Combinatoire énumérative des chemins appliquée aux circuits

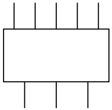
Christophe Cordero

Post-doctorant au LITIS
de l'université de Rouen

5 février 2020

Exemple de circuits

Générateur



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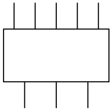
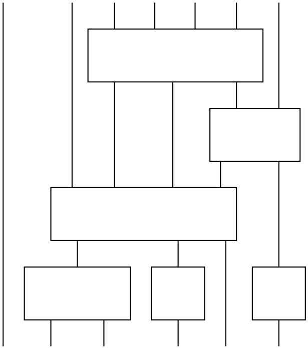


Circuit



Définition des circuits

Un ***circuit***

Définition des circuits

Un ***circuit*** est soit un *générateur* $\begin{array}{c} \overbrace{\mid \;\cdots\; \mid}^{s} \\ \boxed{\quad g \quad} \\ \underbrace{\mid \;\cdots\; \mid}_{e} \end{array}$

Définition des circuits

Un ***circuit*** est soit un *générateur* $\begin{array}{c}\overbrace{|\;\cdots\;|}^{s}\\ \boxed{\quad g\quad}\\ \underbrace{|\;\cdots\;|}_{e}\end{array}$, soit un *fil* $|$

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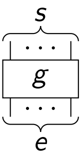
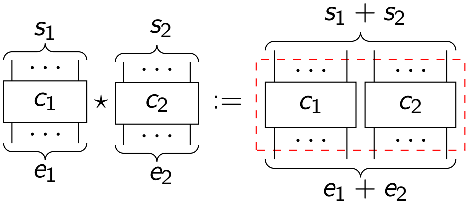
Un ***circuit*** est soit un *générateur* , soit un *fil* $|$,

soit un assemblage :



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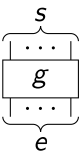
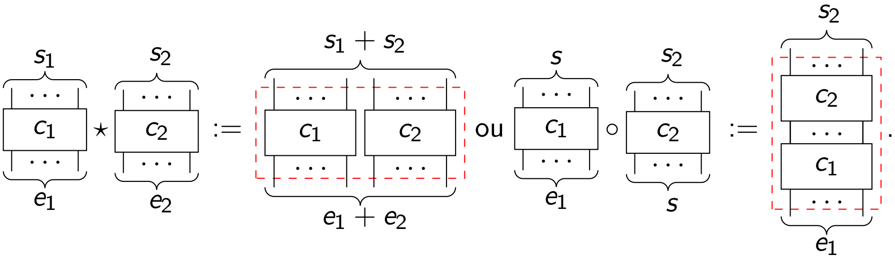
Un ***circuit*** est soit un *générateur* , soit un *fil* | ,

soit un des assemblages :



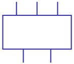
Générateurs : $$\mathbb{G} := \left\{ \begin{array}{c} \overbrace{|\;\cdots\;|}^{\beta_1} \\ \boxed{\;1\;} \\ \underbrace{|\;\cdots\;|}_{\alpha_1} \end{array}, \ldots, \begin{array}{c} \overbrace{|\;\cdots\;|}^{\beta_1} \\ \boxed{\;m_1\;} \\ \underbrace{|\;\cdots\;|}_{\alpha_1} \end{array}, \ldots, \begin{array}{c} \overbrace{|\;\cdots\;|}^{\beta_d} \\ \boxed{\;1\;} \\ \underbrace{|\;\cdots\;|}_{\alpha_d} \end{array}, \ldots, \begin{array}{c} \overbrace{|\;\cdots\;|}^{\beta_d} \\ \boxed{\;m_d\;} \\ \underbrace{|\;\cdots\;|}_{\alpha_d} \end{array} \right\}.$$

Dénombrer les circuits

Problème

$$\mathrm{C}_{e,n,s}(\mathbb{G}) := \left\{ \overbrace{\underbrace{\boxed{n \text{ générateurs}}}_{e}}^{s} \right\}$$

Générateurs : $\mathbb{G} := \left\{ \overbrace{\underbrace{\boxed{1}}_{\alpha_1}}^{\beta_1}, \dots, \overbrace{\underbrace{\boxed{m_1}}_{\alpha_1}}^{\beta_1}, \dots, \overbrace{\underbrace{\boxed{1}}_{\alpha_d}}^{\beta_d}, \dots, \overbrace{\underbrace{\boxed{m_d}}_{\alpha_d}}^{\beta_d} \right\}.$

Dénombrement de $\mathrm{C}_{2,n,n+2}\big(\big\{$  $\big\}\big)$

Cas initiaux

Dénombrement de $C_{2,n,n+2}\left(\left\{\boxed{}\right\}\right)$

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<u>$n = 0$:</u> 		

Dénombrement de $C_{2,n,n+2}\left(\left\{\ \right\}\right)$

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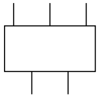
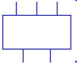
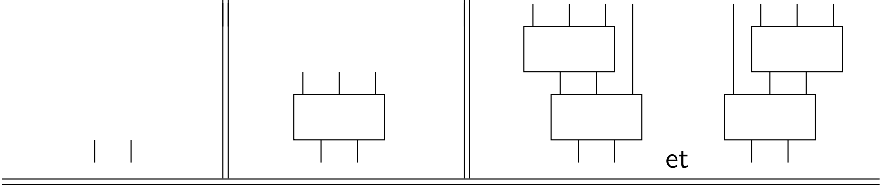


Dénombrement de $C_{2,n,n+2}\left(\left\{ \right.\right.$  $\left.\left.\right\}\right)$

Cas initiaux

$\underline{n = 0 :}$ $\underline{n = 1 :}$ $\underline{n = 2 :}$



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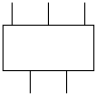
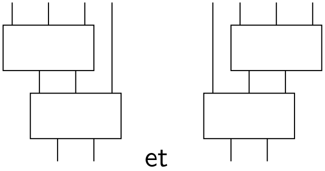
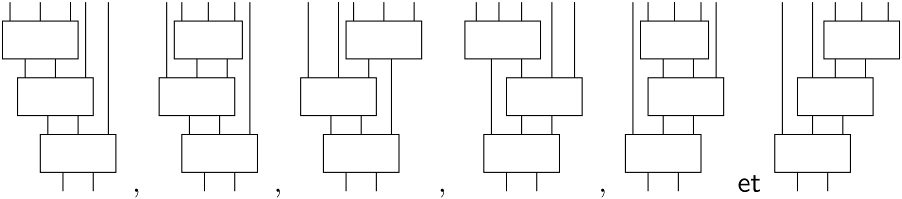
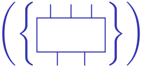


