Generalizations of the associative operad and convergent rewrite systems

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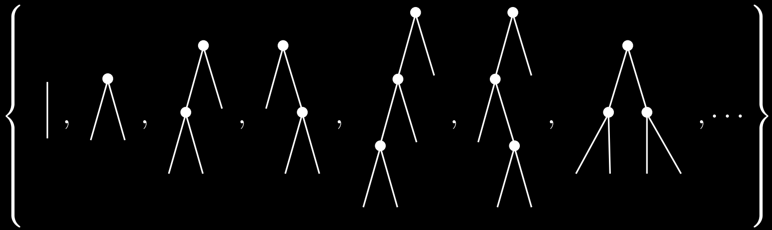
July 7, 2018

Magmatic operad: definition

Set of binary trees:



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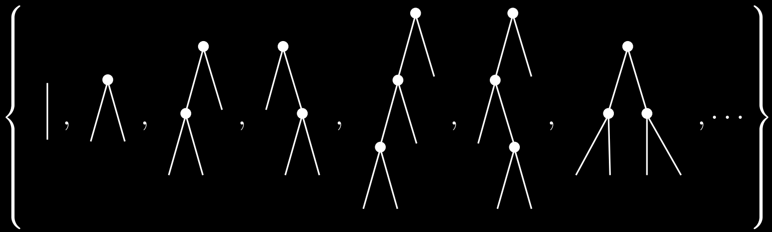
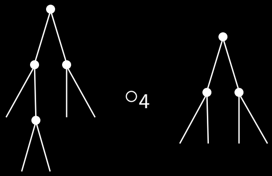


Provided with a set of algebraic operations $\{\circ_1, \circ_2, \circ_3, \cdots\}$:



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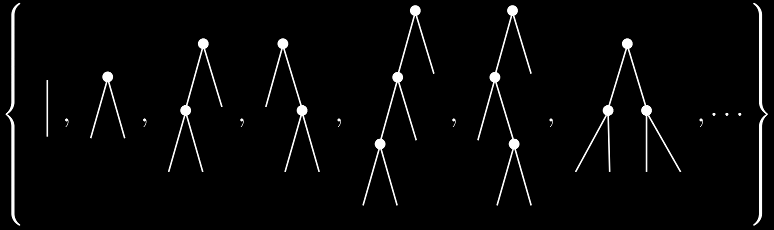

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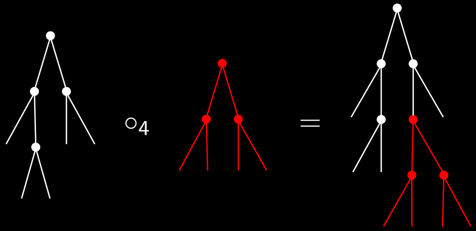

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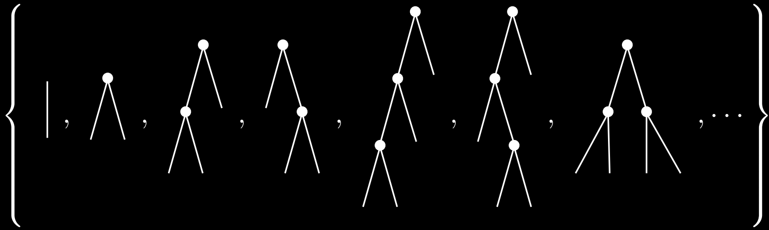

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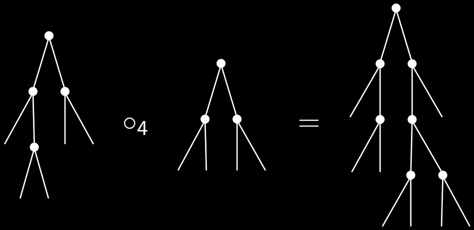
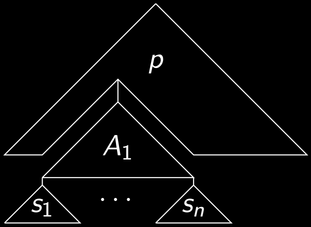


Quotient of Magmatic operad: $\mathsf{Mag}\Big/_{\langle A_1 \equiv A_2 \rangle}$



