Explorations de la conjecture du triangle

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Introduction to Coding Theory

Variable-length code:

$$\mathcal{S} \longrightarrow \mathcal{T}$$

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Table	
11	a
10001	b
01	c
$\vdots$	

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Difficulty: the frame must be uniquely decomposable!

Code

Definition

A set $X \subset \mathcal{A}^*$ is a **code** if and only if for all $\omega \in X^*$ there exist a unique $n \geq 0$ and a unique sequence $x_1, \ldots, x_n \in X$ such that

$$\omega = x_1\, x_2\, \cdots\, x_n.$$

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Example

The set $\{aabb, abaaa, b, ba\}$ is not a code because

$$babaaabb = (b)(abaaa)(b)(b) = (ba)(ba)(aabb).$$

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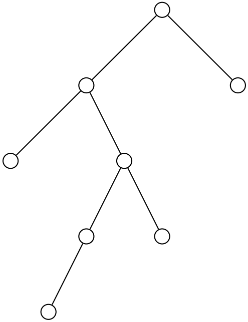
The set $\{b, ab, a^2b, a^3b, a^4b, \dots\}$ is a prefix code.

Proposition

A prefix set different than $\{\varepsilon\}$ is a code.

Why Prefix Code?

Easy to produce!



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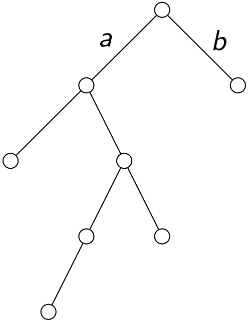

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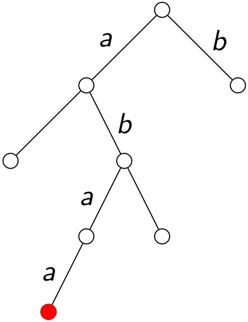

Produce the word *abaa*.

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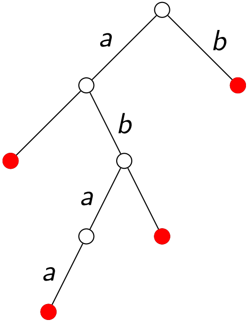

Produce the prefix code $\{aa, abaa, abb, b\}$.

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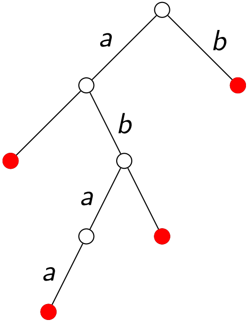

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Easy to decode!

For example: $aaaabaaabbabaabaab$

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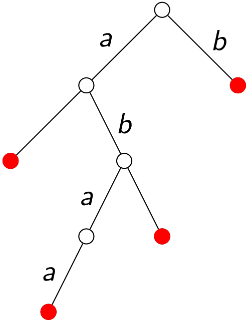

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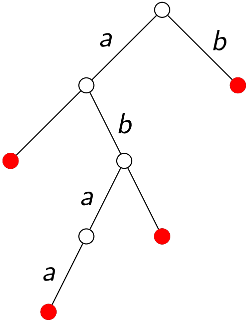

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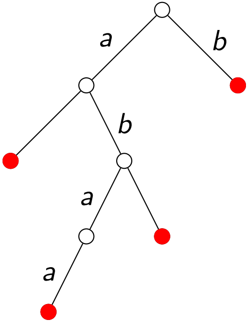

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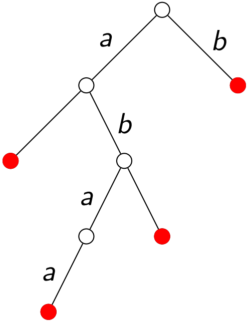

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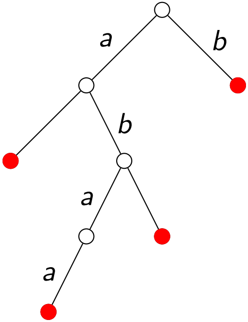