<u>$n = 2$:</u>



<u>$n = 3$:</u>



Dénombrement de $C_{2,n,n+2}$ 

Cas initiaux

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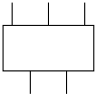

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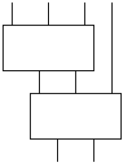
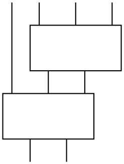
 et 

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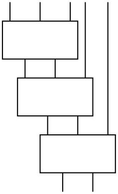
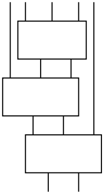
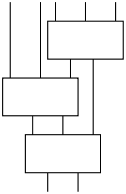
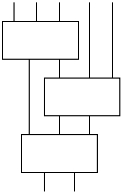
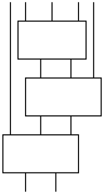
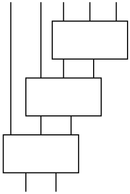
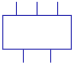
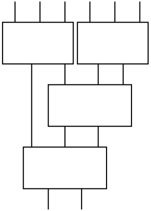
 ,  ,  ,  ,  et 

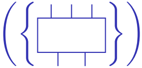
Suite : $1, 1, 2, 6, ?, \ldots$

Dénombrement de $C_{2,n,n+2}\left(\left\{\right.$  $\left.\right\}\right)$

Difficulté : ambiguïté

$\underline{n = 4 :}$



DÉNOMBREMENT DE $\mathrm{C}_{2,n,n+2}$ 

DIFFICULTÉ : AMBIGUÏTÉ

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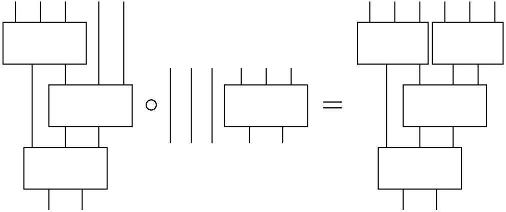

Dénombrement de $C_{2,n,n+2}(\{\ \})$

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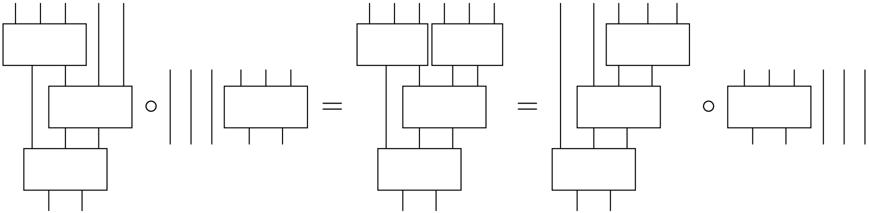
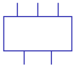
<u>$n = 4$:</u>



Dénombrement de $C_{2,n,n+2}\left(\left\{\right.\right.$$\left.\left.\right\}\right)$

L'encyclopédie OEIS

A264868 Number of rooted tandem duplication trees on n gene segments.

1, 1, 2, 6, 22, 92, 420, 2042, 10404, 54954, 298648, 1660714, 9410772, 54174212, 316038060, 1864781388, 11111804604, 66782160002, 404392312896, 2465100947836, 15116060536540, 93184874448186, 577198134479356, 3590697904513792 (list; graph; refs; listen; history; text; internal format)

REFERENCES Mathematics of Evolution and Phylogeny, O. Gascuel (ed.), Oxford University Press, 2005

Dénombrement de $C_{2,n,n+2}\left(\left\{\boxed{}\right\}\right)$

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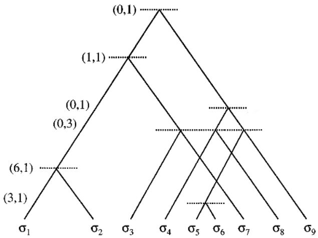
Arbre tandem de duplication



[Image de Gascuel, Hendy, Jean-Marie et Mclachlan]

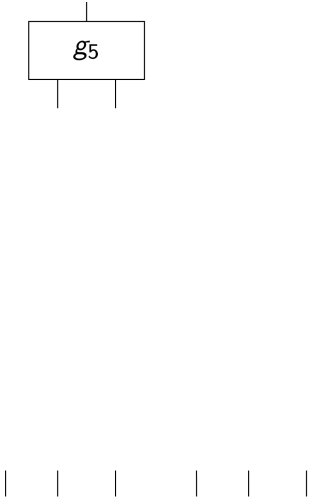
Construction non ambiguë des circuits

« Tetris »

Théorème [C.]

Construction non ambiguë des circuits

« Tetris »



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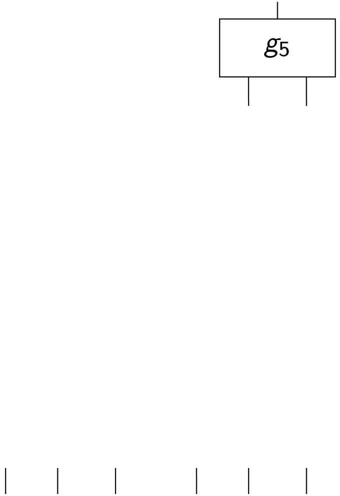

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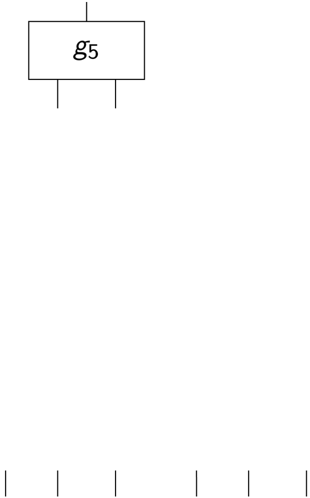

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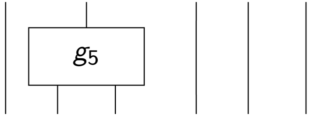

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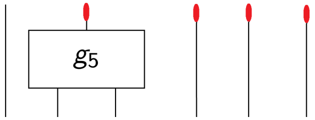

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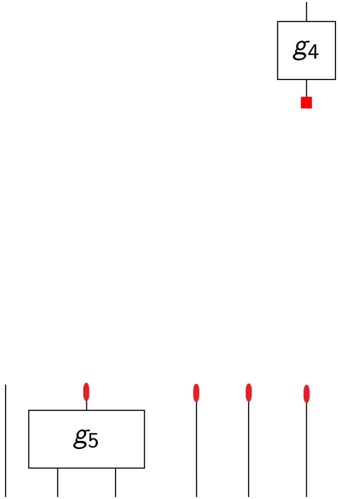

