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Sur l'explosion critique et surcritique pour les équations des ondes et de la chaleur semi-linéaires

Thèse dirigée par Pierre Raphaël
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Résumé

Cette thèse porte sur l'étude des propriétés qualitatives des solutions des équations de la chaleur (NLH) $\partial_t u = \Delta u + u^p$ et des ondes (NLW) $\partial_{tt} u = \Delta u + u^p$ semi-linéaires. Le but est d'introduire, d'expliquer et de donner une preuve rigoureuse des quatre résultats principaux obtenus par l'auteur et ses collaborateurs. Ces travaux sont les suivants.

(i) La construction et la description de solutions devenant lentement (par rapport à la vitesse générique) singulières en temps fini dans le régime dit energie surcritique. Ce sont une famille dénombrable de solutions lisses et localisées qui concentrent l'état stationnaire radial. Pour (NLW) elles sont à symétrie radiale, et l'on démontre leur stabilité conditionnelle. Pour (NLH) elles sont construites dans le cas général non radial d'un domaine lisse et borné avec conditions au bord de Dirichlet.

(ii) La classification complète de la dynamique des solutions non-radiales de (NLH) au voisinage de l'état stationnaire radial dans le régime dit énergie critique en grande dimension. Cela inclut en particulier la construction de la variété instable et la preuve d'un résultat de rigidité la caractérisant.

(iii) La construction, la description précise et la stabilisation de solutions particulières devenant singulières à la vitesse générique pour (NLH) dans le régime énergie surcritique. Elles forment une famille dénombrable de solutions explosives autosimilaires radiales exactes. Leur existence était déjà connue, mais la nouvelle méthode utilisée ici pour leur construction permet de montrer leur stabilité conditionnelle non radiale. En particulier, elles peuvent émerger comme profil à l'explosion pour des solutions lisses et localisées.

Mots clés : Explosion, soliton, équation de la chaleur, équation des ondes, énergie critique, énergie surcritique, auto-similaire, comportement asymptotique, état stationnaire, concentration, stabilité, existence, parabolique, dispersif.

On critical and supercritical blow-up for the semilinear wave and heat equations

Abstract

This thesis is devoted to the study of the qualitative behavior of solutions to the semilinear heat (NLH) $\partial_t u = \Delta u + u^p$ and wave (NLW) $\partial_{tt} u = \Delta u + u^p$ equations. We introduce, explain, and give a rigorous proof of four main results obtained by the author and collaborators. These main results are the following.

(i) The construction and description of solutions becoming slowly (with respect to the generic speed) singular in finite time in the so-called energy supercritical setting. These are a countable family of smooth and well-localized solutions concentrating the radial steady state. For (NLW) they are radially symmetric, and their conditional stability is proven. For (NLH) they are constructed in the non-radial setting, on any smooth and bounded domain with Dirichlet boundary condition.

(ii) The complete classification for (NLH) of possible dynamics for non-radial solutions in the vicinity of the radial steady state for the so-called energy critical setting in large dimensions. This includes the description and the characterization of minimal objects belonging to the unstable manifold.

(iii) The construction, precise description and stabilization of particular solutions becoming singular in finite time with the generic speed for (NLH) in the energy supercritical setting. These are a countable family of exact radial backward self-similar solutions. Their existence was already known, but the new method of construction here allows to prove their non-radial conditional stability. In particular, they can emerge as the blow-up profile from smooth and well-localized solutions.

Key words: Blow-up, soliton, heat equation, wave equation, energy critical, energy supercritical, self-similar, long time dynamics, stationary state, concentration, stability, existence, parabolic, dispersive.

Organization/Organisation

Pour les lecteurs francophones : Le chapitre 0, écrit en français, contient une introduction du domaine de recherche et une présentation du travail de l'auteur. Il est long d'une dizaine de page et est rédigé à un niveau formel accessible à tous. Le reste du document est en langue anglaise.

For English-speaking readers: Due to the length of the proofs, this document is structured as follows in order to satisfy the various kinds of readers.

- Chapter 1 is devoted to a quick presentation of the general context surrounding this work and to a short statement of the results obtained by the author. It is around ten pages long and is aimed at a broad audience.

- In Chapter 2, each one of these results is introduced with details and explained separately, and is stated in its full formulation. Each time, a rather complete sketch of the proof retaining the important arguments is given. It is a bit less than forty pages long and can be read by non-specialists with a background in functional analysis.

- Finally, Chapter 3, Chapter 4, Chapter 5 and Chapter 6 contain the complete proofs of the results obtained by the author.

Remerciements

Je tiens en premier lieu à remercier mon directeur de thèse Pierre Raphaël. Il m'a fait découvrir un domaine de recherche passionnant ; dans mon travail sa grande disponibilité, sa confiance en moi et son aide logistique ont compté. Plus personnellement, il est enrichissant de travailler avec quelqu'un de dynamique et concentré, de transparent dans sa réflexion mathématique, et qui sait prendre du recul quand il le faut.

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Cette thèse utilise le package pour les diagrammes de Paul Taylor, et elle contient des illustrations qui ont été réalisées avec l'aide précieuse de David. Je remercie chaleureusement mon amie Coline Caussade d'avoir accepté de compléter ce document avec une autre lecture de mon travail.

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Mon orientation vers la recherche en analyse des équations aux dérivées partielles doit beaucoup à certains de mes professeurs, le nom de certains étant déjà apparu plus haut. En classe préparatoire il y eut Arnaud Pinguet et Jean-François Le Floc'h, puis Guillaume Carlier, Laure Saint-Raymond, Nicolas Burq, Jose A. Carrillo, Thomas Alazard en tant que tuteur, Filippo Santambrogio, Radu Ignat et Jean-Yves Chemin.

Pour la partie financière, je tiens à remercier principalement l'Ecole Normale Supérieure auprès de laquelle j'ai obtenu mon contrat doctoral, le laboratoire J.A. Dieudonné et l'équipe Analyse Géométrie et Dynamique, l'Université Nice Sophia Antipolis, l'European Research Council via les projets BlowDiSol, SingWave et SCAPDE, le Mathematical Science Research Institute et le Groupe De Recherche sur les Équations aux Dérivées Partielles du Centre National de la Recherche Scientifique.

De nombreuses autres personnes ont eu un lien avec cette thèse même s'il est moins direct. Parmi ces gens, il y a ma famille et les amis qui n'apparaissent pas ici, qui sont ce que j'ai de plus précieux. Il y a vous que peut-être, malheureusement, j'ai oublié de citer. Ne m'en voulez pas, il y aura d'autres occasions pour se rattraper, ces remerciements ayant été écrits avec une certaine hâte de passer à la suite.

Suite à cette lecture de thèse, j'ai voulu proposer une vision et une traduction particulière, en la dotant d'autres mots et d'un langage distant de celui mathématique: un parler imagé.

On pourrait parler d'un cheminement pourvu d'indices à ma compréhension et à mon approche du travail de Charles: la structure qui précède les équations mais aussi la présence de la forme courbe -écho à l'onde- manifeste à chaque proposition. Du reste chaque dessin est légendé de son « jumeau simplifié »: en pointillé, comme le sont les dessins numérotés à relier des enfants. Chacun d'entre eux a vocation à être construit et grandi ou à se perdre: Des « objets imaginaires qui ressemblent à des choses que l'on voit dans la réalité mais en simplifié, (...) et qui changent avec le temps".

We have

"par instabilité d'échelle"

and

"échelle spatiale".

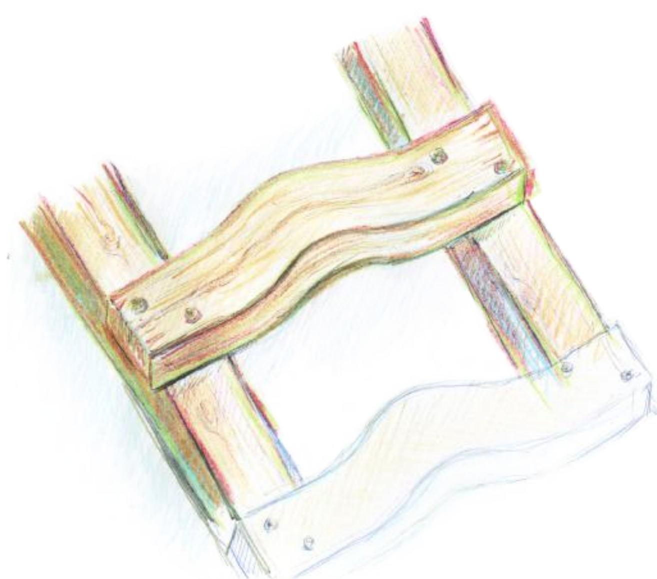
Recall that

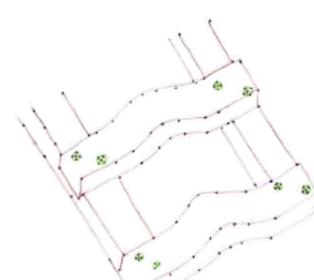
Les échelles des couvreurs sont galbées pour une surface d'appui plus importante

due to

L'instabilité de la surface du toit.

We infer from





and this conclude the proof of

"Une somme de vagues".

Also for

"un cadre adapté"

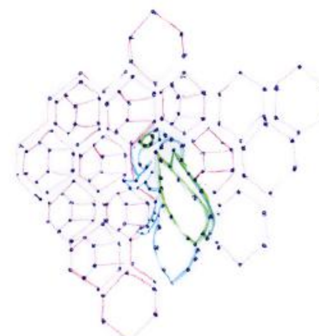
we have

Le cadre est un support matériel parallèlépipèdique fait de matière rigide: un châssis, comme le cadre d'une ruche

and

L'initiateur du cadre F.Huber avait crée une ruche à feuillets adaptée à l'observation de la vie des abeilles. Une amorce de cire permettait aux abeilles de construire dans le sens du cadre et, mal établie, une ruche donnait des cires courbées.

We deduce



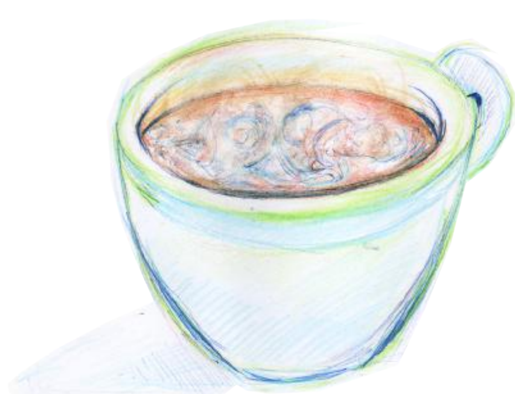
We have

"Le flot de la chaleur".

Recall that

Torahiko Terada décrit la brume de sa tasse de thé chaud comme le résumé miniature du couple océan atmosphère, où lorsqu'une des goutelettes en levitation s'effondre, elle fissure la surface de la mince pellicule blanche qui se fendille et se divise en petits continents.

This yields



We have

"le calcul à la main"

and hence

Le palmu des couturières, utilisé pour mesurer des lignes courbes, est la plus grande longueur matérialisée par la main, soit la distance entre l'extrémité du pouce et l'extrémité de l'auriculaire en position écarté, avec l'index le majeur et l'annulaire repliés.

Wich yields

We have

"Des comportements asymptotiques"

such that

Une branche infinies de courbes tendant vers l'infinitésimal, et qui prend en compte des variables relatives au temps

which is equivalent to

Des modélisations architecturales de croissance des plantes.

This allows us to draw

.

Contents

0

Introduction et résumé détaillé

0.1 Introduction générale

Les équations aux dérivées partielles (EDP) modélisent l'évolution temporelle de certaines quantités définies sur un milieu continu pour des problèmes issus des sciences naturelles et d'autres domaines des mathématiques. Depuis le début de leur étude au XVIII^{e} siècle, en lien avec la naissance du calcul infinitésimal au siècle précédant, elles n'ont cessé d'être étudiées. Avec le développement de l'informatique, les limites de certains aspects de la modélisation ont été repoussées au delà de celles de l'esprit humain. L'étude des EDP s'articule maintenant autour de trois domaines majeurs : les sciences naturelles, le calcul numérique et l'analyse théorique.

Ce dernier domaine a évolué au cours des décénies. Les premiers travaux portaient souvent sur des équations raisonnables dans l'espoir d'obtenir des formules permettant notamment le calcul à la main des solutions. Maintenant, l'aspect le plus important de l'étude de l'existence de solutions est celui de la stabilité plus que celui de la recherche de formules explicites, en liaison avec celle des schémas numériques et l'obtention de solutions faibles dans certains espaces topologiques. Pour bien des équations l'existence de solutions a été obtenue au moins dans un cadre peu raffiné, et il s'agit dorénavant d'étudier les propriétés de ces dernières. L'emphase est placée sur leur description qualitative pour des équations soit canoniques soit fondamentales en sciences. Cette restriction est due aux raisons suivantes : l'aspect quantitatif est abordé le plus souvent par calcul numérique, et comme les théories générales sont rares l'étude est restreinte à un faible nombre d'équations représentatives de l'ensemble. Alors que l'analyse linéaire a connu un développement profond au cours du XX^{e} siècle, l'analyse non linéaire a connu un développement majeur plus récemment. Ses thématiques principales pour les équations d'évolution sont entre autres la meilleure compréhension des modèles de mécanique des fluides, des équations cinétiques, des équations de réaction-diffusion, des ondes, de la relativité générale, de la théorie des champs et des flots géométriques. Les questions centrales sont les suivantes. Peut-on trouver des propriétés universelles dans le comportement des solutions de ces équations? Peut-on fournir un cadre adapté à l'étude et l'explication des phénomènes non linéaires?

De nombreuses stratégies ont été adoptées pour répondre à ces questions. En premier lieu se place l'étude de situations faiblement non-linéaires. Par exemple, l'étude de la dynamique de la perturbation d'une solution spéciale, faisant intervenir des termes principaux obtenus par linéarisation et des termes non-linéaires d'ordre inférieur. Pour aborder des problèmes plus fortement non-linéaires, on peut se placer dans des cas particuliers où l'on dispose d'un cadre fonctionnel permettant de prendre en compte tous les effets de la dynamique ensemble. Par exemple, lorsque les effets non-linéaires peuvent être traités dans le même cadre que les effets linéaires, et où l'on doit alors comprendre les interactions entre plusieurs solutions linéaires. Récemment cette étude des résonances a permis par exemple une meilleure compréhension de la turbulence et de la répartition de l'énergie entre les différentes échelles spatiales pour l'équation de Schrödinger non linéaire, voir [15, 22, 70] et les références mentionnées dans ces travaux. Un deuxième exemple est l'étude d'équations dont la dynamique possède une formulation géométrique particulière dans l'espace des phases dont on peut tirer de nombreuses informations, notamment les équations hamiltoniennes intégrables et leurs perturbations. Le cas de l'équation de Korteweg-de Vries est important car il a été démontré que toute solution se décompose pour des temps grands en une somme de vagues, voir le livre [1]. C'est ce genre de résultats, pour des équations ne possédant pas une telle structure, qui nous intéresse ici. Nous allons en effet nous pencher sur la dernière catégorie de problèmes, celle de ceux qui ne tombent pas dans les deux cas précédents. Les effets non linéaires sont forts, et il n'y a pas a priori de cadre fonctionnel permettant une réduction satisfaisante.

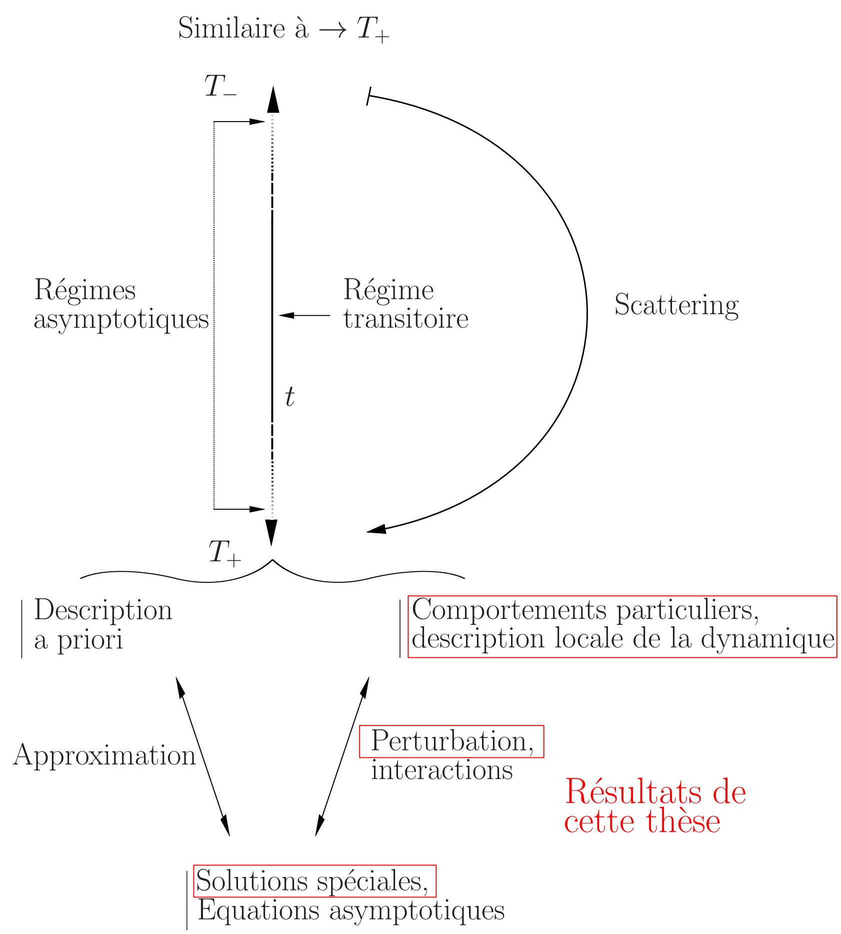
Dans cette dernière classe de problèmes, certains en particulier ont reçu une large attention ces dernières années et possèdent les propriétés suivantes. Il existe une rigidité autour des solutions gardant une certaine cohérence, elles doivent être des solutions spéciales de l'équation : ondes solitaires, solutions périodiques, états stationnaires, solutions autosimilaires etc., pour l'équation originale ou bien pour des équations asymptotiques. Si une solution ne reste pas cohérente, les effets nonlinéaires s'affaiblissent et elle suit alors une dynamique linéaire au voisinage de la solution nulle. Parmi les équations célèbres dans cette catégorie on compte les équations de Schrödinger, de la chaleur et des ondes non linéaires, les équations d'ondes géométriques telles les Schrödinger et wave maps, le flot de la chaleur harmonique, des variantes de l'équation de Korteweg-de Vries, les flots de Ricci, de la courbure moyenne, de Yamabe etc.. Voici la stratégie pour leur étude qui a émergé depuis la deuxième moitié du XX$^{\text{e}}$ siècle.

En notant (T_-, T_+) l'intervalle de temps maximal pour l'existence d'une solution, on classifie en premier séparément les divers comportements asymptotiques près de T_- et T_+ (l'étude près de T_- est en général surtout faite pour des équations réversibles). Dans la plupart des cas c'est un problème ardu à grandes données et fortement non linéaire. Différentes classes de solutions spéciales apparaissent alors pour décrire l'asymptotique de toute solution. L'étape suivante est donc l'étude de ces solutions spéciales. Elles sont souvent la solution d'équations de forme plus simple (mais dont l'étude peut s'avérer tout aussi compliquée), et dont l'analyse nécessite l'aide du calcul des variations et de la théorie elliptique par exemple. Puis, on étudie la dynamique des solutions de l'équation originale au voisinage de ces solutions spéciales, par la perturbation d'une ou l'interaction de plusieurs. Cela permet parfois de construire des exemples particuliers de comportement, et dans le meilleur des cas de décrire complètement le flot au voisinage de ces configurations. Le but ultime est de combiner l'analyse a priori d'une solution générale, ramenant à un cadre perturbatif de ces configurations, à l'analyse précise de ces dernières afin de décrire tous les comportements asymptotiques possibles.

Quand cette situation asymptotique en T_- et T_+ est clarifiée, on s'intéresse alors aux connexions possibles entre les comportements en T_- et ceux en T_+, c'est le scattering au sens que lui donnent les physiciens. Là encore, on peut commencer avec l'étude de cas particulier, et le but final est une description complète de toutes les connexions possibles. Pour finir, on étudie le comportement d'une solution pour des temps intermédiaires, et l'on cherche à décrire ce régime transitoire.

Toutes ces différentes étapes de l'étude du comportement des solutions de l'équation considérée peuvent être réalisées d'une manière relativement indépendante. Parfois, une compréhension générale peut émerger de différents résultats montrés pour des équations appartenant à une même classe, et non nécessairement pour une équation bien précise. On peut également restreindre l'étude, en ne considérant qu'une classe de solutions en particulier (celle des solutions explosives que l'on va décrire plus bas, celle des solutions bornées etc.), ou en quittant le cadre déterministe et en s'intéressant au comportement générique des solutions. Cette dernière approche a reçu un développement conséquent récemment, voir [13, 14] par exemple.

Quand $T_+ \neq +\infty$, la solution est dite explosive. Il se passe alors un évènement pour des temps proches de T_+ qui empêche la solution d'être étendue après ce temps maximal. Ce phénomène peut avoir deux interprétations. S'il existe sans réelle signification physique, il peut être vu comme un défaut du modèle, celui-ci étant peut-être trop simple, et cela est alors en lien avec l'instabilité des schémas numériques correspondants. Il peut également avoir un intérêt physique (concentration en temps fini d'un ensemble de particules, formation de chocs etc.). Aussi, l'on peut penser qu'il existe une certaine rigidité autour des explosions et que, pour des temps proches de T_+, près de la singularité, la solution ne dépend plus vraiment de la donnée initiale et qu'un mécanisme universel est déclenché. La classe des solutions explosives est par conséquent un cas modèle pour la mise en place de la stratégie d'étude décrite plus haut.



Pendant son doctorat, l'auteur de la présente thèse a considéré deux équations d'évolution non linéaires canoniques, et a étudié la dynamique près de solutions particulières rentrant dans la description asymptotique universelle de solutions générales. Les résultats apportés concernent la construction de scénarios explosifs précis obtenus par concentration d'états stationnaires par instabilité d'échelle, ainsi que la description de la dynamique au voisinage d'états stationnaires et de solutions autosimilaires explosives.

0.2 Résumé des résultats obtenus

Nous nous tournons maintenant vers la présentation des travaux réalisés par l'auteur et ses collaborateurs.

0.2.1 Présentation succinte des équations étudiées

Les travaux de cette thèse portent sur les deux équations suivantes. La première est l'équation des ondes semi-linéaire considérée sur l'espace entier

$$(NLW) \quad \begin{cases} u_{tt} = \Delta u + |u|^{p-1}u, \\ u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), \end{cases} \quad (t,x) \in I \times \mathbb{R}^d,$$

où $I \subset \mathbb{R}$ est un intervalle, $d \in \mathbb{N}^*$ est la dimension et $\Delta = \sum_1^d \frac{\partial^2}{\partial x_i^2}$ est le Laplacien. La seconde est l'équation de la chaleur semi-linéaire sur l'espace entier

$$(NLH) \quad \begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \quad (t,x) \in I \times \mathbb{R}^d,$$

et que l'on considérera également parfois posée sur un domaine lisse et borné $\Omega \subset \mathbb{R}^d$

$$(NLH\Omega) \quad \begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u(0,x) = u_0(x), \\ u(t,\cdot) = 0 \ \text{ sur } \partial\Omega, \end{cases} \quad (t,x) \in I \times \mathbb{R}^d,$$

auquel cas est ajoutée la condition au bord de Dirichlet. Dans les deux cas les solutions u sont à valeurs réelles. L'équation des ondes linéaire est la première EDP étudiée par d'Alembert en 1747, et celle de la chaleur linéaire est introduite par Fourier en 1811. Elles modélisent la propagation d'ondes (en électromagnétisme et en acoustique par exemple) pour la première, et des phénomènes diffusifs (diffusion de particules, de température par exemple) pour la seconde. La modélisation précise de ces phénomènes requiert cependant la prise en compte d'effets non linéaires, et (NLW) et (NLH) apparaissent ainsi comme des versions non-linéaires canoniques. Ces deux équations ont attiré l'attention de nombreux mathématiciens et physiciens. Le problème de Cauchy pour des solutions peu régulières a été étudié à partir de la seconde moitié du XX^e siècle, et nous renvoyons aux articles [16, 66, 94, 151, 155] et aux livres [137, 148, 149, 152] pour les questions relatives à l'existence de solutions localement en temps.

Ces équations sont fortement non linéaires car sans la partie linéaire, elles se réduisent toutes deux à des équations différentielles ordinaires considérées point par point, $u_t = |u|^{p-1}u$ et $u_{tt} = |u|^{p-1}u$, pour lesquelles génériquement les solutions tendent vers l'infini en temps fini. Ces deux équations semi-linéaires ont donc été étudiées en tant qu'équations modèles pour l'étude de la formation de singularité pour des équations hyperboliques et dispersives pour (NLW), et pour des équations paraboliques pour (NLH) (en particulier l'équation de Navier-Stokes). Les travaux pionniers concernant l'étude des solutions explosives sont ceux de Fujita, Kaplan et Keller dans les années soixante, puis ceux de Ball, Glassey, John et Levine dans les années soixante-dix pour en citer quelques uns. Depuis les années quatre-vingt, des résultats précis concernant la description a priori de solutions explosives ou globales, ainsi que des dynamiques particulières dans ces deux cas ont été obtenus pour les deux équations et ce document se place dans la lignée de ces travaux.

Ces deux équations possèdent les invariances suivantes. Si $u(t,x)$ est une solution, alors pour toute échelle $\lambda > 0$,

$$\frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right) \quad \text{et} \quad \frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

sont également des solutions de (NLW) et de (NLH) respectivement. Egalement, ces deux équations sont invariantes par translation en espace et étant donné un point $x_0 \in \mathbb{R}$, $u(t, x - x_0)$ est une solution pour les deux équations. Il existe alors deux classes de solutions spéciales, dont l'orbite est contenue dans la classe d'équivalence de la donnée initiale par l'action du groupe de changement d'échelle. La première est celle des états stationnaires

$$u(t,x) = Q(x), \quad \Delta Q + |Q|^{p-1} Q = 0$$

qui sont les mêmes pour (NLH) et (NLW), et la seconde celle des profils autosimilaires explosifs

$$u(t,x) = \frac{1}{(T-t)^{\frac{1}{p-1}}} \psi\left(\frac{x}{\sqrt{T-t}}\right) \quad \text{et} \quad u(t,x) = \frac{1}{(T-t)^{\frac{2}{p-1}}} \psi\left(\frac{x}{T-t}\right)$$

pour (NLH) et (NLW) respectivement, où $T > 0$ est le temps d'explosion (nous notons T_+ par T pour plus de simplicité dorénavant) et où ψ est la solution d'une équation elliptique qui diffère selon l'équation d'évolution considérée. Pour tout exposant $p > 1$, il existe par exemple pour les deux équations une solution autosimilaire explosive constante en espace (où κ_H et κ_W sont des constantes ne dépendant que de d et p)

$$u(t,x) = \kappa_H (T-t)^{-\frac{1}{p-1}} \quad \text{et} \quad u(t,x) = \kappa_W (T-t)^{-\frac{2}{p-1}}.$$

Pour autant, tous les mécanismes explosifs ne dérivent pas de cette explosion liée à l'équation différentielle ordinaire sous-jacente. Il en existe d'autres, dont l'étude est le point de départ de cette thèse. Pour finir avec cette présentation rapide, mentionnons que chacun des deux problèmes possède une structure géométrique particulière. Les deux quantités

$$\frac{1}{2}\int |\nabla u|^2 + u_t^2 - \frac{1}{p+1}\int |u|^{p+1} \quad \text{et} \quad \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1}$$

ont un rôle spécial pour (NLW) et (NLH) respectivement. Dans le premier cas, c'est un hamiltonien qui est donc conservé, et dans le deuxième c'est une fonctionnelle pour laquelle (NLH) est une descente de gradient. Le contrôle local de l'énergie cinétique (à gauche) sur l'énergie d'interaction (à droite après le $-$) conduit alors à la classification suivante des problèmes :

(i) Si $1 < p < p_c$ le problème est dit énergie sous-critique. Ici $p_c = 1 + \frac{4}{d-2}$ pour $d \geq 3$, et $p_c = +\infty$ pour $d = 1, 2$.

(ii) Si $p = p_c$ le problème est dit énergie critique.

(iii) Si $p > p_c$ le problème est dit énergie surcritique.

0.2.2 Résultats antérieurs

Conformément à la stratégie d'étude présentée précédemment, nous nous intéressons à l'existence d'une description asymptotique universelle de toute solution par des fonctions spéciales, ainsi qu'à la dynamique au voisinage de telles solutions et à leurs interactions. Il se trouve que pour l'asymptotique près d'une singularité pour la classe des fonctions explosives, ces fonctions spéciales sont précisement les états stationnaires et les profils autosimilaires que nous venons de décrire. Nous renvoyons à [43, 44, 46, 47, 50] et [97, 98] pour les cas particuliers où cela a été montré dans les régimes énergie critique et surcritique. Cela signifie que toute solution qui explose en temps fini ressemble à ces solutions spéciales, à renormalisation près, au voisinage de la singularité. Cette propriété des équations est principalement due à la présence d'invariances pour l'équation, notamment par changement d'échelle et translation en espace, ainsi qu'à l'existence d'une structure géométrique (hamiltonienne et descente de gradient) ; une présentation plus éclairante est donnée Section 1.5 (en langue anglaise). Dans le cas énergie sous-critique des résultats encore plus aboutis ont été montrés.

Plus précisemment, toutes les explosions dans le régime énergie sous-critique sont décrites par des profils autosimilaires explosifs. Pour (NLH), il a été montré dans la série de travaux [60, 61, 62, 63, 64, 116, 117] que la seule solution autosimilaire est celle constante en espace, et qu'elle décrit au premier ordre toute solution explosive près de la singularité. Pour (NLW), un résultat complet semblable a été obtenu en dimension un dans la série de travaux [118, 119, 121, 122, 32]. En plus grande dimension, des résultats partiels ont été obtenus, et nous renvoyons à [124] et aux références de ce papier.

Le régime sous-critique a été étudié en premier, car la situation est relativement rigide comme nous venons de le mentionner, mais également car plus d'outils techniques sont disponibles dans cette situation. L'étude des régime critiques a démarré par la suite, et celle des régimes surcritiques est encore balbutiante. Dans ces deux derniers régimes, la situation s'enrichit. Tout d'abord, de nouvelles solutions autosimilaires non constantes en espace pour la chaleur existent. Leur étude débute également dans les années quatre-vingt, car ce genre de dynamique est relativement répandu et déjá étudié de manière intensive au XX^{e} siècle en physique mathématique. Une famille dénombrable de solutions lisses, décroissantes à l'infini et radiales est exhibée dans [18, 19, 153]. Il est alors intéressant de savoir si elles peuvent émerger comme profils à l'explosion pour des solutions bien localisées, ce qui est montré par des arguments non constructifs dans [99], et d'étudier leur stabilité, ce qui était un problème ouvert. Notons que la stabilité de la solution explosive autosimilaire constante en espace a été l'objet de nombreux travaux [38, 40, 41, 99, 115], et que des résultats pour des solutions autosimilaires non constantes de l'équation des ondes sont donnés dans [11, 12].

De plus, des états stationnaires apparaissent. Dans le cas radial, tous sont obtenus à partir d'un profil unique à changement d'échelle près, et sont parfois appelés états fondamentaux ou solitons. L'asymptotique de ce dernier est différente selon la valeur de p. Pour $p = p_c$ c'est le profil de Talenti-Aubin

[4, 150], extrémiseur de l'injection de Sobolev de l'espace $\dot{H}^1$ dans l'espace de Lebesgue $L^{\frac{2d}{d-2}}$

$$Q(x) = Q(|x|), \quad Q(r) := \frac{1}{\left(1+\frac{r^2}{d(d-2)}\right)^{\frac{d-2}{2}}}, \quad r = |x|$$

unique à symétrie près parmi les états stationnaires positifs [58, 59]. Ensuite, il existe une valeur particulière de p, l'exposant de Joseph-Lundgren [77]

$$p_{JL} := 1 + \frac{4}{d-4-2\sqrt{d-1}} > p_c, \quad (p_{JL} := +\infty \text{ pour } 1 \leq d \leq 10)$$

telle que pour $p \in (p_c, p_{JL})$ l'état stationnaire oscille à l'infini autour de la solution stationnaire homogène $c_\infty r^{-\frac{2}{p-1}}$,

$$Q(r) = \frac{c_\infty}{r^{\frac{2}{p-1}}} + \frac{a_1 \sin(\omega \log(r) + c)}{|x|^{\frac{d-2}{2}}} + o\left(\frac{1}{r^{\frac{d-2}{2}}}\right) \text{ lorsque } r \to +\infty, \quad a_1 \neq 0, \; c \in \mathbb{R}.$$

Ici $\frac{d-2}{2} > \frac{2}{p-1}$ puisque l'on est dans le régime surcritique, et les constantes sont données par

$$c_\infty := \left[\frac{2}{p-1}\left(d-2-\frac{2}{p-1}\right)\right]^{\frac{1}{p-1}}, \quad \omega := \sqrt{-\triangle} \text{ et } \triangle := (d-2)^2 - 4pc_\infty^{p-1}$$

($\triangle < 0$ dans ce cas). Pour $p > p_{JL}$, l'on a $\triangle > 0$ et les oscillations cessent

$$Q(r) = \frac{c_\infty}{r^{\frac{2}{p-1}}} + \frac{a_2}{r^\gamma} + o(r^{-\gamma}) \text{ lorsque } r \to +\infty, \; a_2 \neq 0,$$

où $\gamma := \frac{1}{2}(d-2-\sqrt{\triangle})$. Si nous donnons une description détaillée de ces asymptotiques à l'infini, c'est car celles-ci sont directement liées aux propriétés de la dynamique des solutions près de Q comme nous allons le voir, et nous aurons besoin de cette numérologie. Mentionnons que la description des états stationnaires non radiaux est, malheureusement, encore trop pauvre, avec quelques résultats en énergie critique [34, 35] et quasiment aucun en surcritique.

La concentration de l'état fondamental Q en temps fini T

$$u = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + \varepsilon, \quad \text{avec } \lambda(t) \to 0 \text{ lorsque } t \to T$$

par instabilité d'échelle permet de générer des explosions plus lentes que celles autosimilaires. Ce nouveau mécanisme explosif est plus délicat à étudier car il repose sur l'existence de perturbations spécifiques de Q amenant l'échelle à se concentrer en temps fini. Ces explosions sont lentes car $\lambda(t)(T-t)^{-\frac{1}{2}} \to 0$ où $(T-t)^{\frac{1}{2}}$ est la vitesse de diffusion apparaissant pour les explosions autosimilaires. Pour l'équation de la chaleur, de telles dynamiques avec un nombre dénombrable de vitesses possibles pour $\lambda(t)$ sont obtenues dans le cas radial. Leur existence est d'abord établie formellement pour $p = p_c$ en petites dimensions et pour $p > p_{JL}$ en toute dimension dans les années quatre-vingt dix par raccordement asymptotique [55, 72]. La preuve rigoureuse [73] des auteurs n'est jamais publiée, et il faut attendre [125] pour un résultat d'existence basé sur une stratégie similaire en surcritique, et [147] en énergie critique (basé sur les travaux dispersifs décrits au prochain paragraphe). Pour $p \in (p_c, p_{JL})$ de telles solutions n'existent pas [97, 131], ce qui montre l'influence dramatique de l'asymptotique du soliton. Enfin, pour $p > p_{JL}$ ces mécanismes d'explosions lentes sont les seuls possibles dans le cas radial [96, 130, 128].

Pour l'équation des ondes semi-linéaires, ce mécanisme de concentration est étudié plus tardivement et pour les problèmes critiques, au début du XXI^{e} siècle, en lien avec l'étude de l'effondrement du soliton pour l'équation de Schrödinger non linéaire [109, 110, 111, 112, 113] puis les travaux pionniers sur les équations géométriques [88, 144] et [114, 138]. Les travaux [74, 89] construisent pour (NLW) des perturbations du soliton conduisant à sa concentration en temps fini mais cette fois-ci la loi de la vitesse de l'échelle peut appartenir à un continuum pour des solutions peu régulières. Cette différence fondamentale est due à l'effet régularisant de l'équation de la chaleur à l'oeuvre pour (NLH) alors que l'équation des ondes propage les singularités.

Dans le même temps, l'intérêt pour l'étude complète de toutes les dynamiques possibles près de Q, et pas seulement celles de concentration, s'accroît. Pour les problèmes critiques, l'étude de sa stabilité à invariances près permet la mise au jour d'une variété centrale [133, 146], et des dynamiques génériques pour lesquelles la solution quitte le voisinage de Q sont mises en évidence [51, 79]. Cela aboutit au résultat de classification des dynamiques possibles près de Q pour les ondes dans [84]. Un résultat de classification est obtenu pour le flot de la chaleur harmonique dans [68], et pour l'équation de Korteweg-de Vries modifiée dans [102, 103, 104, 105]. Deux points importants sont à noter. Lorsqu'il n'y a pas stabilité orbitale du soliton, comme pour l'équation des ondes et celle de Korteweg-de Vries modifiée, le comportement asymptotique précis des solutions quittant un voisinage de Q est en partie méconnu car cela n'entre plus dans le régime perturbatif. Quand il y a une instabilité d'échelle forte comme pour les ondes, la classification de la dynamique précise des solutions restant proche du soliton à invariance près est également ouverte.

0.2.3 Résultats obtenus par l'auteur

Les résultats sont présentés ici d'une manière quelque peu grossière, et nous renvoyons aux Théorèmes 2.2.4, 2.2.5, 2.2.9, 2.3.4, 2.3.5 et 2.4.4 pour un énoncé détaillé (en langue anglaise). Ceux-ci portent sur l'étude dans les régimes énergie critique et surcritique de la dynamique près de solutions stationnaires et autosimilaires.

Le premier résultat concerne l'existence de solutions lisses concentrant l'état stationnaire radial avec une famille dénombrable de vitesses possibles dans le régime surcritique. Ce travail étend l'analyse des problèmes critiques [138, 141], en correspondance avec les résultats [72, 125] et a été réalisé en parallèle de [114] pour l'équation de Schrödinger surcritique. Une extension de la méthode de construction utilisée fut la preuve que les solutions ainsi construites appartiennent à une variété Lipschitz de codimension explicite.

Theorem 0.2.1 (Explosion lente pour (NLW) en surcritique [23]). *Soit $d \geq 11$ et $p > p_{JL}$. Alors il existe une famille dénombrable de vitesses $(c_\ell)_{\ell \geq \ell_0}$ avec $c_\ell > 1$ et $c_\ell \to +\infty$, et des solutions u_ℓ de (NLW)*

radiales, lisses et à support compact explosant en temps fini par concentration de l'état fondamental

$$u_\ell(t,x) \sim \frac{1}{\lambda_\ell(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda_\ell(t)}\right), \quad \lambda_\ell(t) \sim (T-t)^{c_\ell}.$$

De plus, pour chaque ℓ, il existe une variété Lipschitz (dans un espace des phases approprié) de codimension $\ell - 1 \geq 2$ de solutions explosant selon ce scenario.

La stabilité Lipschitz avec codimension $\ell-1$ signifie ce qui suit. Pour tout ℓ, il existe $\ell-1$ profils instables ψ_j, une décomposition de l'espace des phases près de $u_\ell(0)$, $\text{Vect}(\psi_j)_{1\leq j\leq \ell-1} \oplus \text{Vect}(\psi_j)^\perp_{1\leq j\leq \ell-1}$, et des fonctions Lipschitz à valeurs rélles $a_j(\varepsilon)$ sur $\text{Vect}(\psi_j)^\perp_{1\leq j\leq \ell-1}$ vérifiant ce qui suit. Si l'on perturbe u_ℓ correctement avec une donnée initiale de la forme

$$u(0) = u_\ell(0) + \sum_1^{\ell-1} a_j(\varepsilon)\psi_j + \varepsilon, \quad \varepsilon \in \text{Vect}(\psi_j)^\perp_{1\leq j\leq \ell-1}$$

alors la solution va rester proche de u_ℓ et adopter le même comportement. Si initialement la solution ne s'écrit pas sous cette forme, alors elle va quitter un voisinage de u_ℓ en temps fini (mais son devenir est inconnu). Un deuxième travail est l'extension au cadre non radial des méthodes sus-mentionnées pour l'équation de la chaleur, étendant les résultats de [72, 125]. Il est à mentionner que l'approche précédemment considérée dans [72, 125] repose pour beaucoup sur l'utilisation de techniques paraboliques radiales telles le comptage précis des points d'intersections, et ces méthodes semblent difficiles à implémenter dans le cas non radial. Les travaux similaires en dispersif décrits plus haut ont également été réalisés dans le cas radial, et la concentration non radiale du soliton n'est étudiée que dans un cas stable pour Schrödinger [136].

Theorem 0.2.2 (Explosions de type II non radiales pour (NLH) surcritique [26]). *Soient $d \geq 11$, un domaine lisse et borné $\Omega \subset \mathbb{R}^d$ et $p > p_{JL}$. Alors il existe une famille dénombrable de vitesses $(c_\ell)_{\ell \geq \ell_0'}$ avec $c_\ell > \frac{1}{2}$ et $c_\ell \to +\infty$ et des solutions u_ℓ explosant en temps fini par concentration de l'état fondamental en un point $x_0 = x_0(\ell) \in \Omega$*

$$u_\ell(t,x) \sim \frac{1}{\lambda_\ell(t)^{\frac{2}{p-1}}} Q\left(\frac{x-x_0}{\lambda_\ell(t)}\right), \quad \lambda_\ell(t) \sim (T-t)^{c_\ell}.$$

La principale nouveauté dans le travail ci dessus est l'extension des méthodes de construction surcritiques [23, 114], avec la prise en compte de perturbations localisées en harmoniques sphériques, ainsi que le contrôle des effets de bord. Dans un troisième travail l'auteur, en collaboration avec P. Raphaël et F. Merle, obtient la classification complète de la dynamique de (NLH) non radiale près du soliton dans le cas énergie critique et en grande dimension. L'instabilité d'échelle pour des perturbations petites dans la norme critique disparaît, contrairement à la petite dimension. Les mécanismes de concentration du soliton, s'ils existent encore, doivent donc être essentiellement différents.

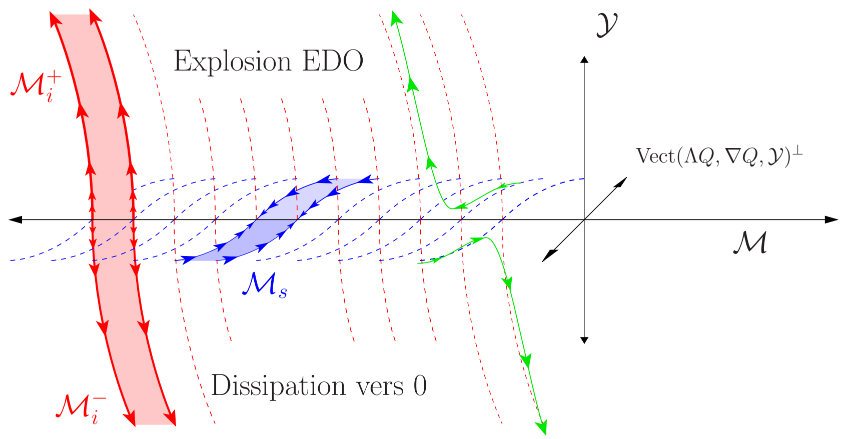
Theorem 0.2.3 (Dynamique de (NLH) critique près de l'état fondamental $d \geq 7$ [24, 25]). *Supposons $d \geq 7$ et $p = p_c$.*

(i) Si une solution est initialement proche de Q dans la norme critique $\dot{H}^1$, alors soit celle-ci explose en temps fini avec pour profil à l'explosion la solution autosimilaire constante en espace, soit celle-ci est globale

et est dissipée vers 0*, soit celle-ci est globale et converge vers* $\lambda_\infty^{2/(p-1)}Q(\lambda_\infty(x-x_\infty))$ *un autre état fondamental . Les solutions associées à ce dernier scénario forment une hypersurface dans* $\dot{H}^1$ *séparant les deux premiers comportements.*

(ii) Si une solution est globale en arrière et reste proche de $(\lambda^{2/(p-1)}Q(\lambda(x-y)))_{\lambda,y}$ *l'ensemble des états fondamentaux, alors soit elle est un état fondamental, soit elle appartient à la variété instable. La variété instable est de dimension* $d+2$*, et possède deux composantes connexes; sur l'une les solutions sont dissipées vers* 0*, sur l'autre les solutions explosent avec la solution autosimilaire constante en espace comme profil à l'explosion.*

L'étape (ii) est en fait fondamentale pour obtenir (i). La variété instable est l'attracteur de toutes les solutions qui quittent un voisinage de Q. Les solutions sur cette variété instable sont des éléments minimaux Q^+ et Q^- dont les propriétés peuvent être obtenues par des arguments paraboliques. Un argument clé dans la preuve est donc le fait que leurs comportements asymptotiques, la dissipation vers 0 et l'explosion avec la solution autosimilaire constante en espace comme profil à l'explosion, sont tous les deux des dynamiques stables.



$$
\begin{aligned}
&\mathcal{M} = \{\lambda^{\frac{2}{p-1}}Q(\lambda(x-z))\}\\
&\mathcal{M}_i^+ = \{\lambda^{\frac{2}{p-1}}Q^+(\lambda^2 t, \lambda(x-z))\}\\
&\mathcal{M}_i^- = \{\lambda^{\frac{2}{p-1}}Q^-(\lambda^2 t, \lambda(x-z))\}\\
&\mathcal{M}_s = \{u, \quad u(t) \to \tilde{Q} \in \mathcal{M} \ \text{ as } t \to +\infty\}
\end{aligned}
$$

Des travaux précédents utilisant cette approche sont [51, 103]; les comportements asymptotiques sur la variété instable ainsi que leur stabilité sont montrés par des techniques paraboliques dans l'esprit de [54, 97]. Dans un quatrième travail, en collaboration avec P. Raphaël et J. Szeftel, l'auteur a donné une construction rigoureuse par raccordement asymptotique des solutions radiales autosimilaires non constantes de [18, 19, 153] en surcritique. Cette technique classique, voir par exemple [8], combinée à des arguments de Sturm-Liouville, permet notamment d'expliciter une structure spectrale puissante dans la zone autosimilaire près de l'explosion (en lien avec [69]), à partir de laquelle sont dérivés des résultats de stabilisation non radiale. En particulier, ces profils peuvent bel et bien apparaître comme profil à l'explosion pour des solutions bien localisées.

Theorem 0.2.4 (Stabilité de solutions autosimilaires excitées pour (NLH) surcritique [27]). *Supposons $d = 3$ et $p > 5$. Il existe $N \in \mathbb{N}$, $N \gg 1$, une famille dénombrable de solutions radiales autosimilaires explosives $(\Phi_n)_{n \geq N}$ et pour chaque $n \geq N$ une variété Lipschitz (dans un espace approprié) de codimension n de solutions explosant par concentration de Φ_n :*

$$u(t,x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} \Phi_n \left(\frac{x - x_0}{\lambda(t)} \right), \quad \lambda(t) \sim \sqrt{T - t}.$$

1

Overview on the asymptotic behavior during singularity formation

The aim of this first chapter is to provide a general introduction to the research area of the author, to the strategy of study that has emerged over the past years, and to the equations at stake. The results obtained during the PhD are then presented in this broad context. Details are avoided and relegated to a more refined presentation in the next chapters.

1.1 General introduction

Evolution partial differential equations (PDEs) model the evolution over time of several quantities defined on a continuum for problems arising from natural sciences or other areas of mathematics. Since the beginning of their study in the eighteenth century, based on the differential calculus that has started the century before, they have been intensively studied. With the expansion of computer sciences, limits of certain aspects of modeling have been repelled beyond that of the human mind. The PDE chain has now three links: natural sciences, numerical calculus and theoretical analysis.

This latter domain has changed over the decades. Early works were often on reasonable equations in the hope of finding formulas to compute solutions by hand. Now, the main issue behind the study of the existence of solutions is more that of the stability, be it for the convergence of numerical scheme or for the finding of topological spaces for weak solutions. For many equations, at least a rough answer to the existence of solutions is available, and it is then the knowledge of their properties that is challenging. The emphasis is currently placed on the qualitative description for either canonical equations or the fundamental ones in natural sciences. This restriction has the following grounds: the quantitative description is most of the time done by numerical simulations, and the low number of equations that are studied is due to the fact that no general approach is available yet. While linear analysis has seen a huge development, nonlinear analysis is a more recent field. Its main concerns for evolution problems are a better theoretical understanding of fluid mechanics, kinetics, reaction-diffusion, waves, general relativity, field theory and geometrical flows to name a few. The central questions are: can one finds universal features for the behavior of the solutions to these equations? Can one provide a insightful framework for the explanation of nonlinear effects?

Many strategies have been adopted to give an answer. First, the investigation of weakly nonlinear cases. For example, near special solutions the dynamics of a perturbation involves a linearized evolution and small nonlinear terms. Second, the situations of strong nonlinearity where there exists an a priori functional framework to catch the various effects of the dynamics altogether. For example, if the nonlinear effects can be incorporated in the same framework as the linear dynamics, and where one then has to understand the interaction of particular linear solutions. Recently this study of resonances has for example given insights on turbulence and on the energy distribution between different spatial scales for the nonlinear Schrödinger equation, see [15, 22, 70] and references therein. As a second example there are cases where the evolution admits a nonlinear geometrical formulation in the phase space that is very descriptive, among them integrable hamiltonian flows and their perturbations. The case of the Korteweg-de Vries equation is notable: any solution decomposes as a sum of waves for large times, see [1] for a review. The third type of configuration are the others, numerous, for which nonlinear effects are strong and there is no a priori insightful functional framework for the study of solutions.

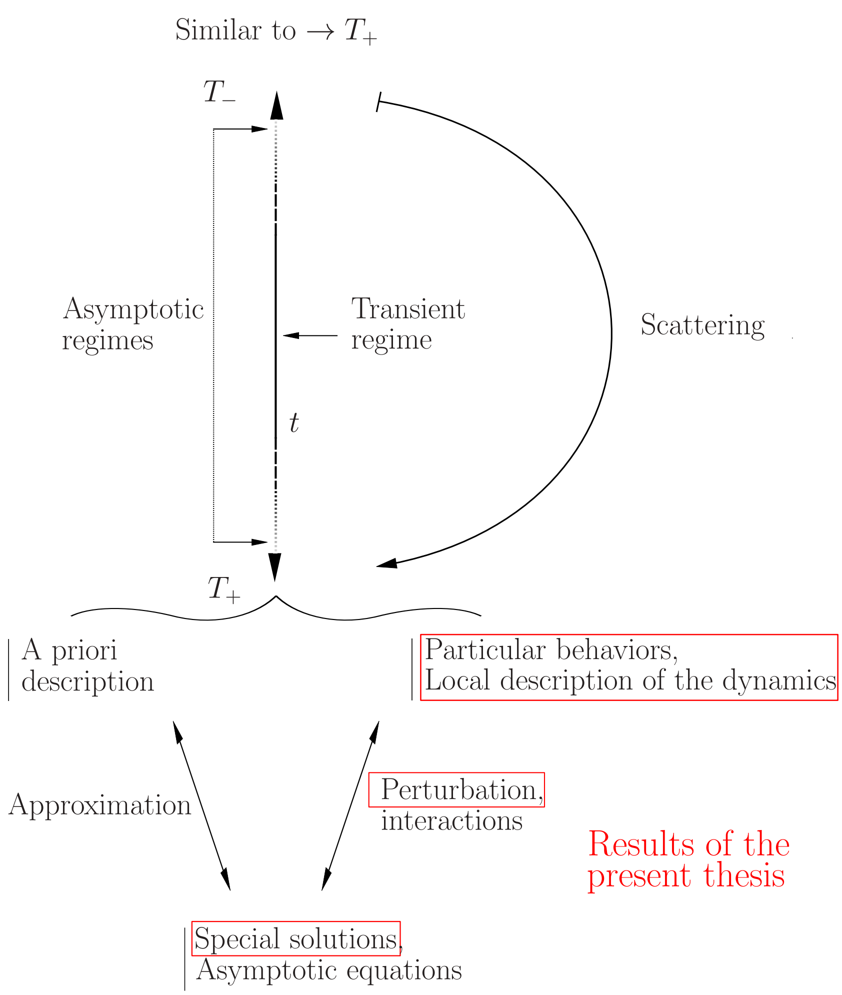
In this document we focus on this last class of problems. Some in particular have the following properties. There is rigidity for solutions staying coherent, they have to be special solutions: traveling, periodic or stationary solutions, breathers or self-similar solutions for example, for the equation or for an asymptotic equation in a special regime. If a solution does not stay coherent, then nonlinear effects are weak and the solution obeys a linear dynamics near the 0 solution. Some famous equations entering this framework are some nonlinear Schrödinger, wave and heat equations, the Schrödinger and wave maps equations, the harmonic heat flow, variants of the Korteweg-de Vries equations, the Ricci, mean curvature and Yamabe flows etc.. We now describe the strategy that has emerged in the second half of the twentieth century to study such situations.

Denoting by (T_-,T_+) the maximal interval of time for which a solution is defined, one classifies a priori the possible asymptotic behaviors near T_+ and T_- separately (the asymptotic study near T_- is mostly done for reversible equations). In most cases this is a highly non-linear large data problem. Different classes of special solutions describing the asymptotic behavior are then identified. The next step is to describe these special solutions. Usually they are solutions of simpler yet challenging equations, requiring the help of elliptic theory and calculus of variations for example. One then study the flow of the original equation near these special solutions, either by perturbation of one or by interaction of several ones. This allows to build particular examples of various behaviors, and in the best cases to describe completely the flow locally near such configurations in the phase space. The final goal is to combine the a priori classification results to the study of these configurations of special solutions, in order to recover the asymptotic behavior of general solutions.

When this picture near T_+ and T_- is clarified, one investigate how solutions connect an asymptotic configuration near T_- to another near T_+: this is scattering (in the physical sense). Again, one can start with particular examples and the final aim is to have the full description of possible connections. Eventually, one studies the behavior of a solution for intermediate times; this is the transient regime.

All these different steps in the study of the equation can be done separately. Sometimes, a general understanding can emerge from a class of equations even if all results are not proven in the very same case. One can also ease the study: by restricting the attention to particular classes of solutions (belonging to a particular phase space, or bounded or localized in some sense), or by leaving the deterministic case and studying the behavior of generic solutions, at the interface with probability theory. This last approach started recently, see [13, 14] for example.

When $T_+\neq+\infty$, the solution is said to blow up in finite time. Blow-up means that a singularity forms that prevents the solution to be extended beyond T_+. This phenomenon can have two interpretations. If it arises in the modeling of a physical phenomenon for which it has no meaning, it signifies that the model is oversimplified; this is linked with the instability of numerical schemes that one could employ. It can also be of physical relevance (finite time collapse of a star, shocks etc.). One can believe that there is rigidity around the mechanisms responsible for blow-up, and that as $t \to T_+$ the solution does really depend on the initial datum anymore. In consequence, the class of blow-up solutions seems to be an interesting particular class of solutions to focus on to employ the above strategy.



During his PhD, the author investigated for some canonical nonlinear evolution equations the dynamics near special solutions entering in the general asymptotic description of arbitrary solutions. The results obtained deal with the construction of particular blow-up behaviors caused by the concentration of stationary states, and the local description of the dynamics near stationary states and backward self-similar solutions.

1.2 The semilinear heat and wave equations

The present work deals with the following canonical nonlinear evolution equations for a monomial nonlinearity of degree $p>1$. The semilinear heat equation (where $\Delta=\sum_{i=1}^d \partial/\partial x_i^2$ is the standard Laplacian)

$$(NLH)\ \ u_t=\Delta u+|u|^{p-1}u$$

is a model parabolic equation. The second is the semilinear wave equation, a model dispersive equation,

$$(NLW)\ \ u_{tt}=\Delta u+|u|^{p-1}u.$$

In each cases the unknown function is real valued $u(x,t)\in\mathbb{R}$ but it could be complex or vector valued. The underlying linear equation for (NLH) is the linear heat equation that models a diffusion process: thermal or particles diffusion, spreading of a species for example. For (NLW) it is the linear wave equation which arises for example in acoustics and electromagnetics. A refined description of such physical phenomena, however, always involves variants of these equations which include nonlinear effects. A great amount of work has then been devoted to the study of the canonical nonlinear versions that are (NLH) and (NLW) where the nonlinearity is a monomial.

The semilinear wave and heat equations have been a subject of study for a long time. The wave equation is the first evolution partial differential equation investigated by d'Alembert in 1747, while the heat equation was introduced by Fourier in 1811. The initial value problem for weak solutions attracted the attention of mathematicians starting from the second half of the twentieth century. Special techniques (maximum principle, De Giorgi-Nash-Moser theorems etc...), were developed at that time to handle stability issues. The study of blow-up solutions started at the same time but mostly from the perspective of the initial value problem (i.e. existence of blow-up dynamics and conditions for blow-up), and we refer for example to the work of Fujita, Kaplan and Keller in the sixties. Then, the asymptotic behavior of the solutions started to be specified (with the works of Ball, Glassey, John and Levine for example in the seventies). The study of these equations was related to that of the Navier-Stokes equations, of certain geometric flows, and more generally of any nonlinear equation involving dissipation, for (NLH), and to that of nonlinear hyperbolic equations, of some models of general relativity and field theory, and more generally of equations with dispersion, for (NLW).

After, the real qualitative behavior of blowing-up solutions was studied, with the investigation of blow-up rates (the "speed" at which relevant quantities tend to their limits), blow-up profiles (the local first order equivalent of the solution during singularity formation) and blow-up sets (the place where the blow-up happens). From that moment the literature becomes too vast for an introduction, and references will be given throughout the present document.

Let us now specify for these two equations the general strategy that we explained in Section 1.1. To ease the analysis, we will only consider the semilinear wave equation, for the special class of blow-up solutions. The special solutions describing the asymptotic behavior of arbitrary solutions are the following. They consist in stationary states that do not change with time

$$u(t,x)=\psi(x)\ \text{ where }\ \Delta\psi+|\psi|^{p-1}\psi=0$$

and in backward self-similar solutions conserving their shape but with a scale shrinking in finite time T at a point x_0

$$u(t,x)=\frac{1}{(T-t)^{\frac{2}{p-1}}}\psi\left(\frac{x-x_0}{T-t}\right)$$

where ψ solves another elliptic equation.

The reasons behind the fact that these should be the building blocks of any solutions are the following (at a very formal level). First, the semilinear wave equation admits invariances. This means that some transformations send a solution onto another solution. Two main invariances are the invariance by space isometry and scale transformation. For any isometry I, if u is a solution then so is $u(t+t_0,I(x))$. For any $\lambda>0$, the function

$$\lambda^{\frac{2}{p-1}}u(\lambda t,\lambda x)$$

is also a solution. We will use the following notation for a function $u:\mathbb{R}^d\to\mathbb{R}$

$$g_I.u=x\mapsto u(I(x)),\quad g_\lambda.u=u_\lambda=x\mapsto\lambda^{\frac{2}{p-1}}u(\lambda x). \tag{1.2.1}$$

These invariances allow one to reduce the study of all solutions to that of solutions in a smaller subset of the phase space. Here, via scale change and space translations, the dynamics of solutions staying coherent can be made "more compact", and the solution is then described by limit profiles up to the above invariances. The fact that the limit profiles are stationary states or self-similar solutions comes from the fact that the equation admits other geometric properties that prevent the existence of quasi-periodic (up to invariances) solutions. Namely, (NLW) can be seen as an infinite dimensional hamiltonian system with energy

$$E(u,u_t):=\frac{1}{2}\int_{\mathbb{R}^d}(|\nabla u|^2+u_t^2)dx-\frac{1}{p+1}\int_{\mathbb{R}^d}|u|^{p+1}dx,\quad E(u(t),u_t(t))=E(u(0),u_t(0)), \tag{1.2.2}$$

and the invariances imply other rigidities via Noether theorem. If a solution is not coherent, i.e. is not compact up to space translations and scale change, that means that all the different space scales are decorrelated. In that case, the nonlinear interactions cancel in average and the solution undergoes dispersion at main order, converging to zero.

We omitted one other crucial invariance. The equation is invariant under the transformations of the Minkowski spacetime (only for (NLW)). For any $\ell\in\mathbb{R}^d$, $|\ell|<1$ the Lorentz transform of u

$$u_\ell(x,t):=u\left(\frac{t-x.\ell}{\sqrt{1-|\ell|^2}},x-\frac{x.\ell}{|\ell|^2}\ell+\frac{\frac{x.\ell}{|\ell|}\frac{\ell}{|\ell|}-\ell t}{\sqrt{1-|\ell|^2}}\right) \tag{1.2.3}$$

is also a solution (where the argument is in the domain of u). In fact, one has to include this transformation to the asymptotic description we just explained: any blow-up solution is believed to be described locally near the singularity by means of stationary and backward self-similar solutions, up to scale change and Lorentz transformations.

We will explain a bit more this resolution in special solutions in Section 1.5.

The relative control of the Dirichlet energy over the interaction energy in (*1.2.2*) leads to the following classification for the problems.

(i) *Energy subcritical case:* this refers to the case $d=1,2$ and $p>1$, or $d\geq 3$ and $1<p<1+\frac{4}{d-2}$. In that case $\|\nabla u\|_{L^2}\gtrsim \|u\|_{L^{p+1}_{\text{loc}}}$ by Sobolev embedding.

(ii) *Energy critical case:* for $d\geq 3$ for the special value $p=p_c$ where

$$p_c := 1+\frac{4}{d-2} \quad (p_c=+\infty \text{ for } d=1,2) \tag{1.2.4}$$

the energy is invariant by scaling $E(u_\lambda)=E(u)$.

(iii) *Energy supercritical case:* for $d\geq 3$ and $p>p_c$, the local L^{p+1} norm of the solution is no more controlled by the Dirichlet energy.

Roughly, this means that the real interplay between nonlinearity and dispersion should occur in critical and supercritical cases. It appears that the subcritical case was studied first since it is the framework where most of the functional analysis tools are available. Notable examples for the heat equation are the works of Bricmont, Filippas, Giga, Herrero, Kohn, Kupiainen and Velazquez, and later Merle and Zaag, to name a few. For the wave equation, after works by Alinhac, Caffarelli, Friedman, Kichenassamy and Littman among many others, Merle and Zaag were able to give a complete description of the blow-up for the one dimensional wave equation. To illustrate all the ideas presented so far, and to introduce the results obtained by the author, we now describe in the next Section this amazing series of work.

The critical and then the supercritical cases received attention later on. In particular, as stationary states only exists for these problems, the blow-up mechanism relying on their concentration started to be studied in the nineties with a real development at the beginning of the twentieth century. As this is the main subject here we do not give references right now: they will be spread throughout this document.

1.3 Presentation of a model case: the one dimensional semilinear wave equation

To illustrate the kind of issues one is interested in and the type of results that one is looking for, we will focus on a model case that has been completely settled. This context is that of the description of blow-up dynamics for the one dimensional semilinear wave equation. We describe here some outstanding results that have been proven in the series of work [118, 119, 121, 122, 32] by Merle and Zaag, and Zaag and Côte for the last one, in light of the general strategy we just explained. We follow here partially [120] and will stay at a rather formal level. Let $p>1$ and consider the equation

$$(NLW1d) \quad \begin{cases} u_{tt}=\Delta u+u^p, \\ u(0,x)=u_0(x),\ \partial_t u(0,x)=u_1(x), \end{cases} \quad (t,x)\in I\times\mathbb{R}.$$

where $I\subset\mathbb{R}$ is a time interval and u is real valued, with smooth compactly supported initial datum $u(0)$ and $u_t(0)$ belonging to $C_0^\infty(\mathbb{R})$.

