalization of the associative operad depending on a nonnegative integer d, called d-comb associative operad, as the quotient of the magmatic operad by the operad congruence identifying the left and the right comb binary trees of degree d. We study the case $d = 3$ and provide an orientation of its space of relations by using rewrite systems on trees and the Buchberger algorithm for operads to obtain a convergent rewrite system.

Introduction

Associative algebras are spaces endowed with a binary product $\star$ satisfying among others the associativity relation $(x_1 \star x_2) \star x_3 = x_1 \star (x_2 \star x_3)$. It is well-known that the algebras of this category are representations of the associative (nonsymmetric) operad As. This operad can be seen as the quotient of the magmatic operad Mag (the free operad of binary trees on the binary generator $\star$) by the operad congruence $\equiv$ satisfying



(0.1)

These two binary trees are the syntax trees of the expressions appearing in the above associativity relation.

In a more combinatorial context and regardless of the theory of operads, the Tamari order is a partial order on the set of the binary trees having a fixed number of internal nodes d. This order is generated by the covering relation consisting in rewriting a tree $\mathfrak{t}$ into a tree $\mathfrak{t}'$ by replacing a subtree of $\mathfrak{t}$ of the form of the left member of (0.1) into a tree of the form of the right member of (0.1). This transformation is known in a computer science context as the right rotation operation [8] and intervenes in algorithms involving binary search trees [1]. The partial order hence generated by the right rotation operation is known as the Tamari order [9] and has a lot of combinatorial and algebraic properties (see for instance [3, 7]).

A first connection between the associative operad and the Tamari order is based upon the fact that the orientation of (0.1) from left to right provides a convergent orientation of the congruence $\equiv$. The normal forms of the underlying rewrite rule of this orientation are right comb binary trees and are hence in one-to-one correspondence with the elements of As.

This work is intended to be a first strike in the study of the eventual links between the Tamari order and some quotients of the operad Mag. In the long run, we would like to study quotients $\mathsf{Mag}/_{\equiv}$ of Mag where $\equiv$ is an operad congruence generated by equivalence classes of trees of a fixed degree. In particular, we would like to know if $\equiv$ is generated by equivalence

classes of trees forming intervals of the Tamari order leads to algebraic properties for $\mathsf{Mag}/_{\equiv}$ (like the description of orientations of its space of relations, nice bases and Hilbert series).

We focus here on one of these quotients $\mathsf{CAs}^{(3)}$ which is the operad describing the category of the algebras equipped with a binary product $\star$ and subjected to the relation $((x_1 \star x_2) \star x_3) \star x_4 = x_1 \star (x_2 \star (x_3 \star x_4))$. This is a kind of associativity relation in higher degree $d = 3$. This operad is generated by an equivalence class of trees which is not an interval for the Tamari order. As preliminary computer experiments show, $\mathsf{CAs}^{(3)}$ has oscillating first dimensions (see (3.13)), what is rather unusual among all known operads. In this paper, we provide an orientation of the space of relations of $\mathsf{CAs}^{(3)}$. For this, we use rewrite systems on trees [2] and the Buchberger algorithm for operads [4].

This text is presented as follows. Section 1 contains preliminaries about the magmatic operad and rewrite rules on trees. In Section 2, we define the operad $\mathsf{CAs}^{(3)}$ as a particular case of a more general construction of generalizations $\mathsf{CAs}^{(d)}$, $d \geqslant 1$, of As. Finally, Section 3 contains the orientation of the space of relations of $\mathsf{CAs}^{(3)}$ (Theorem 3.1). As consequences, we obtain for $\mathsf{CAs}^{(3)}$ the description of one of its Poincaré-Birkhoff-Witt bases (Proposition 3.2) and the description of its Hilbert series (Proposition 3.3).

