DESCRIPTION OF SINGULAR SOLUTIONS TO THE PRANDTL'S EQUATIONS AND THE SEMI-LINEAR HEAT EQUATION: A BRIEF INTRODUCTION

CHARLES COLLOT

1. Introduction

These are short lecture notes written as the author was invited to give a mini-course in the University of Science at Technology of China, in Hefei, China, and he is grateful to the math department for the invitation and the organisation of his stay.

There are merely no books on the detailed description of singularity formation for evolution PDEs, especially concerning the construction and stability of blowup solutions, despite the fascinating aspect of this topic. These notes aim at explaining briefly some key concepts, with a level of difficulty accessible to graduate students/ students in Masters. The aim is to show how, despite the diversity of the PDE world, there are universal features in singularity formation. In particular, we focus on the appearance of self-similarity, and on the implication it has that stability problems in singularity formation have a lot in common. To this aim we focus on two apparently unrelated equations, the semilinear-heat equation

$$\begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u|_{t=0} = u_0 \end{cases} \tag{1.1}$$

describing a scalar field $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ subject to diffusion and nonlinear growth, and the Prandtl's system:

$$\begin{cases} u_t + uu_x + vu_y - u_{yy} = -p_x^E & (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \quad \lim_{y\to\infty} u(t,x,y) = u^E(t,x), \end{cases} \tag{1.2}$$

describing the evolution of the velocity field (u,x) of a fluid in the upper half plane $\mathbb{R} \times [,\infty)$, in the vanishing viscosity limit of the Navier-Stokes equations near a boundary. The first one is a semi-linear equation for which the solution might become unbounded in finite time, creating a singularity, while the second one is a quasilinear transport equation for which the spatial derivatives might become unbounded in finite, creating also a singularity. We shall study in details the inviscid version of the Prandtl's system:

$$\begin{cases} u_t + uu_x + vu_y = -p_x^E & (t,x,y) \in [0,T] \times \mathbb{R} \times \mathbb{R}_+, \\ u_x + v_y = 0, \\ v|_{y=0} = 0, \quad \lim_{y\to\infty} u(t,x,y) = u^E(t,x). \end{cases} \tag{1.3}$$

The key issues we want to underline and explain in these notes are the following.

- Section 1 deals with the existence and properties of solutions (local well-posedness). This issue is a classical one in PDEs, and it will allow us to insist on the key properties of the dynamics. We will briefly encounter the concept of weak solution, and show the existence of solution to (1.1)

using an iterative scheme (fixed point). At the heart of this scheme will lie the regularising effects of the heat equation, and we shall also present the closely related parabolic energy estimates, and parabolic regularisation techniques. To study the Prandtl's system, we shall go over linear and quasilinear transport dynamics, and study finite speed of propagation and the existence of an underlying geometry encoded by the characteristics.

- Section 2 deals with the notion of self-similarity. This key feature of many singularity formations will be explained in details, and we shall see how and why it applies to evolution PDEs. After reviewing basic examples, we will construct self-similar solutions both for the semilinear heat equation and the inviscid Prandtl's system. For both PDEs, blow-up solutions are asymptotically self-similar near the singularity, highlighting why the problem of the stability of self-similar solutions is a universal problem. Asymptotic self-similarity for the semilinear heat equation is mostly a consequence of the existence of a scaling invariance, parabolic regularising effects, and local energy dissipation. Asymptotic self-similarity for the inviscid Prandtl's system is mostly a consequence of the existence of a scaling invariance, the finite speed of propagation of the equation, and the asymptotic self-similarity for the characteristics.

- Section 3 presents a way to perform the stability analysis of a self-similar blow-up. It focuses on the energy supercritical semilinear heat equation in dimension 3. The linearised infinite dimensional dynamics are studied in details. An idea to avoid exponentially growing instabilities using topology is presented. For the full stability problem, we explain how to renormalise the solution in order to zoom at the singularity location, giving rise to modulation equations for the relevant scale and position parameters. We present how to obtain the existence of a perturbation remaining small over time by performing the analysis of solutions in a suitable bootstrap regime.

We focus on key aspects that we believe are relevant for persons that are not familiar with singularity formation. We focus also on the brief explanation of key techniques, which is of interest for graduate students. We thus cannot provide precise bibliographical references (we mainly give references to textbooks). Importantly, there are certain types of blow-up that we do not mention (ODE, type II). These notes were written in a short time interval and could present errors and typos. Despite the fact that these notes focus on particular examples, give sometimes simplified results, and avoid certain proofs in particular when too technical, we hope it can give a little glimpse of relevant issues in this research field.

Contents

2. Local well-posedness and basic properties

2.1. **Introduction.** We give here some results concerning the existence and properties of solutions. Here is the list of the main concepts and ideas, those underlined being examined here:

- *Local well posedness* (LWP) refers to the existence, uniqueness, and continuous dependance of solutions u to a problem with respect to the input in the model: here the initial datum u_0. The very concept of solution is questionable, as the equations can be regarded with different angles. Classical solutions are those u regular enough so that all terms of the equation make sense and that it is satisfied everywhere. Relaxing this condition yields to weak solutions. We give here the example of distributional solutions to the heat equation. An exhausting list of the questions regarding the solution map $u_0 \mapsto u$ on may want to answer is made by Terence Tao in [29], 3.2. What is a Solution. Here: solutions of the semi-linear heat equation become instantaneously classical solutions, and solutions of the Prandtl's system are also classical.
- *Iterative scheme and leading order description* LWP involves suitable function spaces for the initial datum and the solution, and many proofs involve an iterative scheme containing a description to leading order of the solution. This is the case here for the semi-linear heat equation and the main point is the following. In ordinary differential equations, writing $u_t = f(u)$, it is enough to treat u as a perturbation of u_0 and $f(u)$ as a perturbation of $f(u_0)$. For the PDEs here, $f(u)$ involves derivatives and this picture does not hold. A fruitful idea is to write $u_t = f_h(u) + f_l(u)$ where $f_h(u)$ collects the terms with higher order derivatives. The leading order part is then the solution to $u_t = f_h(u)$. Suitable function spaces can be found for LWP issues relying on the understanding of the properties of this approximate dynamics and its stability. The example here where $u_t = \Delta u + u^p$ is treated as a perturbation of $u_t = \Delta u$ is very similar to plenty of other semi-linear equations.
- *Criticality.* The largest possible spaces of datum allowing for LWP can be guessed by a rule of thumb explained here: that of criticality.
- *Properties of the solutions.* LWP results are usually linked with establishing properties of the solution. Here we will examine smoothing properties for the semilinear heat relying on parabolic bootstrap techniques and energy estimates. The inviscid Prandtl's equations enjoy rather a strong locality of the dynamics with finite speed of propagation, and propagation of singularities.
- *Blow-up criteria* Properties of u allow to continue the solution (typically boundedness of some norms). The nonexistence of a solution past the

maximal time of existence is thus detected by blow-up criteria (typically un-boundedness of these norms). A refined understanding of this phenomenon is the purpose of the next section.

2.2. **Low regularity existence theory for the semilinear heat equation.** We construct here solutions to (1.1) for initial data in Lebesgue spaces, and find the best Lebesgue space in which this holds.

2.2.1. *The linear heat equation $u_t = \Delta u$.* We develop here a framework to make sense of the heat equation when the initial datum is not twice differentiable. This is an example of a weak solution. A key property is that we have a formula to represent solutions.

Lemma 2.1 (Representation formula for the linear heat equation). *Assume $u \in C([0,T], L^p(\mathbb{R}^d))$ solves the weak version of the heat equation, that is:*

$$\langle u(t), \phi(t) \rangle - \langle u(0), \phi(0) \rangle = \int_0^t \langle u(s), \phi_s + \Delta\phi \rangle ds$$

for all test functions $\phi \in C^\infty([0,\infty], \mathcal{S}(\mathbb{R}^d))$. Then:

$$u(t) = K_t * u_0, \quad K_t = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \tag{2.1}$$

where $*$ is the convolution product:

$$(K_t * u_0)(x) = \int_{\mathbb{R}^d} u_0(x-y) K_t(y) dy.$$

Proof. We use the adjoint problem. Fix $t > 0$. For $\phi_0 \in \mathcal{S}(\mathbb{R}^d)$ we define for $s \in [0,t]$:

$$\phi(s) := K_{(t-s)} * \phi.$$

In particular, $\phi \in C^\infty([0,\infty], \mathcal{S}(\mathbb{R}^d))$ solves $\phi_s = -\Delta\phi$. Hence from the weak formulation of the heat equation:

$$\langle u(t), \phi(t) \rangle = \langle u(0), \phi(0) \rangle,$$

or in other words:

$$\langle u(t), \phi_0 \rangle = \langle u(0), K_t * \phi_0 \rangle$$

so that $u(t) = K_t * u_0$ and the lemma is proved.

$\square$

Lemma 2.2 (Estimates for the heat kernel). *One has for any $r \in [1,\infty]$, $s \in \mathbb{N}$, $\alpha \in \mathbb{N}^d$ with $\sum_1^d \alpha_i = s$:*

$$\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} K_t \right\|_{L^r(\mathbb{R}^d)} \leq \frac{C}{t^{\frac{d}{2}\left(1-\frac{1}{r}\right)+\frac{|\alpha|}{2}}}. \tag{2.2}$$

where $\partial^{|\alpha|} K_t / \partial x^\alpha = \partial^{|\alpha|} K_t / \partial x_1^{\alpha_1} ... \partial x_d^{\alpha_d}$.

Proof. This is left as an exercise. It is instructive as it enlighten the role of the spatial scale $\sqrt{t}$.

$\square$

We could make sense of solutions to $u_t = \Delta u$ even if u_0 is not regular using Lemma 2.1. Thanks to the representation formula, such solutions are in fact instantaneously classical solutions due to the following smoothing effects.

Lemma 2.3 (Regularising effects I). *For any $u_0 \in L^p(\mathbb{R}^d)$, $q \in [p, \infty]$ and $\alpha \in \mathbb{N}^d$, there holds:*

$$\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha}(K_t * u_0) \right\|_{L^q(\mathbb{R}^d)} \leq \frac{C}{t^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{|\alpha|}{2}}} \|u_0\|_{L^p(\mathbb{R}^d)} \qquad (2.3)$$

Proof. Recall that when differentiating of a convolution product derivatives can be placed on either term so that:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}(K_t * u_0) = \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} K_t \right) * u_0.$$

Recall Young inequality for convolution:

$$\|f * g\|_{L^q(\mathbb{R}^d)} \leq \|f\|_{L^p(\mathbb{R}^d)} \|g\|_{L^r(\mathbb{R}^d)}, \quad \text{for } \frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}.$$

Apply these identities to (2.1) with $f = u_0$, $g = \frac{\partial^{|\alpha|}}{\partial x^\alpha} K_t$ and $1/r = 1 + 1/q - 1/p \in [0, 1]$, and use the estimate (2.2):

$$\left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha}(K_t * u_0) \right\|_{L^q} = \left\| \left(\frac{\partial^{|\alpha|}}{\partial x^\alpha} K_t \right) * u_0 \right\|_{L^q} \leq \|u_0\|_{L^p} \left\| \frac{\partial^{|\alpha|}}{\partial x^\alpha} K_t \right\|_{L^r(\mathbb{R}^d)} \leq \frac{C}{t^{\frac{d}{2}\left(\frac{1}{p}-\frac{1}{q}\right)+\frac{|\alpha|}{2}}} \|u_0\|_{L^p(\mathbb{R}^d)}$$

$\square$

Representation formulas as (2.1) give all necessary informations for the solution. Energy estimates give weaker information, but which are extremely useful for stability issues. For the heat equation they also encode the smoothing effect. Below is a local energy estimate.

Lemma 2.4 (Local solution, energy estimate). *Given $R > 0$, $u_0 \in L^2(B(0, R))$ and $f \in L^2([0, T], H^{-1}(B(0, R)))$, there exists a unique weak solution to:*

$$\begin{cases} u_t = \Delta u + f, & (t, x) \in [0, T] \times \Omega, \\ u(t, x) = 0, & (t, x) \in [0, T] \times \partial\Omega, \end{cases}$$

such that:

$$\int_{[0,T]\times\Omega} |\nabla u|^2 + \sup_{0 \leq t \leq T} \int_\Omega |u(t)|^2 \leq C \int_{[0,T]} \|f(s)\|_{H^{-1}(\Omega)}^2 + \int_\Omega u_0^2. \qquad (2.4)$$

Moreover, if $f \in L^2([0, T] \times L^2(\mathbb{R}^d))$ then for any $0 < \tilde{R} < R$:

$$\sum_{|\alpha|=2} \int_{[0,T]\times B(0,\tilde{R})} \left| \frac{\partial^{|\alpha|} u}{\partial x^\alpha} \right|^2 + |\partial_t u|^2 \leq C(\tilde{R}) \left(\int_{[0,T]\times B(0,R)} f^2 + \int_{B(0,R)} u_0^2 \right). \qquad (2.5)$$

Remark 2.5. The above Lemma is an example of the parabolic regularisation effects whose rule of thumb is the following: solutions to the heat equation gain $+2$ derivatives in space and $+1$ in time with respect to the forcing term.

Proof. For the description of weak solution, the existence and uniqueness, we refer to Ladyzhenskaya and Ural'tseva, Friedman or Lieberman's books [16, 11, 18]. We perform the following energy estimates whose derivations are valid only under regularity assumptions, but that can be showed to hold true for general weak solutions using the framework developed in these books.

Step 1 Assume $f \in L^2 H^{-1}$. First multiply by u and integrate by parts, then using Young inequality $|ab| \le |a|/2\epsilon + \epsilon|b|/2$ for any $\epsilon > 0$:

$$
\begin{aligned}
\frac{1}{2}\int_\Omega u(t)^2 + \int_{[0,t]\times\Omega} |\nabla u|^2 &= \frac{1}{2}\int_\Omega |u_0|^2 + \int_{[0,t]\times\Omega} fu \\
&\le \frac{1}{2}\int_\Omega |u_0|^2 + \frac{1}{2\epsilon}\int_{[0,t]\times\Omega} \|f\|_{H^{-1}(\Omega)}^2 + \frac{\epsilon}{2}\int_{[0,t]\times\Omega} \|u\|_{H^1(\Omega)} \\
&\le \frac{1}{2}\int_\Omega |u_0|^2 + \frac{1}{2\epsilon}\int_{[0,t]\times\Omega} \|f\|_{H^{-1}(\Omega)}^2 + \frac{C\epsilon}{2}\int_{[0,t]\times\Omega} \|\nabla u\|_{L^2(\Omega)}
\end{aligned}
$$

where we used the Poincare inequality $\|u\|_{H^1(\Omega)}^2 \le C\|\nabla u\|_{L^2(\Omega)}^2$ for the last line. Choosing $\epsilon \le 1/C$ gives the desired result (2.4).

Step 2 Assume now $f \in L^2 L^2$. Let χ be a smooth cut-off function with $\chi \equiv 1$ on $B(0, \tilde{R})$ and $\chi = 0$ for $|x| \ge R$. Let $v_i = \partial_{x_i}(\chi u)$. Then v solves:

$$\partial_t v_i = \Delta v_i + \partial_{x_i}(\chi f - 2\nabla\chi.\nabla u - \Delta\chi u)$$

with boundary condition $v|_{\partial\Omega} = 0$, so we can apply the estimate of Step 1 and obtain (2.5).

$\square$

2.2.2. *Weissler's low regularity local well-posedness result.* Prepared with the properties of the linear heat equation given in the previous subsubsection, we are now able to give a local in time existence Theorem for the semilinear heat equation. The main ingredients are the following: the leading order part of the solution is the solution to the linear heat equation, the smoothing effects of the linear heat equation improve the regularity of the nonlinear terms and persist for the full equation, the final solution is in fact a classical one via an improvement of regularity called parabolic regularity bootstrap. We refer to [**?**] for a textbook where these issues are detailed.

Theorem 2.6 ([31, 3]). *Assume $q > d(p-1)/2$ (resp. $q = d(p-1)/2$) and $q \ge 1$ (resp. $q > 1$). Given any $u_0 \in L^q(\mathbb{R}^d)$, there exists a time $T = T(u_0) > 0$ and a unique function $u \in C([0,T], L^q(\mathbb{R}^d))$ with $u|_{t=0} = u_0$ such that*

u is C^1 in t and C^2 in x in $(0,T) \times \mathbb{R}^d$ and satisfies (1.1) on $(0,T) \times \mathbb{R}^d$.

Moreover, one has the following smoothing and continuity estimate:

$$\|u(t) - v(t)\|_{L^q(\mathbb{R}^d)} + t^{\frac{d}{2q}}\|u(t) - v(t)\|_{L^\infty(\mathbb{R}^d)} \le C\|u_0 - v_0\|_{L^q(\mathbb{R}^d)}$$

for $0 < t < \min(T(u_0), T(v_0))$ where $C = C(\|u_0\|_{L^q}, \|v_0\|_{L^q})$. Finally, for any bounded set (resp. compact set) $\mathcal{K}$ of $L^q(\mathbb{R}^d)$ a uniform time $T(\mathcal{K})$ exists such that for any $u_0 \in \mathcal{K}$ the solution of (1.1) exists on $[0, T(\mathcal{K})]$.