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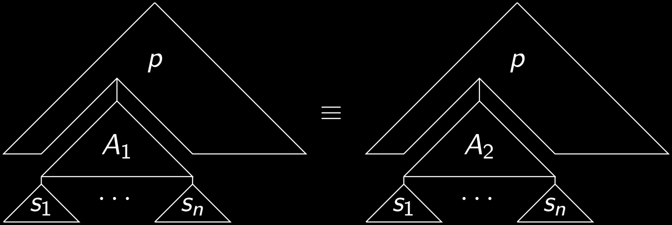
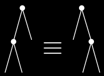


As OPERAD: Mag $\Big/ \Big\langle$  $\Big\rangle$

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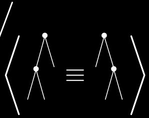
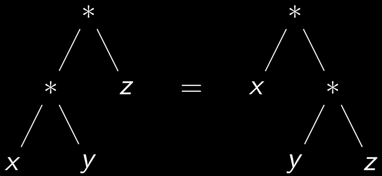
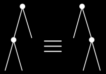
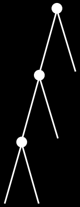
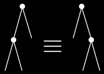


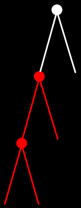


$$(x * y) * z \quad = \quad x * (y * z)$$

As OPERAD: Mag $\Big/ \Big\langle$  $\Big\rangle$



As OPERAD: Mag$\Big/\Big\langle$  $\Big\rangle$



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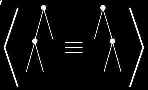
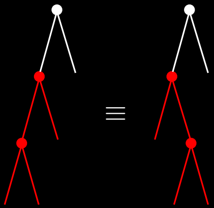
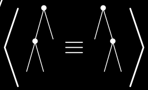
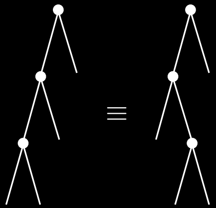




As OPERAD: Mag / 



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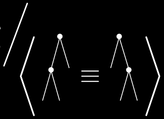
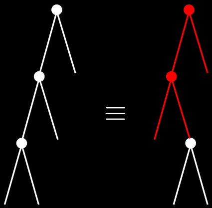
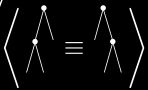
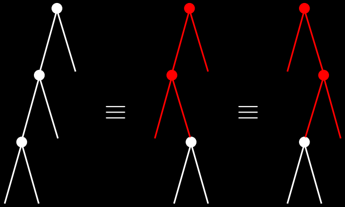




As OPERAD: Mag/ 



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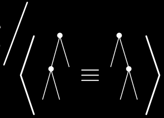

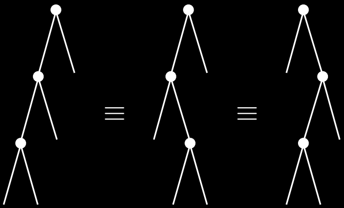

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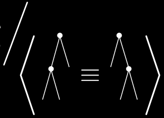
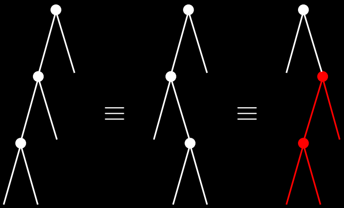
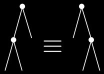
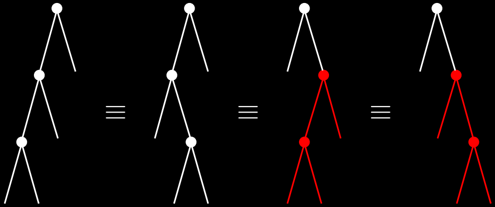
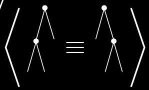
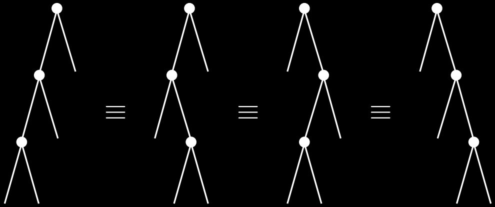