1 The magmatic operad, quotients, and rewrite rules

We consider nonsymmetric set-theoretic operads. Let $\mathcal{O}$ be such an operad. We denote respectively by $\circ_i$ and $\circ$ the partial and complete compositions of $\mathcal{O}$. For any $n \geqslant 1$, $\mathcal{O}(n)$ is the set of the elements x of $\mathcal{O}$ of arity $|x| = n$. We denote by Mag the magmatic operad, that is the free operad over one binary generator $\star$, and we represent the elements of Mag by binary trees. The *arity* $|\mathfrak{t}|$ (resp. *degree* $\deg(\mathfrak{t})$) of a binary tree $\mathfrak{t}$ is its number of leaves (resp. internal nodes). Given a binary tree $\mathfrak{t}$, we denote by $\mathrm{p}(\mathfrak{t})$ the *prefix word* of $\mathfrak{t}$, that is the word on $\{\mathtt{0}, \mathtt{2}\}$ obtained by a left to right depth-first traversal of $\mathfrak{t}$ and by writing $\mathtt{0}$ (resp. $\mathtt{2}$) when a leaf (resp. an internal node) is encountered. The set of all words on $\{\mathtt{0}, \mathtt{2}\}$ is endowed with the lexicographic order $\leqslant$ induced by $\mathtt{0} < \mathtt{2}$.

If $\to$ is a binary relation on Mag such that $\mathfrak{s} \to \mathfrak{s}'$ implies $|\mathfrak{s}| = |\mathfrak{s}'|$, we denote by $\Rightarrow$ the *rewrite rule induced* by $\to$. Formally we have $\mathfrak{t} \circ_i (\mathfrak{s} \circ [\mathfrak{r}_1, \ldots, \mathfrak{r}_n]) \Rightarrow \mathfrak{t} \circ_i (\mathfrak{s}' \circ [\mathfrak{r}_1, \ldots, \mathfrak{r}_n])$, if $\mathfrak{s} \to \mathfrak{s}'$ where $n = |\mathfrak{s}|$, and $\mathfrak{t}$, $\mathfrak{r}_1$, $\ldots$, $\mathfrak{r}_n$ are binary trees. In other words, one has $\mathfrak{t} \Rightarrow \mathfrak{t}'$ if it is possible to obtain $\mathfrak{t}'$ from $\mathfrak{t}$ by replacing a subtree $\mathfrak{s}$ of $\mathfrak{t}$ by $\mathfrak{s}'$ whenever $\mathfrak{s} \to \mathfrak{s}'$. We use here the standard terminology (*terminating*, *confluent*, *convergent*, *branching pair*, *joinable*, *normal form*, *etc.*) about rewrite rules and rewrite systems [2].

Given an operad $\mathcal{O} \simeq \mathsf{Mag}/_{\equiv}$ where $\equiv$ is an operad congruence of Mag, we say that $\to$ is an *orientation* of $\equiv$ if the reflexive, transitive, and symmetric closure of $\Rightarrow$ is $\equiv$. We say that $\to$ is a *convergent orientation* if $\Rightarrow$ is convergent. When $\to$ is a convergent orientation of $\equiv$, the set of all normal forms of $\Rightarrow$ is a *Poincaré-Birkhoff-Witt basis* of the operad $\mathcal{O}$ and its elements are exactly the binary trees avoiding, as subtrees, the trees appearing as left members in $\to$.

We shall use the following criterion to prove that a rewrite rule on Mag is terminating.

Lemma 1.1. *Let $\to$ be a binary relation on Mag. If for any $\mathfrak{t}, \mathfrak{t}' \in \mathsf{Mag}$ such that $\mathfrak{t} \to \mathfrak{t}'$ one has $\mathrm{p}(\mathfrak{t}) > \mathrm{p}(\mathfrak{t}')$, then the rewrite rule induced by $\to$ is terminating.*

Moreover, we shall use the following result appearing in [5] specialized on rewrite rules on Mag to prove that a terminating rewrite rule is convergent.

Lemma 1.2. *Let $\to$ be a binary relation on Mag wherein all trees $\mathfrak{t}$ and $\mathfrak{t}'$ such that $\mathfrak{t} \to \mathfrak{t}'$ have degrees at most ℓ. Then, if the rewrite rule $\Rightarrow$ induced by $\to$ is terminating and all its branching pairs of degrees at most $2\ell - 1$ are joinable, $\Rightarrow$ is convergent.*

2 Generalizations of the associative operad

It is known that the binary relation $\to$ orienting (0.1) from left to right is a convergent orientation of (0.1). Then, a Poincaré-Birkhoff-Witt basis of As is the set of all right comb binary trees.