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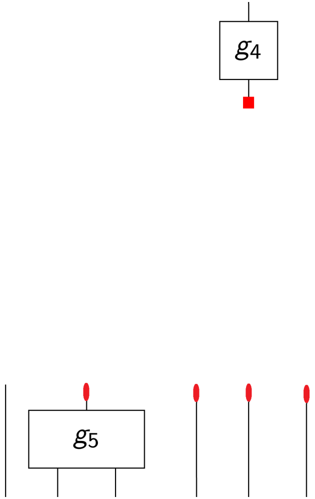

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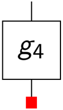

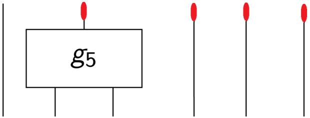

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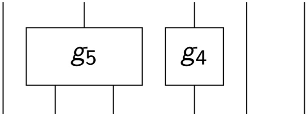

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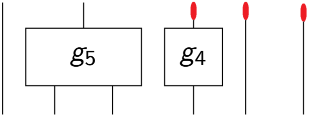

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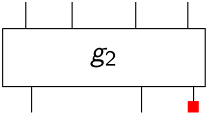

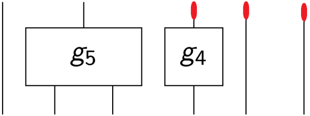

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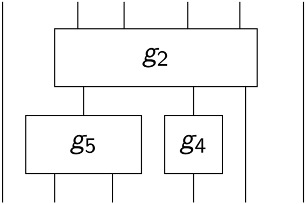

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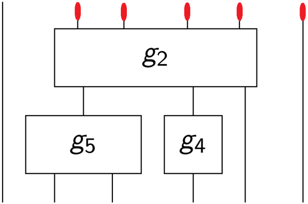

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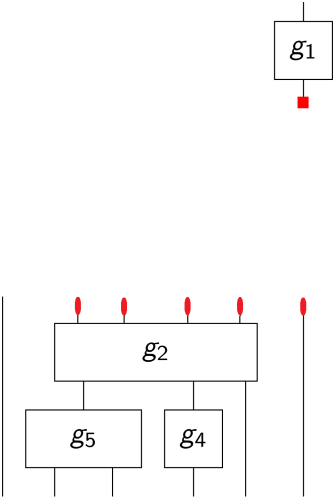

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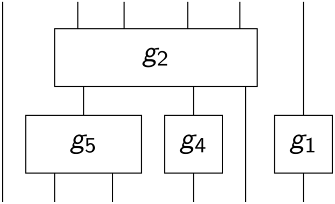

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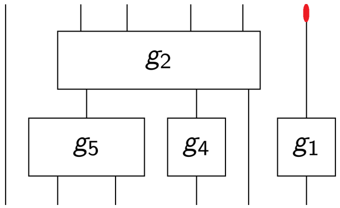

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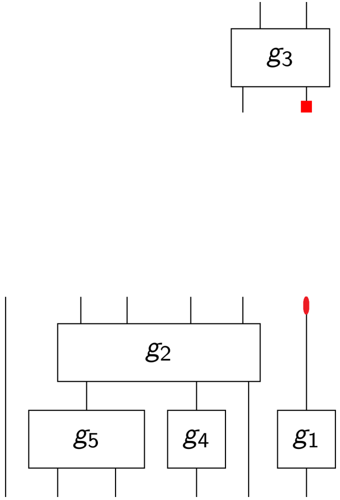

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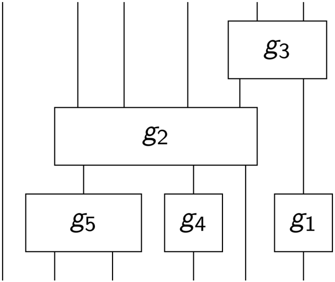

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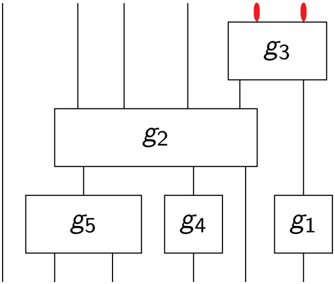

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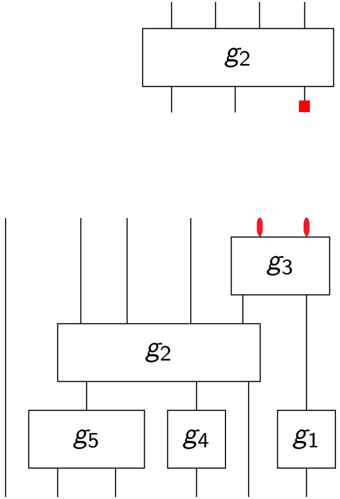

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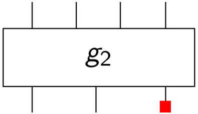

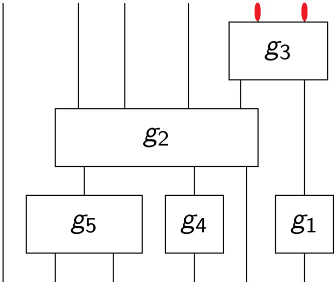

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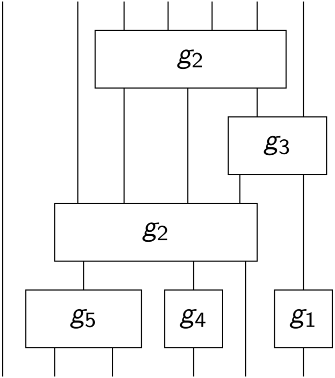


Théorème [C.]