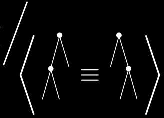


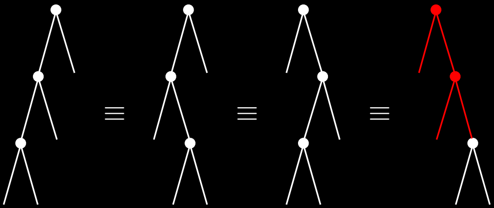
As OPERAD: Mag / ⟨  ⟩

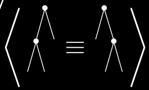


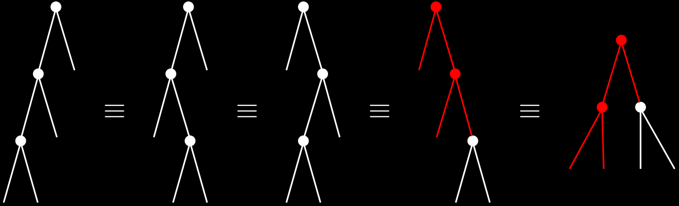
As OPERAD: Mag / 

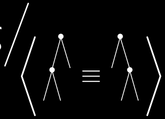


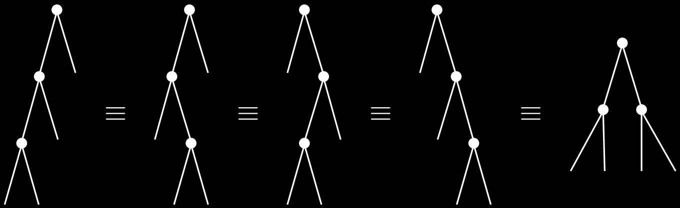
As OPERAD: Mag / 

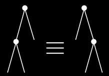


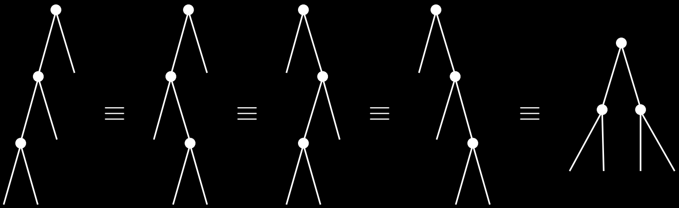
As OPERAD: Mag/ 



As OPERAD: Mag 



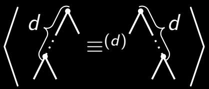
As OPERAD: Mag $/ \langle$  $\rangle$



HILBERT SERIES:
$$\mathcal{H}_{\mathsf{As}}(t) = \sum_{n\geq 1} \#\mathsf{As}(n)\, t^n = t + t^2 + t^3 + t^4 + \cdots = \frac{t}{1-t}$$

Generalization of As operad: $\mathsf{CAs}^{(d)}$

$\mathsf{CAs}^{(d)} := \mathsf{Mag} \big/$



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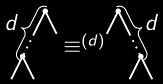
$\mathsf{CAs}^{(d)} := \mathsf{Mag} \Big/ \Big\langle$  $\Big\rangle$

Relation:

$$(\cdots((x_1 * x_2) * x_3) \cdots) * x_{d+1} = x_1 * (\cdots (x_{d-1} * (x_d * x_{d+1})) \cdots)$$

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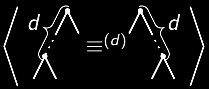
$\mathsf{CAs}^{(d)} := \mathsf{Mag} \Big/$ 

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Remark: $\mathsf{CAs}^{(0)} \cong \mathsf{CAs}^{(1)} \cong \mathsf{Mag}$ and $\mathsf{CAs}^{(2)} = \mathsf{As}$

Generalization of As operad: $\mathsf{CAs}^{(d)}$

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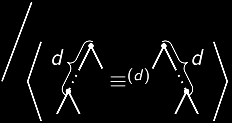

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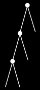
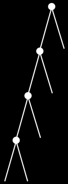
<u>Hilbert series:</u>

$$\begin{aligned}
\mathcal{H}_{\mathsf{Mag}}(t) &= t + t^2 + 2t^3 + 5t^4 + 14t^5 + 42t^6 + 132t^7 + 429t^8 \ldots \\
\mathcal{H}_{\mathsf{As}}(t) &= t + t^2 + t^3 + t^4 + t^5 + t^6 + t^7 + t^8 \ldots \\
\mathcal{H}_{\mathsf{CAs}^{(3)}}(t) &= t + t^2 + 2t^3 + 4t^4 + 8t^5 + 14t^6 + 20t^7 + 19t^8 \ldots
\end{aligned}$$