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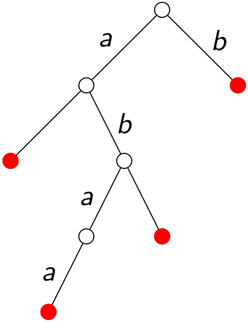

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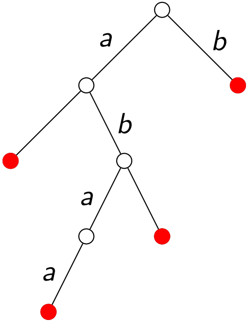

Produce the prefix code $\{aa, abaa, abb, b\}$.

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Commutatively Prefix Conjecture

Definition

A set $X \subset \mathcal{A}^*$ is **commutatively prefix** if there exists a prefix code P such that the multisets

$$\{(|x|_a, |x|_b) \,:\, x \in X\} \text{ and } \{(|p|_a, |p|_b) \,:\, p \in P\}$$

are equal.

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Conjecture from Perrin and Schützenberger (1965)

All finite maximal codes are commutatively prefix.

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Triangle conjecture (Perrin and Schützenberger)

A finite bayonet code is either commutatively prefix or not included in a finite maximal code.

Non-Commutatively Prefix Bayonet Code

(Well known) Proposition

A bayonet code X is commutatively prefix if and only if

$$\left|X \cap \mathcal{A}^{\leq n}\right| \leq n, \text{ for all } n \geq 0.$$

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Example

In 1984, Shor found the bayonet code

$$\begin{array}{lllll} \{ \quad b, & ba, & ba^7, & ba^{13}, & ba^{14}, \\ a^3b, & a^3ba^2, & a^3ba^4, & a^3ba^6, & \\ a^8b, & a^8ba^2, & a^8ba^4, & a^8ba^6, & \\ a^{11}b, & a^{11}ba, & a^{11}ba^2 & \quad \} & \end{array}$$

with 16 elements and included in $\mathcal{A}^{\leq 15}$. Hence, it is a non-commutatively prefix code.

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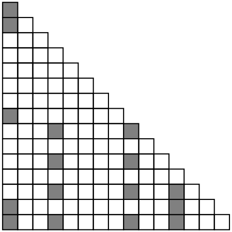

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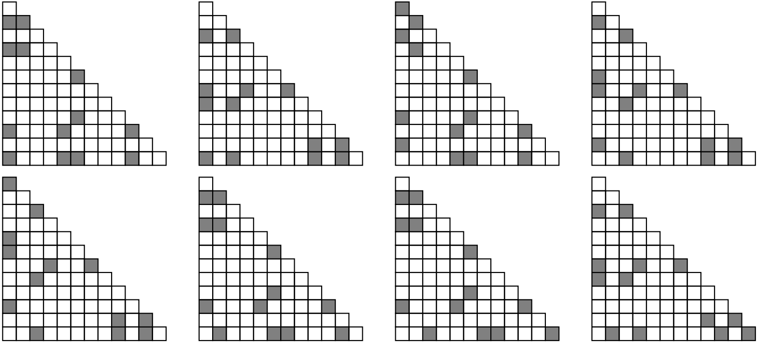

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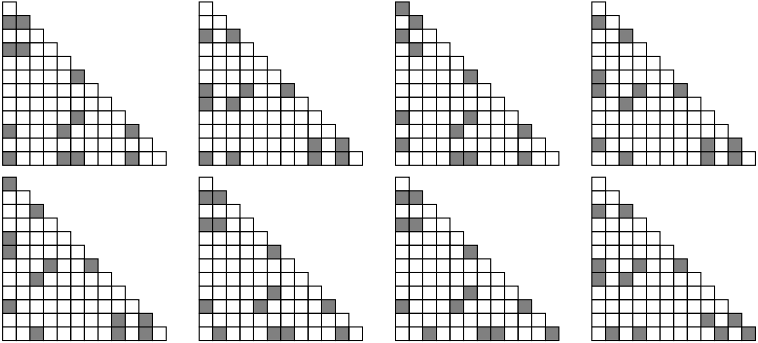