Proof. **Step 1** *Construction of the solution, non-critical case.* We first assume $q > d(p-1)/2$. We look for a solution to (1.1) of the following form using Duhamel formula:

$$u(t) = K_t * u_0 + \int_0^t K_{t-s}(|u(s)|^{p-1}u(s))ds. \tag{2.6}$$

Let $M > 0$ and $T > 0$ to be fixed below, and consider the following mapping Φ which to a function $v : [0,T] \times \mathbb{R}^d$ associates:

$$\Phi(v) = K_t * u_0 + \int_0^t K_{t-s}(|v(s)|^{p-1}v(s))ds.$$

We claim that Φ is a contraction on the following set of functions $u : [0, T] \times \mathbb{R}^d$:

$$K = \left\{ u, \; \|u(t)\|_K := \sup_{0 < t \leq T} \left(\|u\|_{L^q(\mathbb{R}^d)} + t^\alpha \|u(t)\|_{L^{pq}(\mathbb{R}^d)} \right) \leq M \right\},$$

where $\alpha = d(p-1)/2pq$. Indeed, given $v \in K$ we compute using the estimates on the heat kernel (2.3) that for each $t \in [0, T]$, $\Phi(v)(t) \in L^q(\mathbb{R}^d)$ with:

$$
\begin{aligned}
\|\Phi(v)(t)\|_{L^q(\mathbb{R}^d)} &\leq \|K_t * u_0\|_{L^q(\mathbb{R}^d)} + \int_0^t \|K_{t-s} * (|v(s)|^{p-1} v(s))\|_{L^q(\mathbb{R}^d)} ds \\
&\leq C\|u_0\|_{L^q(\mathbb{R}^d)} + C \int_0^t \|v(s)\|_{L^{pq}(\mathbb{R}^d)}^p ds \\
&\leq C\|u_0\|_{L^q(\mathbb{R}^d)} + C\|v\|_K^p \int_0^t s^{-\alpha p} ds \\
&\leq C\|u_0\|_{L^q(\mathbb{R}^d)} + (M+1)^p C \frac{T^{1-\alpha p}}{1 - \alpha p} \\
&\leq \frac{M}{2}
\end{aligned}
$$

if $M > C\|u_0\|_{L^q(\mathbb{R}^d)}$ and T has been chosen small enough depending on M, since $\alpha p < 1$ from the assumption on q. Again, from (2.3) and the definition of α, for $t \in (0, T]$:

$$
\begin{aligned}
t^\alpha \|\Phi(v)(t)\|_{L^{pq}(\mathbb{R}^d)} &\leq t^\alpha \|K_t * u_0\|_{L^{pq}(\mathbb{R}^d)} + t^\alpha \int_0^t \|K_{t-s} * (|v(s)|^{p-1} v(s))\|_{L^{pq}(\mathbb{R}^d)} ds \\
&\leq C\|u_0\|_{L^q(\mathbb{R}^d)} + C t^\alpha \int_0^t (t-s)^{-\alpha} \||v(s)|^p\|_{L^q(\mathbb{R}^d)} ds \\
&\leq C\|u_0\|_{L^q(\mathbb{R}^d)} + C\|v\|_K^p t^\alpha \int_0^t (t-s)^{-\alpha} s^{-\alpha p} ds \\
&\leq C\|u_0\|_{L^q(\mathbb{R}^d)} + (M+1)^p T^{1-\alpha p} \int_0^t (1-\sigma)^{-\alpha} \sigma^{-\alpha p} d\sigma \\
&\leq M + \frac{1}{2}
\end{aligned}
$$

if similarly $M > C\|u_0\|_{L^q(\mathbb{R}^d)}$, and T has been chosen small enough depending on M, because $0 < \alpha < p\alpha < 1$. This shows that Φ maps the set K onto itself. To proceed further, we claim that there exists $C > 0$ such that for any $a, b \in \mathbb{R}$:

$$\left| |a|^{p-1} a - |b|^{p-1} b \right| \leq C|a - b|(|a|^{p-1} + |b|^{p-1}).$$

To show this, assume $a, b \neq 0$ and then let $r = b/a$. Then the above inequality is equivalent to:

$$\left| 1 - |r|^{p-1} r \right| \leq C|1 - r|(1 + |r|^{p-1}).$$

The above inequality can be checked via the Taylor expansion of $r \mapsto |r|^{p-1} r$ at 1 for $r \to 1$, via noticing that both sides have the same power for $r \to \infty$. So such a constant $C > 0$ exists as both quantities are positive in between. This inequality implies in particular via Hölder inequality that:

$$\||v|^{p-1} v - |w|^{p-1} w\|_{L^q(\mathbb{R}^d)} \leq C\|v - w\|_{L^{pq}(\mathbb{R}^d)} \left(\|v\|_{L^{pq}(\mathbb{R}^d)}^{p-1} + \|w\|_{L^{pq}(\mathbb{R}^d)}^{p-1} \right).$$

Therefore, for $v, w \in K$ we can estimate the difference as before using (2.3):

$$
\begin{aligned}
\|\Phi(v)(t) - \Phi(w)(t)\|_{L^q(\mathbb{R}^d)} &= \left\| \int_0^t K_{t-s} * \left(|v(s)|^{p-1} v(s) - |w(s)|^{p-1} w(s) \right) ds \right\|_{L^q} \\
&\leq C \int_0^t \|v - w\|_{L^{pq}(\mathbb{R}^d)} \left(\|v\|_{L^{pq}(\mathbb{R}^d)}^{p-1} + \|w\|_{L^{pq}(\mathbb{R}^d)}^{p-1} \right) ds \\
&\leq C \|v - w\|_K \left(\|v\|_K^{p-1} + \|w\|_K^{p-1} \right) \int_0^t s^{-\alpha p} ds \\
&\leq \|v - w\|_K \frac{C 2 M^{p-1} T^{1-\alpha p}}{1 - \alpha p} \\
&< \|v - w\|_K,
\end{aligned}
$$

for T small enough. Similarly, we show the same estimate for the L^{pq} norm, implying $\|\Phi(v)(t) - \Phi(w)(t)\|_K < \|v - w\|_K$. Hence, Φ is a contraction on K. From Banach fixed point Theorem, it possesses a unique fixed point that we call $u(t)$: this is the solution we are looking for.

Step 2 *The critical case.* In the case where $q = d(p-1)/2$ and $q > 1$, the argument above has to be refined. It uses crucially the improvement $\|K_t * u_0\|_{L^r} = o(t^\alpha \|u_0\|_{L^q})$ where $r > q$ and $\alpha = d(1/q - 1/r)/2$, in comparison with (2.2).

Step 3 *Improving the regularity.* The solution found in Step 1 is in $L^{pq}(\mathbb{R}^d)$ immediately after the initial time. This is a Lebesgue space with exponent greater than q. Note that we obtained this improved regularity by using the smoothing of the heat kernel (2.2) and the Duhamel formulation (2.6). This procedure can be iterated: once the solution is in L^{pq}, it can be shown to be in L^r for later times for some $r > pq$, and so on. A precise quantification of the gain for the exponents shows that iterating this procedure a finite number of times allows to reach the L^∞ regularity, see [3] for the details. Such a procedure is called a *parabolic bootstrap of regularity*. Analogously, differentiability can be showed with a similar reasoning by putting derivatives in the estimate (2.2). This eventually gives that u is differentiable with t and twice differentiable in space on $(0, T] \times \mathbb{R}^d$ as claimed in the Theorem.

Step 4 *Uniqueness.* The uniqueness assuming solely that u is continuous in time with values in L^q, and that u is a classical solution immediately after the initial time, is beyond the scope of these notes. We refer the reader to [3] for the use of a clever duality argument.

$\square$

2.3. **Classical solutions to quasilinear transport equations.** The approach for LWP of quasilinear transport equations like the inviscid Prandtl's system (1.3) is somewhat similar to that taken in the previous subsection for the semilinear heat equation. One needs first to investigate the dynamics produced by the terms involving higher order derivatives: here this is linear transport. The general approach for local in time existence at high regularity for quasilinear transport equations relies on approximation schemes involving linear transport. Prandtl is an exception but the study of linear transport will highlight the role played by the geometry and the characteristics.

2.3.1. *Linear transport.* The following Lemma gives a representation formula for linear transport: the solution is conserved along the integral lines of the flow, called characteristics.

Lemma 2.7. *For Ω a smooth domain, assume $f \in C^1([0, T] \times \overline{\Omega}, \mathbb{R}^d)$ with $f_{|\partial\Omega}$ tangent to $\partial\Omega$, and $u_0 \in C^1(\overline{\Omega})$. Then there exists a unique $C^1([0, T] \times \overline{\Omega})$ solution to:*

$$u_t + f.\nabla u = 0, \tag{2.7}$$

given by the formula:

$$u(t, x) = u_0(\phi_t^{-1}(x)) \tag{2.8}$$

where $(\phi_t)_{t\in[0,T]}$ is the semi-group of diffeomorphism associated with the characteristic ODEs:

$$\partial_t \phi_t(x) = f(t, \phi_t(x)), \quad \phi_0(x) = x.$$

Remark 2.8.   • From the formula (2.10) one sees that in contrast with the linear heat equation, see (2.3), there are no smoothing effects: the solution can initially fail to be C^2 and will never be C^2 for later times.
   • This equation involves finite speed of propagation and is very local: if initially u_0 is supported in a compact set $K \subset \overline{\Omega}$, then from (2.10) $u(t)$ is supported on the compact set $\phi_t(K)$. If $u_0 - v_0 = 0$ on a set $K \subset \overline{\Omega}$, this remains so on $\phi_t(K)$. The local speed of the equation at x is $|f(x)|$.

Proof. We perform a so-called analysis/synthesis reasoning. We first assume that u is a C^1 solution and show it has to be given by (2.10), and then show that this indeed provides with a solution. So let's assume u is a C^1 solution to (2.9). Given each initial particle position $X \in \Omega$ we solve the ode's:

$$\dot{x} = f(x), \quad x(0) = X.$$

This is possible as f is of class C^1, and from Cauchy-Lipschitz theory we obtain that given $t \in [0, T]$, the mapping $\phi_t(\cdot)$ which relates the initial position X to the position at time t is a diffeomorphism. We compute that along a trajectory from (2.9) and the identity above:

$$\frac{d}{dt}u(t, \phi_t(X)) = u_t(t, \phi_t(X)) + \nabla u.\frac{d}{dt}\phi_t(X) = -(f.\nabla u)(t, \phi_t(X)) + (\nabla u.f)(t, \phi_t(X)) = 0.$$

Therefore, the function $u(t, \phi_t(X))$ is constant with time, so that for each $X \in \overline{\Omega}$:

$$u(t, \phi_t(X)) = u_0(X).$$

Setting $X = \phi_t^{-1}x$, we obtain that u is necessary given by the formula (2.10). Assume finally that u is given by the formula (2.10). Then, we have that the identity above holds true for any t, X. Differentiating it with time one obtains:

$$0 = u_t(t, \phi_t(X)) + \nabla u.\frac{d}{dt}\phi_t(t, X) = u_t(t, \phi_t(X)) + \nabla u.f(t, \phi_t(X)).$$

Hence $u_t + f.\nabla u = 0$: u solves (2.9). $\qquad\square$

2.3.2. *Quasilinear transport equations and the Prandtl's system.* In quasilinear transport equation, the velocity field depends on the solution itself yielding an equation of the form $u_t + f(u).\nabla u = 0$ where u can be vector valued.

Lemma 2.9. *For Ω a smooth domain, assume $f, g \in C^1([0,T] \times \overline{\Omega} \times \mathbb{R}^n, \mathbb{R}^d)$ with $f_{|\partial\Omega}$ tangent to $\partial\Omega$, and $u_0 \in C^1(\overline{\Omega}, \mathbb{R}^n)$ with both u_0 and ∇u_0 bounded. Then there exists $0 < \overline{T} \leq T$ and a unique $C^1([0,\overline{T}] \times \overline{\Omega})$ solution to:*

$$u_t + f(t,u).\nabla u = g(t,u), \tag{2.9}$$

given by the formula:

$$u(t,x) = U(t,X). \tag{2.10}$$

Above, given $X \in \overline{\Omega}$, $(\phi_t(X), U(t,X))$ is the unique solution to the ODE:

$$\begin{cases} \partial_t \phi_t(X) = f(t, \phi_t(X), U), \\ \partial_t U = g(t, \phi_t(X), U), \end{cases}$$

and $X = X(t,x) = \phi_t^{-1}(x)$ denotes the inverse mapping of the diffeomorphism $\phi_t : \overline{\Omega} \to \overline{\Omega}$.

Remark 2.10. The structure seen in Lemma 2.7 is preserved: one still has finite speed of propagation and the existence of characteristics.

Proof. The proof is a generalisation of the ideas of the proof of Lemma 2.7; we safely leave it to the reader.

$\qquad\square$

One could wonder wether the regularity requirement for the initial datum as well as for the solution can be lowered. As far as classical solutions are concerned, it is of course not possible. However, weak solutions can be obtained either in L^∞ in the scalar case $n = 1$ and $d \in \mathbb{N}$ [?], or for $(d,n) = (1,2)$, or for functions with bounded variations in one dimension $d = 1$ and $n \in \mathbb{N}$. The techniques involved fail in the general case $d, n \geq 2$, and solutions for quasilinear symmetric systems are obtained in the L^2-based Sobolev space H^s with $s > d/2 + 1$ (which is embedded in C^1 hence does not improve on Lemma 2.9). The aforementioned results and bibliography are well-documented in the textbook [2, 27].

Hence, the general developments for LWP at low regularity involving quasilinear transport do not handle the case of the inviscid Prandtl's system (1.3). More-over, an additional difficulty is that the vertical velocity "looses" a derivative: $v(t,x,y) = \int_0^y u_x(t,x,\tilde{y})d\tilde{y}$. Therefore, assuming u to be C^1 does not give $v \in C^1$ as required by Lemma 2.7. The solution resides in the fact that the equation possesses characteristics, as in Lemma 2.9, with an extra property: volume conservation. Building on this idea, a local in time LWP result for classical solutions was obtained in [13]. We refined it by computing exactly the maximal time of existence in [5]. Wether weaker solutions can be obtained is an open problem, but which is not the one we address here as our interest resides in the breakdown of classical solutions.

The idea is to first solve the characteristics equation, and then to construct a solution. Assuming formally that we have a solution, the characteristics would be defined as:

$$\begin{cases} \dot{x} = u(t,x,y), \\ \dot{y} = v(t,x,y), \qquad (x(0), y(0), u(0)) = (X, Y, u_0(X,Y)). \\ \dot{u} = -p_x^E(t,x), \end{cases}$$

One fundamental feature of the Prandtl's system is that the pressure $p_x^E(t,x)$ does not depend on the vertical variable y. As a consequence, given an initial particle

position $(X, Y) \in \mathbb{R} \times [0, \infty)$ and an initial horizontal velocity $u_0(X, Y)$, we can decide to forget about the vertical dynamics and only solve the horizontal transport:

$$\begin{cases} \dot{x} = u, \\ \dot{u} = -p_x^E(t, x), \end{cases} \quad (x(0), u(0)) = (X, u_0(X, Y)). \quad (2.11)$$

Another feature of the Prandtl's system is the following. Differentiating (1.3) with y one finds that the "vorticity" u_y solves:

$$u_{yt} + u u_{xy} + v u_{yy} = 0.$$

This means that the quantity u_y is transported along the characteristics, and that if in particular it is initially 0, it should remain so. Differentiating (1.3) with x this time one finds that the horizontal derivative u_x solves:

$$u_{xt} + u u_{xx} + v u_{xy} = -u_x^2 - v_x u_y - p_{xx}^E(t, x).$$

Notice also that at the boundary $v = 0$ so that $\dot{y} = 0$ and the characteristics remain at the boundary. If initially $u_y = 0$ or $Y = 0$, which remain true along the characteristics, one sees that u_x would solve:

$$\frac{d}{dt} u_x(t, x(t), y(t)) = -u_x^2(t, x(t), y(t)) - p_{xx}^E(t, x).$$

Such an ODE is said to be a Ricatti-type equation because of the square nonlinearity in the right hand side. Its solution might exist for all times. For example, if $p_{xx}^E \equiv 0$ and $u_x(0, X, Y) = -1$ then $u_x(t, x(t), y(t)) = -1/(1 - t)$ which tends to $-\infty$ as t approaches 1! The solution $u(t, x, y)$ would then encounter a problem and loose regularity. To detect when the above Ricatti equation might explode, we consider for each particle intially at the boundary or in the set of zero vorticity the ODEs:

$$\begin{cases} \dot{x} = u, \\ \dot{u} = -p_x^E(x), \\ \dot{u}_x = -(u_x)^2 - p_{xx}^E(x) \end{cases} \quad (x, u, u_x)(0) = (X, u_0(X, Y), u_{0x}(X, Y)). \quad (2.12)$$

Once the pressure field p^E is given, one can solve the ODEs above and define $T(X, Y)$ as their maximum time of existence. The infimum of these "problematic" times is:

$$T := \min(T_a, T_b), \quad T_a := \min\{T(X, Y), \ u_{0Y}(X, Y) = 0, \ Y > 0\}, \quad T_b := \min\{T(X, Y), \ Y = 0\}. \quad (2.13)$$

It turns out that we can construct solutions to the inviscid Prandtl's system up to the time T defined above. The following result is a refinement of [13].

Theorem 2.11 ([5]). *Let $u_0 \in C^2(\mathbb{R} \times [0, \infty))$ such that $\nabla u_0 \in L^\infty(\mathbb{R} \times [0, \infty))$. Let $(u^E, p_x^E) \in C^2([0, \infty) \times \mathbb{R})$. Then there exists a unique solution $u \in C^1([0, T) \times \mathbb{R} \times [0, \infty))$ of (1.3) where T is defined by (2.13), which satisfies moreover $\|\nabla u\|_{L^\infty([0, \tilde{T}] \times \mathbb{R} \times [0, \infty))} < \infty$ for any $\tilde{T} < T$. If T is finite then the solution satisfies:*

$$\lim_{t \uparrow T} \|u_x\|_{L^\infty(\mathbb{R} \times [0, \infty))} = \infty.$$

If in addition $u_0 \in C^k(\mathbb{R} \times [0, \infty))$ and $(u^E, p_x^E) \in C^k([0, \infty) \times \mathbb{R})$ for some $k \geq 3$, then $u \in C^{k-1}([0, T) \times \mathbb{R} \times [0, \infty))$. The mapping which to u_0 assigns the solution u is strongly continuous from $C^k(\mathbb{R} \times [0, \infty))$ into $C^{k-1}([0, T'] \mathbb{R} \times [0, \infty))$ for any $T' < T$.

Proof. The proof relies on the special structure of the characteristics and uses the Crocco transformation. The existence follows from their nondegeneracy until time T, while the regularity follows from standard regularity theory for level sets of functions.

Step 1 *Existence.* We first solve for the tangential displacement, which we denote by:

$$x(t, X, Y) = \phi_1[t](X, Y), \qquad (2.14)$$

where x above is the solution of (2.11). Notice that (2.11) can always be solved *globally* in time and the above function x is well-defined at any time $t > 0$. We next study the level sets $x = Cte$ in Lagrangian variables. Let us show first that they are non-degenerate. In the first case, assume that (X_0, Y_0) is such that $u_{0Y}(X_0, Y_0) \neq 0$. Then the Crocco transformation $(X, Y) \mapsto (X, u)$ is a well defined local diffeomorphism in a neighbourhood of (X_0, Y_0). The ODE solved by (2.11) is divergence free in the (x, u) phase space. Therefore, at any time $t > 0$, the mapping $(x, u) \mapsto (x(t), u(t))$ is volume preserving, and is in particular a diffeomorphism. Hence, the mapping $(X, Y) \mapsto (x(t), u(t))$ is a local diffeomorphism near (X_0, Y_0). It follows that $\nabla x(t, X_0, Y_0) \neq 0$ for any $t > 0$ in this first case.