CAs$^{(3)}$: Rewriting rule:  $\to^{(3)}$ 



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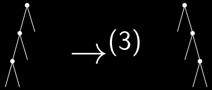

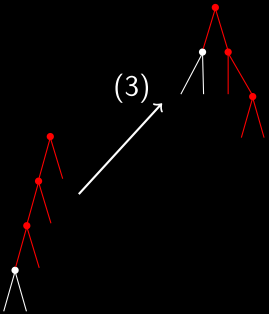

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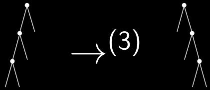

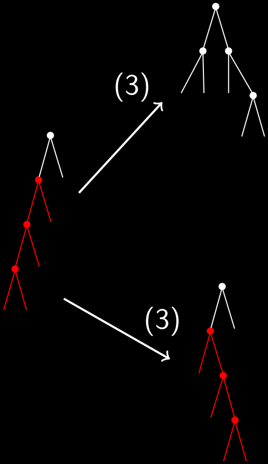

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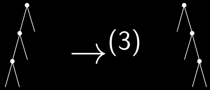
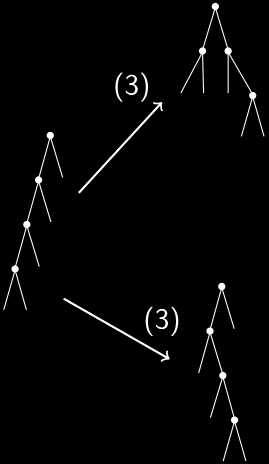
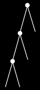
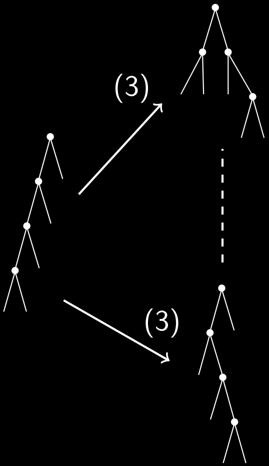


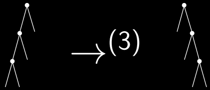


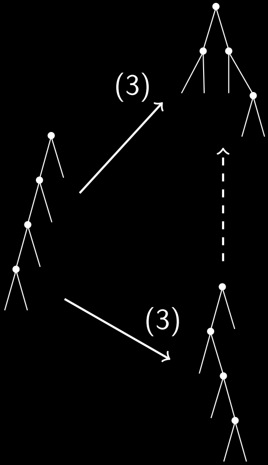
- Not confluent!

CAs$^{(3)}$: Rewriting rule:  $\to^{(3)}$ 



- Not confluent!
- Compute a completion.

CAs$^{(3)}$: Rewriting rule: 



- Not confluent!
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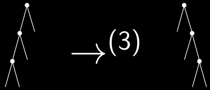
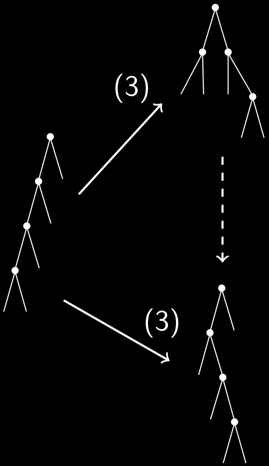
- Need a reduction ordering.

CAs$^{(3)}$: Rewriting rule: 



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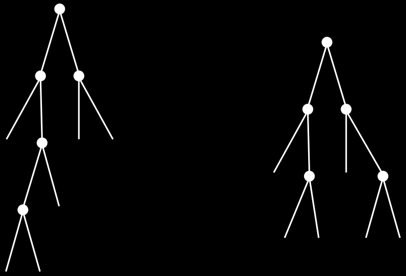
Reduction order: prefix order



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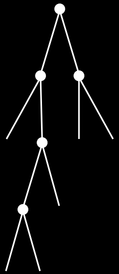

