A Note with Computer Exploration on the Triangle Conjecture

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Introduction to Coding Theory

Variable-length code:

$$\mathcal{S} \longrightarrow \mathcal{T}$$

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10001	b
01	c
$\vdots$	

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⋮	

Difficulty: the frame must be uniquely decomposable!

Code

Definition

A set $X \subset \mathcal{A}^*$ is a **code** if and only if for all $\omega \in X^*$ there exist a unique $n \geq 0$ and a unique sequence $x_1, \ldots, x_n \in X$ such that

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Example

The set $\{aabb, abaaa, b, ba\}$ is not a code because

$$babaaabb = (b)(abaaa)(b)(b) = (ba)(ba)(aabb).$$

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Proposition

A prefix set different than $\{\varepsilon\}$ is a code.

Commutatively Prefix Conjecture

Definition

A set $X \subset \mathcal{A}^*$ is **commutatively prefix** if there exists a prefix code P such that the multisets

$$\{(|x|_a, |x|_b) : x \in X\} \text{ and } \{(|p|_a, |p|_b) : p \in P\}$$

are equal.

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Conjecture from Perrin and Schützenberger (1965)

All finite maximal codes are commutatively prefix.

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Triangle conjecture (Perrin and Schützenberger)

A finite bayonet code is either commutatively prefix or it is not included in a finite maximal code.

Non-Commutatively Prefix Bayonet Code

(Well known) Proposition

A bayonet code X is commutatively prefix if and only if

$$\left|X \cap \mathcal{A}^{\leq n}\right| \leq n, \text{ for all } n \geq 0.$$

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with 16 elements and included in $\mathcal{A}^{\leq 15}$. Hence, it is a <u>non-commutatively prefix code</u>.

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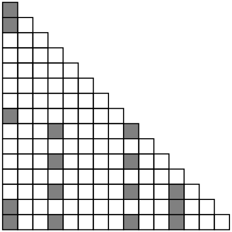

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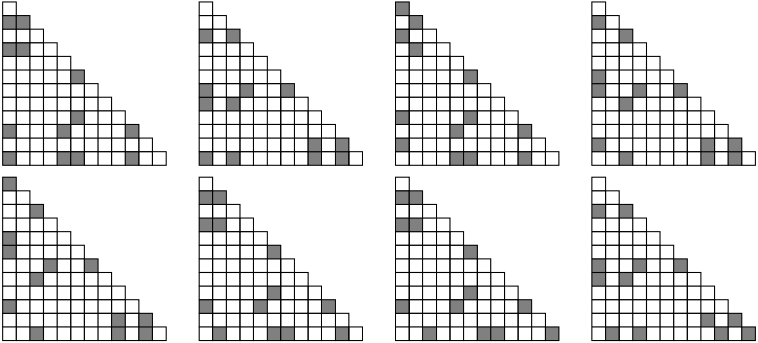

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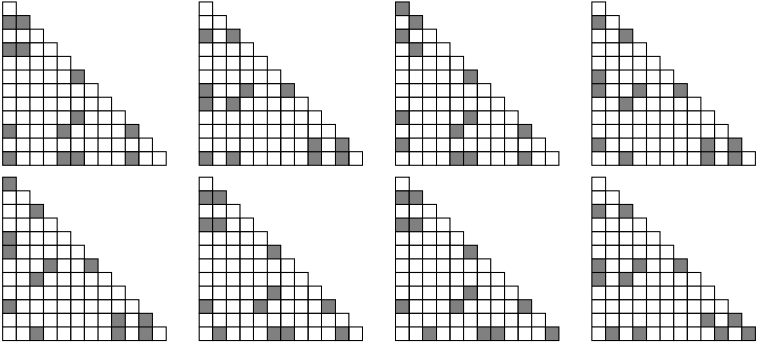

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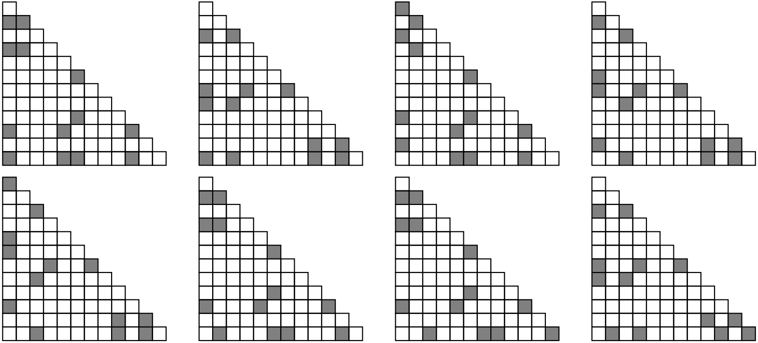