In the second case, we assume that $u_{0Y}(X_0, Y_0) = 0$ or $Y_0 = 0$. Let us consider the set $Y = Y_0$ in Lagrangian variables. At each X close to X_0, the couple (x, u) solves (2.11), so that in particular:

$$\partial_t(\partial_X x) = \partial_X u, \quad \text{implying} \quad \partial_{tt}(\partial_X x) = \partial_X(-p_x^E(t, x)) = -(\partial_X x)p_{xx}^E(t, x).$$

This shows that at each fixed X:

$$\frac{d}{dt}\left(\frac{\partial_t \partial_X x}{\partial_X x}\right) = -\left(\frac{\partial_t \partial_X x}{\partial_X x}\right)^2 - p_{xx}^E(t, x), \quad \frac{\partial_t \partial_X x}{\partial_X x}(0) = \partial_X u_0.$$

In particular, at the point (X_0, Y_0), the quantity $\partial_t \partial_X x / \partial_X x$ is precisely the third component of the ODE system (2.12). Because of the definition of T (2.13), one obtains that the solution to the above differential equation is well defined for $t < T$. Hence $\partial_t \log(\partial_X x)$ is well-defined for $t < T$ which after integration gives that $\partial_X x(X_0, Y_0) > 0$. Hence, $\nabla x(t, X_0, Y_0) \neq 0$ in this second case as well.

We just showed that $\nabla_{X,Y} x \neq 0$ everywhere as long as $t < T$. Hence, in Lagrangian variables, the level sets $x = Cte$ are non-degenerate. At the boundary, the previous discussion implies that $\partial_X x_{|Y=0} \neq 0$. Therefore, the upper half plane is foliated by curves $\Gamma[x]$ corresponding to the level sets $\{x(X, Y) = x\}$. Since u_0, u^E, p_x^E are C^2, solving the ODE (2.11) produces a solution map that is also of class C^2, and $x(t, X, Y)$ is a C^2 function. Hence the curves $\Gamma[x]$ are C^1. This allows us to define an arclength parametrisation s for each of these curves, where $s = 0$ corresponds to the point at the boundary $Y = 0$.

The change of coordinates $(t, X, Y) \mapsto (t, x, s[t, x](X, Y))$ is a C^1 diffeomorphism from $[0, T) \times \mathbb{R} \times [0, \infty)$ onto itself. At a point (X, Y), considering the orthonormal base (v_1, v_2) with $v_1 = \frac{\nabla \phi_1[t](X,Y)}{|\nabla \phi_1[t](X,Y)|}$ and $v_2 = \frac{\nabla^\perp \phi_1[t](X,Y)}{|\nabla \phi_1[t](X,Y)|}$ where $(z_1, z_2)^\perp = (-z_2, z_1)$ one sees that

$$\frac{\partial x}{\partial v_1}_{|t,v_2} = |\nabla \phi_1[t](X, Y)|, \quad \frac{\partial x}{\partial v_2}_{|t,v_2} = 0, \quad \frac{\partial s}{\partial v_2}_{|t,v_2} = 1.$$

This shows the following value for the determinant of the change of variables:

$$\left|\text{Det}\left(\frac{\partial(x, s)}{\partial X}, \frac{\partial(x, s)}{\partial Y}\right)\right| = |\nabla \phi_1[t](X, Y)|.$$

To find the second component of the characteristics, we look for a C^1 mapping $(x, s) \mapsto (x, y)$. It satisfies:

$$\frac{\partial x}{\partial s} = 0, \quad \frac{\partial x}{\partial x} = 1,$$

and hence its determinant is

$$\left| \mathrm{Det} \left(\frac{\partial(x, y)}{\partial x}_{|s}, \frac{\partial(x, y)}{\partial s}_{|x} \right) \right| = \left| \frac{\partial y}{\partial s}_{|x} \right|.$$

Since the mapping $(X, Y) \mapsto (x, y)$ has to preserve volume, from the two determinants above we infer that:

$$\begin{aligned}
1 &= \left| \mathrm{Det} \left(\frac{\partial(x, y)}{\partial X}_{|Y}, \frac{\partial(x, y)}{\partial Y}_{|X} \right) \right| = \left| \mathrm{Det} \left(\frac{\partial(x, y)}{\partial X}_{|Y}, \frac{\partial(x, y)}{\partial Y}_{|X} \right) \right| \left| \mathrm{Det} \left(\frac{\partial(x, y)}{\partial X}_{|Y}, \frac{\partial(x, y)}{\partial Y}_{|X} \right) \right| \\
&= |\nabla \phi[t](X, Y)| \left| \frac{\partial y}{\partial s}_{|x} \right|.
\end{aligned}$$

This and the boundary condition forces the choice

$$\frac{\partial y}{\partial s}_{|x} = \frac{1}{|\nabla \phi[t](X(x, s), Y(x, s))|},$$

yielding the formula for y:

$$y(t, X, Y) = \phi_2[t](X, Y) = \int_0^{s[t, x](X, Y)} \frac{d\tilde{s}}{|\nabla \phi_1[t](\gamma[t, x](\tilde{s}))|}. \qquad (2.15)$$

Note that before T, the denominator in the above integral is uniformly away from 0. The function y above is of class C^1 because γ, s and $\nabla \phi_1$ are. The mapping $(t, X, Y) \mapsto (t, x, y)$ is thus a C^1 diffeomorphism from $[0, T) \times \mathbb{R} \times [0, \infty)$ onto itself. We finally define the solution as $u(t, x, y) = u_0(X, Y)$. Clearly,

$$\frac{\partial x}{\partial t}_{|XY} = u_0(X, Y) = u(t, x, y).$$

Since the mapping $(X, Y) \mapsto (x, y)$ is C^1 and preserves the measure, $\partial_x \frac{\partial x}{\partial t}_{|XY} + \partial_y \frac{\partial y}{\partial t}_{|XY} = 0$, yielding:

$$\frac{\partial y}{\partial t}_{|XY} = -\int_0^y \partial_x u(t, x, \tilde{y}) d\tilde{y}.$$

And since $\partial_t u(t, x(t), y(t)) = -p_x^E(x(t))$ and u is C^1, one deduces that u solves the inviscid Prandtl's equations. Note that the matching condition at infinity in (1.3) are indeed satisfied for the following reason. Initially as $y \to \infty$, $u_0 \to u_0^E$. u^E solves the Bernouilli equation:

$$u_t(t, x) + u(t, x) u_x(t, x) = -p^E(t, x).$$

that has a global solution, and whose characteristics correspond to the tangential displacement (2.11) of the characteristics for u. This gives the desired compatibility.

Step 2 *Regularity.* Assume $u_0 \in C^k$. The formula (2.14) for $x(t, X, Y)$ defines a C^k function since x is obtained as the solution of the ODE (2.11) with a C^k vector field. In the formula (2.15), $\nabla \phi_1[t]$ is C^{k-1}, and s and γ come from the parametrisation of the level sets of a C^k function, hence are also C^{k-1}. Therefore, u is of class C^{k-1}. The continuity of the flow follows from similar arguments.

Step 3 *Uniqueness.* If u is a C^2 solution then uniqueness is straightforward as the characteristics are well defined and have to produce the diffeomorphism constructed

above. In the case where $u \in C^1$ only, let us detail how the normal component of the characteristics and the volume preservation can be obtained. Define the characteristics $(x(t), y(t))$ through:

$$\frac{\partial x}{\partial t} = u(t, x, y), \quad x(0) = X, \quad \frac{\partial y}{\partial t} = -\int_0^{y(t)} u_x(t, x, y), \quad y(0) = Y.$$

One can indeed solve the second equation because the function $\int_0^y u_x(t, x, y)$ is C^1 in the third variable. One obtains characteristics (x, y) such that x is C^1 in (X, Y) and y is only C^1 in t and continuous in the other variables. u then solves $\dot{u} = -p_x^E(x)$ along the characteristics, implying that it is given by the formula (2.14). Moreover, since x is a C^1 function, and y is a C^1 function in t, with $\partial_t y$ being C^1 in y, such that $\partial_y(\partial_t y(t)) = -\partial_x(\partial_t(x(t)))$, an approximation argument using a regularisation procedure gives that the characteristics must preserve volume. The mapping $(X, Y) \mapsto (x, y)$ is then a bijection preserving volume with $x \in C^1$, which can be showed to be necessarily of the form described in Step 1.

Step 4 *Blow-up.* Assume that $T < \infty$. Then by definition T the solutions to the ODEs (2.12) must blow up at time T, which is only possible if $u_x \to -\infty$ as $t \to T$. $\qquad \square$

2.4. **The viscous Prandtl's system.** We saw in the previous Subsection how LWP for the inviscid Prandtl's equations (1.3) may seem at first delicate due to the loss of one derivative in the vertical transport $v = -\int_0^y u_x$. The LWP was therefore strongly relying on the characteristics of the equations whose volume preserving property somehow counterbalance this loss of derivative. Including the transversal viscosity to obtain the original Prandtl's system (1.2) breaks such stability: the LWP only holds in most cases for analytic data and ill-posedness in Sobolev spaces can happen. We refer to the recent result [8] and its bibliography for LWP in the general case, to [20] and references therein for the special case of monotonic flows, and to [23] for a textbook unfortunately prior to recent developments.

2.5. **Criticality.** What are the largest spaces in which LWP can be established? For the semilinear heat equation for example, if $u(t, x)$ is a solution then so is $u_\lambda(t, x) = \lambda^{-2/(p-1)} u(t/\lambda^2, x/\lambda)$ for any $\lambda > 0$. The norm $\|u_0\|_{L^{q_c}(\mathbb{R}^d)}$ is invariant under the transformation $u_0 \mapsto \lambda^{-2/(p-1)} u_0(x/\lambda)$ for $q_c = d(p-1)/2$ which is precisely the exponent appearing in Theorem 2.6. $L^{q_c}(\mathbb{R}^d)$ in this case is the so called critical space. The rule of thumb for LWP in general is the following: given an equation, a scale invariance it possesses, and a scale of functional spaces (H^s spaces for example), one looks at the space invariant by the scaling (some H^{s_c}): it should be the limiting regularity above which LWP holds and below which it fails.

For the semilinear heat equation, ill-posedness below L^{q_c} is easily proved. Indeed, there exist solutions $u(t, x)$ with a finite maximal time of existence T as we will see later. Take $q < q_c$ and such a solution. Consider the solutions u_λ as $\lambda \to 0$: their maximal time of existence $\lambda^2 T$ converge to 0, while since $\|u_\lambda(0)\|_{L^q} = \lambda^{2(q_c-q)/q(p-1)}\|u_0\|_{L^q} \to 0$ their initial datum converge to 0 in L^q. As LWP in L^q would provide stability of the zero solution, the problem is ill posed in L^q.

In general, the rule of thumb provides a lower bound for the regularity needed (the function space has to be able to measure the transition to small scales as dictated by the scaling invariance), but is sometimes not optimal as for the Prandtl's system.

We mentioned indeed that the viscous case requires analyticity. In the inviscid case, the invariances $u \mapsto \lambda u(t, \lambda x, y)$ and $u \mapsto u(t, x, \lambda y)$ predicts for example $u_x \in L^\infty$ and $u \in L^\infty$ for the critical regularity, but one actually needs C^2 as seen in Theorem 2.11.

2.6. **Blow-up.** Theorems 2.6 and Lemma 2.9 give the existence of local in time solutions. The maximal time of existence is thus defined as the time $T \in (0, \infty]$ for which a solution exists on $[0, T)$ but not on $[0, T')$ for any $T' > T$. Theorem 2.6 gives the following blow-up criteria: $T < \infty$ is finite if and only if:

$$\liminf_{t \uparrow T} \|u(t)\|_{L^q(\mathbb{R}^d)} = \infty \tag{2.16}$$

for any $q \in (d(p-1)/2, \infty]$. Indeed, it the above quantity bound remained bounded along a subsequence $t_n \uparrow T$, applying Theorem 2.6 for n large enough with initial datum $u(t_n)$ would prove that a solution exist passed T, a contradiction. For the inviscid Prandtl's system, the LWP in Theorem 2.11 requires $u \in C^2$, but this Theorem also proves the sharper blow-up criteria than just the divergence of the C^2 norm: $\|\nabla u(t)\|_{L^\infty}$ has to diverge as $t \to T$. The next section gives "easy" examples of blow-up solutions: truly backward self-similar blow-up.

For the viscous Prandtl's equations however, no easy example of blow-up solutions has yet been found. Assuming that the solution is odd in x: $u(x) = -u(-x)$, with precise examples of compatible odd and even in x respectively Eulerian field and pressure u^E and p^E, [9, 15] found existence of blow-up solutions using a contradiction argument. The key idea is that the trace of the solution $\xi(t, y) = -u_x(t, 0, y)$ solves:

$$\begin{cases} \xi_t = \xi_{yy} + \xi^2 + (\int_0^y \xi)\xi_y, \\ \xi(t, 0) = 0, \ \lim_{y \to \infty} \xi(t, y) = p_{xx}^E(t, 0). \end{cases} \tag{2.17}$$

This parabolic equation on the half line resembles the semi-linear heat equation (1.1) and we were able in [6] to describe a stable singularity formation scenario.

Theorem 2.12 ([6]). *Let $p^E = 0$. There exists an open set in L^1 of initial data for (2.17) for which solutions blow up in finite time $T(u_0) > 0$ with:*

$$\xi(t, x) = \frac{1}{T - t} \sin^2 \left(\frac{y}{2\mu(T - t)^{-\frac{1}{2}}} \right) \mathbb{1}(0 \le y \le \mu 2\pi(T - t)^{-\frac{1}{2}}) + o_{L^\infty}((T - t)^{-1}).$$

for some $\mu(u_0) > 0$.

A simplification appears asymptotically and ξ resembles a rescaling of the blow-up profile $\sin^2(y/2)\mathbb{1}(0 \le y \le 2\pi)$. This feature, called asymptotic self-similarity, appears in many blow-up phenomena and is described in the next section in details.

3. Self-similarity

We are interested in these notes in the description of singular solutions, and asymptotic self-similarity is one of its essential feature. The first subsections here borrow mostly from [1].

We give here some concepts and results around self-similarity. Here is the list of the main concepts and ideas, those underlined being examined here:

- *Self-similarity* A phenomenon is called self-similar if it properties at one location can be deduced from those at another location by a similarity transformation. The easiest example of a similarity transformation is that of plane geometry, from which the name "self-similar" comes from. What exactly

the "similarity transformation" is is a problem-dependent notion. We shall see simple examples from physics and PDEs. The common feature is that self-similar phenomena are more rigid, and hence usually easier to study.

- *Relation with invariances* One key property permitting the appearance of self-similar phenomena is the existence of invariances. In physics a fundamental example is that of dimensional analysis. In singularity formation for PDEs, the analogue is the existence of a scaling invariance, that we will study for both the semilinear heat equation and the Prandtl's system.

- *Backward self-similar profiles.* These are solutions of a special form keeping the same shape but concentrating to smaller scales as time evolve, leading to the formation of a singularity. They are constructed here for both the semilinear heat equation and the inviscid Prandtl's system. The construction for the semi-linear heat equation provides an example of rigorous matched asymptotics.

- *Asymptotic self-similarity.* Most problem enjoy asymptotic self-similarity in various asymptotic configurations, which explains why this is a central notion. We prove here asymptotic self-similarity for singularities of the semi-linear heat equation for type I blow-up in the radial case, insisting on the role of renormalisation, parabolic regularisation, and energy dissipation. We prove asymptotic self-similarity for the inviscid Prandtl's system for non-degenerate initial data, but this time the main reasons are the finite speed of propagation, the regularity of the initial datum, and the geometry of the problem (the existence of characteristics).

3.1. **The easiest example: differentiability.** The easiest example of self-similar functions are the power type functions $f(x) = ax^k$. In this case, the value of the function at x can be related to that at 1 via the power-type self-similar transformation: $f(x) = x^k f(1)$. The meaning of asymptotic self-similarity in this case is that of differentiability: when we study functions which are differentiable we make the hypothesis of local self-similarity everywhere, $f(x + t) = a + bt$ as $t \to 0$. The asymptotic self-similar form of the function is thus given by the leading order term in its Taylor expansion.

3.2. **An example from physics, dimensional analysis.** Let us consider the following problem: given an ocean of infinite (very large) depth, with a gravity field with gravitational constant g, we consider surface waves having small wave length λ. What is the speed c of these waves as a function of g and λ? To answer to this problem, we look for a sort of Taylor expansion $c = ag^\alpha \lambda^\beta$, where a, α and β are coefficients.

Dimensional analysis provides a way to compute the exponents α and β, so that a only needs to be computed experimentally. The speed c is has meter per second units: ms^{-1}, the gravitational field has units ms^{-2} and the wave length m. Hence when writing $c = ag^\alpha \lambda^\beta$, the left hand side has units ms^{-1}, and the right hand side $m^{\alpha+\beta} s^{-2\alpha}$. The fundamental idea of dimensional analysis is that both sides must have the same units, hence $\alpha = \beta = 1/2$, and we find $c = a\sqrt{g\lambda}$.

Why should it be so? This is an easy but somewhat deep reasoning. The first thing to notice is that physical phenomena are independent of the units chosen to measure them. Consider Physical System 1, with a given value of cms^{-1}, gms^{-2} and λm in the meter/second unit system, so that $c = ag^\alpha \lambda^\beta$. Change now the way you measure, say in meter'/second' with $m' = L_1 m$ and $s' = L_2 s$ where

$L_1, L_2 > 0$ encode the change of units. In this new unit system, the new values are $c' = cL_1^{-1}L_2 m's'^{-1}$, $g' = gL_1^{-1}L_2^2 m's'^{-2}$ and $\lambda' = \lambda L_1^{-1}$. The equality $c = ag^\alpha\lambda^\beta$ then yields that for Physical System 1 measured in units m's', we have $c' = aL_1^{\alpha+\beta-1}L_2^{1-2\alpha}g'^\alpha\lambda'^\beta$.