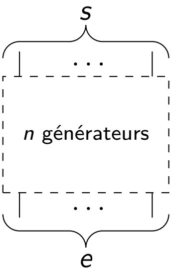
Bijection : circuits $\longleftrightarrow$ chemins

Théorème [C. en 2018]

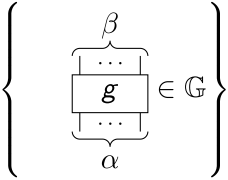
Bijection : circuits $\longleftrightarrow$ chemins

Théorème [C. en 2018]

Circuits



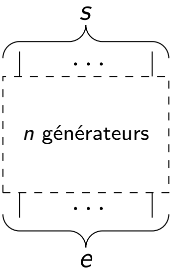
Générateurs



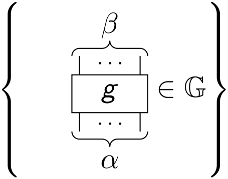
Bijection : circuits $\longleftrightarrow$ chemins

Théorème [C. en 2018]

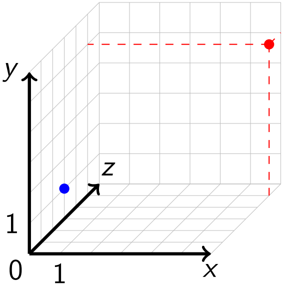
Circuits



Générateurs



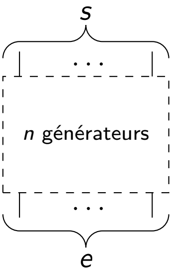
Chemins de $(0, 1, e)$ à (n, s, s)



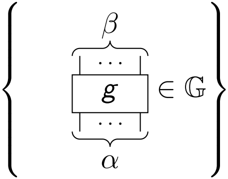
BIJECTION : CIRCUITS $\longleftrightarrow$ CHEMINS

THÉORÈME [C. EN 2018]

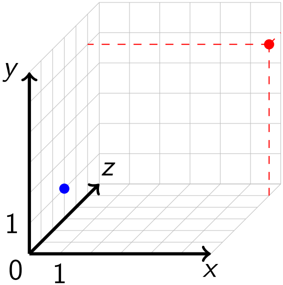
Circuits



Générateurs



Chemins de $(0, 1, e)$ à (n, s, s)

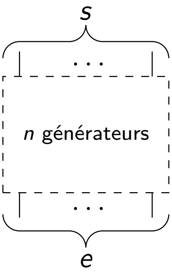


Contrainte : « $1 \leq y \leq z$ »

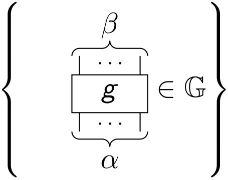
Bijection : circuits $\longleftrightarrow$ chemins

Théorème [C. en 2018]

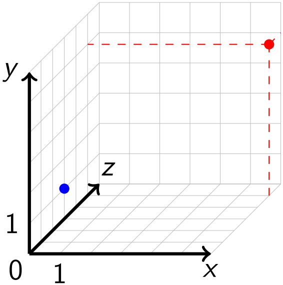
Circuits



Générateurs



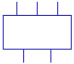
Chemins de $(0, 1, e)$ à (n, s, s)



Contrainte : « $1 \leq y \leq z$ »

Pas

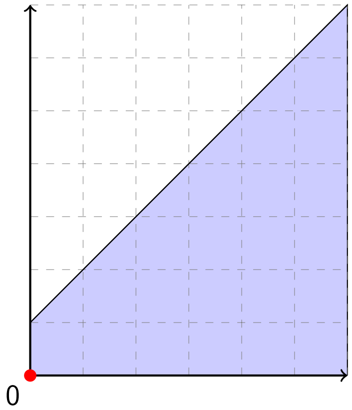
$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \cup \left\{ \ldots, \begin{bmatrix} 1 \\ 1-\alpha \\ \beta-\alpha \end{bmatrix}_g, \ldots \right\}$$

Exemple bijection : $\mathrm{C}_{2,n,n+2}\left(\left\{\right.\right.$  $\left.\left.\right\}\right)$

Exemple bijection : $\mathrm{C}_{2,n,n+2}\left(\left\{\square\right\}\right)$

Pas : $\searrow$ et $\uparrow$.

$\longrightarrow$



| |

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$\longrightarrow$

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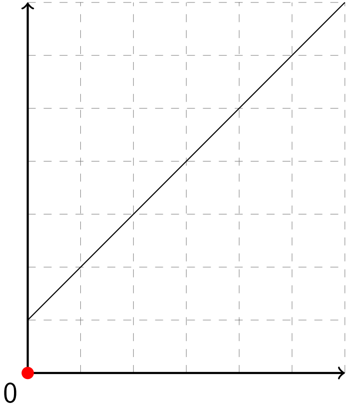

Exemple bijection : $C_{2,n,n+2}\left(\left\{\ \right\}\right)$

$\longrightarrow$

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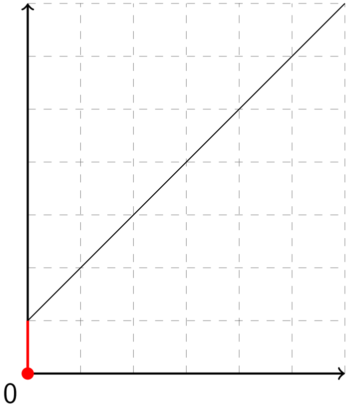
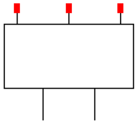
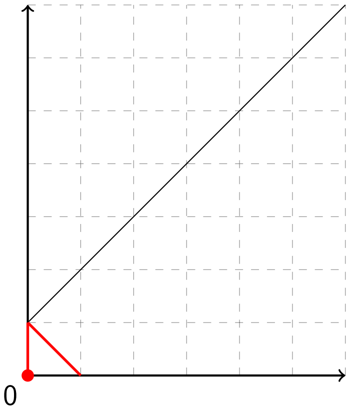


Exemple bijection : $C_{2,n,n+2}\left(\left\{\boxed{}\right\}\right)$

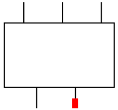
Pas : $\searrow$ et $\uparrow$.



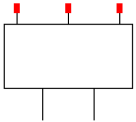
$\longrightarrow$



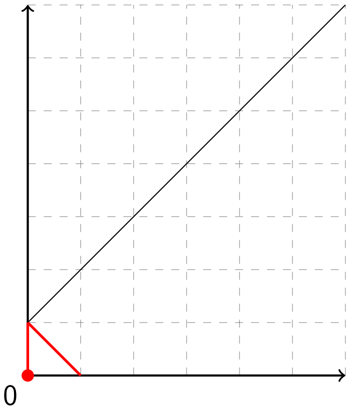
Exemple bijection : $C_{2,n,n+2}\left(\left\{\quad\right\}\right)$

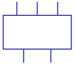


Pas : $\searrow$ et $\uparrow$.