First, let us define the blow-up curve of a solution, which is the set of spacetime points where a singularity can form. The information for (NLW) propagates with finite speed one. To be more precise, if u and v are two solutions on $[0,T]$ such that $(u(0),u_t(0)) = (v(0),v_t'(0))$ on $[x_0-R, x_0+R]$ for some $R>0$, then $u=v$ on $[x_0-R+t, x_0+R-t]$ for $t\in[0,\min(R,T)]$. This property is classical, see the book [52] for instance. Following Alinhac [3] we define

Definition 1.3.1. An open set $\Omega\subset[0,+\infty)\times\mathbb{R}$ is called an influence domain if it contains all backward light cones emanating from its points

$$\forall (t,x)\in\Omega,\quad \{(t,x)\in[0,T]\times\mathbb{R},\quad |x-y|\le T-t\}\subset\Omega.$$

An influence domain Ω can be seen as the subgraph $\Omega=\{(t,x)\in[0,+\infty)\times\mathbb{R},\ t<T(x)\}$ where $T(x)=\sup\{t\in[0,+\infty),\ (t,x)\in\Omega\}$, and it follows from Definition 1.3.1 that either $T(x)$ is $\pm\infty$ everywhere or is a 1-Lipschitz function.

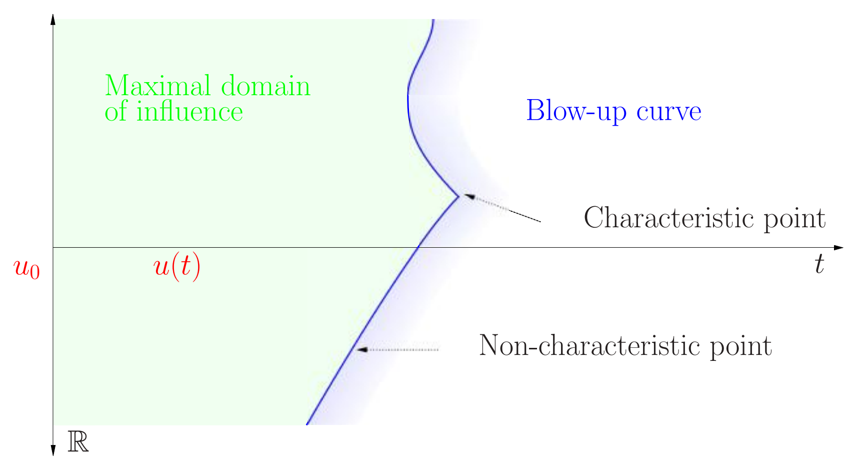
Definition 1.3.2. Let $\Omega_{\max}$ be the union of all influence domains Ω such that there exists a C^∞ solution of (NLH) in Ω with initial datum $(u(0),u_t(0))$. Then Ω is the largest influence domains with this property and is called the maximal influence domain of the solution.

The boundary of the maximal influence domain, $\Gamma:=\Omega_{\max}=\{((T(x),x),\ x\in R\}$ is called the blow-up curve of u and its points are classified in two categories.

Definition 1.3.3. A point $x_0\in\mathbb{R}^d$ is said to be a non-characteristic point if there exist $C_0>1$ and $t_0<T(x_0)$ such that

$$\{(t,x)\in[t_0,T(x_0))\times\mathbb{R}^d,\quad |x-x_0|\le C_0(T(x_0)-t)\}\subset\Omega_{\max}$$

and is said to be characteristic if not.



Let us denote by $\mathcal{R}$ the set of non-characteristic points, and by $\mathcal{S}$ the set of characteristic points. The nature of these sets has been studied and is the following.

Theorem 1.3.4 ([119, 121, 122]). *(i) $\mathcal{R}$ is a non empty open set, and on $\mathcal{R}$ the blow-up curve Γ is of class C^1.*

(ii) $\mathcal{S}$ *is made of isolated points. For* $x_0 \in \mathcal{S}$, $x \mapsto T(x)$ *is left and right differentiable at* x_0 *with* $T'_{left}(x_0) = 1$ *and* $T'_{right}(x_0) = -1$.

One is then interested in the description of a blow-up solution u near Γ, and in finding dynamical differences between characteristic and non-characteristic points. First, we recall that the special solutions describing the asymptotic behavior of a solution during blow-up are conjectured in the general case to be stationary solutions, backward self-similar solutions and Lorentz transforms (defined by (*1.2.3*)) of these two kind of solutions. A particular self-similar solution that always exists is the constant in space blow-up profile

$$u(t,x) = \frac{\kappa_W}{(T-t)^{\frac{2}{p-1}}}, \quad \kappa_W := \left(\frac{2(p+1)}{(p-1)^2}\right)^{\frac{1}{p-1}}. \tag{1.3.1}$$

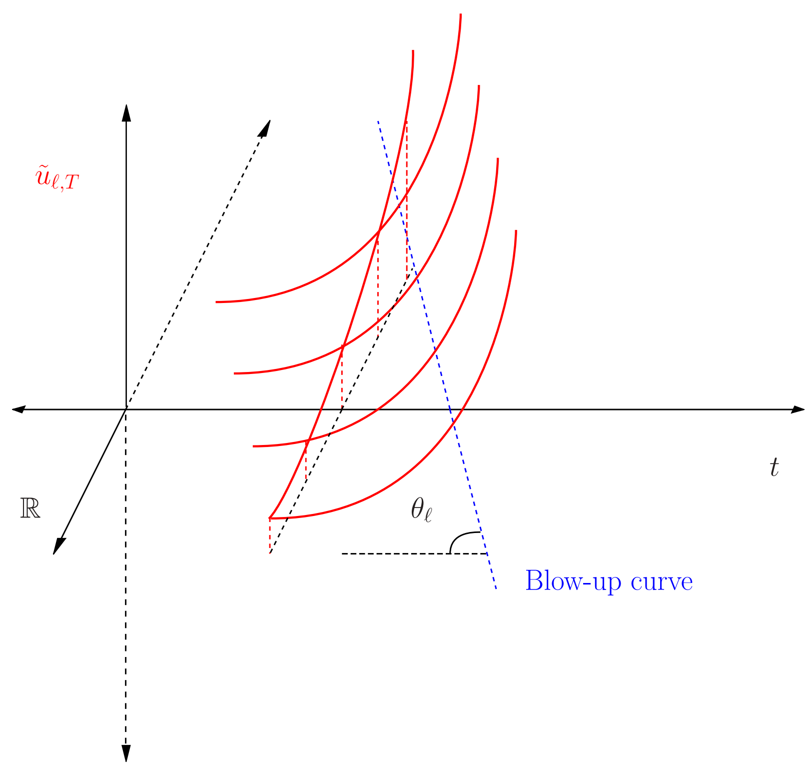
For $(NLW1d)$ this is in fact the only special solution.

Theorem 1.3.5 ([118]). *There is no stationary solution. The only self-similar solutions are Lorentz transforms of the ODE blow-up profile, of the form*

$$\tilde{u}_{\ell,T}(t,x) = \frac{\kappa_W (1-\ell^2)^{\frac{1}{p-1}}}{[T-t+\ell(x-x_0)]^{\frac{2}{p-1}}}$$

for $\ell \in (-1,1)$ *and* $x_0 \in \mathbb{R}$ *(where it is defined).*

Let us insist that though they are not well defined on a strip of the form $[0,T] \times \mathbb{R}$, they are well defined on light cones which is sufficient due to the finite speed of propagation. The blow-up curve of these functions is a straight line given by $T(x) = T + \ell(x - x_0)$. The angle between the blow-up curve and the line $x = 0$ in the spacetime is $\theta(\ell) = \text{Arctan}(\ell^{-1})$. As $|\ell| \to 1$, $|\theta| \to \pi/4$ and the solution concentrates more and more along the blow-up curve.



One therefore conjectures that any blow-up solution is near Γ a perturbation of one or several Lorentz transforms of the ODE blow-up profile in interaction. To state the result let us introduce the similarity variables. This is a suitable way to renormalize in the light cone emanating from $(x_0, T(x_0)) \in \Gamma$ near this point:

$$w_{x_0,T(x_0)}(x,s) = (T(x_0)-t)^{\frac{2}{p-1}} u(t, x_0 + (T(x_0)-t)x), \quad s = -\log(T(x_0)-t).$$

In these variables, the self-similar solutions of Theorem 1.3.5 are

$$\kappa_\ell(x) := \kappa_W \frac{(1-\ell^2)^{\frac{1}{p-1}}}{(1+\ell x)^{\frac{2}{p-1}}}$$

where $\ell \in (-1,1)$. The result is the following.

Theorem 1.3.6 (Blow-up profiles for the one-dimensional (NLW) [118, 121]). *(i)* Non characteristic case: *if $x_0 \in \mathbb{R}$ is non characteristic then there exists $\ell(x_0) \in (-1,1)$ such that $w_{x_0,T(x_0)}$ converges as $s \to +\infty$ to κ_ℓ.*

(ii) Characteristic case: *if $x_0 \in \mathbb{R}$ is characteristic, then there exist $k(x_0) \in \mathbb{N}$, $k(x_0) \geq 2$, and continuous functions $\ell_i(s) \in (-1,1)$ for $i = 1, ..., k(x_0)$ such that $w_{x_0,T(x_0)}$ converges as $s \to +\infty$ to $\sum_1^{k(x_0)} \kappa_{\ell_i(s)}$. Moreover, at least two angles $\theta(\ell_j)$ and $\theta(\ell_{j'})$ are such that $\theta(\ell_j)(s) + \theta(\ell_{j'})(s) \to \pi/2$.*

From the above theorem one obtains numerous informations. Going back to original variables, the results give first order approximation on light cones near the blow-up curve. When a point is characteristic, the solution resembles one only self-similar solution with fixed angle, and the tangent to the blow-up curve is the blow-up curve of this latter. To form a characteristic point, there must be at least two interacting self-similar solutions, whose associated blow-up curves become degenerate and form a right angle. Also, this gives precious growth estimates for the solution near the blow-up curve.

The above results form a complete a priori description of singularity formation. The next step is to investigate the existence of such scenarios by constructing examples.

Theorem 1.3.7 ([32]). *For any $k \geq 1$, there exists a solution that decomposes near a point of the blow-up curve into the sum of k self-similar solutions.*

Finally, let us mention that the stability of the different scenarios has been investigated. Some scattering results have also been obtained. Namely, solutions that are global backward in time and satisfying some growth conditions have to be backward self-similar solutions or 0. Such kind of Liouville type rigidity theorems are extremely useful in the analysis.

1.4 Presentation of the work of the author

We still present here the results in a rather informal way that we think is more suitable to an introduction. References of earlier and related works are given in the next chapter devoted to their detailed presentation.

The results obtained during the PhD of the author deal with the energy critical and supercritical cases for the semilinear heat and wave equations. For that range of nonlinearities, two main difficulties arise. First, the set of special solutions describing the asymptotic of general solutions is bigger. Steady states and backward self-similar solutions that are different from the ODE blow-up exist. The stationary states of (NLW) and (NLH) being the same, we will use the same notation for Q, the radial one that can be proved to be unique up to scale change

$$Q(x)=Q(|x|), \quad Q(0)=1, \quad \Delta Q+|Q|^{p-1}Q=0.$$

It is most of the time called the ground state. New asymptotic behaviors then appear, that are linked to the Joseph-Lundgren exponent

$$p_{JL}:=1+\frac{4}{d-4-2\sqrt{d-1}}>p_c, \quad (p_{JL}:=+\infty \text{ for } 1\leq d\leq 10). \tag{1.4.1}$$

The second difficulty is technical. Critical and supercritical problems having been investigated recently, the tools to handle them are still being invented and perfected. As a consequence, a complete description, as for the one dimensional semilinear wave equation explained in Section 1.3, is still challenging and a great amount of work is currently done in that direction.

In a first work, the possibility of concentration in finite time of the radial stationary state for the supercritical (NLW) was studied. Different speeds of concentration for smooth solutions were found, and the stability of these scenarios was investigated.

Theorem 1.4.1 (Slow blow-up for supercritical (NLW) [23]). *For a range of supercritical exponents $p>p_{JL}$, there exists a countable family of speeds $(c_\ell)_{\ell\geq\ell_0}$ with $c_\ell>1$ and $c_\ell\to+\infty$ and radial smooth and compactly supported solutions u_ℓ blowing up by concentration of the ground state*

$$u_\ell(t,x)\sim\frac{1}{\lambda_\ell(t)^{\frac{2}{p-1}}}Q\left(\frac{x}{\lambda_\ell(t)}\right), \quad \lambda_\ell(t)\sim(T-t)^{c_\ell}.$$

Moreover, for each ℓ there exists a Lipschitz manifold (in a suitable space of radially symmetric functions) of codimension $\ell-1\geq 2$ of solutions blowing up with the same scenario.

The Lipschitz stability with codimension $\ell-1$ means the following. For each ℓ, there exists $\ell-1$ unstable profiles ψ_j, and a decomposition of the phase space near $u_\ell(0)$, $\text{Span}(\psi_j)_{1\leq j\leq\ell-1}\oplus\text{Span}(\psi_j)^\perp_{1\leq j\leq\ell-1}$ and real-valued Lipschitz functions $a_j(\varepsilon)$ on $\text{Span}(\psi_j)^\perp_{1\leq j\leq\ell-1}$ such that the following holds. If one perturbs u_ℓ correctly with an initial datum of the form:

$$u(0)=u_\ell(0)+\sum_1^{\ell-1}a_j(\varepsilon)\psi_j+\varepsilon, \quad \varepsilon\in\text{Span}(\psi_j)^\perp_{1\leq j\leq\ell-1}$$

then the solution has the same behavior. If the initial datum has not this form, then it will escape a neighborhood of u_ℓ (but its behavior after the exit of this neighborhood remains unknown). In a second work, the author constructed a detailed example of a blow-up solution for the supercritical heat equation in the non-radial case.

Theorem 1.4.2 (Type II blow-up for supercritical (NLH) [26]). *Let $p > p_{JL}$, Ω be a smooth bounded domain and consider (NLH) with Dirichlet boundary condition on Ω. There exists a countable family of speeds $(c_\ell)_{\ell \geq \ell_0'}$ with $c_\ell > \frac{1}{2}$ and $c_\ell \to +\infty$ and solutions u_ℓ blowing up by concentration of the ground state at a point $x_0 \in \Omega$*

$$u_\ell(t,x) \sim \frac{1}{\lambda_\ell(t)^{\frac{2}{p-1}}} Q\left(\frac{x-x_0}{\lambda_\ell(t)}\right), \quad \lambda_\ell(t) \sim (T-t)^{c_\ell}.$$

In a third work, the author, in collaboration with P. Raphaël and F. Merle, obtained the complete classification of the dynamics near the ground state for the energy critical heat equation in large dimensions and in the non-radial case. They also classified all solutions that are global backward in time and resembles a ground state as $t \to -\infty$: they must belong to the unstable manifold around Q. This Liouville type theorem can be seen as a scattering result. To state the result let us first recall that (NLH) admits the constant in space backward self-similar solution

$$\tilde{u}_T(t,x) := \frac{\kappa_H}{(T-t)^{\frac{1}{p-1}}}, \quad \kappa_H := \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \tag{1.4.2}$$

which corresponds to the ODE blow-up profile.

Theorem 1.4.3 (Dynamics near the ground state for critical (NLH) in large dimensions [24, 25]). *Let $d \geq 7$ and $p = p_c$.*

(i) If a solution starts close to Q in a key topology for the equation, then either it will blow-up with the ODE blow-up solution (1.4.2) as blow-up profile near the singularity, or it is global and is dissipated to 0, or it is global and converges to another ground state $\lambda_\infty^{\frac{2}{p-1}} Q(\lambda_\infty(x-x_\infty))$. This latter scenario forms a hyper surface separating the first two behaviors.

(ii) If a solution is global backward in time and stays close to the set of ground states $\left(\lambda^{\frac{2}{p-1}} Q(\lambda(x-y))\right)_{\lambda,y}$, then it is a ground state or belongs to the $d+2$-dimensional unstable manifold. This set has two connected components; on one solutions are dissipated to 0, on the other solutions blow-up with type I blow-up.

In a fourth work, the author, in a collaboration with P. Raphaël and J. Szeftel, gave an alternative construction to non-constant self-similar solutions and could study their non-radial stability.

Theorem 1.4.4 (Stability of non-ODE self-similar blow-up for (NLH) [27]). *Let $p \in (p_c, p_{JL})$. There exists $N \in \mathbb{N}$, $N \gg 1$, a countable family of radial self-similar solutions $(\Phi_n)_{n \geq N}$ and for each $n \geq N$ a Lipschitz manifold (in a suitable non-radial functional space) of codimension n of initial data u blowing-up by concentration of Φ_n*

$$u(t,x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} \Phi_n\left(\frac{x-x_0}{\lambda(t)}\right), \quad \lambda(t) \sim \sqrt{T-t}.$$

1.5 More insights on the asymptotic description of blow-up

To end this general introduction, we now present heuristic results that describe the behavior of a blow-up function. This details the strategy described in Section 1.1.

For a blow-up solution, one can believe that as $t \to T$ a universal mechanism for blow-up takes place, and that the solution does not really depend on the initial datum anymore. This universal mechanism should be thought as a first order approximation of the solution during singularity formation, providing an intermediate asymptotic. Intermediate asymptotics have been used extensively in the physics literature. The aim is to provide a simplified description of a system for the study of asymptotic regimes. Most of the time, these simplifications occur thanks to the invariances of the equation under the action of some groups which allow to kill some degrees of freedom. Group renormalization originates from quantum field theory and statistical physics as an other approach to this issue, and the two points of view have been linked [7].

We will now see that some of the invariances described in Section 1.2 are directly responsible for the dynamical description of the solutions. In his book on fluid mechanics [10], G. Birkhoff wrote "I hope that, in the future, the debt of mechanics to the concepts of group theory will be more explicitly recognized." and we shall now go in that direction. Let us mention that an application of group renormalization to the study of blow-up for (NLH) can be found in [17].

Often, a first approach to the description of the asymptotic behavior of solutions of an evolution PDE is to find solutions whose trajectory is obtained from the initial datum by the action of the groups of symmetries (for the original PDE or for an asymptotic formulation), and then one tries to recover true solutions admitting this special one as a first order approximation. This is an inverse method, which does not provide any insight on why this picture should be universal; the belief being that any solution is asymptotically described by such special solutions. We now present a formal computation to illustrate this universality.

1.5.1 A formal computation

Here we formally demonstrate the existence of a rigid asymptotic description linked to the symmetries of the equation, in the case of a particular blow-up solution of (NLH).

Consider $u(t,x) = u(t,|x|) \geq 0$ a radially symmetric and radially decreasing positive solution (these properties being conserved by the flow) of (NLH) blowing up at $T > 0$. One can rigorously prove that any blow-up solution has to be unbounded as $t \to T$ with the lower bound

$$\forall t \in (0,T), \quad \|u(t)\|_{L^\infty} \geq \frac{\kappa_H}{(T-t)^{\frac{1}{p-1}}}. \tag{1.5.1}$$

Consequently $u(t,0) = \|u(t)\|_{L^\infty} \to +\infty$ as $t \to T$ and satisfies the differential inequality

$$\frac{d}{dt}\|u(t)\|_{L^\infty} = \Delta u(t,0) + u(t,0)^p \leq u(t,0)^p = \|u(t)\|_{L^\infty}.$$

We define the scale $\lambda(t) = \|u(t)\|_{L^\infty}^{-\frac{p-1}{2}}$ and assume that $\frac{d}{dt}\|u(t)\|_{L^\infty} \geq 0$, which from the above identity and (*1.5.1*) implies

$$0 \leq -\lambda\lambda_t \leq \frac{p-1}{2} \quad \text{and} \quad \lambda \leq \kappa^{-\frac{p-1}{2}}\sqrt{T-t}. \tag{1.5.2}$$

We renormalize u according to the scaling invariance (*1.2.1*) of the equation by defining

$$u(t,x)=\frac{1}{\lambda^{\frac{2}{p-1}}}v(s,y), \quad y=\frac{x}{\lambda}$$

where the renormalized time s solves the differential equation

$$\frac{ds}{dt}=\frac{1}{\lambda^2}, \quad s(0)=s_0. \tag{1.5.3}$$

The inequality $\frac{ds}{dt}\geq \kappa^{p-1}(T-t)^{-1}$ which comes from (*1.5.2*) ensures that $s(t)\to+\infty$ as $t\to T$. v is a convenient renormalization since

$$\forall s\geq s_0, \quad |v(0,s)|=1 \text{ and } \|v\|_{L^\infty}=1.$$

As u satisfies (NLH), v satisfies the renormalized equation

$$v_s-\frac{\lambda_s}{\lambda}=\Delta v+|v|^{p-1}v. \tag{1.5.4}$$

Now assume that v is not only bounded but also convergent in C^2_{loc} (think of compactness as coming from parabolic regularization, and of the fact that $v_s\to 0$ as an energy dissipation)

$$v\to w \text{ and } \partial_s v\to 0. \tag{1.5.5}$$

The differential bound on the scale (*1.5.2*) becomes in renormalized time (*1.5.3*)

$$0\leq -\frac{\lambda_s}{\lambda}\leq \frac{p-1}{2}.$$

We therefore distinguish between two subcases depending on the asymptotic of the scale λ:

$$\text{Case 1} \quad \lim_{s\to T} -\tfrac{\lambda_s}{\lambda}\in (0,\tfrac{2}{p-1}], \quad (\text{and then } \lambda\sim\sqrt{T-t}) \tag{1.5.6}$$

$$\text{Case 2} \quad \lim_{s\to T} -\tfrac{\lambda_s}{\lambda}=0 \quad (\text{then } \lambda\ll\sqrt{T-t}). \tag{1.5.7}$$

(i) *Case 1: asymptotically self-similar blow-up.* From (*1.5.4*), (*1.5.5*) and (*1.5.6*) the asymptotic profile w must solve:

$$c\Lambda w=\Delta w+|w|^{p-1}w, \quad c=\lim_{s\to T} -\frac{\lambda_s}{\lambda}>0 \tag{1.5.8}$$

(one can change c to $\frac{1}{2}$ by the scale change $w=c^{\frac{1}{p-1}}\tilde{w}(\sqrt{c}x)$). The above equation is called the backward self-similar equation. Any solution of (*1.5.8*) is in correspondance with an exact solution of (NLH) under the form

$$u(t,x)=\frac{1}{(T-t)^{\frac{2}{p-1}}}\tilde{w}\left(\frac{x}{\sqrt{T-t}}\right)$$

which is indeed a solution blowing up at 0 at time T. These self-similar solutions are exact solutions for which the scale shrinks at the universal averaged diffusion speed.

(ii) *Case 2: asymptotically stationary blow-up.* Similarly from (*1.5.4*), (*1.5.5*) and (*1.5.7*) w must solve

$$0=\Delta w+|w|^{p-1}w. \tag{1.5.9}$$

which is the equation for stationary solution of (NLH). The meaning of this situation is the following: for a blow-up that happens slower (*1.5.7*) than the natural blow-up speed (*1.5.6*), at main order diffusion must cancel nonlinear effects (*1.5.9*).

In original variables one retrieves for the solution

$$u(t,x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} w\left(\frac{x}{\lambda(t)}\right), \quad \text{with} \quad \begin{cases} \lambda \sim \sqrt{T-t} \ \text{ and } \ \|u\|_{L^\infty} \sim c(T-t)^{-\frac{1}{p-1}} & \text{in case 1,} \\ \lambda \ll \sqrt{T-t} \ \text{ and } \ \|u\|_{L^\infty} \gg (T-t)^{-\frac{1}{p-1}} & \text{in case 2.} \end{cases}$$

Case 1 and Case 2 are referred to as type I and II blow-up.

1.5.2 The general perspective

We now extend the result of the formal computation we just performed to the general case, from a theoretical but very formal point of view. Fix a phase space X for the Cauchy theory of (NLH) or (NLW). Let G be the group of transformations of X generated by isometries g_I and scaling transformations g_λ defined in (*1.2.1*) and (*1.2.1*). The curves generated by the solution map commute with the action of G. That is to say if $\{u(t),\ t \in [0,T)\} \subset X$ is the curve of a solution, then so is $\{g.u(t),\ t \in [0,T)\}$ for any $g \in G$. Any $u \in X$ generates an equivalence class

$$\tilde{u} = \tilde{u}(u) = \{g.u, \quad g \in G\} \subset X$$

where $\tilde{u}$ should be seen as the renormalization class of u under the action of the group G. The elements of the group G are suitable to produce a "compact" renormalization. That is to say, for $u(t)$ a blow-up solution, there should exist renormalization parameters $g(t) \in G$ such that $g(t).u \to u_\infty$ is convergent as $t \to T$. Informally, this comes from the fact that the lack of compactness in certain functional spaces (such as Lebesgue or Sobolev spaces) can be retrieved through scale changes and space translations. This is the concentration-compactness principle developed by P.-L. Lions [95], and later adapted to nonlinear evolution PDEs in the breakthrough works [5, 79].

To continue with the formal exemple we just developed in Subsection 1.5.1, if the blow-up set of a solution u of (NLH) consists in k points $x_i \in R^d$ for $1 \le i \le k$, then there should exist k scales $\lambda_i(t)$ and asymptotic profiles $w_i : \mathbb{R}^d \to R$ solutions of (*1.5.8*) or (*1.5.9*) such that

$$\lambda_i(t)^{\frac{2}{p-1}} u(t, \lambda_i(t)(x_i - x)) \to w_i(x) \ \text{ as } t \to T.$$

The general formal picture in this case is the following:

$$\begin{tikzcd}
u(t) \in X \arrow[r, "t \to T"] \arrow[d, "\text{renormalization}"'] & u(t,x) \sim \displaystyle\sum_1^k \frac{1}{\lambda_i(t)^{\frac{2}{p-1}}} w_i\left(\frac{x-x_i}{\lambda(t)}\right) + u_\infty \\
\tilde{u}(t) \in X/G \arrow[r, "t \to T", "\text{compactness}"'] & (\tilde{w}_i)_{1 \le i \le k} \in (X/G)^k, \ w_i(t) \in \tilde{w}_i(0) \ \forall t \in [0, T(w_i)) \arrow[u, "\text{nonlinear}", "\text{superposition principle}"']
\end{tikzcd}$$

where u_∞ is the limit profile outside the blow-up set. The above asymptotic decomposition is referred to as the soliton resolution conjecture. Let us mention how the above picture should be modified in the general case.

(i) If a solution of (NLH) blows up on the set $\{x \in \mathbb{R}^d, \ x_1 = x_2 = ... = x_{d'} = 0\}$ for $d' < d$, then at any point y of this set the solution should look like a d'-dimensional blow-up:

$$u(t,x) \sim \frac{1}{\lambda(t)^{\frac{2}{p-1}}} w\left(\frac{\tilde{x}-\tilde{y}}{\lambda}\right), \quad \tilde{x} = (x_1, x_2, ..., x_{d'}) \in \mathbb{R}^{d'}$$

where w solves (*1.5.8*) or (*1.5.9*) in dimension d'. If it blows-up on a $d-d'$-dimensional set, then one should recover this result by a change of space variables.

(ii) For (NLW) the corresponding statement is the existence of a blow-up profile in every light cone emanating from the blow-up curve. For any $x \in \mathbb{R}^d$ such that $T(y) \neq T(x)$ for y close to x, there exists a main order approximation of u near $(x, T(x))$ consisting of a profile w solving either (*1.5.9*) or the analogue of (*1.5.8*) for (NLW), modified by scaling, translation and Lorentz transformations (defined by (*1.2.3*)). If there exists a d'-dimensional set near x for which $T(y) = T(x)$ then one would have an analogue to (i).

(iii) Superposition of blow-up bubbles can in principle exists. These would have the form

$$u(t,x) \sim \sum_1^k \frac{1}{\lambda_i(t)^{\frac{2}{p-1}}} w\left(\frac{x_0 - x}{\lambda_i(t)}\right), \quad \text{with } \lambda_i(t) \to 0 \text{ and } \lim_{t\to T} \frac{\lambda_i(t)}{\lambda_j(t)} + \frac{\lambda_j(t)}{\lambda_i(t)} = +\infty$$

where the conditions on the scales ensure that the pieces are decorraleted in space as $t \to T$. The existence of such solutions is mostly still open.

(iv) The condition of the invariance of the curves of the limit profiles, $w_i(t) \in \tilde{w}_i(0)$ for all $t \in [0, T(w_i))$, should in general be replaced by the fact that w_i is a compact solution. This means that there exists $g_i(t)$ such that $\{g_i(t).w_i, \ t \in [0, T(w_i))\}$ is a relatively compact set of X. The set of compact solutions could in principle contain periodic solutions for exemple (breathers or discretely self-similar solutions) but the conjecture for (NLH) and (NLW) is that it only contains stationary states and self-similar solutions (and Lorentz transforms of these solutions for (NLW)), see [48].

(v) The description of the blow-up bubbles in some cases has to be refined, introducing new intermediate scales. For exemple for the ODE blow-up (*1.4.2*) one has to find how the constant in space profile is localized.

2

Dynamics near steady states and backward self-similar solutions

This chapter is divided in four Sections. The first one introduces some general features of the two equations. Then each one of the other three sections is devoted to a detailed and motivated presentation of the works obtained by the author during his PhD. Sketches of proof of these latter are given, keeping solely the main technical details, to provide more comprehensible insights.

2.1 Preliminaries on the semilinear wave and heat equations

The two equations we are dealing with are the following. The semilinear heat equation will be considered mainly on the whole space in dimension $d \in \mathbb{N}^*$,

$$(NLH) \quad \begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u(0,x) = u_0(x), \end{cases} \quad (t,x) \in I \times \mathbb{R}^d, \tag{2.1.1}$$

where $I \subset \mathbb{R}$ is an interval, but sometimes we will also consider the case of a smooth and bounded space domain $\Omega \subset \mathbb{R}^d$ in which case one adds the Dirichlet boundary condition

$$(NLH\Omega) \quad \begin{cases} u_t = \Delta u + |u|^{p-1}u, \\ u(0,x) = u_0(x), \\ u(t,\cdot) = 0 \ \text{ on } \partial\Omega, \end{cases} \quad (t,x) \in I \times \mathbb{R}^d. \tag{2.1.2}$$

The second equation is the semilinear wave equation on the whole space

$$(NLW) \quad \begin{cases} u_{tt} = \Delta u + |u|^{p-1}u, \\ u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), \end{cases} \quad (t,x) \in I \times \mathbb{R}^d. \tag{2.1.3}$$

Many informations on these equations can be found for example in the reference books [148, 149, 151] for nonlinear waves, and [137] for nonlinear heat equations.

2.1.1 Basic properties

The linear equations. Some of their basic properties can be derived by their linear versions. These are the linear heat and wave equations $u_t = \Delta u$ and $u_{tt} = \Delta u$. The solution of the linear heat equation is

$$S_t^H(u_0) := u(t,x) = K_t * u_0(x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{\frac{|x-y|^2}{4t}} u_0(y)dy, \quad K_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \tag{2.1.4}$$

and that of the linear wave equation is on the Fourier side:

$$S_t^W(u_0,u_1) = u(t,x), \quad \hat{u}(t,\xi) = \frac{1}{2}\left[\hat{u}_0(\xi) - i\frac{\hat{u}_1(\xi)}{|\xi|}\right] e^{i|\xi|t} + \frac{1}{2}\left[\hat{u}_0(\xi) + i\frac{\hat{u}_1(\xi)}{|\xi|}\right] e^{-i|\xi|t}.$$

Let us insist on the following properties. All Sobolev norms are dissipated or conserved respectively,

$$\forall s \geq 0, \quad \|u_0\|_{\dot{H}^s} \geq \|S_t^H u_0\|_{\dot{H}^s} \underset{t\to+\infty}{\longrightarrow} 0, \quad \|(u_0,u_1)\|_{\dot{H}^{s+1}\times\dot{H}^s} = \|S_t^W(u_0,u_1)\|_{\dot{H}^{s+1}\times\dot{H}^s}.$$

Any possibly singular initial datum becomes smooth instantaneously by the linear heat flow

$$\forall u_0 \in L^2(\mathbb{R}^d), \ \forall t>0, \quad K_t * u_0 \in C^\infty(\mathbb{R}^d)$$

which will be referred to as a regularizing effect. There is no such property for the linear wave equation for it propagates singularities. We recall that both the linear and semilinear wave equations have finite speed of propagation, as seen in Section 1.3. The linear heat equation has an infinite speed of propagation. However, it can be interpreted as the evolution of a density probability for particles having Brownian random trajectories, and we will keep in mind that the (averaged) diffusion speed is $\sqrt{t}$. We will see that this property persists in some sense for (NLH) since the nonlinearity is local.

Comparison principle. For (NLH) the usual order on functions is preserved. In particular, since 0 is always a solution, positive initial data lead to positive solutions. For a proof of this fact we refer to [137].

Lemma 2.1.1 (Comparison principle for (NLH)). *(i) Let u and v be smooth solutions of (NLH) on $[0,T]$ such that $u_0\geq v_0$. Then $u(t)\geq v(t)$ for $t\in[0,T]$.*

(ii) Let u be a smooth positive solution of (NLH) on $[0,T]$ such that $u_t(0)$ has constant sign. Then u_t has the same sign for all $t\in[0,T]$.

Energy. For the semilinear heat equation the quantity

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^d} |u|^{p+1} dx \tag{2.1.5}$$

is dissipated by the flow

$$\frac{d}{dt}E(u(t)) = -\int_{\mathbb{R}^d} u_t^2 dx \leq 0 \tag{2.1.6}$$

as $\partial u + |u|^{p-1}u$ is precisely the opposite of the gradient of the functional E on any suitable phase space. For the semilinear wave equation, we recall that the quantity defined in (*1.2.2*) is an hamiltonian for the equation, as $(u_t, \partial u + |u|^{p-1}u)$ is precisely the symplectic gradient of E on any suitable phase space for the symplectic form

$$\omega((u_1,u_2),(v_1,v_2)) = \int_{\mathbb{R}^d} (u_1v_2 - u_2v_1)dx.$$

We will call E the energy in both cases for simplicity.

Invariances and criticality. We already described the invariances of (NLW) in Section 1.3: scale changes, Lorentz transformations and space isometries. This equation is also time reversible. It admits other symmetries (for some values of the parameters a conformal invariance for example) that will not be used here. (NLH) has less invariances but keeps the two main ones. If u is a solution of (NLH) then by scale change, for any $\lambda>0$

$$\frac{1}{\lambda^{\frac{2}{p-1}}} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

is also a solution, and for any isometry I of $\mathbb{R}^d$ $u(t, I(x))$ is another solution too. For both equations, the scale change at a fixed time is given by (*1.2.1*) and is an isometry on the critical homogeneous Sobolev space

$$\forall \lambda > 0, \quad \|u_\lambda\|_{\dot{H}^{s_c}(\mathbb{R}^d)} = \|u\|_{\dot{H}^{s_c}(\mathbb{R}^d)}, \quad s_c := \frac{d}{2} - \frac{2}{p-1}. \tag{2.1.7}$$

As the scale invariances are the same for the space variable for (NLH) and (NLW), the two equations are energy subcritical for $1 < p < p_c$, critical for $p = p_c$, and supercritical for $p > p_c$, where p_c is defined in (*1.2.4*) and where criticality was explained in Section 1.2. Another explaination to this distinction is that $s_c < 1$, $s_c = 1$ or $s_c > 1$ if $p < p_c$, $p = p_c$ or $p > p_c$, and consequently this encodes the relative position between the critical Sobolev space and the space associated to the Dirichlet energy $\dot{H}^1$.

2.1.2 Solutions and maximal time of existence

If for a function, the equation (NLH) or (NLW) holds pointwise at any (t, x), it is said to be a classical solution. There exists a natural way to weaken this formulation of the notion of solution for (NLH) and (NLW) which is relevant for plenty of reasons, and the problem to finding in which topological space the weak problem is well-posed has attracted a great amount of work since the second half of the twentieth century. For more details, in particular the weak formulations of (NLH) and (NLW), we refer to the below-mentioned works. The main results are the following. For (NLH), the regularizing effects imply that weak and classical solutions are almost the same.

Proposition 2.1.2 (Local well posedness of (NLH) [16, 155]). *Let an exponent*

$$q \in \left(\frac{d(p-1)}{2}, +\infty \right] \quad \left(\textit{or } q = \frac{d(p-1)}{2} \right) \quad \textit{and } q \geq 1 \quad (\textit{resp. } q > 1).$$

For any $u_0 \in L^q(\mathbb{R}^d)$ there exists $T > 0$ and a unique weak solution $u \in C([0,T], L^q(\mathbb{R}^d))$ of (NLH). Moreover, $u \in C((0,T], C^2(\mathbb{R}^d)) \cap C^1((0,T], C(\mathbb{R}^d))$ and u is a classical solution on $(0,T]$.

There are two additional features to point out. First, if the nonlinearity is analytic, i.e. $p \in 2\mathbb{N}+1$ then the above solution is C^∞ on $(0,T] \times \mathbb{R}^d$. Second, there is a compatibility for the well-posedness in the different phase spaces: if q_1 and q_2 are two exponents satisfying the conditions of Proposition 2.1.2, if $u_0 \in L^{q_1} \cap L^{q_2}(\mathbb{R}^d)$, and if $u_1 \in C([0,T_1], L^{q_1}(\mathbb{R}^d))$ and $u_2 \in C([0,T_2], L^{q_2}(\mathbb{R}^d))$ are the two corresponding solutions both starting from u_0, then $u_1 = u_2$ on $[0, \min(T_1, T_2)]$. In particular, the maximal time of existence of a solution (that will be defined hereafter) is independent of the chosen phase space. Note that the limit Lebesgue space that appears is the one left invariant by scale change

$$\forall \lambda > 0, \ \forall u \in L^{\frac{d(p-1)}{2}}(\mathbb{R}^d), \quad \|u\|_{L^{\frac{d(p-1)}{2}}(\mathbb{R}^d)} = \|u_\lambda\|_{L^{\frac{d(p-1)}{2}}(\mathbb{R}^d)}.$$

For the semilinear wave equation one has the following result.

Proposition 2.1.3 (Local well posedness of (NLW) [66, 94]). *Assume $p \in 2\mathbb{N}+1$, $p \geq 1 + 4/(d-1)$ and $d \geq 2$. Let $s \geq s_c$ (defined in (2.1.7)). Then for any $u_0 \in H^s(\mathbb{R}^d)$ there exists $T > 0$ and a unique weak solution $u \in C([0,T], H^s(\mathbb{R}^d))$ of (NLW) which satisfies $u \in L^{\frac{(p-1)(d+1)}{2}}([0,T] \times \mathbb{R}^d)$.*

Again the limit Sobolev space that appears is the critical Sobolev space. The results differs in the subconformal range $1 < p < 1 + \frac{4}{d-1}$, where some problems are still open, and we refer to [151] for the most recent work in this setting. Also, if the nonlinearity is not smooth, i.e. $p \notin 2\mathbb{N} + 1$, then one must add some compatibility conditions to ensure the regularity.

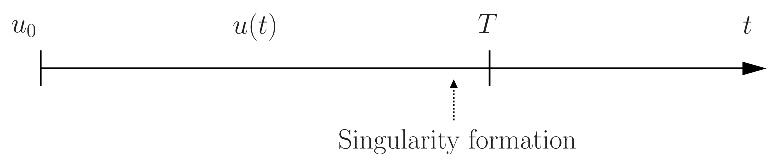
The above range of Lebesgue and Sobolev spaces for the well-posedness is sharp for both Proposition 2.1.2 and Proposition 2.1.3.

Remark 2.1.4. For Proposition 2.1.2 and 2.1.3 the propagation of regularity holds (for $p \in 2\mathbb{N} + 1$ or under suitable conditions). Namely, if u_0 is regular and if $u(t)$ is the weak solution on $[0, T]$ given by these propositions, then $u(t)$ is in fact also regular. We refer to the aforementioned papers for a detailed discussion.

Fix a common notation for the phase spaces of the equations: X for the initial data and $\mathcal{C}([0, \tilde{T}], X) \cap Y$ for the solutions. Given $u_0 \in X$, the maximal time of existence is defined as

$$T = T(u_0) := \sup\{\tilde{T} > 0, \text{ there exists a solution } u \in \mathcal{C}([0, \tilde{T}], X) \cap Y \text{ with } u(0) = u_0\}.$$

Under the assumptions of Lemmas 2.1.2 and 2.1.3, $T > 0$ and there exists a unique maximal solution $u(t) \in C([0, T], X) \cap Y$. We will only consider maximal solutions in this document and T will always refer to the maximal time of existence. If $T = +\infty$ the solution exists for all times and is then said to be a global solution. If $T < +\infty$, the solution is said to blow-up in finite time.



2.1.3 Notions for blow-up issues

For (NLW) we described in Section 1.3 the notion of the blow-up curve for the one dimensional case, and this adapts to the higher dimensional case. For the adaptation to weak solutions we refer to [3]. The blow-up curve is the hypersurface in the spacetime where the solution should become singular. The first time for which there exists a point belonging to the blow-up curve is T, the maximal time of existence described in the above Subsection 2.1.2. At time T, the set of points near which the solution becomes singular can be strongly degenerate, as a Cantor set for example.

Theorem 2.1.5 (Blow-up on any compact set for (NLW), [80]). *Fix $d = 1$ and $p \in 2\mathbb{N} + 1$. Given any compact set $E \subset \mathbb{R}$, there exists a smooth solution of (NLW) on $[0, T) \times \mathbb{R}$ such that the following holds.*

(i) *$u(t, x)$ converges to a smooth function of $x \backslash E$ as $t \to T$.*

(ii) *$u(t, x) \to +\infty$ for all $x \in E$.*

For (NLH), there exists a rather pictorial result for all blow-up solutions: their L^∞ norm explodes as $t \to T$, with a lower bound given by the ODE blow-up (*1.4.2*). The following Lemma is obtained by comparing, thanks to Lemma 2.1.1, a general blow-up solution to the constant in space blow-up solution (*1.4.2*).

Lemma 2.1.6 (Blow-up in L^∞ for (NLH)). *Assume that $p > 1$, that q satisfies the condition of Proposition 2.1.2 and that the maximal solution u in $C([0,T), L^q(\mathbb{R}^d))$ starting from $u_0 \in L^q$ is such that $T < +\infty$. Then*[1]

$$\forall t \in (0,T), \quad \|u(t)\|_{L^\infty} \geq \frac{\kappa_H}{(T-t)^{\frac{1}{p-1}}}. \tag{2.1.8}$$

The lower bound (*2.1.8*) can be saturated by some blow-up solutions, for example the ODE blow-up (*1.4.2*). Blow-up solutions have then been divided in two classes in [97].

Definition 2.1.7 (Type I and type II blow-up). Let u be a solution of (NLH) in the sense of Proposition 2.1.2 and assume $T < +\infty$. One then says that

$$u \text{ blows up with type I if : } \limsup_{t \to T} \|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} < +\infty,$$
$$u \text{ blows up with type II if : } \limsup_{t \to T} \|u(t)\|_{L^\infty}(T-t)^{\frac{1}{p-1}} = +\infty.$$

This distinction is pertinent for the classification of the behaviors for blow-up. In fact, in [98] any radial blow-up solution is shown to have a self-similar solution (resp. a stationary state) as a blow-up profile if the blow-up is of type I (resp. of type II). We refer to Section (1.5) for a formal explanation of this dichotomy. From Lemma 2.1.6, if a solution u of (NLH) blows up at time T its L^∞ norm diverges as $t \to T$. One has a natural definition for the location in space where this happens.

Definition 2.1.8 (Blow-up points and blow-up set). Let u be a solution of (NLH) blowing-up at time $T < +\infty$. $x \in \mathbb{R}^d$ is said to be a blow-up point for u if there exists t_n and x_n such that

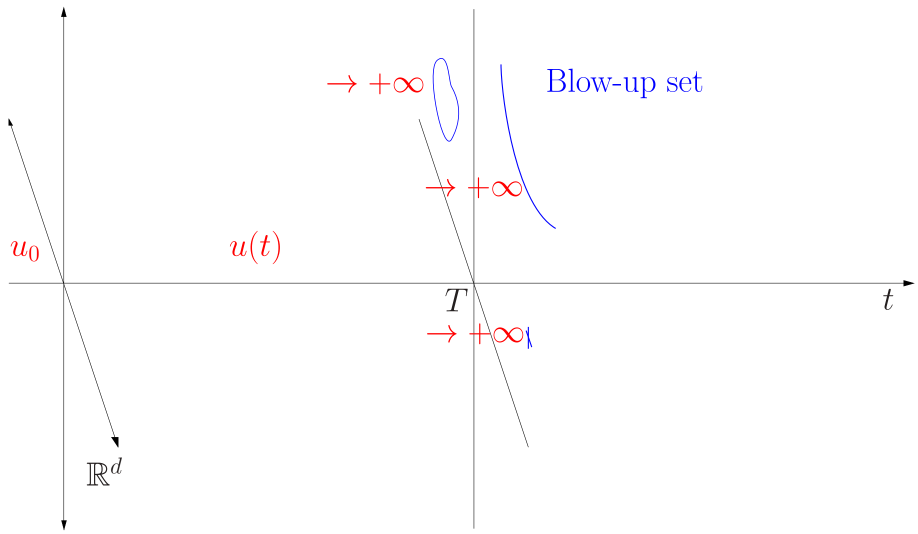
$$x_n \to x, \quad t_n \to T \quad \text{and} \quad |u(t_n, x_n)| \to +\infty \quad \text{as } n \to +\infty.$$

The blow-up set is the set of all blow-up points

$$\mathcal{S} := \{x \in \mathbb{R}^d, \quad x \text{ is a blow-up point for } u\}.$$

A straightforward property of the blow-up set is that it is closed. It cannot be open unless u is the constant in space ODE blow-up solution (*1.4.2*), due to the infinite speed of propagation and the strong nonlinear nature of the equation. The conjecture is that for any $p > 1$, $\mathcal{S}$ is small and well-localized. In the energy subcritical case for solutions starting in H^1, $\mathcal{S}$ was proven to be bounded and with finite $d-1$ Hausdorff measure in [63, 154]. Still in the energy subcritical case, under some nondegeneracy assumption, $\mathcal{S}$ has been proved to be a manifold of class C^2 in [156].

[1]The L^∞ norm being indeed finite for each $t \in (0,T)$ from Proposition 2.1.2.



One can wonder if there exists a way to study the solution after T where it can make sense, as for the wave equation. This is generically not possible due to the infinite speed of propagation; we refer to [99] and references therein for this issue that goes beyond the scope of the present document.

2.1.4 The energy subcritical case

This document being devoted to the energy critical and supercritical cases, we briefly review what is known in the subcritical setting. There is no stationary states (for both (NLW) and (NLH) since they share the same steady states) in the radial or in the positive class from [59]. For (NLH) the only backward self-similar function is the ODE blow-up profile (*1.4.2*), [61]. Any blow-up solution is then attracted by this profile and one has a very good understanding of blow-up. The blow-up profiles and blow-up rates are as follows.

Theorem 2.1.9 (Blow-up for the subcritical (NLH) [60, 61, 62, 63, 64, 116, 117]). *Let $1<p<p_c$. Let u be a solution of (NLH) blowing up at time $T>0$ and $x_0\in\mathbb{R}^d$ be a blow-up point of u. Then the blow-up is of type I and*

$$(T-t)^{\frac{1}{p-1}}u(t,\sqrt{T-t}(x-x_0))\to\pm\kappa_H \quad \textit{in } C^2_{loc}(\mathbb{R}^d) \textit{ as } t\to T$$

where κ_H is given by (1.4.2).

The ODE blow-up profile $\kappa_H(T-t)^{-\frac{1}{p-1}}$ is not decaying, and one can then wonder how to describe the way it is localized near a blow-up point. For this problem and the stability issue of the various ways to localize this profile, we refer to [54, 56, 71, 115, 117].

A particular setting of the subcritical wave equation is the one-dimensional case. We explained in Section 1.3 in the introduction that this problem has been settled. Though the conjecture is that any blow-up is asymptotically self-similar, this is still an open problem for the entire subcritical region. Most of the work has been done in the subconformal case $1<p<1+\frac{4}{d-1}<p_c$, and we refer to [124] and references therein for this problem.

2.2 Finite time concentration of the ground state

In chapter 1 we gave various motivations to study what happens near steady states for (NLH) and (NLW). These profiles can indeed appear as a first order terms in the decomposition of the solution either near the blow-up curve, but also for large times when the solution is global, see the series of works [43, 44, 46, 47, 50] and [49] for a review for the energy-critical (NLW), and [98, 97] for (NLH). We describe in this Section some known results on finite time bubbling of the ground state and present the construction of particular behaviors we made. As no radial stationary state exists for $1<p<p_c$ [59] we focus on the energy critical and energy supercritical cases $d\geq 3$ and $p\geq p_c$ where p_c is given by (*1.2.4*).

First, let us recall the properties of the radial stationary states. They are solution of the elliptic equation

$$\Delta\phi+|\phi|^{p-1}\phi=0 \tag{2.2.1}$$

and received a great attention in connexion with the Yamabe problem and the study of elliptic equations.

Theorem 2.2.1 (Subcritical and critical steady states [4, 58, 59, 150]). *(i)* Energy subcritical case: *let p satisfy $1<p<p_c$. Then there exist no either positive or radially bounded solution to (2.2.1).*

(ii) Energy critical case: *let $p=p_c$. Then all positive solutions to (2.2.1) that decay at infinity are rescaled versions $(Q_\lambda(x-z))_{\lambda,z}$ of the Talenti-Aubin profile*

$$Q(x):=\frac{1}{\left(1+\frac{|x|^2}{d(d-2)}\right)^{\frac{d-2}{2}}} \tag{2.2.2}$$

which minimizes the constant in the Sobolev embedding of $\dot H^1(\mathbb{R}^d)$ into $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$:

$$\frac{\|Q\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}}{\|\nabla Q\|_{L^2(\mathbb{R}^d)}}=\min_{v\in\dot H^1(\mathbb{R}^d)}\frac{\|v\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}}{\|\nabla v\|_{L^2(\mathbb{R}^d)}}.$$

For $p=p_c$, in the non radial case, there exists an infinite number of solutions to (*2.2.1*) with arbitrary energy [35], see also [34]. However, no precise information on the asymptotic behavior at infinity of a general steady states or on the spectrum of the linearized operator near these states have been proven. In the energy supercritical setting $p>p_c$, there also exist a positive radially decaying solution, but its unicity in the positive class is still open.

Theorem 2.2.2 (Radial steady states in the energy supercritical case [35, 67, 93]). *Let $p>p_c$. All smooth radially symmetric solution to (2.2.1) are renormalized versions of a unique solution Q with $Q(0)=1$. Its asymptotic behavior at infinity in space is the following.*

(i) For $p>p_{JL}$ defined in (1.4.1):

$$Q(x)=\frac{c_\infty}{|x|^{\frac{2}{p-1}}}+\frac{a_1}{|x|^\gamma}+o(|x|^{-\gamma}) \ \text{ as } |x|\to+\infty,\ a_1\neq 0, \tag{2.2.3}$$

with

$$c_\infty:=\left[\frac{2}{p-1}\left(d-2-\frac{2}{p-1}\right)\right]^{\frac{1}{p-1}}, \tag{2.2.4}$$

$$\gamma:=\frac12(d-2-\sqrt{\triangle}),\quad \triangle:=(d-2)^2-4pc_\infty^{p-1}\ \ (\triangle>0 \text{ iff } p>p_{JL}). \tag{2.2.5}$$

(i) For $p_c < p < p_{JL}$, defining $\omega := \sqrt{-\triangle}$:

$$Q(x) = \frac{c_\infty}{|x|^{\frac{2}{p-1}}} + \frac{a_2 \sin(\omega \log(r) + c)}{|x|^{\frac{d-2}{2}}} + o\left(\frac{1}{r^{\frac{d-2}{2}}}\right) \text{ as } |x| \to +\infty, \quad a_2 \neq 0, \ c \in \mathbb{R}. \tag{2.2.6}$$

In all cases, given a stationary state Q the action of the symmetry group gives a $d+1$ manifold of stationary states:

$$\left\{ \frac{1}{\lambda^{\frac{2}{p-1}}} Q\left(\frac{x-y}{\lambda}\right), \ \lambda > 0, \ y \in \mathbb{R}^d \right\}.$$

2.2.1 Continuum of blow-up speeds for the wave equation

In low dimensions, for the energy critical problem, the ground state has strong scale instabilities. The construction of concentration dynamics started with the pioneering works on the log-log blow-up for the Schrödinger equation [109, 110, 111, 112, 113] and the seminal works on geometric wave equations [88, 144] and [114, 138]. There exists a continuum of speeds for which it can shrink in finite time.

Theorem 2.2.3 (Slow blow-up for the critical (NLW) [74, 82, 86, 89]). *(i) Let $d = 3$, $p = p_c = 5$ and $T \in \mathbb{R}$. For any $\nu > 0$ there exists a radial solution of (NLW) of regularity[2] $H^{1+\frac{\nu}{2}-}$ under the form*

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + w(t), \quad \lambda(t) \sim (T-t)^{1+\nu}$$

where the remainder $(w(t), \partial_t w(t))$ is convergent as $t \to T$ in $\dot{H}^1 \times L^2$.

(ii) Let $d = 3$ and $p = p_c = 3$. Then the above statement holds for any $\nu > 3$ and $0 < \epsilon_0 \ll 1$ for

$$\lambda \sim (T-t)^{1+\nu} e^{-\epsilon_0 sin(|log(T-t)|)}.$$

(iii) Let $d = 4$ and $p = p_c = 3$. Then the above statement holds but for a smooth radial function for

$$\lambda \sim (T-t) e^{-\sqrt{|log(T-t|}}.$$

The last scenario, (iii), involves a smooth function whereas the solutions of (i) and (ii) are never $\mathcal{C}^\infty$. Indeed, the wave equation propagates the singularities of its solutions. We will see later that this is not the case for (NLH) for which a smoothing effect holds. This shows the dramatic influence of the functional space in which one is studying solutions for blow-up issues. Also, another simpler construction of this type of dynamics is provided in [75].

Let us now move to the energy supercritical setting. Very few results are available for dynamics near Q. In [86] large global solutions near Q are constructed, but they cannot be compactly supported and do not belong to the critical space $\dot{H}^{s_c}$. We obtained in [23] a countable number of concentration scenarios of Q involving smooth and compactly supported solutions, with a detailed asymptotic. The existence of such dynamics rely on the structure at infinity of Q given in Theorem 2.2.2.

[2] i.e. the solution is in $H^{1+\frac{\nu}{2}-\epsilon}$ for any $0 < \epsilon \leq 1 + \frac{\nu}{2}$.

Theorem 2.2.4 (Slow blow up for (NLW) above the Joseph-Lundgren exponent [23]). *Let $d \geq 11$, p_{JL} be given by (1.4.1) and a nonlinearity $p \in 2\mathbb{N}+1$ with $p > p_{JL}$. Let γ be given by (2.2.5) and define*

$$\alpha := \gamma - \frac{2}{p-1}. \tag{2.2.7}$$

Assume moreover

$$\left(\frac{d}{2} - \gamma\right) \notin \mathbb{N}. \tag{2.2.8}$$

For any integer $\ell \in \mathbb{N}$ with $\ell > \alpha$ and for a large enough regularity exponent

$$s_+ \in \mathbb{N}, \quad s_+ \geq s(\ell) \quad (s(\ell) \to +\infty \quad \text{as} \quad \ell \to +\infty),$$

there exists a smooth radially symmetric and compactly supported initial data $(u_0, u_1) \in H^{s_+} \times H^{s_+-1}(\mathbb{R}^d)$ such that the corresponding solution to (NLH) blows up in finite time $0 < T < +\infty$ by concentrating the ground state

$$u(t,r) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(Q+\varepsilon)\left(\frac{r}{\lambda(t)}\right)$$

with the following features. (i) Blow up speed*:*

$$\lambda(t) = c(u_0)(1 + o_{t\uparrow T}(1))(T-t)^{\frac{\ell}{\alpha}}, \quad c(u_0) > 0; \tag{2.2.9}$$

(iii) Asymptotic stability above scaling in renormalized variables*:*

$$\lim_{t\uparrow T} \|\varepsilon(t,\cdot), \lambda(\partial_t u)_\lambda(t,\cdot)\|_{\dot{H}^s \times \dot{H}^{s-1}} = 0 \quad \textit{for all} \quad s_c < s \leq s_+; \tag{2.2.10}$$

(iv) Boundedness below scaling*:*

$$\limsup_{t\uparrow T} \|u(t), \partial_t u(t)\|_{\dot{H}^s \times \dot{H}^{s-1}} < +\infty \quad \textit{for all} \quad 1 \leq s < s_c; \tag{2.2.11}$$

(v) Behavior of the critical norms*:*

$$\|u(t)\|_{\dot{H}^{s_c}} = \left[c(d,p)\sqrt{\ell} + o_{t\uparrow T}(1)\right]\sqrt{|\log(T-t)|}, \tag{2.2.12}$$

$$\limsup_{t\uparrow T} \|\partial_t u(t)\|_{\dot{H}^{s_c-1}} < +\infty. \tag{2.2.13}$$

In the same paper, the present author investigated the topological structure of the set of initial data associated to these blow-up scenarios. Their stability is given by the following result.

Theorem 2.2.5 (Lipschitz manifold structure for Theorem 2.2.4 [23]). *We keep the notations and assumptions of Theorem 2.2.4. Let a slightly supercritical regularity exponent $\sigma = \sigma(\ell)$ satisfy*

$$0 < \sigma - s_c \ll 1 \quad \textit{small enough.}$$

There exists a locally Lipschitz manifold of codimension $\ell - 1$ in $\dot{H}^\sigma \cap \dot{H}^{s_+}(\mathbb{R}^d) \times \dot{H}^{\sigma-1} \cap \dot{H}^{s_+-1}(\mathbb{R}^d)$ of radially symmetric functions of initial data leading to the blow up scenario described by Theorem 2.2.4. We point out that as $\alpha > 2$, the codimension satisfies $\ell - 1 \geq 2$.

Comments: The assumption (*2.2.8*) is technical. It is here to avoid some logarithmic corrections in Hardy type inequalities used in the analysis. The growth of the critical Sobolev norm (*2.2.12*) is coherent with [45] where solutions with bounded Sobolev norms were proved to be global and scattering to 0. This is a key difference between the energy critical case where solutions concentrating the ground state can remain bounded in the critical space. The analysis behind the two above Theorems originates from the related works [114, 138, 141].

Open problems: We just state here the radial ones, the non-radial case being considered later on after Theorem 2.2.9. In both the energy critical and supercritical problems the concentration speed seems linked with the regularity of the solution. It would then be interesting to classify all possible blow-up rates for smooth solutions. The smooth solutions blowing-up with the ℓ-th speed ($\ell > \alpha$) constructed by the author have been proven to be stable with codimension $\ell - 1$. The conjecture of the author is that they are instable with $\ell - 1 - E[\alpha]$ directions of instability leading to the faster concentration speeds, and $E[\alpha]$ directions leading to ODE blow-up, scattering towards 0 or scattering toward the ground state. For $p_c < p < p_{JL}$, concentration of the ground state is impossible for (NLH), see Theorem 2.2.7. The method of the proof cannot be applied here and it is then important to investigate this range of exponents. For $p = p_{JL}$ the asymptotic of Q is different but known and this case should be considered as well. Finally, the finite time concentration of several ground states all centered at the origin in the radial case is an important open problem, see [76] for a related result.

The final aim is of course to classify all dynamics and their topological properties near the ground state and this issue is the subject of the next Section where related open problems will be given.

2.2.2 Quantization of blow-up speeds for the semilinear heat equation

Historically, the concentration dynamics near the ground state for (NLH) were investigated first compared to (NLW). The different blow-up behaviors were conjectured using matched asymptotics. Two other important nonlinear parabolic evolution equations are closely related: the harmonic heat flow and the Patlak-Keller-Segel equation. The first rigorous construction of the full sequence of quantized blow-up speed for a critical problem was done for the harmonic heat flow [141] in connexion with the related works [139, 140]. Due to the regularizing effects and the infinite speed of propagation of the heat equation, blow-up results for (NLH) are more rigid. In fact, the speeds at which the scale of a ground state can shrink or extend are no longer a continuum as in Theorems 2.3.1 and 2.2.3 for non-smooth solutions of (NLW): a quantization appears.

In [97, 98], the authors proved that in the radial case if the blow-up is of type II in the sense of Definition 2.1.7 then it must concentrate the ground state. We will then adopt the L^∞ point of view of type II blow-up to describe the results of the literature, as they were all done in the radial setting. We first start with the radial energy critical case $p = p_c$. In [55] the authors proved formally the existence of a countable family of type II blow-up solutions with different blow-up rates for $d = 3, 4, 5$. The rigorous existence of one of these solutions in dimension 4 has been done in [147].

Theorem 2.2.6 (Type II blow-up for critical (NLH) in low dimension [55, 147]). *Let* $p = p_c$.

(i) d=3,5. *For any $\ell \in \mathbb{N}^*$ there exists formally a radial type II blow-up solutions with $\|u\|_{L^\infty} \sim (T-t)^\ell$.*

(ii) d=4. *For any $\ell \in \mathbb{N}^*$ there exists formally a radial type II blow-up solutions with the asymptotic growth $\|u\|_{L^\infty} \sim (T-t)^\ell |log(T-t)|^{\frac{2\ell+2}{2\ell+1}}$. For $\ell = 1$ the existence of such a solution has been proved rigorously.*

In [55] it is predicted for $d=6$ the existence of one solution with a degenerate asymptotic. As the formal solution there does not seem to obey the universal lower bound (*2.1.8*) and as its construction involves the matching with a type I blow-up solution, the author of the present paper has some doubts regarding the existence of such a solution.

Let us know move to what is known for the dynamics around Q in the energy supercritical case. The asymptotic behavior of the ground state Q is different according to the position of p relatively to p_{JL}, see Theorem 2.2.2. For $p_c < p < p_{JL}$, Q has oscillations at infinity and ΛQ changes sign infinitely many times. Using among other tools some parabolic arguments for the number of intersections of solutions of (NLH), nonexistence of type II blow-up was proved in this range.

Theorem 2.2.7 (Non-existence of type II blow-up on (p_c, p_{JL}) [97, 98, 131]). *Let $p \in (p_c, p_{JL})$ and u be a radial solution blowing up at time T with $\lim_{r \to +\infty} u(0,r) = 0$. Then u blows up with type I.*

Note that we slightly modified the statement of [131] in view of the uniform decay estimate provided by Proposition C.1 in [99]. For $p > p_{JL}$, Q ceases to oscillate at infinity. A countable family of radial solutions blowing-up at different quantized rates was predicted formally in [72]. The authors gave a rigorous proof in an unpublished paper [73], before it was done in [125]. What is remarkable is that these blow-up rates were proven to be the only possible ones among radial solutions in [96, 128, 130]. We recall that γ is defined by (*2.2.5*).

Theorem 2.2.8 (Classification of type II blow-up on $(p_{JL}, +\infty)$ [97, 98, 131]). *Let $p > p_{JL}$ and recall that α is defined by (2.2.7).*

(i) There exists a sequence $(u_\ell)_{\ell > \frac{\alpha}{2}}$ of radial solutions of (NLH) blowing-up with type II blow-up:

$$\|u_\ell(t)\|_{L^\infty} \sim \frac{c_\ell}{(T-t)^{\frac{2\ell}{\alpha(p-1)}}}.$$

(ii) If $\gamma \notin 2\mathbb{N}$ and u is a type II blow-up solution, under some technical assumptions it blows-up with one of the above rates.