Consider Physical System 2, with given values of $\bar{c}ms^{-1}$, $\bar{g}ms^{-2}$ and $\bar{\lambda}m$ in the meter/second unit system, so that $\bar{c} = a\bar{g}^\alpha\bar{\lambda}^\beta$. There exists a unique system of units m', s' in which the values of the gravity coefficient and the wavelength of Physical System 1 equal those of Physical system 2 in the m, s system of units: going from m,s to m',s' is given by the above transformation with $L_1 = \lambda/\bar{\lambda}$ and $L_2 = \sqrt{\lambda\bar{g}/\bar{\lambda}g}$.

Let us know compare Physical System 1 in m',s' units, with values $g'm's'^{-2}$ and $\lambda'm'$ with $g' = \bar{g}$ and $\lambda' = \bar{\lambda}$, and Physical System 2, with values $\bar{g}ms^{-2}$ and $\bar{\lambda}m$ in m,s units. These systems are indistinguishable! Let us quote Barrendblatt: "All physical laws can be represented in a form equally valid for all observers (i.e. not depending on the units chosen). All systems within a given class are equivalent, i.e. there are no distinguished, somehow preferred, systems among them.". Hence we get that the speed of Physical System 1 in units m',s', must equal that of Physical System 2, in units m,s, so that: $aL_1^{\alpha+\beta-1}L_2^{1-2\alpha}\bar{g}^\alpha\bar{\lambda}^\beta = a\bar{g}^\alpha\bar{\lambda}^\beta$, and necessarily $\alpha + \beta - 1 = 0$ and $1 - 2\alpha = 0$.

3.3. **Self-similarity in PDEs, group invariance.** How to extend self-similarity as described above to PDEs? The problem of the velocity of the wave described above has a group invariance. Consider the group $(0,\infty)^2$ acting on functions $c : (g, \lambda) \mapsto c(g, \lambda)$ via the formula: $(L_1, L_2).c : (g, \lambda) \mapsto L_1 L_2^{-1}c(L_1^{-1}L_2^2 g, L_1^{-1}\lambda)$. If $c : (0,\infty)^2 \mapsto (0,\infty)$ was a solution to the problem, dimensional analysis implied that $(L_1, L_2).c$ also was a solution. Since there is a unique solution, c is invariant under the action of the group (a fixed point of the action). This gave its particular form.

Many PDEs also enjoy group invariances, and a solution will be called self-similar if it is invariant by the action of the group. We have already encountered a self-similar solution, the heat kernel in (2.1). Indeed, the heat equation is invariant under the following action of the group $(0,\infty)$ on functions $u : (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$:

$$u(t, x) \mapsto \frac{1}{\lambda^d}u(\frac{t}{\lambda^2}, \frac{x}{\lambda}) = u_\lambda(t, x), \quad (3.1)$$

as if u solves $u_t = \Delta u$ then u_λ also solves $u_{\lambda,t} = \Delta u_\lambda$. Moreover, the Dirac delta is invariant under the transformation

$$u(x) \mapsto \frac{1}{\lambda^d}u(\frac{x}{\lambda}).$$

Hence, if u solves:

$$u_t = \Delta u, \quad u(0) = \delta_0$$

then for all $\lambda > 0$, u_λ also solves:

$$\partial_t u_\lambda = \Delta u_\lambda, \quad u_\lambda(0) = \delta_0.$$

There is a unique solution to the above problem, and thus $u = u_\lambda$ for every $\lambda > 0$. We obtain that the heat kernel $K(t, x)$ satisfies $K = K_\lambda$. Note that this allows to eliminate one variable and to find the formula for K_t. Indeed, fixing t and x and

setting $\lambda = \sqrt{t}$ gives:

$$K(t,x) = \frac{1}{t^{\frac{d}{2}}} K(1, \frac{x}{\sqrt{t}}) = \frac{1}{t^{\frac{d}{2}}} \psi(\frac{x}{\sqrt{t}}).$$

The equation $K_t = \Delta K$ is transformed into the following equation for ψ:

$$-\frac{d}{2} - \frac{y}{2}.\nabla \psi = \Delta \psi, \quad \int \psi = 1,$$

whose solution, as ψ must be radial, is $\psi(y) = \frac{1}{(4\pi)^{\frac{d}{2}}} e^{-\frac{|y|^2}{4}}$ by a direct check.

3.4. **Self-similar profiles for the heat and inviscid Prandtl's equations.**
The semilinear heat equation (1.1) and the Prandtl's system (1.3) both possess self-similar solutions which are blowing up in finite time, both related to the invariances of these equations. The semilinear heat equation (1.1) admits the following scaling invariance: if $u(t,x)$ is a solution, then:

$$\frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

is a solution for any $\lambda > 0$. Assume $u : (-\infty, 0) \times \mathbb{R}^d \to \mathbb{R}$ is a solution that is invariant under the above transformation. Fixing t and x and setting $\lambda = \sqrt{-t}$ gives:

$$u(t,x) = \frac{1}{(-t)^{\frac{2}{p-1}}} u(-1, \frac{x}{\sqrt{t}}) = \frac{1}{(-t)^{\frac{2}{p-1}}} \psi(\frac{x}{\sqrt{t}}). \tag{3.2}$$

where ψ is called the self-similar profile. An example of such solution is the space-independent solution:

$$u(t,x) = \frac{\kappa}{(-t)^{\frac{2}{p-1}}}, \quad \psi(y) = \kappa = (\frac{1}{p-1})^{\frac{1}{p-1}}.$$

There are also space dependent solutions, but we however don't have any formula most of the time. Note that such solutions are unbounded as t approaches 0: they are blow-up solutions. Self-similarity diminishes the number of independent variables, and simplifies the problem at hand. Indeed, u given by (3.2) is a solution of (1.1) if and only if Ψ solves the *stationary* self-similar equation:

$$\Delta \Psi + |\Psi|^{p-1} \Psi = \frac{1}{2} \Lambda \Psi, \quad \Lambda = \frac{2}{p-1} + y.\nabla \tag{3.3}$$

The following Theorem gives the existence of solutions to this elliptic equation, and hence of backward self-similar blowup solutions for the semilinear heat equation. The result holds in any dimension for supercritical nonlinearities below the Joseph-Lundgren exponent, but we restrict ourself to the dimension 3 for clarity.

Theorem 3.1 ([30, 17, 4, 7]). *Assume $d = 3$ and $p > 5$. Then there exists a countable family $(\Psi_L)_{L \in \mathbb{N}}$ of bounded radial solutions to (3.3)*

Proof. We solely highlight the main points here, as the technical details are lengthy. The key idea is that of *matched asymptotics*. In the radial setting $\Psi(y) = \Psi(r)$, $r = |y|$, (3.3) is equivalent to:

$$\Psi'' + \frac{2}{r} \Psi' + |\Psi|^{p-1} \Psi = \frac{1}{p-1} \Psi + r \Psi' \tag{3.4}$$

which is an ODE of degree two. If two solutions Ψ_{in} and Ψ_{out} are such that $\Psi_{in}(r_0) = \Psi_{out}(r_0)$ and $\Psi'_{in}(r_0) = \Psi'_{out}(r_0)$ at some point r_0 then $\Psi_{in} = \Psi_{out}$ and these solutions coincide everywhere. By constructing two bounded solutions

Ψ_{in} on $[0, r_0]$ and Ψ_{out} on $[r_0, \infty)$ satisfying this matching condition at r_0, one then ensures that a bounded solution on $[0, \infty)$ exists.

Step 1 *Inner and outer solutions.* The construction of Ψ_{in} relies on the following bifurcation argument. Ψ_{in} solves (3.4) on $[0, r_0)$ if and only if $\Phi_{in}(\rho) = \mu^{1/(p-1)}\Psi_{in}(\sqrt{\mu}\rho)$ solves:

$$\Phi'' + \frac{2}{r}\Phi' + |\Phi|^{p-1}\Phi = \mu\left(\frac{1}{p-1}\Phi + r\Phi'\right) \tag{3.5}$$

on $[0, r_0/\sqrt{\mu})$. As $\mu \to 0$, this equation converges to the equation

$$\Phi'' + \frac{2}{r}\Phi' + |\Phi|^{p-1}\Phi = 0 \tag{3.6}$$

which is that of stationary states of the semilinear heat equation. This equation is an even simpler ODE and is well understood (see [12] for example) for example: all solutions are of the form $\lambda^{-2/(p-1)}Q(\rho/\lambda)$, where Q is the solution with $Q(1) = 1$. Moreover, it behaves as $\rho \to \infty$ as:

$$Q(\rho) \sim \frac{c_\infty}{\rho^{\frac{2}{p-1}}} + \frac{c_1\sin(\omega\log\rho + c_2)}{\rho^{\frac{1}{2}}} + o(\rho^{-\frac{1}{2}}) \quad \text{as } \rho \to \infty. \tag{3.7}$$

where $c_\infty = [(d - 2 - 2/(p-1))2/(p-1)]^{1/(p-1)}$, and $\omega, c_1 \neq 0$ are two numbers depending on p as well. To construct the solution Φ_{in}, one sees (3.5) as a bifurcation of (3.6) with bifurcation parameter $\mu \to 0$. We find for each μ small enough a solution $\Phi_{in}[\mu]$ close to Q using perturbation of ODEs techniques. This gives an inner solution on $[0, r_0]$ of the form:

$$\Psi_{in}[\mu](r) = \frac{1}{\mu^{\frac{1}{p-1}}}Q\left(\frac{r}{\sqrt{\mu}}\right) + hot. \tag{3.8}$$

The construction of Ψ_{out} relies on an easier perturbation argument. First, the solution

$$\Psi^*(r) = \frac{c_\infty}{r^{\frac{2}{p-1}}}$$

is a solution to (3.4). The linearised equation around Ψ^* is:

$$\tilde{\Psi}'' + \frac{2}{r}\tilde{\Psi}' + \frac{pc_\infty^{p-1}}{r^2}\tilde{\Psi} = \frac{1}{p-1}\tilde{\Psi} + r\tilde{\Psi}'.$$

With a change of variables this linear second order differential equation can be related to special functions: hypergeometric functions. One fundamental solution is unbounded at infinity, while the second one is bounded and behaves near the origin like

$$\tilde{\Psi}_1 \sim \frac{\sin(\omega\log r + c_3)}{r^{\frac{1}{2}}} \quad \text{as } r \to 0. \tag{3.9}$$

For each $|\epsilon| < 0$ small enough, a solution $\Psi_{out}[\epsilon]$ on $[r_0, \infty)$ is constructed such that:

$$\Psi_{out}[\epsilon](r) = \Psi^*(r) + \epsilon\Psi_1(r) + hot. \tag{3.10}$$

Step 2 *The matching.* We take r_0 to be fixed small, such that $\omega\log r_0 + c_3 = \pi/2$ to simplify the expression of the exterior term. The conditions $\Psi_{in}(r_0) = \Psi_{out}(r_0)$

and $\Psi'_{in}(r_0) = \Psi'_{out}(r_0)$ are then equivalent, from (3.8), (3.10), (3.7) and (3.9) to:

$$\begin{cases} \frac{1}{\mu^{\frac{1}{p-1}}} Q\left(\frac{r_0}{\sqrt{\mu}}\right) = \Psi^*(r_0) + \epsilon\Psi_1(r_0) + hot, \\ \frac{1}{\mu^{\frac{1}{p-1}+\frac{1}{2}}} Q'\left(\frac{r_0}{\sqrt{\mu}}\right) = \Psi'^*(r_0) + \epsilon\Psi'_1(r_0) + hot \end{cases}$$

$$\Leftrightarrow \begin{cases} -\mu^{\frac{1}{2}-\frac{1}{p-1}} c_1 \sin\left(\frac{\omega}{2}\log\mu\right) = \epsilon + hot, \\ -\omega c_1 \mu^{\frac{1}{2}-\frac{1}{p-1}} \cos\left(\frac{\omega}{2}\log\mu\right) = 0 + hot \end{cases}$$

We now look, given each $0 < \mu \ll 1$, for a suitable value of ϵ so that the above system is satisfied. What we see is that this is not always possible: the first equation fixes the value of ϵ and can always be solved, while the second equation is only satisfied for a sequence of values μ_k with asymptotic geometric form $\mu_k \sim C\alpha^k$, $\alpha = e^{2\pi/\omega}$. The rigorous proof of the existence of the sequence μ_k can be established with suitable bounds for the hot in the system above, and the application of the intermediate value theorem using the oscillations of the left hand side.

$\square$

The inviscid Prandtl's equations (1.3) with zero outer flow $u^E = 0 = p^E$ admit the following scaling invariance: if $u(t,x,y)$ is a solution, then:

$$\frac{\mu}{\lambda} u\left(\frac{t}{\lambda}, \frac{x}{\mu}, \frac{y}{\nu}\right) \tag{3.11}$$

is a solution for any $(\lambda, \mu, \nu) \in (0, \infty)^3$. The scaling group is a three dimensional Lie group. Self-similar solutions are in this case solutions that are invariant under the action of a one-dimensional subgroup. The following Proposition gives the fundamental example. Note that the Jacobian of the solution is unbounded as t approaches 0: it is a blow-up solution.

Proposition 3.2. *The mapping* $(a,b) \mapsto (\mathcal{X}, \mathcal{Y})$ *given by:*

$$\Phi(a,b) = \left(a + b^2 + a^3, \int_{-\infty}^{b} \frac{d\tilde{b}}{1 + 3\Psi_1^2\left(a + a^3 + b^2 - \tilde{b}^2\right)}\right), \tag{3.12}$$

where Ψ_1 *is the inverse function of* $\mathcal{X} \mapsto -\mathcal{X} - \mathcal{X}^3$, *defines a volume preserving diffeomorphism between* $\mathbb{R}^2$ *and the subset of the upper half plane* $\{0 < \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X})\}$ *where*

$$\mathcal{Y}^*(\mathcal{X}) = \int_{-\infty}^{0} \frac{d\tilde{b}}{1 + 3\Psi_1^2\left(a + a^3 - \tilde{b}^2\right)} \tag{3.13}$$

The opposite of the tangential component of its inverse:

$$\Theta := \begin{array}{ll} -\Phi_1^{-1}: & \{(\mathcal{X},\mathcal{Y}) \in \mathbb{R} \times (0,\infty), \ 0 < \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X})\} \rightarrow \mathbb{R}^2 \\ & (\mathcal{X}, \mathcal{Y}) \mapsto -a, \end{array} \tag{3.14}$$

is a self-similar profile, that is, the following function $u : (-\infty, 0) \times \mathbb{R} \times (0, \infty) \rightarrow \mathbb{R}$ *is a solution of (1.3) with* $u^E = p^E = 0$:

$$u(t,x,y) = (-t)^{\frac{1}{2}} \Theta\left(\frac{x}{(-t)^{\frac{3}{2}}}, \frac{y}{(-t)^{-\frac{1}{4}}}\right). \tag{3.15}$$

Proof. This is a consequence of the volume preserving property of Φ and of the formula for the characteristics. The fact that Φ is volume preserving diffeomorphism, i.e. that, writing $\Phi = (\Phi^1, \Phi^2)$:

$$\partial_{\mathcal{X}}\Phi^1 \partial_{\mathcal{Y}}\Phi^2 - \partial_{\mathcal{Y}}\Phi^1 \partial_{\mathcal{X}}\Phi^2 = 1,$$

is a direct computation. We now study its image. Let us fix $\mathcal{X} \in \mathbb{R}$ and study the curve of equation $a + b^2 + a^3 = \mathcal{X}$, that we parametrise with the variable b. The vertical component of the image is:

$$\mathcal{Y} = \int_{-\infty}^{b} \frac{d\tilde{b}}{1 + 3\Psi_1^2 \left(\mathcal{X} - \tilde{b}^2 \right)},$$

which is an increasing function of b, hence, the set $\{\Phi(a,b), \; \Phi_1 = \mathcal{X}\}$ consists of the interval $(0 < \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X}))$ with $\mathcal{Y}^*$ given by (3.13). We now prove that Θ is a self-similar profile. Let us denote by $\tilde{\Theta}$ the vertical component of the inverse:

$$\Phi^{-1} = (-\Theta, \tilde{\Theta}).$$

Let us solve the inviscid Prandtl's system with initial datum $u_0(X,Y) = \Theta(X,Y)$. We recall that Prandtl's inviscid equations are solved via the characteristics, as detailed in the proof of Theorem 2.11. These characteristics $(X,Y) \mapsto (x(t), y(t))$ define at each fixed time $t > 0$ a volume preserving diffeomorphism whose tangential component is given by:

$$x = X + t\Theta(X,Y).$$

because $p^E = 0$. Let us perform the change of variables $(a,b) \mapsto (X,Y)$, with $(a,b) = (-\Theta(X,Y), \tilde{\Theta}(X,Y))$, so that $(X,Y) = \Phi(a,b)$ by definition. The definition of Φ gives:

$$X = a + b^2 + p^2 a^3 \quad \text{so that} \quad x = a + b^2 + p^2 a^3 + t\Theta(X,Y) = a(1-t) + b^2 + p^2 a^3.$$

Moreover the mapping $(a,b) \mapsto (x,y)$ is volume preserving. This change of variables is summarised below:

$$\begin{array}{ccc} & (X,Y) & \\ \begin{smallmatrix} vol.pres. \\ X = a+a^3+b^2 \end{smallmatrix} \nearrow & & \searrow \begin{smallmatrix} vol.pres. \\ x = X - ta \end{smallmatrix} \\ (a,b) & \longrightarrow & (x,y). \end{array}$$

Hence the mapping $(a,b) \mapsto (x,y)$ is volume preserving and satisfies $x = a(1-t) + a^3 + b^2$. One can then retrieve the formula for $y(a,b)$ as we did in the proof of Theorem 2.11. Indeed, the set $\{(\tilde{a}, \tilde{b}), \; x(\tilde{a}, \tilde{b}) = x(a,b)\}$ is given by the equation

$$a(1-t) + b^2 + a^3 = \tilde{a}(1-t) + \tilde{b}^2 + \tilde{a}^3$$

and corresponds to a curve which, parametrised by $\tilde{b}$, is given by:

$$\left\{ \left(-(1-t)^{\frac{1}{2}} \Psi_1 \left(\frac{a(1-t) + a^3 + b^2 - \tilde{b}^2}{(1-t)^{\frac{3}{2}}} \right), \tilde{b} \right), \; \tilde{b} \in \mathbb{R} \right\}.$$