Let us now define for any $d \geqslant 1$ the *d-comb associative operad* $\mathsf{CAs}^{(d)}$ as the quotient operad $\mathsf{Mag}/_{\equiv^{(d)}}$ where $\equiv^{(d)}$ is the smallest operad congruence of Mag satisfying

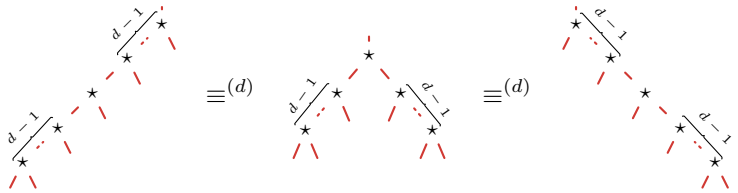
$$\underbrace{(\dots(\star \circ_1 \star) \circ_1 \dots) \circ_1 \star}_{d \text{ operands}} \equiv^{(d)} \underbrace{\star \circ_2 (\cdots \circ_2 (\star \circ_2 \star) \dots)}_{d \text{ operands}}. \tag{2.1}$$

In words, (2.1) says that the left and the right comb binary trees of degree d are equivalent for $\equiv^{(d)}$. Notice that $\equiv^{(1)}$ is trivial so that $\mathsf{CAs}^{(1)} = \mathsf{Mag}$ and that $\equiv^{(2)}$ is the operad congruence defined by (0.1) so that $\mathsf{CAs}^{(2)} = \mathsf{As}$.

As shown by the following statement, the operads $\mathsf{CAs}^{(d)}$ are related to each other.

Proposition 2.1. *For any $d \geqslant 3$, $\mathsf{CAs}^{(d)}$ is a quotient operad of $\mathsf{CAs}^{(2d-1)}$ and $\mathsf{CAs}^{(2)}$ is a quotient operad of $\mathsf{CAs}^{(d)}$.*

Proof. Since

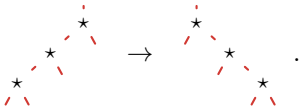


(2.2)

where a dotted edge between two internal nodes denotes a left or a right comb tree of degree $d-1$ (hence, the trees of (2.2) are of degree $2d-1$), the relation $\mathfrak{t} \equiv^{(2d-1)} \mathfrak{t}'$ implies $\mathfrak{t} \equiv^{(d)} \mathfrak{t}'$ for any trees $\mathfrak{t}$ and $\mathfrak{t}'$. Hence, $\equiv^{(2d-1)}$ is finer than $\equiv^{(d)}$, whence the first part of the statement of the proposition. The second part of the statement of the proposition is a consequence of the fact that the relation $\mathfrak{t} \equiv^{(d)} \mathfrak{t}'$ implies $\mathfrak{t} \equiv^{(2)} \mathfrak{t}'$ for any trees $\mathfrak{t}$ and $\mathfrak{t}'$. □

3 The 3-comb associative operad

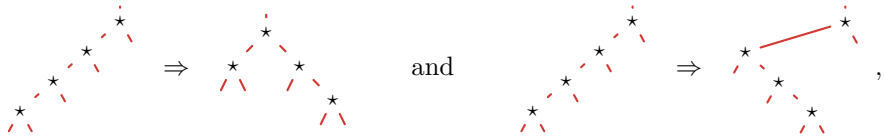
We now focus on the study of the operad $\mathsf{CAs}^{(3)}$. By definition, this operad is the quotient of Mag by the operad congruence spanned by the relation



(3.1)

This binary relation is compatible with the lexicographic order on prefix words presented at the beginning of Section 1 in the sense that the prefix word of the left member of (3.1) is lexicographically greater than the prefix word of the right one.

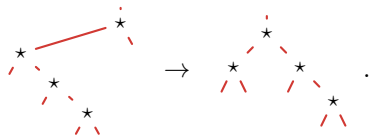
However, the rewrite rule $\Rightarrow$ induced by $\to$ is not confluent. Indeed, we have



(3.2)

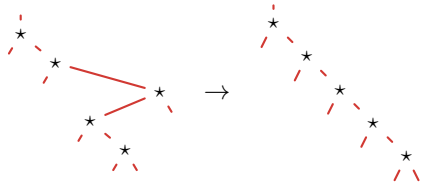
and the two right members of (3.2) form a branching pair which is not joinable.