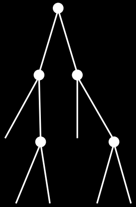

Prefix traversal: 22022000200 22020020200

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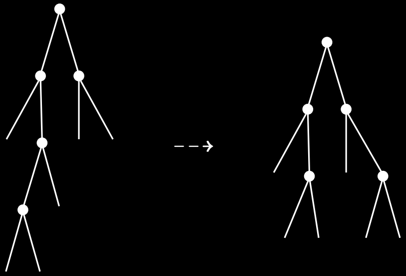
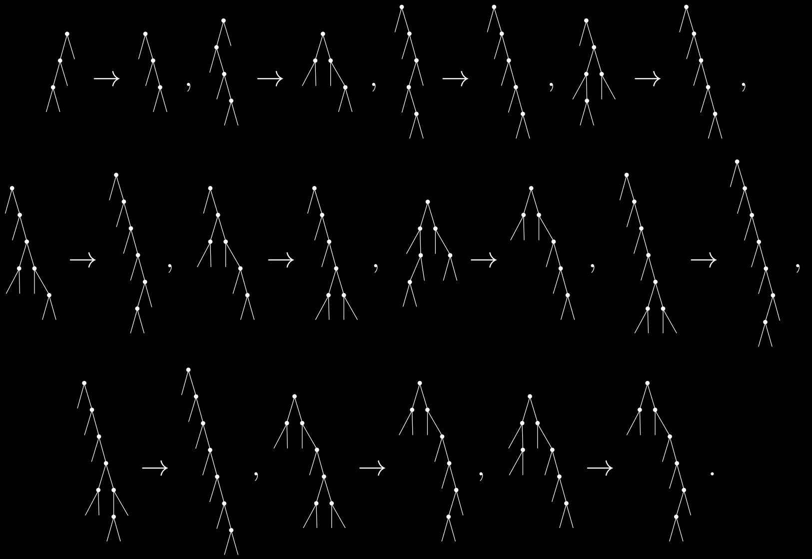


Prefix traversal: 22022000200 $>_{lex}$ 22020020200

Confluent rewriting system of CAs$^{(3)}$



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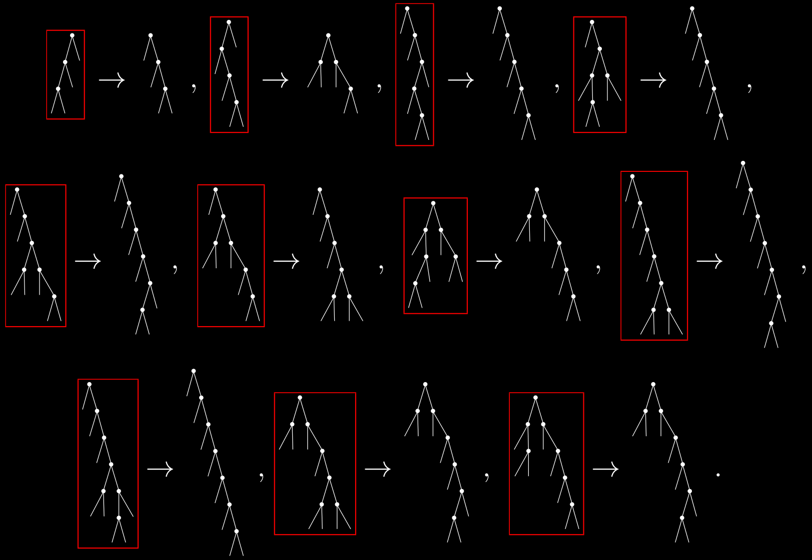
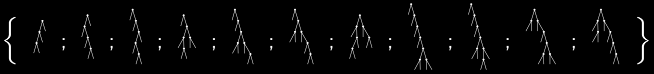


Hilbert series of $\mathsf{CAs}^{(3)}$

$$\mathcal{H}_{\mathsf{CAs}^{(3)}}(t) = G\frac{}{\{\;\cdots\;\}}$$



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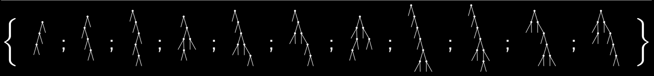

$$\mathcal{H}_{\mathsf{CAs}^{(3)}}(t) = \frac{t\left(1-t+t^2+t^3+2t^4+2t^5-7t^7-2t^8+t^9+2t^{10}+t^{11}\right)}{(1-t)^2}$$

$$= t + t^2 + 2t^3 + 4t^4 + 8t^5 + 14t^6 + 20t^7 + 19t^8 + 16t^9 + 14t^{10} + \sum_{n \geq 11} (n+3)t^n$$

Rewriting system for $\mathsf{CAs}^{(d \geq 4)}$?