As $\partial_a x = (1-t) + 3a^2$, one deduces from (2.15) and performing a change of variables that

$$\begin{aligned} y(a,b) &= \int_{-\infty}^{b} \frac{d\tilde{b}}{(1-t) + 3\tilde{a}^2(x(a,b))} = \frac{1}{1-t} \int_{-\infty}^{b} \frac{d\tilde{b}}{1 + 3\Psi_1^2 \left(\frac{a(1-t)+a^3+b^2-\tilde{b}^2}{(1-t)^{\frac{3}{2}}} \right)} \\ &= \frac{d\tilde{b}}{(1-t)^{\frac{1}{4}}} \int_{-\infty}^{b} \frac{1}{1 + 3\Psi_1^2 \left(\frac{a(1-t)+a^3+b^2}{(1-t)^{\frac{3}{2}}} - \tilde{b}^2 \right)}. \end{aligned}$$

Therefore, the mapping $(a, b) \mapsto (x, y)$ is given by:

$$(x, y) = \left(a(1-t) + b^2 + a^3, \frac{1}{(1-t)^{\frac{1}{4}}} \int_{-\infty}^{b} \frac{1}{1 + 3\Psi_1^2 \left(\frac{a(1-t)+a^3+b^2}{(1-t)^{\frac{3}{2}}} - \tilde{b}^2 \right)} \right).$$

A direct consequence of the formula (3.12) for Φ and the fact that $(-\Theta, \tilde{\Theta}) = \Phi^{-1}$ is that the inverse of the above mapping is:

$$(a, b) = \left(-(1-t)^{\frac{1}{2}} \Theta \left(\frac{x}{(1-t)^{\frac{3}{2}}}, \frac{y}{(1-t)^{-\frac{1}{4}}} \right), (1-t)^{\frac{3}{4}} \tilde{\Theta} \left(\frac{x}{(1-t)^{\frac{3}{2}}}, \frac{y}{(1-t)^{-\frac{1}{4}}} \right) \right).$$

As $a = -\Theta(X, Y) = -u_0(X, Y)$, and as along the characteristics $u(t, x, y) = u_0(X, Y)$ because $p^E = 0$, one gets that:

$$u(t, x, y) = \Theta(X, Y) = (1-t)^{\frac{1}{2}} \Theta \left(\frac{x}{(1-t)^{\frac{3}{2}}}, \frac{y}{(1-t)^{-\frac{1}{4}}} \right).$$

This gives the desired formula (3.15) by a time translation of 1. $\qquad\square$

3.5. Asymptotic self-similarity.
We have seen in the previous section examples of self-similar solutions for the heat equation, the semi-linear heat equation and the inviscid Prandtl's equations. These are special solutions, but what makes them important is that they appear most of the time in asymptotic configurations, especially for singularity formation. What are the reasons for this appearance? The first reason is the existence of group invariances as (3.1) and (3.11): as the solution blows up it is escaping to infinity in the phase space, but the dynamics there is similar to dynamics of order 1 thanks to this symmetry. To be put rigorously, this involves renormalisation. The second reason is that the flow, up to such renormalisation, is compact. This compactness comes from parabolic regularity and energy dissipation for the semi-linear heat equation. This compactness has a different origin for the inviscid Prandtl's equations, it comes from the regularity of the solution and from finite speed of propagation.

The following result comes from and shows asymptotic self-similarity for singularities of the heat equation. It is a simplified version of what is obtained in [22] and borrows from [21].

Proposition 3.3. *Assume $u_0 \in L^\infty(\mathbb{R}^d)$ is radially symmetric, nonnegative, attains its maximum at the origin and satisfies $u(r) = o(r^{-2/(p-1)})$ as $r \to \infty$. Assume the corresponding solution u to the semilinear heat equation (1.1) blows up at time $T > 0$, with the upper bound:*

$$\|u(t)\|_{L^\infty} \leq \frac{C}{(T-t)^{\frac{1}{p-1}}}, \quad C > 0.$$

Then there exists a nonzero C^2 function Ψ such that:

$$(T-t)^{\frac{1}{p-1}} u(t, y\sqrt{T-t}) \to \Psi(y)$$

locally uniformly on compact sets of the variable y.

Proof. **Step 1** *Lower bound on the blow-up rate.* First let us prove that there exists $c > 0$ such that:

$$\frac{c}{(T-t)^{\frac{1}{p-1}}} \leq \|u(t,x)\|_{L^\infty} \leq \frac{C}{(T-t)^{\frac{1}{p-1}}}. \tag{3.16}$$

Take the function $w(t,x) = 0$. Then it solves the semilinear heat equation $w_t = \Delta w + |w|^{p-1}w$. Hence the weighted difference $u' = e^{-Mt}(u-w)$, for any $M > 0$, solves:

$$\partial_t u' = \Delta u' - M'u' + u'\frac{|v|^{p-1}v - |w|^{p-1}w}{v-w} = \Delta u' + u'(f(t,x) - M).$$

For any fixed $t_0 \in (0,T)$, the function $f(t,x)$ is bounded on $[0,t_0] \times \mathbb{R}^d$ so that $f(t,x) - M \leq 0$ for M large enough. We can then apply the maximum principle for linear parabolic equations (see for example the textbook [10]): the negative part of u' attains its maximum at $t = 0$. Since $u_0 \geq 0$, this maximum is actually 0 so that $u' \geq 0$ on $[0,t_0]$ implying $u \geq 0$ on $[0,t_0]$. As t_0 is arbitrary we obtain $u \geq 0$ on $[0,T) \times \mathbb{R}^d$.

Take now explicitly $c = (p-1)^{-1/(p-1)}$ and introduce the functions $v_T(t,x) = c(T-t)^{-1/(p-1)}$. Then v_T solves the semilinear heat equation: $\partial_t v_T = \Delta v_T + v_T^p$. Assume by contradiction that at some time $t_0 \in [0,T)$ one has $\|u(t_0)\|_{L^\infty} < v_T(t_0)$. Then this means that for some $\delta > 0$ small enough, for all $x \in \mathbb{R}^d$:

$$u(t_0,x) \leq v_{T+\delta}(t_0,x).$$

Define now $u' = v_{T+\delta} - u$. It is the difference of solutions to the semilinear heat equation, and satisfies $u'(t_0) \geq 0$. By the same argument above used to compare u with w, we obtain that $u'(t) \geq 0$ for $t \geq t_0$. This means that for $t_0 \leq t < T$:

$$0 \leq u(t,x) \leq \frac{c}{(T+\delta-t)^{\frac{1}{p-1}}}.$$

In particular, $\|u(t)\|_{L^\infty}$ does not diverge as $t \to T$, so that u does not blow up from the blow-up criterion (2.16).

Step 2 *Renormalisation and local energy dissipation* We now rescale the equation introducing the so called parabolic self-similar variables:

$$y = \frac{x}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad v(s,y) = (T-t)^{\frac{1}{p-1}}u(t,x).$$

Then v solves:

$$\partial_s v = \Delta v + |v|^{p-1}v + \frac{1}{2}\Lambda v, \quad \Lambda = \frac{2}{p-1} + y.\nabla. \tag{3.17}$$

Let us define the following local energy functional:

$$E(v) = \int_{\mathbb{R}^d}\left(\frac{1}{2}|\nabla v|^2 + \frac{1}{2(p-1)}w^2 - \frac{1}{p+1}|v|^{p+1}\right)\rho dy, \quad \rho(y) = \frac{1}{(4\pi)^{\frac{d}{2}}}e^{-\frac{|y|^2}{4}}.$$

Then a direct computation shows:

$$\frac{d}{ds}(E(v(s))) = -\int_{\mathbb{R}^d}|v_s|^2\rho dy. \tag{3.18}$$

(In fact, equation (3.17) is a gradient descent for the functional E). In particular, $E(v)$ is decreasing with time. We now claim that for all times $s \geq -\log T$:

$$E(v) \geq \frac{p-1}{p+1}\left(\int w^2\rho\right)^{\frac{p+1}{2}} \tag{3.19}$$

Indeed, we compute that for any time $s \geq s^* \geq s_0$:

$$\frac{1}{2}\frac{d}{ds}\int v^2\rho = \int\left(-|\nabla v|^2 - \frac{1}{p-1}v^2 + |v|^{p+1}\right)\rho dy$$

$$= -2E(v) + \frac{p-1}{p+1}\int|v|^{p+1}\rho dy$$

$$\geq -2E(v(s^*)) + \frac{p-1}{p+1}\left(\int|v|^2\rho dy\right)^{\frac{p+1}{2}}.$$

We above that $E(v(s)) \leq E(v(s^*))$ and Jensen's inequality. Assume by contradiction now that (3.19) is violated at some time s^*. Then for any $s \geq s^*$:

$$\frac{1}{2}\frac{d}{ds}\int v^2(s)\rho > \frac{p-1}{p+1}\left(\left(\int v^2(s)\rho\right)^{\frac{p+1}{2}} - \left(\int v^2(s^*)\rho\right)^{\frac{p+1}{2}}\right).$$

For the quantity $a(s) = \int v^2(s)\rho$ this is a convex differential inequality: $\partial_s a > c(a^q(s) - a^q(s^*))$ with $c = 2(p-1)/(p+1)$ and $q = (p+1)/2 > 1$. As a consequence, a must tend to infinity in finite time S. This is a contradiction as in time s, v is a global solution. Reintegrating the energy dissipation (3.18), with the bound (3.19) gives:

$$\int_{s_0}^{\infty}\int_{\mathbb{R}^d}|v_s|^2\rho dy ds \leq E(v(s_0)) - E(v(s)) \leq E(v(s_0)). \tag{3.20}$$

Step 3 *Existence of limit profiles.* From the bound (3.16) we obtain that v satisfies for any $s \geq s_0$:

$$\|v(s)\|_{L^\infty} \leq C, \quad c \leq v(s,0).$$

Recall parabolic regularisation, Lemma 2.4. Applying this Lemma to v and iteratively to its derivatives, we obtain that v_s, as well as first order and second order spatial derivatives of v are uniformly locally bounded in L^2. To avoid technicalities here, let us mention that parabolic regularity as in Lemma 2.4 also works in general L^p spaces, and that as a result, v_s and spatial derivatives of v up to order 2 (and higher order derivatives[1]) are uniformly bounded. From this boundedness for higher order derivatives and the integrability (3.20) of v_s, we obtain that v_s tends to zero uniformly on compact sets as $s \to \infty$. Let us now consider any sequence of times $s_n \to \infty$ and the sequence of solutions $v_n(s,y) = v(s + s_n, y)$. Then v_n also solves the renormalised heat equation (3.17). Moreover, $\partial_s v_n(0)$ converges to zero uniformly on compact sets as $n \to 0$. Hence $\Delta v_n(0) - |v_n|^{p-1}(0)v_n(0)$ converges to zero uniformly on compact sets. This implies that as n converges to ∞, $v_n(0)$ converges to a solution Ψ to $\Delta\Psi - |\Psi|^{p-1}\Psi = 0$. Moreovoer, since $v_n(0,0) \geq c > 0$, Ψ is not the zero solution.

Step 4 *Conclusion.* We obtained in Step 3 that for any $s_n \to \infty$, $v(s_n)$ approaches the set of self-similar profiles as $n \to \infty$. Let ω denote the ω-limit set of v:

$$\omega := \{\Psi, \ \exists s_n \to \infty \text{ such that } v(s_n,\cdot)\underset{n\to\infty}{\to}\Psi\}.$$

Step 3 showed that ω is a subset of the set of stationary solutions to (3.17). To conclude we need to show that ω contains only one element. Assume by contradiction it has at least two elements. Then from uniform boundedness of v and of its derivatives we obtain that ω must be connected and hence contain an infinite number of

---

[1] We always gain a little bit more than one derivative in time and two in space, but it actually depends on the value of p

elements. Now the equation $\Delta \Psi + |\Psi|^{p-1}\Psi = \frac{1}{2}\Lambda\Psi$, in the radial setting, is a second order ordinary differential equation. Moreover, at the origin $\partial_r \Psi = 0$. Hence by uniqueness, Ψ is determined uniquely by its value at the origin $\Psi(0)$. Therefore, ω contains at least three elements, Ψ_1, Ψ_2, Ψ_3 such that $\Psi_1(0) < \Psi_3(0) < \Psi_2(0)$. As v must tend to Ψ_1 and Ψ_2 along different subsequences, this implies that there exists a sequence $s_n \to \infty$ such that $v(s_n, 0) - \Psi_3(0) = 0$. Let us consider the intersection number between v and Ψ_3 defined as:

$$Z(s) = \{\#r \geq 0, \quad v(s, r) - \Psi_3(r) = 0\}.$$

Then from [4] $\Psi_3(r) \sim cr^{2/(p-1)}$ as $r \to \infty$. Hence $Z(s_0)$ is finite from the decay assumption on u_0. Next, from [24], Z is a decreasing function of time and each time $v(s_n, 0) - \Psi_3(0) = 0$, Z must decrease strictly at $s = s_n$. We see a contradiction here: an integer valued quantity cannot decrease strictly infinitely many times. Hence ω only contained one element.

$\square$

The following result comes from and shows asymptotic self-similarity for singularities of the Prandtl's system equation. It is a simplified version of what is obtained in [5].

Proposition 3.4. *Assume $u^E = p^E = 0$, $u_0 \in C^4(\mathbb{R} \times [0, \infty))$ is such that on the set where $\partial_y u_0 = 0$, the function $\partial_x u_0$ attains its minimum at $(0, Y_0)$ for some $Y_0 > 0$, and has the following Taylor expansion:*

$$u_0(X, Y_0 + Y) = -X + (Y - X)^2 + a_1 X^3 + a_2 X^2 Y + a_3 XY^2 + a_4 Y^3 + hot, \quad (3.21)$$

with

$$a_1 + a_2 + a_3 + a_4 = 1.$$

Then u blows up at time $T = 1$, and one has:

$$(T - t)^{-\frac{1}{2}} u(t, \mathcal{X}(T - t)^{\frac{3}{2}}), \mathcal{Y}(T - t)^{-\frac{1}{4}}) \to \Theta(\mathcal{X}, \mathcal{Y})$$

locally uniformly on compact sets of $\{(\mathcal{X}, \mathcal{Y}) \in \mathbb{R} \times [0, \infty), \; 0 < \mathcal{Y} < 2\mathcal{Y}^*(\mathcal{X})\}$.

Remark 3.5. The hypotheses can be weakened, and Θ can be showed to be the profile of the generic singularity formation, see [5]. The validity of the profile Θ also holds further than compact sets of self-similar variables. The formulation above however allows for easier computations.

Proof. We recall that the solution indeed blows up at time 1 from Theorem 2.11.

Step 1 *Lagragian and Eulerian self-similar variables.* Let us change variables and write:

$$(X, Y_0 + Y) = \left(a(T - t)^{\frac{1}{2}} - \frac{b}{2}(T - t)^{\frac{3}{4}}, Y_0 + a(T - t)^{\frac{1}{2}} + \frac{b}{2}(T - t)^{\frac{3}{4}} \right)$$

The variables (a, b) allow to zoom near $(0, Y_0)$. As $p^E = 0$, we get from the ODEs (2.11) $x = X + tu_0(X, Y)$. Hence, as $T = 1$, from the Taylor expansion (3.21):

$$u_0(t, X, Y_0 + Y) = -X + (T - t)^{\frac{3}{2}} b^2 + (a_1 + a_2 + a_3 + a_4)(T - t)^{\frac{3}{2}} a^3 + O((T - t)^{\frac{7}{4}}).$$

As $a_1 + a_2 + a_3 + a_4 = 1$ this gives:

$$x(a, b) = (T - t)^{\frac{3}{2}}(a + a^3 + b^2) + O((T - t)^{\frac{7}{4}}).$$

We define the Eulerian self-similar variables:

$$\mathcal{X} = \frac{x}{(T - t)^{\frac{3}{2}}}, \quad \mathcal{Y} = \frac{y}{(T - t)^{-\frac{1}{4}}}.$$

From the equation above:

$$\mathcal{X}(a,b) = a + a^3 + b^2 + O((T-t)^{\frac{1}{4}}).$$

The changes of variables as summarised by the following diagram:

$$
\begin{array}{ccc}
(a,b) & \xrightarrow{\;vol.pres\;} & (\mathcal{X},\mathcal{Y}) \\
\downarrow & & \uparrow \\
(X,Y) & \xrightarrow{\;vol.pres\;} & (x,y)
\end{array}
$$

Indeed, the determinant of $(a,b) \mapsto (X,Y)$ is $(T-t)^{5/4}$ and that of $(x,y) \mapsto (\mathcal{X},\mathcal{Y})$ is $(T-t)^{-5/4}$.