In the critical and supercritical case, all the aforementioned works rely on parabolic tools that are easier to use for radial parabolic problems. In particular, for the case of the semilinear heat equation in a bounded domain $(NLH\Omega)$, type II blow-up was shown to exist without a detailed description only for radial solutions in the ball and via non constructive techniques [99, 126]. In general, there was only one known result [136] of concentration dynamics of a periodic state for any semilinear equation, be it of parabolic or dispersive type. In this paper and on other non-radial works regarding self-similar blow-up, the underlying dynamics on the whole space were already stable which eases the analysis. The author of the present paper was able to give a more detailed construction of localized versions of the type II blow-up solutions of Theorem 2.2.8 for $(NLH\Omega)$ without parabolic tools. This is the main result of [26]. A technical assumption involves the following numbers for $n\in\mathbb{N}$ ($\triangle_n>0$ if $p>p_{JL}$):

$$-\gamma_n:=\frac{-(d-2)+\sqrt{\triangle_n}}{2},\ \triangle_n:=(d-2)^2-4cp_\infty+4n(d+n-2). \tag{2.2.14}$$

Theorem 2.2.9 (Non radial type II blow-up for the supercritical (NLH) [26]). *Let $p\in 2\mathbb{N}+1$ with $p>p_{JL}$ and $\epsilon>0$. Let $\Omega\subset\mathbb{R}^d$ be a smooth open bounded domain. For $x_0\in\Omega$ let $\chi(x_0)$ be a smooth cut-off function around x_0 with support in Ω. Pick $\ell\in\mathbb{N}$ satisfying $2\ell>\alpha$. Then, there exists a large enough regularity exponent:*

$$s_+=s_+(\ell)\in 2\mathbb{N},\ 4s_+\gg 1$$

such that under the non degeneracy condition:

$$\left(\frac{d}{2}-\gamma_n\right)\notin 2\mathbb{N} \text{ for all } n\in\mathbb{N} \text{ such that } d-2\gamma_n\le s_+, \tag{2.2.15}$$

there exists a solution u of $(NLH\Omega)$ with $u_0\in H^{s_+}(\Omega)$ (which can be chosen smooth and compactly supported) blowing up in finite time $0<T<+\infty$ by concentration of the ground state at a point $x_0'\in\Omega$ with $|x_0'-x_0|\le\epsilon$:

$$u(t,x)=\chi_{x_0}(x)\frac{1}{\lambda(t)^{\frac{2}{p-1}}}Q\left(\frac{x-x_0'}{\lambda(t)}\right)+v \tag{2.2.16}$$

with: (i) *x_0' is the only blow-up point of u.*

(ii) Blow-up speed*: (where α is defined in (2.2.7))*

$$\|u\|_{L^\infty}=c(u_0)(T-t)^{-\frac{2\ell}{\alpha(p-1)}}(1+o(1)),\quad \text{as } t\to T,\ c(u_0)>0, \tag{2.2.17}$$

$$\lambda(t)=c'(u_0)(1+o_{t\to T}(1))(T-t)^{\frac{\ell}{\alpha}},\quad \text{as } t\to T,\ c'(u_0)>0. \tag{2.2.18}$$

(iii) Asymptotic stability above scaling in renormalized variables*:*

$$\lim_{t\to T}\left\|\lambda(t)^{\frac{2}{p-1}}w\left(t,x_0+\lambda(t)x\right)\right\|_{H^s(\lambda(t)^{-1}(\Omega-\{x_0\})}=0\ \text{ for all }\ s_c<s\le s_+. \tag{2.2.19}$$

(iv) Boundedness below scaling*:*

$$\limsup_{t\to T}\|u(t)\|_{H^s(\Omega)}<+\infty,\quad \text{for all } 0\le s<s_c. \tag{2.2.20}$$

(v) Asymptotic of the critical norm*:*

$$\|u(t)\|_{H^{s_c}(\Omega)}=c(d,p)\sqrt{\ell}\sqrt{|\log(T-t)|}(1+o(1)),\quad \text{as } t\to T,\ c(d,p)>0. \tag{2.2.21}$$

Comments: In the above Theorem, the assumption (*2.2.15*) is technical to avoid logarithmic corrections in Hardy inequalities used in the proof. The analysis used in the proof originates from [23, 114, 138, 141]. Concentration of the ground state for the radial supercritical harmonic heat flow has been recently investigated in [9]. The non-radial analysis behind Theorem 2.2.9 is the first step to other non-radial problems, such as the interaction of several ground states, see [29, 117, 106] for related works.

Open problems: We only mention the non-radial problems, the radial being stated after Theorem 2.2.5. It would be very interesting to know more on non-radial stationary states and to investigate the dynamics near these solutions to enter more in the non-radial setting. However, no other solution than Q has been constructed in the energy supercritical setting. By combining the Lipschitz continuity result on type II blow-up solutions obtained for the wave equation, Theorem 2.2.5, and the non-radial construction started here, one could be able to construct a solution blowing-up by concentration of the ground state at k prescribed points, extending the previous result [107]. This last project is one of the current works of the author. Eventually, the interaction of several ground states in the energy supercritical setting is another very interesting question.

2.2.3 Sketch of the proof of Theorems 2.2.4, 2.2.5 and 2.2.9

We now give the main ideas behind the proof of Theorems 2.2.4, 2.2.5 and 2.2.9. The complete proofs are the subject of Chapter 3 for the first two theorems and of Chapter 4 for the last one. We start by sketching the proof for the wave setting, and then we turn to the differences when one treats the heat equation.

Notations for the wave equation

It is more convenient to use the vectorial formulation for (NLW)

$$(NLW)\ \begin{cases} \partial_t \boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}), \\ \boldsymbol{u}_{|t=0} = \boldsymbol{u}_0 \end{cases} \qquad (t,x)\in\mathbb{R}^+\times\mathbb{R}^d,\ \boldsymbol{u}(t,x):\mathbb{R}^d\to\mathbb{R}\times\mathbb{R}$$

for $\boldsymbol{u}=(u,u_t)$, and where

$$\boldsymbol{u}=\begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix},\quad \boldsymbol{F}(\boldsymbol{u}):=\begin{pmatrix} u^{(2)} \\ \Delta u^{(1)}+(u^{(1)})^p \end{pmatrix}.$$

The corresponding scalar product is $(\boldsymbol{u},\boldsymbol{v}) = \int u^{(1)}v^{(1)}+u^{(2)}v^{(2)}$. The scaling invariance for this equation is for $\lambda>0$

$$\boldsymbol{u}_\lambda(x):=\begin{pmatrix} \lambda^{\frac{2}{p-1}}u^{(1)}(\lambda y) \\ \lambda^{\frac{2}{p-1}+1}u^{(2)}(\lambda y) \end{pmatrix}.$$

We let the infinitesimal generator of the scaling group be:

$$\boldsymbol{\Lambda u}:=\begin{pmatrix} \Lambda^{(1)}u^{(1)} \\ \Lambda^{(2)}u^{(2)} \end{pmatrix}:=\begin{pmatrix} \left(\frac{2}{p-1}+y.\nabla\right)u^{(1)} \\ \left(\frac{2}{p-1}+1+y.\nabla\right)u^{(2)} \end{pmatrix}.$$

We keep the notation for the stationary state

$$\boldsymbol{Q}:=\begin{pmatrix} Q \\ 0 \end{pmatrix}.$$

We define, where $E[x]$ denotes the integer part of x:

$$\begin{cases} k_0:=E[\frac{d}{2}-\gamma]>1, \\ \delta_0:=\frac{d}{2}-\gamma-k_0,\ 0<\delta_0<1. \end{cases}$$

because we are assuming $\left(\frac{d}{2}-\gamma\right)\notin\mathbb{N}$. For $L\gg 1$ a large integer we define

$$s_L := k_0+1+L.$$

We introduce a generic radial, C^∞ cut-off function:

$$\chi\equiv 1 \ on \ \mathcal{B}^d(1),\ \chi\equiv 0 \ on \ \mathbb{R}^d\backslash\mathcal{B}^d(2) \ \text{ and } \ \chi_B : y\mapsto \chi\left(\frac{y}{B}\right) \tag{2.2.22}$$

for $B>0$, and we use the notation $\chi_B\boldsymbol{u}=(\chi_B u^{(1)},\chi_B u^{(2)})$. Given $b_1>0$ we introduce two particular zones for $0<\eta\ll 1$

$$B_0 := b_1^{-1},\ \ B_1 := b_1^{-1-\eta}.$$

We use the notation $a\lesssim b$ if $a\leq Cb$ for C an independent constant which can change from lines to lines.

Outline of the proof for the wave equation

Step 1 *Linear analysis.* Thanks to the scaling invariance of the equation, the linearized operator can be considered without loss of generality at scale 1, i.e. near $\boldsymbol{Q}$. This linear operator is

$$\boldsymbol{H\varepsilon} := \begin{pmatrix} -\varepsilon^{(2)} \\ -\Delta\varepsilon^{(1)}-pQ^{p-1}\varepsilon^{(1)}\end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \mathcal{L} & 0\end{pmatrix}\boldsymbol{\varepsilon},\ \ \mathcal{L}:=-\Delta-pQ^{p-1}.$$

The auxiliary linear operator $\mathcal{L}$ that appears is self-adjoint and positive on $\dot H^1$ (see [78])

$$\int u\mathcal{L}u\geq c\|u\|_{\dot H^1},\ \ c>0. \tag{2.2.23}$$

Hence the instabilities near Q are not consequences of linearly unstable eigenfunctions, but are due to an accumulation of the spectrum of $\mathcal{L}$ near 0. To be more precise the linear flow $\boldsymbol{\varepsilon}_t=-\boldsymbol{H\varepsilon}$ admits the following conserved quantities

$$\mathcal{E}_k := \int \varepsilon^{(1)}\mathcal{L}^s\varepsilon^{(1)}+\varepsilon^{(2)}\mathcal{L}^{s-1}\varepsilon^{(2)} \tag{2.2.24}$$

which are positive for smooth and compactly supported functions thanks to (*2.2.23*), in particular for $s=1$ thanks to (*2.2.23*) one gets that the $\dot H^1\times L^2$ norm of the solution is conserved. But from the Cauchy theory Proposition 2.1.3 one knows that these are the critical and supercritical norms that are important, and the inequality (*2.2.23*) ceases to be true for iterates of $\mathcal{L}$ and thus these norms are not dispersed well. The obstruction is that in the Sobolev space $\dot H^s$, $\mathcal{L}^k$ admits a certain number of zeros. These zeros can be computed almost explicitely, indeed the generalized kernel of $\mathcal{L}$ is

$$\{f \text{ radial and smooth},\ \exists j\in\mathbb{N},\ \mathcal{L}^j f=0\} = \text{Span}\,(T_i)_{i\in\mathbb{N}},$$

where T_i is radial, with

$$T_0=\Lambda Q,\ \ \mathcal{L}T_0=0,\ \ \mathcal{L}T_{i+1}=-T_i,\ \ T_i(r)\sim r^{-\gamma+2i}. \tag{2.2.25}$$

As the potential in $\mathcal{L}$ is radial, such a result is obtained by standard applications of ODE theory. The generalized kernel of $\boldsymbol{H}$ is accordingly

$$\{\boldsymbol{f} \text{ radial},\ \exists j\in\mathbb{N},\ \boldsymbol{H}^j\boldsymbol{f}=0\} = \text{Span}\,(\boldsymbol{T}_i)_{i\in\mathbb{N}},\ \text{ with }\ \boldsymbol{T}_{2i}=\begin{pmatrix} T_{2i}\\ 0\end{pmatrix},\ \boldsymbol{T}_{2i+1}=\begin{pmatrix} 0 \\ T_{2i+1}\end{pmatrix}.$$

and one has $\boldsymbol{H}\boldsymbol{T}_{i+1}=-\boldsymbol{T}_i$. Therefore, $(\boldsymbol{T}_i)_{i\in\mathbb{N}}$ can be considered, roughly because they do not decay well, as the ordered generators of the tangent space to the center manifold near $\boldsymbol{Q}$. $\boldsymbol{H}$ then enjoys a Hardy type coercivity estimate outside a truncated and localized version of this set:

Proposition 2.2.10. *There exists $C>0$ and compactly supported profiles $(\boldsymbol{\Phi}_i)_{0\leq i\leq L}$ satisfying*

$$(\boldsymbol{T}_i,\boldsymbol{\Phi}_j)=\delta_{i,j}$$

such that for any $\varepsilon\in\dot{H}^\sigma\cap\dot{H}^{s_L}\times\dot{H}^{\sigma-1}\cap\dot{H}^{s_L-1}$ with $\varepsilon\in$ Span$(\boldsymbol{\Phi}_i)_{0\leq i\leq L}^\perp$ there holds

$$\sum_{k=0}^{s_L}\int\frac{|\nabla^k\varepsilon^{(1)}|^2}{1+|x|^{2(s_L-k)}}+\sum_{k=0}^{s_L-1}\int\frac{|\nabla^k\varepsilon^{(2)}|^2}{1+|x|^{2(s_L-1-k)}}\leq C\mathcal{E}_{s_L} \qquad (2.2.26)$$

where $\mathcal{E}_{s_L}$ is defined by (2.2.24).

Here the localized orthogonalities $\boldsymbol{\varepsilon}\perp\boldsymbol{\Phi}_i$ for $0\leq i\leq L$ ensure that $\boldsymbol{\varepsilon}$ is not one of the elements $(\boldsymbol{T}_i)_{0\leq i\leq L}$ of the generalized kernel of $\boldsymbol{H}$ for which this estimate fails. The proof of this estimate relies on the manipulation of hardy type inequalities and on the minimization of suitable functionals. Such a coercivity estimate is linked to some dispersion for $\boldsymbol{\varepsilon}$ (in particular, a Morawetz type estimate will hold). Indeed, these are the estimates that one obtains without orthogonality conditions for the standard linear wave equation where $\mathcal{L}$ is replaced by $-\Delta$ for $s\in[1,...,\frac{d}{2})$.

Hence, a localized version of the subspace Span$(T_i)_{1\leq i\leq L}$ is the part of a certain Sobolev space of functions that do not decay well via the linear flow. Moreover, as

$$\boldsymbol{H}\boldsymbol{T}_1=-\boldsymbol{\Lambda}\boldsymbol{Q}=-\frac{\partial}{\partial\lambda}(\boldsymbol{Q}_\lambda)_{|\lambda=1}$$

this space contains the profile which, at the linear level, makes the scale of the ground state change. We now investigate roughly the dynamics on this space, a combination of linear flow and scale change, before refining the analysis.

Step 2 *Formal computation of the dynamical system for the coordinates on the truncated center manifold.* Since $\boldsymbol{\Lambda}\boldsymbol{Q}=\boldsymbol{T}_0$, in view of Proposition 2.2.10 we look for a solution under the form

$$\boldsymbol{u}(t)=\left(\boldsymbol{Q}+\sum_1^L b_i(t)\boldsymbol{T}_i+\boldsymbol{\varepsilon}(t)\right)_{\frac{1}{\lambda(t)}},\quad \boldsymbol{\varepsilon}\in\text{Span}(\boldsymbol{\Phi}_i)_{0\leq i\leq L}^\perp,$$

decomposing a priori between a part $\sum_1^L b_i(t)\boldsymbol{T}_i$ that is supposed to decay slowly and interact with Q, and a remainder $\boldsymbol{\varepsilon}$ that is supposed to decay faster. Introducing the renormalize time

$$\frac{ds}{dt}=\frac{1}{\lambda(t)},\quad s(0)=s_0, \qquad (2.2.27)$$

the renormalized function

$$\boldsymbol{v}=\boldsymbol{u}_{\lambda(t)},\quad \boldsymbol{v}=\boldsymbol{Q}+\sum_1^L b_i(t)\boldsymbol{T}_i+\boldsymbol{\varepsilon}(t),\quad y:=\frac{r}{\lambda(t)} \qquad (2.2.28)$$

then solves the renormalized flow equation

$$\boldsymbol{v}_s - \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda v} = \boldsymbol{F}(\boldsymbol{v}) \tag{2.2.29}$$

which given the above decomposition can be rewritten as

$$\sum_1^L b_{i,s}\boldsymbol{T}_i + \boldsymbol{\varepsilon}_s - \frac{\lambda_s}{\lambda}\sum_1^L b_i \boldsymbol{\Lambda T}_i - \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda\varepsilon} - \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda Q} = \sum_1^L b_i \boldsymbol{T}_{i-1} - \boldsymbol{H\varepsilon} + \boldsymbol{NL}$$

where $\boldsymbol{N}L$ stands for the nonlinear terms that we expect of lower order. Identifying the terms in this equation leads to $-\frac{\lambda_s}{\lambda} = b_1$. The asymptotic behavior (*2.2.25*) gives $\boldsymbol{\Lambda T}_i \sim (2i-\alpha)\boldsymbol{T}_i$ and therefore neglecting the effects of the nonlinear terms in the above equation yields the following finite dimensional dynamical system for the parameters[3]

$$\begin{cases} \frac{\lambda_s}{\lambda} &= -b_1, \\ b_{i,s} &= -(i-\alpha)b_1 b_i + b_{i+1} \ \text{ for } 1 \le i \le L. \end{cases} \tag{2.2.30}$$

The natural question is: what type of special solutions does the approximate dynamics possess ? For $\ell > \alpha$, there exists a solution $(\lambda^e(s), b^e(s))$ of (*2.2.30*) such that coming back to original time variables with (*2.2.27*), $\lambda^e(t)$ goes to 0 in finite time T with asymptotics $\lambda^e \sim (T-t)^{\frac{\ell}{\alpha}} \sim s^{-\frac{\ell}{\ell-\alpha}}$ and $s(t) \to +\infty$ as $t \to T$. This is the blow-up dynamics we are going to construct rigorously. In renormalized time s these profiles are given by:

$$\begin{cases} b_i^e(s) = \frac{c_i}{s^i} \ \text{ for } \ 1 \le i \le \ell, \\ b_i^e \equiv 0 \ \text{ for } \ i > \ell. \end{cases} \tag{2.2.31}$$

Moreover, in renormalized variables, the light cone $r \lesssim T-t$ corresponds to the zone $y \lesssim s$.

Step 3 *The approximate blow-up profile.* Following the formal computation done in (ii), we want to take as approximate blow-up profile $(\boldsymbol{Q} + \sum_1^L b_i^e \boldsymbol{T}_i)$. There are two problems however. First, for i large enough the profile $\boldsymbol{T}_i$ is unbounded at infinity from (*2.2.25*). We therefore need to localize it, but where? A priori, we know that the important zone in original variables is the light cone $r \lesssim T-t$ or equivalently $y \lesssim s$ and we are going to cut slightly after this area: at $y \sim s^{1+\eta}$ for some $0 < \eta \ll 1$. The second problem is that the nonlinear terms will affect the dynamics. To deal with it, one inverts successive elliptic equations to correct the Ansatz $(\boldsymbol{Q} + \sum_1^L b_i^e \boldsymbol{T}_i)$ by a lower order term $\boldsymbol{S}$ which pushes the error outside the light cone. This uses ODE techniques and a careful treatment of all the parameters at stake.

Proposition 2.2.11 (The approximate blow-up profile). *There is a constant $g(d,p) > 0$ such that the following holds. Let $I = (s_0, s_1)$ and $(b_i(s))_{1 \le i \le L}$ be C^1 real-valued functions on I such that*

$$|b_i| \lesssim s^{-i}, \ \ 0 < b_1 \sim s^{-1} \ \ \text{and} \ \ |b_{1,s}| \lesssim s^{-2}. \tag{2.2.32}$$

There exists a profile $\boldsymbol{Q}_b$ given by

$$\boldsymbol{Q}_b = \boldsymbol{Q} + \chi_{B_1}\boldsymbol{\alpha}_b, \ \ \boldsymbol{\alpha}_b := \sum_1^L b_i \boldsymbol{T}_i + \boldsymbol{S}(b, y)$$

[3] With the convention that $b_{L+1} = 0$.

with the following properties. The correction $\chi_{B_1}\boldsymbol{S}(b,y)$ satisfies for $j=1,2$:

$$|\chi_{B_1}S^{(j)}(b,y)|\leq C(1+y)^{-\gamma-g-(j-1)}, \quad |\chi_{B_1}\frac{\partial}{\partial b_i}S^{(j)}(b,y)|\leq C(1+y)^{-\gamma-g-(j-1)+i} \tag{2.2.33}$$

$$|\boldsymbol{S}(b,y)|\lesssim s^{-2} \ \ \textit{and} \ \ |\frac{\partial}{\partial_{b_i}}\boldsymbol{S}(b,y)|\lesssim s^{-1} \ \textit{on compact sets}^4. \tag{2.2.34}$$

One has the identity

$$\partial_s(\boldsymbol{Q}_b)-\boldsymbol{F}(\boldsymbol{Q}_b)+b_1\boldsymbol{\Lambda}\boldsymbol{Q}_b=\boldsymbol{\psi}+\chi_{B_1}\boldsymbol{Mod} \tag{2.2.35}$$

where the modulation term is

$$\boldsymbol{Mod}=\sum_{i=1}^{L}[b_{i,s}+(i-\alpha)b_1b_i-b_{i+1}]\left[\boldsymbol{T}_i+\frac{\partial \boldsymbol{S}}{\partial b_i}\right]. \tag{2.2.36}$$

For $0<\eta<\eta^*(d,p,L)\ll 1$ small enough and $s_0\gg 1$ large enough one has the estimates on the error $\boldsymbol{\psi}=\boldsymbol{\psi}(b)$:

$$\|\boldsymbol{\psi}\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\lesssim s^{-1-g}, \quad \int_{\mathbb{R}^d}\psi^{(1)}\mathcal{L}^{s_L}\psi^{(1)}+\psi^{(2)}\mathcal{L}^{s_L-1}\psi^{(2)}\lesssim s^{-1-L-(1-\delta_0)(1+\eta)}, \tag{2.2.37}$$

and[5]

$$|\psi|\lesssim s^{-L-3} \ \ \textit{on compact sets.} \tag{2.2.38}$$

The interpretation of Proposition 2.2.11 is the following.

(i) The a priori bound (*2.2.32*) on the parameters is natural since it holds for the formal blow-up profile found in (ii).

(ii) $\boldsymbol{S}$ is indeed a lower order correction. From the asymptotic behavior of $\boldsymbol{T}_i$ (*2.2.25*), the size of the parameters (*2.2.32*), the bounds on $\boldsymbol{S}$ (*2.2.33*) and (*2.2.34*), $\boldsymbol{S}$ and $\frac{\partial}{\partial b_i}\boldsymbol{S}$ are lower order compared to $\sum b_i\boldsymbol{T}_i$ and $\frac{\partial}{\partial b_i}\sum b_j\boldsymbol{T}_j=T_i$ and on compact sets ($O(s^{-1})\ll 1$ since $s_0\ll 1$ versus $O(1)$) and at infinity (this is the $O(|y|^{-g})$ gain).

(iii) The excitation $\chi_{B_1}\alpha_b$ is located on the slightly enlarged light cone $y\lesssim s^{1+\eta}$ from (*2.2.38*) and since $b_1\sim s$ from (*2.2.32*). It does not appear clearly here but the error ψ_b is essentially localized outside this slightly enlarged light cone.

(iv) The gain on the error is of size $s^{-\eta(1-\delta_0)}$ and obtained by cutting slightly after the light cone. Let us explain formally this. The main order pieces of the approximate blow-up profile being the excitations of the form $\chi_{B_1}b_i\boldsymbol{T}_i$, to see that the error is of lower order one has to compare the cumulated error it will produce for the dynamics $\int_{s_0}^s|\boldsymbol{\psi}_b|ds$ with the size of these excitations. As $\boldsymbol{\psi}_b$ is very small on compact sets the main part of the error is located at the boundary of the light cone $y\sim s$. There, a computation using (*2.2.25*), (*2.2.26*), (*2.2.37*) and Cauchy-Schwarz gives for exemple for the first component:

$$\forall i\leq\frac{L}{2}, \quad \int_{y=s}^{2s}b_{2i}|T_{2i}|\sim s^{d-\gamma}, \quad \int_{s_0}^{s}\int_{y=s'}^{2s'}|\psi_b^{(1)}|ds'\lesssim s^{d-\gamma-\eta(1-\delta_0)}$$

(and similarly for the second).

[5] Here on compact sets means for a fixed compact set and for s_0 large enough.

(v) This proposition provide a L-dimensional manifold relying on L parameters for which the evolution by (NLW) is almost explicit. Indeed, (*2.2.35*) and (*2.2.36*) can be rewritten[6] as

$$\boldsymbol{F}(\boldsymbol{Q}_b)=\sum_1^L(b_{i+1}-(i-\alpha)b_1b_i)\frac{\partial}{\partial_{b_i}}(\boldsymbol{Q}_b)+b_1\frac{\partial}{\partial\lambda}[(\boldsymbol{Q}_b)_\lambda]_{|\lambda=1}-\boldsymbol{\psi}_b$$

which means that at main order the parameters should evolve via the finite dimensional dynamical system (*2.2.30*) as predicted by the formal computation.

Step 4 *The trapped regime.* We now fix $\ell\in\mathbb{N}$, $\ell>\alpha$ and $L\gg\ell$. We would like to state that there exists a solution of (NLW) that stays until the blow-up time close to the approximate blow-up profile $(\boldsymbol{Q}_{b^e})_{\frac{1}{\lambda^e}}$. Rather than working in original variables (the solutions might not blow-up at the same time for exemple), we will work with the renormalized flow. For that, we need to know at which scale to renormalize a solution and to provide a suitable decomposition of a solution close to the manifold of ground states $(\boldsymbol{Q}_\lambda)_{\lambda>0}$. A solution $\boldsymbol{u}$ will be in the suitable neighborhood for that purpose if it satisfies the condition

$$\exists\tilde\lambda>0,\quad\left|\begin{array}{l}\|u^{(1)}-Q_{\frac{1}{\tilde\lambda}}\|_{L^\infty}+\tilde\lambda\|u^{(2)}\|_{L^\infty}<\frac{\kappa}{\tilde\lambda^{\frac{2}{p-1}}}\text{ and}\\ \|u_{\tilde\lambda}-Q\|_{L^\infty}<(\boldsymbol{u}_{\tilde\lambda}-\boldsymbol{Q},\boldsymbol{\Phi}_1)\end{array}\right.\qquad(2.2.39)$$

for a small constant κ. As an application of the implicit function one obtains:

Lemma 2.2.12 (Geometrical decomposition). *There exist $\kappa,K>0$ such that for any solution of (NLW) $\boldsymbol{u}\in\mathcal{C}^1([0,T),(L^\infty\times L^\infty))$ satisfying (2.2.39) on $t\in[0,T)$ there exist a unique choice of the parameters $\lambda:[0,T)\to(0,+\infty)$ and $b:[0,T)\to\mathbb{R}^L$ such that $b_1>0$ and*

$$u=(\boldsymbol{Q}_b+\boldsymbol{\varepsilon})_{\frac{1}{\lambda}},\quad\boldsymbol{\varepsilon}\in\mathit{Span}(\boldsymbol{\Phi}_i)^\perp_{0\le i\le L},\quad\sum_1^L|b_i|+\|\boldsymbol{\varepsilon}\|_{L^\infty\times L^\infty}\le K\kappa.\qquad(2.2.40)$$

Moreover, λ, $b_1,...,b_L$ and z are $\mathcal{C}^1$ in time functions.

For any solution satisfying (*2.2.39*) on some time interval, the renormalized flow (*2.2.27*) and (*2.2.28*) associated to the scale λ given by the above Lemma is well-defined. We now focus on a class of solutions staying close to $\boldsymbol{Q}_b$ for this renormalized flow. Note that from now on we omit some technicalities for the sake of clarity.

Definition 2.2.13 (Trapped solution). A solution is said to be trapped on $[0,\tilde T)$ if it satisfies the following. It starts in the following initial neighborhood of the approximate blow-up profile:

$$\boldsymbol{u}(0)=\boldsymbol{Q}_{b(0)}+\boldsymbol{\varepsilon}(0),\quad\boldsymbol{\varepsilon}(0)\in\mathrm{Span}(\boldsymbol{\Phi}_i)^\perp_{0\le i\le L},\qquad(2.2.41)$$

with[7], for $s_0\gg1$ and $0<\tilde\eta_2\ll\tilde\eta_1\ll\eta$,

$$\|\boldsymbol{\varepsilon}(0)\|_{\dot H^\sigma\times\dot H^{\sigma-1}}+\sqrt{\mathcal{E}_{s_L}(0)}\ll s_0^{-L-(1-\delta_0)(1+\tilde\eta_1)},\quad\sum_1^L s_0^{i+\tilde\eta_2}|b_i(0)-b_i^e(s_0)|\ll1,\quad\lambda(0)=1.\qquad(2.2.42)$$

[6]Up to an error that has the same size as the previous one $\boldsymbol{\psi}_b$, and still with the convention that $b_{L+1}=0$.
[7]The renormalization factor s^{-i} is precisely the size of b_i^e.

It satisfies on $[0,\tilde{T})$ the conditions of the decomposition Lemma 2.2.12. For the renormalized flow (*2.2.27*) and (*2.2.28*) associated to the scale $\lambda(t)$ provided by this Lemma, the decomposition (*2.2.40*) for all $[s_0, s(\tilde{T}))$ lives in a bigger neighborhood than the initial one[8]:

$$\|\boldsymbol{\varepsilon}(s)\|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \lesssim s^{-(\sigma-s_c)\frac{\ell}{\ell-\alpha}}, \quad \sqrt{\mathcal{E}_{s_L}(s)} \lesssim s^{-L-(1-\delta_0)(1+\tilde{\eta}_1)}, \quad \sum_1^L s^{i+\tilde{\eta}_2}|b_i(s)-b_i^e(s)| \lesssim 1, \quad (2.2.43)$$

$$\lambda(s) \sim s^{-\frac{\ell}{\ell-\alpha}}. \quad (2.2.44)$$

Step 5 *Solutions trapped forever are solutions described by Theorem 2.2.4.* If $\boldsymbol{u}$ is a solution that is trapped on its maximal interval of existence $[0,T)$, from the estimate (*2.2.44*) on the scale and the definition of the renormalized flow (*2.2.27*), it must blow-up $T<+\infty$ with the asymptotic $\lambda(t)\sim(T-t)^{\frac{\ell}{\alpha}}$ for the scale and the renormalized flow is global $\lim_{t\to T} s(t) = +\infty$. Moreover, using the bounds (*2.2.43*) and (*2.2.44*) satisfied by trapped solutions, such solutions indeed blow up with the refined asymptotics (*2.2.9*), (*2.2.10*), (*2.2.11*),(*2.2.12*) and (*2.2.13*) of Theorem 2.2.4.

Therefore, to end the proof of Theorem 2.2.4 we will show that there exist solutions that are trapped forever. For that, one needs to investigate the behavior of trapped solutions.

Step 6 *Analysis in the trapped regime.* The evolution of a trapped solution is computed by plugging the decomposition *2.2.40* in (NLW) using the approximate dynamics (*2.2.35*):

$$\begin{array}{rcll}\boldsymbol{\varepsilon}_s - \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda\varepsilon} + \boldsymbol{H}(\boldsymbol{\varepsilon}) & = & -\boldsymbol{Mod}(t) + (\frac{\lambda_s}{\lambda}+b_1)\boldsymbol{\Lambda Q}_b - \boldsymbol{\psi}_b & \\ & & +\boldsymbol{F}(\boldsymbol{Q}_b+\boldsymbol{\varepsilon}) - \boldsymbol{F}(\boldsymbol{Q}_b) + \boldsymbol{H}_b(\boldsymbol{\varepsilon}) & \} := \mathbf{NL}(\varepsilon) \\ & & +\boldsymbol{H}(\boldsymbol{\varepsilon}) - \boldsymbol{H}_b(\boldsymbol{\varepsilon}) & \} := \mathbf{L}(\varepsilon)\ ,\end{array} \quad (2.2.45)$$

where $NL^{(1)}=0$ and $NL^{(2)}=\sum_{k=2}^p C_k^p Q_b^{p-k}(\varepsilon^{(1)})^k$ is the nonlinear term, $\boldsymbol{H}_b$ is the linearized operator near $\boldsymbol{Q}_b$ and $\boldsymbol{L}$ is a lower order potential:

$$\boldsymbol{H}_b = \begin{pmatrix} 0 & -1 \\ -\Delta - pQ_b^{p-1} & 0 \end{pmatrix}, \quad \boldsymbol{L} = \begin{pmatrix} 0 & 0 \\ p(Q_b^{p-1}-Q^{p-1}) & 0 \end{pmatrix}.$$

The dynamics near the approximate blow-up profile is then described by how the scale λ, the parameters b_i and the remainder ε evolve and interact for a trapped solution. To compute the evolution of the parameters, one takes the scalar product between (*2.2.45*) and the orthogonality profiles $\boldsymbol{\Phi}_i$ and estimate all terms. The contribution of the error ψ_b is estimated by (*2.2.38*) and that of the remainder ε by the a priori bound (*2.2.43*) as this norm controls ε on compact sets from the coercivity (*2.2.26*). The nonlinear term's influence is controlled by the fact that as ε is controlled at a low and a high regularity it has a L^∞ bound. For the last term b_L however, as ε "contains" the next term in the generalized kernel T_{L+1}, the analysis requires a technical flux computation at the boundary of the light cone which we avoid to talk about here.

To control the infinite dimensional remainder $\boldsymbol{\varepsilon}$, one uses energy estimates built on the linearized operator (*2.2.24*) because of the coercivity (*2.2.26*). The main ingredients are the use of dispersion through a

[8]In original variables, the first bound just means that $\boldsymbol{\varepsilon}_{\frac{1}{\lambda}}$ is bounded in $\dot{H}^\sigma\times\dot{H}^{\sigma-1}$.

Morawetz type estimate to manage some local interactions, the manipulation of Hardy type inequalities and Sobolev embeddings for the nonlinear term. Also the renormalized flow is dezooming the blow-up, what creates a damping for ε at supercritical regularities. The main force term being ψ_b, the differential bounds will involve the quantities (*2.2.37*). The different modulation estimate for b_L requires in fact the more technical study of a modified energy estimate for the sum "$\boldsymbol{\varepsilon} + b_L\chi_{B_1}\boldsymbol{T}_L$" which we avoid to talk about here as well.

Lemma 2.2.14 (Boostrap analysis). *Let u be a trapped solution on $[s_0, s_1)$, then it enjoys:*

(i) Modulation estimates[9]

$$\left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_1^L |b_{i,s} + (i-\alpha)b_1 b_i - b_{i+1}| \lesssim \|\boldsymbol{\psi}\|_{loc} + \|\boldsymbol{\varepsilon}\|_{loc} \lesssim s^{-L-1-(1-\delta_0)\tilde{\eta}_1} \tag{2.2.46}$$

(ii) Lyapunov monotonicity for the remainder

$$\frac{d}{ds}\|\boldsymbol{\varepsilon}\|^2_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \ll s^{-1-g}, \quad \frac{d}{ds}\mathcal{E}_{s_L} \lesssim s^{-1-2L-2(1-\delta_0)(1+\eta)}. \tag{2.2.47}$$

Step 7 *Exit of the trapped regime.* Any solution starting close to the approximate blow-up profile $\boldsymbol{Q}_b$ in the sense of (*2.2.41*) is in the trapped regime at least for small times. Let then $T_{\text{exit}}(\boldsymbol{u}_0) > 0$ be the maximal time on which the solution is trapped, and $S_{\text{exit}} = s(T_{\text{exit}})$. We now investigate why a solution leaves the trapped regime, and what happens at time S_{exit}. First, from the definition, at time $s(T_{\text{exit}})$ one of the bounds in (*2.2.43*) or the bound (*2.2.44*) must be violated.

To know precisely which bound fails, we reintegrate the bootstrap differential bounds (*2.2.46*) and (*2.2.47*). For the remainder ε, (*2.2.47*) implies that this part of the solution is stable and that in fact a better bound than the one for ε in (*2.2.43*) holds (as $\tilde{\eta}_1 \ll \eta$). Therefore the solution cannot escape the trapped regime because the remainder has grown large. For the scale λ, similarly, (*2.2.46*) implies that (*2.2.44*) is always verified and so the exit cannot happen because the scale behaves badly. Therefore, it is the bound in (*2.2.43*) concerning the parameters that must fail. But the parameters almost evolve according to the approximate differential system (*2.2.30*) thanks to (*2.2.46*). Therefore, it suffices to check the instabilities of the dynamical system (*2.2.30*) near the special solution b^e.

Lemma 2.2.15 (Exit of the trapped regime). *Let $U_i(s) = s^{-i}[b_i(s) - b_i^e(s)]$ for $1 \leq i \leq L$. Then the linearization of (2.2.30) close to $(b_i^e)_{1\leq i\leq L}$ possesses $\ell - 1$ directions of instability involving only the ℓ-th first parameters. More precisely, there exists a linear change of variables*

$$(U_1, ..., U_\ell) \mapsto (V_1(s), ..., V_\ell(s))$$

and $\ell - 1$ positive numbers $\mu_i > 0$ for $2 \leq i \leq \ell$ such that a solution escapes the trapped regime if and only if at time S_{exit} there holds

$$\sum_2^\ell S_{exit}^{\tilde{\eta}_2} |V_i(S_{exit})| = 1, \tag{2.2.48}$$

and moreover, for any trapped solution the parameters V_i for $2 \leq i \leq \ell$ evolve according to the unstable dynamics

$$V_{i,s} = \frac{\mu_i}{s} V_i + O(s^{-1-2\tilde{\eta}_2}). \tag{2.2.49}$$

[9] With the convention $b_{L+1} = 0$ and where $\|\cdot\|_{\text{loc}}$ denotes a local norm.

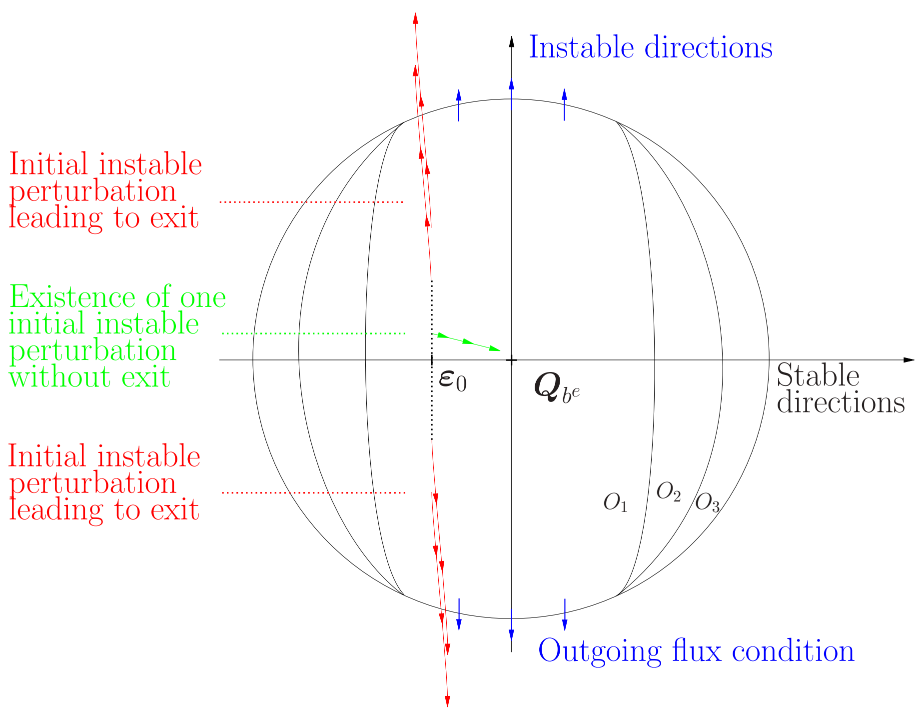
Step 8 *Existence of a solution staying trapped until its maximal time of existence.* Fix $V_1(0)$, $U_i(0)$ for $\ell+1 \leq i \leq L$ and $\varepsilon(0)$ satisfying the initial conditions (*2.2.41*) and (*2.2.42*) and let only $(V_2(0), ..., V_\ell(0))$ vary. We look at all solutions starting from (*2.2.41*) corresponding to all the choices of $(V_2(0), ..., V_\ell(0))$ such that (*2.2.42*) holds. If they leave the trapped regime then we define the function

$$f[V_1(0), U_{\ell+1}(0), ..., U_L(0), \varepsilon(0)](V_2(0), ..., V_\ell(0)) := \frac{S_{\text{exit}}^{\tilde{\eta}_2}}{s_0^{\tilde{\eta}_2}} V_i(S_{\text{exit}}).$$

The domain of $f[V_1(0), U_{\ell+1}(0), ..., U_L(0), \varepsilon(0)]$ is a subset of the $\ell-1$-dimensional ball $B(0, s_0^{-\tilde{\eta}_2})$ and this function takes values in its boundary $S(0, s_0^{\tilde{\eta}_2})$ thanks to (*2.2.48*). Moreover, from a continuity argument, using the outgoing condition for the flux (*2.2.49*), f is continuous, the sphere $S(0, s_0^{\tilde{\eta}_2})$ belongs to its domain of definition and f is the identity on this set. As an application of Brouwer's continuity theorem, one obtains that the domain of f cannot be the entire ball $B(0, s_0^{\tilde{\eta}_2})$. The immediate consequence of this fact is the following.

Proposition 2.2.16 (Existence of solutions staying trapped forever). *Given $U_i(0)$ for $\ell+1 \leq i \leq L$, $V_1(0)$ and ε satisfying the initial conditions (2.2.41) and (2.2.42), there exists $(V_2(0), ..., V_\ell(0))$ satisfying (2.2.42) such that the corresponding solution starting from (2.2.41) is trapped until its maximal time of existence.*

From (v) the proof of Theorem 2.2.4 is over. The next pictures illustrate the bootstrap analysis we performed.



Bootstrap: as long as a solution starting in O_1 belongs to O_3 it lives in fact in the smaller neighborhood O_2 and therefore can only escape via the instable directions.

Step 9 *The manifold construction and proof of Theorem 2.2.5.* The existence of the special type II blow-up solutions is implied by the existence of solutions staying trapped for all times given by Proposition 2.2.16.

To show that there exists a Lipschitz manifold of codimension $\ell - 1$ of solutions blowing up according to the dynamics of Theorem 2.2.4, we will then show that there exists a Lipschitz manifold of initial data satisfying (*2.2.41*) and (*2.2.42*) such that the corresponding solution stays trapped for all times.

To this aim, we show that the parameters along the unstable directions $(V_2(0), ..., V_\ell(0))$ associated to the initial stable perturbation $V_1(0)$, $U_i(0)$ for $\ell + 1 \leq i \leq L$ and ε, given by Proposition 2.2.16, are unique and depend in a Lipschitz way on the initial stable perturbation. The main result behind the Lipschitz manifold structure is the following.

Proposition 2.2.17. *Let $V_1(0)$, $V_1'(0)$, $U_i(0)$ and $U_i'(0)$ for $\ell + 1 \leq i \leq L$, and ε, ε' satisfying the initial conditions (2.2.41) and (2.2.42), and let $(V_2(0), ..., V_\ell(0))$ and $(V_2'(0), ..., V_\ell'(0))$ be the parameters given by Proposition 2.2.16. Then*

$$\begin{aligned}\sum_2^\ell |V_i(0) - V_i'(0)| \quad &\lesssim \quad |V_1(0) - V_1'(0)| + \sum_{\ell+1}^L |U_i(0) - U_i'(0)| + \|\varepsilon(0) - \varepsilon'(0)\|_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \\ &\quad + \|\varepsilon(0) - \varepsilon'(0)\|_{\dot{H}^{s_L} \times \dot{H}^{s_L - 1}}\end{aligned}$$

To show the above proposition, we analyse the evolution equations for the differences of parameters $b_i - b_i'$ and errors $\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}'$. We find that $\ell - 1$ differences of parameters evolve according to an unstable linear dynamics (from Lemma 2.2.15), and that the dynamics of the $L - \ell + 1$ others and the difference of errors is stable. The differences of the stable parameters and errors only have a small feedback on the time evolution of the unstable parameters. Thus, if the initial difference of the unstable parameters is too big compared to the initial differences of the stable parameters and errors, the unstable linear dynamics wins and expels the differences of unstable parameters away from 0. Hence one of the two solutions cannot blow up according to our scenario, yielding a contradiction. Quantifying this reasoning gives precisely the result of the above Proposition.

Some technicalities arise as well, as one has to compare the two solution at different times (one can blow-up before the other). This difference of times implies that one has a good dissipative structure for the difference of errors only if one works at a lower regularity level $\dot{H}^{s_L - 1} \times \dot{H}^{s_L - 2}$ to have an a priori estimate for a higher regularity error term. One then needs to go from to trapped regime (Definition 2.2.13) associated to the decomposition on the truncated manifold at order L $(T_i)_{1 \leq i \leq L}$ to that of order $(T_i)_{1 \leq i \leq L-1}$ to be able to use the bootstrap analysis framework.

After that, one uses standard differential geometry to show the Lipschitz manifold structure. Note that the scaling transformation $\lambda \mapsto \boldsymbol{u}_\lambda$ is not continuous on a Sobolev space unless one assumes some a priori bounds on higher order derivatives. This is another reason why we need the previous lower order decomposition which ensures such an estimate.

This ends the sketch of the proofs of Theorems 2.2.4 and 2.2.5. We now turn to Theorem 2.2.9 for the case of the semilinear heat equation.

Notations for the heat equation

Between the setting of (NLW) and that of (NLH), some different objects play the same role and we will keep the same notation.

General notations for the analysis. The linearized operator close to Q is:

$$Hu := -\Delta u - pQ^{p-1}u = -\Delta + V, \quad V := -pQ^{p-1}$$

We let

$$F(u) := \Delta u + f(u), \quad f(u) := |u|^{p-1}u.$$

Given a strictly positive real number $\lambda > 0$ and function $u : \mathbb{R}^d \to \mathbb{R}$, we define the rescaled function:

$$u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x).$$

This semi-group has the infinitesimal generator:

$$\Lambda u := \frac{\partial}{\partial \lambda}(u_\lambda)_{|\lambda=1} = \frac{2}{p-1}u + x.\nabla u.$$

For $z \in \mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$, the translation of vector z of u is denoted by:

$$\tau_z u(x) := u(x-z).$$

This group has the infinitesimal generator:

$$\left[\frac{\partial}{\partial z}(\tau_z u)\right]_{|z=0} = -\nabla u.$$

The original space variable will be denoted by $x \in \Omega$ and the renormalized one by y, related through $x = z + \lambda y$.

Supercritical numerology: Recall the γ_n is defined in (*2.2.14*) and define

$$\alpha_n := \gamma_n - \frac{2}{p-1}.$$

It is worth noting that

$$\gamma_0 = \gamma, \ \gamma_1 = \frac{2}{p-1} + 1, \ \gamma_n < \frac{2}{p-1} \text{ for } n \geq 2 \text{ and } \gamma_n \sim -n, \tag{2.2.50}$$

and in particular γ_n is decreasing and negative for large n. In addition $\alpha_0 = \alpha$, $\alpha_1 = 1$ and $\alpha_n < 0$ for $n \geq 2$. For $n \in \mathbb{N}$ we define[10]:

$$m_n := E\left[\frac{1}{2}(\frac{d}{2} - \gamma_n)\right]$$

and denote by δ_n the positive real number $0 \leq \delta_n < 1$ such that:

$$d = 2\gamma_n + 4m_n + 4\delta_n.$$

[10] $E[x]$ stands for the entire part: $x - 1 < E[x] \leq x$.

For $1 \ll L$ a very large integer we define the Sobolev exponent:

$$s_L := m_0 + L + 1$$

In this paper we assume the technical condition (*2.2.8*) for $s_+ = s_L$ which means:

$$0 < \delta_n < 1$$

for all integer n such that $d - 2\gamma_n \leq 4s_L$ (there is only a finite number of such integers from (*2.2.50*)). We let n_0 be the last integer to satisfy this condition:

$$n_0 \in \mathbb{N}, \quad d - 2\gamma_{n_0} \leq 4s_L \quad \text{and} \quad d - 2\gamma_{n_0+1} > 4s_L$$

and we define:

$$\delta_0' := \max_{0 \leq n \leq n_0} \delta_n \in (0,1).$$

For all integer $n \leq n_0$ we define the integer:

$$L_n := s_L - m_n - 1$$

and in particular $L_0 = L$.

Non-radial analysis: the number of spherical harmonics of degree n is:

$$k(0) := 1, \ k(1) := d, \ k(n) := \frac{2n+p-2}{n} \binom{n+p-3}{n-1} \text{ for } n \geq 2.$$

The Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}(1)$ is self-adjoint with compact resolvent and its spectrum is $\{n(d+n-2), \ n \in \mathbb{N}\}$. For $n \in \mathbb{N}$ the eigenvalue $n(d+2-n)$ has geometric multiplicity $k(n)$, and we denote by $(Y^{(n,k)})_{n \in \mathbb{N}, \ 1 \leq k \leq k(n)}$ an associated orthonormal Hilbert basis of $L^2(\mathbb{S}^d)$:

$$L^2(\mathbb{S}^{d-1}(1)) = \overset{+\infty \perp}{\underset{n=0}{\oplus}} \text{Span}\left(Y^{(n,k)}, \ 1 \leq k \leq k(n)\right),$$

$$\Delta_{S^{d-1}(1)} Y^{(n,k)} = n(d+n-2) Y^{(n,k)}, \quad \int_{S^{d-1}(1)} Y^{(n,k)} Y^{(n',k')} = \delta_{(n,k),(n',k')},$$

with the special choices:

$$Y^{(0,1)}(x) = C_0, \quad Y^{1,k}(x) = -C_1 x_k$$

where C_0 and C_1 are two renormalization constants. The action of H on each spherical harmonics is described by the family of operators on radial functions

$$H^{(n)} := -\partial_{rr} - \frac{d-1}{r}\partial_r + \frac{n(d+n-2)}{r^2} - pQ^{p-1} \tag{2.2.51}$$

for $n \in \mathbb{N}$ as for any radial function f they produce the identity

$$H\left(x \mapsto f(|x|) Y^{(n,k)}\left(\frac{x}{|x|}\right)\right) = x \mapsto (H^{(n)}(f))(|x|) Y^{(n,k)}\left(\frac{x}{|x|}\right).$$

For two strictly positive real numbers $b_1^{(0,1)} > 0$ and $\eta > 0$ we define the scales:

$$B_0 = |b_1^{(0,1)}|^{-\frac{1}{2}}, \quad B_1 = B_0^{1+\eta}.$$

The blow-up profile for the proof of Theorem 2.2.9 is an excitation of several direction of stability and instability around the soliton Q. Each one of these directions of perturbation, denoted by $T_i^{(n,k)}$ will be associated to a triple (n,k,i), meaning that it is the i-th perturbation located on the spherical harmonics of degree (n,k). For each (n,k) with $n \leq n_0$, there will be $L_n + 1$ such perturbations for $i = 0, ..., L_n$ except for the cases $n = 0$, $k = 1$, and $n = 1$, $k = 1, ..., d$, where there will be L_n perturbations for $i = 1, ..., L_n$ ($n = 1, 2$). Hence the set of triple (n,k,i) used in the analysis is:

$$\begin{aligned} \mathcal{I} \ := \ & \{(n,k,i) \in \mathbb{N}^3, \ 0 \leq n \leq n_0, \ 1 \leq k \leq k(n), \ 0 \leq i \leq L_n\} \\ & \backslash(\{(0,1,0)\} \cup \{(1,1,0), ..., (1,d,0)\}) \end{aligned}$$

with cardinal $\#\mathcal{I} := \sum_{n=0}^{n_0} k(n)(L_n + 1) - d - 1 < +\infty$. To localize some objects we will use a radial cut-off function $\chi \in C^\infty(\mathbb{R}^d)$:

$$0 \leq \chi \leq 1, \ \ \chi(|x|) = 1 \text{ for } |x| \leq 1, \ \ \chi(|x|) = 0 \text{ for } |x| \geq 2$$

and for $B > 0$, χ_B will denote the cut-off around $\mathcal{B}^d(0,B)$:

$$\chi_B(x) := \chi\left(\frac{x}{B}\right).$$

Outline of the proof for the heat equation

The construction of the type II blow-up solutions shares similarities with the one for the semilinear wave equation, Theorem 2.2.4. We already sketched the proof of this result is in Subsubsection 2.2.3. Therefore, we shall go faster and only point out the main novelties.

Without loss of generality, via scale change and translation in space one can assume that $x_0 = 0$ and $\mathcal{B}^d(7) \subset \Omega$.

Step 1 *Truncated center-manifold and approximate blow-up profile.* As we study a concentration, hence localized, phenomenon, we start by neglecting the influence of the boundary and investigate the non-radial case on the whole space $\Omega = \mathbb{R}^d$. The linearized operator near Q is H and its generalized kernel, even on non-radial functions, can be computed almost explicitly using ODE techniques as the potential is radial. Namely,

$$\{f, \ \exists j \in \mathbb{N}, \ H^j f = 0\} = \text{Span}\left(T_i^{(n,k)}\right)_{(n,i)\in\mathbb{N}^2, \ 1\leq k\leq k(n)},$$

where $T_i^{(n,k)}(x) = T_i^{(n)}(|x|)Y^{(n,k)}\left(\frac{x}{|x|}\right)$, $T_i^{(n)}$ being radial, is located on the spherical harmonics of degree (n,k), with the first zeroes being given by the invariances of the equation

$$T_0^{(0,1)} = \Lambda Q, \ \ T_0^{(1,k)} = \partial_{x_k} Q, \ \ HT_0^{(n,k)} = 0, \ \ HT_{i+1}^{(n,k)} = -T_i^{(n,k)}. \tag{2.2.52}$$

The blow-up profile for the wave equation of Subsubsection 2.2.3 relied on three pieces: an approximate blow-up profile built as an excitation on the suitably truncated and localized generalized kernel and a lower order correction, and a remainder orthogonal to the excitation. For the present non-radial case, one has to truncate and localize the full non-radial generalized kernel by incorporating higher spherical harmonics.

Proposition 2.2.18 (The approximate blow-up profile). *There is a constant $g(d,p)>0$ such that the following holds. Let $I=(s_0,s_1)$ be an interval, and $b_i^{(n,k)}$ for $(n,k,i)\in\mathcal{I}$ be C^1 functions with*

$$|b_i^{(n,k)}|\lesssim s^{-\left(\frac{\gamma-\gamma_n}{2}+i\right)}, \quad 0<b_1^{(0,1)}\sim s^{-1}, \quad |b_{1,s}^{(0,1)}|\lesssim s^{-2}.$$

There exist a profile Q_b where $b=(b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ of the form

$$Q_b:=Q+\chi_{B_1}\alpha_b, \quad \alpha_b:=\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)}+S(b). \tag{2.2.53}$$

$S(b)$ is a correction satisfying

$$|\chi_{B_1}S(b)|\leq C(1+y)^{-\gamma-g}, \quad \left|\chi_{B_1}\frac{\partial}{\partial b_i}S(b)\right|\leq C(1+y)^{-\gamma-g+i}$$

$$|S(b,y)|\lesssim s^{-2} \text{ and } \left|\frac{\partial}{\partial_{b_i}}S(b,y)\right|\lesssim s^{-1} \text{ on compact sets.}$$

Q_b satisfies the identity

$$\partial_s Q_b-F(Q_b)+b_1^{(0,1)}\Lambda Q_b+b_1^{(1,\cdot)}.\nabla Q_b=\psi_b+\chi_{B_1}\boldsymbol{Mod}$$

where $b_1^{(1,\cdot)}=(b_1^{(1,1)},...,b_1^{(1,d)})$ and where the modulation term is (with the convention $b_{L_n+1}^{(n,k)}=0$)

$$\boldsymbol{Mod}=\sum_{(n,k,i)\in\mathcal{I}}[b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}]\left[T_i^{(n,k)}+\frac{\partial S}{\partial b_i^{(n,k)}}\right].$$

ψ_b is an error term satisfying for $0<\eta\ll 1$ small and $s_0\gg 1$ large enough

$$\|\psi_b\|_{\dot{H}^\sigma}\lesssim s^{-1-g}, \quad \int_{\mathbb{R}^d}\psi_b H^{s_L}\psi_b\lesssim s^{-1-L-(1-\delta_0)(1+\eta)},$$

$$|\psi_b|\lesssim s^{-L-3} \text{ on compact sets.}$$

The zone $|y|\lesssim B_1$ appearing as the support of the excitation $\chi_{B_1}\alpha_b$ will correspond for our blow-up profile to the slightly enlarged self-similar zone $|x-z(t)|\lesssim\sqrt{T-t}$. The generalized kernel is cut in a special way, according to the definition of the set of parameters $\mathcal{I}$, and we recall that $\mathcal{I}$ is finite. The profile S is constructed by a careful treatment of all the lower order interaction terms emanating from the different spherical harmonics, and by inversion of elliptic equations on spherical harmonics. The key point is that as the product of two spherical harmonics projects only onto finitely many spherical harmonics, there is just a finite number of spherical harmonics to consider for the whole procedure.

The way to truncate the generalized kernel is pertinent thanks to the following coercivity estimate on the complement of this subspace.

Proposition 2.2.19. *There exists $C>0$ and compactly supported profiles $(\Phi_i^{(n,k)})_{(n,k,i)\in\mathcal{I}\cup\{(0,1,0),(1,1,0),...,(1,d,0)\}}$ satisfying*

$$(T_i^{(n,k)},\Phi_j^{(n,k)})=\delta_{i,j}$$

such that for any $\varepsilon\in\dot{H}^\sigma\cap\dot{H}^{s_L}$ with

$$\varepsilon\in \boldsymbol{Span}(\Phi_i^{(n,k)})^{\perp}_{(n,k,i)\in\mathcal{I}\cup\{(0,1,0),(1,1,0),...,(1,d,0)\}}$$

there holds

$$\sum_{k=0}^{s_L}\int\frac{|\nabla^k\varepsilon|^2}{1+|x|^{2(s_L-k)}}\leq C\int\varepsilon H^{s_L}\varepsilon=:\mathcal{E}_{s_L}. \tag{2.2.54}$$

The key point behind this coercivity estimate is that on spherical harmonics, H is modified by the addition of the potential $\frac{n(d+n-2)}{r^2}$, see (*2.2.51*). This potential becomes more and more positive as n grows, meaning that perturbations on higher degree spherical harmonics are dissipated faster. As a consequence, there is only a finite number of profiles on a finite number of spherical harmonics that dissipate slowly and that one has to avoid for the coercivity (*2.2.26*) to hold.

Step 2 *Decomposition of a solution.* We now consider solutions of $(NLH\Omega)$ with the Ansatz

$$u=\chi(Q_b)_{z,\frac{1}{\lambda}}+w \tag{2.2.55}$$

and decompose the remainder w according to:

$$w_{\text{int}}:=\chi_3 w,\quad w_{\text{ext}}:=(1-\chi_3)w,\quad \varepsilon:=(\tau_{-z}w_{\text{int}})_\lambda. \tag{2.2.56}$$

w_{ext} is the remainder outside the blow-up zone, w_{int} the remainder inside the blow-up zone, and ε is the renormalization of the remainder inside the blow-up zone corresponding to the scale and central point of the ground state $Q_{z,\frac{1}{\lambda}}$. w is orthogonal to the suitably truncated center manifold:

$$\varepsilon\in\text{Span}(\Phi_i^{(n,k)})^{\perp}_{(n,k,i)\in\mathfrak{I}\cup\{(0,1,0),(1,1,0),...,(1,d,0)\}} \tag{2.2.57}$$

where $\Phi_i^{(n,k)}$ is defined in Proposition 2.2.19. Such a decomposition is ensured by a technical application of the implicit function lemma for the case of a domain. With the scale λ it provides we define the renormalized time

$$\frac{ds}{dt}=\frac{1}{\lambda^2},\quad s(0)=s_0.$$

Step 3 *The approximate blow-up profile.* A formal computation, based on Proposition 2.2.18, gives that the parameters should evolve at first order according to

$$\begin{cases}\frac{\lambda_s}{\lambda}=-b_1^{(0,1)},\quad \frac{z_s}{\lambda}=-b_1^{(1,\cdot)},\\ b_{i,s}^{(n,k)}=-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)},\quad\forall(n,k,i)\in\mathfrak{I}\end{cases}$$

with the convention $b_{L_n+1}^{(n,k)}=0$. The above finite dimensional dynamical system admits for all $\ell\in\mathbb{N}$ with $\ell>\frac{\alpha}{2}$ the special solution

$$\bar{b}_i^{(0,1)}=\frac{c_i}{s^i}\ \text{ for } i=1,...,\ell\ \text{ and else }\ \bar{b}_i^{(n,k)}\equiv 0$$

where $c_i=c_i(\ell)$ are constants and $c_1=\frac{\ell}{2\ell-\alpha}$. Such a special solution concentrates the scale in finite time T with $\lambda(t)\sim(T-t)^{\frac{\ell}{\alpha}}$.

Step 4 *Trapped solutions.* We now fix $\ell\in\mathbb{N}$, $\ell>\frac{\alpha}{2}$ and focus on solutions that are close to the approximate blow-up profile Q_{b^e}. More precisely we say that a solution is trapped if under the decomposition (*2.2.55*) and (*2.2.56*), the parameters satisfy

$$|b_i^{(n,k)}(s)-\bar{b}_i^{(n,k)}(s)|\lesssim s^{-(\frac{\gamma-\gamma_n}{2}+i)-\tilde{\eta}_2},\quad \lambda\sim s^{-\frac{\ell}{2\ell-\alpha}}$$

and the remainder satisfy

$$\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}+\|w_{\text{ext}}\|^2_{H^\sigma(\Omega)}\lesssim 1,\quad \frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s_L-s_c)}}+\|w_{\text{ext}}\|^2_{H^{s_L}(\Omega)}\lesssim \frac{1}{\lambda^{2(2s_L-s_c)}s^{2L+2(1-\delta_0)+\tilde\eta_1}}, \tag{2.2.58}$$

where $0<\tilde\eta_2\ll\tilde\eta_1\ll\eta$ are fixed constants.