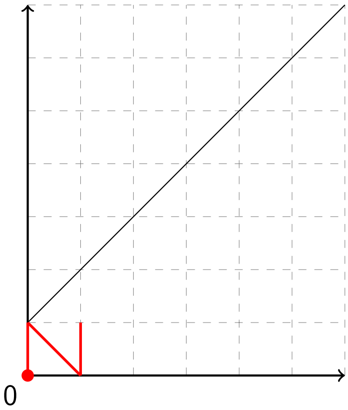
$\longrightarrow$



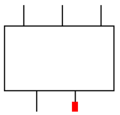
Exemple bijection : $C_{2,n,n+2}$({})

$\longrightarrow$

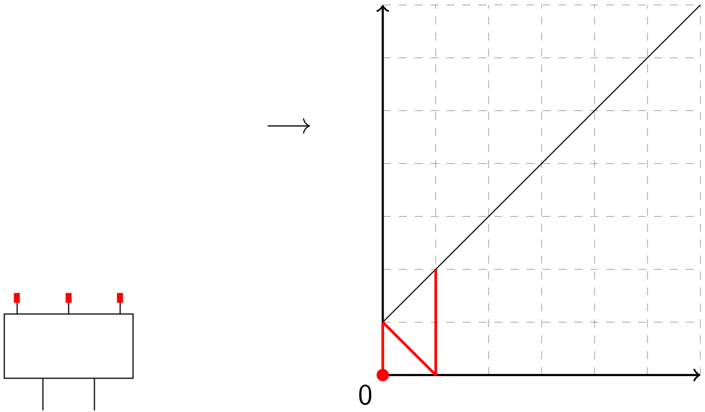
Pas : $\searrow$ et $\uparrow$.

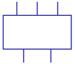


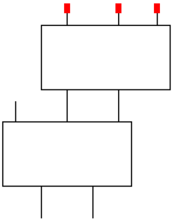
EXEMPLE BIJECTION : $C_{2,n,n+2}\left(\left\{ \quad \right\}\right)$



Pas : $\searrow$ et $\uparrow$.



Exemple bijection : $C_{2,n,n+2}\left(\left\{\right.\right.$  $\left.\left.\right\}\right)$



$\longrightarrow$

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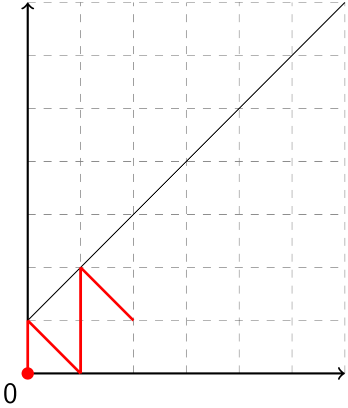
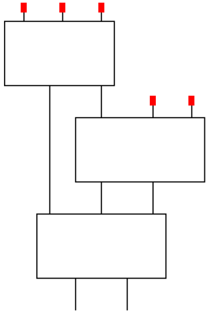
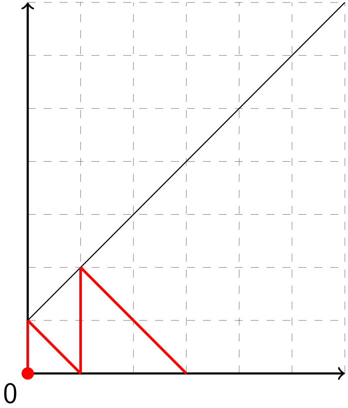


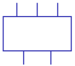
Exemple bijection : $C_{2,n,n+2}\left(\left\{\square\right\}\right)$

Pas : $\searrow$ et $\uparrow$.

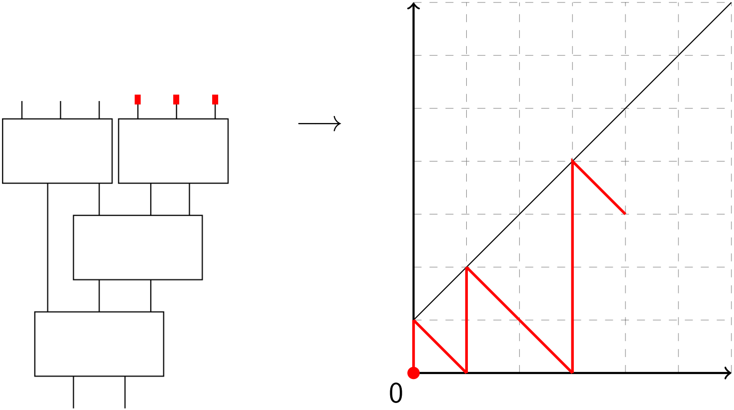


$\longrightarrow$



Exemple bijection : $\mathrm{C}_{2,n,n+2}\Big(\Big\{$  $\Big\}\Big)$

Pas : $\searrow$ et $\uparrow$.



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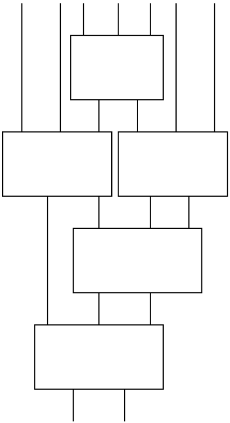

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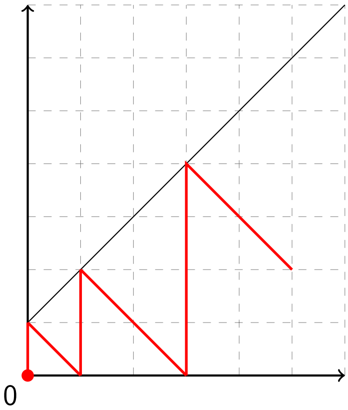
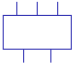
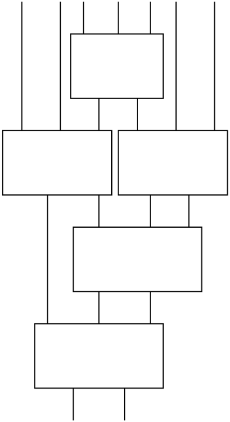


Exemple bijection : $C_{2,n,n+2}\left(\left\{\right.\right.$$\left.\left.\right\}\right)$



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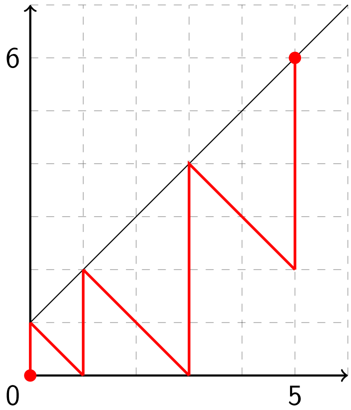

Formule de récurrence : $c_n := \left|\mathrm{C}_{2,n,n+2}\left(\left\{\square\right\}\right)\right|$

Proposition

$$c_n = \begin{cases} 1 & \text{si } n = 0, \\ \sum_{k=1}^{n} (-1)^{k+1} \binom{2+n-2k}{k} c_{n-k} & \text{sinon.} \end{cases}$$