In order to transform the rewrite rule induced by (3.1) into a convergent one, we apply the Buchberger algorithm for operads [4, Section 3.7] with respect to the lexicographic order on prefix words. Following this algorithm, we need to put the right members of (3.2) in relation by $\to$. To respect the lexicographic property of the prefix words, this leads to the new relation

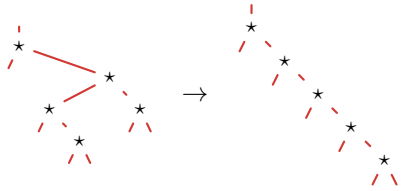


(3.3)

The Buchberger algorithm applied on binary trees of degrees 5, 6, and 7 provides the new relations



(3.4)'



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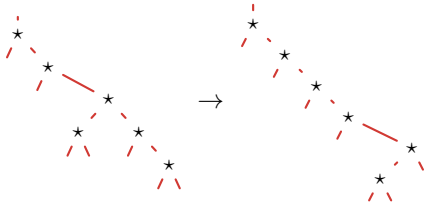

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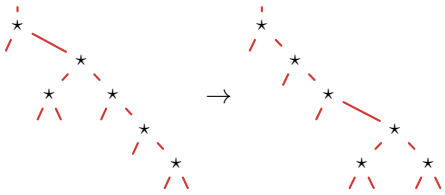

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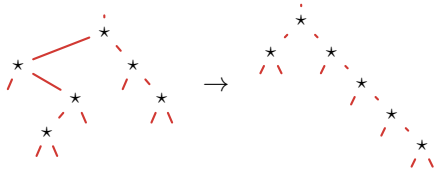

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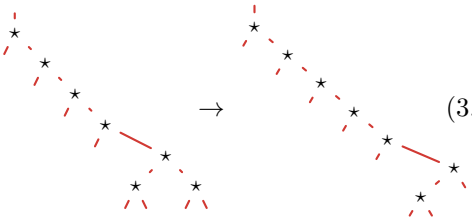

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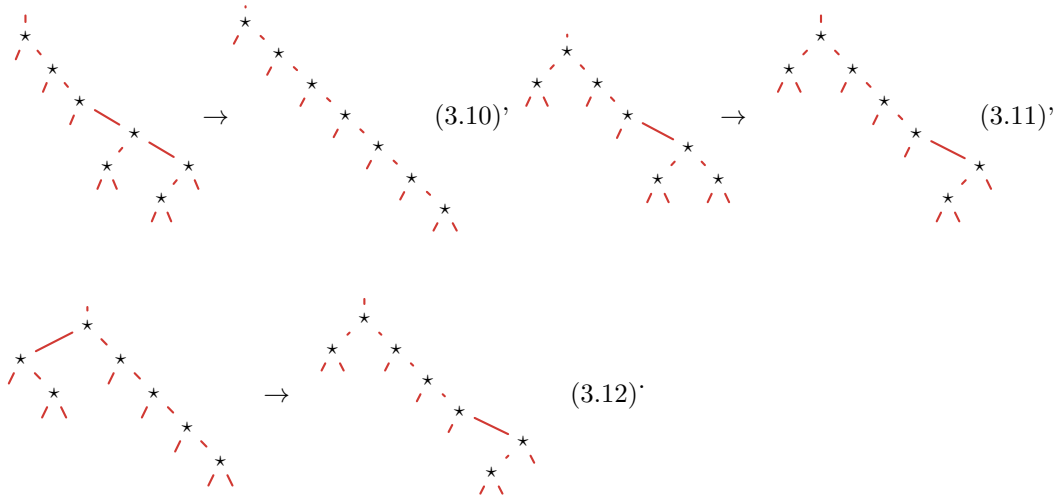

We claim that the rewrite rule $\Rightarrow$ induced by binary relation $\to$ satisfying (3.1), (3.3), (3.4)—(3.12) is convergent. First, for every relation $\mathfrak{t} \to \mathfrak{t}'$, we have $\mathrm{p}(\mathfrak{t}) > \mathrm{p}(\mathfrak{t}')$. Therefore, by Lemma 1.1, $\Rightarrow$ is terminating. Moreover, the greatest degree of a tree appearing in $\to$ is 7 so that, from Lemma 1.2, to show that $\Rightarrow$ is convergent, it is enough to prove that each tree of degree at most 13 admits exactly one normal form. Equivalently, this amounts to show that the number of normal forms of trees of arity n is equal to $\#\mathsf{CAs}^{(3)}(n)$. By computer exploration, we get the same sequence

$$1, 1, 2, 4, 8, 14, 20, 19, 16, 14, 14, 15, 16, 17 \tag{3.13}$$

for $\#\mathsf{CAs}^{(3)}(n)$ and for the numbers of normal forms of arity n, when $1 \leqslant n \leqslant 14$. Hence, we get our following main result.