Evolution equations. From (*2.2.53*), under the decomposition (*2.2.55*) and (*2.2.56*), the evolution of a trapped solution has the form

$$\partial_t w_{\text{ext}}=\Delta w_{\text{ext}}+\Delta\chi_3 w+2\nabla\chi_3.\nabla w+(1-\chi_3)w^p, \tag{2.2.59}$$

since w has been cut into w_{int} and w_{ext} away from the approximate blow-up profile, and

$$\begin{aligned}\partial_t w_{\text{int}}+H_{z,\frac1\lambda}w_{\text{int}} \;&=\; -\tfrac{1}{\lambda^2}\chi\tau_z\left(\chi_{B_1}\text{Mod}-\left(\tfrac{\lambda_s}{\lambda}+b_1^{(0,1)}\right)\Lambda Q_b-\left(\tfrac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla Q_b\right)_{\frac1\lambda}\\ &\quad -\tfrac{1}{\lambda^2}\chi\tau_z\psi_{b,\frac1\lambda}+L(w_{\text{int}})+NL(w_{\text{int}})+\tilde L+\tilde{NL}+\tilde R\end{aligned} \tag{2.2.60}$$

where $H_{z,\frac1\lambda}$, $\text{NL}(w_{\text{int}})$, $L(w_{\text{int}})$ are the linearized operator, the non linear term and the small linear terms resulting from the interaction with a non cut approximate blow up profile:

$$H_{z,\frac1\lambda}:=-\Delta-p\left(\tau_z(\tilde Q_{\frac1\lambda})\right)^{p-1},\quad H_{b,z,\frac1\lambda}:=-\Delta-p\left(\tau_z(\tilde Q_{b,\frac1\lambda})\right)^{p-1}$$

$$\begin{aligned}\text{NL}(w_{\text{int}})&:=F\left(\tau_z(\tilde Q_{b,\frac1\lambda})+w_{\text{int}}\right)-F\left(\tau_z(\tilde Q_{b,\frac1\lambda})\right)+H_{b,\frac1\lambda}(w_{\text{int}}),\\ L(w_{\text{int}})&:=H_{z,\frac1\lambda}w_{\text{int}}-H_{b,z,\frac1\lambda}w_{\text{int}}=\tfrac{p}{\lambda^2}\tau_z(\chi_{B_1}^{p-1}\alpha_b^{p-1})_{\frac1\lambda}\end{aligned}$$

and the last terms are the boundary terms induced by the localization in Ω of the approximate blow up profile (in $B(0,2)$) and the remainder (in $B(0,6)$):

$$\tilde L:=-\Delta\chi_3 w-2\nabla\chi_3.\nabla w+p\tau_z Q_{\frac1\lambda}^{p-1}(\chi^{p-1}-\chi_3)w, \tag{2.2.61}$$

$$\tilde{NL}:=\sum_{k=2}^{p}C_k^p\tau_z Q_{\frac1\lambda}^{p-k}(\chi^{p-k}-\chi_3^{k-1})\chi_3 w^k, \tag{2.2.62}$$

$$\tilde R:=\Delta\chi\tau_z Q_{\frac1\lambda}+2\nabla\chi\nabla\tau_z Q_{\frac1\lambda}+\chi\tau_z Q_{\frac1\lambda}^p(\chi^{p-1}-1). \tag{2.2.63}$$

Modulation and Control of the remainder. For a trapped solution there holds the following estimates on the evolution of the pieces in the decomposition:

$$\begin{aligned}&\left|\tfrac{\lambda_s}{\lambda}+b_1^{(0,1)}\right|+\left|\tfrac{z_s}{\lambda}+b_1^{(1,\cdot)}\right|+\sum_{(n,k,i)\in\mathcal{I}}\left|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_i^{(n,k)}b_1^{(0,1)}+b_{i+1}^{(n,k)}\right|\\ \lesssim\;&\sqrt{\mathcal{E}_{2s_L}}+s^{-L-3}\end{aligned}$$

(With the convention $b_{L_n+1}^{(n,k)}=0$) and

$$\frac{d}{dt}\left(\frac{1}{\lambda^{2(\sigma-s_c)}}\mathcal{E}_\sigma+\|w_{\text{ext}}\|_{H^\sigma(\Omega)}\right)\lesssim\frac{1}{\lambda^2 s^{1+g}}+\frac{1}{\lambda^{(\sigma-s_c)}}\sqrt{\mathcal{E}_\sigma}\|\nabla^\sigma\psi\|_{L^2},$$

$$\begin{aligned}\tfrac{d}{dt}\left(\tfrac{1}{\lambda^{2(2s_L-s_c)}}\mathcal{E}_{2s_L}+\|w_{\text{ext}}\|_{H^{2s_L}(\Omega)}\right)\;&\lesssim\;\tfrac{1}{\lambda^{2(2s_L-s_c)+2}s^{2L+3-2\delta_0+2\tilde\eta_1+\kappa}}\\ &\quad+\tfrac{1}{\lambda^{2s_L-s_c}}\sqrt{\mathcal{E}_{2s_L}}\|H_{z,\frac1\lambda}^{s_L}\psi\|_{L^2},\end{aligned}$$

where $\kappa > 0$ represents a gain. The obtention of these estimates, given the heavy form of the evolution equations, is technical and we only sketch what is new compared to Subsubsection 2.2.3.

The modulation equations are obtained by computing the projection of the dynamics onto relevant directions inside the blow-up zone. Therefore, in their form, they do not see the localizations and the fact that we work on a domain. All the last directions of perturbations on the truncated generalized kernel on each spherical harmonics, $T_{L_n}^{(n,k)}$, are more influenced by the remainder, and an improved modulation equation for these terms require a careful flux computation at the border of the self-similar zone that we do not detail here.

To control the remainder inside the blow-up zone w_{int}, one needs to control the interactions with the approximate blow-up profile thanks to the non-radial coercivity of adapted norms and non-radial Hardy type inequalities. The control of a slightly supercritical norm and another high regularity norm (*2.2.58*) allows to control precisely the energy transfer between low and high frequencies and to control the nonlinear terms and partly the terms at the boundary of the blow-up zone. The dissipation in (*2.2.59*) and (*2.2.60*) (for the second equation it is a consequence of the coercivity (*2.2.54*)) absorbs the rest of these boundary terms and smaller order local interactions. One also needs to take into account technicalities coming from the special modulation estimates for the last parameters $b_{L_n}^{(n,k)}$.

For the remainder outside the blow-up zone w_{ext}, one uses parabolic techniques adapted to the Dirichlet problem allowing to treat the boundary terms on $\partial\Omega$. Then, the terms created by the localization as well as the nonlinear terms are controlled via the two a priori estimates at low and high regularity for w in the trapped regime and by dissipation.

Finally, we consider an already very concentrated initial approximate blow-up profile to work on a very small time. This way, the localization of the blow-up profile and the boundary terms do not alter the blow-up dynamics.

Step 5 *Existence via a topological argument:.* As in Subsubsection 2.2.3, the existence of a solution staying trapped on its maximal time of existence follows by a topological argument. Indeed, the linear instabilities of the approximate blow-up profile are in finite number, and only a finite number of directions can be associated to a 0 eigenvalue in which case they undergo only nonlinear effects that are under control and are harmless, which allows to apply Brouwer's fixed point argument. A solution staying trapped forever is then proved to be the solution of Theorem 2.2.9 we are looking for, by manipulating the a priori bounds for such solutions.

2.3 Classification of the dynamics near the ground state

One has seen in the previous Section 2.2 the existence of a particular type of dynamics near the ground state: its concentration in finite time. Other asymptotic behaviors are however possible for solutions starting close to Q. A first one is the possibility to converge towards Q or one of its renormalized version. We describe here some other typical behaviors and some classification results that have been obtained.

2.3.1 A vast range of behaviors

The scale instability can also produce concentration or expansion of Q in infinite time for global solutions. In the energy critical setting for the wave equation, one has again a continuum of scale speeds.

Theorem 2.3.1 (Non-scattering global solution for the critical (NLW) [39]). *Let $d = 3$ and $p = 5$. There exists $\epsilon_0 > 0$ such that for any $\delta > 0$ and $\nu \in [-\epsilon_0, \epsilon_0]$ there exists $t_0 \geq 1$ and a radial solution u of (NLW) on $[t_0, +\infty)$ of the form*

$$u(t,x) = \lambda(t)^{\frac{2}{p-1}} Q(\lambda(t)x) + v(t,x) + \eta(t,x), \quad \lambda(t) \sim t^{\nu}$$

where v solves the linear wave equation (LW) with $\|v, v_t\|_{\dot{H}^1 \times L^2} \leq \delta$ and $\|\eta, \eta_t\|_{\dot{H}^1 \times L^2} \to 0$ as $t \to +\infty$.

Note that we modified slightly the Theorem of [39] in view of the result in [47]. Since the two above theorems are perturbative results, it is interesting to know the structure of the linearized operator near Q. The linearized dynamics are stable in a codimension 1 subspace, and linearly unstable in a dimension 1 subspace. In view of this spectral structure, these solutions live in a center stable manifold near the manifold of ground states $(\lambda^{\frac{2}{p-1}} Q(\lambda x))_{\lambda > 0}$ of codimension 1 for which the linear instability does not take control, and are thus unstable.

Theorem 2.3.2 (Partial classification for the radial (NLW) [85]). *Let $d = 3$ and $p = p_c = 5$. There exist constants $0 < \epsilon < \delta < 1 < C$ and a connected C^1 manifold $\mathcal{M}$ with codimension 1 of radial functions in $\dot{H}^1 \times L^2$ satisfying the following. $\mathcal{M}$ is invariant by the flow and by the scaling transform (1.2.1); it contains the set of stationary states $(\lambda^{\frac{2}{p-1}} Q(\lambda x))_{\lambda > 0}$. Let u be any solution with $E(u(0), u_t(0)) < E(Q) + \epsilon^2$ with maximal time of existence T. Then we have only one of the following scenarios.*

(i) Scattering: $(u(0), u_t(0)) \notin \pm\mathcal{M}$, $T = +\infty$ *and* $\lim_{t \to +\infty} \|(u, u_t) - (v, v_t)\|_{\dot{H}^1 \times L^2} = 0$ *for v a solution of (LW).*

(ii) Blow-up away from the ground state: $(u(0), u_t(0)) \notin \pm\mathcal{M}$, $T < +\infty$ *and*

$$\liminf_{t \to T} \inf_{\lambda > 0} \|(u, u_t) - (\lambda^{\frac{2}{p-1}} Q(\lambda x), 0)\|_{\dot{H}^1 \times L^2} > \delta > C\epsilon.$$

(iii) Blow-up by concentration of the ground state: $(u(0), u_t(0)) \in \pm\mathcal{M}$, $T < +\infty$ *and u admits a decomposition*

$$u = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + v(t,x)$$

with $\lambda(t) \ll (T - t)$ and (v, v_t) is convergent in $\dot{H}^1 \times L^2$ as $t \to T$.

(iv) Global dynamics near the branch of ground states: $(u(0), u_t(0)) \in \pm\mathcal{M}$, $T = +\infty$ *and u admits a decomposition*

$$u = \frac{2}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + v(t,x) + \eta(t,x)$$

with $\lambda(t) \ll t$ where v is a solution of (LW) and $\eta \to 0$ as $t \to T$ in $\dot{H}^1 \times L^2$.

We said in the title "partial" because the exact nature of blow-up when it is away from the ground state is not known, and that the dynamics on the center-manifold $\mathcal{M}$ has not been completely understood. We refer to [85] for more details, especially for the topology of the corresponding sets and the non-radial case. Here again we modified a bit the Theorem of [85] in view of the result in [47]. Q is in fact a threshold in term of energy and kinetic energy for the behaviors; this study started in the fundamental work [79] and was continued in [51, 84]. In [51] notably the author constructed the unstable manifold associated to the unique unstable eigenfunction of the linearized operator. Other notable related works were the stabilization of the ground state for the Schrödinger equation [146] and the classification of the dynamics in its vicinity for the Klein Gordon equation [132], see the book [133].

For the semilinear heat equation, concentration in infinite time of ground states in interaction is possible in the energy critical case [29]. For only one ground state, concentration in infinite time has been shown in the supercritical setting. Though the ground state is not precisely mentioned in the paper where the following result is taken, the ground state should be the main profile appearing at a suitable scale since the solution breaks the usual self-similar scaling.

Theorem 2.3.3 (Concentration in infinite time on $(p_{JL},+\infty)$ [127]). *Let $p>p_{JL}$. For any $\ell\in 2\mathbb{N}^*$, there exists a global solution of (NLH) such that:*

$$\|u(t)\|_{L^\infty}\sim t^{\frac{d-2\gamma+2\ell}{(p-1)(\gamma-\frac{2}{p-1})}} \quad \textit{as } t\to+\infty.$$

Let us mention that the following behavior is impossible for well-localized initial data, for example compactly supported [126]. The condition for ℓ to be even should be technical, as it is assumed for the use of intersection number techniques.

A classification of possible dynamics near Q in the supercritical case, even in a partial sense, is still open. We now turn to the energy critical heat equation and look for a classification result in the spirit of Theorem 2.3.2. It is worth noting that solutions concentrating the ground state are only known to exist in low dimensions in the energy critical case both for (NLW) and (NLH). In all these cases their rigorous construction involved small perturbation in the energy topology $\dot H^1$ or $\dot H^1\times L^2$. For $d\geq 7$ the author, in a joint work with F. Merle and P. Raphaël, proved that such solutions cannot exist and classified all possible dynamics near in the vicinity of the ground state in the non-radial setting.

Theorem 2.3.4 (Classification of the flow near Q for $d\geq 7$ [24]). *Let $d\geq 7$. There exists $0<\eta\ll 1$ such that the following holds. Let $u_0\in\dot H^1(\mathbb{R}^d)$ with*

$$\|u_0-Q\|_{\dot H^1}<\eta,$$

then the corresponding solution to (NLH) follows one of the three regimes:
1. Soliton*: the solution is global and asymptotically attracted by the ground state*

$$\exists(\lambda_\infty,z_\infty)\in\mathbb{R}_+^*\times\mathbb{R}^d \textit{ such that } \lim_{t\to+\infty}\left\|u(t,\cdot)-\frac{1}{\lambda_\infty^{\frac{d-2}{2}}}Q\left(\frac{\cdot-z_\infty}{\lambda_\infty}\right)\right\|_{\dot H^1}=0.$$

Moroever,

$$|\lambda_\infty - 1| + |z_\infty| \to 0 \ \text{ as } \ \eta \to 0.$$

2. Dissipation*: the solution is global and dissipates*

$$\lim_{t\to+\infty} \|u(t,\cdot)\|_{\dot{H}^1} = 0, \quad \lim_{t\to+\infty} \|u(t,\cdot)\|_{L^\infty} = 0.$$

3. Type I blow up*: the solution blows up in finite time* $0 < T < +\infty$ *in the ODE type I self similar blow up regime near the singularity*

$$\|u(t,\cdot)\|_{L^\infty} \sim \kappa_H (T-t)^{-\frac{d-2}{4}}.$$

There exist solutions associated to each scenario. Moreover, the scenario (Dissipation) and (Type I blow up) are stable in the energy topology.

There could still exist solutions concentrating the ground state which are large in the critical space, but the blow-up mechanism should be different. An important result linked to the previous Theorem was the construction and characterization of the unstable manifold near the branch of ground states. This is the following Liouville type result.

Theorem 2.3.5 (The instable manifold around Q [24]). *Let* $d \geq 7$.

1. Existence of backwards minimal elements*: There exist two strictly positive,* $\mathcal{C}^\infty$ *radial solutions of* (NLH), Q^+ *and* Q^-*, defined on* $(-\infty, 0] \times \mathbb{R}^d$*, which are minimal backwards in time*

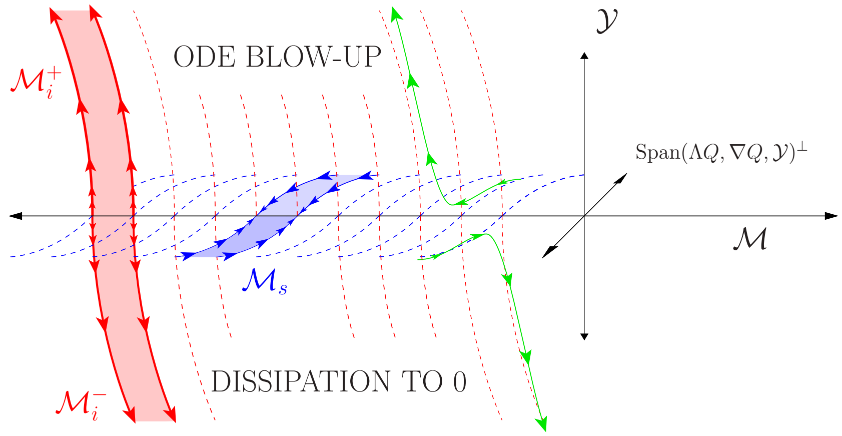
$$\lim_{t\to-\infty} \|Q^\pm - Q\|_{\dot{H}^1} = 0$$

and have the following forward behaviour: Q^+ *explodes according to type I blow up with profile* κ_H *at some finite later time, and* Q^- *is global and dissipates* $Q^- \to 0$ *as* $t \to +\infty$ *in* $\dot{H}^1(\mathbb{R}^d)$.
2. Rigidity*: Moreover, there exists* $0 < \delta \ll 1$ *such that if* u *is a solution of* (NLH) *on* $(-\infty, 0]$ *such that:*

$$\sup_{t\leq 0} \ \inf_{\lambda>0,\ z\in\mathbb{R}^d} \|u(t) - Q_{z,\lambda}\|_{\dot{H}^1} \leq \delta$$

then $u = Q^\pm$ *or* $u = Q$ *up to the symmetries of the flow.*

Solutions dissipating towards a ground state, scenario (1) in Theorem 2.3.4, appear as a codimensional 1 threshold between type I blow-up and dissipation to 0. These solutions should form a regular enough manifold and the corresponding dynamics in the phase space should be as depicted in the following picture.



$$\begin{array}{|l} \mathcal{M} = \{\lambda^{\frac{2}{p-1}} Q(\lambda(x-z))\} \\ \mathcal{M}_i^+ = \{\lambda^{\frac{2}{p-1}} Q^+(\lambda^2 t, \lambda(x-z))\} \\ \mathcal{M}_i^- = \{\lambda^{\frac{2}{p-1}} Q^-(\lambda^2 t, \lambda(x-z))\} \\ \mathcal{M}_s = \{u, \quad u(t) \to \tilde{Q} \in \mathcal{M} \ \text{ as } t \to +\infty\} \end{array}$$

Comments: A similar classification result, where the equivalent of the "dissipative" scenario is still open, is the critical generalized Korteweg de Vries equation [102, 103, 104]. There, the manifold of solutions converging to the branch of ground states was rigorously constructed [105]. An important step in all these works is the characterization of the instable manifold [51], started in and linked to the study of minimal elements initiated in [108]. For the energy critical harmonic heat flow from $\mathbb{R}^2$ to $\mathbb{S}^2$, if one restricts the study to the class of corotational solutions, the homotopy degree plays a similar role to the one of the dimension in Theorem 2.3.5, the ground states are orbitally stable and classification results were established when scale instability is not too strong [68].

Open problems: The rigorous manifold construction for solutions dissipating to the ground state for the critical (NLH) in large dimensions could be done following [105]. This project is a current direction of work of the author. The final goal is to extend this result in a setting where there is strong scale instability, as in low dimensions, and to include all possible scale behaviors in the classification. Another interesting question is the existence or non-existence of type II blow-up in the same setting as Theorem 2.3.5. If it exists it must be either a large (since large in the critical space) perturbation of Q and the construction results done so far cannot be applied in a straightforward way. Another direction is to look at the interaction of between Q and other objects.

2.3.2 Sketch of the proof of Theorems 2.3.4 and 2.3.5

Our aim is to explain how the energy critical and supercritical cases are drastically different since the ground state has different asymptotics at infinity in space in these cases. Indeed, from Theorems 2.2.1 and 2.2.2, in the critical case it has the asymptotic of the Green function of the Laplace equation $r^{-4/(p_c-1)} = r^{-(d-2)}$ whereas in the supercritical case it matches the homogeneous self-similar solution $c_\infty r^{-2/(p-1)}$.

Notations

The linearized operator close to Q is:

$$Hu := -\Delta u - pQ^{p-1}u = -\Delta + V$$

where we introduced the potential $V := -pQ^{p-1}$. We let

$$F(u) := \Delta u + f(u), \quad f(u) := |u|^{p-1}u.$$

Given a strictly positive real number $\lambda > 0$ and function $u : \mathbb{R}^d \to \mathbb{R}$, we define the rescaled function:

$$u_\lambda(x) = \lambda^{\frac{2}{p-1}} u(\lambda x).$$

This semi-group has the infinitesimal generator:

$$\Lambda u := \frac{\partial}{\partial \lambda}(u_\lambda)_{|\lambda=1} = \frac{2}{p-1}u + x.\nabla u.$$

For $z \in \mathbb{R}^d$ and $u : \mathbb{R}^d \to \mathbb{R}$, the translation of vector z of u is denoted by:

$$\tau_z u(x) := u(x - z).$$

This group has the infinitesimal generator:

$$\left[\frac{\partial}{\partial z}(\tau_z u)\right]_{|z=0} = -\nabla u.$$

The original space variable will be denoted by $x \in \Omega$ and the renormalized one by y, related through $x = z + \lambda y$.

Sketch of the proof of Theorems 2.3.4 and 2.3.5 for $d \geq 11$

As in our case $p_c \in (1, 2)$ is not an integer, solutions are only partially regular since the nonlinearity is not smooth. To be more precise we first recall the following local well-posedness result including regularizing effects.

Proposition 2.3.6 (Local well posedness of critical (NLH) in $\dot{H}^1$ and regularizing effects). *Let $d \geq 7$. For any $u_0 \in \dot{H}^1(\mathbb{R}^d)$ there exists $T(u_0) > 0$ and a weak solution $u \in \mathcal{C}([0, T(u_0)), \dot{H}^1(\mathbb{R}^d))$ of (NLH). In addition the following regularizing effects hold:*

(i) $u \in C^{(\frac{3}{2},3)}((0, T_{u_0}) \times \mathbb{R}^d)$, *$u$ is a classical solution of (NLH) on $(0, T_{u_0}) \times \mathbb{R}^d$.*

(ii) $u \in C((0, T(u_0)), W^{3,\infty}(\mathbb{R}^d))$.

(iii) $u \in C((0, T(u_0)), \dot{H}^3(\mathbb{R}^d))$, $u \in C^1((0, T(u_0)), \dot{H}^1)$.

For any $0 < t_1 < t_2 < T_{u_0}$ the solution mapping is continuous from $\dot{H}^1$ into $C^{(\frac{3}{2},3)}([t_1, t_2] \times \mathbb{R}^d)$, $C([t_1, t_2], W^{3,\infty})$, $C((t_1, t_2), \dot{H}^1 \cap \dot{H}^3)$ and $C^1((t_1, t_2), \dot{H}^1)$ at u_0.

Therefore, one can use derivatives of u of order at most 3 in our analysis. As mentioned in Theorem 2.2.6, the concentration dynamics for Q are allowed in low dimensions, with $d = 6$ being a threshold. In fact, the analysis near Q becomes simpler for the case $d \geq 11$ and we will fix d in that range and $p = p_c$ from now on. For $d \in [7, 10]$, the complications are purely technical.

Step 1 *The linearized operator and geometrical decomposition.* The first thing to do is to study the linearized dynamics, involving the operator

$$H := -\Delta - pQ^{p-1}.$$

H is a nice self-adjoint Schrödinger operator with a smooth radial potential decaying at infinity with rate

$$V(x) = -\frac{d+2}{d-2}\frac{1}{\left(1+\frac{|x|^2}{d(d-2)}\right)^2} = O\left((1+|x|)^{-4}\right).$$

Its structure was already known [145] (except for the coercivity). Since the potential is radial, ODE techniques can be applied to study the action of H even on non-radial functions, in particular via Sturm Liouville arguments. Using some techniques from calculus of variation one has the following result.

Proposition 2.3.7 (Structure of the linearized operator). $H := -\Delta - pQ^{p-1}$ *has the following spectral structure .*

(i) It has only one negative eigenvalue $-e_0$ *with multiplicity one, associated to a smooth strictly positive and exponentially decaying eigenfunction* $\mathcal{Y}$.

(ii) Its kernel is given by the natural invariances

$$\mathit{Ker}(H) = \mathit{Span}(\Lambda Q, \partial_{x_1}Q, ..., \partial_{x_d}Q).$$

(iii) Coercivity: there exists $C > 0$ *such that for* $i = 1, 2, 3$ *for all function* $\varepsilon \in \dot{H}^i$ *satisfying the orthogonality* $\varepsilon \in \mathit{Span}(\mathcal{Y}, \Lambda Q, \partial_{x_1}Q, ..., \partial_{x_d}Q)^{\perp}$ *(well-defined by Sobolev embedding) there holds*

$$\frac{1}{C}\int \varepsilon H^i \varepsilon \leq \|\varepsilon\|^2_{\dot{H}^i} \leq C\int \varepsilon H^i \varepsilon, \quad i = 1, 2, 3. \tag{2.3.1}$$

Therefore, there is one direction associated to a well-localized linear instability, and, when restricted to the orthogonal of the manifold of stationary states and of this instability, H dissipates like the standard Laplacian (*2.3.1*). The second step is to decompose any solution close to Q according to the above spectral structure. As we work in the non-radial setting one has to consider solutions close to the manifold of ground states

$$\mathcal{M} := \left\{\frac{1}{\lambda^{\frac{2}{p-1}}}Q\left(\frac{x-z}{\lambda}\right), \quad \lambda > 0, \ z \in \mathbb{R}^d\right\}.$$

The following lemma is a consequence of the implicit function theorem.

Lemma 2.3.8. *Any* $u \in \dot{H}^1$ *satisfying*

$$\underset{\bar{z}\in\mathbb{R}^d, \ \bar{\lambda}>0}{\mathit{inf}} \left\| u - \frac{1}{\bar{\lambda}^{\frac{2}{p-1}}}Q\left(\frac{x-\bar{z}}{\bar{\lambda}}\right)\right\|_{\dot{H}^1} \ll 1,$$

can be written in a unique way:

$$u = \frac{1}{\lambda^{\frac{2}{p-1}}}(Q + a\mathcal{Y} + \varepsilon)\left(\frac{x-z}{\lambda}\right), \quad a \in \mathbb{R}, \tag{2.3.2}$$

$$\varepsilon \in \dot{H}^1, \quad \varepsilon \in \textit{Span}(\mathcal{Y}, \Lambda Q, \partial_{x_1} Q, ..., \partial_{x_d} Q)^{\perp}, \tag{2.3.3}$$

$$|a| + \|\varepsilon\|_{\dot{H}^1} \lesssim \inf_{\bar{z}\in\mathbb{R}^d,\ \bar{\lambda}>0} \left\| u - \frac{1}{\bar{\lambda}^{\frac{2}{p-1}}} Q\left(\frac{x-\bar{z}}{\bar{\lambda}}\right)\right\|_{\dot{H}^1}. \tag{2.3.4}$$

Step 2 *Analysis of solutions near $\mathcal{M}$: preliminaries.* We now investigate in a quantitative way how a solution evolves near the manifold of ground states.

Definition 2.3.9 (Trapped solutions). Let $I \subset \mathbb{R}$ be a time interval containing 0. A solution u of (NLH) is said to be trapped at distance $0 < \eta \ll 1$ on I if:

$$\sup_{t\in I} \inf_{\bar{z}\in\mathbb{R}^d,\ \bar{\lambda}>0} \left\| u(t) - \frac{1}{\bar{\lambda}^{\frac{2}{p-1}}} Q\left(\frac{x-\bar{z}}{\bar{\lambda}}\right)\right\|_{\dot{H}^1} \leq \eta.$$

For trapped solutions, the decomposition Lemma 2.3.8 applies and to the scale λ it provides we associate the renormalized time

$$\frac{ds}{dt} = \frac{1}{\lambda(t)^2}, \quad s(0) = s_0.$$

In renormalized variables, injecting the decomposition (*2.3.2*) in (NLH) one finds the evolution equation

$$\partial_s \varepsilon + a_s \mathcal{Y} - \frac{x_s}{\lambda}.\nabla(Q + a\mathcal{Y} + \varepsilon) - \frac{\lambda_s}{\lambda}\Lambda(Q + a\mathcal{Y} + \varepsilon) = -H\varepsilon + e_0 a\mathcal{Y} + NL \tag{2.3.5}$$

where NL is the nonlinear term

$$NL := |Q + a\mathcal{Y} + \varepsilon|^{p-1}(Q + a\mathcal{Y} + \varepsilon) - Q^{p-1} - pQ^{p-1}(a\mathcal{Y} + \varepsilon).$$

To establish the modulation equations for the parameters, we take the scalar product between (*2.3.5*) and $\mathcal{Y}$, ΛQ and $\partial_{x_i} Q$ for $1 \leq i \leq d$, with the help of the orthogonality conditions (*2.3.3*). To control the part of the solution on the infinite dimensional subspace ε, we use energy methods that are adapted at the linear level (*2.3.1*). Eventually, the dissipation of the energy (*2.1.5*) and (*2.1.6*) provides an a priori space time estimate for all trapped solution. The result is the following.

Lemma 2.3.10. *Let u be a trapped solution in the sense of Definition 2.3.9. Then there holds*

(i) Modulation

$$|a_s - e_0 a| + \left|\frac{z_s}{\lambda}\right| + \left|\frac{\lambda_s}{\lambda}\right| \lesssim |a|^2 + \|\varepsilon\|_{\dot{H}^2}^2, \tag{2.3.6}$$

(ii) Lyapunov monotonicity

$$\frac{d}{ds}\left(\|\varepsilon\|_{\dot{H}^1}^2\right) \approx \frac{d}{ds}\left(\int \varepsilon H \varepsilon\right) \lesssim -\|\varepsilon\|_{\dot{H}^2}^2 + O(|a|^4),$$

$$\frac{d}{ds}\left(\|\varepsilon\|_{\dot{H}^2}^2\right) \approx \frac{d}{ds}\left(\int \varepsilon H^2 \varepsilon\right) \lesssim -\|\varepsilon\|_{\dot{H}^3}^2 + O(|a|^4 + \|\varepsilon\|_{\dot{H}^2}^4).$$

(iii) Energy dissipation

$$|E(u)-E(Q)| \lesssim \inf_{\bar z\in\mathbb{R}^d,\ \bar\lambda>0} \left\| u-\frac{1}{\bar\lambda^{\frac{2}{p-1}}}Q\left(\frac{x-\bar z}{\bar\lambda}\right)\right\|^2_{\dot H^1}, \quad \frac{d}{ds}\left(E(u)\right) \lesssim -a^2 - \| \varepsilon \|^2_{\dot H^2},$$

$$\int_{s_0}^{s_1}\left(a^2+\| \varepsilon \|^2_{\dot H^2}\right) ds \lesssim \sup_{s\in[s_0,s_1]} \inf_{\bar z\in\mathbb{R}^d,\ \bar\lambda>0} \left\| u(s)-\frac{1}{\bar\lambda^{\frac{2}{p-1}}}Q\left(\frac{x-\bar z}{\bar\lambda}\right)\right\|^2_{\dot H^1} \lesssim \eta^2 \qquad (2.3.7)$$

The interpretation of the above estimates is clear: the instable part evolves according to a linear unstable dynamics plus nonlinear terms, the stable part dissipates at the linear level and undergo nonlinear effects, and the scale and the central points are only affected by nonlinear effects. The direct consequence of (*2.3.6*) and (*2.3.7*) is that the scale cannot move

$$\forall s \geq s_0, \quad \left|\frac{\lambda(s)}{\lambda(s_0)}-1\right| \lesssim \eta^2$$

which prevents concentration of the ground state.

Let us stress the key differences in comparison with the low dimensional case. If one were to perform the blow-up constructions as done in the proofs of the result of [147], one would consider a suitable localization of the generalized kernel of H Span$(T_i)_{i\geq 1}$ where $T_0=\Lambda Q$ and $T_{i+1}=H^{-1}T_i$. For any $d\geq 6$ one has $T_i\sim r^{2i-2}$ as soon as $i\geq 1$, and so as the dimension increases these profiles lives in worst and worst functional spaces. For $d\geq 7$ suitable localizations of these profiles leading to the concentration of Q are unreachable from small perturbations starting small in $\dot H^1$.

Another point of vue is that the perturbation is located at a larger scale than the one of the ground state since it dissipates towards it. Consequently the perturbation interacts mostly with the far away tail of the ground state. As the dimension increases, the ground state decreases faster and faster and these interactions are smaller and controlled by dissipation. This corresponds to the following estimate:

$$\left|\int_{s_0}^{s}\int_{\mathbb{R}^d} Q|NL(s)|\right| \lesssim \int_{s_0}^{s}\int_{\mathbb{R}^d}\frac{\varepsilon^2+a^2\mathcal{Y}^2}{1+|x|^4}dxds \lesssim \int_{s_0}^{s}(a^2+\|\varepsilon\|^2_{\dot H^2})ds<+\infty$$

and such an a priori estimate on the nonlinear term fails in low dimensions.

Step 3 *The instable manifold.* As for finite dimensional systems, the presence of the linear instability $-e_0$ associated to the profile $\mathcal{Y}$ implies that there exist two particular solutions emanating from the ground state from $-\infty$ of the approximate form $Q\pm e^{e_0 t}\mathcal{Y}$ (i.e. matching the linear dynamics for perturbations along the direction $\mathcal{Y}$). As the rest of the spectrum is stable, such solutions are the only one being able to approach the ground state backwards in time as $t\to-\infty$.

The existence and unicity of solutions having such a behavior as $t\to-\infty$ follows from a fixed point argument involving the estimates of Lemma 2.3.10. Then, any solutions staying close to Q backward in time must have such a behavior, using dissipation and energy arguments and must then be one of these special solution. Their forward in time behavior is studied using comparison principles, parabolic regularizing effects and convexity for the blow-up. As Q^+ is a positive radial blow-up solution, it has to blow-up with type I from [97].

Theorem 2.3.11. *There exist two strictly positive radial solutions Q^+ and Q^- defined at least on $(-\infty, t_0] \times \mathbb{R}^d$ for some $t_0 \in \mathbb{R}$ such that:*

$$Q^\pm = Q \pm e^{e_0 t}\mathcal{Y} + O(e^{2e_0 t}) \quad \textit{on } (-\infty, t_0].$$

Q^+ blows up with self-similar blow-up with profile κ_H forward in time. Q^- is global and dissipates toward 0. Moreover there exists $\eta > 0$ such that if u is a solution of (NLH) that is trapped at distance η in the sense of Definition 2.3.9 then $u = Q^+$ or $u = Q$ or $u = Q^-$ up to the symmetries of the flow.

The behaviors associated to Q^+ and Q^- are moreover stable. The stability of dissipation is rather easy to show but the stability of the self-similar blow-up with profile κ_H is more involved and adapts to the energy critical setting an argument from [54]. This has been the subject of the work [25] and we will sketch the proof of this fact after the end of the current proof. To end the proof of Theorem 2.3.4, we now show that for any solution starting close to Q, either the linear instability $a(t)\mathcal{Y}$ dominates and makes the solution exit a universal neighborhood of Q close to Q^+ or Q^-, or it never takes control, meaning that the solution is located on the stable infinite dimensional subspace (*2.3.3*) and undergoes dissipation toward Q.

Lemma 2.3.12. *There exists a large enough constant $K > 0$ such that the following holds. Let u be a trapped solution in the sense of Definition 2.3.9, and $[0, \tilde{T})$, with $0 < \tilde{T} < T$, be the largest time interval starting at 0 on which it is trapped.*

(i) If for all $t \in [0, T)$,

$$|a(t)| < K \parallel \varepsilon(t) \parallel_{\dot{H}^2}^2$$

then u is in a dissipative regime, is global $\tilde{T} = T = +\infty$ and converges in $\dot{H}^1$ toward a renormalized stationary state $\frac{1}{\lambda_\infty^{\frac{2}{p-1}}} Q\left(\frac{x - z_\infty}{\lambda}\right)$.

(ii) If for some instability time $T_{ins} \in (0, \tilde{T})$,

$$|a(T_{ins})| = K \parallel \varepsilon(T_{ins}) \parallel_{\dot{H}^2}^2, \tag{2.3.8}$$

then u enters an instable regime, and there exists $T_{exit} > T_{ins}$ such that[11]:

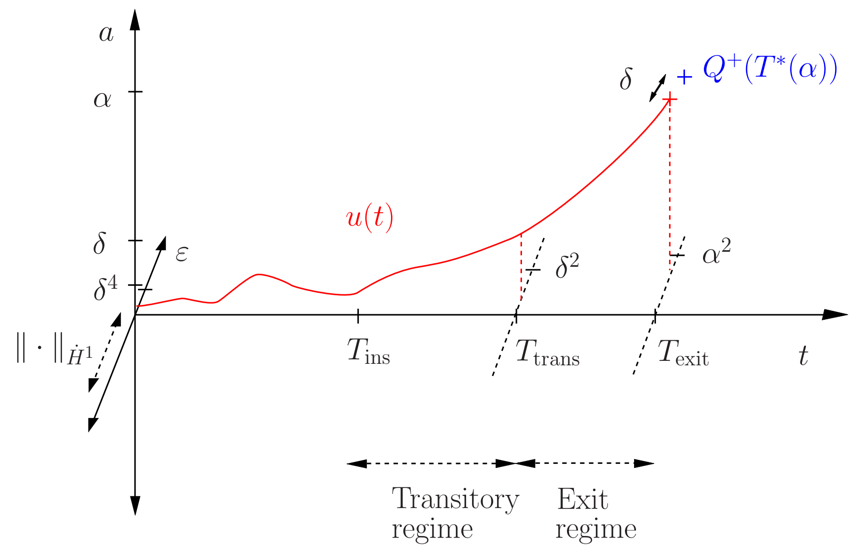
$$\textit{either} \quad \parallel u(T_{exit}) - Q^+ \parallel_{\dot{H}^1} \ll 1 \quad \textit{or} \parallel u(T_{exit}) - Q^- \parallel_{\dot{H}^1} \ll 1.$$

If a solution is in scenario (ii), the instability at T_{ins} is of quadratic order compared to the stable part from (*2.3.8*). The instability dynamics happen the following way: there exists a transition regime $[T_{\text{ins}}, T_{\text{trans}}]$ such that at time T_{trans}, the stable part did not grow, but the instable part is now of order its square root

$$\|\varepsilon(T_{\text{trans}}\|_{\dot{H}^1} + \|\varepsilon(T_{\text{trans}}\|_{\dot{H}^2} \lesssim |a(T_{\text{trans}}|^2, \quad |a(T_{\text{trans}}| = |a(T_{\text{ins}}|^{\frac{1}{4}}.$$

The picture is as follows.

[11] We made a simplification here. The correct statement is that there exists a universal α such that if the solution starts initially at distance δ^4 of Q, it exits a neighborhood of size α of $\mathcal{M}$ at distance δ of a renormalization (involving a compact set of parameters depending only of α) of $Q^\pm$.



The proof of Lemma 2.3.12 involves a study of each pieces of the decomposition (*2.3.2*) in the three different regimes, the dissipative one, or the transition regime and then the exit regime. In the dissipative regime, the convergence to a ground state requires the careful treatment of estimates that are critical for the equation. In the transition regime $[T_{\text{ins}}, T_{\text{trans}}]$ one uses a bootstrap analysis with the bounds on trapped solutions from Lemma 2.3.10 to show that the domination of the instability is increasing. In the exit regime $[T_{\text{trans}}, T_{\text{exit}}]$, one shows that the solution is in a regime with exponential growth for the perturbation driven by the instability, which allows to compare it with a suitable renormalized version of Q.

If a solution enters the unstable regime, it will then have the same behavior as Q^+ or Q^- since they are stable for the $\dot H^1$ topology, and the proof of Theorem 2.3.4 is complete.

The fact that the dissipation to 0 is a stable behavior, i.e. that this set is open in $\dot H^1$ (thanks to regularizing effects this statement can be made for other topologies and one obtains convergence to 0 in L^∞ for example), is direct thanks to local continuity of the flow map in $\dot H^1$ and dissipation to 0 for all small initial data. The stability of type I blow-up is however a harder issue and we now sketch its proof.

Sketch of proof of the stability of type I blow-up in the critical setting

We are interested in proving the following theorem.

Theorem 2.3.13 (Stability of critical ODE blow-up [25]). *For $p=p_c$, the set of solutions blowing-up with type I is open in L^∞ and solutions have $\pm\kappa_H$ for blow-up profile.*

The same result was proven in the subcritical case in [54]. As the proof relied on other results using subcritical tools, we decided to write a clear proof of this fact, adapting the strategy of [54] and incorporating critical arguments when needed.

We fix $d\geq 3$ and $p=p_c$. This theorem and the ideas behind the proof adapt that of the same result in the work [54] in the subcritical case. We shall go fast since the full proof is already short. The key fact behind the proof is that there is no other self-similar solution than the constant in space ODE blow-up profile, [61]. Let us first recall some known results concerning type I blow-up solutions.

Proposition 2.3.14 (Liouville type theorem fot type I blow up [116, 117]). *Let $1 < p \leq p_c$ and u be a solution of (NLH) on $(-\infty, 0] \times \mathbb{R}^d$ such that $\|u\|_{L^\infty} \leq C(-t)^{\frac{1}{p-1}}$ for some constant $C > 0$, then there exists $T \geq 0$ such that $u = \pm \frac{\kappa_H}{(T-t)^{\frac{1}{p-1}}}$.*

Proposition 2.3.15 (Description of type I blow up [63, 116, 117]). *Let $1 < p \leq p_c$ and u solve (NLH) with $u_0 \in W^{2,\infty}(\mathbb{R}^d)$ blowing up at $T > 0$. The two following properties are equivalent:*

$$\begin{array}{ll} (i) & \textit{The blow-up is of type } I. \\ (ii) & \exists K > 0, \ \ |\Delta u| < \frac{1}{2}|u|^p + K \textit{ on } \mathbb{R}^d \times [0, T) \end{array}$$

Proposition 2.3.14 states that there is rigidity for solutions satisfying the type I estimate globally backward in time: they must be the constant in space blow-up profile. Proposition 2.3.15 states the equivalence between blowing-up with type I and satisfying pointwise an ODE type differential bound

$$\frac{d}{dt}(|u|) \geq \frac{1}{2}|u|^p - K.$$

We now fix u a solution blowing-up with type I at time T. One way to prove Theorem 2.3.13 would have been to identify for u the blow-up set and the corresponding refined blow-up profiles, then another close enough solution would have been shown to enter the basin of attraction of one of these latter self-similar solutions. It is rather unclear if such a strategy can work for numerous reasons.

Instead, we reason in terms of limiting objects. Assume by contradiction that there is a sequence $u_n(0) \to u(0)$ in $W^{2,\infty}(\mathbb{R}^d)$ that do not blow-up with type I blow-up. We will construct a limit object having contradictory properties. First, from the equivalence between (i) and (ii) of Proposition 2.3.15 (naming K the constant associated to u), one has that for each n (ii) with constant $2K$ must fail at some (t_n, x_n) for u_n. Without loss of generality one can assume

$$\forall (t,x) \in [0, t_n] \times \mathbb{R}^d, \ \ |\Delta u_n| < \frac{1}{2}|u_n|^p + 2K, \ \ \text{andă} \ \ |\Delta u(t_n, x_n)| = \frac{1}{2}|u|^p + 2K. \qquad (2.3.9)$$

Roughly, in view of Proposition 2.3.15, the above estimate states that before t_n, u_n behaves like a type I blow-up solution, and leaves this regime at time t_n. Some refined continuity argument imply that u_n must blow-up at time $T_n \to T$ and that $t_n \to T$. Therefore, at time t_n the singularity is closer and closer and we renormalize the solution using the scaling of the equation. Let

$$\lambda_n := \left(\frac{1}{\|u_n(t_n)\|_{L^\infty}} \right)^{\frac{p-1}{2}} \quad \text{and } v_n(\tau, y) := \lambda_n^{\frac{2}{p-1}} u_n \left(t_n + \lambda_n^2 \tau, x_n + \lambda_n y \right)$$

for $(\tau, y) \in \left[-\frac{t_n}{\lambda_n^2}, \frac{T_n - t_n}{\lambda_n^2} \right) \times \mathbb{R}^d$. Then $\|v_n(0)\|_{L^\infty} = 1$ from the definition. Moreover, since u_n resembles a type I blow-up solution until t_n, one can prove that it grows until this time implying $\lambda_n \to 0$. Hence the lower bound of the renormalized time interval, $-\frac{\tilde{t}_n}{\lambda_n}$, goes to $-\infty$. Moreover, by a direct computation, the bound (*2.3.9*) yields a similar bound for v_n

$$\forall \tau \in \left[-\frac{t_n}{\lambda_n}, 0 \right], \ \ |\Delta v_n| < \frac{1}{2}|v_n|^p + 2K\lambda_n^{\frac{2p}{p-1}}, \ \ |\Delta v_n(0,0)| = \frac{1}{2}|v_n(0,0)|^p + 2K\lambda_n^{\frac{2p}{p-1}} \qquad (2.3.10)$$

on $\left[-\frac{\tilde{t}_n}{\lambda_n},0\right]$. The bound (*2.3.9*), after some calculations, imply that v_n satisfies a uniform bound

$$\forall \tau \in \left[-\frac{\tilde{t}_n}{\lambda_n},0\right], \quad \|v_n(\tau)\|_{L^\infty} \leq C(1-\tau)^{-\frac{1}{p-1}}$$

for C independent of n. Regularizing effects transform this boundedness into compactness and v_n converges to some limit object v which solves (NLH) at least on $(-\infty,0]\times R^d$, and the above bound implies that $|v|\leq C(1-\tau)^{-\frac{1}{p-1}}$. Hence v satisfies the type I bound of Proposition 2.3.14, and therefore is constant in space from this Proposition $v=\kappa_H(\tilde{T}-t)^{-\frac{1}{p-1}}$, for $\tilde{T}\in[1,+\infty]$. However, the equality at time t_n (*2.3.10*) gives at the limit $|\Delta v(0,0)|=\frac{1}{2}|v_n(0,0)|^p$ which is a contradiction if $\tilde{T}\neq+\infty$.

The above picture describes essentially the proof in the case $\tilde{T}\in\mathbb{R}$ for the definition of $\tilde{v}$. However, when $\tilde{T}=+\infty$, one needs to refine the previous argument and to make a similar extraction of a contradictory limiting object but before t_n. We refer to the full proof in Chapter 5 for that case.

2.4 Stability of backward self-similar solutions for (NLH)

We study here the stability of backward self-similar solutions of (NLH). We recall that these solutions are exact solutions of (NLH) of the form

$$\frac{1}{(T-t)^{\frac{1}{p-1}}}w\left(\frac{x}{\sqrt{T-t}}\right).$$

A direct computation shows that a solution of (NLH) can be written this cay if and only if w solves the stationary elliptic equation

$$\frac{1}{2}\Lambda w=\Delta w+|w|^{p-1}w, \quad \Lambda=\frac{2}{p-1}+x.\nabla, \quad x\in\mathbb{R}^d. \tag{2.4.1}$$

These profiles appear naturally as a class of blow-up profiles for (NLH) as discussed in Chapter 1.1, see Theorem 2.1.9 for a subcritical result, and [98] for the radial critical and supercritical cases. Especially, they are the blow-up profiles for type I blow-up solutions in the sense of Definition 2.1.7. Their stability and the study of the dynamics in their vicinity is then an important issue.

Self-similar blow-up was a mechanism Leray raised the attention to for solutions to the three dimensional Navier-Stokes equations [92]. Their study has then attracted a great amount of work, see [20, 134] for example. For the wave equation, the stability of the ODE blow-up profile (*1.3.1*) in dimensions greater than one has attracted a lot of attention, see [40, 41] and references therein. In certain settings a countable family of other self-similar solutions have been found [11, 12]. For other geometrical equations, such as the harmonic heat flow or the harmonic maps, these issues have also been considered, see for example [30, 28, 37, 53]. Backward and forward self-similar solutions can be at the heart of a mechanism explaining non-uniqueness for global weak solutions obtained by compactness methods, for equations with subcritical coercive conservation laws. A localized self-similar blow-up happens, and the extension after the blow-up might then not be unique, see [8, 57].

2.4.1 On the stability of type I ODE and non-ODE blow-up for (NLH)

The ODE blow-up profile $\kappa_H(T-t)^{-\frac{1}{p-1}}$ is expected to be the generic blow-up profile. In Theorem 2.1.9 we saw that any blow-up is of type I in the energy subcritical case $1<p<p_c$. Moreover, as soon as a solution resembles the constant in space blow-up profile, it is attracted by it towards the same behavior.

Theorem 2.4.1 (Stability of subcritical ODE blow-up [54]). *For $1<p<p_c$, any blow-up of (NLH) is of type I with $\pm\kappa_H$ as blow-up profile, and the set of blow-up solutions is open in $L^\infty(\mathbb{R}^d)$.*

The topology here is not very important due to regularizing effects. Indeed, since two solutions close to each other in L^q, $q\geq \frac{d(p-1)}{2}$ will be close in L^∞ at a small later time from Proposition 2.1.2, this openness result yields the same result for all the other topologies where (NLH) is well-posed. The above result still holds in the critical case $p=p_c$ where we adapted the proof of [54], [25].

In the energy supercritical case, the ODE blow-up has still not been proven to be stable, but there are two results in this direction. First, if one localizes the ODE blow-up, one can find a stable dynamics. The following result is due to Merle and Zaag (though the result is stated in the energy subcritical case it should adapt to the supercritical case as well from a discussion with H. Zaag). The zone to localize the blow-up profile is a logarithmic correction to the self-similar zone, using the correction

$$f(z):=\frac{1}{\left(p-1+\frac{(p-1)^2}{4p}|z|^2\right)^{\frac{1}{p-1}}},\quad z\in\mathbb{R}^d. \tag{2.4.2}$$

Theorem 2.4.2 (Stable ODE blow-up solution for (NLH) [115]). *Let $1<p<p_c$. There exists $T_0>0$ such that for each $T\in(0,T_0]$ and $g\in W^{1,p+1}(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ with $\|g\|_{L^\infty}\leq |log(T)|^{-2}$, there exist $d_0\in\mathbb{R}$ and $d_1\in\mathbb{R}^d$ such that the solution of (NLH) with initial data*

$$u_0(x)=\frac{1}{T^{\frac{1}{p-1}}}\left[f(z)\left(1+\frac{d_0+d_1.z}{p-1+\frac{(p-1)^2}{4p}|z|^2}\right)+g(z)\right]$$

where f is defined by (2.4.2) satisfies uniformly in $z\in\mathbb{R}^d$

$$\lim_{t\to T}(T-t)^{\frac{1}{p-1}}u\left(t,(T-t)^{\frac12}|log(T-t)|^{\frac12}z\right)=f(z) \tag{2.4.3}$$

and

$$\forall R>0,\qquad \sup_{(t,x)\in[0,T)\times\mathbb{R}^d\setminus B(0,R)}|u(t,x)|<+\infty.$$

Then, if one consider positive radial solutions, one has the following stability of ODE blow-up along special curves of initial data.

Theorem 2.4.3 (Partial stability of supercritical ODE blow-up [99]). *Let $p>p_c$ and $v_0\in L^\infty\cap H^1$ be a radial positive function. For $\lambda>0$ denote by v_λ the solution of (NLH) with initial datum λv_0. Then there exists $k(v_0)$ and $0<\lambda^*<\lambda_1<...<\lambda_k$ such that for $0<\lambda<\lambda^*$ v_λ is global, for $\lambda\geq\lambda^*$ it blows-up in finite time, and moreover if $\lambda\neq\lambda^*,\lambda_1,...,\lambda_k$ the blow-up is of type I with κ_H as blow-up profile.*

For $\lambda = \lambda^*, \lambda_1, ..., \lambda_k$, the blow-up of v_λ is not an ODE blow-up: it is either a type II blow-up as seen in Section 2.3, or a type I blow-up with a self-similar blow-up profile different from κ_H. It is then interesting to know when such solutions exist. In the supercritical range of parameters $p_c < p < p_{JL}$ where p_{JL} is defined in (*1.4.1*), there exists a countable family of positive radial self-similar solutions [18, 19, 153]. There exists another exponent, the Lepin exponent

$$p_L := 1 + \frac{6}{d-10} > p_{JL} \ \ (p_L := +\infty \ if \ 1 \leq d \leq 10)$$

such that on (p_{JL}, p_L) there still exists a finite number of radial positive self-similar solutions [91], but they cease to exist above p_L [129]. A key property of these profiles is that they do not belong to the critical space $\dot{H}^{s_c}$ neither in H^1, and one can wonder if they can effectively appear as blow-up profiles. As for $p_c < p < p_{JL}$ there is no type II blow-up from Theorem 2.2.7, the existence of λ^* provides a non-constructive argument: some can be attained. Also, they must be strongly instable since they lay at the border of the set of blow-up solutions and global solutions. In [27], the author, in a joint work with P. Raphaël and J. Szeftel, investigated the stability of these solutions and their emergence from well-localized initial data.

We recall that χ is a cut-off function and that $\chi_A = \chi(x/A)$.

Theorem 2.4.4 (Stability of non-constant type I blow-up [27]). *Let* $d = 3$ *and* $p > p_c = 5$. *There exists* $N \gg 1$ *and a family* $(\Phi_n)_{n \geq N}$ *of smooth radially decaying solutions to* (*2.4.1*). *For any* $n \geq N$ *there exists a Lipschitz codimension* n *manifold*[12] *of possibly non radial initial data with finite energy under the form*

$$u_0 = \chi_{A_0} \Phi_n + w_0$$

where $A_0 \gg 1$ *is large enough and* w_0 *is small enough*

$$\| w_0 \|_{H^2} \ll 1, \tag{2.4.4}$$

such that the corresponding solution to (NLH) *blows up in finite time* $0 < T < +\infty$ *with a decomposition*

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} (\Phi_n + v) \left(t, \frac{x - x(t)}{\lambda(t)} \right)$$

where:

1. Control of the geometrical parameters*: the blow up speed is self similar*

$$\lambda(t) = \sqrt{(2 + o(1))(T-t)} \ \ as \ \ t \to T$$

and the blow up point converges

$$x(t) \to x(T) \ \ as \ \ t \to T. \tag{2.4.5}$$

2. Behaviour of Sobolev norms*: there holds the asymptotic stability of the self similar profile above scaling*

$$\lim_{t \to T} \| v(t) \|_{\dot{H}^s} = 0 \ \ for \ \ s_c < s \leq 2, \tag{2.4.6}$$

[12] see Proposition 6.4.9 for a precise statement of the Lipschitz regularity.

the boundedness of norms below scaling

$$\limsup_{t\to T} \|u(t)\|_{\dot{H}^s} < +\infty \ \ \textit{for} \ \ 1 \le s < s_c, \tag{2.4.7}$$

and the logarithmic growth of the critical norm

$$\|u(t)\|_{\dot{H^{s_c}}} = c_n(1+o_{t\to T}(1))\sqrt{|\log(T-t)|}, \ \ c_n \neq 0. \tag{2.4.8}$$

Comments: Note that $p_{JL}(3) = +\infty$ thus there is no upper bound for the range of p. The result is stated and proven in dimension 3 but it should propagate to higher dimensions. The main problem is that for d large one has $(p_c, p_{JL}) \subset (1,2)$ and the lack of regularity of the nonlinearity prevents from using derivatives in the analysis. The main novelty in (2.4.4) is the obtention of the stability of the profiles in a rather direct way by the knowledge of the spectral structure of the linearized operator in the self-similar zone. This robust analysis for parabolic problems could be propagated to other problems, in more non-radial settings such as the interaction of several solitons.

Open problems: The ODE blow-up should still be stable in the energy supercritical case, but this is still an open problem. An interesting question would be to know to which behavior the instabilities of non-constant self-similar solutions of (2.4.4) lead to. The conjecture of the author is that around Φ_n, there is a codimension 1 manifold of solutions being attracted to Φ_j for $1 \le j \le n$ separating two stable regions where solutions undergo ODE blow-up or dissipation to 0. The extension of this analysis to other non-radial stability of self-similar solutions (harmonic heat flow, harmonic maps...) is also an interesting direction of work.

2.4.2 Sketch of the proof of Theorem 2.4.4

We fix $d = 3$ and $p > 5$, $p \in 2\mathbb{N}+1$ for simplicity. The existence of a countable family of radial positive self-similar solutions Φ_n was already known [18, 19, 153]. However, to be able to study the dynamics near these solutions, one needs more information about them, and in particular the structure of the linearized operator near them. To obtain a refined description, we use a matched asymptotic procedure.

Step 1 *Precise description of the self-similar solutions.*
Exact solution at infinity. At infinity, the solution is seen as a perturbation of the homogeneous self-similar profile

$$\Phi^* := c_\infty r^{-\frac{2}{p-1}}$$

where c_∞ is defined in (*2.2.4*). Indeed, it is a zero of the self-similar equation (*2.4.1*), and a perturbation of this profile $\Phi^* + w$ still solving this equation yields

$$-\Delta w - p\Phi^* w + \Lambda w = NL$$

where NL is a nonlinear term. The linear operator here, $-\Delta - p\Phi^* + \Lambda$, studied on $[r_0, +\infty)$ for some $0 < r_0 \ll 1$, admits a unique well-decaying zero ψ and one can produce a solution of the self-similar equation (*2.4.1*) resembling $\Phi^* + \epsilon\psi$ for $\epsilon \in [0, \epsilon^*)$ on $[r_0, +\infty)$ by a fixed point argument and ODE techniques.

Exact solution at the origin. At the origin, the solution is seen as a perturbation of a rescaled ground state $(Q+v)_{\frac{1}{\lambda}}$, $0<\lambda\ll 1$, because this profile solves (*2.4.1*) on $[0,r_0)$ if and only if

$$-\Delta v - pQ^{p-1}v = -\lambda^2\Lambda(Q+v)+NL \ \text{ on } \left[0,\frac{r_0}{\lambda}\right].$$

Again one can construct a solution of the self-similar equation (*2.4.1*) resembling $(Q)_{\frac{1}{\lambda}}$ for $\lambda\in[0,\lambda^*)$ on $[0,r_0]$ by a fixed point argument and ODE techniques.

Matching and quantization. We obtain a true solution of (*2.4.1*) by matching the two above solutions at the point r_0, the interior one $\approx (Q)_{\frac{1}{\lambda}}$ on $[0,r_0]$ and the exterior one $\approx \Phi^*+\epsilon_1\psi$ on $[r_0,+\infty)$. By Cauchy theory for ODE, the exterior solution extends the interior one if and only if the two solutions and their derivatives are equal at r_0. This writes approximately

$$\begin{cases} (Q)_{\frac{1}{\lambda}}(r_0)\approx(\Phi^*+\epsilon\psi)(r_0),\\ \partial_r((Q)_{\frac{1}{\lambda}}(r_0)\approx\partial_r(\Phi^*+\epsilon\psi)(r_0). \end{cases}$$

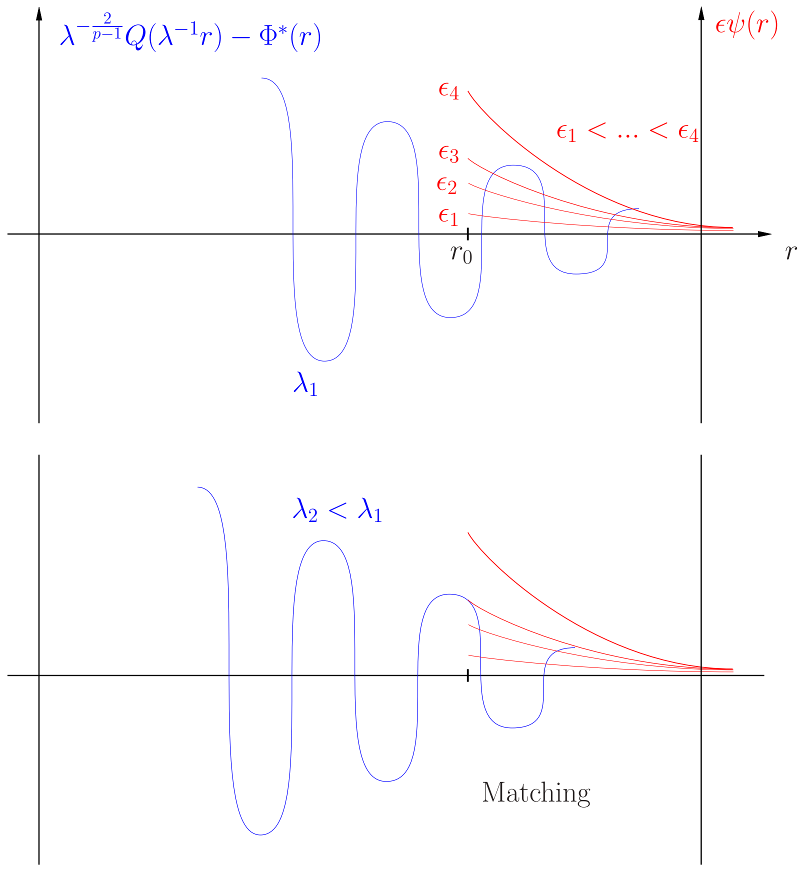
Injecting the asymptotic behavior (*2.2.6*) of Q for $p_c<p<p_{JL}$ in the above equation yields the condition[13]

$$\lambda\left[-\frac{d-2}{2}\sin[\omega\log(r_0\lambda^{-1})+c]+\omega\cos[\omega\log(r_0\lambda^{-1})]\right]-\frac{\partial_r\psi(r_0)}{\psi(r_0)}r_0\sin[\omega\log(r_0\lambda^{-1})+c]\approx 0$$

which for λ small amounts to ask for $\sin[\omega\log(r_0\lambda^{-1})+c]\approx 0$. Such values of λ are given by the law $\lambda=r_0e^{-\omega^{-1}(k\pi-c)}$ for $k\in\mathbb{N}$, $k\gg 1$ large enough. This is the quantization: only for a discrete set of values of λ one can construct a solution of (*2.4.1*).

The key point behind the above matching procedure is that the outer solution is monotone at r_0 with respect to ϵ, and that the inner solution oscillates infinitely many times at r_0 as $\lambda\to 0$. Whenever this solution makes one oscillation, one can find a new full solution.

[13]One can always chose r_0 such that $\frac{\partial_r\psi(r_0)}{\psi(r_0)}\neq 0,+\infty$.



This method allows to count the numbers of oscillations and the result is the following.

Proposition 2.4.5 (Existence and asymptotic of excited self similar solutions). *Let $d = 3$ and $p > 5$. For all $n > N$ large enough, there exist a smooth radially symmetric solution to the self similar equation (2.4.1) such that*

$$\Lambda\Phi_n \ \ \textit{vanishes exactly n times on} \ \ (0, +\infty).$$

Moreover, there exists a small enough constant $r_0 > 0$ independent of n such that:

(i) Behavior at infinity*:*

$$\lim_{n\to+\infty} \sup_{r\geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_*(r)| = 0. \tag{2.4.9}$$

(ii) Behaviour at the origin*: there exists a sequence $\mu_n > 0$ with $\mu_n \to 0$ as $n \to +\infty$ such that*

$$\lim_{n\to+\infty} \sup_{r\leq r_0} \left| \Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n}\right) \right| = 0. \tag{2.4.10}$$

Step 2 *Linearized dynamics.* We now look for solutions of (NLH) close to $(T-t)^{-\frac{1}{p-1}}\Phi_n(x(T-t)^{-\frac{1}{2}})$ of the form

$$u = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(\phi_n + v)\left(\frac{x-x(t)}{\lambda(t)}\right)$$

and we expect $\lambda(t) \sim \sqrt{T-t}$. Defining the renormalized flow

$$\frac{ds}{dt} = \frac{1}{\lambda^2}, \quad s(0) = s_0 \tag{2.4.11}$$

v must then solve the equation

$$v_s - \Delta v - p\Phi_n^{p-1} + \Lambda v = \left(-\frac{\lambda_s}{\lambda} + 1\right)\Lambda v + NL$$

because we expect $\frac{\lambda_s}{\lambda} \approx -1$ as for the unperturbed solution. The linearized operator is therefore

$$\mathcal{L}_n := -\Delta - p\Phi_n^{p-1} + \Lambda.$$

It is self-adjoint for the measure $e^{-\frac{|y|^2}{2}}$, associated to the localized modified Dirichlet energy

$$\int v\mathcal{L}ve^{-\frac{|y|^2}{2}} = \int |\nabla v|^2 e^{-\frac{|x|^2}{2}} - \int p\Phi_n^{p-1}v^2 e^{-\frac{|x|^2}{2}},$$

bounded from below, and with compact resolvant. Therefore, its spectrum consists of a unbounded sequence of eigenvalues. As the potential is radial, one can apply Sturm-Liouville theory, and the negative eigenvalues are dictated by the number of zeros of the gauge mode

$$\mathcal{L}\Lambda\Phi_n = -2\Lambda\Phi_n$$

which is n by Proposition 2.4.5. Therefore, there are n negative eigenvalues below -2. Above -2 after some computations relying again on the almost explicit knowledge of Φ_n and on the slight use of numerical computations, the only negative eigenvalues are given by the translation invariances and the rest of the spectrum is strictly positive. Let us introduce the spaces

$$H^k_\rho := \left\{ v \in H^k_{\text{loc}}, \ \int |\nabla v|^j e^{-\frac{|x|^2}{2}} < +\infty \ \text{ for } 0 \le j \le k \right\}.$$

and the scalar product $(u,v)_\rho = \int uve^{-\frac{|x|^2}{2}}$.