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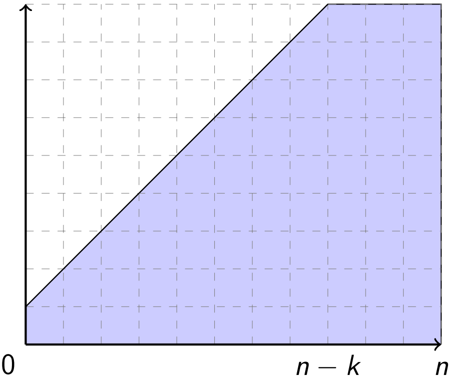
Démonstration

Soit E_k l'ensemble des chemins du type



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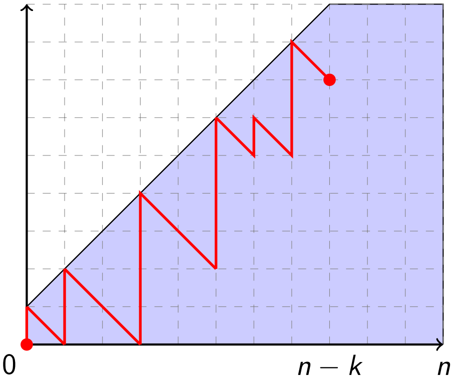

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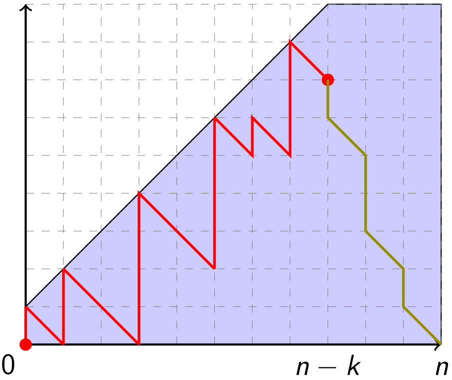

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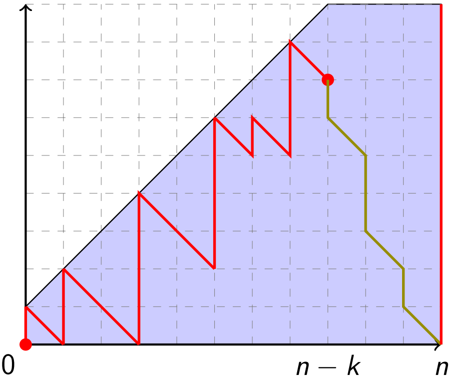

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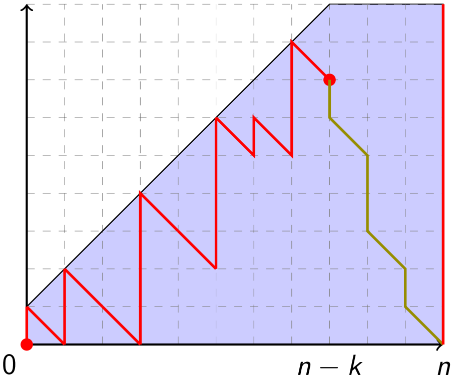
Démonstration

Soit E_k l'ensemble des chemins du type



Remarque : $c_n = |E_0|$

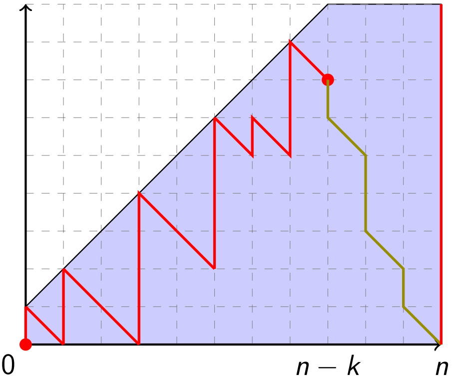
Formule de récurrence : $c_n := \left|\mathrm{C}_{2,n,n+2}\left(\left\{\right\}\right)\right|$

Proposition

$$c_n = \begin{cases} 1 & \text{si } n = 0, \\ \sum_{k=1}^{n} (-1)^{k+1} \binom{2+n-2k}{k} c_{n-k} & \text{sinon.} \end{cases}$$

Démonstration

Soit E_k l'ensemble des chemins du type



Remarque : $c_n = |E_0|, |E_n| = 0$

Formule de récurrence : $c_n := \left|\mathrm{C}_{2,n,n+2}\left(\left\{\quad\right\}\right)\right|$

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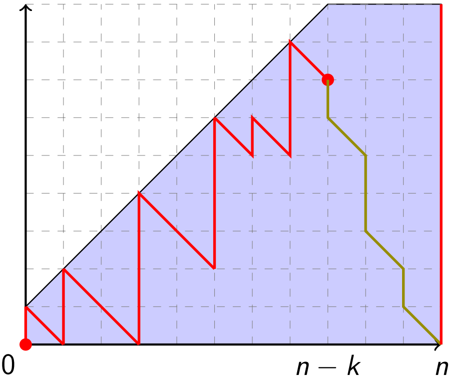
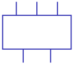


Remarque : $c_n = |E_0|, |E_n| = 0$

et $|E_k| = |E_k \sqcup E_{k+1}| - |E_{k+1}|$.

Formule de récurrence : $c_n := \left|\mathrm{C}_{2,n,n+2}\left(\left\{\right.\right.\right.$  $\left.\left.\left.\right\}\right)\right|$

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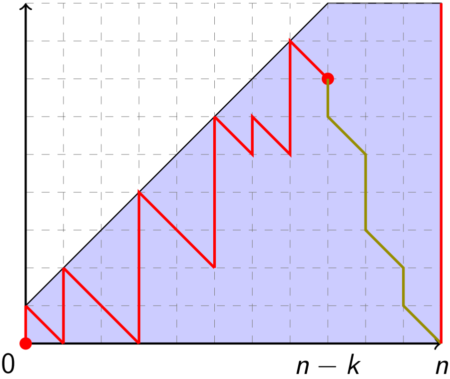

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Donc $c_n = \sum_{k=1}^{n} (-1)^{k+1} |E_k \sqcup E_{k+1}|$.