Step 2 *Stability of self-similar characteristics.* In Step 1 we obtained that $(a,b) \mapsto (\mathcal{X},\mathcal{Y})$ is a volume preserving diffeomorphism such that $\mathcal{X} = a + a^3 + b^2 + O((T-t)^{1/4})$. Hence we expect it to be close to Φ given by Proposition 3.2. Let us show this precisely. Let us introduce:

$$\mathcal{X}^{\Theta}(a,b) = a + a^3 + b^2, \quad \mathcal{Y}^{\Theta}(a,b) = \int_{-\infty}^{b} \frac{d\tilde{b}}{1 + 3\Psi_1^2(a + a^3 + b^2 - \tilde{b}^2)}.$$

We have showed above (the same being true for derivatives as well):

$$\mathcal{X}(a,b) = \mathcal{X}^{\Theta} + O((T-t)^{\frac{1}{4}}), \quad \partial_a \mathcal{X}(a,b) = \partial_a \mathcal{X}^{\Theta} + O((T-t)^{\frac{1}{4}}), \quad \partial_b \mathcal{X}(a,b) = \partial_b \mathcal{X}^{\Theta} + O((T-t)^{\frac{1}{4}}).$$

We recall the formula giving the y variable:

$$y(X,Y) = \int_{\Gamma} \frac{ds}{|\nabla x|}$$

where Γ is the curve $\{x(\tilde{X},\tilde{Y}) = x(X,Y)\}$ and ds denotes the curve length. Consider the box $\{|a| \le \delta^{1/3}(T-t)^{-1/2}, |b| \le \delta(T-t)^{-3/4}\}$. Then, as $\delta \to 0$, in (X,Y) variables this box is a small rectangle B of size $\delta^{1/3} \times \delta$ near $(0, Y_0)$. We split:

$$y(X,Y) = \int_{\Gamma \cap B^c} \frac{ds}{|\nabla x|} + \int_{\Gamma \cap B} \frac{ds}{|\nabla x|}.$$

In the exterior of the rectangle B, ∇x is uniformly away from 0 as $t \to T$, and nothing singular happens:

$$\left| \int_{\Gamma \cap B^c} \frac{ds}{|\nabla x|} \right| \le C.$$

In the interior of the rectangle, we change variables and use the (a,b) variables. The curve we are looking for can be shown to enter this rectangle via the $b = -\delta(T-t)^{-3/4}$ side. After a computation one finds the following identity:

$$\int_{\Gamma \cap B} \frac{ds}{|\nabla x|} = \frac{1}{(T-t)^{\frac{1}{4}}} \int_{\delta(T-t)^{-\frac{3}{4}}} \frac{d\tilde{b}}{\partial_a \mathcal{X}(a(\tilde{b}), \tilde{b})}$$

where $(a(\tilde{b}), \tilde{b})$ is a parametrisation of the curve $\mathcal{X}(\tilde{a}, \tilde{b}) = \mathcal{X}(a,b)$. Let us look more carefully at this parametrisation. Fix $|a|, |b| \le M$ for a large constant $M > 0$ and $(\tilde{a}, \tilde{b}) \in \{|a| \le \delta^{1/3}(T-t)^{-1/2}, |b| \le \delta(T-t)^{-3/4}\}$. Fix $|\mathcal{X}| \le K$ and $\kappa \le \mathcal{Y} \le 2\mathcal{Y}^*(\mathcal{X}) - \kappa$ for some large enough constant $K > 0$ and small enough constant $\kappa > 0$. Then from the Taylor expansion of u at $(0, Y_0)$:

$$\mathcal{X}(a,b) = a + a^3 + b^2 + O((T-t)^{\frac{3}{4}}), \quad \mathcal{X}(\tilde{a}, \tilde{b}) = (\tilde{a} + \tilde{a}^3)(1 + O((T-t)^{\frac{1}{4}} + \delta)) + b^2(1 + O((T-t)^{\frac{1}{4}} + \delta)).$$

and

$$\partial_a \mathcal{X}(\tilde{a}, \tilde{b}) = (1 + 3\tilde{a}^2)(1 + O((T-t)^{\frac{1}{4}} + \delta)).$$

Therefore, for K fixed δ small enough and t close enough to T, $\mathcal{X}(\tilde{a}, \tilde{b}) = \mathcal{X}$ implies that $|b| \leq L(1 + |a|^{3/2})$ and $|a| \leq L(1 + |b|^{2/3})$ for some $L = L(K)$. This allows to go back once more in the Taylor expansion of u and to be able to write the refined formulas:

$$\mathcal{X} = \mathcal{X}(\tilde{a}, \tilde{b}) = (\tilde{a} + \tilde{a}^3)(1 + O((T - t)^{\frac{1}{4}} + |\tilde{b}|^{\frac{1}{3}}(T - t)^{-\frac{1}{4}}) + \tilde{b}^2 + O((T - t)^{\frac{1}{4}}),$$

$$\partial_a \mathcal{X}(\tilde{a}, \tilde{b}) = (1 + 3\tilde{a}^2)(1 + O((T - t)^{\frac{1}{4}} + |\tilde{b}|^{\frac{1}{3}}(T - t)^{-\frac{1}{4}})).$$

We recall that $\Psi_1(z)$ is the inverse of $z \mapsto -z - z^3$ and satisfies $\Psi(z) \sim \mp|z|^{1/3}$ as $z \to \pm\infty$ and $\Psi(z) = -z + z^3$ as $z \to 0$. After basic estimates involving these asymptotic expansions, we find that the solution of the above equation is:

$$\tilde{a} = -\Psi_1(\mathcal{X} - 4\tilde{b}^2 + O(T - t)^{\frac{1}{4}})(1 + O((T - t)^{\frac{1}{4}} + |\tilde{b}|^{\frac{1}{3}}(T - t)^{-1/4}).$$

Replacing this in the expression for $\partial_a \mathcal{X}$, and using the identity for $\mathcal{X}(a, b)$, we finally obtain that on the curve $\mathcal{X}(\tilde{a}, \tilde{b}) = \mathcal{X}$ there holds:

$$\begin{aligned}
\partial_a \mathcal{X}(\tilde{a}, \tilde{b}) &= (1 + \Psi_1^2(\mathcal{X} - 4\tilde{b}^2 + O(T - t)^{\frac{1}{4}})(1 + O((T - t)^{\frac{1}{4}} + |\tilde{b}|^{\frac{1}{3}}(T - t)^{-1/4})) \\
&\quad \times (1 + O((T - t)^{\frac{1}{4}} + |\tilde{b}|^{\frac{1}{3}}(T - t)^{-\frac{1}{4}})) \\
&= (1 + \Psi_1^2(a + a^3 + b^2 - 4\tilde{b}^2))(1 + O((T - t)^{\frac{1}{4}} + |\tilde{b}|^{\frac{1}{3}}(T - t)^{-\frac{1}{4}})).
\end{aligned}$$

Going back to the integral we had to compute:

$$\begin{aligned}
\int_{\Gamma \cap B} \frac{ds}{|\nabla x|} &= \frac{1}{(T - t)^{\frac{1}{4}}} \int_{-\delta(T-t)^{-\frac{3}{4}}}^{b} \frac{d\tilde{b}}{1 + 3\Psi_1^2(a + a^3 + b^2 - \tilde{b}^2)}(1 + O((T - t)^{\frac{1}{4}} + |\tilde{b}|^{\frac{1}{3}}(T - t)^{-\frac{1}{4}})) \\
&= \frac{\mathcal{Y}^{\Theta}(a, b)}{(T - t)^{\frac{1}{4}}}\left(1 + O(T - t)^{\frac{1}{4}}\right).
\end{aligned}$$

Hence we have shown that:

$$\mathcal{Y}(a, b) = \mathcal{Y}^{\Theta}(a, b) + O(T - t)^{\frac{1}{4}}.$$

The same computation also shows that:

$$\partial_a \mathcal{Y}(a, b) = \partial_a \mathcal{Y}^{\Theta}(a, b) + O(T - t)^{\frac{1}{4}}, \quad \partial_b \mathcal{Y}(a, b) = \partial_b \mathcal{Y}^{\Theta}(a, b) + O(T - t)^{\frac{1}{4}}.$$

We are now ready to invert the characteristics. We look for a solution of the form $(a, b) = (\bar{a} + h_1, \bar{b} + h_2)$ to $(\mathcal{X}, \mathcal{Y})(a, b) = (\mathcal{X}, \mathcal{Y})$, where

$$(\bar{a}, \bar{b}) = \left(a^{\Theta}(\mathcal{X}, \mathcal{Y}), b^{\Theta}(\mathcal{X}, \mathcal{Y})\right).$$

Then $(\bar{a}, \bar{b})$ belongs also to a compact zone, included in $[-M, M]^2$ for M large enough so that our previous computations apply. Consider then the mapping:

$$\Phi : (h_1, h_2) \mapsto \left(\mathcal{X}\left(\bar{a} + h_1, \bar{b} + h_2\right), \mathcal{Y}\left(\bar{a} + h_1, \bar{b} + h_2\right)\right)$$

From the estimates on the derivatives done above, there holds for $|h_1|, |h_2| = O((T - t)^{1/12})$:

$$\begin{pmatrix} \partial_{h_1}\Phi_1 & \partial_{h_2}\Phi_1 \\ \partial_{h_1}\Phi_2 & \partial_{h_2}\Phi_2 \end{pmatrix} = \begin{pmatrix} \partial_a \mathcal{X}^{\Theta}(\bar{a}, \bar{b}) & \partial_b \mathcal{X}^{\Theta}(\bar{a}, \bar{b}) \\ \partial_a \mathcal{Y}^{\Theta}(\bar{a}, \bar{b}) & \partial_b \mathcal{Y}^{\Theta}(\bar{a}, \bar{b}) \end{pmatrix} + O((T - t)^{\frac{1}{4}}). \qquad (3.22)$$

Also, again from the computations performed above:

$$\Phi(0, 0) - (\mathcal{X}, \mathcal{Y}) = O((T - t)^{\frac{1}{4}}).$$

Note that, as $\bar{a}$ and $\bar{b}$ vary in the compact zone $[-M, M]^2$, the leading order term matrices in (3.22) belong to a compact set of invertible matrices. Hence we can invert the above equation, uniformly as $t \to T$ in the zone that we consider: there

exists $(h_1, h_2) = O((T-t)^{\frac{1}{4}})$ such that $\Phi(h_1, h_2) = (\mathcal{X}, \mathcal{Y})$. Hence we inverted the characteristics and:

$$a = \bar{a} + O\left((T-t)^{\frac{1}{4}}\right), \quad b = \bar{b} + O((T-t)^{\frac{1}{4}}).$$

Using the Taylor expansion of u and the fact that $|a|, |b| \lesssim 1$:

$$\begin{aligned}
u(t, x, y) &= u_0(t, X, Y) = -a(T-t)^{\frac{1}{2}} + O((T-t)^{\frac{3}{4}}) \\
&= (T-t)^{\frac{1}{2}} \Theta\left(\frac{x}{(T-t)^{\frac{3}{2}}}, \frac{y}{(T-t)^{-\frac{1}{4}}}\right) + O((T-t)^{\frac{3}{4}}).
\end{aligned}$$

This proves the proposition.

$\square$

4. Stability of (truly) self-similar blowup

We investigate here the following stability problem: given a backward self-similar profile as presented in the previous section, what happens to the solution under small perturbation? Here is the list of the main concepts and ideas, those underlined being examined here:

- *Linearised dynamics* As long as the perturbation remains small, its evolution is governed by the linearised dynamics. We give here a detailed description of the semi-group generated by the self-adjoint linearised operator.
- *Prevention of instabilities* The linearised dynamics makes the instable eigenmodes grow exponentially with time. Such instabilities can be prevented by tuning suitably the initial datum. This is presented in details in the finite dimensional case, and the proof uses fundamentally a topological argument (an application of Brouwer's degree theory).
- *Decomposition of the solution.* To capture the time of the blow-up, as well as its spatial location, we use a dynamical renormalisation. This involves the dynamical determination of key parameters such as the scale and the position, resulting in so-called modulation equations.
- *Bootstrap regime* The existence of an initial datum such that the renormalised perturbation converges to 0 asymptotically relies on the study of a bootstrap regime. If one assumes small a priori bounds on the perturbation on a time interval $[s_0, s_1]$, one can show that they in fact hold on $[s_0, s_1 + \delta]$ for some universal $\delta > 0$. This shows that the small a priori bounds are in fact true for all times $[s_0, \infty)$.

Theorem 4.1 (Finite codimensional stability of Ψ_L [7])**.** *Assume $d = 3$ and $p \in 2\mathbb{N} + 1$ is such that $p \geq 7$. Let $L \in \mathbb{N}$ large enough. There exists a Lipschitz codimension L manifold in $H^2(\mathbb{R}^d)$ of initial data such that the corresponding solution to (1.1) blows up in finite time $0 < T < +\infty$ with a decomposition*

$$u(t, x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} (\Psi_L + v)\left(t, \frac{x - x(t)}{\lambda(t)}\right)$$

where Ψ_L is the self-similar profile given by Theorem 3.1 and:
1. *Control of the geometrical parameters: the blow up speed is self similar*

$$\lambda(t) = \sqrt{T-t}(1 + o(1)) \quad as \quad t \to T$$

and the blow up point converges

$$x(t) \to x(T) \quad as \quad t \to T. \tag{4.1}$$

2. *Behaviour of Sobolev norms: there holds the asymptotic stability of the self similar profile above scaling*

$$\lim_{t\to T} \|v(t)\|_{\dot{H}^s} = 0 \quad for \quad s_c < s \le 2, \tag{4.2}$$

the boundedness of norms below scaling

$$\limsup_{t\to T} \|u(t)\|_{\dot{H}^s} < +\infty \quad for \quad 1 \le s < s_c, \tag{4.3}$$

and the logarithmic growth of the critical norm

$$\|u(t)\|_{\dot{H}^{s_c}} = c_n(1 + o_{t\to T}(1))\sqrt{|\log(T-t)|}, \quad c_n \ne 0. \tag{4.4}$$

4.1. **The linear dynamics.** The first intuition about the dynamics of a perturbation of the self-similar profile Ψ_L is that it should be described to leading order by the linear dynamics. We therefore first describe completely the linear dynamics in this Subsection. We study solutions v to:

$$\partial_s v + \mathcal{L}v = 0, \quad v_0 \in L^2(\rho), \tag{4.5}$$

where $\mathcal{L}$ is the linearised operator

$$\mathcal{L} := -\Delta - p\Psi_L^{p-1} + \frac{1}{p-1} + y.\nabla$$

arising in the linearisation of (3.3) around Ψ_L. The properties of $\mathcal{L}$ are the following. They are related to the Hilbert space

$$L_\rho^2 := \left\{ u, \text{ such that } \|u\|_{L_\rho^2}^2 = \int_{\mathbb{R}^d} |u|^2 \rho(y)dy < \infty \right\}, \quad \rho(y) = \frac{1}{(4\pi)^{\frac{d}{2}}} e^{-\frac{|y|^2}{4}}$$

and its corresponding Sobolev spaces:

$$H_\rho^s := \left\{ u, \text{ such that } \sum_{|\alpha|\le s} \left\| \frac{\partial^{|\alpha|}u}{\partial x^\alpha} \right\|_{L_\rho^2}^2 = \|u\|_{H_\rho^s}^2 < \infty \right\}.$$

Proposition 4.2 (Spectral gap for $\mathcal{L}_n$ [7]). *The operator $\mathcal{L}$ is essentially self-adjoint from $H^2(\rho)$ into $L^2(\rho)$, with compact resolvant. For $L \gg 1$ large enough one has the following properties:*
1. *Eigenvalues. The spectrum of $\mathcal{L}$ is given by*

$$-\mu_{L+1} < \cdots < -\mu_3 < -\mu_2 = -1 < -\mu_1 = -\frac{1}{2} < 0 < \tilde{\mu}_1 < \tilde{\mu}_2 < \ldots \tag{4.6}$$

The eigenvalues $(-\mu_j)_{1\le j\le L+1}$ are simple and associated to spherically symmetric eigenvectors

$$\varphi_j, \quad \|\varphi_j\|_{L_\rho^2} = 1, \quad \varphi_1 = \frac{\Lambda\Psi_L}{\|\Lambda\Psi_L\|_\rho},$$

and the eigenspace for $-\mu_1$ is spanned by $\partial\Psi_L/\partial_{x_k}$. Moreover, there holds as $r \to +\infty$

$$|\partial_k\varphi_j(r)| \lesssim (1+r)^{-\frac{2}{p-1}-\mu_j-k}, \quad 1 \le j \le L+1, \quad k \ge 0. \tag{4.7}$$

2. *Spectral gap. There holds for some constant $\mu^* > 0$:*

$$\forall \varepsilon \in H_\rho^1, \ (\mathcal{L}_n\varepsilon, \varepsilon)_\rho \ge \mu^*\|\epsilon\|_{H_\rho^1}^2 - \frac{1}{\mu^*}\left[\sum_{j=1}^{L+1}(\varepsilon, \varphi_j)_\rho^2 + \sum_{k=1}^{3}(\varepsilon, \partial_{x_k}\Psi_L)_\rho^2\right]. \tag{4.8}$$

We do not give the proof of the above proposition here. We solely mention that $\mathcal{L}$ is a perturbation of the well understood Schrödinger operator $-\Delta + 1/(p-1) + y.\nabla$ by the radial potential $p\Psi_L^{p-1}$. This operator can be studied on spherical harmonics, and its eigenvalues are counted using Sturm oscillation principles.

Let us recall some basic facts for solving linear evolution equations as (4.5). We can solve it using the abstract framework of either dissipative operators (Hille-Yosida-Philipps Theorem, see [26]) or using the Spectral Theorem (see [25]) and the associated diagonalisation of $\mathcal{L}$ directly. In any ways, one has the following properties:

- For any $v_0 \in L^2(\rho)$ there exists a unique solution $v \in C([0,\infty), L^2(\rho))$.
- At any fixed $s \geq 0$ the solution map $v_0 \mapsto v(s)$ is continuous from $L^2(\rho)$ into itself.
- If v_0 belongs to the domain of $\mathcal{L}$, then $v(s)$ belongs to the domain for all s and the trajectory is continuous with respect to the natural topology on the domain. Hence in particular $v \in C^1(L^2(\rho))$ and the equation $\partial_s v + \mathcal{L}v = 0$ is satisfied in $L^2(\rho)$.

The following Lemma describes the linear dynamics generated by (4.5): instable modes grow exponentially, while the remaining part of the perturbation ε on the stable spectrum of $\mathcal{L}$ decreases and satisfy a global energy estimate.