Proposition 2.4.6 (Spectral gap for $\mathcal{L}_n$). *Let $n \ge N$ with $N \gg 1$ large enough, then the following holds:*
1. Eigenvalues. *The spectrum of $\mathcal{L}_n$ is given by*

$$-\mu_{n+1,n} < \cdots < -\mu_{2,n} < -\mu_{1,n} = -2 < -\mu_{-1,n} = -1 < 0 < \lambda_{0,n} < \lambda_{1,n} < \dots$$

with

$$\lambda_{j,n} > 0 \text{ *for all* } j \ge 0 \text{ *and* } \lim_{j\to+\infty} \lambda_{j,n} = +\infty.$$

The eigenvalues $(-\mu_{j,n})_{1\le j\le n+1}$ are simple and associated to radial eigenvectors

$$\psi_{j,n}, \ \ \|\psi_{j,n}\|_{L^2_\rho} = 1, \ \ \psi_{1,n} = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_\rho},$$

and the eigenspace for $\mu_{-1,n}$ is spanned by

$$\psi^k_{-1,n} = \frac{\partial_k \Phi_n}{\|\partial_k \Phi_n\|_\rho}, \quad 1 \le k \le 3.$$

Moreover, the instable eigenfunctions are well-localized with an explicit asymptotic

$$|\psi_{j,n}(r)| \lesssim (1+r)^{-\frac{2}{p-1}-\mu_{j,n}}, \quad 1 \le j \le n+1.$$

2. Spectral gap. *There holds for some constant $c_n > 0$:*

$$\forall \varepsilon \in H^1_\rho, \ (\mathcal{L}_n \varepsilon, \varepsilon)_\rho \ge c_n \|\varepsilon\|^2_{H^1_\rho} - \frac{1}{c_n}\left[\sum_{j=1}^{n+1} (\varepsilon, \psi_{j,n})^2_\rho + \sum_{k=1}^{3} (\varepsilon, \psi^k_{0,n})^2_\rho\right]. \tag{2.4.12}$$

Step 3 *Trapped regime and existence by topological argument.* Functions near Φ_n can be decomposed as

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(\Phi_n + \psi + \varepsilon)(s,y), \quad y = \frac{x - x(t)}{\lambda(t)} \tag{2.4.13}$$

with, in view of Proposition 2.4.6,

$$\psi = \sum_{j=2}^{n+1} a_j \psi_j, \quad (\varepsilon, \psi_j)_\rho = (\varepsilon, \partial_k \Phi_n)_\rho = 0, \quad 1 \le j \le n+1, \quad 1 \le k \le 3. \tag{2.4.14}$$

We introduce another decomposition

$$v = \psi + \varepsilon, \quad \Phi_n + v = \chi_{\frac{1}{\lambda}} \Phi_n + w.$$

As long as a solution stays close to a renormalized version of Φ_n the above decomposition exists thanks to an application of the implicit function theorem, and the renormalized time s associated to $\lambda(t)$ by (*2.4.11*) is well-defined. We now consider solutions that are trapped near Φ_n in the sense that for the parameters:

$$\lambda(s) \sim e^{-s},$$
$$\sum_{j=2}^{n+1} |a_j|^2 \le e^{-2\mu s}, \tag{2.4.15}$$

where $\mu = \frac{c_n}{8}$, c_n being given by (*2.4.12*), and for the remainder

$$\|\varepsilon\|_{H^2_\rho} < K e^{-\mu s}, \quad \|\Delta v\|_{L^2} < K e^{-\mu s} \ \text{ and } \ \|w\|_{\dot{H}^{s_c}} \ll 1. \tag{2.4.16}$$

A trapped solution satisfies the equation

$$\partial_s \varepsilon + \mathcal{L}_n \varepsilon = F - \text{Mod}$$

where the modulation term is

$$\text{Mod} = \sum_{j=2}^{n+1} [(a_j)_s - \mu_j a_j]\, \psi_j - \left(\frac{\lambda_s}{\lambda} + 1\right)(\Lambda \Phi_n + \Lambda \psi) - \frac{x_s}{\lambda} \cdot (\nabla \Phi_n + \nabla \psi)$$

and where the force term is (NL being the nonlinear terms)

$$F = L(\varepsilon) + NL, \quad L(\varepsilon) = \left(\frac{\lambda_s}{\lambda} + 1\right)\Lambda \varepsilon + \frac{x_s}{\lambda}.\nabla \varepsilon.$$

Its evolution is then described by the computation of modulation equations for the parameters and of energy type estimates involving the exponentially localized norm $e^{-|y|^2/2}$ which in original variables describes the self-similar zone.

Proposition 2.4.7. *For a trapped solution, for s_0 large enough there holds for the parameters*

$$\left|\frac{\lambda_s}{\lambda}+1\right|+\left|\frac{x_s}{\lambda}\right|+\sum_{j=2}^{n+1}|(a_j)_s-\mu_j a_j|\lesssim \|\varepsilon\|^2_{H^1_\rho}+\|\Delta v\|^2_{L^2}+\sum_{j=2}^{n+1}|a_j|^2. \tag{2.4.17}$$

There holds for the remainder, with $c_n>0$ given by (2.4.12),

$$\frac{d}{ds}\|\varepsilon\|^2_{L^2_\rho}+c_n\|\varepsilon\|^2_{H^1_\rho}\lesssim \sum_{j=2}^{n+1}|a_j|^4+\|\Delta v\|^4_{L^2}+\|v\|^2_{L^\infty}\left[\|\Delta v\|^2_{L^2}+\sum_{j=2}^{n+1}|a_j|^2\right],$$

$$\begin{aligned}\frac{d}{ds}\|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho}+c_n\|\mathcal{L}_n\varepsilon\|^2_{H^1_\rho} &\lesssim \|\varepsilon\|^2_{H^1_\rho}+\sum_{j=2}^{n+1}|a_j|^4+\|\Delta v\|^4_{L^2}\\ &+ \|v\|^2_{L^\infty}\left[\|\Delta v\|^2_{L^2}+\|\varepsilon\|^2_{H^1_\rho}+\sum_{j=2}^{n+1}|a_j|^2\right],\end{aligned}$$

$$\frac{d}{ds}\left[\frac{1}{\lambda^{4-\delta-2s_c}}\int|\Delta v|^2dy\right]+\frac{1}{\lambda^{4-\delta-2s_c}}\int|\nabla\Delta v|^2dy\lesssim\frac{1}{\lambda^{4-2s_c-\delta}}\left[\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2\right]$$

$$\frac{d}{ds}\int|\nabla^{s_c}w|^2dy+\int|\nabla^{s_c+1}w|^2dy\lesssim\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2+\lambda^{\delta(2-s_c)}+\|\Delta v\|^\delta_{L^2}.$$

for some small enough universal constant $0<\delta=\delta(p)\ll 1$.

The interpretation is as follows. The main blow-up dynamics are computed in the self-similar zone in the spaces H^k_ρ where the linearized operator is diagonalized. There, the instabilities are linear and well-localized, and the remainder on the stable eigenspace is linearly decaying. The good control on the nonlinear term allows us to prove that its influence is of lower order.

What happens beyond the self-similar zone is less important, and one just needs to check that the whole perturbation stays under control there. This outer perturbation only interacts with the blow-up dynamics inside the self-similar zone at the nonlinear level. To be able to control the nonlinear interaction we control a critical $\dot H^{s_c}$ and a supercritical $\dot H^2$ norm in the far away zone in renormalized variables. Since $d=3$ this gives a L^∞ bound on the perturbation by Sobolev embedding. Outside the self-similar zone, the critical and supercritical norms of the whole perturbation are dissipated, and the localization of the profile Φ_n in the zone $|y|\sim\lambda^{-1}$ is harmless since this corresponds to a fixed localization $|x|\sim 1$ in original variables, and since the initial solution is already very concentrated.

Step 4 *Bootstrap analysis and exit of the trapped regime.* Reintegrating the differential equations in the above Proposition gives that as long as the solution is trapped, the stable part of the perturbation ε enjoys in fact better bounds than (2.4.16) and this part cannot grow big. Therefore, a solution exits the trapped regime if and only if the bound on the parameters (2.4.15) is violated. However, 0 is a linear repulsive equilibrium for these parameters from (2.4.17). With an application of Brouwer's fixed point theorem one gets the existence of a solution staying trapped forever.

Proposition 2.4.8. *For any $\lambda_0 = e^{-s_0}$ and ε_0 satisfying (2.4.14) and*

$$\|(1-\chi_{\frac{1}{\lambda_0}})\Phi_n + \varepsilon_0\|_{\dot{H}^{s_c}} + \|\varepsilon_0\|_{H^2_\rho} + \|\Delta \varepsilon_0\|_{L^2} \leq e^{-2\mu s_0},$$

there exists $(a_2(0), ..., a_{n+1}(0))$ with $\sum_{j=2}^{n+1} |a_j|^2 \leq e^{-2\mu s_0}$ such that the corresponding solution of (NLH) starting from (2.4.13) is trapped on its maximal interval of existence.

If a solution is trapped forever, the direct reintegration of the modulation bounds (*2.4.17*) and the use of the a priori bounds (*2.4.15*) and (*2.4.16*) imply that it blows up with the asymptotic described by Theorem 2.4.4. As for the manifold construction in Subsection 2.2.3, the set of initial data leading to solutions that are trapped forever is then proved to be a Lipschitz manifold with codimension n, ending the proof of the Theorem 2.4.4. The key fact behind this topological structure is that the parameters (2.4.8) are unique and depend in a Lipschitz way on ε_0.

To prove this last result, one needs to study the difference of two trapped solutions. Using all the a priori bounds on trapped solutions (in particular the L^∞ a priori bound to control the nonlinear term), the evolution of such a difference is considered in the self-similar zone where it is almost linear and decoupled. The difference along the instable directions $\sum_2^{n+1}(a_i - a_i')\psi_i$ evolves according to a linearly instable dynamics, while the difference along the stable directions $\varepsilon - \varepsilon'$ evolves according to a stable one. Therefore, if the two solutions are trapped forever, the two differences must stay small for all time, which is possible only if the nonlinear feedback from the stable part to the instable one is large enough. This is possible only if the initial difference of the stable remainders $\varepsilon_0 - \varepsilon_0'$ is comparable in size with the initial difference of the instable perturbation $\sum_2^{n+1}(a_i(0) - a_i'(0))\psi_i$. Quantifying precisely this fact gives the desired Lipschitz dependance.

3

Concentration of the ground state for the energy supercritical semilinear wave equation

3.1 Introduction, organization and notations

In this chapter we prove Theorems 2.2.4 and 2.2.5. This work is to appear in the Memoirs of the American Mathematical Society [23]. We gave a detailed sketch of the proof of this Theorem in Subsection 2.2.3.

We gave various motivations to the results of Theorems 2.2.4 and 2.2.5 in Section 2.2 of the previous Chapter 2. We now give a rigorous proof of these two theorems. As certain objects here and in other chapters are different but play a similar role in the analysis, we use the same notation for them. As a consequence, we start by describing all the notations that are specific to this chapter, and the reader is invited to come back here whenever he or she has some doubts.

The chapter is divided in three parts. The first part is devoted is devoted to the proof of Theorem 2.2.4. In section 3.2 we present the main tools to understand the linearized operator near the soliton. After that we are able to construct or primary approximate blow-up profile in Proposition 3.2.12. We then localize this profile in a slightly enlarged light cone emanating backward from the origin and the blow-up time, and estimate the remainder of the approximate dynamics in Proposition 3.2.14. We end this section by studying the approximate dynamics governing the finite number of parameters describing the approximate blow-up profile. The existence of special solutions for (*2.2.30*) is done in Lemma 3.2.16, their linear stability is studied in Lemma 3.2.17. In section 3.3 we implement our bootstrap method near the approximate blow-up profile and state our main existence result in Proposition 3.3.2. First we explain how to "project" the full (NLW) on the manifold of approximate solutions in Lemma 3.3.1. Then we estimate the impact of the remainder in the decomposition on the dynamics of the parameters by computing the modulations equations in Lemmas 3.3.3 and 3.3.5. In the second part we estimate the size of the remainder. We start by deriving the monotonicity formula for a low regularity Sobolev norm in Proposition 3.3.6, then we do it for a high regularity norm built on the linearized operator in Proposition 3.3.7, which is the main result of the section. We end the section with deriving a Morawetz identity to control some local terms that appeared earlier in the computations in Proposition 3.3.9. In section 3.4 we end the proof of the main Proposition 3.3.2. We show that in fact better bounds hold for the remainder in Lemma 3.4.2. We then examine the dynamics for the parameters in Lemmas 3.4.4 and 3.4.6, we show the existence of a true blow-up solution by topological arguments. For the completeness of the result we study the behavior of Sobolev norms in subsection 3.4.2.

The second part is devoted to the proof of Theorem 2.2.5. In Section 3.5 we investigate the topological properties of the set of initial data leading to the above blow-up scenario. In Proposition 3.5.2 we show that for such solutions starting at the same scale with some additional regularity, we have Lipschitz dependence in adapted variables. We remove the extra assumptions in Proposition 3.5.13, which allows us to prove that the set of initial data staying in our blow up scenario is a Lipschitz manifold whose codimension is explicit.

The third part is the Appendix. In Section 3.A we prove the nondegeneracy of the first coefficient in the expansion of the soliton. In Section 3.B we prove the equivalence between norms built on the linearized operator and weighted norms on usual derivatives. Then we recall some Hardy type inequalities in Section 3.C. These two previous sections allow to prove the coercivity of an adapted norm in Section 3.D under some orthogonality conditions. Eventually we give some useful norms on solutions trapped near the approximate blow-up profile in Section 3.E.

Specific notations

Super critical numerology. Given $d \geq 11$, $p > p_{JL}$ (defined in (*1.4.1*)), we let α and α_2 be the roots of the polynomial $X^2 - (d-2-\frac{4}{p-1})X + 2(d-2-\frac{2}{p-1})$ satisfying $\alpha < \alpha_2$. One can check that the condition $p > p_{JL}$ ensures the reality of α and α_2, and that they are not equal (see Lemma 3.2.2). This definition is coherent with the formula (*2.2.7*). We define[1]:

$$\begin{cases} k_0 := E[\frac{d}{2} - \gamma] > 1, \\ \delta_0 := \frac{d}{2} - \gamma - k_0, \ 0 < \delta_0 < 1. \end{cases} \tag{3.1.1}$$

because we are assuming $\left(\frac{d}{2} - \gamma\right) \notin \mathbb{N}$ from (*2.2.8*), so that

$$d = 2\gamma + 2k_0 + 2\delta_0. \tag{3.1.2}$$

We let

$$g := \min(\alpha, \alpha_2 - \alpha) - \epsilon > 0 \tag{3.1.3}$$

and

$$g' := \min(g, 2, 1 + \delta_0 - \epsilon) > 0 \tag{3.1.4}$$

be the two real numbers that will quantify some gain in the asymptotics of our objects later on. ϵ stands for a very small constant $0 < \epsilon \ll 1$ that can be chosen independently of the sequel. The presence of $-\epsilon$ and $1 + \delta_0$ is just a way to simplify the writing of results later on.

Notations for the analysis: For the sake of simplicity, we will use the following equivalent formulation for the focusing nonlinear wave equation (NLW):

$$(NLW) \begin{cases} \partial_t \boldsymbol{u} = \boldsymbol{F}(\boldsymbol{u}), \\ \boldsymbol{u}_{|t=0} = \boldsymbol{u}_0 \end{cases} \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d, \ \boldsymbol{u}(t,x) : \mathbb{R}^d \to \mathbb{R} \times \mathbb{R}. \tag{3.1.5}$$

We will consider radial solutions: $\boldsymbol{u}(x) = \boldsymbol{u}(r)$ where $r = |x|$. We refer to the coordinates of a function $\boldsymbol{u}$ as $u^{(1)}$ and $u^{(2)}$:

$$\boldsymbol{u} = \begin{pmatrix} u^{(1)} \\ u^{(2)} \end{pmatrix}. \tag{3.1.6}$$

[1]where we recall the definition of the integer part for $x \in \mathbb{R}$, $E[x] \leq x < E[x] + 1$, $E(x) \in \mathbb{Z}$.

We let the expression $\boldsymbol{F}$ be:

$$\boldsymbol{F}(\boldsymbol{u}) := \begin{pmatrix} u^{(2)} \\ \Delta u^{(1)} + f(u^{(1)}) \end{pmatrix}, \ f(t) := |t|^{p-1}t. \tag{3.1.7}$$

The bold notations will always refer to vectors. We make an abuse of notation (regarding (*3.1.6*)) by still denoting the stationnary state introduced earlier by $\boldsymbol{Q}$:

$$\boldsymbol{Q} := \begin{pmatrix} Q \\ 0 \end{pmatrix}.$$

Given a large integer $L \gg 1$, we define the Sobolev exponent:

$$s_L := k_0 + 1 + L. \tag{3.1.8}$$

We will use the standard scalar product on $L^2(\mathbb{R}^d)$ and $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$:

$$\langle u, v\rangle := \int_{\mathbb{R}^d} uv \ \text{ and } \ \langle \boldsymbol{u}, \boldsymbol{v}\rangle := \int_{\mathbb{R}^d} u^{(1)}v^{(1)} + \int_{\mathbb{R}^d} u^{(2)}v^{(2)}.$$

Let $0 < \lambda$, we denote the renormalized function by:

$$\boldsymbol{u}_\lambda(x) := \begin{pmatrix} \lambda^{\frac{2}{p-1}} u^{(1)}(\lambda y) \\ \lambda^{\frac{2}{p-1}+1} u^{(2)}(\lambda y) \end{pmatrix}. \tag{3.1.9}$$

The rescaled coordinates are then[2]:

$$\boldsymbol{u}_\lambda := \begin{pmatrix} u_\lambda^{(1)} \\ u_\lambda^{(2)} \end{pmatrix}. \tag{3.1.10}$$

We let the generator of the scaling be:

$$\boldsymbol{\Lambda u} := \begin{pmatrix} \Lambda^{(1)} u^{(1)} \\ \Lambda^{(2)} u^{(2)} \end{pmatrix} := \begin{pmatrix} \left(\frac{2}{p-1} + y.\nabla\right) u^{(1)} \\ \left(\frac{2}{p-1} + 1 + y.\nabla\right) u^{(2)} \end{pmatrix}.$$

We introduce the renormalized space variable:

$$y := \frac{r}{\lambda}.$$

Given $b_1 > 0$, we define:

$$B_0 := \frac{1}{b_1}, \ B_1 := B_0^{1+\eta} \tag{3.1.11}$$

where η is a small number $0 < \eta \ll 1$ which will be choosen later. For any $B > 0$, χ a cut-off function and $\boldsymbol{u}$ a function, we will use the notation

$$\chi_B \boldsymbol{u} := \begin{pmatrix} \chi_B u^{(1)} \\ \chi_B u^{(2)} \end{pmatrix} \tag{3.1.12}$$

Analysis near the ground state. The linearized operator near $\boldsymbol{Q}$ of equation (*3.1.5*) is given by:

[2] Notice that the subscript λ does not mean the same renormalization for the first and the second coordinates. There should be no confusion in the sequel as the superscripts (1) or (2) will always be presents.

$$\boldsymbol{H}\boldsymbol{\varepsilon} := \begin{pmatrix} -\varepsilon^{(2)} \\ -\Delta\varepsilon^{(1)} - pQ^{p-1}\varepsilon^{(1)} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -\Delta - pQ^{p-1} & 0 \end{pmatrix} \boldsymbol{\varepsilon}, \tag{3.1.13}$$

so that:

$$\boldsymbol{F}(\boldsymbol{Q}+\boldsymbol{\varepsilon}) = -\boldsymbol{H}\boldsymbol{\varepsilon} + \boldsymbol{NL}. \tag{3.1.14}$$

Here $\boldsymbol{NL}$ stands for the purely nonlinear term:

$$\boldsymbol{NL} := \begin{pmatrix} 0 \\ f(Q+\varepsilon^{(1)}) - f(Q) - pQ^{p-1}\varepsilon^{(1)} \end{pmatrix}. \tag{3.1.15}$$

We define:

$$\mathcal{L} := -\Delta - pQ^{p-1}, \tag{3.1.16}$$

so that:

$$\boldsymbol{H} = \begin{pmatrix} 0 & -1 \\ \mathcal{L} & 0 \end{pmatrix}. \tag{3.1.17}$$

Eventually, we note the potential:

$$V := pQ^{p-1}. \tag{3.1.18}$$

The notations in this chapter are heavy. To ease understanding, we only use subscripts and superscripts that we separate with comas. For exemple, $T^{(2)}_{i,\frac{1}{\lambda}}$ denotes the second coordinates of the vector function $\mathbf{T}_i$ where i denotes an indice, renormalized at the scale λ. The meaning should however always be clear from the context.

3.2 The linearized dynamics and the construction of the approximate blow-up profile

To understand the dynamics close to the 1-parameter family of ground states $\left(\boldsymbol{Q}_{\frac{1}{\lambda}}\right)_{\lambda>0}$ we study first its linearization. We start by the presentation of appropriate notions, and technical lemmas about the linearized operator $\boldsymbol{H}$. Once we have these tools, we are able to create an approximate blow up profile in the second part of this section.

3.2.1 The stationnary state and its numerology

Previous details on Q were given in Lemma 2.2.2, but for the analysis here we need a more complete description. Almost all the properties below are known ones, see [93, 78].

Lemma 3.2.1 (Asymptotic expansion of the ground state). *Let $p > p_{JL}$ (defined in (1.4.1)). We recall that $g > 0$, c_∞ and γ are defined in (2.2.5), (2.2.4) and (3.1.3). One has (with the corresponding asymptotic for the derivatives):*

(i) Asymptotics at infinity:

$$Q = \frac{c_\infty}{y^{\frac{2}{p-1}}} + \frac{a_1}{y^\gamma} + O\left(\frac{1}{y^{\gamma+g}}\right), \text{ as } y \to +\infty, \tag{3.2.1}$$

for a non null constant $a_1 \neq 0$ and Q stays below the singular self-similar profile

$$0 < Q(y) < \frac{c_\infty}{|y|^{\frac{2}{p-1}}}. \tag{3.2.2}$$

(ii) Degeneracy:

$$\Lambda^{(1)} Q = -\alpha \frac{a_1}{y^\gamma} + O\left(\frac{1}{y^{\gamma+k}}\right), \quad \textit{as } y \to +\infty, \tag{3.2.3}$$

$$\frac{d}{d\lambda}[(Q_\lambda)^{p-1}]_{|\lambda=1} = O\left(\frac{1}{r^{2+\alpha}}\right) \quad \textit{as } r \to +\infty. \tag{3.2.4}$$

(iii) Positivity of $\mathcal{L}$:

$$-\frac{(d-2)^2}{4|y|^2} + \frac{\delta(p)}{|y|^2} \leq V(y) < 0. \tag{3.2.5}$$

(iv) Positivity of $\Lambda^{(1)} Q$:

$$\Lambda^{(1)} Q > 0. \tag{3.2.6}$$

Proof of lemma 3.2.1

Only the fact that $a_1 \neq 0$ is not proven in the references we quoted. To prove it, we have to enter in details in their proof of the asymptotic expansion. This is done in Lemma 3.A.1 of Appendix A.

□

We now state important properties of the numbers attached to the asymptotic expansion of the ground state. A proof can be found in [114], Lemma A1.

Lemma 3.2.2 (supercritical numerology). *Let $d \geq 11$, p_{JL} and, α be given by (1.4.1) and (2.2.7). Then:*

(i) the condition $p > p_{JL}$ is equivalent to:

$$2 + \sqrt{d-1} < s_c < \frac{d}{2}.$$

(ii) α is real if and only if $p > p_{JL}$. In that case there holds the bounds:

$$2 < \alpha < \frac{d}{2} - 1.$$

3.2.2 factorization of $\mathcal{L}$

The positivity of $\Lambda^{(1)} Q$ (3.2.6) implies from a direct calculation the factorization of this operator.

Lemma 3.2.3 (Factorization of $\mathcal{L}$). *Let:*

$$W := \partial_y(log(\Lambda^{(1)} Q)), \tag{3.2.7}$$

and define the first order operators on radial functions:

$$A : u \mapsto -\partial_y u + W u, \ A^* : u \mapsto \frac{1}{y^{d-1}} \partial_y (y^{d-1} u) + W u. \tag{3.2.8}$$

Then we have:

$$\mathcal{L} = A^* A. \tag{3.2.9}$$

Remark 3.2.4. The adjunction is taken with respect to the radially symmetric Lebesgue measure:

$$\int_{y>0}(Au)vy^{d-1}dy=\int_{y>0}u(A^*v)y^{d-1}dy.$$

Proof of Lemma 3.2.3

This factorization relies on the fact that $\Lambda^{(1)}Q>0$, and then it is a standard property of Schrödinger operators with a non-vanishing zero eigenfunction. One can compute:

$$A^*Au=-\Delta u+(\frac{d-1}{y}W+\partial_y W+W^2)u.$$

Then the result follows from:

$$\frac{d-1}{y}W+\partial_y W+W^2=\frac{\Delta\Lambda^{(1)}Q}{\Lambda^{(1)}Q}=\frac{-\mathcal{L}\Lambda^{(1)}Q-V\Lambda^{(1)}Q}{\Lambda^{(1)}Q}=-V,$$

where we used the fact that $\mathcal{L}\Lambda^{(1)}Q=0$.

□

We collect here the informations about the asymptotic behavior of the potentials V and W which will be used many times in the sequel. These results are a direct implication of the previous Lemma 3.2.1.

Lemma 3.2.5. *(Asymptotic behavior of the potentials:)*
There holds:

(i) Asymptotics:

$$\partial_y^k V=\begin{cases} O(1)\ as\ y\rightarrow 0\\ \frac{c_k}{y^{2+k}}+O\left(\frac{1}{y^{2+\alpha+k}}\right)\ \ as\ y\rightarrow+\infty\end{cases}, \tag{3.2.10}$$

$$\partial_y^k W=\begin{cases} O(1)\ as\ y\rightarrow 0\\ \frac{c_k'}{y^{1+k}}+O\left(\frac{1}{y^{1+g+k}}\right)\ \ as\ y\rightarrow+\infty\end{cases}, \tag{3.2.11}$$

with $c_k\neq 0$, $c_k'\neq 0$ and $c_1'=-\gamma$.

(ii) Degeneracy:

$$\partial_y^k\left(\frac{d}{d\lambda}[(Q_\lambda)^{p-1}]_{|\lambda=1}\right)=O\left(\frac{1}{y^{2+\alpha+k}}\right)\ \ as\ y\rightarrow+\infty. \tag{3.2.12}$$

3.2.3 Inverting H on radially symmetric functions

We first start by inverting $\mathcal{L}$. We are only considering radially symmetric functions, so $\Delta=\partial_{yy}+(d-1)\frac{\partial_y}{y}$, and we can apply basic results from ODE theory. We will do this thanks to the explicit knowledge of the kernel of $\mathcal{L}$. Indeed from the rewriting:

$$A:u\mapsto-\Lambda^{(1)}Q\partial_y\left(\frac{u}{\Lambda^{(1)}Q}\right),\ A^*:u\mapsto\frac{1}{y^{d-1}\Lambda^{(1)}Q}\partial_y(y^{d-1}\Lambda^{(1)}Qu), \tag{3.2.13}$$

we note that:

$$Au=0\text{ iff }u\in\text{Span}(\Lambda^{(1)}Q),\ A^*u=0\text{ iff }u\in\text{Span}\left(\frac{1}{y^{d-1}\Lambda^{(1)}Q}\right). \tag{3.2.14}$$

It implies that for radially symetric functions:

$$\mathcal{L}u = 0 \text{ iff } u \in \text{Span}(\Lambda^{(1)}Q, \Gamma), \tag{3.2.15}$$

with:

$$\Gamma(y) := \Lambda^{(1)}Q(y) \int_1^y \frac{dx}{x^{d-1}(\Lambda^{(1)}Q(x))^2}. \tag{3.2.16}$$

We already knew $\Lambda^{(1)}Q$ was in the kernel of $\mathcal{L}$ since it is the tangent vector to the branch of stationnary solutions $(Q_\lambda)_{\lambda>0}$. We just found the second vector in the kernel: Γ. From the asymptotic behavior (*3.2.3*) of $\Lambda^{(1)}Q$, we deduce the following asymptotic for Γ:

$$\Gamma \underset{y\to 0}{\sim} \frac{-c}{y^{d-2}} \text{ and } \Gamma \underset{y\to+\infty}{\sim} \frac{c'}{y^\gamma}, \tag{3.2.17}$$

c and c' being two positive constants. Both results are obtained from (*3.2.16*), with the fact that $\Lambda^{(1)}Q > 0$ and the asymptotic (*3.2.3*) that implies:

$$0 < \int_1^{+\infty} \frac{dx}{x^{d-1}(\Lambda^{(1)}Q)^2} \leq C \int_1^{+\infty} \frac{dx}{x^{d-1-2\gamma}} < +\infty,$$

where we used the relation from (*2.2.5*): $d-1-2\gamma$>1.

Now that we know the Green's functions of $\mathcal{L}$ we can introduce the formal inverse:

$$\mathcal{L}^{-1}f := -\Gamma(y)\int_0^y f\Lambda^{(1)}Qx^{d-1}dx + \Lambda^{(1)}Q(y)\int_0^y f\Gamma x^{d-1}dx. \tag{3.2.18}$$

One can check that for f smooth and radial we have indeed $\mathcal{L}(\mathcal{L}^{-1}f) = f$. As we do not have uniqueness for the equation $\mathcal{L}u = f$, one may wonder if this definition is the "right" one. The answer is yes because this inverse has the good asymptotic behavior at the origin and $+\infty$, see Lemma 3.2.8. To compute easily the asymptotic, we will use the following computational lemma.

Lemma 3.2.6. *(Inversion of $\mathcal{L}$:) Let f be a C^∞ radially symmetric function, and denote by u its inverse by $\mathcal{L}$: $u = \mathcal{L}^{-1}f$ given by (3.2.18), then:*

$$Au = \frac{1}{y^{d-1}\Lambda^{(1)}Q}\int_0^y f\Lambda^{(1)}Qx^{d-1}dx, \; u = -\Lambda^{(1)}Q\int_0^y \frac{Au}{\Lambda^{(1)}Q}dx. \tag{3.2.19}$$

This lemma says that to compute $u = \mathcal{L}^{-1}f$, we can do it in a rather easy way in two times: first we compute Au, then we compute u knowing Au.

Proof of Lemma 3.2.6 We compute from the definition of Γ (*3.2.16*):

$$A\Gamma = -\partial_y\Gamma + \frac{\partial_y(\Lambda^{(1)}Q)}{\Lambda^{(1)}Q}\Gamma = -\frac{1}{y^{d-1}\Lambda^{(1)}Q}.$$

We therefore apply A to the definition of u given by (*3.2.18*), and using the cancellation $A(\Lambda Q) = 0$, we find:

$$Au = \frac{1}{y^{d-1}\Lambda^{(1)}Q}\int_0^y f\Lambda^{(1)}Qx^{d-1}dx.$$

which, together with the definition of A (*3.2.13*) gives:

$$u = -\Lambda^{(1)}Q\int_0^y \frac{Au}{\Lambda^{(1)}Q}dx + c_u\Lambda^{(1)}Q,$$

c_u being an integration constant. But from (*3.2.18*) we see that: $u = O(y^2)$ and $Au = O(y)$ as $y \to 0$. From that we deduce the nullity of the constant: $c_u = 0$, which establishes the formula. $\square$

Knowing how to invert $\mathcal{L}$, we define the inverse of $\boldsymbol{H}$ by the following formula:

$$\boldsymbol{H}^{-1} := \begin{pmatrix} 0 & \mathcal{L}^{-1} \\ -1 & 0 \end{pmatrix}. \tag{3.2.20}$$

3.2.4 Adapted derivatives, admissible and homogeneous functions

The usual derivatives, that is to say the ∇^k ones, are not fit for the study of (NLW) close to the family of ground states $(\boldsymbol{Q}_\lambda)_{\lambda>0}$, because they do not commute with the linearized operator $\mathcal{L}$. In this subsection we describe the adapted derivatives we will use. The asymptotic behavior of the adapted derivatives of the profiles, at the origin and at infinity, is going to play an important role. The second significant property is the vectorial position (when a function $\boldsymbol{f}$ has only one of its coordinate being non null). For the profiles we will use later, these informations are contained in the notion of admissible function. Given a radial function $f(x) = f(|x|)$, we define the sequence:

$$f_k = \mathcal{A}^k f$$

of adapted derivatives of f by induction:

$$f_0 := f \text{ and } f_{k+1} := \begin{cases} Af_k \text{ for } k \text{ even,} \\ A^* f_k \text{ for } k \text{ odd.} \end{cases} \tag{3.2.21}$$

Definition 3.2.7. (Admissible functions:) Let p_1 be a positive integer, p_2 be a real number, and ι an indice $\iota \in \{0;1\}$.
We say that a vector of functions $\boldsymbol{f} = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix}$ of two C^∞ radially symmetric functions is admissible of degree (p_1, p_2, ι) if:

(i) ι *is the position:*

$$\boldsymbol{f} = \begin{pmatrix} f^{(1)} \\ 0 \end{pmatrix} \text{ (ie } f^{(2)} = 0) \text{ if } \iota = 0, \quad \text{and } \boldsymbol{f} = \begin{pmatrix} 0 \\ f^{(2)} \end{pmatrix} \text{ (ie } f^{(1)} = 0) \text{ if } \iota = 1. \tag{3.2.22}$$

We will then write indifferently f to denote $f^{(1)}$ or $f^{(2)}$ in the two cases.

(ii) p_1 *describes the behavior near* 0*:*

$$\forall 2p \geq p_1, \; f(y) = \sum_{k=p_1-\iota, \; k \text{ even}}^{2p} c_k y^k + O(y^{2p+2}), \text{ as } y \to 0. \tag{3.2.23}$$

(iii) p_2 *describes the behavior at infinity:*

$$\forall k \in \mathbb{N}, \quad |f_k(y)| = O(y^{p_2-\gamma-\iota-k}) \text{ as } y \to +\infty. \tag{3.2.24}$$

The actions of $\boldsymbol{H}$ and $\boldsymbol{H}^{-1}$ on admissible functions enjoy the following properties:

Lemma 3.2.8. *(Action of $\boldsymbol{H}$ and $\boldsymbol{H}^{-1}$ on admissible functions:) Let $\boldsymbol{f}$ be an admissible function of degree (p_1, p_2, ι), with $p_2 \geq -1$ then:*

(i) $\forall i \geq 0$, $\boldsymbol{H}^i \boldsymbol{f}$ *is admissible of degree* $(max(p_1 - i, \iota), p_2 - i, \iota + i mod 2)$.

(ii) $\forall i \geq 0$, $\boldsymbol{H}^{-i} \boldsymbol{f}$ *is admissible of degree* $(p_1 + i, p_2 + i, \iota + i mod 2)$.

Proof of Lemma 3.2.8 Action of $\boldsymbol{H}$: We compute:

$$\boldsymbol{H}^{2k} = (-1)^k \begin{pmatrix} \mathcal{L}^k & 0 \\ 0 & \mathcal{L}^k \end{pmatrix}, \text{ and } \boldsymbol{H}^{2k+1} = (-1)^k \begin{pmatrix} 0 & -\mathcal{L}^k \\ \mathcal{L}^{k+1} & 0 \end{pmatrix}. \tag{3.2.25}$$

So that the property we claim holds by a direct check at the definitions of adapted derivatives and admissible functions.

Action of $\boldsymbol{H}^{-1}$: We are going to prove the property by induction on i. We will prove it for $\iota = 0$, the proof being the same for $\iota = 1$. We can suppose without loss of generality that p_1 is even. The property is true, of course, for $i = 0$. Suppose now it is true for i. If i is even, then:

$$\boldsymbol{H}^{-(i+1)} \boldsymbol{f} = \boldsymbol{H}^{-1} \boldsymbol{H}^{-i} \boldsymbol{f} = \begin{pmatrix} 0 & \mathcal{L}^{-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} (H^{-i} f)^{(1)} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -(H^{-i} f)^{(1)} \end{pmatrix}.$$

The induction hypothesis for $\boldsymbol{H}^{-i} \boldsymbol{f}$ implies that the function $\boldsymbol{H}^{-(i+1)} \boldsymbol{f}$ is of degree $(p_1 + i + 1, p_2 + i + 1, 1)$. Suppose now i is odd. Then we have:

$$\boldsymbol{H}^{-(i+1)} \boldsymbol{f} = \begin{pmatrix} 0 & \mathcal{L}^{-1} \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ (H^{-i} f)^{(2)} \end{pmatrix} = \begin{pmatrix} \mathcal{L}^{-1} (H^{-i} f)^{(2)} \\ 0 \end{pmatrix}.$$

We write $u = \mathcal{L}^{-1} (H^{-i} f)^{(2)})$. We have from the induction hypothesis:

$$(H^{-i} f)^{(2)} = \sum_{k = p_1 + i - 1, \ k \text{ even}}^{2p} c_k y^k + O(y^{2p+2}), \ as \ y \to 0.$$

From (3.2.18) one can see the gain:

$$u = \sum_{k = p_1 + i + 1, \ k \text{ even}}^{2p} c'_k y^k + O(y^{2p+2}), \ as \ y \to 0,$$

and since $\iota(\boldsymbol{H}^{-(i+1)} \boldsymbol{f}) = 0$, we get $p_1(\boldsymbol{H}^{-(i+1)} \boldsymbol{f}) = p_1 + 1$.

From the induction hypothesis for $\boldsymbol{H}^{-i} \boldsymbol{f}$, and the relation $u_k = (H^{-i} f)^{(2)}_{k-2}$ for $k \geq 2$, the asymptotic (3.2.24) at $+\infty$ for u is true for $k \geq 2$. One only needs to check the asymptotic at $+\infty$ for $k = 0$ and $k = 1$. We use the computational Lemma 3.2.6:

$$\begin{aligned} Au = \frac{1}{y^{d-1} \Lambda^{(1)} Q} \int_0^y (H^{-i} f)^{(2)} \Lambda^{(1)} Q x^{d-1} dx &= O\left(\frac{1}{y^{d-1-\gamma}} \int_0^y x^{p_2 + i - 1 - 2\gamma + d - 1} dx \right) \\ &= O(y^{p_2 + i - \gamma}), \end{aligned}$$

where we used the asymptotic (3.2.3) of $\Lambda^{(1)} Q$. Indeed the integral in the right hand side is divergent from:

$$p_2 + i - 1 - 2\gamma + d = p_2 + i + \sqrt{\triangle} + 1 > 0.$$

We then do the same for u:

$$u = -\Lambda^{(1)} Q \int_0^y \frac{Au}{\Lambda^{(1)} Q} dx = O\left(y^{-\gamma} \int_0^y x^{p_2+i-\gamma+\gamma} \right) = O(y^{p_2+i+1-\gamma}),$$

and from $\iota(\boldsymbol{H}^{-1} f) = 0$ we deduce $p_2(\boldsymbol{H}^{-1} f) = p_2 + i + 1$. □

This notion of admissible function will be helpful to construct the approximate blow-up profile. The building blocks of this profile are the generators of the kernel of the iterates of $\boldsymbol{H}$.

Lemma 3.2.9. *(*Generators of the kernel of $\boldsymbol{H}^i$*:) We recall that the numbers α and g' are defined in* (*2.2.7*), (*3.1.4*). *Let $(\boldsymbol{T}_i)_{i\in\mathbb{N}}$ denote the sequence of profiles given by:*

$$\boldsymbol{T}_0 := \boldsymbol{\Lambda Q}, \ \ \boldsymbol{T}_{i+1} := -\boldsymbol{H}^{-1} \boldsymbol{T}_i, \ i \in \mathbb{N}. \tag{3.2.26}$$

Let $(\boldsymbol{\Theta}_i)_{i\in\mathbb{N}}$ be the associated sequence defined by:

$$\boldsymbol{\Theta}_i := \boldsymbol{\Lambda T}_i - (i-\alpha) \boldsymbol{T}_i, \ i \in \mathbb{N}. \tag{3.2.27}$$

Then:

(i) $\boldsymbol{T}_i$ *is admissible of degree* $(i, i, i \,mod\, 2)$.

(ii) $\boldsymbol{\Theta}_i$ *is admissible of degree* $(i, i-g', i, i \,mod\, 2)$.

This lemma states that the T_i's and Θ_i's have only one coordinate being non null, depending on the parity of i. We will then make the following abuse of notation (with respect to (*3.1.6*)):

$$\boldsymbol{T}_{2i} = \begin{pmatrix} T_{2i} \\ 0 \end{pmatrix}, \ \boldsymbol{T}_{2i+1} = \begin{pmatrix} 0 \\ T_{2i+1} \end{pmatrix}, \ \boldsymbol{\Theta}_{2i} = \begin{pmatrix} \Theta_{2i} \\ 0 \end{pmatrix} \text{ and } \boldsymbol{\Theta}_{2i+1} = \begin{pmatrix} 0 \\ \Theta_{2i+1} \end{pmatrix} \tag{3.2.28}$$

Proof of Lemma 3.2.9 From the degenerescence (*3.2.3*) and the fact that $A\Lambda^{(1)} Q = 0$, $\boldsymbol{\Lambda Q}$ is admissible of degree $(0,0,0)$. Hence due to the properties of the action of $\boldsymbol{H}^{-1}$ on admissible functions, the previous Lemma 3.2.8, we get that $\boldsymbol{T}_i$ is admissible of degree $(i, i, i\text{mod}2)$.

To prove the second part about the $\boldsymbol{\Theta}_i$'s we will procede by induction. The asymptotic behavior of the solitary wave (*3.2.3*) ensures that the property is true for $\boldsymbol{\Theta}_0 = \boldsymbol{\Lambda}(\boldsymbol{\Lambda Q}) + \alpha \boldsymbol{\Lambda Q}$. For i odd we have:

$$\begin{aligned} \boldsymbol{\Theta}_i &= \begin{pmatrix} 0 \\ \Lambda^{(2)} T_i^{(2)} - (i-\alpha) T_i^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ -\left((\Lambda^{(1)} + 1) T_{i-1}^{(1)} - (i-1+1-\alpha) T_{i-1}^{(1)} \right) \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ -\Theta_{i-1}^{(1)} \end{pmatrix}. \end{aligned}$$

So if the property is true for i even, it is true for $i+1$ from a direct check at the definition of the degree. Let us now assume that i is even, $i \geq 2$. We compute the following relation:

$$\mathcal{L}(\Lambda^{(1)} u) = 2\mathcal{L} u + \Lambda^{(1)} \mathcal{L} u + (2V + y.\nabla V) u. \tag{3.2.29}$$

The asymptotic behavior of the potential (Lemma 3.2.5), implies the improved decay:

$$2V + y.\nabla V = O\left(\frac{1}{y^{2+\alpha}} \right). \tag{3.2.30}$$

We then compute:

$$\mathcal{L}(\Theta_i^{(1)}) = -\Theta_{i-2}^{(1)} + (2V + y.\nabla V)T_i^{(1)}. \tag{3.2.31}$$

The induction hypothesis, together with the decay property of the potential and the degree of $\boldsymbol{T}_i$ give that $\boldsymbol{H\Theta}_i$ is of degree $(i-1, i-1-g', 1)$. As $0 < g' \leq 2$ we have that $p_2(\boldsymbol{H\Theta}_i) = i-1-g' \geq -1$ and we can apply the inversion Lemma 3.2.8 about admissible functions: $\boldsymbol{H}^{-1}(\boldsymbol{H\Theta}_i)$ is of degree $(i, i-g', 0)$. One has $\mathcal{L}^{-1}\mathcal{L}(\Theta_i) = \Theta_i + a\Lambda^{(1)}Q + b\Gamma$, with a and b two integration constants. From the asymptotics $\Theta_i(y) \underset{y\to 0}{\to} 0$, $\mathcal{L}^{-1}\mathcal{L}(\Theta_i) \underset{y\to 0}{\to} 0$, $\Lambda^{(1)}Q(y) \underset{y\to 0}{\to} c > 0$ and $\Gamma(y) \underset{y\to 0}{\to} +\infty$ one deduces $a = b = 0$. This means that $\Theta_i = \mathcal{L}^{-1}\mathcal{L}(\Theta_i)$ is of degree $(i, i-g', 0)$. $\square$

In the following, we will have to deal with polynomial functions of the coefficients b_i. Knowing in advance that $b_i \approx b_1^i$ for the approximate blow-up profile[3], we have that $\prod b_i^{J_i} \approx b_1^{\sum iJ_i}$. Given a L-tuple J of integers, we define:

$$|J|_1 = \sum_1^L J_i, \text{ and } |J|_2 = \sum_1^L iJ_i. \tag{3.2.32}$$

Definition 3.2.10 (Homogeneous functions). b denotes a L-tuple $(b_i)_{1\leq i\leq L}$. p_1 is an integer, p_2 is a real number, ι is an indice $\iota \in \{0;1\}$ and p_3 is an integer. We say that a function $\boldsymbol{S}(b,y)$ is homogeneous of degree (p_1, p_2, ι, p_3) if it can be written as a finite sum:

$$\boldsymbol{S} = \sum_{J\in\mathcal{J},\ |J|_2 = p_3} \left(\prod_{i=1}^L b_i^{J_i} \boldsymbol{S}_J(y) \right),$$

$\#\mathcal{J} < +\infty$, where for each J, $\boldsymbol{S}_J$ is an admissible function of degree (p_1, p_2, ι).

Because of the asymptotics of the potential W, see (3.2.5), asking that $\mathcal{A}^k f$ behave like $y^{-\gamma+k+p_2}$ at infinity is equivalent to say that $\partial_y^k f$ behaves the same way. As a consequence, the asymptotics can be multiplied, differentiate etc... which is the object of the following computational lemma. It is a straightforward application of Lemma 3.B.1 from the Appendix.

Lemma 3.2.11 (Calculus on homogeneous functions:). *Let* $\boldsymbol{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$, $\boldsymbol{g} = \begin{pmatrix} g \\ 0 \end{pmatrix}$ *be homogeneous of degree*[4] $(p_1, p_2, 0, p_3)$ *and* $(p_1', p_2', 0, p_3')$ *(p_1 and p_1' even). Then:*

(i) Multiplication: *the product* $\boldsymbol{fg} := \begin{pmatrix} fg \\ 0 \end{pmatrix}$ *is an homogeneous profile of degree* $(p_1 + p_1', p_2 + p_2' - \gamma, 0, p_3 + p_3')$.

(ii) Multiplication by the potentials involved in the analysis: $\boldsymbol{fQ}^k := \begin{pmatrix} fQ^k \\ 0 \end{pmatrix}$ *is an homogeneous profile of degree* $(p_1, p_2 - k\frac{2}{p-1}, 0, p_3)$

[3] see Lemma 3.2.16.

[4] we just state the result for $\iota = 0$ as in (NLW) the nonlinearity only acts on the first coordinate.

3.2.5 Slowly modulated blow profiles and growing tails

We now construct an approximate blow up profile using the tools we previously displayed. First, we construct an approximate blow-up profile generating a blow up locally around the origin, but far away nonetheless it is irrelevant because it has polynomial growth (Proposition 3.2.12). Secondarily we cut this profile in a relevant zone to avoid this problem (Proposition 3.2.14). This cutting procedure creates additional error terms which will be estimated.

To manipulate the topological properties of the dynamics we will make use of the following adapted norms for $k \in \mathbb{N}$:

$$\begin{aligned} \|\boldsymbol{u}\|_k^2 &= \|u^{(1)}_{k_0+1+k}\|^2_{L^2} + \|u^{(2)}_{k_0+k}\|^2_{L^2} \\ &= \int u^{(1)}\mathcal{L}^{k_0+1+k}u^{(1)} + \int u^{(2)}\mathcal{L}^{k_0+k}u^{(2)}, \end{aligned} \tag{3.2.33}$$

involving the k-th adapted derivative of u defined in (*3.2.21*). We will also the local version of these norms:

$$\|\boldsymbol{u}\|^2_{k,(y\leq M)} = \|u^{(1)}_{k_0+1+k}\|^2_{L^2(|y|\leq M)} + \|u^{(2)}_{k_0+k}\|^2_{L^2(|y|\leq M)}. \tag{3.2.34}$$

As the scale λ of our solution is changing with time, we want to work with the appropriate space variable $y = \frac{r}{\lambda}$. The appropriate renormalized time is:

$$s(t) = s_0 + \int_{t_0}^{t} \frac{1}{\lambda(\tau)} d\tau. \tag{3.2.35}$$

Let $\boldsymbol{u}$ be a solution of (NLW) on the time interval $[0, T[$, and $\lambda : [0, T[\to \mathbb{R}_+^*$ be a C^1 function. We define the associated renormalized solution by:

$$\boldsymbol{v}(y, s) = \boldsymbol{u}_{\lambda(t)}(y, t).$$

The time evolution of $\boldsymbol{v}$ is then given by:

$$\boldsymbol{v}_s = \boldsymbol{F}(\boldsymbol{v}) + \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda v}. \tag{3.2.36}$$

It is often easier to work with this renormalized flow.

In the next proposition we state the existence of a primary blow up profile. This construction is related to the so-called center manifolds. The idea is to construct a manifold, tangent to the vector space of the generalized kernel of the linearized operator at the point $\boldsymbol{Q}_\lambda$, displaying a special dynamics. At the linear level, this dynamics is driven by the linearized operator. At the quadratic level it is driven by the scaling. The non linear terms only affect the dynamics at higher order, thus being invisible as we work in a perturbative setting[5]. The dynamics on this manifold is then easy to write down.

Proposition 3.2.12. (Construction of the approximate profile) *Let a very large odd integer* [6]*:*

$$L \gg 1 \tag{3.2.37}$$

[5] this point will be made clearer when studying the full non-linear dynamics.

[6] we take L to be odd just to know the coordinates of the objects we are manipulating, but it is not important.

and let $b=(b_1,...,b_L)$ denote a L-tuple of real numbers, with $b_1>0$. There exists a L-dimensional manifold of C^∞ radially symmetric functions $(Q_b)_{b\in\mathbb{R}_+^*\times\mathbb{R}^{L-1}}$ satisfying the following identity:

$$\boldsymbol{F}(\boldsymbol{Q}_b)=b_1\boldsymbol{\Lambda}\boldsymbol{Q}_b+\sum_{i=1}^{L}(-(i-\alpha)b_1b_i+b_{i+1})\frac{\partial\boldsymbol{Q}_b}{b_i}-\boldsymbol{\psi}_b, \tag{3.2.38}$$

where we used the convention $b_{L+1}=0$. $\boldsymbol{\psi}_b$ stands for a higher order remainder term situated on the second coordinate:

$$\boldsymbol{\psi}_b=\begin{pmatrix}0\\ \psi_b\end{pmatrix}. \tag{3.2.39}$$

Let B_1 be defined by (3.1.11). In the regime in which $|b_i|\lesssim|b_1|^i$, $0<b_1\ll 1$, it enjoys the following estimates (the adapted norm is defined by (3.2.34)):

(i) Global[7] bounds: *For $0\leq j\leq L$:*

$$\|\boldsymbol{\psi}_b\|^2_{j,(y\leq 2B_1)}\leq C(L)b_1^{2j+2+2(1-\delta_0)+2g'-C\eta} \tag{3.2.40}$$

(ii) Local improved bounds:

$$\forall j\geq 0,\ \forall B>1,\ \int_{y\leq B}|\nabla^j\psi_b^{(1)}|^2+|\nabla^j\psi_b^{(2)}|^2\leq C(j,L)B^{C(j,L)}b_1^{2L+6}. \tag{3.2.41}$$

The profile $\boldsymbol{Q}_b$ is of the form:

$$\boldsymbol{Q}_b:=\boldsymbol{Q}+\boldsymbol{\alpha}_b,\ \boldsymbol{\alpha}_b:=\sum_{i=1}^{L}b_i\boldsymbol{T}_i+\sum_{i=2}^{L+2}\boldsymbol{S}_i, \tag{3.2.42}$$

where $\boldsymbol{T}_i$ is given by Lemma 3.2.9, and the $\boldsymbol{S}_i$'s is are homogeneous functions in the sense of definition 3.2.10:

$$\begin{cases}\boldsymbol{S}_i:=\boldsymbol{S}_i(b,y),\ 1\leq i\leq L+2\\ \boldsymbol{S}_1=0\end{cases},$$

with:

$$\begin{cases}deg(\boldsymbol{S}_i)=(i,i-g',i\ mod 2,i)\\ \frac{\partial\boldsymbol{S}_i}{\partial b_j}=0\ for\ 2\leq i\leq j\leq L\end{cases}. \tag{3.2.43}$$

Remark 3.2.13. Because of the form (3.2.42) of the profile $\boldsymbol{Q}_b$, including its time evolution in (3.2.38) yields:

$$\boldsymbol{Q}_{b,s}-\boldsymbol{F}(\boldsymbol{Q}_b)+b_1\boldsymbol{\Lambda}\boldsymbol{Q}_b=\boldsymbol{Mod}(t)+\boldsymbol{\psi}_b, \tag{3.2.44}$$

where:

$$\boldsymbol{Mod}(t)=\sum_{i=1}^{L}[b_{i,s}+(i-\alpha)b_1b_i-b_{i+1}]\left[\boldsymbol{T}_i+\sum_{j=i+1}^{L+2}\frac{\partial\boldsymbol{S}_j}{\partial b_i}\right]. \tag{3.2.45}$$

From the homogeneity property of the $\boldsymbol{S}_i$'s (3.2.43), we have the following position depending on the parity of i, and make the abuse of notation (regarding (3.1.6)):

$$\boldsymbol{S}_{2i}=\begin{pmatrix}S_{2i}\\ 0\end{pmatrix},\ \boldsymbol{S}_{2i+1}=\begin{pmatrix}0\\ S_{2i+1}\end{pmatrix}. \tag{3.2.46}$$

[7]here the zone $y\leq B_1$ is called global because we will cut the profile Q_b in the next section at this precise location.

Proof of Proposition 3.2.12 Step 1: Computation of the error. We take a profile having the form (*3.2.42*) and compute the following identity:

$$-\boldsymbol{F}(Q_b)+b_1\boldsymbol{\Lambda Q}_b=\boldsymbol{A}_1-\boldsymbol{A}_2,$$

with:

$$\begin{aligned}\boldsymbol{A}_1&:=b_1\boldsymbol{\Lambda Q}+\textstyle\sum_{i=1}^{L}[\boldsymbol{T}_i+b_i\boldsymbol{HT}_i+b_1b_i\boldsymbol{\Lambda T}_i]+\sum_{i=2}^{L+2}[\boldsymbol{HS}_i+b_1\boldsymbol{\Lambda S}_i],\\ \boldsymbol{A}_2&:=\begin{pmatrix}0\\ f(Q+\alpha_b^{(1)})-f(Q)+f'(Q)\alpha_b^{(1)}\end{pmatrix}.\end{aligned}$$

Knowing in advance the fact that $S_i\sim b_1^i$ and $b_i\sim b_1^i$ we rearange all the term according to the power of b_1:

$$\begin{array}{rcl}\boldsymbol{A}_1&=&b_1(\boldsymbol{\Lambda Q}+\boldsymbol{HT}_1)+\sum_1^{L-1}[b_1b_i\boldsymbol{\Lambda T}_i+b_{i+1}\boldsymbol{HT}_{i+1}+\boldsymbol{HS}_{i+1}+b_1\boldsymbol{\Lambda S}_i]\\ &&+b_1b_L\boldsymbol{\Lambda T}_L+\boldsymbol{HS}_{L+1}+b_1\boldsymbol{\Lambda S}_L+b_1\boldsymbol{\Lambda S}_{L+1}+\boldsymbol{HS}_{L+2}+b_1\boldsymbol{\Lambda S}_{L+2}\\ &=&\sum_1^{L-1}[b_1b_i\boldsymbol{\Lambda T}_i-b_{i+1}\boldsymbol{T}_i+\boldsymbol{HS}_{i+1}+b_1\boldsymbol{\Lambda S}_i]\\ &&+b_1b_L\boldsymbol{\Lambda T}_L+\boldsymbol{HS}_{L+1}+b_1\boldsymbol{\Lambda S}_L+b_1\boldsymbol{\Lambda S}_{L+1}+\boldsymbol{HS}_{L+2}+b_1\boldsymbol{\Lambda S}_{L+2}.\end{array}$$

Because we have assumed p to be an integer, and from the localization of the $\boldsymbol{T}_i$'s (*3.2.28*), we can expand[8] $\boldsymbol{A}_2$ as a sum of polynomials of order higher or equal to 2:

$$A_2^{(2)}=\sum_{j=2}^{p}C_jQ^{p-j}(\alpha_b^{(1)})^j=\sum_{j=2}^{p}C_jQ^{p-j}\left(\sum_{i=2,\ i\text{ even}}^{L-1}b_iT_i+\sum_{i=2}^{L+2}S_i^{(1)}\right)^j.$$

Again, we reorder these polynomials according to:

$$A_2^{(2)}=\sum_{i=2}^{L+2}P_i+R.$$

where:

$$P_i=\sum_{j=2}^{p}C_jQ^{p-j}\left(\sum_{J,|J|_1=j,|J|_2=i}\ \prod_{k=2,\ k\text{ even}}^{L-1}b_i^{J_k}T_k^{J_k}\prod_{k=2}^{L+2}(S_k^{(1)})^{\tilde J_k}\right),$$

where here $J=(J_2,...,J_{L-1},\tilde J_2,...,\tilde J_{L+2})$ and the way to count the powers of b_1 is: $|J|_2=\sum_{k=1}^{\frac{L-1}{2}}2kJ_{2k}+\sum_{k=1}^{L+2}k\tilde J_k$. The remainder is:

$$R=\sum_{j=2}^{p}C_jQ^{p-j}\sum_{J,|J|_1=j,|J|_2\geq L+3}\left(\prod_{k=2,\ k\text{ even}}^{L-1}b_k^{J_k}T_k^{J_k}\prod_{k=1}^{L+2}(S_k^{(1)})^{\tilde J_k}\right).$$

We make an abuse of notation by denoting $\boldsymbol{P}_i:=\begin{pmatrix}0\\P_i\end{pmatrix}$ and $\boldsymbol{R}:=\begin{pmatrix}0\\R\end{pmatrix}$. The error term $\boldsymbol{\psi}_b$ has then the following expression (anticipating that $\frac{\partial\boldsymbol{S}_j}{\partial b_i}=0$ for $j\leq i$):

$$\begin{array}{rcl}\boldsymbol{\psi}_b&=&\sum_{i=1}^{L}(-(i-\alpha)b_1b_i+b_{i+1})\frac{\partial\boldsymbol{Q}_b}{\partial b_i}+\boldsymbol{A}_1-\boldsymbol{A}_2\\ &=&\sum_1^L(-(i-\alpha)b_1b_i+b_{i+1})\left[\boldsymbol{T}_i+\sum_{j=i+1}^{L+2}\frac{\partial\boldsymbol{S}_j}{\partial b_i}\right]+\boldsymbol{A}_1-\boldsymbol{A}_2\\ &=&\sum_1^L[\boldsymbol{H}(\boldsymbol{S}_{i+1})+b_1b_i\boldsymbol{\Theta}_i+b_1\boldsymbol{\Lambda S}_i+\boldsymbol{P}_{i+1}+\sum_{j=2}^{i-1}((j-\alpha)b_1b_j-b_{j+1})\frac{\partial\boldsymbol{S}_i}{\partial b_j}]\\ &&+\boldsymbol{H}(\boldsymbol{S}_{L+2})+b_1\boldsymbol{\Lambda S}_{L+1}+\boldsymbol{P}_{L+2}+\sum_{j=2}^{L}(-(j-\alpha)b_1b_j+b_{j+1})\frac{\partial\boldsymbol{S}_{L+1}}{\partial b_i}\\ &&+b_1\boldsymbol{\Lambda S}_{L+2}+\sum_{j=2}^{L}(-(j-\alpha)b_1b_j+b_{j+1})\frac{\partial\boldsymbol{S}_{L+2}}{\partial b_i}+\boldsymbol{R}_1.\end{array}\qquad(3.2.47)$$

[8]For the moment we include all the $S_i^{(1)}$ because we still have not proved their localization.

Step 2: Expression of the $\boldsymbol{S}_i$'s, simplification of ψ_b. We define the $\boldsymbol{S}_i$'s by induction, in order to cancel the terms with a power of b_1 less than $L+2$ in (*3.2.47*):

$$\begin{cases} \boldsymbol{S}_1 = 0, \\ \boldsymbol{S}_i = -\boldsymbol{H}^{-1}(\boldsymbol{\Phi}_i) \text{ for } 2 \leq i \leq L+2, \end{cases} \tag{3.2.48}$$

with the following expression for the profiles $\boldsymbol{\Phi}_i$:

$$\begin{cases} \boldsymbol{\Phi}_{i+1} = b_1 b_i \boldsymbol{\Theta}_i + b_1 \boldsymbol{\Lambda} \boldsymbol{S}_i + \boldsymbol{P}_{i+1} + \sum_{j=1}^{i-1} (-(j-\alpha) b_1 b_j - b_{j+1}) \frac{\partial \boldsymbol{S}_i}{\partial b_j} \text{ for } 1 \leq i \leq L, \\ \boldsymbol{\Phi}_{L+2} = b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+1} + \boldsymbol{P}_{L+2} + \sum_{j=1}^{L-1} (-(j-\alpha) b_1 b_j - b_{j+1}) \frac{\partial \boldsymbol{S}_{L+1}}{\partial b_j}. \end{cases} \tag{3.2.49}$$

The $\boldsymbol{S}_i$'s being defined by (*3.2.48*), ψ_b has now the following expression:

$$\psi_b = b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+2} + \sum_{j=1}^{L} (-(j-\alpha) b_1 b_j + b_{j+1}) \frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i} + \boldsymbol{R}. \tag{3.2.50}$$

Step 3: Properties of the $\boldsymbol{S}_i$'s. We claim the following facts (we recall that the homogeneity is defined in Definition 3.2.10):

(i) $\boldsymbol{S}_i$ is homogeneous of degree $(i, i-g', i \text{ mod} 2, i)$

(ii) $\boldsymbol{P}_i = 0$ for i odd,

(iii) the condition $\frac{\partial \boldsymbol{S}_j}{\partial b_i} = 0$ for $j \leq i$ is fullfiled.

The proof of the fact that $\boldsymbol{P}_i = 0$ for i odd is an easy induction left to the reader. We will also prove the two other facts by induction. For $i = 2$ we have:

$$\boldsymbol{S}_2 = \boldsymbol{H}^{(-1)} (b_1^2 \boldsymbol{\Theta}_1 + \boldsymbol{P}_2),$$

and it is straightforward to check that $\boldsymbol{P}_2 = 0$. Hence from the result about the $\boldsymbol{\Theta}_i$'s given by Lemma 3.2.9, we have that $\boldsymbol{S}_2$ is of degree $(2, 2-g', 0, 2)$. It is also clear from the previous identity that $\frac{\partial S_2}{\partial b_i} = 0$ for $2 \leq i \leq L$.