Formule de récurrence : cas général

Pour le plaisir des yeux

Générateurs

$$\mathbb{G} := \left\{ \underbrace{\overbrace{\begin{array}{|c|} \cdots \\ \hline \boxed{\;1\;} \\ \hline \cdots \end{array}}^{\beta_1}}_{\alpha_1}, \ldots, \underbrace{\overbrace{\begin{array}{|c|} \cdots \\ \hline \boxed{m_1} \\ \hline \cdots \end{array}}^{\beta_1}}_{\alpha_1}, \ldots, \underbrace{\overbrace{\begin{array}{|c|} \cdots \\ \hline \boxed{\;1\;} \\ \hline \cdots \end{array}}^{\beta_d}}_{\alpha_d}, \ldots, \underbrace{\overbrace{\begin{array}{|c|} \cdots \\ \hline \boxed{m_d} \\ \hline \cdots \end{array}}^{\beta_d}}_{\alpha_d} \right\}$$

Formule de récurrence : cas général

Pour le plaisir des yeux

Générateurs

$$\mathbb{G} := \left\{ \begin{array}{c} \overbrace{\begin{array}{|c|} \cdots \end{array}}^{\beta_1} \\ \boxed{1} \\ \underbrace{\begin{array}{|c|} \cdots \end{array}}_{\alpha_1} \end{array}, \ldots, \begin{array}{c} \overbrace{\begin{array}{|c|} \cdots \end{array}}^{\beta_1} \\ \boxed{m_1} \\ \underbrace{\begin{array}{|c|} \cdots \end{array}}_{\alpha_1} \end{array}, \ldots, \begin{array}{c} \overbrace{\begin{array}{|c|} \cdots \end{array}}^{\beta_d} \\ \boxed{1} \\ \underbrace{\begin{array}{|c|} \cdots \end{array}}_{\alpha_d} \end{array}, \ldots, \begin{array}{c} \overbrace{\begin{array}{|c|} \cdots \end{array}}^{\beta_d} \\ \boxed{m_d} \\ \underbrace{\begin{array}{|c|} \cdots \end{array}}_{\alpha_d} \end{array} \right\}$$

Théorème [C. en 2018]

Le nombre $|\mathrm{C}_{e,\,n,\,s}(\mathbb{G})|$ vérifie la relation de récurrence :

$$\begin{cases} 1 & \text{si } n = 0 \text{ et } s = e, \\ \displaystyle\sum_{\ell=1}^{n} (-1)^{\ell+1} \sum_{k_1+\ldots+k_d=\ell} \binom{\ell}{k_1,\,\ldots,\,k_d} \binom{s+\ell-\sum_{i=1}^{d} k_i\beta_i}{\ell} m_1^{k_1}\ldots m_d^{k_d} \left| \mathrm{C}_{e,n-\ell,s-\sum_{i=1}^{d} k_i(\beta_i-\alpha_i)}(\mathbb{G}) \right| & \\ & \text{si } n, s \geq 1, \\ 0 & \text{sinon.} \end{cases}$$

Formule close
Déterminant

Générateurs

$$\mathbb{G} := \left\{ \begin{array}{c} \overbrace{\mid \cdots \mid}^{\beta} \\ \boxed{\;1\;} \\ \underbrace{\mid \cdots \mid}_{\alpha} \end{array}, \ldots, \begin{array}{c} \overbrace{\mid \cdots \mid}^{\beta} \\ \boxed{\;m\;} \\ \underbrace{\mid \cdots \mid}_{\alpha} \end{array} \right\}$$

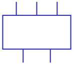
Formule close
Déterminant

Générateurs

$$\mathbb{G} := \left\{ \underbrace{\overbrace{\begin{array}{c} | \cdots | \\ \boxed{\;1\;} \\ | \cdots | \end{array}}^{\beta}}_{\alpha}, \ldots, \underbrace{\overbrace{\begin{array}{c} | \cdots | \\ \boxed{\;m\;} \\ | \cdots | \end{array}}^{\beta}}_{\alpha} \right\}$$

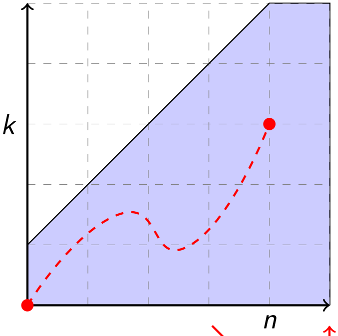
Théorème [C. en 2018]

$$|\mathrm{C}_{e,n,s}(\mathbb{G})| = m^n \det(\mathcal{B}), \text{ où } \mathcal{B}_{i,j} := \binom{e - i(\alpha - 1) + (j-1)(\beta - 1)}{i - j + 1}.$$

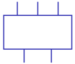
Formule raffinée de récurrence : $C_{2,n,n+2}\Big(\Big\{$  $\Big\}\Big)$

Définition

Soit $t(n,k)$ le nombre de chemins



n'utilisant que les pas $\searrow$ et $\uparrow$.

Formule raffinée de récurrence : $C_{2,n,n+2}\Big(\Big\{$  $\Big\}\Big)$

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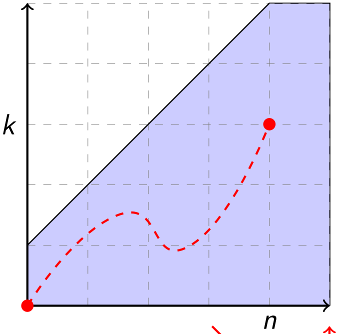

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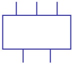
Formule raffinée de récurrence : $\mathrm{C}_{2,n,n+2}\Big(\Big\{$  $\Big\}\Big)$

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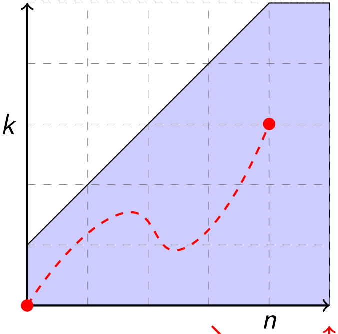

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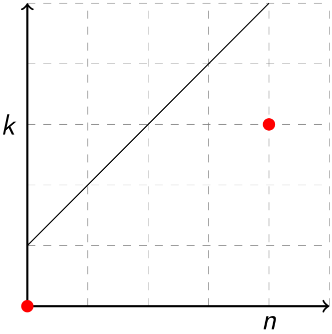
Formule raffinée de récurrence : $C_{2,n,n+2}(\{\ \})$

Démonstration



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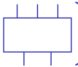
Formule raffinée de récurrence : $C_{2,n,n+2}\left(\left\{ \right.\right.$  $\left.\left.\right\}\right)$

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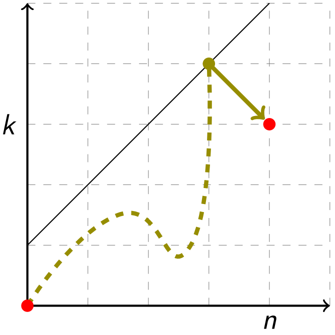