Remark: Almost 3.000 rules at degree 40 provided by completion for $\mathsf{CAs}^{(4)}$ using prefix order.

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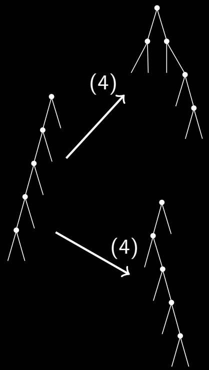

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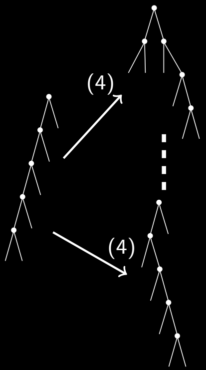

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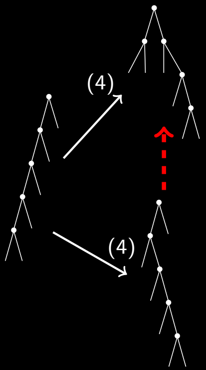

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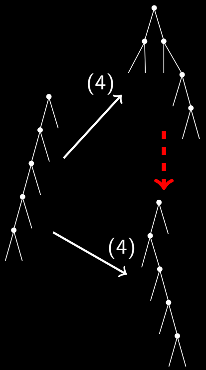

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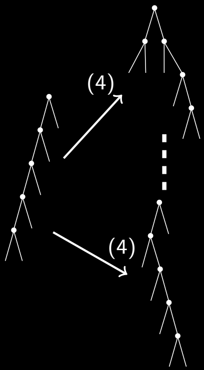

Conjecture: There is no finite, confluent and terminating rewriting system of $\mathsf{CAs}^{(d)}$ with one generator when $d \geq 4$.

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Proof: $f\Big($  $\Big) =$ 

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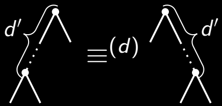
Remark: f exists iff 

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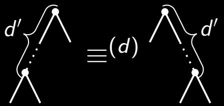
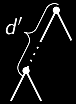
Remark: f exists iff 

Proof: if $d' = 1 + k(d-1)$ then



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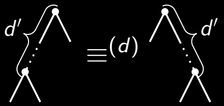

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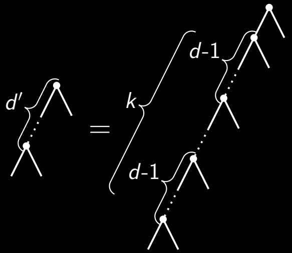

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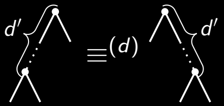

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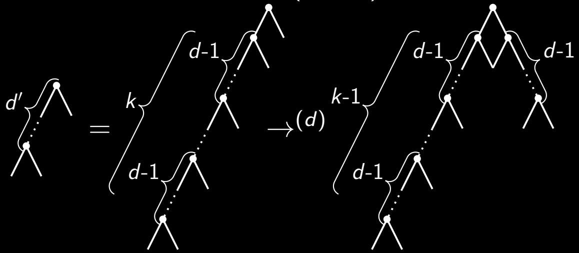

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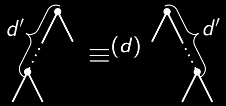

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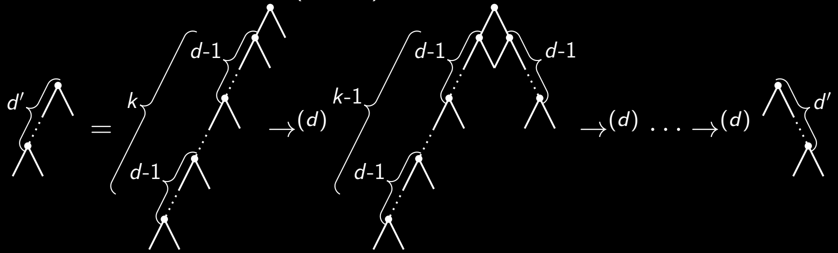

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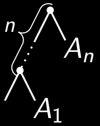
Definition: $g\left(\right.$ 

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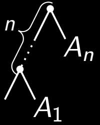
Definition: $g\left(\right.$ 