We now suppose $i \geq 3$, and that the properties (i) and (iii) are true for all $2 \leq j < i$, which is our induction hypothesis. We look at all the terms in the right hand side of (*3.2.49*). $b_1 b_{i-1} \boldsymbol{\Theta}_{i-1}$ is of degree $(i-1, i-1-g', i-1 \text{mod} 2, i)$. By the induction hypothesis, $b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{i-1}$ is of degree $(i-1, i-1-g', i-1 \text{ mod} 2, i)$, and so is the profile $(-(j-\alpha) b_1 b_j - b_{j+1}) \frac{\partial \boldsymbol{S}_{i-1}}{\partial b_j}$. If i is odd, $\boldsymbol{P}_i = 0$ and there is nothing to prove. If i is even, from the position of the $\boldsymbol{T}_i$'s (*3.2.28*), and the position (*3.2.46*) of the $\boldsymbol{S}_j$'s for $j < i$ given by the induction hypothesis (i), $\boldsymbol{P}_i$ is a linear combination of terms of the form:

$$Q^{p-j} \prod_{k<i,\ k \text{ even}} b_k^{J_k} T_k^{J_k} \prod_{k<i,\ k \text{ even}} S_k^{\tilde{J}_k},$$

for $2 \leq j \leq p$, $|J|_1 = j$ and $|J|_2 = i$. From the induction hypothesis and the Calculus Lemma for admissible functions 3.2.11, we deduce the asymptotics:

$$\begin{aligned} Q^{p-j} \prod_{k<i,\ k \text{ even}} b_k^{J_k} T_k^{J_k} \prod_{k<i,\ k \text{ even}} S_k^{\tilde{J}_k} &= O\left(b_1^i \frac{1}{1+y^{(p-j)\frac{2}{p-1} + \sum J_k(\gamma-k) + \sum \tilde{J}_k(\gamma-k+g')}} \right) \\ &= O\left(b_1^i \frac{1}{1+y^{2+\frac{2}{p-1}+j\alpha+\sum \tilde{J}_k g' - i}} \right) \\ &= O\left(b_1^i \frac{1}{1+y^{2+\gamma+(j-1)\alpha+\sum \tilde{J}_k g' - i}} \right), \end{aligned}$$

which adapts for higher derivatives (ie deriving k times the left hand side amounts to divide the right hand side by y^k). As $j \geq 2$ and $\alpha \geq 2 \geq g'$ we conclude that $\boldsymbol{P}_i$ is of degree $(i-1, i-1-g', 1, i)$ (the expansion at the origin can be checked the same way). In this step, so far, we have proven that $\boldsymbol{\Phi}_i$ is of degree $(i-1, i-1-g', i-1 \text{ mod} 2, i)$, hence from the inversion Lemma 3.2.8 $\boldsymbol{S}_i$ is of degree $(i, i-g', i\text{mod}2, i)$.

Step 4: Bounds for the error term. We now turn to the expression of the error ψ_b given by (*3.2.50*), and estimate all terms in the right hand side. We showed in step 3 that $\boldsymbol{S}_{L+2}$ is of degree $(L+2, L+2-g', L+2 \text{ mod}2, L+2)$. As L is odd, and as $\boldsymbol{R}$ is situated on the second coordinate we obtain the localization of $\boldsymbol{\psi}_b$:

$$\boldsymbol{\psi}_b = \begin{pmatrix} 0 \\ b_1 \Lambda^{(2)} S^{(2)}_{L+2} + \sum_{j=1}^{L} (-(j-\alpha) b_1 b_j + b_{j+1}) \frac{\partial S^{(2)}_{L+2}}{\partial b_i} + R \end{pmatrix}.$$

We start by estimating the first two terms. We already know that $b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+2}$ and $\sum_{j=1}^{L} (-(j-\alpha) b_1 b_j + b_{j+1}) \frac{\partial \boldsymbol{S}^{(2)}_{L+2}}{\partial b_i}$ are of degree $(L+2, L+2-g', 1, L+3)$. This leads to the following estimates (the local adapted norm was defined in (*3.2.34*)):

$$\begin{array}{rc} & \| b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+2} + \sum_{j=1}^{L} (-(j-\alpha) b_1 b_j + b_{j+1}) \frac{\partial \boldsymbol{S}^{(2)}_{L+2}}{\partial b_i} \|^2_{j,(\leq B_1)} \\ \leq & C(L) \int_0^{B_1} \left| \frac{|b_1|^{L+3}}{y^{\gamma-(L+2-g')+1+k_0+j}} \right|^2 y^{d-1} dy \\ = & C(L) b_1^{2L+6} \int_0^{B_1} y^{2\delta_0 - 2g' + 2L + 2 - 2j - 1} dy = C(L) b_1^{2j+2+2(1-\delta_0)+2g'}. \end{array}$$

The integral in the right hand side is always divergent as $j \leq L$, and as $1 + \delta_0 - g' \geq 0$ (see the definition of g' (*3.1.4*), the presence of $1+\delta_0$ was made to produce this result). We now prove the local estimates. We recall that we proved in step 3 that $b_1 \boldsymbol{\Lambda} \boldsymbol{S}_{L+2} + \sum_{j=1}^{L} (-(j-\alpha) b_1 b_j + b_{j+1}) \frac{\partial \boldsymbol{S}_{L+2}}{\partial b_i}$ is homogeneous of degree $p_3 = L+3$. This means that:

$$b_1 \boldsymbol{\Lambda} S_{L+2} + \sum_{j=1}^{L} (-(j-\alpha) b_1 b_j + b_{j+1}) \frac{\partial S_{L+2}}{\partial b_i} = \sum_{|J|_2 = L+3} b^J f_J,$$

for a finite number of functions f_J such that $|\partial_y^k f_J| \lesssim y^{-\gamma+L+2-1-g'-k}$ at infinity, and with $b^J = \prod b_i^{J_i}$. Hence the brute force upper bound:

$$\left| \partial_y^k \left(b_1 \Lambda S_{L+2} + \sum_{j=1}^{L} (-(j-\alpha) b_1 b_j + b_{j+1}) \frac{\partial S^{(2)}_{L+2}}{\partial b_i} \right) \right| \lesssim b_1^{L+3} (1+y)^{-\gamma+L+2-1-g'-k}$$

which implies the local bound (*3.2.41*) for this term. We now turn to the bounds for the $\boldsymbol{R}$ term. Thanks to the homogeneity property of the $\boldsymbol{S}_i$'s, R is of the form:

$$R = \sum_{|J|_2 \geq L+3} \prod_{i=1}^{L} b_i^{J_i} g_J,$$

for a finite number of functions g_J whose derivatives have polynomial growth at infinity. This directly implies the local bounds (*3.2.41*) for this term. For the global bounds, we rewrite R as a linear sum of terms of the form:

$$Q^{p-j} \left(\prod_{i=2,\ i \text{ even}}^{L} b_i^{J_i} T_i^{J_i} \prod_{i=2,\ i \text{ even}}^{L} S_i^{\tilde{J}_i} \right),$$

for $|J|_2 \geq L+3$ and $2 \leq j \leq p$. Using again the Calculus Lemma for admissible functions 3.2.11, each term has the asymptotic behavior:

$$\begin{aligned} Q^{p-j}\left(\prod b_i^{J_i} T_i^{J_i} \prod S_i^{\tilde{J}_i}\right) &= O\left(\frac{b_1^{|J|_2}}{1+y^{\frac{2}{p-1}(p-j)+\sum(\gamma-J_i)+\sum(\gamma-\tilde{J}_i+g')}}\right) \\ &= O\left(\frac{b_1^{|J_2|}}{1+y^{2+\gamma+(j-1)\alpha+(\sum\tilde{J}_i)g'-|J_2|}}\right). \end{aligned}$$

For all $k \in \mathbb{N}$:

$$\partial_y^k\left(Q^{p-j}\left(\prod b_i^{J_i} T_i^{J_i} \prod S_i^{\tilde{J}_i}\right)\right) = O\left(\frac{b_1^{|J_2|}}{1+y^{2+\gamma+(j-1)\alpha+(\sum\tilde{J}_i)g'-|J_2|+k}}\right).$$

From the fact that $(j-1)\alpha > 2 \geq g'$ we conclude that the global estimates of the term $\boldsymbol{R}$ are in all cases better (ie with a higher power of b_1, b_1 being small $0 < b_1 \ll 1$) than the ones for $b_1 \boldsymbol{\Lambda S}_{L+2} + \sum_{j=1}^L (-(j-\alpha)b_1 b_j + b_{j+1})\frac{\partial \boldsymbol{S}_{L+2}^{(2)}}{\partial b_i}$, which concludes the proof. $\square$

As we have seen with the previous estimates of the error term $\boldsymbol{\psi}_b$, we have a good approximate dynamics for $y \leq B_1$. However, as

$$T_i \sim y^{-\gamma+i-\delta_{i \text{ odd}}} \to +\infty \text{ as } y \to +\infty \text{ (as soon as } i > \gamma+1),$$

the approximate dynamic is irrelevant far away of the origin. Consequently, we will now localise the profiles of Proposition 3.2.12 in the zone $y \leq B_1$, where $\frac{b_{2i}T_{2i}}{\Lambda^{(1)}Q}$ is nearly of order 1. To do this, we will simply multiply by a cut-off function. This cut will create additional error terms that we will estimate in the next proposition. We recall that our cut-off function χ is defined by (*2.2.22*). We denote by $\chi_{B_1}\boldsymbol{\alpha}_b$:

$$\chi_{B_1}\boldsymbol{\alpha}_b := \begin{pmatrix} \chi_{B_1}\alpha_b^{(1)} \\ \chi_{B_1}\alpha_b^{(2)} \end{pmatrix}. \qquad (3.2.51)$$

Proposition 3.2.14 (Localization of the approximate profile). *We use the assumptions and notations of Proposition 3.2.12. Let $I =]s_0, s_1[$ denote a renormalized time interval, and*

$$\begin{aligned} b : I &\to \mathbb{R}^L \\ s &\mapsto (b_i(s))_{1\leq i\leq L} \end{aligned}$$

be a C^1 map such that: $|b_i| \lesssim b_1^i$ with $0 < b_1 \ll 1$. Assume the a priori bound:

$$|b_{1,s}| \lesssim b_1^2. \qquad (3.2.52)$$

Let $\tilde{\boldsymbol{Q}}_b$ denote the localized profile, given by:

$$\tilde{\boldsymbol{Q}}_b = \boldsymbol{Q} + \chi_{B_1}\boldsymbol{\alpha}_b. \qquad (3.2.53)$$

Then for $0 < \eta \ll 1$ small enough one has the following identity ($\boldsymbol{Mod}(t)$ being defined by (3.2.45)):

$$\tilde{\boldsymbol{Q}}_{b,s} - \boldsymbol{F}(\tilde{\boldsymbol{Q}}_b) + b_1 \boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b = \tilde{\boldsymbol{\psi}}_b + \chi_{B_1}\boldsymbol{Mod}(t). \qquad (3.2.54)$$

$\tilde{\boldsymbol{\psi}}_b$, the new error term, satisfies (the adapted norm being defined in (3.2.34)):

(i) Global weighted bounds:

$$\forall 0 \leq j \leq L-1, \ \|\tilde{\boldsymbol{\psi}}_b\|_j^2 \leq C(L) b_1^{2j+2+2(1-\delta_0)-C_j\eta}, \tag{3.2.55}$$

$$\textit{for } j = L, \ \|\tilde{\boldsymbol{\psi}}_b\|_L^2 \leq C(L) b_1^{2L+2+2(1-\delta_0)(1+\eta)}. \tag{3.2.56}$$

(ii) Local improved bounds: *For* $x \leq \frac{B_1}{2}$, $\tilde{\boldsymbol{\psi}}_b(x) = \boldsymbol{\psi}_b(x)$, *where* $\boldsymbol{\psi}_B$ *is the former error term of Proposition 3.2.12. Hence* $\forall j \geq 0, \ \forall 1 \leq B \leq \frac{B_1}{2}$:

$$\int_{|y|\leq B} |\nabla^j \tilde{\psi}_b^{(1)}|^2 + |\nabla^j \tilde{\psi}_b^{(2)}|^2 = \int_{|y|\leq B} |\nabla^j \psi_b^{(2)}|^2 \lesssim C(L,j) B^{C(L,j)} b_1^{2L+6}. \tag{3.2.57}$$

Remark 3.2.15. When comparing the estimates given by this proposition, and the ones given in the proposition 3.2.12, we note a loss. Indeed the first non cut profile creates an error seen on the corrective terms $\boldsymbol{S}_{L+2}$ and $\boldsymbol{R}$ which enjoy additional gains $y^{-g'}$ or $y^{-\alpha}$ away from the origin compared to the $\boldsymbol{T}_i$'s. When cutting, we see in the additional error term the profiles $\boldsymbol{T}_i$'s, giving a worst estimate as they do not have this additional gain.

However, the error created in the zone $\leq B_1$ is left unperturbed by the cut. The fact that the error enjoys two different estimates: a good one in the zone $y \leq B_1$ and a bad one in the zone $B_1 \leq y \leq 2B_1$ will be helpful in the analysis later.

Proof of Proposition 3.2.14 We compute the error in localizing:

$$\begin{aligned} \tilde{\boldsymbol{Q}}_{b,s} - \boldsymbol{F}(\tilde{\boldsymbol{Q}}_b) + b_1 \boldsymbol{\Lambda} \tilde{\boldsymbol{Q}}_b \ &= \ \chi_{B_1} \boldsymbol{\psi}_b + \chi_{B_1} \boldsymbol{Mod}(t) \\ &\quad + \chi_{B_1,s} \tilde{\boldsymbol{\alpha}}_b + b_1 (\boldsymbol{\Lambda} \tilde{\boldsymbol{Q}}_b - \chi_{B_1} \boldsymbol{\Lambda} \boldsymbol{Q}_b) \\ &\quad - (\boldsymbol{F}(\tilde{\boldsymbol{Q}}_b) - \boldsymbol{F}(\boldsymbol{Q}) - \chi_{B_1} (\boldsymbol{F}(\boldsymbol{Q}_b) - \boldsymbol{F}(\boldsymbol{Q})). \end{aligned}$$

So we have the following expression for the new error term:

$$\begin{aligned} \tilde{\boldsymbol{\psi}}_b \ &= \ \chi_{B_1} \boldsymbol{\psi}_b + \chi_{B_1,s} \boldsymbol{\alpha}_b + b_1 (\boldsymbol{\Lambda} \tilde{\boldsymbol{Q}}_b - \chi_{B_1} \boldsymbol{\Lambda} \boldsymbol{Q}_b) \\ &\quad - (\boldsymbol{F}(\tilde{\boldsymbol{Q}}_b) - \boldsymbol{F}(\boldsymbol{Q}) - \chi_{B_1} (\boldsymbol{F}(\boldsymbol{Q}_b) - \boldsymbol{F}(\boldsymbol{Q})), \end{aligned} \tag{3.2.58}$$

and we aim at estimating all these terms in global and local norms.

Local bounds: From (*3.2.58*) we clearly see that $\tilde{\boldsymbol{\psi}}_b \equiv \boldsymbol{\psi}_b$ for $|y| \leq \frac{B_1}{2}$, because the new error terms appearing when cutting are created in the zone $B_1 \leq |y| \leq 2B_1$. Therefore the local bounds are a direct consequence of the local ones established in (*3.2.41*).

Global bounds: We recall that $\|\boldsymbol{f}\|_j^2 = \|f_j^{(1)}\|_{L^2}^2 + \|f_{j-1}^{(2)}\|_{L^2}^2$ where the j-th adapted derivative of a function is defined by (*3.2.21*). We will now compute this norm for all the terms in the right hand side of (*3.2.58*).

- $\chi_{B_1} \boldsymbol{\psi}_b$ *term:* When applying the differential operators A or A^* to any product $\chi_{B_1} f$, we have:

$$\begin{aligned} A(\chi_{B_1} f) \ &= \ \chi_{B_1} f_1 - b^{1+\eta} \partial_y \chi \left(\frac{y}{B_1} \right) f, \\ A^* A(\chi_{B_1} f) \ &= \ \chi_{B_1} f_2 + b^{1+\eta} \partial_y \chi \left(\frac{y}{B_1} \right) f_1 \\ &\quad - \left[b^{2+2\eta} \partial_y^2 \chi (\tfrac{y}{B_1}) + b^{1+\eta} \partial_y \chi \left(\tfrac{y}{B_1} \right) \left(2W + \tfrac{d-1}{y} \right) \right] f. \end{aligned} \tag{3.2.59}$$

And so on for higher powers of A and A^*. Because of the asymptotic of W, see Lemma 3.2.5, the general expression is of the form:

$$(\chi_{B_1} f)_i = \chi_{B_1} f_i + \mathbb{1}_{B_1 \leq y \leq 2B_1} \sum_{j=1}^{i} a_j f_j,$$

where $a_i(y) = O(y^{-(i-j)})$. It means that deriving χ_{B_1} amounts to dividing by B_1 and localizing in the zone $B_1 \leq y \leq 2B_1$. Hence for $0 \leq j \leq L$:

$$\begin{array}{lll} \| \chi_{B_1} \boldsymbol{\psi}_b \|_j^2 & = & \int \left| (\chi_{B_1} \psi_b^{(2)})_{k_0+j} \right|^2 \\ & \leq & C(L) \sum_{i=1}^{k_0+j} \int_{B_1 \leq |y| \leq 2B_1} b^{2(1+\eta)i} |\psi_{b,k_0+j-i}^{(2)}|^2 + \int_{|y| \leq 2B_1} |\psi_{b,k_0+j}^{(2)}|^2 \\ & \leq & C(L) \| \psi_b \|_{j, \leq 2B_1}^2 + C(L) \sum_{i=1}^{k_0+j} \int b^{2(1+\eta)i} \left| \frac{b^{L+3}}{y^{\gamma-(L+2)+1+g'+k_0+j-i}} \right|^2 \\ & \leq & C(L) b^{2j+2+2(1-\delta_0)(1+\eta)}, \end{array} \quad (3.2.60)$$

thanks to the Proposition 3.2.12.

• $\chi_{B_1,s} \boldsymbol{\alpha}_b$ *term:* We have from the assumption $|b_{1,s}| \lesssim b_1^2$:

$$\chi_{B_1,s} = (1+\eta) b^\eta b_s y \partial_y \chi(\frac{y}{B_1}) \lesssim b_1 b_1^{1+\eta} y \partial_y \chi(\frac{y}{B_1}).$$

Again, deriving $y \partial_y \chi \frac{y}{B_1}$ amounts to dividing by B_1, we get:

$$\begin{array}{lll} \| \chi_{B_1,s} \boldsymbol{\alpha}_b \|_j^2 & = & \int |(\chi_{B_1,s} \alpha_b^{(1)})_{k_0+j+1}|^2 + |(\chi_{B_1,s} \alpha_b^{(2)})_{k_0+j}|^2 \\ & \leq & C(L) b_1^2 \int_{B_1 \leq y \leq 2B_1} |\alpha_{b,k_0+j+1}^{(1)}|^2 + |\alpha_{b,k_0+j}^{(2)}|^2. \end{array} \quad (3.2.61)$$

We estimate the two terms using the asymptotic of the T_i's from Lemma 3.2.9 and (*3.2.43*) for the S_i's:

$$\begin{array}{lll} \int_{B_1 \leq |y| \leq 2B_1} |\alpha_{b,k_0+j+1}^{(1)}|^2 & \leq & \int_{B_1 \leq |y| \leq 2B_1} \sum_{i=2,\ i \text{ even}}^{L-1} |b_i T_{i,k_0+j+1}|^2 \\ & & + \int_{B_1 \leq |y| \leq 2B_1} \sum_{i=2,\ i \ even}^{L+1} |S_{i,k_0+j+2}|^2 \\ & \leq & C(L) \sum_{i=2,\ i \text{ even}}^{L-1} b_1^{2i} \int_{B_1}^{2B_1} \frac{1}{y^{2\gamma-2i+2k_0+2j+2}} y^{d-1} dy \\ & & + C(L) \sum_{i=2,\ i \text{ even}}^{L+1} b_1^{2i} \int_{B_1}^{2B_1} \frac{1}{y^{2\gamma-2i+2k_0+2j+2+2g'}} y^{d-1} dy \\ & = & C(L) \sum_{i=2,\ i \text{ even}}^{L-1} b_1^{2i} \int_{B_1}^{2B_1} y^{2\delta_0-2+2i-2j-1} dy \\ & & + C(L) \sum_{i=2,\ i \text{ even}}^{L+1} b_1^{2i} \int_{B_1}^{2B_1} y^{2\delta_0-2+2i-2j-2g'-1} dy. \end{array} \quad (3.2.62)$$

Similarly:

$$\begin{array}{lll} \int_{B_1}^{2B_1} |\alpha_{b,k_0+j}^{(2)}|^2 & \leq & C(L) \sum_{i=1,\ i \text{ odd}}^{L} b_1^{2i} \int_{B_1}^{2B_1} y^{2\delta_0-2+2i-2j-1} dy \\ & & + C(L) \sum_{i=3,\ i \text{ odd}}^{L+2} b_1^{2i} \int_{B_1}^{2B_1} y^{2\delta_0-2+2i-2j-2g'-1} dy. \end{array} \quad (3.2.63)$$

The first upper bound (*3.2.61*), combined with the two we just proved, (*3.2.62*) and (*3.2.63*), lead to (because $0 < \delta_0 < 1$ avoids a possible log-term in the first sum):

$$\begin{array}{lll} \| \chi_{B_1,s} \boldsymbol{\alpha}_b \|_j^2 & \leq & C(L) \sum_{i=1}^{L} b_1^{2i} B_1^{2\delta_0-2+2i-2j} \\ & & + C(L) \sum_{i=2}^{L+2} b_1^{2i} B_1^{2\delta_0-2+2i-2j-2g'} \log(B_1) \\ & \leq & C(L) \sum_1^L b^{2j+2(1+\eta)(1-\delta_0)-\eta(2i-2j)} \\ & & + C(L) \sum_2^{L+2} b^{2j+2(1+\eta)(1-\delta_0)-\eta(2i-2j-2g')+2g'} \log(B_1) \\ & \leq & \left\{ \begin{array}{l} C(L) b^{2j+2(1-\delta_0)-C_j\eta} \text{ for } j \leq L-1, \\ C(L) b^{2L+2(1+\eta)(1-\delta_0)} \text{ for } j = L, \end{array} \right. . \end{array} \quad (3.2.64)$$

for η small enough.

• $\boldsymbol{F}(\tilde{\boldsymbol{Q}}_b)-\boldsymbol{F}(\boldsymbol{Q})-\chi_{B_1}(\boldsymbol{F}(\boldsymbol{Q}_b)-\boldsymbol{F}(\boldsymbol{Q}))$ *term:* We compute:

$$\begin{array}{c}\boldsymbol{F}(\tilde{\boldsymbol{Q}}_b)-\boldsymbol{F}(\boldsymbol{Q})-\chi_{B_1}(\boldsymbol{F}(\boldsymbol{Q}_b)-\boldsymbol{F}(\boldsymbol{Q}))\\ =\begin{pmatrix}0\\ \Delta(\chi_{B_1}\alpha_b^{(1)})-\chi_{B_1}\Delta(\alpha_b^{(1)})+f(\tilde{Q}_b)-f(Q)-\chi_{B_1}(f(Q_b)-f(Q))\end{pmatrix}.\end{array} \tag{3.2.65}$$

We estimate the two terms in the right hand side of (3.2.65):

$$\begin{array}{rcl}\Delta(\chi_{B_1}\alpha_b^{(1)})-\chi_{B_1}\Delta(\alpha_b^{(1)}) & = & \partial_y(\chi_{B_1})\partial_y(\alpha_b^{(1)})+\Delta(\chi_{B_1})\alpha_b^{(1)}\\ & = & b^{1+\eta}\partial_y\chi(\frac{y}{B_1})\partial_y(\alpha_b^{(1)})+b^{2(1+\eta)}\Delta\chi(\frac{y}{B_1})\alpha_b^{(1)}.\end{array}$$

Considering the asymptotics of $\alpha_b^{(1)}$ we have:

$$\begin{array}{rl} & \int|(\Delta(\chi_{B_1}\alpha_b^{(1)})-\chi_{B_1}\Delta(\alpha_b^{(1)}))_{k_0+j}|^2\\ \leq & C(L)b^{2(1+\eta)}\int_{B_1}^{2B_1}\left(\sum_{i=2}^{L-1}b^{2i}y^{2\delta_0-2j+2i-2}+\sum_{i=2,L+1}b^{2i}y^{2\delta_0-2j+2i-2-2g'}\right)dy\\ + & C(L)b^{4(1+\eta)}\int_{B_1}^{2B_1}\left(\sum_{i=2}^{L-1}b^{2i}y^{2\delta_0-2j+2i}+\sum_{i=2,L+1}b^{2i}y^{2\delta_0-2j+2i-2g'}\right)dy\\ \leq & \left\{\begin{array}{l}C(L)b^{2+2j+2(1-\delta_0)-C_j\eta}\ for\ 0\leq j\leq L,\\ C(L)b^{2+2L+2(1-\delta_0)(1+\eta)}\ for\ j=L.\end{array}\right.\end{array} \tag{3.2.66}$$

because $i<L-1$ in the sum concerning the $\boldsymbol{T}_i$'s and because of the gain $g'>0$ in the one of the $\boldsymbol{S}_i$'s. The second term is:

$$f(\tilde{Q}_b)-f(Q)-\chi_{B_1}(f(Q_b)-f(Q))=\chi_{B_1}\sum_{k=2}^{p}C_kQ^{p-k}(\chi_{B_1}^{k-1}-1)\alpha_b^{(1)k}.$$

For each $2\leq k\leq p$, we can expand the polynomial and we have a linear sum of terms of the form:

$$\chi_{B_1}Q^{p-k}(\chi_{B_1}^{k-1}-1)\prod_{i=2,\ i\text{ even}}^{L-1}(b_iT_i)^{J_i}\prod_{i=2,\ i\text{ even}}^{L+1}(S_i)^{\tilde{J}_i},$$

for $|J|_1=k$. According to the calculus Lemma 3.2.11 for homogeneous functions:

$$\begin{array}{rcl}\partial_y^l\left(Q^{p-k}\prod_{i=2,\ i\text{ even}}^{L-1}(b_iT_i)^{J_i}\prod_{i=2,\ i\text{ even}}^{L+1}(S_i)^{\tilde{J}_i}\right) & \underset{y\to+\infty}{=} & O\left(\frac{b_1^{|J|_2}}{y^{(p-k)\frac{2}{p-1}+k\gamma+\sum\tilde{J}_ig'-|J|_2+l}}\right)\\ & \underset{y\to+\infty}{=} & O\left(\frac{b_1^{|J|_2}}{y^{2+\gamma+(k-1)\alpha-|J|_2+\sum\tilde{J}_ig'+l}}\right).\end{array}$$

As we have seen before, the presence of the term χ_{B_1} does not affect the computation (deriving χ_{B_1} amounts to divide by y):

$$\begin{array}{rl} & \int_{B_1}^{2B_1}|(Q^{p-k}\prod_{i=2,\ i\text{ even}}^{L-1}(b_iT_i)^{J_i}\prod_{i=2,\ i\text{ even}}^{L+1}(S_i)^{\tilde{J}_i})_{k_0+j}|^2\\ \leq & C(L)\int_{B_1}^{2B_1}\frac{b_1^{2|J|_2}}{y^{4+2\gamma+2(k-1)\alpha-2|J|_2+2\sum\tilde{J}_ig'+2k_0+2j}}y^{d-1}dy\\ \leq & C(L)\int_{B_1}^{2B_1}b_1^{2|J|_2}y^{-4+2\delta_0-2(k-1)\alpha+2|J|_2-2\sum\tilde{J}_ig'-2j-1}dy\\ \leq & C(L)b^{2+2j+2(1-\delta_0)(1+\eta)}\text{ for }0\leq j\leq L.\end{array} \tag{3.2.67}$$

because of the gain $(k-1)\alpha>\alpha>2$. The bound (3.2.67) then implies the bound for $1\leq j\leq L$:

$$\int|(f(\tilde{Q}_b)-f(Q)-\chi_{B_1}(f(Q_b)-f(Q)))_{j+k_0}|^2\leq C(L)b^{2+2j+2(1-\delta_0)(1+\eta)}\text{ for }0\leq j\leq L. \tag{3.2.68}$$

The primary decomposition (3.2.65), with the bounds (3.2.66) and (3.2.68) implies the bound we were looking for:

$$\| \boldsymbol{F}(\tilde{\boldsymbol{Q}}_b)-\boldsymbol{F}(\boldsymbol{Q})-\chi_{B_1}(\boldsymbol{F}(\boldsymbol{Q}_b)-\boldsymbol{F}(\boldsymbol{Q}))\|_j^2 \leq \begin{cases} C(L)b^{2+2j+2(1-\delta_0)-C_j\eta} \text{ for } 0\leq j\leq L, \\ C(L)b^{2+2L+2(1-\delta_0)(1+\eta)} \text{ for } j=L. \end{cases} \tag{3.2.69}$$

- $b_1(\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b-\chi_{B_1}\boldsymbol{\Lambda}\boldsymbol{Q}_b)$ *term:* We compute:

$$\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b-\chi_{B_1}\boldsymbol{\Lambda}\boldsymbol{Q}_b=(1-\chi)\boldsymbol{\Lambda}\boldsymbol{Q}+y\partial_y(\chi_{B_1})\boldsymbol{\alpha}_b.$$

We have that:

$$y\partial_y(\chi_{B_1})=b_1^{1+\eta}y\partial_y\chi(\frac{y}{B_1}).$$

So the term $y\partial_y(\chi_{B_1})\boldsymbol{\alpha}_b$ behaves the same way as the term $\chi_{B_1,s}\boldsymbol{\alpha}_b$ previously treated and enjoys the same estimations. Finally we estimate the soliton contribution, because of which we had to differentiate k_0 times at least in order to have integrability. We again use the fact that deriving k times χ_{B_1} amounts to divide by y^k and to localize in the zone $B_1\leq y\leq 2B_1$.

$$\begin{array}{rcl} \int |b_1(1-\chi_{B_1})\Lambda^{(1)}Q_{k_0+j+1}|^2 & \leq & C(L)b_1^2\int_{B_1}^{\infty}y^{-2\gamma-2k_0-2-2j+d-1}dy \\ & \leq & C(L)b_1^{2+2j+2(1-\delta_0)+(2j+2(1-\delta_0))\eta}. \end{array}$$

So that finally:

$$\| b_1(\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b-\chi_{B_1}\boldsymbol{\Lambda}\boldsymbol{Q}_b)\|_j^2 \lesssim \begin{cases} C(L)b^{2j+2(1+\eta)(1-\delta_0)-C_j\eta} \text{ for } j\leq L-1, \\ C(L)b^{2L+2(1+\eta)(1-\delta_0)} \text{ for } j=L, \end{cases}. \tag{3.2.70}$$

The decomposition (3.2.58), with the bounds for each term (3.2.60), (3.2.64), (3.2.69) and (3.2.70) give the global bounds (3.2.55) and (3.2.56) we had to prove. □

3.2.6 Study of the dynamical system driving the evolution of the parameters $(b_i)_{1\leq i\leq L}$

We have constructed in the preceding propositions 3.2.12 and 3.2.14 a manifold of functions near the solitary wave such that:

$$\boldsymbol{F}(\tilde{\boldsymbol{Q}}_b)\sim b_1\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b+\sum_{i=1}^{L}(-(i-\alpha)b_1b_i+b_{i+1})\frac{\partial\tilde{\boldsymbol{Q}}_b}{\partial b_i}.$$

By applying scaling, and the identity $\frac{\partial(\boldsymbol{f}_\lambda)}{\partial\lambda}=\frac{1}{\lambda}\boldsymbol{\Lambda}\boldsymbol{f}_\lambda$ we have that:

$$\boldsymbol{F}(\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}})\sim\frac{b_1}{\lambda}(\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b)_\lambda+\sum_{i=1}^{L}\frac{1}{\lambda}(-(i-\alpha)b_1b_i+b_{i+1})\frac{\partial\tilde{\boldsymbol{Q}}_b}{\partial b_i}.$$

Hence approximately a solution of (NLW) on this manifold gives:

$$\begin{array}{rcl} -\frac{\lambda_t}{\lambda}\boldsymbol{\Lambda}(\tilde{\boldsymbol{Q}}_b)_{\frac{1}{\lambda}}+\sum b_{i,t}\left(\frac{\partial\tilde{\boldsymbol{Q}}_b}{\partial b_i}\right)_{\frac{1}{\lambda}} & = & \partial_t(\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}}) \\ & = & \boldsymbol{F}(\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}}) \\ & \sim & \frac{b_1}{\lambda}(\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b)_{\frac{1}{\lambda}}+\sum(-(i-\alpha)b_1b_i+b_{i+1})\left(\frac{\partial\tilde{\boldsymbol{Q}}_b}{\partial b_i}\right)_{\frac{1}{\lambda}}. \end{array}$$

By identifying the terms we obtain:

$$\left\{\begin{array}{l} \lambda_t = -b_1, \\ b_{i,t} = \frac{1}{\lambda}(-(i-\alpha)b_1 b_i + b_{i+1}) \text{ for } 1 \leq i \leq L+1, \\ b_{L,t} = -\frac{1}{\lambda}(L-\alpha)b_1 b_L. \end{array}\right. \tag{3.2.71}$$

We thus want to study the behavior of the solutions of this dynamical system in order to understand the behavior of a real solution close to the manifold of approximate solutions. Writing it in renormalized variables (the renormalized time being defined by (*3.2.35*)), the evolution of the b_i's is given by:

$$\left\{\begin{array}{l} b_{i,s} = -(i-\alpha)b_1 b_i + b_{i+1} \text{ for } 1 \leq i \leq L-1, \\ b_{L,s} = -(L-\alpha)b_1 b_L. \end{array}\right. \tag{3.2.72}$$

We show in this section that this dynamical system admits exceptional solutions leading to an explosive scenario, and that the stability of such solutions can be explicitly computed.

Lemma 3.2.16. *(Special solutions for the dynamical system:) Let $\ell \in \mathbb{N}$ such that $\alpha < \ell$. Then*[9] $b^e :]0, +\infty[\to \mathbb{R}^L$ *given by:*

$$\left\{\begin{array}{l} b_i^e(s) = \frac{c_i}{s^i} \text{ for } 1 \leq i \leq \ell, \\ b_i^e \equiv 0 \text{ for } \ell < i, \end{array}\right. \tag{3.2.73}$$

with the constant c_i given by:

$$c_1 = \frac{\ell}{\ell - \alpha} \text{ and } c_{i+1} = -\frac{\alpha(\ell - i)}{\ell - \alpha} c_i \text{ for } 1 \leq i \leq \ell - 1, \tag{3.2.74}$$

is a solution of (3.2.72). Moreover, if the renormalized time s and the scaling satisfy:

$$\frac{ds}{dt} = \frac{1}{\lambda}, \ s(0) = s_0 > 0, \ \frac{d}{dt}\lambda = -b_1, \ \lambda(0) = 1,$$

then there exists $T > 0$ with $s(t) \to +\infty$ as $t \to T$, and there holds:

$$\lambda(t) \underset{t \to T}{\sim} (T-t)^{\frac{\ell}{\alpha}}.$$

We do not write here the proof as it is a direct computation. When dealing with the real equation (NLW), we want these special solutions to persist. A real solution will imply a corrective term "orthogonal" to the manifold $\left(\tilde{Q}_{b,\lambda}\right)_{b,\lambda}$ and a corrective term for the parameters. Therefore, to understand the time evolution of the part of the error on the manifold $\left(\tilde{Q}_{b,\lambda}\right)_{b,\lambda}$, we have to understand the dynamics of (*3.2.72*) close to the special solution $(b^e(s))_{s>0}$.

Lemma 3.2.17. *(Linearization around the special trajectories) Let us denote a perturbated solution around b^e by:*

$$b_k(s) = b_k^e(s) + \frac{U_k(s)}{s^k}, \text{ for } 1 \leq k \leq L, \tag{3.2.75}$$

and note $U = (U_1, ..., U_L)$ the perturbation. Suppose b is a solution of (3.2.72), then the evolution of U is given by:

$$U_s = \frac{1}{s} A_\ell U + O\left(\frac{|U|^2}{s}\right), \tag{3.2.76}$$

[9] We forget the dependence with ℓ and write b^e to avoid additional notations, as ℓ will be fixed throughout the chapter

with:

$$
A_\ell = \begin{pmatrix}
-(1-\alpha)c_1+\alpha\frac{\ell-1}{\ell-\alpha} & 1 & & & & & & & \\
. & . & . & & & & & & \\
-(i-\alpha)c_i & & \alpha\frac{\ell-i}{\ell-\alpha} & 1 & & & & & \\
. & & & . & . & & (0) & & \\
-(\ell-\alpha)c_\ell & & & & 0 & 1 & & & \\
0 & & & & & \alpha\frac{-1}{\ell-\alpha} & . & & \\
. & & & & & & . & 1 & \\
0 & & (0) & & & & & \alpha\frac{\ell-i}{\ell-\alpha} & . \\
. & & & & & & & & . \quad 1 \\
0 & & & & & & & & \alpha\frac{(\ell-i)}{\ell-\alpha}
\end{pmatrix} \qquad (3.2.77)
$$

A_ℓ is diagonalizable into the matrix diag$(-1, \frac{2\alpha}{\ell-\alpha}, ., \frac{i\alpha}{\ell-\alpha}, ., \frac{\ell\alpha}{\ell-\alpha}, \frac{-1}{\ell-\alpha}, ., \frac{\ell-L}{\ell-\alpha})$. We denote the eigenvector associated to the eigenvalue -1 by v_1 and the eigenvectors associated to the unstable modes $\frac{2\alpha}{\ell-\alpha}, ..., \frac{\ell\alpha}{\ell-\alpha}$ by $v_2, ..., v_\ell$. They are a linear combination of the ℓ first components only. That is to say there exists a $L\times L$ matrix coding a change of variables:

$$
P_\ell := \begin{pmatrix} P'_\ell & 0 \\ 0 & Id_{L-\ell} \end{pmatrix}, \qquad (3.2.78)
$$

with P'_ℓ an invertible $\ell\times\ell$ matrix and $Id_{L-\ell}$ the $L-\ell\times L-\ell$ identity such that:

$$
P_\ell A_\ell P_\ell^{-1} = \begin{pmatrix}
-1 & & (0) & & q_1 & & & \\
& \frac{2\alpha}{\ell-\alpha} & & & q_2 & & & \\
& & . & & & & & \\
& & & \frac{\ell\alpha}{\ell-\alpha} & q_\ell & & (0) & \\
& & & & \frac{-\alpha}{\ell-\alpha} & 1 & & \\
& & & & & . & . & \\
& & (0) & & & & . & 1 \\
& & & & & & & \alpha\frac{\ell-L}{\ell-\alpha}
\end{pmatrix}. \qquad (3.2.79)
$$

with q_i being some coefficient $q_i\in\mathbb{R}$ for $1\le i\le\ell$.

Proof of Lemma 3.2.17 Step 1: Linearization. We compute:

$$
\begin{array}{rcl}
0 & = & b_{k,s}+(k-\alpha)b_1b_k-b_{k+1} \\
& = & \frac{1}{s^{k+1}}[s(U_{k,s}-kU_k+(k-\alpha)c_1U_k(k-\alpha)c_kU_1-U_{k+1}+O(U_1U_k)] \\
& = & \frac{1}{s^{k+1}}[s(U_{k,s}+\alpha\frac{k-\ell}{\ell-\alpha}U_k+(k-\alpha)c_kU_1-U_{k+1}+O(U_1U_k)].
\end{array}
$$

which gives the expression of A_ℓ.

Step 2: Diagonalization. We will compute by induction the characteristic polynomial. The case $\ell=3$ can be done by hand. We now assume $\ell\ge 4$ and let:

$$
\mathcal{P}_\ell(X)=\det(A_\ell-XId).
$$

We first notice that: $\mathcal{P}_\ell(X) = \det(A'_\ell - XId)\det(A''_\ell - XId)$ where A'_ℓ stands for the $\ell \times \ell$ matrix on the top left corner, and A''_ℓ for the $(L-\ell)\times(L-\ell)$ matrix on the bottom right corner:

$$A'_\ell = \begin{pmatrix} -(1-\alpha)c_1 + \alpha\frac{\ell-1}{\ell-\alpha} & 1 & & & (0) \\ . & . & . & & \\ -(i-\alpha)c_i & & \alpha\frac{\ell-i}{\ell-\alpha} & 1 & \\ . & (0) & & . & . \\ -(\ell-\alpha)c_l & & & & 0 \end{pmatrix}, \tag{3.2.80}$$

$$A''_\ell = \begin{pmatrix} -\frac{\alpha}{\ell-\alpha} & 1 & & & (0) \\ & . & . & & \\ & & -\alpha\frac{i-\ell}{\ell-\alpha} & 1 & \\ & & & . & . \\ (0) & & & & -\alpha\frac{L-\alpha}{\ell-\alpha} \end{pmatrix}. \tag{3.2.81}$$

We have:

$$\det(A''_\ell - XId) = \prod_{i=\ell+1}^{L} (-1)\left(X + \frac{(i-\ell)\alpha}{\ell-\alpha}\right). \tag{3.2.82}$$

We write $\mathcal{P}'_\ell = \det(A'_\ell - XId)$. We develop this determinant with respect to the last row and iterate this process. It gives for $\mathcal{P}'_\ell$ an expression of the form:

$$\begin{aligned} \mathcal{P}'_\ell = (-1)^{\ell+1}(-1)(\ell-\alpha)c_\ell \quad &+ \quad (-X)\Big[(-1)^{\ell}(-1)(\ell-1-\alpha) + (\tfrac{\alpha}{\ell-\alpha} - X) \\ &\times \quad \Big[(-1)^{\ell-1}(-1)(\ell-2-\alpha)c_{\ell-2} + (\tfrac{2\alpha}{\ell-\alpha} - X)[...]\Big]\Big]. \end{aligned}$$

We let for $1 \leq i \leq \ell$:

$$A_i := (-1)^{\ell+2-i}(-1)(\ell+1-i-\alpha)c_{\ell+1-i}, \tag{3.2.83}$$

and

$$B_i := (i-1)\frac{\alpha}{\ell-\alpha} - X. \tag{3.2.84}$$

We then rewrite:

$$\mathcal{P}'_\ell = A_1 + B_1\left(A_2 + B_2\left[A_3 + B_3\left[...\right]\right]\right).$$

We now let for $1 \leq i \leq \ell - 1$:

$$C_i := (-1)^{\ell+1-i}(X(\ell-i-\alpha)c_{\ell-i} + \frac{\ell-\alpha}{i}c_{\ell-i+1}). \tag{3.2.85}$$

We have the following relation for $1 \leq i \leq \ell - 2$:

$$C_i + B_1B_2A_{i+2} = B_{i+2}C_{i+1}. \tag{3.2.86}$$

Indeed we compute:

$$\begin{aligned}
C_i + B_1B_2A_{i+2} &= (-1)^{\ell+1-i}(X(\ell-i-\alpha)c_{\ell-i} + \tfrac{\ell-\alpha}{i}c_{\ell-i+1}) \\
&\quad +(-X)(\tfrac{\alpha}{\ell-\alpha} - X)(-1)^{\ell-i}(-1)(\ell-i-1-\alpha)c_{\ell-i-1} \\
&= (-1)^{\ell-i}(-X(\ell-i-\alpha)c_{\ell-i} - \tfrac{\ell-\alpha}{i}c_{\ell-i+1} \\
&\quad +X(\tfrac{(i+1)\alpha}{\ell-\alpha} - i\tfrac{\alpha}{\ell-\alpha} - X)(\ell-i-1-\alpha)c_{\ell-i-1}) \\
&= B_{i+2}(-1)^{\ell-i}(\ell-i-1-\alpha)c_{\ell-i-1} \\
&\quad +(-1)^{\ell-i}\big[-X(\ell-i-\alpha)c_{\ell-i} - \alpha c_{\ell-i} \\
&\quad -i\tfrac{\alpha}{\ell-\alpha}X(\ell-i-1-\alpha)(-\tfrac{\ell-\alpha}{\alpha(i+1)}c_{\ell-i})\big] \\
&= B_{i+2}(-1)^{\ell-i}(\ell-i-1-\alpha)c_{\ell-i-1} \\
&\quad +(-1)^{\ell-i}c_{\ell-i}(-X(\ell-i-\alpha)+\alpha+\tfrac{i}{i+1}(\ell-i-1-\alpha)X) \\
&= B_{i+2}(-1)^{\ell-i}(\ell-i-1-\alpha)c_{\ell-i-1} + (-1)^{\ell-i}c_{\ell-i}(-\tfrac{\ell-\alpha}{i+1}X+\alpha) \\
&= B_{i+2}(-1)^{\ell-i}(\ell-i-1-\alpha)c_{\ell-i-1} + (-1)^{\ell-i}\tfrac{\ell-\alpha}{i+1}c_{\ell-i}B_{i+2} \\
&= B_{i+2}C_{i+1}.
\end{aligned}$$

We also have:

$$A_1 + B_1A_2 = C_1.$$

By iterations we get:

$$\begin{aligned}
\mathcal{P}'_\ell &= A_1 + B_1A_2 + B_1B_2A_3 + B_1B_2B_3(A_4 + B_4(...)) \\
&= C_1 + B_1B_2A_3 + B_1B_2B_3(A_4 + B_4(...)) \\
&= C_2B_3 + B_1B_2B_3(A_4+B_4(...)) = B_3(C_2 + B_1B_2(A_4+B_4(...)) \\
&= B_3(B_4C_3 + B_1B_2B_4(A_5+B_5(...)) = B_3B_4(C_3 + B_1B_2(A_5+B_5(...)) \\
&\dots \\
&= B_3...B_\ell(C_{\ell-1} + B_1B_2).
\end{aligned}$$

We compute the last polynomial:

$$C_{\ell-1} + B_1B_2 = X(1-\alpha)c_1 + \frac{\ell-\alpha}{\ell-1}c_2 + (-X)\left(\frac{\alpha}{\ell-\alpha} - X\right) = (X+1)\left(X - \frac{\alpha\ell}{\ell-\alpha}\right).$$

So:

$$\mathcal{P}'_\ell = (X+1)\prod_{i=2}^{\ell}\left(\frac{i\alpha}{\ell-\alpha} - X\right).$$

This result, together with the result concerning $\mathcal{P}''_\ell$ shows that A_ℓ is diagonalizable and that its eigenvalues are: $(-1, \frac{2\alpha}{\ell-\alpha}, ..., \frac{\ell\alpha}{\ell-\alpha}, \frac{-\alpha}{\ell-\alpha}, ..., \frac{(L-\ell)\alpha}{\ell-\alpha})$.

In addition, from the form of A_ℓ, one sees that the ℓ first components do not affect the $L-\ell$ last ones: $\mathbb{P}_{(\ell+1,L)}A\mathbb{P}_{(1,\ell)} = 0$ where $\mathbb{P}_{(\ell+1,L)}$ and $\mathbb{P}_{(1,\ell)}$ are the projectors:

$$\mathbb{P}_{(\ell+1,L)}(U_1,...,U_L) = (0,...,0,U_{\ell+1},...,U_L),\ \mathbb{P}_{(1,\ell)}(U_1,...,U_L) = (U_1,...,U_\ell,0,...,0).$$

This gives the last result stated in the lemma. The v_i's are a linear combination of the ℓ first components only. $\square$

3.3 The trapped regime

In this section we are considering a real solution of (NLW). We fix $1 \ll L$ odd and $\alpha < \ell$. Our aim is to show that the approximate solution $(\tilde{\boldsymbol{Q}}_{b^e})_{\frac{1}{\lambda^e}}$ constructed in the last section does persist. That is to say that there exists an orbit of the (NLW) equation that stays asymptotically (with respect to renormalized time s) close to the family of special approximate solutions $(\tilde{\boldsymbol{Q}}_{b^e})_{\frac{1}{\lambda}}$. Note that we do not prescribe in advance the behavior of the scaling λ, but it will be shown to have the same asymptotical behavior as λ^e.

In order to do that, we need to understand how the full dynamics affects the approximate one we exhibited in the last section. We decompose a true solution under the form $\boldsymbol{u}(t) = (\tilde{\boldsymbol{Q}}_b + \boldsymbol{\varepsilon})_{\frac{1}{\lambda}}$. We aim at estimating the contribution of the error $\boldsymbol{\varepsilon}$ on the parameters dynamics, and at estimating the size of $\boldsymbol{\varepsilon}$ in adapted norms.

The special approximate solutions $(\tilde{\boldsymbol{Q}}_{b^e})_{\frac{1}{\lambda}}$ for $\lambda \sim \lambda^e$, generate a reasonable error term, because as $|b_i^e| \lesssim s^{-i} \approx (b_1^e)^i$ the estimates on the error term $\boldsymbol{\psi}_b$ in Proposition 3.2.14 apply. But they are not stable along the unstable directions $(v_2, ..., v_\ell)$ (defined in Lemma 3.2.17), and if the parameters b_i's move too much, the error term in the approximate dynamics grows too big, consequently making a control over $\boldsymbol{\varepsilon}$ impossible. Therefore we cannot work close to the full approximate manifold $\left(\tilde{\boldsymbol{Q}}_{b,\lambda}\right)_{b,\lambda}$: we are restricted to work close to the subset of these approximate trajectories $\left(\tilde{\boldsymbol{Q}}_{b^e(s),\lambda}\right)_{s>0,\lambda>0}$. We work in a neighborhood of these approximate trajectories, study all the real trajectories starting from that neighborhood, and show that at least one must stay in that neighborhood for all time. We make a proof based on a bootstrap technique. We in particular argue "forward" in time what allows us to measure precisely the stabilities and instabilities.

The fact that staying in an appropriate neighborhood of a special approximate solution leads to a blow-up, whose blow-up rate and asymptotic behavior can be computed, will be shown in the next section.

3.3.1 Setting up the bootstrap

We are now going to define in which neighborhood of the family of approximate solutions $\left(\tilde{\boldsymbol{Q}}_{b^e(s),\frac{1}{\lambda}}\right)_{s,\lambda}$ we want to work. We start by defining how we decompose our solution into the sum $\boldsymbol{u} = (\tilde{\boldsymbol{Q}}_b + \boldsymbol{\varepsilon})_{\frac{1}{\lambda}}$. After that we describe the neighborhood and state the main Proposition of the chapter claiming the existence of an orbit staying inside.

3.3.1.1 Projection onto the approximate solutions manifold

Close to $\boldsymbol{Q}$, the manifold $\left(\boldsymbol{Q}_{b,\lambda}\right)_{b,\lambda}$ is tangent to the vector space $Span(\boldsymbol{T}_i)$. It is consequently appealing to ask $\langle \boldsymbol{T}_i, \boldsymbol{\varepsilon}\rangle = 0$ for all i. However, the $\boldsymbol{T}_i$'s are not in appropriate functional spaces, and in particular cannot be used to generate orthogonality conditions. Instead, we will create a sequence of profiles with compact support that approximate such orthogonality conditions. We let the adjoint of $\boldsymbol{H}$ be the operator:

$$\boldsymbol{H}^* = \begin{pmatrix} 0 & \mathcal{L} \\ -1 & 0 \end{pmatrix}. \tag{3.3.1}$$

We have the following relations: $\langle \boldsymbol{H}\boldsymbol{u}, \boldsymbol{v}\rangle = \langle \boldsymbol{u}, \boldsymbol{H}^*\boldsymbol{v}\rangle$, and

$$\boldsymbol{H}^{*2i} = \begin{pmatrix} (-1)^i\mathcal{L}^i & 0 \\ 0 & (-1)^i\mathcal{L}^i \end{pmatrix}, \ \boldsymbol{H}^{*(2i+1)} = \begin{pmatrix} 0 & (-1)^i\mathcal{L}^{i+1} \\ (-1)^{i+1}\mathcal{L}^i & 0 \end{pmatrix}. \tag{3.3.2}$$

We recall that L is an odd, large integer. We let $M \gg 1$ be a large constant, and define:

$$\boldsymbol{\Phi}_M = \sum_{p=0}^{L} c_{p,M}\boldsymbol{H}^{*p}(\chi_M \boldsymbol{\Lambda Q}), \tag{3.3.3}$$

with the constants $c_{p,M}$ for $0 \leq p \leq L$ defined by:

$$c_{0,M} = 1 \ \text{ and } \ c_{k,M} = (-1)^{k+1}\frac{\sum_{p=0}^{k-1} c_{p,M}\langle \boldsymbol{H}^{*p}(\chi_M\boldsymbol{\Lambda Q}), \boldsymbol{T}_k\rangle}{\langle \chi_M \boldsymbol{\Lambda Q}, \boldsymbol{\Lambda Q}\rangle}. \tag{3.3.4}$$

Lemma 3.3.1. *(Generation of orthogonality conditions:) The profile $\boldsymbol{\Phi}_M$ is located on the first coordinate:*

$$\boldsymbol{\Phi}_M = \begin{pmatrix} \Phi_M \\ 0 \end{pmatrix}, \tag{3.3.5}$$

because for $1 \leq k = 2i+1 \leq L$ an odd integer one has $c_{k,M} = 0$. Moreover the following bounds hold:

$$\begin{cases} |\langle \Phi_M, \Lambda Q\rangle| \sim cM^{2k_0+2\delta_0}, \\ |c_{p,M}| \leq CM^p, \\ \int \Phi_M^2 \leq CM^{2k_0+2\delta_0}. \end{cases} \tag{3.3.6}$$

for two positive constants $c, C > 0$. In addition, the following orthogonality conditions are met for $1 \leq j \leq L$ and $i \in \mathbb{N}$:

$$\langle \boldsymbol{\Phi}_M, \boldsymbol{H}^i\boldsymbol{T}_j\rangle = \langle \chi_M\boldsymbol{\Lambda Q}, \boldsymbol{\Lambda Q}\rangle \delta_{i,j}. \tag{3.3.7}$$

Proof of Lemma 3.3.1 Proof of the orthogonality conditions:

$$\begin{array}{rcl} \langle \boldsymbol{\Phi}_M, \boldsymbol{\Lambda Q}\rangle & = & c_{0,M}\langle \chi_M\boldsymbol{\Lambda Q}, \boldsymbol{\Lambda Q}\rangle + \sum_{p=1}^{L} c_{p,M}\langle \chi_M\boldsymbol{\Lambda Q}, \boldsymbol{H}^p(\boldsymbol{\Lambda Q})\rangle \\ & = & \langle \chi_M\boldsymbol{\Lambda Q}, \boldsymbol{\Lambda Q}\rangle \\ & \sim & cM^{d-2\gamma}, \end{array}$$

$c > 0$, from the asymptotic $\Lambda^{(1)}Q \sim \frac{c'}{y^\gamma}$, $c' \neq 0$. This proves the first property of (*3.3.6*). The orthogonality with respect to the $\boldsymbol{T}_i$'s is created on purpose by the definition of the constants $c_{p,M}$:

$$\langle \boldsymbol{\Phi}_M, \boldsymbol{T}_k\rangle = \sum_{p=0}^{k-1} c_{p,M}\langle \boldsymbol{H}^{*p}(\chi_M\boldsymbol{\Lambda Q}), \boldsymbol{T}_k\rangle + c_{k,M}\langle \chi_M\boldsymbol{\Lambda Q}, \boldsymbol{H}^k\boldsymbol{T}_k\rangle = 0.$$

Hence by duality:

$$\langle \boldsymbol{\Phi}_M, \boldsymbol{H}^i\boldsymbol{T}_j\rangle = \langle \chi_M\boldsymbol{\Lambda Q}, \boldsymbol{\Lambda Q}\rangle \delta_{i=j}.$$

This proves (*3.3.7*).

Bounds on the constants: We notice by induction that $c_{p,M}=0$ for p odd. This implies that $\Phi_M^{(2)}=0$. We prove the estimate on the constants $c_{p,M}$ by induction. Since $c_0=1$, the estimation is true for $k=0$. We assume now k to be even. By definition we have:

$$\begin{array}{rcl} |c_{k,M}| & = & \frac{|\sum_{p=0}^{k-1}\langle \boldsymbol{H}^{*p}(\chi_M \boldsymbol{\Lambda Q}),\boldsymbol{T}_k\rangle|}{|\langle \chi_M \boldsymbol{\Lambda Q},\boldsymbol{\Lambda Q}\rangle|} \\ & \leq & CM^{-d+2\gamma}\sum_{p=0}^{k-1}|c_{p,M}||\langle \boldsymbol{H}^{*p}(\chi_M\boldsymbol{\Lambda Q}),\boldsymbol{T}_k\rangle| \\ & = & CM^{-d+2\gamma}\sum_{p=0}^{k-1}|c_{p,M}||\langle \chi_M\boldsymbol{\Lambda Q},\boldsymbol{T}_{k-p}\rangle|. \end{array}$$

In the sum, for $k-p$ odd this term equals 0. So we have $k-p\geq 2$. Using the asymptotics $\Lambda^{(1)}Q\sim cy^{-\gamma}$ and $T_{k-p}\sim cy^{-\gamma+k-p}$ the integral in the scalar product is divergent and we estimate:

$$|\langle \chi_M\boldsymbol{\Lambda Q},\boldsymbol{T}_{k-p}\rangle|\sim cM^{d-2\gamma+k-p}.$$

Using the induction hypothesis we get:

$$M^{-d+2\gamma}|c_{p,M}||\langle \boldsymbol{H}^{*p}(\chi_M\boldsymbol{\Lambda Q},\boldsymbol{T}_k\rangle\leq CM^k,$$

and so the estimate is true for $c_{k,M}$. We have proven the second assertion of (*3.3.6*).

L^2estimate: $\int|\Phi_M|^2$ is a finite sum of terms of the following form enjoying the bound (from the asymptotic (*3.2.3*)):

$$\begin{array}{rcl} & & |\langle c_{p_1,M}\boldsymbol{H}^{*p_1}(\chi_M\boldsymbol{\Lambda Q}),c_{p_2,M}\boldsymbol{H}^{*p_2}(\chi_M\boldsymbol{\Lambda Q})\rangle| \\ \leq & CM^{p_1+p_2}|\langle\mathcal{L}^{\frac{p_1+p_2}{2}}(\chi_M\Lambda Q),\chi_M\Lambda Q\rangle|\leq & CM^{-2\gamma+d}, \end{array}$$

because we assumed $\frac{d}{2}-\gamma$ not to be an integer. It implies the last bound in (*3.3.6*) □

3.3.1.2 Modulation:

We want to decompose a function $\boldsymbol{u}$ close to $\boldsymbol{Q}_\lambda$ as a unique sum $\boldsymbol{u}=(\boldsymbol{Q}_b+\boldsymbol{\varepsilon})$, with $\boldsymbol{\varepsilon}$ "orthogonal" to the manifold $(\boldsymbol{Q}_{b,\lambda})_{b,\lambda}$. We make the following change of variable for the parameter b: $\tilde{b}_1:=(b_1,0,...,0)$ and $\tilde{b}_i=(b_1,0...,0,b_i,0,...,0)$ and introduce the application $\phi:(\lambda,b)\mapsto(\langle\tilde{\boldsymbol{Q}}_b,\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle)_{0\leq i\leq L}$. We denote by $D\phi$ the jacobian matrix of ϕ at the point $(1,(0,...,0))$ in the $(\lambda,\tilde{b})$ basis. From the properties (*3.3.6*) and (*3.3.7*) of the profile $\boldsymbol{\Phi}_M$ that we previously established, one has:

$$D\phi=\langle\boldsymbol{\Lambda Q},\chi_M\boldsymbol{\Lambda Q}\rangle\begin{pmatrix} 1 & 0 & 0 & . & 0 \\ & 1 & 1 & . & 1 \\ & & 1 & & (0) \\ & & & . & \\ (0) & & & & 1 \end{pmatrix}.$$

This proves that ϕ is a local diffeomorphism around $(1,(0,...,0))$. The implicit function theorem gives for $\boldsymbol{u}$ close enough[10] to $\boldsymbol{Q}$ the existence of a unique decomposition:

$$\boldsymbol{u}=(\tilde{\boldsymbol{Q}}_b)_{\frac{1}{\lambda}}+\boldsymbol{w}=(\tilde{\boldsymbol{Q}}_b+\boldsymbol{\varepsilon})_{\frac{1}{\lambda}}, \tag{3.3.8}$$

[10]the closeness assumption is described in the next subsection and is compatible with what we are saying here.

with ε verifying the $L+1$ orthogonality conditions:

$$\langle \boldsymbol{\varepsilon}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle = 0, \ for\ 0 \leq i \leq M. \tag{3.3.9}$$

Hence for a real solution to (NLW) starting close enough to $\boldsymbol{Q}$, and by scaling argument, we have as long as $\boldsymbol{u}$ is close enough to $\boldsymbol{Q}_\lambda$ a decomposition:

$$\boldsymbol{u} = (\tilde{\boldsymbol{Q}}_{b(t)} + \boldsymbol{\varepsilon})_{\frac{1}{\lambda(t)}}, \tag{3.3.10}$$

with b and λ being C^1 in time[11], and ε satisfying (3.3.9).

3.3.1.3 Adapted norms:

We quantify the smallness of ε through the following norms:

(i) *High order Sobolev norm adapted to the linearized operator:* Remember that $s_L = L + k_0 + 1$ and that the k-th adapted derivative of a function f, f_k, is defined in (3.2.21). We define:

$$\begin{aligned} \mathcal{E}_{s_L} &:= \int |\varepsilon^{(1)}_{k_0+L+1}|^2 + \int |\varepsilon^{(2)}_{k_0+L}|^2 \\ &= \int \varepsilon^{(1)} \mathcal{L}^{k_0+L+1} \varepsilon^{(1)} + \int \varepsilon^{(2)} \mathcal{L}^{k_0+L} \varepsilon^{(2)}, \end{aligned} \tag{3.3.11}$$

which is coercive thanks to the result of Lemma 3.D.3. In particular:

$$\mathcal{E}_{s_L} \gtrsim \| \boldsymbol{\varepsilon} \|^2_{\dot{H}^{s_L} \times \dot{H}^{s_L-1}} .$$

As we will see later on in this chapter, a local part of this norm will have to be treated separately. Let $N > 0$, we define[12]:

$$\mathcal{E}_{s_L,\text{loc}} := \int_{y \leq N} |\varepsilon^{(1)}_{k_0+L+1}|^2 + \int_{y \leq N} |\varepsilon^{(2)}_{k_0+L}|^2. \tag{3.3.12}$$

(ii) *Low order slightly supercritical Sobolev norm:* We choose a real number σ such that:

$$0 < \sigma - s_c \ll 1, \tag{3.3.13}$$

and we define:

$$\mathcal{E}_\sigma := \int |\nabla^\sigma \varepsilon^{(1)}|^2 + \int |\nabla^{\sigma-1} \varepsilon^{(2)}|^2. \tag{3.3.14}$$

Estimates we want to bootstrap and main Proposition:

Let s_0 denote a large enough real number $s_0 \gg 1$. We recall the definition of the renormalized variables:

$$y = \frac{r}{\lambda(t)}, \ s(t) = s_0 + \int_0^t \frac{d\tau}{\lambda(\tau)}. \tag{3.3.15}$$

We introduce notations for the decomposition of the solution in both real and renormalized time:

$$\boldsymbol{u} = \tilde{\boldsymbol{Q}}_{b(t),\frac{1}{\lambda(t)}} + \boldsymbol{w} = (\tilde{\boldsymbol{Q}}_{b(s)} + \boldsymbol{\varepsilon}(s))_{\frac{1}{\lambda(s)}}. \tag{3.3.16}$$

[11]As the dynamic will be smooth enough.

[12]Here by $\int |\nabla^s f|^2$ we mean $\int |\xi|^{2s} |\hat{f}|^2$ where $\hat{f}$ is the Fourier transform of f.