Theorem 3.1. *The binary relation $\to$ satisfying (3.1), (3.3), (3.4)—(3.12) is a convergent orientation of the congruence $\equiv^{(3)}$ of $\mathsf{CAs}^{(3)}$.*

The binary relation $\to$ has, arity by arity, the cardinalities

$$0, 0, 0, 1, 1, 2, 3, 4, 0, \dots \ . \tag{3.14}$$

We obtain from Theorem 3.1 also the following consequences.

Proposition 3.2. *The set of the trees avoiding as subtrees the ones appearing as left members of $\to$ is a Poincaré-Birkhoff-Witt basis of $\mathsf{CAs}^{(3)}$.*

From Proposition 3.2, and by using a result of [6] describing a system of equations for the generating series of syntax trees avoiding some sets of subtrees, we obtain the following result.

Proposition 3.3. *The Hilbert series of $\mathsf{CAs}^{(3)}$ is*

$$\mathcal{H}_{\mathsf{CAs}^{(3)}}(t) = \frac{t}{(1-t)^2}\left(1 - t + t^2 + t^3 + 2t^4 + 2t^5 - 7t^7 - 2t^8 + t^9 + 2t^{10} + t^{11}\right). \tag{3.15}$$

For $n \leqslant 10$, the dimensions of $\mathsf{CAs}^{(3)}(n)$ are provided by Sequence (3.13) and for all $n \geqslant 11$, the Taylor expansion of (3.15) shows that $\#\mathsf{CAs}^{(3)}(n) = n + 3$.

Perspectives

Our first axis of perspectives consists in collecting properties about the operads $\mathsf{CAs}^{(d)}$. A natural question consists in finding all the morphisms between the operads $\mathsf{CAs}^{(d)}$. Some surjective morphisms are described by Proposition 2.1 and we can hope to a full description of these, as well as some possible injections. Moreover, we can try to obtain a convergent orientation of $\equiv^{(d)}$ and general expressions of the Hilbert series of $\mathsf{CAs}^{(d)}$ when $d \geqslant 4$. By computer exploration, we have the sequence

$$1, 1, 2, 5, 13, 35, 96, 264, 724, 1973, 5355, 14390 \tag{3.16}$$

for the first dimensions for $\mathsf{CAs}^{(4)}$. By applying the Buchberger algorithm on trees of degrees until 10, we obtain that a convergent orientation of $\equiv^{(4)}$ has, arity by arity, the sequence $0, 0, 0, 0, 1, 1, 0, 3, 4, 5, 18, 22$ for its first cardinalities. Moreover, for $\mathsf{CAs}^{(5)}$, we get the sequence

$$1, 1, 2, 5, 14, 41, 124, 384, 1210, 3861, 12440 \tag{3.17}$$

of dimensions and the first cardinalities $0, 0, 0, 0, 0, 1, 1, 0, 0, 4, 5$ for any convergent orientation of $\equiv^{(5)}$. Finally, for $\mathsf{CAs}^{(6)}$, we get the sequence

$$1, 1, 2, 5, 14, 42, 131, 420, 1375, 4576, 15431 \tag{3.18}$$

of dimensions and the first cardinalities $0, 0, 0, 0, 0, 0, 1, 1, 0, 0, 0$ for any convergent orientation of $\equiv^{(6)}$. We can notice that only $\mathsf{CAs}^{(3)}$ seems to have oscillating first dimensions.

A second axis concerns a complete understanding of $\mathsf{CAs}^{(3)}$. We can try to construct an explicit basis of this operad. Proposition 3.2 describe a basis in terms of trees avoiding some patterns but, we can hope to find a simpler description. This includes the description of a family of combinatorial objects forming a basis of $\mathsf{CAs}^{(3)}$ and an adequate definition of a partial composition map $\circ_i$ on these. Moreover, a natural question is to explore the suboperads $\mathsf{CAs}^{(3)}$ in the category of vector spaces.

In a last axis, we can consider further generalizations of As being quotients of Mag by congruences defined by identifying certain binary trees of a same fixed degree. A possible question is, as presented in the introduction, to investigate if combinatorial properties of the trees belonging to a same equivalence class imply algebraic properties on the obtained operads.

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