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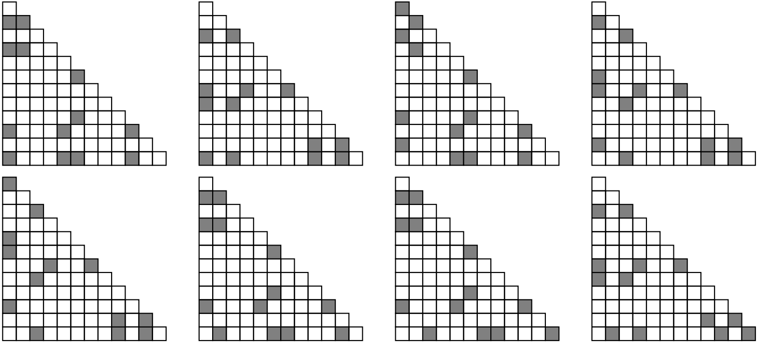

$n = 13, 14$: 0 code.

$n = 15$: 76 codes.

$n = 16$: at least 50 codes...

$n = 17$: at least 6 codes...

Shor Inequality

A consequence of our computing

Question from Shor (1984)

What is the maximum value of $\frac{|X|}{n}$ where X is a code belonging to $a^*ba^* \cap \mathcal{A}^{\leq n}$ and n an integer?

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Partial answer from Shor and Hansel

This value in between $\frac{16}{15}$ and $1 + \frac{1}{\sqrt{2}}$.

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This value in between $\frac{13}{12}$ and $1 + \frac{1}{\sqrt{2}}$.

Factorisations of Cyclic Groups

Definition

Given $n \geq 1$, the ordered pair $(L, R) \subset [0, n[^2$ is a **factorisation** of $\mathbb{Z}/n\mathbb{Z}$ if

$$\forall k \in [0, n[, \ \exists!(\ell, r) \in L \times R \text{ such that } k = \ell + r \bmod n.$$

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The ordered pair $(\{1, 3, 5\}, \{1, 2, 7, 8\})$ is a factorisation of $\mathbb{Z}/12\mathbb{Z}$.

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Link Between Factorisations and Codes

Theorem from Restivo, Salemi, and Sportelli (1989)

If X is a finite maximal code such that $b, a^n \in X$ then (L, R) is a factorisation of $\mathbb{Z}/n\mathbb{Z}$, where

$$L := \left\{ k \bmod n : a^k b^+ \in X \right\} \text{ and } R := \left\{ k \bmod n : b^+ a^k \in X \right\}.$$

Such a factorisation is called a **factorisation associated** to X.

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$$(L \supseteq \{0, 3, 8, 11\}, R \supseteq \{0, 1, 7, 13, 14\})$$

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We do not know any of these factorisations.

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Theorem from Sands (2000)

If (L, R) is a factorisation of $\mathbb{Z}/n\mathbb{Z}$ and p is an integer relatively prime to $|L|$ then (pL, R) is a factorisation of $\mathbb{Z}/n\mathbb{Z}$.

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Recall: the factorisation of $\mathbb{Z}/n\mathbb{Z}$ associated to Shor's code is of the form

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Notice that $(L, 3R)$, $(L, 5R)$, $(L, 8R)$, and $(L, 11R)$ are not factorisations. Thus $3|n$, $5|n$, $2|n$, and $11|n$.
Hence n is a multiple of $2 \times 3 \times 5 \times 11 = 330$.

Results About Factorisations

Recall: we found 54 non-commutatively prefix code containing b.

Number of codes	Order of the letter a
4	$2 \times 3 \times 5 \times k = 30k$, with $k \geq 3$
12	$2 \times 3 \times 11 \times k = 66k$, with $k \geq 3$
4	$2 \times 3 \times 5 \times 11 \times k = 330k$, with $k \geq 4$
8	$2 \times 3 \times 5 \times 13 \times k = 390k$, with $k \geq 4$
8	$2 \times 3 \times 5 \times 13 \times k = 390k$, with $k \geq 3$
4	$2 \times 5 \times 13 \times k = 130k$, with $k \geq 3$

Lower bound

A corollary of a theorem from Perrin and Schützenberger (1977)

To be included in a finite maximal code, a bayonet code must be included in a bayonet code $X \subseteq a^{<n}ba^{<n}$ such that $|X| = n$ and $\{a^n\} \cup X$ is a code, for an integer n.

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THANK YOU!