The parameters b_i are chosen as a perturbation of the solution b^e:

$$b_i(s) = b_i^e(s) + \frac{U_i(s)}{s^i}. \tag{3.3.17}$$

To treat the stable and unstable modes separately, we employ the change of variables coded by the matrix P_ℓ defined by (3.2.78). Instead of $U_1, ..., U_\ell$ we consider:

$$V_i := (P_\ell U)_i \ \text{ for } 1 \le i \le \ell. \tag{3.3.18}$$

We assume initially[13]:

(i) Smallness of the unstable modes: Let $0 < \tilde{\eta}$ be a constant to be defined later.

$$(V_2(s_0), ..., V_\ell(s_0)) \in \mathcal{B}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right). \tag{3.3.19}$$

(ii) Smallness of the stable modes[14], for $(\epsilon_i)_{\ell+1 \le i \le L}$ strictly positive constants to be chosen later on:

$$V_1(s_0) \le \frac{1}{10 s_0}, \ \text{and } |b_i(s_0)| \le \frac{\epsilon_i}{10 s_0^{(i-\alpha)c_1}} \ \text{for } \ell + 1 \le i \le L. \tag{3.3.20}$$

(iii) Smallness of the initial perturbation in high and low Sobolev norms:

$$\mathcal{E}_{s_L}(s_0) + \mathcal{E}_\sigma(s_0) < \frac{1}{s_0^{2L+2+2(1-\delta_0)(1+\eta)}}. \tag{3.3.21}$$

(iv) Normalization: up to a fix rescaling, we may always assume:

$$\lambda(s_0) = 1. \tag{3.3.22}$$

Proposition 3.3.2. (Existence of an initial datum for which the solution stays in yhe trapped regime:) *There exists universal constants for the analysis:*

$$\begin{array}{l} 0 < \eta = \eta(d,p,L) \ll 1, \ M = M(d,p,L) \gg 1, \ N = N(d,p,L,M) \gg 1, \\ K_i = K_i(d,p,L,M) \gg 1, \ \textit{for}\, i = 1,2, \ s_0 = s_0(l,d,p,L,M,K) \gg 1, \end{array} \tag{3.3.23}$$

and constants for smallness:

$$0 < \epsilon_i \ for \ \ell + 1 \le i \le L \ \textit{and}\, 0 < \tilde{\eta} \quad (all \ \ll 1), \tag{3.3.24}$$

such that the following fact holds. Given $\varepsilon(s_0)$ satisfying (3.3.9), (3.3.21), and stable parameters $V_1(s_0)$, $(b_{\ell+1}(s_0), ..., b_L(s_0))$ satisfying (3.3.20), there exists initial conditions for the unstable parameters $(V_2(s_0), ...V_\ell(s_0))$ satisfying (3.3.19) for which the solution to (NLW) with initial data $\tilde{\boldsymbol{Q}}_{b(s_0)} + \varepsilon(s_0)$ with:

$$\begin{array}{rcl} b(s_0) & = & b^e(s_0) + (0, ..., 0, b_{\ell+1}(s_0), ..., b_L(s_0)) \\ & & + \left(\frac{(P_\ell^{-1}(V_1(s_0),...,V_\ell(s_0),0,...,0))_1}{s_0}, ..., \frac{(P_\ell^{-1}(V_1(s_0),...,V_\ell(s_0),0,...,0))_\ell}{s_0^\ell}, 0, ..., 0 \right), \end{array}$$

admits the following bounds for all $s \ge s_0$:

[13] the choice of the constants is done in the next proposition.

[14] the $\frac{1}{10}$ is arbitrary: we just want the initial condition to be smaller than the information we want to bootstrap, see Proposition 3.3.2.

- control of the part on the approximate profiles manifold: *for the unstable modes:*

$$(V_2(s), ...V_\ell(s)) \in \mathcal{B}^{\ell-1}\left(\frac{1}{s^{\tilde{\eta}}}\right). \tag{3.3.25}$$

for the stable modes:

$$|V_1(s)| \leq \frac{1}{s^{\tilde{\eta}}},\ |b_k(s)| \leq \frac{\epsilon_k}{s^{k+\tilde{\eta}}},\ \textit{for } \ell+1 \leq k \leq L. \tag{3.3.26}$$

- control of the error term:

$$\begin{aligned} \mathcal{E}_{s_L}(s) &\leq K_1 b_1^{2L+2(1-\delta_0)(1+\eta)}, \\ \mathcal{E}_\sigma(s) &\leq K_2 b_1^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}}. \end{aligned} \tag{3.3.27}$$

Let us now sketch the proof of Proposition 3.3.2. We argue by contradiction and suppose that for all initial data of the unstable modes $(V_2, ..., V_\ell) \in \mathcal{B}^{\ell-1}(s_0^{-\tilde{\eta}})$, the conditions are not met for all time and define the exit time:

$$\begin{aligned} s^* &= s^*(\varepsilon(s_0), s_0, V_1(s_0), ..., V_\ell(s_0), b_{\ell+1}(s_0), ..., b_L(s_0)) \\ &= sup\{s \geq s_0 \text{ such that } (3.3.27),\ (3.3.25) \text{ and } (3.3.26) \text{ hold on} [s_0, s]\} \\ &< +\infty. \end{aligned} \tag{3.3.28}$$

By continuity of the flow and the smallness of the initial perturbation, we know that $s^* > 0$. We perform a three steps reasoning to prove the contradiction:

(i) First we show that as long as ε is controlled by the estimates (*3.3.27*), it does not perturb too much the dynamical system (*3.2.72*). That is to say we have a sufficient control over the evolution of the b_i's to show that the perturbation U of the trajectory b^e evolves according to the linearisation at the leading order.

(ii) (i) has given us control over the part of the solution on the approximate manifold, this allows us to compute the evolution of the scale λ. Under the bootstrap conditions we know the size of the error term $\tilde{\psi}_b$ generated by the approximate dynamics. Once we know the behavior of $\tilde{\psi}_b$ and λ, we can look for better informations about ε. Indeed we apply an energy method and find out that we control the time evolution of $\mathcal{E}_{s_L}$ and $\mathcal{E}_\sigma$. As ε is a stable perturbation, we find that we have in fact a better estimate for this term: ε is smaller than the estimate given by (*3.3.27*). Hence at time s^*:

$$\begin{aligned} \mathcal{E}_{s_L}(s^*) &< K_1 b_1^{2L+2(1-\delta_0)(1+\eta)}, \\ \mathcal{E}_\sigma(s^*) &< K_2 b_1^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}}. \end{aligned} \tag{3.3.29}$$

This implies that the exit of the trapped regime is only when the parameters do not satisfy the estimates (*3.3.25*) and (*3.3.26*) anymore.

(iii) With the estimates we have found regarding the parameters dynamics in (i) we are able to say that this is impossible. Indeed, the stable parameters cannot go away because their dynamics is stable. It is possible for some unstable parameters to go away, but they cannot all leave the ball $\mathcal{B}^{\ell-1}\left(\frac{1}{(s^*)^{\tilde{\eta}}}\right)$ in finite time. We have seen in Lemma 3.2.17 that the V_i's for $2 \leq i \leq \ell$ evolve as a linearized system around a repulsive equilibrium. The true dynamics, adding a small error term to their time evolution, preserves this structure. The dynamics in our case cannot expulse all the orbits away from the equilibrium point: we will show how in that case it would be a contradiction to Brouwer's fixed point theorem.

3.3.2 Evolution equations for ε and w:

We recall that we are studying a solution under the form:

$$\boldsymbol{u} = \tilde{\boldsymbol{Q}}_{b(t),\frac{1}{\lambda(t)}} + \boldsymbol{w} = (\tilde{\boldsymbol{Q}}_{b(s)} + \boldsymbol{\varepsilon}(s))_{\frac{1}{\lambda(s)}},$$

where $\tilde{\boldsymbol{Q}}_b$ is defined by (*3.2.53*) and ε satisfies the orthogonality conditions (*3.3.9*), this decomposition being explained in Subsubsection 3.3.1.2. The evolution of ε and $\boldsymbol{w}$ is given by, introducing a notation for the non linear and small linear terms in renormalized variables:

$$\begin{array}{rcll} \boldsymbol{\varepsilon}_s - \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda}\boldsymbol{\varepsilon} + \boldsymbol{H}(\boldsymbol{\varepsilon}) & = & -\boldsymbol{Mod}(t) + (\frac{\lambda_s}{\lambda} + b_1)\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b - \tilde{\boldsymbol{\psi}}_b & \\ & & +\boldsymbol{F}(\tilde{\boldsymbol{Q}}_b + \boldsymbol{\varepsilon}) - \boldsymbol{F}(\tilde{\boldsymbol{Q}}_b) + \boldsymbol{H}_b(\boldsymbol{\varepsilon}) & \} := \mathbf{NL}(\varepsilon) \\ & & +\boldsymbol{H}(\boldsymbol{\varepsilon}) - \boldsymbol{H}_b(\boldsymbol{\varepsilon}) & \} := \mathbf{L}(\varepsilon)\ , \end{array} \qquad (3.3.30)$$

where $\boldsymbol{H}_b$ denotes the linearization close to $\tilde{\boldsymbol{Q}}_b$:

$$\boldsymbol{H}_b := \begin{pmatrix} 0 & -1 \\ -\Delta - p\tilde{Q}_b^{p-1} & 0 \end{pmatrix}, \qquad (3.3.31)$$

and, introducing a notation for the non linear and small linear terms in original variables:

$$\begin{array}{rcll} \boldsymbol{w}_t + \boldsymbol{H}_{\frac{1}{\lambda}}\boldsymbol{w} & = & \frac{1}{\lambda}(-\boldsymbol{Mod}(t) + (\frac{\lambda_s}{\lambda} + b_1)\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b)_{\frac{1}{\lambda}} - \frac{1}{\lambda}\tilde{\boldsymbol{\psi}}_{b,\frac{1}{\lambda}} & \\ & & +\boldsymbol{F}(\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}} + \boldsymbol{w}) - \boldsymbol{F}(\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}}) + \boldsymbol{H}_{b,\frac{1}{\lambda}}\boldsymbol{w} & \} := \mathbf{NL}(\mathbf{w}) \\ & & +\boldsymbol{H}_{\frac{1}{\lambda}}\boldsymbol{w} - \boldsymbol{H}_{b,\frac{1}{\lambda}}\boldsymbol{w} & \} := \mathbf{L}(\mathbf{w})\ , \end{array} \qquad (3.3.32)$$

where:

$$\boldsymbol{H}_{\frac{1}{\lambda}} := \begin{pmatrix} 0 & -1 \\ -\Delta - p(Q_{\frac{1}{\lambda}})^{p-1} & 0 \end{pmatrix}, \text{ and } \boldsymbol{H}_{b,\frac{1}{\lambda}} := \begin{pmatrix} 0 & -1 \\ -\Delta - p(\tilde{Q}_{b,\frac{1}{\lambda}})^{p-1} & 0 \end{pmatrix}. \qquad (3.3.33)$$

We notice that the $\boldsymbol{NL}$ and $\boldsymbol{L}$ terms are situated on the second coordinate:

$$\boldsymbol{NL}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ NL(\varepsilon) \end{pmatrix}, \ \boldsymbol{NL}(\boldsymbol{w}) = \begin{pmatrix} 0 \\ NL(w) \end{pmatrix}, \ \boldsymbol{L}(\boldsymbol{\varepsilon}) = \begin{pmatrix} 0 \\ L(\varepsilon) \end{pmatrix}, \ \boldsymbol{L}(\boldsymbol{w}) = \begin{pmatrix} 0 \\ L(w) \end{pmatrix}. \qquad (3.3.34)$$

We let the new modulation term that now includes the scale change be:

$$\tilde{\boldsymbol{Mod}}(t) := \boldsymbol{Mod}(t) - \left(\frac{\lambda_s}{\lambda} + b_1\right)\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b. \qquad (3.3.35)$$

3.3.3 Modulation equations

In this section we compute the influence of ε on the equations governing the evolution of the parameters λ and b.

Lemma 3.3.3 (Modulation estimates). *Assume that all the constants involved in Proposition 3.3.2 are fixed in their range*[15]*, except* s_0*. Then for* s_0 *large enough there holds the bounds for* $s_0 \leq s < s^*$*:*

$$\begin{array}{rl} & \left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_{i=1}^{L-1} |b_{i,s} + (i-\alpha)b_1 b_i + b_{i+1}| \\ \leq & C(M)b_1^{L+3} + C(L,M)b_1\sqrt{\mathcal{E}_{s_L}}, \end{array} \qquad (3.3.36)$$

$$|b_{L,s} + (L-\alpha)b_1 b_L| \leq C(M)\sqrt{\mathcal{E}_{s_L}} + C(M)b_1^{L+3}. \qquad (3.3.37)$$

[15]It means that, for example, if we wrote $0 < C \ll 1$ that C is fixed very small

Remark 3.3.4. Under the assumption on the smallness of ε (*3.3.27*) This implies in particular that:

$$\frac{\lambda_s}{\lambda}=-b_1+O(b_1^2)$$

and

$$b_{i,s}=-(i-\alpha)b_1b_i+b_{i+1}+O(b_1^{i+2})$$

for $1\leq i\leq L-1$. If we had also $b_{L,s}=-(L-\alpha)b_1b_L+O(b_1^{L+1+c})$ for a small constant $c>0$, this would be enough to conclude that the dynamics of the parameters is given at the first order by (*3.2.72*). Unfortunately this last condition is not met. We will see how to skirt this problem in the next Lemma 3.3.5.

Proof of Lemma 3.3.3 We let:

$$D(t)=\left|\frac{\lambda_s}{\lambda}+b_1\right|+\sum_{i=1}^{L}|b_{i,s}+(i-\alpha)b_1b_i-b_{i+1}|. \tag{3.3.38}$$

For $0\leq i\leq L$ we take the sclar product of (*3.3.30*) with $\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M$:

$$\begin{array}{rcl}\langle \tilde{\boldsymbol{Mod}}(t),\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle & = & \langle -\boldsymbol{H}(\boldsymbol{\varepsilon}),\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle+\langle\frac{\lambda_s}{\lambda}\boldsymbol{\Lambda\varepsilon},\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle-\langle\tilde{\boldsymbol{\psi}}_b,\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle\\ & & +\langle \boldsymbol{NL}(\boldsymbol{\varepsilon}),\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle+\langle \boldsymbol{L}(\boldsymbol{\varepsilon}),\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle.\end{array} \tag{3.3.39}$$

Step 1: *law for* λ. We take $i=0$ in the preceding equation (*3.3.39*) and compute all the terms. As $\boldsymbol{\Phi}_M$ is located on the first coordinate, see (*3.3.6*), it gives:

$$\langle \boldsymbol{NL}(\boldsymbol{\varepsilon}),\boldsymbol{\Phi}_M\rangle=\langle \boldsymbol{L}(\boldsymbol{\varepsilon}),\boldsymbol{\Phi}_M\rangle=0. \tag{3.3.40}$$

$\boldsymbol{\Phi}_M$ is of compact support in $|y|\leq 2M$ and situated on the first coordinate. For b_1 small enough one has in this zone $\tilde{\boldsymbol{\psi}}_b(y)=\boldsymbol{\psi}_b(y)$, and $\boldsymbol{\psi}_b$ is situated on the second coordinate from (*3.2.39*). Hence:

$$\langle\tilde{\boldsymbol{\psi}}_b,\boldsymbol{\Phi}_M\rangle=0. \tag{3.3.41}$$

The linear term is equal to 0 because of the orthogonality conditions (*3.3.9*):

$$\langle -\boldsymbol{H}(\boldsymbol{\varepsilon}),\boldsymbol{\Phi}_M\rangle=0. \tag{3.3.42}$$

The left hand side, the modulation term, is the one catching the evolution of λ_s:

$$\begin{array}{rcl}\langle \tilde{\boldsymbol{Mod}}(t),\boldsymbol{\Phi}_M\rangle & = & (\frac{\lambda_s}{\lambda}+b)\langle\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b,\boldsymbol{\Phi}_M\rangle\\ & & +\sum_{i=1}^L(b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})\langle\boldsymbol{T}_i+\sum_{j=i+1}^L\frac{\partial\boldsymbol{S}_j}{\partial b_i},\boldsymbol{\Phi}_M\rangle\\ & = & (\frac{\lambda_s}{\lambda}+b_1)\langle\boldsymbol{\Lambda Q},\boldsymbol{\Lambda Q}\rangle+O(b_1D(t)).\end{array} \tag{3.3.43}$$

We now estimate the scaling term:

$$\begin{array}{rcl}|\langle\frac{\lambda_s}{\lambda}\boldsymbol{\Lambda\varepsilon},\boldsymbol{\Phi}_M\rangle| & \leq & |\frac{\lambda_s}{\lambda}+b_1||\langle\Lambda^{(1)}\varepsilon^{(1)},\Phi_M\rangle|+b_1|\langle\Lambda^{(1)}\varepsilon^{(1)},\Phi_M\rangle|\\ & \leq & (b_1+D(t))|\parallel\Lambda\varepsilon^{(1)}\parallel_{L^2(\leq M)}\parallel\Phi_M\parallel_{L^2}.\end{array}$$

We use the coercivity estimate from Corollary 3.D.4 to relate the L^2 norm on the compact set $y\leq M$ to $\mathcal{E}_{s_L}$:

$$\int_{y\leq M}|\varepsilon^{(1)}|^2=\int_{y\leq M}(1+y)^{2k_0+2L+2}\frac{|\varepsilon^{(1)}|^2}{1+y^{2k_0+2L+2}}\leq C(M)\mathcal{E}_{s_L},$$

$$\int_{y\leq M}|y\partial_y\varepsilon^{(1)}|^2\leq\int_{y\leq M}(1+y)^{2k_0+2L+2}\frac{|\partial_y\varepsilon^{(1)}|^2}{1+y^{2k_0+2L}}\leq C(M)^{2(k_0+L+1)}\mathcal{E}_{s_L}.$$

This gives:

$$|\langle\frac{\lambda_s}{\lambda}\Lambda^{(1)}\varepsilon^{(1)},\Phi_M\rangle|\leq C(M)(b_1+D(t))\sqrt{\mathcal{E}_{s_L}}. \tag{3.3.44}$$

Now that we have computed all the terms in (*3.3.39*) for $i=0$, in (*3.3.40*), (*3.3.41*), (*3.3.42*), (*3.3.43*) and (*3.3.44*), we end up with:

$$\left|\frac{\lambda_s}{\lambda}+b_1\right|=O(b_1D(t))+O((b_1+D(t))C(M)\sqrt{\mathcal{E}_{s_L}}). \tag{3.3.45}$$

Step 2: *law of* b_i *for* $1\leq i\leq L-1$. We take again equation (*3.3.39*) and do the same computations. The $\tilde{\boldsymbol{Mod}}$ term represents the approximate dynamics:

$$\langle\tilde{\boldsymbol{Mod}}(t),\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle=\langle\boldsymbol{\Lambda Q},\boldsymbol{\Phi}_M\rangle(b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})+O(b_1D(t)). \tag{3.3.46}$$

The linear term still disappears because of the orthogonality conditions:

$$\langle-\boldsymbol{H}(\boldsymbol{\varepsilon}),\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle=0. \tag{3.3.47}$$

For the scale changing term, as before, thanks to the coercivity of $\mathcal{E}_{s_L}$ and to (*3.3.45*):

$$|\langle\frac{\lambda_s}{\lambda}\boldsymbol{\Lambda\varepsilon},\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle|\leq(b_1+D(t))C(M)\sqrt{\mathcal{E}_{s_L}}. \tag{3.3.48}$$

The error contribution, as $\tilde{\boldsymbol{\psi}}_b=\boldsymbol{\psi}_b$ for $y\leq 2M$ (for s_0 small enough) is estimated thanks to Proposition 3.2.12:

$$|\langle\tilde{\boldsymbol{\psi}}_b,\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle|\leq C(M)b_1^{L+3}. \tag{3.3.49}$$

We now want to estimate the nonlinear contribution. Since $\boldsymbol{NL}$ is a linear sum of terms of the form $\tilde{Q}_b^{p-k}\varepsilon^{(1)k}$ for $k\geq 2$ we estimate using Cauchy-Schwarz, the L^∞ estimate given in Lemma 3.E.1, and again the coercivity estimate:

$$\begin{array}{rcl}\langle\tilde{\boldsymbol{Q}}_b^{p-k}\boldsymbol{\varepsilon}^{(1)k},\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle & \leq & C(M)\parallel\epsilon^{(1)}\parallel_{L^\infty}^{k-2}\mathcal{E}_{s_L}\\ & = & o(b_1\sqrt{\mathcal{E}_{s_L}}),\end{array} \tag{3.3.50}$$

in the regime (*3.3.27*). Because $(\tilde{Q}_b^{(1)})^{p-1}-Q^{p-1}=O(b_1)$ there holds for the small linear term:

$$|\langle L(\varepsilon),H^{*i}\Phi_M\rangle|\leq b_1C(M)\sqrt{\mathcal{E}_{s_L}}. \tag{3.3.51}$$

We have estimated all the terms in (*3.3.39*) for $1\leq i\leq L-1$, in (*3.3.46*), (*3.3.47*), (*3.3.48*), (*3.3.49*), (*3.3.50*) and (*3.3.51*), it yields:

$$|b_{i,s}-(i-\alpha)b_1b_i|\leq O(b_1D(t))+C(M)b_1^{L+3}+C(M)b_1\sqrt{\mathcal{E}_{s_L}}. \tag{3.3.52}$$

Step 3: *the law of* b_L. We compute:

$$\langle\tilde{\boldsymbol{Mod}}(t),\boldsymbol{H}^{*L}\boldsymbol{\Phi}_M\rangle=O(b_1D(t))+(b_{L,s}+(L-\alpha)b_1b_L)\langle\boldsymbol{\Lambda Q},\boldsymbol{\Phi}_M\rangle.$$

The terms that we previously estimated still admits the same bounds. But the linear term does not disappear in this case. We recall that we have chosen L odd. From the identity (*3.2.25*) relating $\boldsymbol{H}^k$ to $\mathcal{L}$:

$$|\langle \boldsymbol{H}(\boldsymbol{\varepsilon}), \boldsymbol{H}^{*L}\boldsymbol{\Phi}_M\rangle| = |\langle \boldsymbol{H}^{L+1}\boldsymbol{\varepsilon}, \boldsymbol{\Phi}_M\rangle| = |\int \mathcal{L}^{\frac{L+1}{2}}\varepsilon^{(1)}\Phi_M| \leq C(M)\sqrt{\mathcal{E}_{s_L}}.$$

This gives:

$$\left|\frac{\langle \boldsymbol{H}(\boldsymbol{\varepsilon}), \boldsymbol{H}^{*L}\boldsymbol{\Phi}_M\rangle}{\langle \boldsymbol{\Phi}_M, \boldsymbol{\Lambda Q}\rangle}\right| \lesssim M^{-\delta_0}\sqrt{\mathcal{E}_{s_L}}. \tag{3.3.53}$$

We then conclude that:

$$|b_{L,s} - (L-\alpha)b_1 b_L| \leq C(M)(b_1 D(t) + b_1^{L+3}) + C(M)\sqrt{\mathcal{E}_{s_L}}. \tag{3.3.54}$$

Step 4: *reinjection of the bounds.* Summing (*3.3.54*), (*3.3.52*) and (*3.3.45*) we find that:

$$D(t) = O(\sqrt{\mathcal{E}_{s_L}} + b_1^{L+3}). \tag{3.3.55}$$

This allows us to go back to the previous estimate of the law of λ (*3.3.45*), of the b_i's (*3.3.52*), and of b_L (*3.3.54*) to obtain the desired estimates (*3.3.36*) and (*3.3.37*). □

3.3.4 Improved modulation equation for b_L

We have seen in remark 3.3.4 that the control over the evolution of b_L we found in the last Lemma 3.3.3 is not sufficient. In fact, this is because our orthogonality conditions approximate a true orthogonal decomposition (which would have been to ask $\langle \boldsymbol{\varepsilon}, \boldsymbol{T}_i\rangle = 0$ and would have implied the vanishing of the bad term $\langle \boldsymbol{H\varepsilon}, \boldsymbol{T}_L\rangle = \langle \boldsymbol{\varepsilon}, -\boldsymbol{T}_{L-1}\rangle = 0$). Nevertheless, we are able to determine which part of $\boldsymbol{\varepsilon}$ contributes in the worst way to the evolution of b_L and to control it. This is the subject of the following lemma:

Lemma 3.3.5 (Improved modulation equation for b_L:). *We recall that B_0 is given by (3.1.11). Assume all the constants involved in Proposition 3.3.2 are fixed in their range except s_0. Then for s_0 large enough there holds*[16] *for $s_0 \leq s < s^*$:*

$$\begin{aligned} &\left| b_{L,s} + (L-\alpha)b_1 b_L - \frac{d}{ds}\left[\frac{\langle \boldsymbol{H}^L\boldsymbol{\varepsilon}, \chi_{B_0}\boldsymbol{\Lambda Q}\rangle}{\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}}\left(\frac{\partial S_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\right]\right| \\ \leq\ & \frac{1}{B_0^{\delta_0}}C(L,M)\left[\sqrt{\mathcal{E}_{s_L}} + b_1^{L+1-\delta_0+g'}\right], \end{aligned} \tag{3.3.56}$$

where g' is the gain in the asymptotic of the profiles S_i defined by (3.1.4).

Proof of Lemma 3.3.5 Step 1: Expression of the time derivative of the numerator. We first compute the time evolution of the numerator of the new term we introduced in (3.3.5): $\langle \boldsymbol{H}^L\boldsymbol{\epsilon}, \chi_{B_0}\boldsymbol{\Lambda Q}\rangle$. From the evolution equation for $\boldsymbol{\varepsilon}$ given by (*3.3.30*):

$$\frac{d}{ds}\left(\langle \boldsymbol{H}^L\boldsymbol{\varepsilon}, \chi_{B_0}\boldsymbol{\Lambda Q}\rangle\right) = \langle \boldsymbol{H}^L\boldsymbol{\varepsilon}_s, \chi_{B_0}\boldsymbol{\Lambda Q}\rangle + \langle \boldsymbol{H}^L\boldsymbol{\varepsilon}, b_{1,s}y\partial_y\chi(\frac{y}{B_0})\boldsymbol{\Lambda Q}\rangle. \tag{3.3.57}$$

[16] The denominator being non null from (*3.3.69*).

We will now compute each term in the right hand side. We first estimate the second term. From the modulation equation (*3.3.36*), and under the bootstrap assumptions (*3.3.27*) one has $|b_{1,s}|\leq Cb_1^2$. We use the expression of $\boldsymbol{H}^L$ given by (*3.2.25*), L being odd, and the coercivity of $\mathcal{E}_{s_L}$, see Corollary 3.D.4:

$$\begin{array}{rcl} \left|\langle \boldsymbol{H}^L\boldsymbol{\varepsilon}, b_{1,s}\partial_y\chi(\frac{y}{B_0})\boldsymbol{\Lambda Q}\rangle\right| & = & \left|\int (-1)^{\frac{L+1}{2}}\mathcal{L}^{\frac{L-1}{2}}\varepsilon^{(2)}b_{1,s}y\partial_y\chi(\frac{y}{B_0})\Lambda^{(1)}Q\right| \\ & \leq & Cb_1^2\int_{B_0}^{2B_0}|\varepsilon^{(2)}_{L-1}|\frac{y}{y^\gamma}=Cb_1^2\int_{B_0}^{2B_0}\frac{|\varepsilon^{(2)}_{L-1}|}{y^{k_0+1}}y^{k_0-\gamma+2} \\ & \leq & C(M)b_1^2\sqrt{\mathcal{E}_{S_L}}\left(\int_{B_0}^{2B_0}y^{2k_0-2\gamma+4}\right)^{\frac12} \\ & \leq & C(M)b_1^2\sqrt{\mathcal{E}_{S_L}}b_1^{-(2k_0+\delta_0+2)} \\ & \leq & C(M)\sqrt{\mathcal{E}_{S_L}}b_1^{-(2k_0+\delta_0)} \end{array} \qquad (3.3.58)$$

where we used the asymptotic (*3.2.3*) of $\Lambda^{(1)}Q$ (and we recall that f_k stands for the k-th adapted derivative of f given by (*3.2.21*)). We now aim at estimating the other term in the right hand side of (*3.3.57*). We compute using again the expression of $\boldsymbol{H}^L$ given by (*3.2.25*) and the fact that L is odd:

$$\begin{array}{rl} & (-1)^{\frac{L+1}{2}}\langle \boldsymbol{H}^L\boldsymbol{\varepsilon}_s,\chi_{B_0}\boldsymbol{\Lambda Q}\rangle=\int\mathcal{L}^{\frac{L-1}{2}}\varepsilon_s^{(2)}\Lambda^{(1)}Q \\ = & \int\chi_{B_0}\Lambda^{(1)}Q\left(-\mathcal{L}\varepsilon^{(1)}+\frac{\lambda_s}{\lambda}\Lambda^{(2)}\varepsilon^{(2)}-\tilde{Mod}(t)^{(2)}-\tilde{\psi}_b^{(2)}+NL(\varepsilon)+L(\varepsilon)\right)_{L-1}, \end{array} \qquad (3.3.59)$$

and we now estimate all the terms in the right hand side.

• $\mathcal{L}\varepsilon^{(1)}$ ***term:*** There holds using coercivity and the fact that $A(\Lambda^{(1)}Q)=0$:

$$\begin{array}{rcl} \left|\int\chi_{B_0}\Lambda^{(1)}Q(\mathcal{L}\varepsilon^{(1)})_{L-1}\right| & \leq & C\int_{B_0}^{2B_0}\frac{1}{y^{\gamma+1}}|\varepsilon^{(1)}_L|\leq C\int_{B_0}^{2B_0}\frac{\varepsilon^{(1)}_L}{y^{k_0+1}}y^{k_0-\gamma} \\ & \leq & C(M)\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)}. \end{array} \qquad (3.3.60)$$

•$\Lambda^{(2)}\varepsilon^{(2)}$ ***term:*** Again, using the same arguments, as $\frac{|\lambda_s|}{\lambda}\leq Cb_1$ from (*3.3.36*):

$$\begin{array}{rcl} \left|\int\chi_{B_0}\Lambda^{(1)}Q\frac{\lambda_s}{\lambda}(\Lambda^{(2)}\varepsilon^{(2)})_{L-1}\right| & \leq & Cb_1\int_{B_0}^{2B_0}\frac{1}{y^{\gamma+1}}|\epsilon^{(2)}_{L-1}|\leq C(M)b_1\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+1+\delta_0)} \\ & \leq & C(M)\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)}. \end{array} \qquad (3.3.61)$$

• $\tilde{\psi}_b$ ***term:*** Because we are in the zone $\sim B_0$ we do not see the bad tail. We can then use the improved bound of Proposition 3.2.12:

$$\begin{array}{rcl} \left|\int\chi_{B_0}\Lambda^{(1)}Q(\tilde{\psi_b}^{(2)})_{L-1}\right| & = & \left|\int\chi_{B_0}\Lambda^{(1)}Q(\psi_b^{(2)})_{L-1}\right| \\ & \leq & \parallel\Lambda^{(1)}Q\parallel_{L^2(\leq 2B_0)}\parallel\psi^{(2)}_{b,L-1}\parallel_{L^2(\leq 2B_0)} \\ & \leq & Cb_1^{L+1-2k_0-2\delta_0+g'}. \end{array} \qquad (3.3.62)$$

• $NL(\varepsilon)$ ***term:*** By duality we put all the derivatives on $\Lambda^{(1)}Q$:

$$\left|\int\chi_{B_0}\Lambda^{(1)}Q(NL(\varepsilon))_{L-1}\right| = \left|\int(\chi_{B_0}\Lambda^{(1)}Q)_{L-1}NL(\varepsilon)\right|\leq C\int_{B_0}^{2B_0}\frac{1}{y^{\gamma+L-1}}|NL(\varepsilon)|.$$

We know that $NL(\varepsilon)$ is a sum of terms of the form: $Q^{p-k}\varepsilon^{(1)k}$ for $k>2$. So from the asymptotic (*3.2.1*) of Q and using coercivity:

$$\begin{array}{rcl} \left|\int_{B_0}^{2B_0}\frac{1}{y^{\gamma+L-1}}Q^{p-k}\varepsilon^{(1)k}\right| & \leq & C\parallel\varepsilon^{(1)}\parallel_{L^\infty}^{k-1}\int_{B_0}^{2B_0}\frac{|\varepsilon^{(1)}|}{y^{\gamma+L-1+\frac{2}{p-1}(p-k)}} \\ & \leq & C(M)\parallel\varepsilon^{(1)}\parallel_{L^\infty}^{k-1}\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)}b_1^{-2+\frac{2}{p-1}(p-k)}. \end{array}$$

We now use the estimate provided by Lemma 3.E.1:

$$\begin{array}{rcl} \| \varepsilon^{(1)} \|_{L^\infty} & \leq & C(M,K_1,K_2)\sqrt{\mathcal{E}_\sigma} b_1^{\frac{d}{2}-\sigma+\frac{\frac{2}{p-1}\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)} \\ & \leq & C(M,K_1,K_2)\left(\frac{\mathcal{E}_\sigma}{b_1^{\sigma-s_c}}\right) b_1^{\frac{2}{p-1}+\frac{\frac{2}{p-1}\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}. \end{array}$$

Therefore:

$$\left|\int_{B_0}^{2B_0} \frac{Q^{p-k}\varepsilon^{(1)k}}{y^{\gamma+L-1}}\right| \leq C(M,K_1,K_2)\left(\frac{\mathcal{E}_\sigma}{b_1^{\sigma-s_c}}\right)^{k-1}\sqrt{\mathcal{E}_{s_L}} b_1^{-(2k_0+\delta_0)+\frac{2(k-1)\alpha}{(p-1)L}+O\left(\frac{\sigma-s_c}{L}\right)}.$$

Under the bootstrap estimate, for s_0 small enough this gives:

$$\left|\int \chi_{B_0}\Lambda^{(1)}QNL(\varepsilon^{(1)})_{L-1}\right| \leq \sqrt{\mathcal{E}_{s_L}} b_1^{-(2k_0+\delta_0)}. \tag{3.3.63}$$

Indeed, the constant s_0 being chosen after all the other constants, we can increase s_0 to erase the dependence on the other constant in the preceding equation.

- $L(\varepsilon)$ ***term:***

$$\left|\int \chi_{B_0}\Lambda^{(1)}Q(L(\varepsilon))_{L-1}\right| \leq C\int_{B_0}^{2B_0}\frac{1}{y^{\gamma+L-1}}|Q_b^{p-1}-Q^{p-1}||\varepsilon^{(1)}|.$$

We use the degeneracy of the potential: $Q_b^{p-1}-Q^{p-1}\leq \frac{C}{1+y^{2+\alpha}}$ to estimate:

$$\begin{array}{rcl} \left|\int \chi_{B_0}\Lambda^{(1)}Q(L(\varepsilon))_{L-1}\right| & \leq & C\int_{B_0}^{2B_0}\frac{|\varepsilon^{(1)}|}{y^{\gamma+L-1+2+\alpha}} \\ & \leq & C(M)\sqrt{\mathcal{E}_{s_L}} b_1^{-(2k_0+\delta_0)} b_1^{\alpha}. \end{array} \tag{3.3.64}$$

- $M\tilde{od}(t)^{(2)}$ ***term:*** From the localization of the T_i' and S_i's ((*3.2.28*) and (*3.2.43*)):

$$\begin{array}{rl} & \int M\tilde{od}(t)^{(2)}_{L-1}\chi_{B_0}\Lambda^{(1)}Q \\ = & \int\left(\sum_{i=1}^L(b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})(T_i\delta_{i\text{mod}2,1}+\sum\limits_{j=i+1 \text{ odd}}^{L+2}\frac{\partial S_j}{\partial b_i})\right)_{L-1}\chi_{B_0}\Lambda^{(1)}Q \\ & -\int(\frac{\lambda_s}{\lambda}+b_1)\Lambda^{(2)}\tilde{\alpha}_b^{(2)})_{L-1}\chi_{B_0}\Lambda^{(1)}Q \\ = & (b_{L,s}+(L-\alpha)b_1b_L)\int(T_L+\frac{\partial S_{L+2}}{\partial b_L})_{L-1}\chi_{B_0}\Lambda^{(1)}Q \\ & +\int\left(\sum_{i=1}^{L-1}(b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})(T_i\delta_{i\text{mod}2,1}+\sum\limits_{j=i+1 \text{ odd}}^{L+2}\frac{\partial S_j}{\partial b_i})\right)_{L-1}\chi_{B_0}\Lambda^{(1)}Q \\ & -\int(\frac{\lambda_s}{\lambda}+b_1)(\Lambda^{(2)}\tilde{\alpha}_b^{(2)})_{L-1}\chi_{B_0}\Lambda^{(1)}Q. \end{array}$$

We compute from the fact that $\boldsymbol{H}(\boldsymbol{T_L})=(-1)^L\boldsymbol{\Lambda Q}$:

$$\int(T_L)_{L-1}\chi_{B_0}\Lambda^{(1)}Q=(-1)^{\frac{L-1}{2}}\int|\Lambda^{(1)}Q|^2\chi_{B_0}.$$

For $i<L$, as $(T_i)_{L-1}=0$ we have:

$$\begin{array}{rl} & \left|\int(T_i\delta_{i\text{mod}2,1}+\sum_{j\geq i+1 \text{ odd}}^{L+2}\frac{\partial S_j^{(2)}}{\partial b_i})_{L-1}\chi_{B_0}\Lambda^{(1)}Q\right| \\ = & \left|\int\sum_{j\geq i+1}^{L+2}\left(\frac{\partial S_j^{(2)}}{\partial b_i}\right)_{L-1}\chi_{B_0}\Lambda^{(1)}Q\right| \leq Cb_1^{L-i+g'}b_1^{-(2k_0+2\delta_0)}. \end{array}$$

And for the last term there holds the bound:

$$\left|\int(\Lambda^{(2)}\tilde{\alpha}_b^{(2)})_{L-1}\chi_{B_0}\Lambda^{(1)}Q\right| \leq Cb_1^{L-(2k_0+2\delta_0)}$$

We then conclude, using the majoration obtained in the previous Lemma 3.3.3 for the evolution of the b_i's and λ, that for the $\tilde{Mod}(t)$ term:

$$\begin{array}{rl} & \int \tilde{Mod}(t)^{(2)}_{L-1}\chi_{B_0}\Lambda^{(1)}Q \\ = & (b_{L,s}+(L-\alpha)b_1b_L)\left((-1)^{\frac{L-1}{2}}\int(\Lambda^{(1)}Q)^2\chi_{B_0}+\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\chi_{B_0}\Lambda^{(1)}Q\right) \\ & +O(\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)}+b_1^{L+3-(2k_0+\delta_0)}) \end{array} \quad (3.3.65)$$

(From now on we use the $O()$ notation, the constants hidden depending only on M). We now collect all the estimates (*3.3.60*), (*3.3.61*), (*3.3.62*), (*3.3.63*), (*3.3.64*) and (*3.3.65*), inject them in (*3.3.59*) to find that the first term in the right hand side of (*3.3.57*) is:

$$\begin{array}{rl} & \langle \boldsymbol{H}^L\boldsymbol{\varepsilon}_s,\chi_{B_0}\boldsymbol{\Lambda Q}\rangle+O(\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)})+O(b_1^{L+1-2k_0-2\delta_0+g'}). \\ = & (b_{L,s}+(L-\alpha)b_1b_L)\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle \end{array} \quad (3.3.66)$$

With the two computations (*3.3.66*) and (*3.3.58*), the time evolution of the numerator given by (*3.3.57*) is now:

$$\begin{array}{rl} & \frac{d}{ds}\langle \boldsymbol{H}^L\boldsymbol{\varepsilon},\chi_{B_0}\boldsymbol{\Lambda Q}\rangle+O(\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)})+O(b_1^{L+1-2k_0-2\delta_0+g'}). \\ = & (b_{L,s}+(L-\alpha)b_1b_L)\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle \end{array} \quad (3.3.67)$$

<u>Step 2:</u> end of the computation. We have thanks to our previous estimate (*3.3.67*):

$$\begin{array}{rl} & \frac{d}{ds}\left[\frac{\langle \boldsymbol{H}^L\boldsymbol{\varepsilon},\chi_{B_0}\boldsymbol{\Lambda Q}\rangle}{\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\right] \\ = & (b_{L,s}+(L-\alpha)b_1b_L)+\frac{O(\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)}+b_1^{L+1-(2k_0+2\delta_0)+g'})}{\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle} \\ & -\frac{\langle \boldsymbol{H}^L\boldsymbol{\varepsilon},\chi_{B_0}\boldsymbol{\Lambda Q}\rangle\times\frac{d}{ds}\left[\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle\right]}{\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle^2}. \end{array} \quad (3.3.68)$$

From the asymptotic of $\Lambda^{(1)}Q$ and S_{L+2}, the denominator has the following size:

$$\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle\sim Cb_1^{-2k_0-2\delta_0}, \quad (3.3.69)$$

for some constant $C>0$. So the second term in the right hand side of (*3.3.68*) is:

$$\left|\frac{O(\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)}+b_1^{L+1-(2k_0+2\delta_0)+g'})}{\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\right|\leq C(M)\left(b_1^{-\delta_0}\sqrt{\mathcal{E}_{s_L}}+b_1^{L+1+g'}\right) \quad (3.3.70)$$

We now estimate the third term in the right hand side of (*3.3.68*). We have by coercivity of the adapted norm:

$$|\langle \boldsymbol{H}^L\boldsymbol{\varepsilon},\chi_{B_0}\boldsymbol{\Lambda Q}\rangle|\leq C\int_{B_0}^{2B_0}\frac{\varepsilon(2)}{y^{\gamma+L-1}}\leq C(M)\sqrt{\mathcal{E}_{s_L}}b_1^{-(2k_0+\delta_0)-1}. \quad (3.3.71)$$

As $\frac{\partial S^{(2)}_{L+2}}{\partial b_L}$ does not depend on b_L, we obtain using the modulation bound (*3.3.36*) for b_1,...,b_{L-1}:

$$\left| \frac{\frac{d}{ds}\left[\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}} \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L} \right)_{L-1} \right\rangle \right]}{\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}} \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L} \right)_{L-1} \right\rangle} \right| \leq C(M)b_1.$$

The third term in the right hand side of (*3.3.68*) then admits the bound:

$$\begin{array}{ll} & \left| \frac{\langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, \chi_{B_0}\boldsymbol{\Lambda Q}\rangle \times \frac{d}{ds}\left[\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}} \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L} \right)_{L-1} \right\rangle \right]}{\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}} \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L} \right)_{L-1} \right\rangle^2} \right| \\ \leq & C(M) b_1^{-\delta_0} \sqrt{\mathcal{E}_{s_L}}. \end{array} \qquad (3.3.72)$$

The identity (*3.3.68*), with the bounds on the terms (*3.3.70*) and (*3.3.72*), gives:

$$\begin{aligned} \frac{d}{ds}\left[\frac{\langle \boldsymbol{H}^L \boldsymbol{\epsilon}, \chi_{B_0}\boldsymbol{\Lambda Q}\rangle}{\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}} \left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L} \right)_{L-1} \right\rangle} \right] &= (b_{L,s} + (L-\alpha)b_1 b_L) \\ &\quad + O(\sqrt{\mathcal{E}_{s_L}} b_1^{\delta_0} + b_1^{L+1+g'}), \end{aligned}$$

the constant hidden in the $O()$ depending on M (and L of course but we do not track the dependence on this constant anymore). $\square$

3.3.5 Lyapunov monotonicity for the low Sobolev norm:

As it appeared in the previous subsections, the key estimate in our analysis is the one concerning the high Sobolev norm. Nonetheless, to have an idea on how the lower derivatives behave, and to close an estimate for the nonlinear term in the next section, we start by computing an energy estimate on the low Sobolev norm. We define:

$$\nu := \frac{\alpha}{\ell - \alpha}, \qquad (3.3.73)$$

so that $1 + \nu = \frac{\ell}{\ell - \alpha}$ and that the condition (*3.3.27*) for $\mathcal{E}_\sigma$ can be rewriten as:

$$\mathcal{E}_\sigma \leq K_2 b_1^{2(\sigma - s_c)(1+\nu)} \qquad (3.3.74)$$

Proposition 3.3.6. (Lyapunov monotonicity for the low Sobolev norm:) *Assume all the constants involved in Proposition 3.3.2 are fixed in their range, except* s_0 *and* η. *Then for* s_0 *large enough and* η *small enough there holds for* $s_0 \leq s < s^*$:

$$\frac{d}{dt}\left\{ \frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma - s_c)}} \right\} \leq \frac{b_1 \sqrt{\mathcal{E}_\sigma} b_1^{(\sigma - s_c)(1+\nu)}}{\lambda^{2(\sigma - s_c)+1}} \left[b_1^{\frac{\alpha}{2L} + O\left(\frac{\sigma - s_c}{L}\right)} + b_1^{\frac{\alpha}{2L} + O\left(\frac{\sigma - s_c}{L}\right)} \sum_{k=2}^{p} \left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma - s_c}} \right)^{k-1} \right] \qquad (3.3.75)$$

(the norm $\mathcal{E}_\sigma$ *was defined in (3.3.14)).*

Proof of Proposition 3.3.6 To prove this proposition we will compute the derivative with respect to time of $\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}$ and estimate it in the trapped regime using (*3.3.27*) and the size of the error given by Proposition 3.2.14. From the evolution of $\boldsymbol{w}$ given by (*3.3.32*) we first compute the following identity:

$$\begin{array}{rl} & \frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} = \frac{d}{dt}\left\{\int |\nabla^\sigma w^{(1)}|^2 + \int |\nabla^{\sigma-1} w^{(2)}|^2\right\} \\ = & \int \nabla^\sigma w^{(1)}.\nabla^\sigma (w^{(2)} + \frac{1}{\lambda}(-\tilde{Mod}(t)^{(1)}_{\frac{1}{\lambda}} - \tilde{\psi}_b{}^{(1)}_{\frac{1}{\lambda}})) \\ & + \int \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1}(-\mathcal{L}w^{(1)} + \frac{1}{\lambda}(-\tilde{Mod}(t)^{(2)}_{\frac{1}{\lambda}} - \tilde{\psi}_b{}^{(2)}_{\frac{1}{\lambda}}) + NL(w) + L(w)). \end{array} \qquad (3.3.76)$$

Step 1: estimate on each term. We will now estimate everything in the right hand side of (*3.3.76*).

• *Linear terms:* Because the norm we are using is adapted to a wave equation we have:

$$\begin{array}{rcl} \int \nabla^\sigma w^{(1)}.\nabla^\sigma w^{(2)} - \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1}\mathcal{L}w^{(1)} & = & \int \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1}(pQ^{p-1}_{\frac{1}{\lambda}} w^{(1)}) \\ & \leq & \parallel \nabla^\sigma w^{(2)} \parallel_{L^2} \parallel \nabla^{\sigma-2}(Q^{p-1}_{\frac{1}{\lambda}} w^{(1)}) \parallel_{L^2} . \end{array}$$

We now use the asymptotic behavior $Q^{p-1} \sim \frac{c}{x^2}$ $(c>0)$ and the weighted Hardy estimate from Lemma 3.C.2:

$$\parallel \nabla^{\sigma-2}(Q^{p-1}_{\frac{1}{\lambda}} w^{(1)}) \parallel_{L^2} \leq C \parallel \nabla^\sigma w^{(1)} \parallel_{L^2} = C\frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{\sigma-s_c}}.$$

The other term is estimated by interpolation. Indeed as $\parallel \nabla^{s_L-1}\varepsilon^{(2)} \parallel^2_{L^2} \leq c\mathcal{E}_{s_L}$ from Corollary 3.D.4:

$$\parallel \nabla^\sigma w^{(2)} \parallel_{L^2} \leq \frac{C(M)}{\lambda^{\sigma-s_c+1}} \sqrt{\mathcal{E}_\sigma}^{1-\frac{1}{s_L-\sigma}} \sqrt{\mathcal{E}_{s_L}}^{\frac{1}{s_L-\sigma}}$$

We have the following estimate under the bootstrap conditions (*3.3.27*) :

$$\sqrt{\mathcal{E}_\sigma}^{1-\frac{1}{s_L-\sigma}} \sqrt{\mathcal{E}_{s_L}}^{\frac{1}{s_L-\sigma}} \leq C(K_1,K_2,M) b_1^{(\sigma-s_c)(1+\nu)} b_1^{(\frac{1}{s_L-\sigma})(L+(1-\delta_0)(1+\eta)-(\sigma-s_c)(1+\nu))}$$

and from: $\frac{L+(1-\delta_0)(1+\eta)-(\sigma-s_c)(1+\nu)}{s_L-\sigma} = 1 + \frac{(1-\delta_0)\eta+\alpha}{L} + O(\frac{\sigma-s_c}{L})$ we conclude that:

$$\begin{array}{rl} & \left|\int \nabla^\sigma w^{(1)}.\nabla^\sigma w^{(2)} - \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1}\mathcal{L}w^{(1)}\right| \\ \leq & C(K_1,K_2,M)\frac{\sqrt{\mathcal{E}_\sigma} b_1^{(\sigma-s_c)(1+\nu)}}{\lambda^{2(\sigma-s_c)}} \frac{b_1}{\lambda} b_1^{\frac{\alpha}{L}+O(\frac{\sigma-s_c}{L})}. \end{array} \qquad (3.3.77)$$

• $\tilde{\boldsymbol{Mod}}(t)$ *terms:* We only treat the $\tilde{Mod}(t)^{(2)}$ terms, the computation being the same for the first coordinate.

$$\left|\frac{1}{\lambda}\int \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1}\tilde{Mod}(t)^{(2)}_{\frac{1}{\lambda}}\right| \leq \frac{1}{\lambda^{2(\sigma-s_c)}}\frac{1}{\lambda}\sqrt{\mathcal{E}_\sigma} \parallel \nabla^{\sigma-1}\tilde{Mod}(t) \parallel_{L^2} .$$

We compute thanks to the previous estimate on the modulation, see Lemma 3.3.3:

$$\begin{array}{rl} & \parallel \nabla^{\sigma-1}\tilde{Mod}^{(2)} \parallel_{L^2} \\ \lesssim & (\sqrt{\mathcal{E}_{s_L}} + b^{L+3})\left(\sum_{i<j\leq L+2} \parallel \nabla^{\sigma-1}\left(\chi_{B_1}\frac{\partial S_j^{(2)}}{\partial b_i}\right) \parallel_{L^2} + \sum_0^L \parallel \chi_{B_1}\nabla^{\sigma-1} T_i^{(2)} \parallel_{L^2}\right) \\ = & (\sqrt{\mathcal{E}_{s_L}} + b^{L+3}) b_1^{(1+\eta)(-k_0-\delta_0-L+\sigma)} \\ \leq & C(M) b_1^{\alpha+(1-\delta_0)+(\sigma-s_c)+\eta(1-\delta_0+\alpha+(\sigma-s_c)-L)}. \end{array}$$

Hence, treating similarly the other coordinate:

$$\left|\frac{1}{\lambda}\int \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1}\tilde{Mod}(t)^{(2)}_{\frac{1}{\lambda}} + \nabla^\sigma w^{(1)}.\nabla^\sigma \tilde{Mod}(t)^{(1)}_{\frac{1}{\lambda}}\right| \leq C(M)\frac{b_1\sqrt{\mathcal{E}_\sigma} b_1^{(\sigma-s_c)+\alpha}}{\lambda^{2(\sigma-s_c)+1}}. \qquad (3.3.78)$$

• $\tilde{\psi}_b$ *term:* Again we just compute for the first coordinate $\tilde{\psi}_b^{(1)}$, because we can treat the second one exactly the same way.

$$\left|\frac{1}{\lambda}\int \nabla^\sigma w^{(1)}.\nabla^\sigma \tilde{\psi}^{(1)}_{b,\frac{1}{\lambda}}\right| \leq \frac{1}{\lambda^{2(\sigma-s_c)}}\frac{1}{\lambda}\sqrt{\mathcal{E}_\sigma} \parallel \nabla^\sigma \tilde{\psi}_b^{(1)} \parallel_{L^2} .$$

We can estimate using proposition 3.2.14:

$$\parallel \nabla^\sigma \tilde{\psi}_b^{(1)} \parallel_{L^2} \leq Cb_1^{(1-\delta_0)+\sigma-k_0-C\eta} = Cb_1^{(\sigma-s_c)+\alpha-C\eta+1}.$$

Hence for η small enough:

$$\left|\frac{1}{\lambda}\int \nabla^\sigma w^{(1)}.\nabla^\sigma \tilde{\psi}^{(1)}_{b,\frac{1}{\lambda}}\right| \leq \frac{C}{\lambda^{(\sigma-s_c)}}\frac{b_1}{\lambda}\sqrt{\mathcal{E}_\sigma} b_1^{\sigma-s_c} b_1^{\frac{3\alpha}{4}}.$$

The same computation for the second coordinate gives the same result, hence the error's contribution is:

$$\left|\frac{1}{\lambda}\int \nabla^\sigma w^{(1)}.\nabla^\sigma \tilde{\psi}^{(1)}_{b,\frac{1}{\lambda}} + \frac{1}{\lambda}\int \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1} \tilde{\psi}^{(2)}_{b,\frac{1}{\lambda}}\right| \leq \frac{C}{\lambda^{(\sigma-s_c)}}\frac{b_1}{\lambda}\sqrt{\mathcal{E}_\sigma} b_1^{\sigma-s_c} b_1^{\frac{3\alpha}{4}}. \tag{3.3.79}$$

• $L(w)$ *term:* First using Cauchy-Schwarz:

$$\left|\int \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1}(L(w))\right| \leq \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+1}} \parallel \nabla^{\sigma-1}L(w) \parallel_{L^2} .$$

Now we have that $L(w) = (pQ^{p-1} - p\tilde{Q}_b^{p-1})w^{(1)}$. From the asymptotics of the profiles T_i and S_i, the potential here enjoys the following bounds:

$$\left|\partial_y^k (Q^{p-1} - \tilde{Q}_b^{p-1})\right| \leq Cb_1 \frac{1}{y^{1+\alpha-C(L)\eta}} \tag{3.3.80}$$

It allows us to use the fractionnal Hardy estimate from Lemma 3.C.2:

$$\parallel \nabla^{\sigma-1}L(w) \parallel_{L^2} \leq Cb_1 \parallel \nabla^{\sigma+\frac{1}{p-1}} w^{(1)} \parallel_{L^2},$$

because $\sigma + \frac{1}{p-1} < \frac{d}{2}$, and because for η small enough one has: $\alpha - C(L)\eta \geq \frac{1}{p-1}$ (as $\alpha > 2$). In the trapped regime one has by interpolation:

$$\begin{aligned}\parallel \nabla^{\sigma+\frac{1}{p-1}} w^{(1)} \parallel_{L^2} \quad &\leq \quad C(M)\sqrt{\mathcal{E}_\sigma}^{1-\frac{1}{(p-1)(s_L-\sigma)}} \sqrt{\mathcal{E}_{s_L}}^{\frac{1}{(p-1)(s_L-\sigma)}} \\ &\leq C(K_1,K_2,M) \quad b_1^{(\sigma-s_c)(1+\nu)} b_1^{\frac{1}{p-1}+O\left(\frac{1}{L}\right)}.\end{aligned}$$

Therefore we end up with the following bound on the small linear term:

$$\left|\int \nabla^{\sigma-1} w^{(2)}.\nabla^{\sigma-1}(L(w))\right| \leq C(K_1,K_2,M)\frac{b_1\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+1}} b_1^{(\sigma-s_c)(1+\nu)+\frac{1}{p-1}+O\left(\frac{1}{L}\right)} \tag{3.3.81}$$

• NL *term:* We start by integrating by parts and using Cauchy-Schwarz:

$$\begin{aligned}&\left|\int \nabla^{\sigma-1} w^{(2)} \nabla^{\sigma-1} NL(w)\right| \\ \leq \quad & \tfrac{1}{\lambda^{2(\sigma-s_c)+1}} \parallel \nabla^{\sigma-(k-1)(\sigma-s_c)}\varepsilon^{(2)} \parallel_{L^2} \parallel \nabla^{\sigma-2+(k-1)(\sigma-s_c)} NL(\varepsilon) \parallel_{L^2} .\end{aligned} \tag{3.3.82}$$

The first term is estimated via interpolation, and gives under the bootstrap conditions:

$$\begin{aligned}\parallel \nabla^{\sigma-(k-1)(\sigma-s_c)}\varepsilon^{(2)} \parallel_{L^2} \quad &\leq \quad C(M)\sqrt{\mathcal{E}_\sigma}^{1-\frac{1-(k-1)(\sigma-s_c)}{s_L-\sigma}} \sqrt{\mathcal{E}_{s_L}}^{\frac{1-(k-1)(\sigma-s_c)}{s_L-\sigma}} \\ &\leq \quad C(M,K_1,K_2) b_1^{(\sigma-s_c)(1+\nu)+1-(k-1)(\sigma-s_c)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}.\end{aligned} \tag{3.3.83}$$

We now estimate the second one. We know that $NL(\varepsilon)$ is a linear combination of terms of the form: $\tilde{Q}_b^{(1)(p-k)}\varepsilon^{(1)k}$ for $2\leq k\leq p$. We know also that here we have: $\partial_y^j \tilde{Q}_b^{(1)(p-k)} \leq \frac{c}{y^{\frac{2}{p-1}(p-k)+j}}$. So using the weighted and fractional hardy estimate of Lemma 3.C.2:

$$\parallel \nabla^{\sigma-2+(k-1)(\sigma-s_c)}(Q^{p-k}\varepsilon^{(1)k}) \parallel \leq C \parallel \nabla^{\sigma-2+\frac{2}{p-1}(p-k)+(k-1)(\sigma-s_c)}(\varepsilon^{(1)k}) \parallel_{L^2} .$$

We let $\tilde{\sigma}=E[\sigma-2+\frac{2}{p-1}(p-k)+(k-1)(\sigma-s_c)]$ so that:

$$\sigma-2+\frac{2}{p-1}(p-k)+(k-1)(\sigma-s_c)=\tilde{\sigma}+\delta_\sigma,$$

with $0\leq \delta_\sigma<1$. Developing the integer part of the derivative yields:

$$\parallel \nabla^{\sigma-2+\frac{2}{p-1}(p-k)+(k-1)(\sigma-s_c)}(\varepsilon^{(1)k}) \parallel_{L^2} = \parallel \nabla^{\delta_\sigma}(\nabla^{\tilde{\sigma}}(\varepsilon^{(1)k})) \parallel_{L^2} .$$

We develop the $\nabla^{\tilde{\sigma}}(v^{(1)k})$ term: it is a linear combination of terms of the form:

$$\prod_{j=1}^k \nabla^{l_j}\varepsilon^{(1)},$$

for $\sum_{j=1}^k l_j=\tilde{\sigma}$. We recall the standard commutator estimate:

$$\parallel \nabla^{\delta_\sigma}(uv) \parallel_{L^q} \leq C \parallel \nabla^{\delta_\sigma}u \parallel_{L^{p_1}} \parallel v \parallel_{L^{p_2}} + C \parallel \nabla^{\delta_\sigma}v \parallel_{L^{p_1'}} \parallel u \parallel_{L^{p_2'}},$$

for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_1'}+\frac{1}{p_2'}=\frac{1}{q}$, provided $1<q,p_1,p_1'<+\infty$ and $1\leq p_2,p_2'\leq+\infty$. So by iteration we have that:

$$\parallel \nabla^{\sigma-2}(Q^{p-k}\varepsilon^{(1)k}) \parallel_{L^2} \leq C\sum_{j=1}^k\prod_{i=1}^k \parallel \nabla^{l(j)_i}\varepsilon^{(1)} \parallel_{L^{p(j)_i}},$$

with $l(j)_i=l_i+\delta_\sigma\delta_{i=j}$ and with $\sum\frac{1}{p(j)_i}=\frac{1}{2}$. We have for any i and j: $l(j)_i<\sigma$. Hence we can use Sobolev injection to find:

$$\nabla^{l(j)_i}\varepsilon^{(1)}\in L^{p(j)_i^*},$$

for $p(j)_i^*=\frac{2d}{d-2\sigma+2l(j)_i}$. We compute (the strategy was designed to obtain this):

$$\frac{1}{p(j)^*}:=\sum_{i=1}^k\frac{1}{p(j)_i^*}=\frac{1}{2}.$$

So we take $p(j)_i=p(j)^*i$. We then have:

$$\parallel \nabla^{\sigma-2+\frac{2}{p-1}(p-k)+(k-1)(\sigma-s_c)}(NL(\varepsilon)) \parallel_{L^2} \leq C\sqrt{\mathcal{E}_\sigma}^k. \tag{3.3.84}$$

The Cauchy-Schwarz inequality (*3.3.82*), with the estimates for the two terms (*3.3.83*) and (*3.3.84*) give eventually:

$$\begin{aligned}&\left|\int \nabla^{\sigma-1}w^{(2)}.\nabla^{\sigma-1}(NL(w))\right|\\ \leq\ & \frac{C(K_1,K_2,M)b_1\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+1}}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-1} b_1^{(\sigma-s_c)(1+\nu)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}.\end{aligned} \tag{3.3.85}$$

Step 2: Gathering the bounds. We have made the decomposition (*3.3.76*) and have found an upper bound for all terms in the right hand side in (*3.3.77*), (*3.3.78*), (*3.3.79*), (*3.3.81*) and (*3.3.85*). So we get:

$$\begin{array}{rcl} \frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} & \leq & \frac{C(K_1,K_2,M)}{\lambda^{2(\sigma-s_c)}}\frac{b_1}{\lambda}\sqrt{\mathcal{E}_\sigma}b_1^{(\sigma-s_c)(1+\nu)}\times\Big(b_1^{\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}+b_1^{\alpha-\nu(\sigma-s_c)} \\ & & +b_1^{\frac{3}{4}\alpha-\nu(\sigma-s_c)}+b_1^{\frac{1}{p-1}+O\left(\frac{1}{L}\right)}+b_1^{\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sum_{k=2}^p\left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-1}\Big), \end{array} \quad (3.3.86)$$

We see that if one choose $\sigma-s_c$ small enough there holds:

$$\frac{\alpha}{2L}<\min\left(\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right),\alpha-\nu(\sigma-s_c),\frac{3}{4}\alpha-\nu(\sigma-s_c),\frac{1}{p-1}+O\left(\frac{1}{L}\right)\right). \quad (3.3.87)$$

In the trapped regime we recall that $b_1\sim\frac{c_1}{s}$ is small, so that $b_1^a\ll b_1^b$ if $b<a$. Consequently by taking s_0 big enough to "erase" the constants, (*3.3.86*) combined with (*3.3.87*) give the result of the proposition. □

3.3.6 Lyapunov monotonicity for the high Sobolev norm:

We have seen that in order to control the evolution of the parameters, we need to control the high Sobolev norm $\mathcal{E}_{s_L}$. Indeed, the law of b_L is computed when projecting the dynamics onto $\boldsymbol{H}^{*L}\boldsymbol{\Phi}_M$, which involves at least to control L derivative. Why do we look at the k_0+1+L-th derivative? Because it is only when deriving at least k_0+1 more times that we gain something on the error term $\tilde{\boldsymbol{\psi}}_b$: the η gain (see proposition 3.2.14)[17]. However, if we look at a higher order derivative ($>k_0+L+1$) we loose the control of the solution by lack of Hardy inequalities (Corollary 3.D.4 does not work at a higher level of regularity). For these reasons, the choice $L+k_0+1$ is sharp.

We state here a control on the evolution of $\mathcal{E}_{s_L}$, and prove it. We will not be able to estimate it directly, a local part will require the study of a Morawetz type quantity. This is the subject of the following subsection.