Lemma 4.3. *For any $s \geq 0$ there exists a unique decomposition:*

$$v(s,y) = \sum_{k=1}^{L+1} a_k(s)\varphi_k(y) + \sum_{i=1}^{d} b_i(s)\partial_{y_i}\Psi_L(y) + \varepsilon(s,y),$$

where $\langle \varepsilon, \varphi_k \rangle = 0$ and $\langle \varepsilon \partial_{y_i}\Psi_L \rangle = 0$ for all $1 \leq k \leq L+1$ and $1 \leq i \leq d$. Moreover, one has:

$$a_k(s) = a_k(0)e^{\mu_k s}, \quad b_i(s) = b_i(0)e^{\frac{1}{2}s}$$

and

$$\int_{\mathbb{R}^d} \varepsilon^2(s)\rho dy \leq e^{-2\mu^* s}\int \varepsilon^2(0)\rho dy, \quad \int_0^\infty e^{2\mu^* s}\int_{\mathbb{R}^d}|\nabla\varepsilon(s)|^2\rho dy ds \leq \frac{1}{2\mu^*}\int_{\mathbb{R}^d}\varepsilon^2(0)\rho dy.$$

Proof. The existence and uniqueness of the decomposition follows from the existence and uniqueness of the orthogonal projection onto closed subspaces in Hilbert spaces. We first assume v_0 belongs to the domain of $\mathcal{L}$. Then $v \in C^1([0,\infty), L^2(\rho))$. The parameters a_k and b_i are then differentiable with time and $\varepsilon \in C^1([0,\infty), L^2(\rho))$ so that one has:

$$\partial_s v = \sum_{k=1}^{L+1} \partial_s a_k(s)\varphi_k(y) + \sum_{i=1}^{d} \partial_s b_i(s)\partial_{y_i}\Psi_L(y) + \partial_s\varepsilon(s,y) \qquad (4.9)$$

As said before the Lemma the equation (4.5) is truly satisfied, and v belongs to the domain of $\mathcal{L}$ so that:

$$\mathcal{L}v = \sum_{k=1}^{L+1} \mu_k a_k(s)\varphi_k(y) + \sum_{i=1}^{d} \frac{1}{2}b_i(s)\partial_{y_i}\Psi_L(y) + \mathcal{L}\varepsilon(s,y).$$

Therefore we write:

$$\partial_s\varepsilon + \mathcal{L}\varepsilon = -\sum_{k=1}^{L+1}(\partial_s a_k - \mu_k a_k)\varphi_k - \sum_{1}^{d}(\partial_s b_i - \frac{1}{2}b_i)\partial_{y_i}\Psi_L. \qquad (4.10)$$

We project the above equation onto φ_k which yields, the functions φ_k and $\partial_{y_i} \Psi_L$ being orthonormal:

$$\langle \partial_s \varepsilon, \varphi_k \rangle + \langle \mathcal{L}\varepsilon, \varphi_k \rangle = -(\partial_s a_k - \mu_k a_k).$$

As $\langle \varepsilon, \varphi_k \rangle = 0$ for all $s \geq 0$ and $\varepsilon \in C^1([0,\infty), L^2(\rho))$, we can differentiate this relation and obtain:

$$\langle \partial_s \varepsilon, \varphi_k \rangle = 0.$$

Moreover, as $\mathcal{L}$ is self-adjoint: $\langle \mathcal{L}\varepsilon, \varphi_k \rangle = \langle \varepsilon, \mathcal{L}\varphi_k \rangle = \mu_k \langle \varepsilon, \varphi_k \rangle = 0$. We therefore obtain:

$$\partial_s a_k = \mu_k a_k,$$

implying $a_k = a_k(0)e^{\mu_k s}$. The same reasoning shows $b_i = e^{\frac{1}{2}s}b_i(0)$. We turn to the control of ε. We take the scalar product of (4.10) with ε. This gives (the terms in the right hand side vanishing by the same argument as above):

$$\partial_s \left(\int_{\mathbb{R}^d} \varepsilon^2(s)\rho dy \right) = -2\langle \varepsilon, \mathcal{L}\varepsilon \rangle.$$

Using the coercivity (4.8) we obtain that:

$$\partial_s \left(\int_{\mathbb{R}^d} \varepsilon^2(s)\rho dy \right) \leq -2\mu^* \int_{\mathbb{R}^d} \varepsilon^2(s)\rho dy - 2\mu^* \int_{\mathbb{R}^d} |\nabla \varepsilon|^2 \rho dy.$$

This implies:

$$\partial_s \left(e^{2\mu^* s} \int_{\mathbb{R}^d} \varepsilon^2(s)\rho dy \right) + 2\mu^* e^{2\mu^* s} \int_{\mathbb{R}^d} |\nabla \varepsilon|^2 \rho dy \leq 0.$$

Integrating this inequality gives for any $s \geq 0$:

$$e^{2\mu^* s} \int_{\mathbb{R}^d} \varepsilon^2(s)\rho dy + 2\mu^* \int_0^\infty e^{2\mu^* s} \int_{\mathbb{R}^d} |\nabla \varepsilon|^2 \rho dy ds \leq \int_{\mathbb{R}^d} \varepsilon^2(0)\rho dy.$$

This proves the estimate on ε claimed in the Lemma. We have therefore proved that the Lemma is true if v_0 belongs to the domain of $\mathcal{L}$. For a general initial datum $v_0 \in L^2(\rho)$, we let $(v_n)_{n \in \mathbb{N}}$ be a sequence of elements of the domain of $\mathcal{L}$ converging to v_0 in $L^2(\rho)$ (this is possible as the domain is dense). Then from the continuity of the solution map, we deduce that the information on a_k, b_k and $\int \varepsilon^2 \rho$ goes to the limit. Notice that for each n:

$$\int_0^\infty e^{2\mu^* s} \int_{\mathbb{R}^d} |\nabla \varepsilon_n(s)|^2 \rho dy ds \leq \frac{1}{2\mu^*} \int_{\mathbb{R}^d} \varepsilon_n^2(0)\rho dy,$$

Above, the right hand side converges to $\int_{\mathbb{R}^d} \varepsilon^2(0)\rho$ as $n \to \infty$. $\nabla \varepsilon_n$ is then uniformly bounded in $L^2([0,\infty), L^2(\rho))$. There is therefore a weak limit in this space which has to be $\nabla \varepsilon$, and from lower semi-continuity we get:

$$
\begin{aligned}
\int_0^\infty e^{2\mu^* s} \int_{\mathbb{R}^d} |\nabla \varepsilon(s)|^2 \rho dy ds &\leq \liminf \int_0^\infty e^{2\mu^* s} \int_{\mathbb{R}^d} |\nabla \varepsilon_n(s)|^2 \rho dy ds \\
&\leq \liminf \frac{1}{2\mu^*} \int_{\mathbb{R}^d} \varepsilon_n^2(0)\rho dy \\
&= \frac{1}{2\mu^*} \int_{\mathbb{R}^d} \varepsilon^2(0)\rho dy,
\end{aligned}
$$

which ends the proof of the Lemma.

$\square$

4.2. **Codimensional stability in finite dimension.** We have seen in Lemma 4.3 that the linearised dynamics display a finite number of instabilities, while the infinite dimensional remainder decreases over time. One fundamental question is: how can one avoid the instabilities for the full nonlinear problem, and obtain a perturbation that converges to 0 as $s \to \infty$? The argument for the PDE uses an adaptation to infinite dimension of the following finite dimensional result. The heart of the proof is a topological argument (the nonexistence of a retractation of the closed ball $\mathbb{B}^n$ onto the sphere $\mathbb{S}^{n-1}$).

$$\partial_s u = Au + f(u), \quad A = \begin{pmatrix} \mu_1 & & & & & \\ & \ldots & & & (0) & \\ & & \mu_n & & & \\ & & & -\tilde{\mu}_1 & & \\ & (0) & & & \ldots & \\ & & & & & -\tilde{\mu}_{\tilde{n}} \end{pmatrix} \tag{4.11}$$

where $u \in \mathbb{R}^{n+\tilde{n}}$, $f \in C^1(\mathbb{R}^{n+\tilde{n}}, \mathbb{R}^{n+\tilde{n}})$, and such that $f(0) = 0$, $Jf(0) = 0$ so that for some $C_f > 0$:

$$\tilde{\mu}^* := \min(\tilde{\mu}_1, ..., \tilde{\mu}_{\tilde{n}}) > 0, \quad \mu_i > 0 \ \text{ for } i = 1, ..., n, \quad |f(u)| \leq C_f |u|^2$$

where $|u|$ is the standard Euclidean norm on $\mathbb{R}^{n+\tilde{n}}$. We write:

$$u = \begin{pmatrix} u_1 \\ \ldots \\ u_{n+\tilde{n}} \end{pmatrix} = \begin{pmatrix} v_1 \\ \ldots \\ v_n \\ w_1 \\ \ldots \\ w_{\tilde{n}} \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \ldots \\ v_n \end{pmatrix} \in \mathbb{R}^n, \quad w = \begin{pmatrix} w_1 \\ \ldots \\ w_{\tilde{n}} \end{pmatrix} \in \mathbb{R}^{\tilde{n}},$$

$$A = \begin{pmatrix} B & (0) \\ (0) & \tilde{B} \end{pmatrix}, \quad B = \begin{pmatrix} \mu_1 & & (0) \\ & \ldots & \\ (0) & & \mu_n \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} \tilde{\mu}_1 & & (0) \\ & \ldots & \\ (0) & & \tilde{\mu}_{\tilde{n}} \end{pmatrix}.$$

$$f(u) = \begin{pmatrix} f_1(u) \\ \ldots \\ f_{n+\tilde{n}}(u) \end{pmatrix} = \begin{pmatrix} g_1(u) \\ \ldots \\ g_n(u) \\ h_1(u) \\ \ldots \\ h_{\tilde{n}}(u) \end{pmatrix}, \quad h(u) = \begin{pmatrix} g_1(u) \\ \ldots \\ g_n(u) \end{pmatrix} \in \mathbb{R}^n, \quad h = \begin{pmatrix} h_1 \\ \ldots \\ h_{\tilde{n}} \end{pmatrix} \in \mathbb{R}^{\tilde{n}},$$

Lemma 4.4. *Assume $n, \tilde{n} \geq 1$. There exists $K, \delta > 0$ such that the following holds. For any $w_0 \in \mathbb{R}^{\tilde{n}}$ satisfying $|w_0| \leq \delta$, there exists a unique $v_0 \in \mathbb{R}^n$ satisfying $|v_0| \leq K|w_0|^2$ such that the solution $u(s)$ to (4.11) with initial datum $u_0 = (v_0, w_0)$ satisfies the bound:*

$$|u(s)| \leq 2|w_0|e^{-\tilde{\mu}s}.$$

Proof. We first define the bootstrap regime and the exit time at which a solution leaves the bootstrap regime. Fix $K, \delta > 0$ and $|w_0| \leq \delta$. For any $v_0 \in \mathbb{R}^n$ satisfying $|v_0| \leq K|w_0|^2$ define the exit time of the bootstrap regime:

$$s^*(v_0) = \sup\{s \geq 0, \quad \text{such that } |v(s')| \leq K|w(s')|^2 \ \text{ for all } \ 0 \leq s' \leq s\}.$$

Step 1 We claim that if for some v_0 one has $s^* = \infty$, then this solution satisfies the conclusion of the Lemma. We now prove this claim and assume $s^* = \infty$. For all $s \geq 0$ one has:

$$|u(s)| \leq |v(s)| + |w(s)| \leq |w(s)|(1 + K|w(s)|).$$

In particular, if $K\delta \leq 1/8$ then:

$$|u(0)| \leq \frac{9}{8}|w_0|. \tag{4.12}$$

One then computes, using the above identity, the definition of $\tilde{\mu}^*$ and the bound on f:

$$
\begin{aligned}
\partial_s\left(\frac{1}{2}|w(s)|^2\right) &= \langle w, \tilde{B}w\rangle + \langle w, h(u)\rangle \\
&\leq -\tilde{\mu}^*|w|^2 + |w||f(u)| \\
&\leq -\tilde{\mu}^*|w|^2 + C_f|w|^3(1 + K|w(s)|).
\end{aligned}
$$

Let
$$S = \sup\{s \geq 0, \text{ such that } |u(s')| \leq 2|w_0|e^{-\tilde{\mu}s} \text{ for all } 0 \leq s' \leq s\}.$$
By continuity and the above control on $|u_0|$, $S > 0$. Then for any $0 \leq s < S$, one has from the identity above, if $K\delta \leq 1/8$:

$$\partial_s\left(\frac{1}{2}|w(s)|^2\right) \leq -\tilde{\mu}^*|w|^2 + 8|w_0|^3 e^{-3\tilde{\mu}^*s}C_f(1 + K2|w_0|e^{-\tilde{\mu}s}) \leq -\tilde{\mu}^*|w|^2 + |w_0|^2 e^{-3\tilde{\mu}^*s}\delta C_f 10$$

We then obtain the differential inequality:

$$\partial_s\left(e^{2\tilde{\mu}^*s}|w(s)|^2\right) \leq C_f 20\delta|w_0|^2 e^{-\tilde{\mu}^*s}.$$

Reintegrating the above inequality between 0 and $s < S$ gives:

$$
\begin{aligned}
|w(s)|^2 &\leq e^{-2\tilde{\mu}^*s}|w_0|^2 + e^{-2\tilde{\mu}^*s}\int_0^s C_f 20\delta|w_0|^2 e^{-\tilde{\mu}^*s'}ds' \leq e^{-2\tilde{\mu}^*s}|w_0|^2 + e^{-2\tilde{\mu}^*s}\frac{C_f 20\delta}{\tilde{\mu}^*}|w_0|^2 \\
&\leq 2|w_0|^2 e^{-2\tilde{\mu}^*s}
\end{aligned}
$$

if δ has been chosen small enough depending on C_f and $\tilde{\mu}^*$. Coming back to the inequality for $|u|$ above, for $0 \leq s < S$:

$$|u| \leq |w(s)|(1 + K|w(s)|) \leq \sqrt{2}|w_0|e^{-\tilde{\mu}^*s}\frac{5}{4} \leq c|w_0|e^{-\tilde{\mu}^*s}$$

where $c = 5\sqrt{2}/4 < 2$. By a continuity argument, this implies that $S = \infty$. This ends the proof of the claim.

Step 2 We claim that there exists $|v_0| \leq K|w_0|^2$ such that $s^* = \infty$. We reason by contradiction and assume that $s^* < \infty$ for all $|v_0| \leq K|w_0|^2$. Then this allows us to define the escape mapping from the unit ball to the unit sphere:

$$
\begin{aligned}
\Phi: \quad B_{\mathbb{R}^n}(0,1) &\to \mathbb{S}^{n-1} \\
V_0 &\mapsto \frac{1}{K|w(s^*)|^2}v\left(s^*(K|w_0|^2 V_0)\right)
\end{aligned}
$$

This mapping is well defined. Indeed, by continuity, at time s^* one must have $|v(s^*)| = K|w(s^*)|^2$. Also, again by a continuity argument for the ODE (4.11), the function Φ is continuous. We claim that it is surjective. Indeed, let $V_0 \in \mathbb{S}^{d-1}$, and $v_0 = K|w_0|^2 V_0$. We then compute that $|v|$ is growing at the initial time from (4.12):

$$
\begin{aligned}
\partial_s(|v|^2)_{|s=0} &= \langle v_0, Bv_0\rangle + \langle v_0, g(u_0)\rangle \\
&\geq \min(\mu_1, ..., \mu_n)|v_0|^2 - |v_0|C_f|u_0|^2 \\
&\geq \min(\mu_1, ..., \mu_n)K^2|w_0|^4|V_0|^2 - K|w_0|^2|V_0|C_f\left(\frac{9}{8}\right)^2|w_0|^2 \\
&\geq \frac{1}{2}\min(\mu_1, ..., \mu_n)K^2|w_0|^4
\end{aligned}
$$

if K has been chosen large enough depending on C_f. One computes similarly as in Step 1 that if $K\delta \leq 1/8$:

$$|\partial_s(|w(s)|^2)_{|s=0}| \leq 0$$

for some constant $C > 0$ independent of K and δ. Hence:

$$\partial_s \left(\frac{|v(s)|^2}{K^2|w(s)|^4} \right)_{s=0} = \frac{\partial_s(|v|^2)|w|^2 - 2|v|^2\partial_s(|w|^2)}{K^2|w(s)|^6} > 0.$$

Therefore, from the very definition of s^* we obtain $s^* = 0$, so that $\Phi(V_0) = V_0$. The mapping Φ is hence surjective. We have proved that Φ is a continuous and surjective mapping from the unit ball onto the unit sphere. But such mappings do not exist from Brouwer's Theorem (classical Theorem that you can find in many textbooks).

Step 3 *Uniqueness.* We leave uniqueness as an exercise. Hint: you can employ a similar strategy as in Steps 1 and 2 to study the difference of solutions.   □

4.3. **Full nonlinear stability.** We perform now the full stability analysis of the semilinear heat equation near the backward self-similar profile Ψ_L, having in mind the properties of the linearised dynamics described in Lemma 4.3, and the control of instabilities by the tuning of the initial datum given in finite dimension in Lemma 4.4. The first step is to renormalise dynamically the equation using its symmetries, in order to zoom appropriately at the singularity, as explained in the next two Lemmas.

Lemma 4.5. *There exist $K, \delta > 0$ such that the following holds true. Given any $w \in L^2(\rho)$ with $\|v\|_{L^2(\rho)} \leq \delta$, there exists a unique couple $(\lambda, x_0) \in (0, \infty) \times \mathbb{R}^d$ with $|\lambda - 1| \leq K\delta$ and $|x_0| \leq K\delta$ such that:*

$$(\Psi_L + w)(x) = \frac{1}{\lambda^{\frac{2}{p-1}}}(\Psi_L + v)\left(\frac{x - x_0}{\lambda} \right)$$

where $\langle v, \Lambda\Psi_L \rangle = \langle v, \partial_{y_1}\Psi_L \rangle = ... = \langle v, \partial_{y_d}\Psi_L \rangle = 0$. Moreover, the parameters (λ, x_0), seen as a function from L^2_ρ into $\mathbb{R}^{1+d}$, are differentiable.

Proof. The proof relies on a classical use of the implicit function theorem. Define the mapping

$$\Phi : (v, \lambda, x_0) \mapsto \left(\langle \tilde{v}, \Lambda\Psi_L \rangle, \langle \tilde{v}, \partial_{y_1}\Psi_L \rangle, ..., \langle \tilde{v}, \partial_{y_d}\Psi_L \rangle \right),$$

where:

$$\tilde{v}(y) = \lambda^{\frac{2}{p-1}}(\Psi_L + v)(x_0 + \lambda y) - \Psi_L(y).$$

Φ is a C^2 mapping on $L^2_\rho \times (0, \infty) \times \mathbb{R}^d$. Moreover, one computes that its differential at $(0, 1, 0)$ is, because $\Lambda\Psi_L$ and $\partial_{y_i}\Psi_L$ are orthogonal:

$$J\Phi(0, 0, 1, 0)$$

$$= \begin{pmatrix} \langle \cdot, \Lambda\Psi_L \rangle & \|\Lambda\Psi_L\|^2_{L^2_\rho} & & \\ \langle \cdot, \partial_{y_1}\Psi_L \rangle & & \|\partial_{y_1}\Psi_L\|^2_{L^2_\rho} & (0) \\ ... & & & ... \\ \langle \cdot, \partial_{y_d}\Psi_L \rangle & (0) & & \|\partial_{y_d}\Psi_L\|^2_{L^2_\rho} \end{pmatrix}.$$

Hence, the restriction of the differential to $\{0\} \times \mathbb{R}^{1+d}$ is clearly invertible. By a standard application of the implicit function Theorem, there exist $\delta, K > 0$ such that for any $\|v\|_{L^2_\rho} \leq \delta$, a unique couple $|\lambda - 1| + |x_0| \leq K\delta$ exists such that $\Phi(v, \lambda, x_0) = 0$. These are the desired parameters λ, x. Also, the application of the

implicit Theorem also provides directly that the parameters λ and x_0 obtained this way are C^1 functions of $v \in L^2_\rho$. □

Let us consider the set for some $\delta > 0$ the following neighbourhood of the family of self-similar solutions:

$$\mathcal{O} := \left\{ \frac{1}{\lambda^{\frac{2}{p-1}}} (\Psi_L + v) \left(\frac{x - x_0}{\lambda} \right), \quad (\lambda, x_0) \in (0, \infty) \times \mathbb{R}^d \text{ and } \|\varepsilon\|_{L^\infty} < \delta \right\}.$$

Lemma 4.6. *There exist $\delta, K > 0$ small enough, there exist unique C^1 functions for the L^∞ topology $\lambda : \mathcal{O} \to (0, \infty)$ and $x_0 : \mathcal{O} \to \mathbb{R}^d$ such that any $u \in \mathcal{O}$ can be written under the form:*

$$u(x) = \frac{1}{\lambda^{\frac{2}{p-1}}} (\Psi_L + v) \left(\frac{x - x_0}{\lambda} \right),$$

with v satisfying:

$$\|v\|_{L^\infty} \le K\delta, \quad v \perp_{L^2_\rho} \Lambda \Psi_L, \partial_{y_1} \Psi_L, ..., \partial_{y_d} \Psi_L.$$

Remark 4.7. Lemma 4.6 has to be understood as the orthogonal projection of a function onto the manifold of self-similar solutions $\left(\frac{1}{\lambda^{\frac{2}{p-1}}} \Psi_L \left(\frac{x - x_0}{\lambda} \right) \right)_{\lambda, x_0}$.