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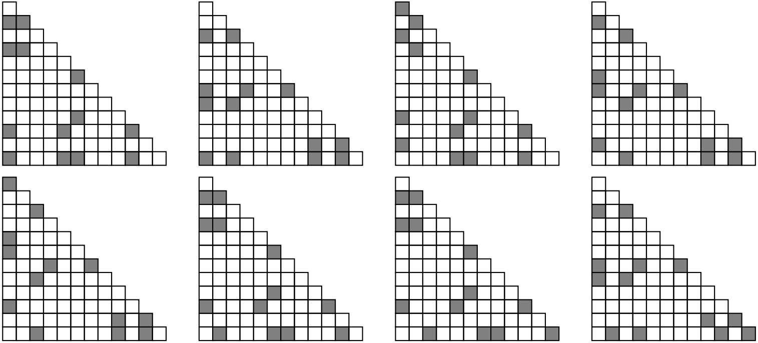

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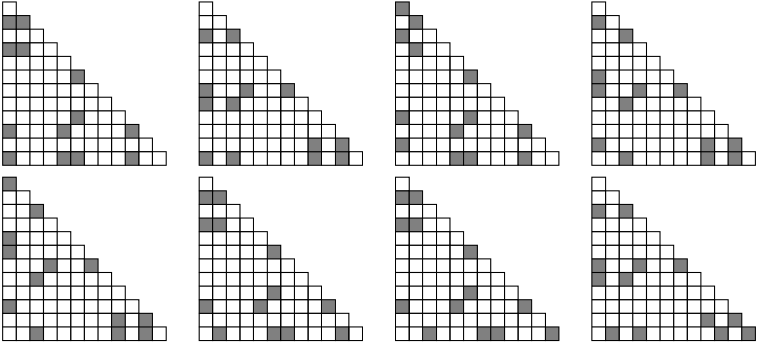

$n = 13, 14$: 0 code.

$n = 15$: 76 codes.

$n = 16$: at least 50 codes...

$n = 17$: at least 6 codes...

Shor Inequality

A consequence of our computing

Question from Shor

What is the maximum value of $\frac{|X|}{n}$ where X is a code belonging to $a^*ba^* \cap \mathcal{A}^{\leq n}$ and n an integer?

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Partial answer from Shor and Hansel

This value in between $\frac{16}{15}$ and $1 + \frac{1}{\sqrt{2}}$.

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(Well known) Proposition

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Remark

The order of a letter is unique because

$$\left(a^i\right)\left(a^j\right) = \left(a^j\right)\left(a^i\right).$$

Factorisations of Cyclic Groups

Definition

Given $n \geq 1$, the ordered pair (L, R) such that $L, R \subset [0, n[$ is a **factorisation** of $\mathbb{Z}/n\mathbb{Z}$ if

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Link Between Factorisations and Codes

Theorem from Restivo, Salemi, and Sportelli (1989)

If X is a finite maximal code such that $b, a^n \in X$ then (L, R) is a factorisation of $\mathbb{Z}/n\mathbb{Z}$, where

$$L := \left\{ k \bmod n : a^k b^+ \in X \right\} \text{ and } R := \left\{ k \bmod n : b^+ a^k \in X \right\}.$$

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A factorisation associated to Shor's code is of the form

$$(L \supseteq \{0, 3, 8, 11\}, R \supseteq \{0, 1, 7, 13, 14\})$$

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We do not know any of these factorisations.

Link Between Factorisations and Codes

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If X is a finite maximal code such that $b, a^n \in X$ then (L, R) is a factorisation of $\mathbb{Z}/n\mathbb{Z}$, where

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Example (one of our codes)

$$\begin{array}{lllll} \{ & b, & ba^2, & ba^8, & ba^{10}, \\ & aba^8, & aba^{10}, & & \\ & a^4b, & a^4ba^2, & & \\ & a^5b, & a^5ba^3, & a^5ba^6, & \\ & a^9b, & a^9ba^2 & \} & \end{array}$$

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Example (one of our codes)

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We do know some of these factorisations.