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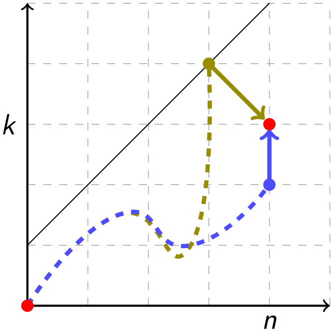

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Formule raffinée de récurrence : cas général

Générateurs (rappel)

$$\mathbb{G} := \left\{ \begin{array}{c} \overbrace{\mid \cdots \mid}^{\beta_1} \\ \boxed{\;1\;} \\ \underbrace{\mid \cdots \mid}_{\alpha_1} \end{array}, \ldots, \begin{array}{c} \overbrace{\mid \cdots \mid}^{\beta_1} \\ \boxed{m_1} \\ \underbrace{\mid \cdots \mid}_{\alpha_1} \end{array}, \ldots, \begin{array}{c} \overbrace{\mid \cdots \mid}^{\beta_d} \\ \boxed{\;1\;} \\ \underbrace{\mid \cdots \mid}_{\alpha_d} \end{array}, \ldots, \begin{array}{c} \overbrace{\mid \cdots \mid}^{\beta_d} \\ \boxed{m_d} \\ \underbrace{\mid \cdots \mid}_{\alpha_d} \end{array} \right\}$$

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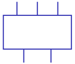
Théorème [C. en 2018]

$$t(n,k,s) = \begin{cases} 1 & \text{si } n = 0,\ k = 1 \text{ et } s = e, \\ t(n,k-1,s) + \displaystyle\sum_{g \in \mathbb{G}(\alpha,\beta)} t(n-1,k-1+\alpha,s-\beta+\alpha) & \\ & \text{si } n \geq 0 \text{ et } 1 \leq k \leq s, \\ 0 & \text{sinon.} \end{cases}$$

Équation fonctionnelle raffinée : $\mathrm{C}_{2,n,n+2}\Big(\Big\{\quad\Big\}\Big)$

Définition

Soit $T(x,y) := \sum_{n,k\geq 0} t(n,k)\, x^n y^k$.

Équation fonctionnelle raffinée : $\mathrm{C}_{2,n,n+2}\Big(\Big\{$$\Big\}\Big)$

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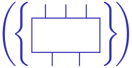
Nos chemins sont récursivement engendrés par

ÉQUATION FONCTIONNELLE RAFFINÉE : $\mathrm{C}_{2,n,n+2}$ 

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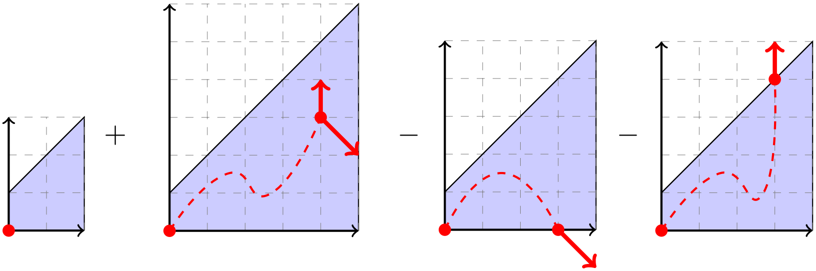

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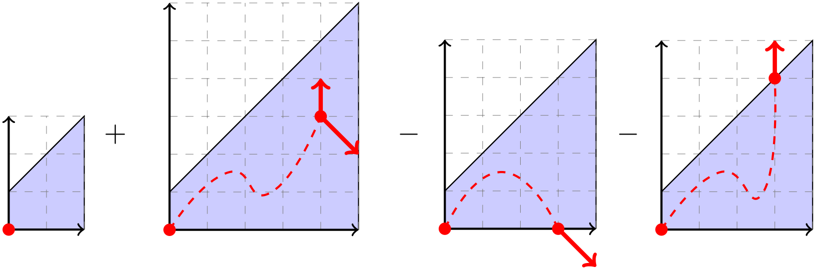
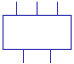


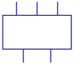
$$T(x,y) = 1 \quad + \quad \left(y + xy^{-1}\right) T(x,y) \quad - \quad xy^{-1}\, T(x,0) \quad - \quad y^2 D(xy)$$

Équation fonctionnelle : $\mathrm{C}_{2,n,n+2}\Big(\Big\{$  $\Big\}\Big)$

Méthode du noyau [Knuth]

Proposition

$$\left(-x + y - y^2\right) T(x,y) = y - xT(x,0) - y^3 D(xy)$$

Équation fonctionnelle : $\mathrm{C}_{2,n,n+2}\Big(\Big\{$  $\Big\}\Big)$

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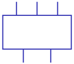
Équation fonctionnelle : $C_{2,n,n+2}\left(\left\{\right.\right.$$\left.\left.\right\}\right)$

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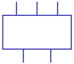
Soit $y_1 := \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} x^{n+1}$ et $y_2 := 1 - y_1$ les racines de $-x + y - y^2$.

Équation fonctionnelle : $C_{2,n,n+2}\left(\left\{\right.\right.$$\left.\left.\right\}\right)$

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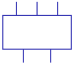
Équation fonctionnelle : $C_{2,n,n+2}\left(\left\{\right.\right.$  $\left.\left.\right\}\right)$

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Proposition

$$\boxed{y_1 - y_1^3 D(xy_1) = y_2 - y_2^3 D(xy_2)}$$

Perspectives

Natures des séries génératrices

Rationnelle ? Algébrique ? Holonome ? . . .

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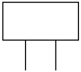
Exemple : Temperley–Lieb, dont les générateurs sont  et .

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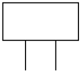
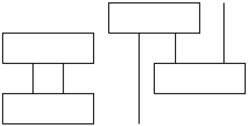
Exemple : Temperley–Lieb, dont les générateurs sont  et .



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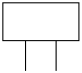
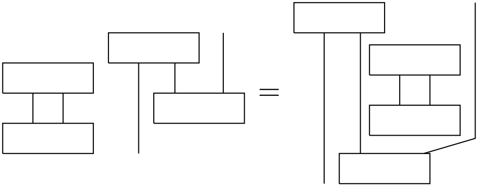
Exemple : Temperley–Lieb, dont les générateurs sont  et  .



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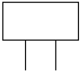
Exemple : Temperley–Lieb, dont les générateurs sont  et  .