Proof. First note that L^∞ embeds continuously in L^2_ρ. Let K be fixed from Lemma 4.5.

Step 1 *Closeness of two decompositions.* We claim that there exists $C > 0$ such that for δ small enough, if $u \in \mathcal{O}$ can be written in two different ways:

$$u(x) = \frac{1}{\lambda^{\frac{2}{p-1}}} (\Psi_L + v) \left(\frac{x - x_0}{\lambda} \right) = \frac{1}{\lambda'^{\frac{2}{p-1}}} (\Psi_L + v') \left(\frac{x - x_0'}{\lambda'} \right), \quad \text{with } \|v\|_{L^\infty}, \|v'\|_{L^\infty} \le K\delta$$

then $|\lambda/\lambda' - 1| + |x_0 - x_0'|^2/\lambda^2 \le C\delta$. Indeed, as Ψ_L attains its maximum at the origin, for δ small the maximum attained for the first decomposition is $\lambda^{-2/(p-1)}(\Psi_L(0) + O(K\delta))$, while the maximum attained for the second decomposition is $\lambda'^{-2/(p-1)}(\Psi_L(0) + O(K\delta))$. The two being equal: $\lambda/\lambda' - 1 = O(K\delta)$. We now rewrite the equality as:

$$(\Psi_L + v)(x) = \left(\frac{\lambda}{\lambda'} \right)^{\frac{2}{p-1}} (\Psi_L + v') \left(\frac{\lambda}{\lambda'} x + \frac{x_0 - x_0'}{\lambda'} \right)$$

Again, as Ψ_L attains its maximum at the origin with $\Delta \Psi_L(0) < 0$, and $|\lambda/\lambda' - 1| \le C\delta$, we obtain that $(x_0 - x_0')/\lambda' = o(1)$ as $\delta \to 0$. We can therefore Taylor expand at 0, so that for some $c > 0$:

$$\Psi_L(0) + O(\delta K) = \left(\frac{\lambda}{\lambda'} \right)^{\frac{2}{p-1}} \left(\Psi_L(0) - c \frac{|x_0 - x_0'|^2}{\lambda'^2} (1 + o(1)) \right) + O(\delta K).$$

which gives: Hence, one must have $|x_0 - x_0'|/\lambda' \le C\sqrt{\delta}$. This proves the claim.

Step 2 *End of the proof.* We define the following mapping: to any $u \in \mathcal{O}$ we pick any decomposition:

$$u = \frac{1}{\lambda_1^{\frac{2}{p-1}}} (\Phi + v_1) \left(\frac{x - x_1}{\lambda_1} \right), \quad \|v_1\|_{L^\infty} \le \delta$$

coming from the very definition of $\mathcal{O}$. Next, we denote by (λ_2, x_2) the functions given by Lemma 4.5. We also change notations and use the notation $\bar{\delta}$ instead of the

constant δ involved in Lemma 4.5. We then take: $v = v_2(\Phi + v_1)$, $\lambda = \lambda_1 \lambda_2(\Phi + v_1)$ and $x_0 = x_1 + \lambda_1 x_2$. One indeed has this way that:

$$(\Phi + v_1)(y) = \frac{1}{\lambda_2^{\frac{2}{p-1}}}(\Psi_L + v)\left(\frac{x - x_2}{\lambda_2}\right)$$

and hence v satisfies the desired orthogonality conditions and size assumptions and:

$$
\begin{aligned}
u &= \frac{1}{\lambda_1^{\frac{2}{p-1}}}(\Phi + v_1)\left(\frac{x - x_1}{\lambda_1}\right) = \frac{1}{(\lambda_1\lambda_2)^{\frac{2}{p-1}}}(\Phi + v)\left(\frac{\frac{x-x_1}{\lambda_1} - x_2}{\lambda_2}\right) \\
&= \frac{1}{(\lambda_1\lambda_2)^{\frac{2}{p-1}}}(\Phi + v)\left(\frac{x - x_1 - \lambda x_2}{\lambda_1\lambda_2}\right) = \frac{1}{\lambda^{\frac{2}{p-1}}}(\Phi + v)\left(\frac{x - x_0}{\lambda}\right),
\end{aligned}
$$

so that (λ, x_0, v) satisfies the conclusion of the Lemma. Pick now another decomposition satisfying the conclusions of the Lemma:

$$u = \frac{1}{\lambda'^{\frac{2}{p-1}}}(\Phi + v')\left(\frac{x - x_0'}{\lambda'}\right), \quad \|v'\|_{L^\infty} \leq K\delta, \quad v' \perp \Lambda\Psi_L, \nabla\Psi_L.$$

Then one has the following decomposition for $\Psi_L + v$:

$$(\Psi_L + v)(x) = \left(\frac{\lambda}{\lambda'}\right)^{\frac{2}{p-1}}(\Psi_L + v')\left(\frac{\lambda}{\lambda'}x + \frac{x_0 - x_0'}{\lambda'}\right),$$

where v and v' both satisfies the orthogonality conditions. From step 1 we deduce $|\lambda/\lambda' - 1| + |x_0 - x_0'|^2/\lambda^2 \leq C\delta$. For δ small enough depending on $\bar{\delta}$, we can apply the uniqueness part of Lemma 4.5: the decomposition is unique. Hence $\lambda = \lambda'$, $x_0 = x_0'$ and $v = v'$. Hence the way to construct λ, x and v given at the beginning of the Step is well defined and does not depend on the first decomposition (λ_1, x_1, v_1) that is picked. This also provides differentiability as λ_2 and x_2 are differentiable functions from Lemma 4.5. This ends the proof of the Lemma.

$\square$

Let us now consider a solution to $u_t = \Delta u + u^p$ with initial condition:

$$u_0 = \frac{1}{\lambda^{\frac{2}{p-1}}}(\Psi_L + w_0)(\frac{x - x_0}{\lambda}), \quad \|w_0\|_{W^{2,\infty}(\mathbb{R}^d)} \ll \delta.$$

Then from parabolic regularity, see[2] Lemmas 2.3 and 2.4, one has $u \in C^1([0, T), L^\infty)$. Hence, as long as $u \in \mathcal{O}$, Lemma 4.6 gives a C^1 in time parameters $x(t)$ and $\lambda(t)$ such that:

$$u(t, x) = \frac{1}{\lambda^{\frac{2}{p-1}}(t)}(\Psi_L + v)\left(\frac{x - x(t)}{\lambda(t)}\right), \quad v \perp_\rho \Lambda\Psi_L, \partial_{y_1}\Psi_L, ..., \partial_{y_d}\Psi_L.$$

We thus define the renormalised flow by defining the renormalised variables:

$$y = \frac{x - x(t)}{\lambda(t)}, \quad s = s_0 + \int_0^t \frac{dt}{\lambda^2(t)},$$

and the renormalised unknown:

$$\tilde{u}(s, y) = \lambda^{\frac{2}{p-1}}(t)u(t, x).$$

It is then an instructive exercise to show that v solves the renormalised flow equation:

$$\partial_s \tilde{u} = \Delta\tilde{u} + \tilde{u}^p + \frac{\lambda_s}{\lambda}\Lambda\tilde{u} + \frac{x_s}{\lambda}.\nabla\tilde{u}.$$

---

[2]These Lemmas are easy examples of parabolic regularity, we need here stronger results which nonetheless rely on the same ideas.

Remark 4.8. The above transformation provides a suitable zoom at the singularity. It also transforms the problem of stability of singularity formation, a finite-time phenomenon, into a global in time stability problem.

Moreover, from the above decomposition of u:

$$\tilde{u} = \Psi_L + v.$$

The perturbation v then solves the renormalised perturbation equation:

$$\partial_s v + \mathcal{L}v = \left(\frac{\lambda_s}{\lambda} + \frac{1}{2}\right)\Lambda(\Psi_L + v) + \frac{x_s}{\lambda}.\nabla(\Psi_L + v) + NL$$

where NL designate the nonlinear terms:

$$NL := (\Psi_L + v)^p - \Psi_L^p - p\Psi_L^{p-1}v.$$

Having Lemma 4.3 in mind, we solely need here to decompose v onto the L-th first unstable eigenmodes:

$$v(s,y) = \sum_{k=1}^{L} a_k(s)\varphi_k(y) + \varepsilon = \varphi + \varepsilon, \quad \varphi = \sum_{k=1}^{L} a_k\varphi_k.$$

where ε is orthogonal to the negative eigenmodes:

$$\varepsilon \perp_\rho \Lambda\Psi_L, \nabla\Psi_L, \varphi_2, ..., \varphi_{L+1}. \tag{4.13}$$

One can then compute the equation for ε, and it is instructive to put it into the following form:

$$\varepsilon_s + \mathcal{L}\varepsilon = F - \text{Mod}$$

where F carries higher order terms, a small linear term in ε and nonlinear terms (defined above):

$$F = L\varepsilon + NL, \quad L\varepsilon = \left(\frac{\lambda_s}{\lambda} + \frac{1}{2}\right)\Lambda\varepsilon + \frac{x_s}{\lambda}.\nabla\varepsilon,$$

and where Mod are the so-called modulation terms:

$$\text{Mod} = \sum_{k=2}^{L+1}(\partial_s a_k - \mu_k a_k)\varphi_k - \left(\frac{\lambda_s}{\lambda} + \frac{1}{2}\right)(\Psi_L + \varphi) + \frac{x_s}{\lambda}(\Psi_L + \varphi).$$

Remark 4.9. Notice the difference between the ε above and the previous linear evolution for ε (4.9). There are mainly two differences. The first one is that the nonlinear terms are now appearing in the equation. The second one is that ε has no more component along the directions $\Lambda\Psi_L$ and $\nabla\Psi_L$. Instead, these have been traded for the obtention of the parameters $\lambda(t)$ and $x(t)$. This is why these "instability" directions are not true instabilities: we can kill these directions by renormalising suitably the solution.

In view of the Toy model problems seen in Lemmas 4.3 and 4.6, you should have an intuition about the fact that we can prove Theorem 4.1 by relying on the following bootstrap Proposition. To state it, we introduce χ a smooth cut-off function with $\chi(y) = 1$ for $|y| \leq 1$ and $\chi(y) = 0$ for $|y| \geq 2$, and its rescaling for $A > 0$:

$$\chi_A(y) = \chi(\frac{y}{A}).$$

Proposition 4.10 (Bootstrap). *There exist universal constants $0 < \mu, \eta \ll 1$, $K, C \gg 1$ such that for all $s_0 \geq s_0(K, C, \mu, \eta) \gg 1$ large enough the following holds. For any λ_0 and ε_0 satisfying $\lambda_0 = e^{-s_0/2}$, the orthogonality (4.13) and*

$$\|(1 - \chi_{\frac{1}{\lambda_0}})\Psi_L + \varepsilon_0\|_{\dot{H}^{s_c}} + \|\varepsilon_0\|_{H_\rho^2} + \|\Delta\varepsilon_0\|_{L^2} \leq e^{-2\mu s_0}, \tag{4.14}$$

there exist $(a_2(0), \ldots, a_{n+1}(0))$ satisfying:

$$\sum_{j=2}^{n+1} |a_j|^2 \leq K e^{-2\mu s_0}$$

such that the solution starting from u_0, written with the decomposition described above satisfies for all $s \geq s_0$:

- *control of the scaling:*

$$\frac{1}{2} e^{-\frac{1}{2}s} < \lambda(s) < 2 e^{-\frac{1}{2}s}; \tag{4.15}$$

- *control of the unstable modes:*

$$\sum_{j=2}^{n+1} |a_j|^2 \leq K e^{-2\mu^* s}; \tag{4.16}$$

- *control of the exponentially weighted norm:*

$$\|\varepsilon\|_{H_\rho^2} < C e^{-\mu^* s}; \tag{4.17}$$

- *control of a Sobolev norm above scaling:*

$$\|\Delta v\|_{L^2} < C e^{-\mu s}; \tag{4.18}$$

- *control of the critical norm:*

$$\|w\|_{\dot{H}^{s_c}} < \eta. \tag{4.19}$$

References

[1] Barenblatt, G. I., & Isaakovich, B. G. (1996). Scaling, self-similarity, and intermediate asymptotics: dimensional analysis and intermediate asymptotics (Vol. 14). Cambridge University Press.

[2] Bressan, A. (2000). Hyperbolic systems of conservation laws: the one-dimensional Cauchy problem (Vol. 20). Oxford University Press on Demand.

[3] Brezis, Haïm, and Thierry Cazenave. "A nonlinear heat equation with singular initial data." Journal D'Analyse MathÈmatique 68.1 (1996): 277-304.

[4] Budd, C. J., & Qi, Y. W. (1989). The existence of bounded solutions of a semilinear elliptic equation. Journal of differential equations, 82(2), 207-218.

[5] Collot, C., Ghoul, T. E., & Masmoudi, N. (2019). Unsteady separation for the inviscid two-dimensional Prandtl's system. arXiv preprint arXiv:1903.08244.

[6] Collot, C., Ghoul, T. E., Ibrahim, S., & Masmoudi, N. (2018). On singularity formation for the two dimensional unsteady Prandtl's system. arXiv preprint arXiv:1808.05967.

[7] Collot, C., Raphael, P., & Szeftel, J. (2019). On the stability of type I blow up for the energy super critical heat equation. American Mathematical Soc..

[8] Dietert, H., & Gerard-Varet, D. (2019). Well-posedness of the prandtl equations without any structural assumption. Annals of PDE, 5(1), 8.

[9] Weinan, E., & Engquist, B. (1997). Blowup of solutions of the unsteady Prandtl?s equation. Comm. Pure Appl. Math, 50(12), 1287-1293.

[10] Evans, L.C., Partial Differential Equations, Springer-Verlag (1998).

[11] Friedman, Avner. Partial differential equations of parabolic type. Courier Dover Publications, 2008.

[12] Gui, C., Ni, W. M., & Wang, X. (1992). On the stability and instability of positive steady states of a semilinear heat equation in Rn. Communications on pure and applied mathematics, 45(9), 1153-1181.

[13] Hong, L., & Hunter, J. K. (2003). Singularity formation and instability in the unsteady inviscid and viscous Prandtl equations. Communications in Mathematical Sciences, 1(2), 293-316.

[14] Kato, Tosio. "The Cauchy problem for quasi-linear symmetric hyperbolic systems." Archive for Rational Mechanics and Analysis 58.3 (1975): 181-205.

[15] Kukavica, I., Vicol, V., & Wang, F. (2017). The van Dommelen and Shen singularity in the Prandtl equations. Advances in Mathematics, 307, 288-311.

[16] Ladyzhenskaya, Olga A., and Nina N. Ural'tseva. Linear and quasilinear elleptic equations. Academic Press, 1968.

[17] Lepin, L. A. (1990). Self-similar solutions of half-linear heat equation. Matematicheskoe modelirovanie, 2(3), 63-74.

[18] Lieberman, Gary M. Second order parabolic differential equations. World scientific, 1996.

[19] Majda, Andrew. Compressible fluid flow and systems of conservation laws in several space variables. Vol. 53. Springer Science & Business Media, 2012.

[20] Masmoudi, N., & Wong, T. K. (2015). Local?in?time existence and uniqueness of solutions to the Prandtl equations by energy methods. Communications on Pure and Applied Mathematics, 68(10), 1683-1741.

[21] Matano, H., & Merle, F. (2004). On nonexistence of type II blowup for a supercritical nonlinear heat equation. Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences, 57(11), 1494-1541.

[22] Matano, H., & Merle, F. (2009). Classification of type I and type II behaviors for a supercritical nonlinear heat equation. Journal of Functional Analysis, 256(4), 992-1064.

[23] Oleinik, O. A., & Samokhin, V. N. (1999). Mathematical models in boundary layer theory (Vol. 15). CRC Press.

[24] Polacik, P., & Chen, X. Y. (1996). Asymptotic periodicity of positive solutions of reaction diffusion equations on a ball. Journal f¸r die reine und angewandte Mathematik, 472, 17-52.

[25] Reed, M., & Simons, B. (2012). Methods of modern mathematical physics volume 1: Functional analysis. Elsevier.

[26] Reed, M., & Simons, B. (2012). Methods of modern mathematical physics volume 3: Fourier Analysis, Self-Adjointness. Elsevier.

[27] Serre, D. (1999). Systems of Conservation Laws 1: Hyperbolicity, entropies, shock waves. Cambridge University Press.

[28] Quittner, P., & Souplet, P. (2007). Superlinear parabolic problems. Blow-up, Global Existence and Steady States, Birkhuser Advanced Texts, Birkhuser, Basel.

[29] Tao, T. (2006). Nonlinear dispersive equations: local and global analysis (No. 106). American Mathematical Soc..

[30] Troy, W. C. (1987). The existence of bounded solutions of a semilinear heat equation. SIAM journal on mathematical analysis, 18(2), 332-336.

[31] Weissler, Fred B. "Local existence and nonexistence for semilinear parabolic equations in L p." Indiana University Mathematics Journal 29.1 (1980): 79-102.