Factorisation Associated to our Code

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Some solutions

For $n \geq 2$,

$$\left(\{0,4,5,9\}, \bigsqcup_{0 \leq i < n} \{8i, 8i+2\}\right)$$

is a factorisation of $\mathbb{Z}/8n\mathbb{Z}$ associated to our code.

Sands Theorem

Theorem from Sands (2000)

If (L, R) is a factorisation of $\mathbb{Z}/n\mathbb{Z}$ and p is an integer relatively prime to $|L|$ then (pL, R) is a factorisation of $\mathbb{Z}/n\mathbb{Z}$.

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Hence n is a multiple of $2 \times 3 \times 5 \times 11 = 330$.

Results About Factorisations

Recall: we found 54 non-commutatively prefix code containing b.

Number of codes	Order of the letter a
4	$2 \times 3 \times 5 \times k = 30k$, with $k \geq 3$
12	$2 \times 3 \times 11 \times k = 66k$, with $k \geq 3$
4	$2 \times 3 \times 5 \times 11 \times k = 330k$, with $k \geq 4$
8	$2 \times 3 \times 5 \times 13 \times k = 390k$, with $k \geq 4$
8	$2 \times 3 \times 5 \times 13 \times k = 390k$, with $k \geq 3$
4	$2 \times 5 \times 13 \times k = 130k$, with $k \geq 3$

Complete Modular Bayonet Code

Definition

We call a n-**modular bayonet code** a bayonet code $X \subseteq a^{<n}ba^{<n}$ such that $\{a^n\} \cup X$ is a code.
We said that it is **complete** if $|X| = n$.

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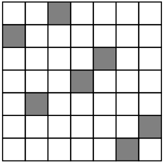
Example



is a complete n-modular bayonet code.

CMBC: Lower Bound

Theorem (Perrin and Schützenberger (1977))

Let X be a finite maximal code. Let $x \in \mathcal{A}$ be a letter and let n be the order of x. For all $\omega \in \mathcal{A}^*$, the set

$$C_x(\omega) := \left\{(i \ mod\ n, j \ mod\ n) : x^i \omega x^j \in X^*\right\}$$

has cardinal n.

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To be included in a finite maximal code, a bayonet code must be included in a complete n-modular bayonet code.

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To be included in a finite maximal code, a bayonet code must be included in a complete n-modular bayonet code.

Computer exploration

None of the 140 non-commutatively prefix bayonet codes satisfies this condition for $n \leq 32$.

CMBC: an Other Consequence

Theorem

The code

$$\left\{b, ba^2, ba^8, ba^{10}, aba^8, aba^{10}, a^4b, a^4ba^2, a^5b, a^5ba^3, a^5ba^6, a^9b, a^9ba^2\right\} \cup \left\{a^{16}\right\}$$

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Remark

It is the smallest known code of this type.

Perspectives

- Conjecture

  If X is a complete n-modular bayonet code then

$$\varphi_q(X) := \left\{ a^{qi \bmod n} b a^j : a^i b a^j \in X \right\}$$

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THANK YOU!

Annex: is it a Bayonet Code?

Algorithm

Given a set X, we define the graph $\mathcal{G}(X)$ by the vertices $[0, n[$ and by the edges

$$\boxed{|i-k|} \longrightarrow \boxed{|j-\ell|},$$

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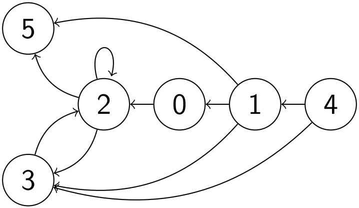

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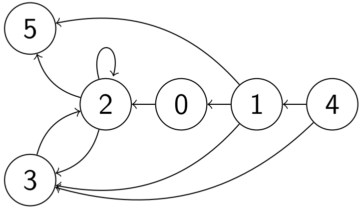

Thus X is a code.