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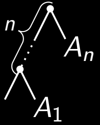
Definition: $g\Bigg($ 

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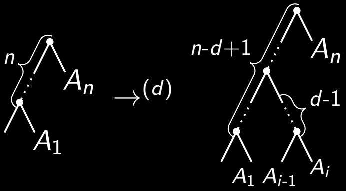
Proof: 

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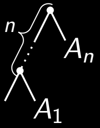
Definition: $g\left(\right.$ 

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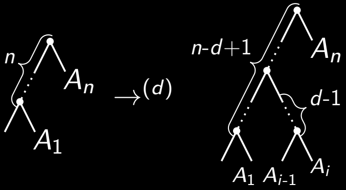
Proof: 

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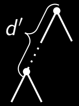
So $g\Bigg($ 

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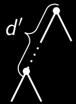
So $g\Bigg($ 

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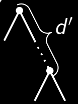
However $g\Bigg($ 

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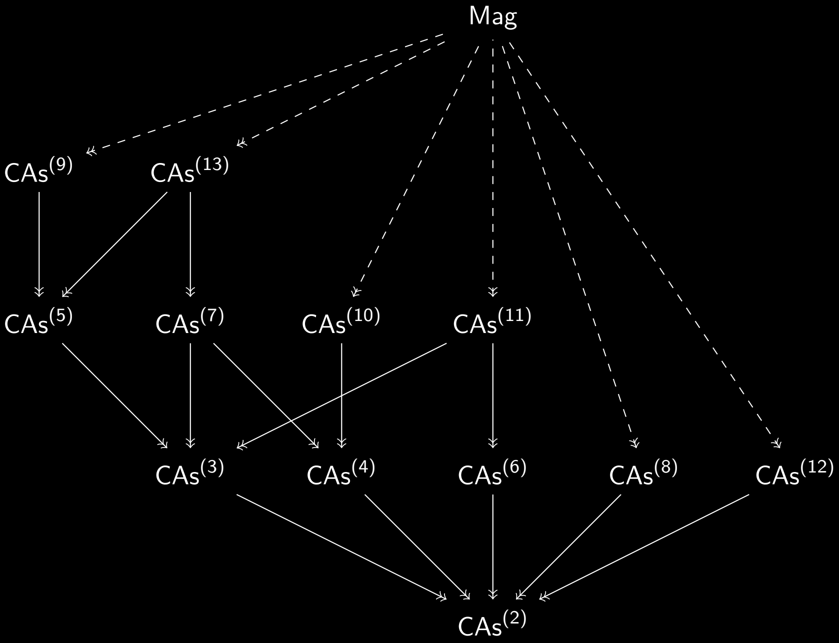
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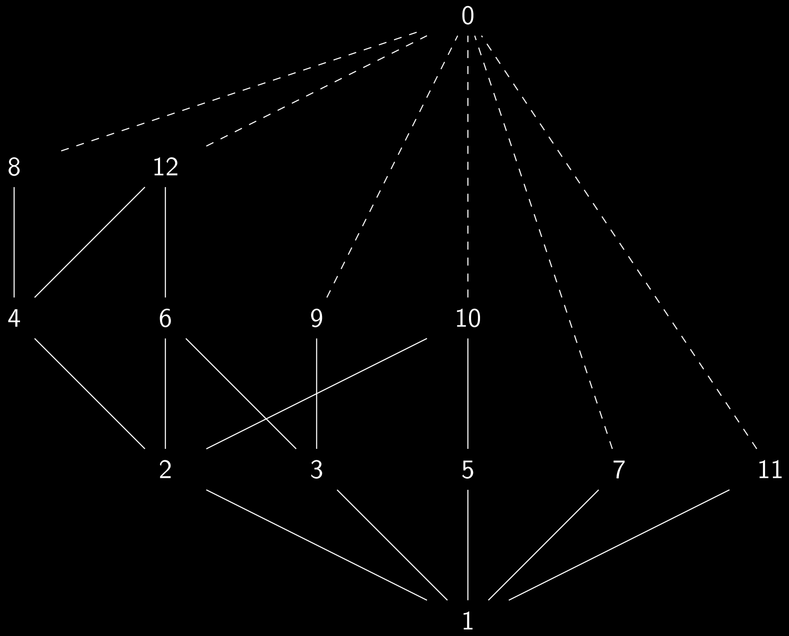
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