Proposition 3.3.7. (Lyapunov monotonicity for the high Sobolev norm:) *Suppose all the constants of Proposition 3.3.2 are fixed, except s_0 and η. Then for s_0 large enough and η small enough there holds for $s_0\leq s<s^*$:*

$$\begin{array}{rcl} \frac{d}{dt}\left\{\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}+O\left(\frac{\mathcal{E}_{s_L}b_1^{\eta(1-\delta_0)}}{\lambda^{2(s_L-s_c)}}\right)\right\} & \leq & \frac{C(M)}{\lambda^{2(s_L-s_c)}}\frac{b_1}{\lambda}\Big[\mathcal{E}_{s_L}b_1^{\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c}{L}\right)}\sum_{k=2}^{p}\left[\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right]^{k-1} \\ & & +C(N)\mathcal{E}_{s_L,loc}+\frac{\mathcal{E}_{s_L}}{N^{\frac{\delta_0}{2}}}+\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+\eta)}\Big] \end{array} \quad (3.3.88)$$

the constant hidden in the $O()$ in the left hand side depending on M (the norms $\mathcal{E}_{s_L}$ and $\mathcal{E}_{s_L,loc}$ are defined by (3.3.11) and (3.3.12)).

[17] this is the reason why we need or approximate profile to expand till the zone $y\sim B_1$.

Proof of Proposition 3.3.7: First we compute the time evolution of $\mathcal{E}_{s_L}$:

$$\begin{array}{rl} & \frac{d}{dt}\left(\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}\right)=\frac{d}{dt}\left(\int |w^{(1)}_{k_0+1+L}|^2+|w^{(2)}_{k_0+L}|^2\right) \\ = & \frac{d}{dt}\left(\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}w^{(1)}+w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}w^{(2)}\right) \\ = & 2\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}w^{(1)}_t+w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}w^{(2)}_t \\ & +\sum_{i=1}^{k_0+L+1}\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1-i}w^{(1)} \\ & +\sum_{i=1}^{k_0+L}\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L-i}w^{(2)} \\ = & 2\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}(w^{(2)}-\frac{1}{\lambda}\tilde{\psi}^{(1)}_{b,\frac{1}{\lambda}}-\frac{1}{\lambda}\tilde{Mod}_{\frac{1}{\lambda}}(t)^{(1)}) \\ & +2\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}(-\mathcal{L}_{\frac{1}{\lambda}}w^{(1)}-\frac{1}{\lambda}\tilde{\psi}^{(2)}_{b,\frac{1}{\lambda}}-\frac{1}{\lambda}\tilde{Mod}(t)^{(2)}_{\frac{1}{\lambda}}+L(w)+NL(w)) \\ & +\sum_{i=1}^{k_0+L+1}\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1-i}w^{(1)} \\ & +\sum_{i=1}^{k_0+L}\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L-i}w^{(2)}. \end{array} \tag{3.3.89}$$

We aim at computing the effect of everything in the right hand side.

Step 1: Terms that can be estimated directly. We claim that the quadratic term, the error term and the non-linear term can be estimated directly, transforming (*3.3.89*) into:

$$\begin{array}{l} \frac{d}{dt}\left(\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}\right) \\ =2\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}(-\frac{1}{\lambda}\tilde{Mod}_{\frac{1}{\lambda}}(t)^{(1)})+w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}(-\frac{1}{\lambda}\tilde{Mod}(t)^{(2)}_{\frac{1}{\lambda}}+L(w)) \\ \quad +\sum_{i=1}^{k_0+L+1}\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1-i}w^{(1)} \\ \quad +\sum_{i=1}^{k_0+L}\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L-i}w^{(2)} \\ +\frac{b_1}{\lambda^{2(s_L-s_c)+1}}\left[O(\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+\eta)})+O\left(\mathcal{E}_{s_L}b_1^{\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sum_{k=2}^{p}\left[\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right]^{k-1}\right)\right]. \end{array} \tag{3.3.90}$$

where the constant hidden in the first $O()$ does not depend on K_1 and K_2. We now prove this intermediate estimate.

• *The linear term:* Because this norm is adapted to the flow of the wave equation we have the fundamental cancellation:

$$\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}w^{(2)}+w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}(-\mathcal{L}_{\frac{1}{\lambda}}w^{(1)})=0. \tag{3.3.91}$$

• *the $\tilde{\psi}_b$ term:* It is this term that gives the eventual estimate for $\mathcal{E}_{s_L}$ we want to prove. We recall that f_j, the j-th adapted derivative of a function f, is defined in (*3.2.21*). We just use Cauchy-Schwarz and the estimate provided in Proposition 3.2.14:

$$\begin{array}{rl} & \left|\frac{1}{\lambda}\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}\tilde{\psi}^{(1)}_{b,\frac{1}{\lambda}}+w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}\tilde{\psi}^{(2)}_{b,\frac{1}{\lambda}}\right| \\ = & \left|\frac{1}{\lambda}\int w^{(1)}_{k_0+1+L}\left(\tilde{\psi}^{(1)}_{b,\frac{1}{\lambda}}\right)_{k_0+1+L}+w^{(2)}_{k_0+L}\left(\tilde{\psi}^{(2)}_{b,\frac{1}{\lambda}}\right)_{k_0+L}\right| \\ \leq & C\frac{1}{\lambda}\frac{1}{\lambda^{2(s_L-s_c)}}\sqrt{\mathcal{E}_{s_L}}b_1^{1+L+(1-\delta_0)(1+\eta)}. \end{array} \tag{3.3.92}$$

for a constant C depending on L only.

• *The non linear term:* We begin by Cauchy-Schwarz inequality and by doing a change on the scaling:

$$\left|\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}NL(w)\right|\leq\frac{1}{\lambda^{2(s_L-s_c)+1}}\sqrt{\mathcal{E}_{s_L}}\parallel NL(\varepsilon)_{k_0+L}\parallel_{L^2}.$$

We aim at estimating the last term in the right hand side. We know that $NL(\varepsilon)$ is a sum of terms of the form $\tilde{Q}_b^{(1)(p-k)}\varepsilon^{(1)k}$ for $2\leq k\leq p$. So by now we have to study quantities of the form: $(\tilde{Q}_b^{(1)(p-k)}\varepsilon^{(1)k})_{k_0+L}$. For $l=(l_0,...,l_k)$ we recall the notation: $|l|_1=\sum_{i=0}^k l_i$. Close to the origin, we have from the equivalence between Sobolev norms and adapted norms (Lemma 3.B.2), and because $H^{s_L}(y\leq 1)$ is an algebra:

$$\int_{y\leq 1}(NL(\varepsilon)_{k_0+L})^2\leq C\sum_{k=2}^{p}\parallel \varepsilon^{(1)}\parallel_{H^{s_L}(y\leq 1)}^{2k}\leq C(M)\mathcal{E}_{s_L}\leq C(M)\sqrt{\mathcal{E}_{s_L}}b_1^2.$$

For $y\geq 1$ we notice that when applying A and A^*:

$$(\tilde{Q}_b^{(1)(p-k)}\varepsilon^{(1)k})_{k_0+L}=\sum_{|l|_1=k_0+L}f_{\tilde{l_0}}\partial_y^{l_0}(\tilde{Q}_b^{(1)(p-k)})\prod_{i=1}^k\partial_y^{l_i}\varepsilon^{(1)}.$$

with $f_{\tilde{l_0}}\sim\frac{1}{1+y^{\tilde{l_0}}}$. We have the following asymptotic for the potential:

$$\partial_y^{l_0}(\tilde{Q}_b^{(1)(p-k)})\leq\frac{C}{1+y^{\frac{2}{p-1}(p-k)+l_0}}.$$

So, putting together the decay given by $\partial_y^{l_0}\tilde{Q}_b^{(1)(p-k)}$ and $f_{\tilde{l_0}}$ and renaming $l_0:=l_0+\tilde{l_0}$ we need to study integrals of the following form:

$$\int_{y\geq 1}|NL(\varepsilon)_{s_L-1}|^2\lesssim\sum_{k=2}^{p}\sum_{|l|_1=k_0+L}\int_{y\geq 1}\frac{\prod_{i=1}^k|\partial_y^{l_i}\varepsilon^{(1)}|^2}{1+y^{\frac{4}{p-1}(p-k)+2l_0}}, \tag{3.3.93}$$

for $\sum_{i=0}^k l_i=s_l-1$. We order the coefficient l_i for $1\leq i\leq k$ by increasing order: $l_1\leq l_2\leq ...\leq l_k$.

∘ *Case 1:* we suppose that: $\frac{2}{p-1}(p-k)+l_0+l_k\leq s_L$. It implies the integrability $\frac{\partial_y^{l_k}\varepsilon^{(1)}}{1+y^{l_0+\frac{2}{p-1}(p-k)}}\in L^2(y\geq 1)$ by the improved Hardy inequality from Lemma 3.E.1. There also holds in that case for all $1\leq i\leq k-1$ that $l_i<s_L-\frac{d}{2}$ which implies $\partial_y^{l_i}\varepsilon^{(1)}\in L^\infty(y\geq 1)$. We then estimate:

$$\left\|\frac{\prod_{i=1}^k|\partial_y^{l_i}\varepsilon^{(1)}|^2}{1+y^{\frac{4}{p-1}(p-k)+2l_0}}\right\|_{L^2(y\geq 1)}\leq C\left\|\frac{\partial_y^{l_k}\varepsilon^{(1)}}{1+y^{\frac{2}{p-1}(p-k)+l_0}}\right\|_{L^2(\geq 1)}\prod_{i=1}^{k-1}\left\|\partial_y^{l_i}\varepsilon^{(1)}\right\|\|_{L^\infty(y\geq 1)}.$$

For $1\leq i\leq k-1$, from the equivalence between Laplace and ∂_y derivatives for $y\geq 1$:

$$\partial_y^{l_i}\varepsilon^{(1)}=\sum_{j=0}^{l_i}f_jD^j\varepsilon^{(1)},$$

with $\partial_y^n f_j=O\left(\frac{1}{1+x^{l_i-j+n}}\right)$ for $y\geq 1$, we deduce:

$$\begin{array}{rcl}\parallel\partial_y^{l_i}\varepsilon^{(1)}\parallel_{L^\infty(y\geq 1)} & \leq & C\sum_{j=0}^{l_i}\parallel\frac{D^j\varepsilon^{(1)}}{1+x^{l_i-j}}\parallel_{L^\infty}\\ & \leq & C\sqrt{\mathcal{E}_\sigma}^{\frac{s_L-l_i-\frac{d}{2}}{s_L-\sigma}}\sqrt{\mathcal{E}_{s_L}}^{\frac{l_i+\frac{d}{2}-\sigma}{s_L-\sigma}}.\end{array}$$

We used Sobolev injection, interpolation and coercivity. For $i=k$ from Lemma 3.E.1:

$$\left\|\frac{\partial_y^{l_k}\varepsilon^{(1)}}{1+y^{\frac{2}{p-1}(p-k)+2l_0}}\right\|_{L^2(y\geq 1)}\leq C(M)\sqrt{\mathcal{E}_\sigma}^{\frac{s_L-l_k-l_0-\frac{2}{p-1}(p-k)}{s_L-\sigma}}\sqrt{\mathcal{E}_{s_L}}^{\frac{l_k+l_0+\frac{2}{p-1}(p-k)-\sigma}{s_L-\sigma}}.$$

So that when combining the last two estimates we find:

$$\begin{aligned} & \left\| \frac{\prod_{i=1}^k |\partial_y^{l_i}\varepsilon^{(1)}|^2}{1+y^{\frac{4}{p-1}(p-k)+2l_0}} \right\|_{L^2(y\geq 1)} \\ \leq & \; C\sqrt{\mathcal{E}_\sigma}^{\sum_{i=1}^{k-1}\left(\frac{s_L-l_i-\frac{d}{2}}{s_L-\sigma}\right)+\frac{s_L-l_k-l_0-\frac{2}{p-1}(p-k)}{s_L-\sigma}} \sqrt{\mathcal{E}_{s_L}}^{\sum_{i=1}^{k-1}\left(\frac{l_i+\frac{d}{2}-\sigma}{s_L-\sigma}\right)+\frac{l_k+l_0+\frac{2}{p-1}(p-k)-\sigma}{s_L-\sigma}} \end{aligned}$$

C depending on M. We can calculate the coefficients:

$$\sum_{i=2}^k \left(\frac{s_L-l_i-\frac{d}{2}}{s_L-\sigma}\right)+\frac{s_L-l_k-l_0-\frac{2}{p-1}(p-k)}{s_L-\sigma} = \frac{(k-1)(s_L-\frac{d}{2}+1-\frac{2}{p-1}(p-k)}{s_L-\sigma},$$

$$\sum_{i=1}^{k-1} \left(\frac{l_i+\frac{d}{2}-\sigma}{s_L-\sigma}\right)+\frac{l_k+l_0+\frac{2}{p-1}(p-k)-\sigma}{s_L-\sigma} = 1+\frac{1-(k-1)(\sigma-s_c)}{s_L-\sigma}.$$

Under the bootstrap assumptions (*3.3.27*) it gives:

$$\left\| \frac{\prod_{i=1}^k |\partial_y^{l_i}\varepsilon^{(1)}|^2}{1+y^{\frac{4}{p-1}(p-k)+2l_0}} \right\|_{L^2} \leq C(K_1,K_2,M)b_1\sqrt{\mathcal{E}_{s_L}}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-1} b_1^{\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}. \qquad (3.3.94)$$

○ *Case 2:* if the last condition does not hold, it implies that $l_k+l_0=s_L-1$ with $\frac{2}{p-1}(p-k)-1>0$, and that consequently for $1\leq i\leq k-1$, $l_i=1$. It means that we have to estimate an integral of the following form:

$$\int_{y\geq 1} |\varepsilon^{(1)}|^{2(k-1)} \frac{|\partial_y^{l_k}\varepsilon^{(1)}|^2}{1+y^{\frac{4}{p-1}(p-k)+2l_0}}.$$

We rewrite it as:

$$\int_{y\geq 1} |\varepsilon^{(1)}|^{2(k-2)} \frac{|\varepsilon^{(1)}|^2}{1+y^{\frac{4}{p-1}(p-k)-2}}\frac{|\partial_y^{l_k}\varepsilon^{(1)}|^2}{1+y^{2+2l_0}}.$$

The L^∞ norm of $\varepsilon^{(1)}$ is estimated in Lemma 3.E.1:

$$\| \varepsilon^{(1)} \|_{L^\infty} \leq C(M,K_1,K_2)\sqrt{\mathcal{E}_\sigma} b_1^{\left(\frac{d}{2}-\sigma\right)+\frac{2\alpha}{(p-1)L}+O\left(\frac{\sigma-s_c}{L}\right)}$$

We use the improved Hardy estimate from Lemma 3.E.1 to estimate:

$$\left\| \frac{\partial_y^{l_k}\varepsilon^{(1)}}{1+y^{1+l_0}} \right\|_{L^2(y\geq 1)} \leq C(M)\sqrt{\mathcal{E}_{s_L}}.$$

And finally we use the weighted L^∞ estimate (still from Lemma 3.E.1):

$$\left\| \frac{|\varepsilon^{(1)}|}{1+y^{\frac{2}{p-1}(p-k)-1}} \right\|_{L^\infty} \leq C(M,K_1,K_2)\sqrt{\mathcal{E}_\sigma} b_1^{\frac{2(p-k)}{p-1}-1+\left(\frac{d}{2}-\sigma\right)+\frac{2\alpha}{(p-1)L}+O\left(\frac{(\sigma-s_c)}{L}\right)}.$$

With these last three estimates we have:

$$\begin{aligned} & \left\| |\varepsilon^{(1)}|^{(k-2)}\frac{|\varepsilon^{(1)}|}{1+y^{\frac{2}{p-1}(p-k)-1}}\frac{|\partial_y^{l_k}\varepsilon^{(1)}|}{1+y^{1+l_0}} \right\|_{L^2(y\geq 1)} \\ \lesssim & \; \sqrt{\mathcal{E}_{s_L}}\sqrt{\mathcal{E}_\sigma}^{(k-2)+1} b_1^{(k-2)\left(\frac{d}{2}-\sigma\right)+\frac{2(k-2)\alpha}{(p-1)L}} b_1^{\frac{2(p-k)}{p-1}-1+\left(\frac{d}{2}-\sigma\right)+\left(\frac{2}{p-1}+\frac{2(p-k)}{p-1}-1\right)\frac{\alpha}{L}+O\left(\frac{(\sigma-s_c)}{L}\right)} \qquad (3.3.95) \\ \leq & \; C(M,K_1,K_2)\sqrt{\mathcal{E}_{s_L}}b_1\left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-1} b_1^{\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}. \end{aligned}$$

We now come back to (*3.3.93*) and inject the bounds we have found. Putting together the result obtained in case 1, (*3.3.94*) and the result obtained in the second case, (*3.3.95*), gives for the non linear term:

$$\left|\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}(NL(w))\right| \leq \frac{C(K_1,K_2,M)}{\lambda^{2(s_L-s_c)}}\frac{b_1}{\lambda}\mathcal{E}_{s_L}\left[\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-1}\right] b_1^{\frac{\alpha}{L}+O\left(\frac{(\sigma-s_c)}{L}\right)}. \quad (3.3.96)$$

We now recapitulate: we have found directs bounds for the quadratic term (*3.3.91*), for the error term (*3.3.92*), and for the non linear term (*3.3.96*). We inject them in (*3.3.89*) to obtain the intermediate identity (*3.3.90*), wich we claimed in this step 1.

Step 2: Terms for which only a local part is problematic. The small linear term and the scale changing term involve a potential that, in both cases, has a better decay than $\frac{1}{y^2}$ far away of the origin. So away from the origin we can control them directly. Unfortunately, close to the origin we cannot. This is why we will have to use an additional tool, the study of a Morawetz type quantity, which will be done in the next subsection. We claim that (*3.3.90*) yields:

$$\begin{array}{l} \frac{d}{dt}\left(\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}\right) = 2\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}(-\frac{1}{\lambda}\tilde{Mod}_{\frac{1}{\lambda}}(t)^{(1)}) + w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}(-\frac{1}{\lambda}\tilde{Mod}(t)_{\frac{1}{\lambda}}^{(2)}) \\ +\frac{b_1}{\lambda^{2(s_L-s_c)+1}}\left[O(\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+\eta)}) + O\left(\mathcal{E}_{s_L}b_1^{\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sum_{k=2}^{p}\left[\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right]^{k-1}\right)\right] \\ +\frac{b_1}{\lambda^{2(s_L-s_c)+1}}O\left(\frac{\mathcal{E}_{s_L}}{N^\delta}+C(N)\mathcal{E}_{s_L,\text{loc}}\right). \end{array} \quad (3.3.97)$$

We are now going to prove this identity (*3.3.97*) by establishing bounds on the small linear term and the scale changing term in (*3.3.90*).

• *The $L(w)$ term:* We start by rescaling and using Cauchy-Schwarz:

$$\left|\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}(L(w))\right| \leq \frac{1}{\lambda^{2(s_L-s_c)+1}}\sqrt{\mathcal{E}_{s_L}}\parallel (L(\varepsilon))_{k_0+L}\parallel_{L^2}.$$

We have: $L(\varepsilon)=p(Q^{p-1}-\tilde{Q}_b^{(1)(p-1)})\varepsilon^{(1)}$. From the asymptotic of the the profiles T_i and S_i there holds the degeneracy:

$$|\partial_y^j(Q^{p-1}-\tilde{Q}_b^{(1)(p-1)})| \leq C(L)\frac{b_1}{1+y^{1+\alpha+j-C(L)\eta}},$$

Let[18] $\delta=\frac{\delta_0}{2}$. We first estimate the integral close to the origin. $H^{s_L-1}(y\leq 1)$ is an algebra, from the equivalence between Laplace based derivatives and adapted ones (see Lemma 3.B.2), and from the weighted coercivity (Lemma 3.D.3):

$$\int_{y\leq 1}(L(\varepsilon))_{s_L-1}^2 \leq Cb_1^2\int_{y\leq 1}\sum_{i=0}^{s_L}|D^i\varepsilon^{(1)}|^2 \leq C(M)b_1^2\int\frac{|\varepsilon_{s_L}^{(1)}|}{1+y^{2\delta}}.$$

Away from the origin we estimate using the weighted coercivity and the equivalence between ∂_y derivatives and adapted derivatives (Lemma 3.B.1).

$$\begin{array}{rcl} \parallel (L(\varepsilon^{(1)})_{k_0+1}\parallel_{L^2(y\geq 1)}^2 & \leq & C\sum_{i=0}^{s_L-1}\parallel \frac{b_1|\varepsilon_i^{(1)}|}{1+y^{1+\alpha+s_L-1-i-C\eta}}\parallel_{L^2(y\geq 1)}^2 \\ & \leq & C(M)b_1^2\parallel \frac{\varepsilon_{s_L}^{(1)}}{1+y^\delta}\parallel_{L^2}^2. \end{array}$$

[18]We cannot expect to gain the weight $y^{-\alpha}$ because if α is too big the weighted coercivity does not apply. The limiting case is δ_0 hence our choice for δ.

With the two estimates, close and away from the origin, we have shown:

$$\| (L(\varepsilon))_{s_L-1} \|_{L^2}^2 \lesssim b_1^2 \left\| \frac{\varepsilon_{s_L}^{(1)}}{1+y^\delta} \right\|_{L^2}^2 . \tag{3.3.98}$$

We now split the term of the right hand side in two parts, one before N and the other after, where $N>0$ is the large constant used in the definition of the local adapted norm (see (*3.3.12*)):

$$\left\| b_1 \frac{\varepsilon_{s_L}^{(1)}}{1+y^\delta} \right\|_{L^2} \leq b_1 \| \varepsilon_{s_L}^{(1)} \|_{L^2(\leq N)} + b_1 \frac{1}{N^\delta} \| \varepsilon_{s_L}^{(1)} \|_{L^2(\geq N)} .$$

Finally:

$$\left| \int w^{(2)} \mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}(L(w)) \right| \lesssim \frac{C(M)}{\lambda^{2(s_L-s_c)}} \frac{b_1\sqrt{\mathcal{E}_{s_L}}}{\lambda} \left(\frac{\sqrt{\mathcal{E}_{s_L}}}{N^\delta} + C(N)\sqrt{\mathcal{E}_{s_L,\text{loc}}} \right).$$

We now use Youngs inequality to reformulate it as:

$$\left| \int w^{(2)} \mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}(L(w)) \right| \leq \frac{C(M)}{\lambda^{2(s_L-s_c)}} \frac{b_1}{\lambda} \left(\frac{\mathcal{E}_{s_L}}{N^\delta} + C(N)\mathcal{E}_{s_L,\text{loc}} \right). \tag{3.3.99}$$

• *The scale changing term:* The same reasoning applies to the scale changing term. Indeed one has:

$$\frac{d}{dt}(\mathcal{L}_{\frac{1}{\lambda}}) = -\frac{\lambda_s}{\lambda^2} pQ_{\frac{1}{\lambda}}^{p-2}(\Lambda^{(1)}Q)_{\frac{1}{\lambda}} = -\frac{\lambda_s}{\lambda^4}\tilde{V}\left(\frac{y}{\lambda}\right)$$

where the potential $\tilde{V}$ satisfies an improved decay property:

$$\left|\partial_y^j V\right| \leq \frac{C}{1+\lambda y^{2+\alpha+j}}.$$

Consequently, as $-\frac{\lambda_s}{\lambda} \approx b_1$ from the modulation equations, we have the same gain of a weight $y^{-\alpha}$ we had for the small linear term. Using verbatim the same techniques one obtains:

$$\begin{array}{ll} & \left| \int \sum_{i=1}^{s_L} w^{(1)} \mathcal{L}_{\frac{1}{\lambda}}^{i-1} \frac{d}{dt}(\mathcal{L}_{\frac{1}{\lambda}}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L-i} w^{(1)} + \int \sum_{i=1}^{s_L-1} w^{(2)} \mathcal{L}_{\frac{1}{\lambda}}^{i-1} \frac{d}{dt}(\mathcal{L}_{\frac{1}{\lambda}}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L-1-i} w^{(2)} \right| \\ \leq & \frac{C(M)}{\lambda^{2(s_l-s_c)}} \frac{b_1}{\lambda} \left(\frac{\mathcal{E}_{s_L}}{N^\delta} + C(N)\mathcal{E}_{s_L,\text{loc}} \right), \end{array} \tag{3.3.100}$$

We now come back to the identity (*3.3.90*) established in step 1, and inject the bounds on the small linear term (*3.3.99*) and on the scale changing term (*3.3.100*). This gives the identity (*3.3.97*) we claimed in this step 2.

Step 3: Managing the modulation term. Eventually, we have to estimate the influence of the modulation term on (*3.3.97*). We claim that:

$$\begin{array}{ll} & \int w^{(1)} \mathcal{L}_{\frac{1}{\lambda}}^{s_L} \frac{1}{\lambda} \tilde{Mod}(t)_{\frac{1}{\lambda}}^{(1)} + \int w^{(2)} \mathcal{L}_{\frac{1}{\lambda}}^{s_L-1} \frac{1}{\lambda} \tilde{Mod}(t)_{\frac{1}{\lambda}}^{(2)} \\ = & \frac{d}{dt} O\left[\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}} b_1^{\eta(1-\delta_0)} \right] + O\left(\frac{b_1\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)+1}} b_1^{\eta(1-\delta_0)} + \frac{b_1\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)+1}} b_1^{L+(1-\delta_0)(1+2\eta)} \right). \end{array} \tag{3.3.101}$$

Once this bound is proven, we can finish the proof of the proposition by injecting it in (*3.3.97*). So to finish to proof, we will now prove (*3.3.101*). For $1 \leq i \leq L-1$, the bound (*3.3.36*) we found for the modulation equations provides a sufficient estimate for the terms $(b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})(\boldsymbol{T}_i+\sum\frac{\partial \boldsymbol{S}_j}{\partial b_i})$. Indeed, pick an indice $1\leq i\leq L-1$ and suppose it is even (the odd case being exactly the same). We calculate:

$$\begin{array}{rl} & \left|\frac{1}{\lambda}\int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}((b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})\chi_{B_1}(T_i+\sum_{j=i+1,\ j\text{ even}}^{L+2}\frac{\partial S_j}{\partial b_i}))_{\frac{1}{\lambda}}\right| \\ & +\left|\frac{1}{\lambda}\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}((b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})\chi_{B_1}(\sum_{j=i+1,\ j\text{ odd}}^{L+2}\frac{\partial S_j}{\partial b_i}))_{\frac{1}{\lambda}}\right| \\ \leq & \frac{C(M)\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)}}(b_1\sqrt{\mathcal{E}_{s_L}}+b_1^{L+3})\left\|\left(\chi_{B_1}\left(T_i+\sum_{j=i+1,\ j\text{ even}}^{L+2}\frac{\partial S_j}{\partial b_i}\right)\right)_{s_L}\right\|_{L^2} \\ & +\frac{C(M)\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)}}(b_1\sqrt{\mathcal{E}_{s_L}}+b_1^{L+3})\left\|\left(\chi_{B_1}\left(\sum_{j=i+1,\ j\text{ odd}}^{L+2}\frac{\partial S_j}{\partial b_i}\right)\right)_{s_L-1}\right\|_{L^2}. \end{array}$$

Since:

$$\left\|\left(\chi_{B_1}\Big(T_i+\sum_{j=i+1,\text{ even}}^{L+2}\frac{\partial S_j}{\partial b_i}\Big)\right)_{s_L}\right\|_{L^2}+\left\|\left(\chi_{B_1}\Big(\sum_{j=i+1,\text{ odd}}^{L+2}\frac{\partial S_j}{\partial b_i}\Big)\right)_{s_L-1}\right\|_{L^2}\leq Cb_1^{(L-i)}$$

and that we assumed $i<L$, this bound implies the following identity for the modulation term:

$$\begin{array}{rl} & \int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\frac{1}{\lambda}\tilde{Mod}(t)^{(1)}_{\frac{1}{\lambda}}+\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\frac{1}{\lambda}\tilde{Mod}(t)^{(2)}_{\frac{1}{\lambda}} \\ = & \frac{1}{\lambda}\int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}((b_{L,s}+(L-\alpha)b_1b_L)\chi_{B_1}(\frac{\partial S_{L+1}}{\partial b_L}))_{\frac{1}{\lambda}}+\frac{b_1O\big(b_1\mathcal{E}_{s_L}+\sqrt{\mathcal{E}_{s_L}}b_1^{L+3}\big)}{\lambda^{2(s_L-s_c)+1}} \\ & +\frac{1}{\lambda}\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}((b_{L,s}+(L-\alpha)b_1b_L)\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}} \end{array} \qquad (3.3.102)$$

The bad term is the last one for $i=L$. But we know by the improved bound for the evolution of b_L, see Lemma 3.3.5 that $b_{L,s}+(L-\alpha)b_1b_L$ is small enough up to the derivative in time of the projection of ε onto $\boldsymbol{H}^{*L}\chi_{B_1}\boldsymbol{\Lambda Q}$. Let[19]:

$$\begin{array}{rl} \boldsymbol{\xi} & := \frac{\langle \boldsymbol{H}^L\boldsymbol{\varepsilon},\chi_{B_0}\boldsymbol{\Lambda Q}\rangle}{\left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\left[\chi_{B_1}\left(\boldsymbol{T}_L+\frac{\partial \boldsymbol{S}_{L+1}}{\partial b_L}+\frac{\partial \boldsymbol{S}_{L+2}}{\partial b_L}\right)\right]_{\frac{1}{\lambda}} \\ & := C(\xi)\left[\chi_{B_1}\left(\boldsymbol{T}_L+\frac{\partial \boldsymbol{S}_{L+1}}{\partial b_L}+\frac{\partial \boldsymbol{S}_{L+2}}{\partial b_L}\right)\right]_{\frac{1}{\lambda}} \end{array} \qquad (3.3.103)$$

We claim that the bad part of the L-th modulation term can be integrated in time the following way:

$$\begin{array}{rl} & \frac{d}{dt}\left(\int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\xi^{(1)}+\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\xi^{(2)}+\frac{1}{2}\int \xi^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\xi^{(1)}+\frac{1}{2}\int \xi^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\xi^{(2)}\right) \\ = & \frac{1}{\lambda}\int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}((b_{L,s}+(L-\alpha)b_1b_L)\chi_{B_1}(\frac{\partial S_{L+1}}{\partial b_L}))_{\frac{1}{\lambda}} \\ & +\frac{1}{\lambda}\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}((b_{L,s}+(L-\alpha)b_1b_L)\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}} \\ & +\frac{b_1}{\lambda^{2(s_L-s_c)+1}}O(\mathcal{E}_{s_L}b_1^{\eta(1-\delta_0)})+\frac{b_1}{\lambda^{2(s_L-s_c)+1}}O(\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1+2\eta)(1-\delta_0)}) \end{array} \qquad (3.3.104)$$

We will prove this identity at the end of this step 3. Once it is established, it allows us to prove the identity (*3.3.101*). Indeed, (*3.3.102*) can be rewritten as:

$$\begin{array}{rl} & \int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\frac{1}{\lambda}\tilde{Mod}(t)^{(1)}_{\frac{1}{\lambda}}+\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\frac{1}{\lambda}\tilde{Mod}(t)^{(2)}_{\frac{1}{\lambda}} \\ = & \frac{d}{dt}\left(\int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\xi^{(1)}+\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\xi^{(2)}-\frac{1}{2}\int \xi^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\xi^{(1)}-\frac{1}{2}\int \xi^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\xi^{(2)}\right) \\ & +\frac{b_1\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)+1}}O\left(b_1^{\eta(1-\delta_0)}\sqrt{\mathcal{E}_{s_L}}+b_1^{L+(1-\delta_0)(1+2\eta)}\right) \end{array} \qquad (3.3.105)$$

[19] $\boldsymbol{\xi}$ can be seen as the coordinate of $\boldsymbol{\varepsilon}$ along the vector $\chi_{B_0}T_L$.

We just have to check the gain obtained by the time integration. From the two estimates (*3.3.69*) and (*3.3.71*) we used in the proof of the improved modulation equation, one has the following size for the coefficient $C(\xi)$:

$$|C(\xi)| \lesssim \sqrt{\mathcal{E}_{s_L}}b_1^{\delta_0-1}. \qquad (3.3.106)$$

From the construction of the profiles S_i in Proposition 3.2.12, one has the following asymptotics:

$$\left|\partial_y^j\left(\frac{\partial S_{L+1}}{\partial b_L}\right)\right| \leq \frac{C(L)b_1}{1+y^{\gamma-L-1+g'+j}}, \ \text{and} \ \left|\partial_y^j\left(\frac{\partial S_{L+2}}{\partial b_L}\right)\right| \leq \frac{C(L)b_1^2}{1+y^{\gamma-L-1+g'+j}}. \qquad (3.3.107)$$

The cancellation $\mathcal{L}^{\frac{L+1}{2}}T_L=0$ implies that the support of $(\chi_{B_1}T_L)_{s_L-1}$ is in the zone $B_1\leq y\leq 2B_1$, hence $\|(\chi_{B_1}T_L)_{s_L-1}\|_{L^2}\lesssim b_1^{(1-\delta_0)(1+\eta)}$. The two last estimates then imply:

$$\begin{aligned} & \left|\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}\xi^{(1)}+w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}\right| \\ \leq & \frac{\sqrt{\mathcal{E}_{s_L}}|C(\xi)|(\|(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{s_L}\|_{L^2}+\|(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{s_L-1}\|_{L^2})}{\lambda^{2(s_L-s_c)}} \leq C(M)\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_l-s_c)}}b_1^{\eta(1-\delta_0)}, \end{aligned} \qquad (3.3.108)$$

For the same reasons:

$$\left|\frac{1}{2}\int\xi^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}\xi^{(1)}+\frac{1}{2}\int\xi^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}\right| \lesssim \frac{1}{\lambda^{2(s_l-s_c)}}\mathcal{E}_{s_L}b_1^{2\eta(1-\delta_0)}, \qquad (3.3.109)$$

The injection of these last bounds (*3.3.108*) and (*3.3.109*) in the previous identity (*3.3.105*) yields the identity (*3.3.101*) we claimed in this step 3. To end the proof of the proposition, it just remains to prove (*3.3.104*), what we are now going to do. Using the improved modulation bound (*3.3.56*) for $b_{L,s}$ one calculates:

$$\begin{aligned} & \frac{d}{dt}\left(\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}+\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}\xi^{(1)}\right) \\ = & \frac{1}{\lambda}\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}((b_{L,s}+(L-\alpha)b_1b_L)\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}} \\ & +\frac{1}{\lambda}\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}((b_{L,s}+(L-\alpha)b_1b_L)\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}} \\ & +\frac{O(b_1^{\delta_0}\sqrt{\mathcal{E}_{s_L}}+b_1^{L+1+g'})}{\lambda}[\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}}+w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}}] \\ & +\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}C(\xi)\partial_t\left(\mathcal{L}_{\frac{1}{\lambda}}^{s_L}\left[\chi_{B_1}\left(\frac{\partial S_{L+1}}{\partial b_L}\right)\right]_{\frac{1}{\lambda}}\right) \\ & +\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}C(\xi)\partial_t\left(\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\left[\chi_{B_1}\left(T_L+\frac{\partial S_{L+2}}{\partial b_L}\right)\right]_{\frac{1}{\lambda}}\right) \\ & +\int w_t^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}+\int w_t^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}\xi^{(1)}. \end{aligned} \qquad (3.3.110)$$

We show that all the other terms are small enough. From the modulation equations (*3.3.36*) for b_i for $i<L$ one has: $|\lambda_s\lambda^{-1}|\lesssim b_1$, $|b_{i,s}|\lesssim b_1^{i+1}$. As $\boldsymbol{\xi}$ does not depend on b_L, this gives us the following bounds when the time derivative applies to $\boldsymbol{\xi}$ or $\boldsymbol{\mathcal{L}}$:

$$\begin{aligned} & \Big|\frac{O(b_1^{\delta_0}\sqrt{\mathcal{E}_{s_L}}+b_1^{L+1+g'})}{\lambda}[\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}}+\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}}] \\ & +\int w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L}C(\xi)\partial_t\left(\mathcal{L}_{\frac{1}{\lambda}}^{s_L}\left[\chi_{B_1}\left(\frac{\partial S_{L+1}}{\partial b_L}\right)\right]_{\frac{1}{\lambda}}\right) \\ & +\int w^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}C(\xi)\partial_t\left(\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\left[\chi_{B_1}\left(T_L+\frac{\partial S_{L+2}}{\partial b_L}\right)\right]_{\frac{1}{\lambda}}\right)\Big| \\ \leq & C(L,M)\frac{b_1\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)+1}}(\sqrt{\mathcal{E}_{s_L}}b_1^{\eta(1-\delta_0)}+b_1^{L+(1-\delta_0)(1+\eta)+g'}), \end{aligned} \qquad (3.3.111)$$

where we used coercivity, (*3.3.106*) and (*3.3.107*) and the fact that $\partial_t(\mathcal{L}^{s_L-1}\chi_{B_1}T_L)$ has its support in $B_1\leq y\leq 2B_1$. We have now to estimate the terms involving $\boldsymbol{w}_t$ in (*3.3.110*). We do exactly the same things we did to the proof of Lemma 3.3.5. For the sake of simplicity we will only do it for the second coordinate, the first one being the same. We first compute the expression:

$$\begin{array}{rcl} \int w_t^{(2)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)} & = & \int -\mathcal{L}_{\frac{1}{\lambda}}w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}+\int -\frac{1}{\lambda}(\tilde{\psi}_b^{(2)}+\tilde{Mod}(t)^{(2)})_{\frac{1}{\lambda}}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)} \\ & & +\int (L(w)+NL(w))\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}. \end{array} \tag{3.3.112}$$

We use the bootstrap assumptions to put an upper bound on everything except the $b_{L,s}$ term. For the linear term one has the bound:

$$\left|\int -\mathcal{L}_{\frac{1}{\lambda}}w^{(1)}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}\right| \leq \frac{\sqrt{\mathcal{E}_{s_L}}}{\lambda^{(s_l-s_c)}}\parallel (\xi^{(2)})_{s_L}\parallel_{L^2}\leq C(M)\frac{b_1}{\lambda^{2(s_l-s_c)+1}}\mathcal{E}_{s_L}b_1^{\eta(1-\delta_0)}. \tag{3.3.113}$$

Using the bounds on the error $\tilde{\psi}_b$ from Proposition 3.2.14:

$$\begin{array}{rcl} \left|\int -\frac{1}{\lambda}(\tilde{\psi}_b^{(2)})_{\frac{1}{\lambda}}\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}\right| & \leq & \frac{1}{\lambda^{s_L-s_c+1}}\parallel (\tilde{\psi}_b^{(2)})_{s_L-1}\parallel_{L^2}\parallel \xi_{s_L-1}^{(2)}\parallel_{L^2} \\ & \leq & \frac{C(M)b_1}{\lambda^{2(s_L-s_c)+1}}\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+2\eta)}. \end{array} \tag{3.3.114}$$

The small linear term gives the same estimate as the linear one:

$$\left|\int L(w)\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}\right| \leq \frac{C(M)b_1}{\lambda^{2(s_l-s_c)+1}}\mathcal{E}_{s_L}b_1^{\eta(1-\delta_0)}. \tag{3.3.115}$$

Finally, we start by decomposing the nonlinear term as a sum of term of the form: $\tilde{Q}_{b,\frac{1}{\lambda}}^{(1)(p-k)}w^{(1)k}$ for $2\leq k\leq p$. For each term we let all the derivatives on $\xi^{(2)}$:

$$\left|\int NL(w)\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi^{(2)}\right| \lesssim \frac{\sqrt{\mathcal{E}_{s_L}}b_1^{\delta_0-1}}{\lambda^{2(s_L-s_c)+1}}\int \frac{|\varepsilon|^{(1)}|^k}{1+y^{\frac{2}{p-1}(p-k}}(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial_{b_L}}))_{2s_L-2}.$$

We know from their construction that $(T_L+\frac{\partial S_{L+2}}{\partial_{b_L}}))_{2s_L-2}=O\left(\frac{1}{1+y^{\gamma+L+1+2k_0}}\right)$, and by using the coercivity of the adapted norm and the L^∞ estimate for $w^{(1)}$:

$$\begin{array}{rcl} \left|\int \frac{|\varepsilon|^{(1)}|^k(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial_{b_L}}))_{2s_L-2}}{1+y^{\frac{2}{p-1}(p-k}}\right| & \leq & C\int \frac{|\varepsilon^{(1)}|^k}{1+y^{\frac{2}{p-1}(p-k)+\gamma+L+2k_0+1}} \\ & \leq & C(M)b_1^{-L+\gamma-1+\frac{2}{p-1}(p-k)}\mathcal{E}_{s_L}\parallel \varepsilon^{(1)}\parallel_{L^\infty}^{k-2} \\ & \leq & C(M,K_1,K_2)\mathcal{E}_{s_L}b_1^{-L+1+\alpha+O(\frac{1}{L})}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-2} \end{array}$$

where the integral in y we used with the Cauchy-Schwarz inequality was indeed divergent. Under the bootstrap assumptions it leads to:

$$\frac{\sqrt{\mathcal{E}_{s_L}}b_1^{1-\delta_0}}{\lambda^{2(s_L-s_c)+1}}\int \frac{|\varepsilon|^{(1)}|^k}{1+y^{\frac{2}{p-1}(p-k)}}(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial_{b_L}}))_{2s_L-2}\leq \frac{b_1\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)+1}}b_1^{\eta(1-\delta_0)+\frac{\alpha}{2}}$$

(as $C(M,K_1,K_2)b_1^\alpha\leq b_1^{\frac{\alpha}{2}}$ for s_0 large). Therefore for the non linear term we have:

$$\left|\int NL(w)\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\xi\right| \leq \frac{b_1\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)+1}}b_1^{\eta(1-\delta_0)+\frac{\alpha}{2}}. \tag{3.3.116}$$

We now treat the modulation terms, preserving the L-th one. With the bound (*3.3.36*) on the modulation for $1\leq i\leq L-1$, one has:

$$\begin{array}{rl} & \left| \int \frac{1}{\lambda}\tilde{Mod}^{(2)}_{\frac{1}{\lambda}}\mathcal{L}^{s_L-1}\xi^{(2)}-\int \frac{1}{\lambda}(b_{L,s}+(L-\alpha)b_1b_L)(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}}\mathcal{L}^{s_L-1}\xi^{(2)}\right| \\ \leq & C(M)\frac{b_1\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)}}(\sqrt{\mathcal{E}_{s_L}}b_1^{\eta(1-\delta_0)}+b_1^{L+3}). \end{array} \tag{3.3.117}$$

We come back to the expression (*3.3.112*) of the term involving $w_t^{(2)}$, inject the bounds we have found for each term (*3.3.113*), (*3.3.114*), (*3.3.115*) and (*3.3.116*), yielding:

$$\begin{array}{rcl} \int w_t^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\xi^{(2)} & = & \int \frac{1}{\lambda}(b_{L,s}+(L-\alpha)b_1b_L)\left(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L})\right)_{\frac{1}{\lambda}}\mathcal{L}^{s_L-1}\xi^{(2)} \\ & & +\frac{b_1}{\lambda^{2(s_L-s_c)+1}}O\left(\mathcal{E}_{s_L}b_1^{\eta(1-\delta_0)}+\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+2\eta)}\right). \end{array} \tag{3.3.118}$$

As we said, the same computation can be done using verbatim the same techniques for the first coordinate, yielding:

$$\begin{array}{rcl} \int w_t^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\xi^{(1)} & = & \int \frac{1}{\lambda}(b_{L,s}+(L-\alpha)b_1b_L)\left(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L}\right)_{\frac{1}{\lambda}}\mathcal{L}^{s_L}\xi^{(1)} \\ & & +\frac{b_1}{\lambda^{2(s_L-s_c)+1}}O\left(\mathcal{E}_{s_L}b_1^{\eta(1-\delta_0)}+\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+2\eta)}\right). \end{array} \tag{3.3.119}$$

Now we look back at the identity (*3.3.110*). We estimated all terms in the right hand side in (*3.3.111*), (*3.3.118*) and (*3.3.119*). Therefore it gives the intermediate identity:

$$\begin{array}{rl} & \frac{d}{dt}\left(\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\xi^{(2)}+\int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\xi^{(1)}\right) \\ = & \frac{1}{\lambda}\int w^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}((b_{L,s}+(L-\alpha)b_1b_L)\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}} \\ & +\frac{1}{\lambda}\int w^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}((b_{L,s}+(L-\alpha)b_1b_L)\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}} \\ & -(b_{L,s}+(L-\alpha)b_1b_L)\left[\int(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}}\mathcal{L}^{s_L}\xi^{(1)}+(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}}\mathcal{L}^{s_L-1}\xi^{(2)}\right] \\ & +O\left(\frac{b_1\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)+1}}\left(\sqrt{\mathcal{E}_{s_L}}b_1^{\eta(1-\delta_0)}+b_1^{L+(1-\delta_0)(1+2\eta)}\right)\right). \end{array} \tag{3.3.120}$$

We will now integrate in time the remaining term involving $b_{L,s}+(L-\alpha)b_1b_L$. From the improved modulation equation (*3.3.56*) for b_L, one compute using (*3.3.120*):

$$\begin{array}{rl} & \frac{d}{dt}\left(\frac{1}{2}\int\xi^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\xi^{(1)}+\frac{1}{2}\int\xi^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\xi^{(2)}\right)=\int\xi^{(1)}_{s_L}\partial_t(\xi^{(1)}_{s_L})+\int\xi^{(2)}_{s_L-1}\partial_t(\xi^{(2)}_{s_L-1}) \\ = & O(b_1^{\delta_0}\sqrt{\mathcal{E}_{s_L}}+b_1^{L+1+g'})[\int\xi^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}}+\xi^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L})_{\frac{1}{\lambda}}] \\ & +(b_{L,s}+(L-\alpha)b_1b_L)(\int\xi^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}}+\xi^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L})_{\frac{1}{\lambda}}) \\ & +\frac{C(\xi)}{2}\int\xi^{(1)}\partial_t\left(\mathcal{L}^{s_L}_{\frac{1}{\lambda}}(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}}\right)+\frac{C(\xi)}{2}\int\xi^{(2)}\partial_t\left(\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}))_{\frac{1}{\lambda}}\right) \end{array}$$

Using verbatim the same techniques employed throughout this step 3 we estimate the remaining terms in this identity and end up with:

$$\begin{array}{rl} & \frac{d}{dt}\left(\frac{1}{2}\int\xi^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\xi^{(1)}+\frac{1}{2}\int\xi^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\xi^{(2)}\right) \\ = & (b_{L,s}+(L-\alpha)b_1b_L)\left(\int\xi^{(1)}\mathcal{L}^{s_L}_{\frac{1}{\lambda}}(\chi_{B_1}\frac{\partial S_{L+1}}{\partial b_L})_{\frac{1}{\lambda}}+\xi^{(2)}\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L})_{\frac{1}{\lambda}}\right) \\ & +\frac{b_1\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)+1}}O(\sqrt{\mathcal{E}_{s_L}}b_1^{2\eta(1-\delta_0)}+b_1^{L+(1-\delta_0)(1+2\eta)+g'}). \end{array} \tag{3.3.121}$$

We can now end the proof: combing the intermediate estimates (*3.3.121*) and (*3.3.120*) yields the identity (*3.3.104*) □

3.3.7 Control from a Morawetz type quantity:

As will be clear when we reintegrate the bootstrap equation in the next section, the term we still do no control in the monotonicity formula for the high regularity norm is the local one. We control it here via the study of a Morawetz type quantity. This term contributes to the time evolution of a bounded quantity (compared with $\mathcal{E}_{s_L}$), so when we integrate it with respect to time it should remain small. For $A>0$ and $\delta>0$ let:

$$\phi_A(x) := \int_0^x \chi_A(x')x'^{(1-\delta)}dx' \tag{3.3.122}$$

be the primitive of the function $\chi_A(x)x^{1-\delta}$ and we still denote by ϕ_A its radial extension. The quantity we will now study is (we recall that the adapted derivative f_k of a function is defined in (*3.2.21*)):

$$\mathcal{M} = -\int [\nabla\phi_A.\nabla\varepsilon^{(1)}_{s_L-1} + (1-\delta)\frac{\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}]\varepsilon^{(2)}_{s_L-1}. \tag{3.3.123}$$

From coercivity (Corollary 3.D.4), it is controlled by the high Sobolev norm:

$$|\mathcal{M}| \leq C(A,M)\mathcal{E}_{s_L} \tag{3.3.124}$$

We start by a lemma describing how this quantity controls the local norm $\mathcal{E}_{s_L,\text{loc}}$ thanks to the fact that $\mathcal{L}>0$ on $\dot{H}^1$.

Lemma 3.3.8. (control from the Morawetz identity at the linear level) *For A big enough, δ small enough, there holds the following control:*

$$\begin{array}{rc} & \int [\nabla\phi_A.\nabla\varepsilon^{(1)}_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}]\mathcal{L}\varepsilon^{(1)}_{s_L-1} \\ & -\int [\nabla\phi_A.\nabla\varepsilon^{(2)}_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(2)}_{s_L-1}]\varepsilon^{(2)}_{s_L-1} \\ \geq & C\frac{\delta}{N^\delta}\mathcal{E}_{s_L,loc} - \frac{C(M)}{A^\delta}\mathcal{E}_{s_L}, \end{array} \tag{3.3.125}$$

for some constant $C>0$ that does not depend on the other constants.

Proof of Lemma 3.3.8 We calculate each term in the left hand side of (*3.3.125*). For the second one we have:

$$-\int [\nabla\phi_A.\nabla\varepsilon^{(2)}_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(2)}_{s_L-1}]\varepsilon^{(2)}_{s_L-1} = \delta\int \frac{\Delta\phi_A}{2}|\varepsilon^{(2)}_{s_L-1}|^2.$$

As $\Delta\phi_A = \frac{(d-\delta)\chi_A}{y^\delta} + \frac{\partial_y\chi(\frac{y}{A})}{Ay^{\delta-1}}$ we get a control over the second coordinate:

$$-\int [\nabla\phi_A.\nabla\varepsilon^{(2)}_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(2)}_{s_L-1}]\varepsilon^{(2)}_{s_L-1} \geq \delta\int\frac{\chi_A|\varepsilon^{(2)}_{s_L-1}|^2}{y^\delta} + O\left(\frac{\mathcal{E}_{s_L}}{A^\delta}\right) \tag{3.3.126}$$

We now turn to the first term in (*3.3.125*). We start by calculating:

$$\begin{array}{rl} & -\int [\nabla\phi_A.\nabla\varepsilon^{(1)}_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}](-\mathcal{L}\varepsilon^{(1)}_{s_L-1}) \\ = & \int (\partial_y^2\phi_A - \frac{\delta\Delta\phi_A}{2}|\nabla\varepsilon^{(1)}_{s_L-1}|^2 - \frac{1-\delta}{4}\int\Delta^2\phi_A|\varepsilon^{(1)}_{s_L-1}|^2 + \int\frac{\nabla V.\nabla\phi_A+\delta\Delta\phi_A V}{2}|\varepsilon^{(1)}_{s_L-1}|^2. \end{array} \tag{3.3.127}$$

We are now going to show that locally, the first term of the right hand side is bigger than the two others and control the first coordinate. We have $\partial_y^2(\psi_A) = \frac{(1-\delta)\chi_A}{y^{-\delta}} + \frac{y^{1-\delta}}{A}\partial_y\chi(\frac{y}{A})$ which leads to:

$$\int (\partial_y^2\phi_A - \frac{\delta\Delta\phi_A}{2}|\nabla\varepsilon^{(1)}_{s_L-1}|^2 = (1-O(\delta))\int\chi_A\frac{|\nabla\varepsilon^{(1)}_{s_L-1}|^2}{y^\delta} + O\left(\frac{1}{A^\delta}\mathcal{E}_{s_L}\right). \tag{3.3.128}$$

We claim the following weighted Hardy inequality for radial functions:

$$\int \frac{\chi_A}{y^{-\delta}}|\nabla u|^2 \geq \frac{(d-2-\delta)^2}{4}\int \chi_A \frac{u^2}{y^{2+\delta}} - C\int \frac{|y\partial \chi(\frac{y}{A})|}{y^{2+\delta}}u^2. \quad (3.3.129)$$

We prove this general inequality now. For smooth radial functions we compute, performing integration by parts:

$$\int \frac{\chi_A}{y^{1+\delta}}u\partial_y u = -\frac{d-2-\delta}{2}\int \frac{u^2}{y^{2+\delta}}\chi_A - \frac{1}{2}\int \frac{u^2}{y^{2+\delta}}\frac{y\partial_y \chi(\frac{y}{A})}{A}. \quad (3.3.130)$$

We can control the left hand side by using Cauchy-Schwarz and Young's inequality:

$$\left|\int \frac{\chi_A}{y^{1+\delta}}u\partial_y u\right| \leq \frac{\epsilon}{2}\int \frac{\chi_A}{y^{2+\delta}}u^2 + \frac{1}{2\epsilon}\int \frac{\chi_A}{y^\delta}|\nabla u|^2. \quad (3.3.131)$$

Combining the two equations (3.3.130) and (3.3.131) with the choice $\epsilon = \frac{d-2-\delta}{2}$ gives the analysis bound (3.3.129) we claimed. We now come back to the identity (3.3.128), which gives the following control thanks to the Hardy inequality (3.3.129) we just proved:

$$\begin{array}{rcl}\int (\partial_y^2\phi_A - \frac{\delta\Delta\phi_A}{2})|\nabla \varepsilon^{(1)}_{s_L-1}|^2 & \geq & \delta\int \chi_A \frac{|\nabla \varepsilon^{(1)}_{s_L-1}|^2}{y^\delta} + (1-O(\delta))^2\frac{(d-2-\delta)^2}{4}\int \chi_A\frac{|\varepsilon^{(1)}_{s_L-1}|^2}{y^{2+\delta}} \\ & & +O\left(\frac{1}{A^\delta}\mathcal{E}_{s_L}\right).\end{array} \quad (3.3.132)$$

With this control coming from the "gradient" part, the equation (3.3.127) can be rewriten as:

$$\begin{array}{rl} & -\int [\nabla \phi_A.\nabla \varepsilon^{(1)}_{s_L-1} + \frac{\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}](-\mathcal{L}\varepsilon^{(1)}_{s_L-1}) \\ \geq & \delta\int \chi_A\frac{|\nabla \varepsilon^{(1)}_{s_L-1}|^2}{y^\delta} + (1-O(\delta))\frac{(d-2-\delta)^2}{4}\int \chi_A \frac{|\varepsilon^{(1)}_{s_L-1}|^2}{y^{2+\delta}} + O\left(\frac{1}{A^\delta}\varepsilon_{s_L}\right) \\ & -\frac{1-\delta}{4}\int \Delta^2\phi_A|\varepsilon^{(1)}_{s_L-1}|^2 + \frac{1}{2}\int (\nabla V.\nabla\phi_A + \delta\Delta\phi_A V)|\varepsilon^{(1)}_{s_L-1}|^2.\end{array} \quad (3.3.133)$$

We now prove that the last two terms are controled by the two first ones. We calculate:

$$-\Delta^2(\phi_A) = \frac{\delta(d-\delta)(d-2-\delta)}{2}\frac{\chi_A}{y^{2+\delta}} + O\left(\frac{1}{A^\delta}1_{A\leq y\leq 2A}\right). \quad (3.3.134)$$

For the term involving the potential we have that because $\Lambda^{(1)}Q, Q > 0$:

$$\begin{array}{rcl}\frac{1}{2}y\partial_y V & = & \frac{y}{2}p(p-1)Q^{p-2}\partial_y Q = \frac{p}{2}(p-1)Q^{p-2}\Lambda^{(1)}Q - pQ^{p-1} \\ & \geq & -pQ^{p-1} \\ & \geq & -\frac{\frac{(d-2)^2}{4}-\delta_p}{y^2},\end{array} \quad (3.3.135)$$

for some $\delta_p > 0$, because the potential is strictly smaller than the Hardy potential from Lemma 3.2.1. The expressions (3.3.134) and (3.3.135) imply that (3.3.133) can be rewriten as:

$$\begin{array}{rl} & \int \partial_y^2\phi_A|\nabla \varepsilon^{(1)}_{s_L-1}|^2 - \frac{1-\delta}{4}\int \Delta^2\phi_A \varepsilon^{(1)2}_{s_L-1} + \frac{1}{2}\int (\nabla V.\nabla \phi + \delta\Delta\phi_A V)\varepsilon^{(1)2}_{s_L-1} \\ \geq & \delta\int \frac{\chi_A|\nabla \varepsilon^{(1)}_{s_L-1}|}{y^\delta} + (\delta_p - O(\delta))\int \frac{|\varepsilon^{(1)}_{s_L-1}|^2}{y^{2+\delta}} + O\left(\frac{\mathcal{E}_{s_L}}{A^\delta}\right).\end{array}$$

Hence the identity (3.3.127) becomes:

$$\begin{array}{rl} & -\int [\nabla\phi_A.\nabla\varepsilon^{(1)}_{s_L-1} + \frac{\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}](-\mathcal{L}\varepsilon^{(1)}_{s_L-1}) \\ \geq & \delta\int \frac{\chi_A|\nabla\varepsilon^{(1)}_{s_L-1}|}{y^\delta} + (\delta_p - O(\delta))\int \frac{\chi_A|\varepsilon^{(1)}_{s_L-1}|^2}{y^{2+\delta}} + O\left(\frac{\mathcal{E}_{s_L}}{A^\delta}\right).\end{array} \quad (3.3.136)$$

We now come back to the left hand side of (3.3.125). We have estimated the two terms in (3.3.126) and (3.3.136). For $\delta \ll \delta_p$ this gives the identity (3.3.125) we had to prove. □

We can now state the control in the full nonlinear wave equation:

Proposition 3.3.9. (Control of the local term by the Morawetz identity) *We suppose all the parameters of Proposition 3.3.2 are fixed in their range, except s_0. For s_0 and A large enough, there holds for $s_0 \leq s < s^*$:*

$$\frac{d}{ds}\mathcal{M} \geq \frac{\delta}{2N^\delta}\mathcal{E}_{s_L,loc} - \frac{C(M)}{A^\delta}\mathcal{E}_{s_L} - C(A)\sqrt{\mathcal{E}_{s_L}}b_1^{L+3}, \tag{3.3.137}$$

($\mathcal{E}_{s_L}$ and $\mathcal{E}_{s_L,loc}$ were defined in (3.3.11) and (3.3.12)).

Remark 3.3.10. As:

$$\frac{d}{dt}\frac{\mathcal{M}}{\lambda^{2(s_L-s_c)}} = 2(s_L-s_c)\frac{b_1\mathcal{M}}{\lambda^{2(s_L-s_c)+1}} + \frac{1}{\lambda^{2(s_L-s_c)+1}}\frac{d}{ds}\mathcal{M},$$

from the control *3.3.124*, the result of the lemma implies (remember $b_1 \leq \frac{1}{s_0}$ in the bootstrap regime, and that s_0 is chosen in last so than b_1 can be arbitrarily small compared to the other constants) :

$$\frac{d}{dt}\left(\frac{\mathcal{M}}{\lambda^{2(s_L-s_c)}}\right) \geq \frac{1}{\lambda^{2(s_L-s_c)+1}}\left(\frac{\delta}{2N^\delta}\mathcal{E}_{s_L,\text{loc}} - \frac{C(M)}{A^\delta}\mathcal{E}_{s_L} - C(A,M)\sqrt{\mathcal{E}_{s_L}}b_1^{L+3}\right).$$

This is because the impact of the scale changing in the estimate we want to prove is of lower order, so we can work both at level ε or $\boldsymbol{w}$.

Proof of Proposition 3.3.9 The control comes from the previous lemma, and the new terms in the full (NLW) will be showed to be negligible. The time evolution of $\mathcal{M}$ is (f_k being the k-th adapted derivative of f defined in (*3.2.21*)):

$$\begin{array}{c} \frac{d}{ds}\mathcal{M} = -\int \nabla\phi_A.\nabla[(-\frac{\lambda_s}{\lambda}\Lambda^{(1)}\varepsilon^{(1)} + \varepsilon^{(2)} - \tilde{\psi}_b^{(1)} - \tilde{Mod}(t)^{(1)})]_{s_L-1}\varepsilon^{(2)}_{s_L-1} \\ -\int \frac{(1-\delta)\Delta\phi_A}{2}(-\frac{\lambda_s}{\lambda}\Lambda^{(1)}\varepsilon^{(1)} + \varepsilon^{(2)} - \tilde{\psi}_b^{(1)} - \tilde{Mod}(t)^{(1)})_{s_L-1}\varepsilon^{(2)}_{s_L-1} \\ +\int \nabla\phi_A.\nabla\varepsilon^{(1)}_{s_L-1}[\mathcal{L}\varepsilon^{(1)} + \frac{\lambda_s}{\lambda}\Lambda^{(2)}\varepsilon^{(2)} + \tilde{\psi}_b^{(2)} + \tilde{Mod}(t)^{(2)} - L(\varepsilon) - NL(\varepsilon)]_{s_L-1} \\ +\int \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}[\mathcal{L}\varepsilon^{(1)} + \frac{\lambda_s}{\lambda}\Lambda^{(2)}\varepsilon^{(2)} + \tilde{\psi}_b^{(2)} + \tilde{Mod}(t)^{(2)} - L(\varepsilon) + NL(\varepsilon)]_{s_L-1}. \end{array} \tag{3.3.138}$$

And we aim at computing the effect of everything in the right hand side. The linear part produces exactly the control we want thanks to the previous Lemma 3.3.8:

$$\begin{array}{cc} & \int[\nabla\phi_A.\nabla\varepsilon^{(1)}_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}]\mathcal{L}\varepsilon^{(1)}_{s_L-1} \\ & -\int[\nabla\phi_A.\nabla\varepsilon^{(2)}_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(2)}_{s_L-1}]\varepsilon^{(2)}_{s_L-1} \\ \geq & \frac{\delta}{2N^\delta}\mathcal{E}_{s_L,\text{loc}} - \frac{C}{A^\delta}\mathcal{E}_{s_L}. \end{array} \tag{3.3.139}$$

We are now going to show that all the other terms are of smaller order. As we work on a compact support, from the coercivity (*3.D.24*) and the fact that $\frac{\lambda_s}{\lambda} \sim -b_1$ from (*3.3.36*):

$$\begin{array}{cc} & \left|\int[\nabla\phi_A.\nabla(\frac{\lambda_s}{\lambda}\Lambda^{(1)}\varepsilon^{(1)}_{s_L-1}) + \frac{(1-\delta)\Delta\phi_A}{2}\frac{\lambda_s}{\lambda}\Lambda^{(1)}\varepsilon^{(1)}_{s_L-1}]\varepsilon^{(2)}_{s_L-1}\right| \\ + & \left|\int[\nabla\phi_A.\nabla(\varepsilon^{(1)}_{s_L-1}) + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}]\frac{\lambda_s}{\lambda}\Lambda^{(2)}\varepsilon^{(2)}_{s_L-1}\right| \\ \leq & b_1C(A)\mathcal{E}_{s_L}, \end{array} \tag{3.3.140}$$

so with b_1 small enough it is negligeable. Still from the compactness of the support of ϕ_A, for b_1 small enough we do not see the bad tail of $\tilde{\boldsymbol{\psi}}_b$ (remember that for $y \leq B_1$, $\tilde{\boldsymbol{\psi}}_b = \boldsymbol{\psi}_b$). Hence:

$$\begin{array}{cc} & \left|\int[\nabla\phi_A.\nabla(\tilde{\psi}^{(1)}_{b,s_L-1}) + \frac{(1-\delta)\Delta\phi_A}{2}\tilde{\psi}^{(1)}_{b,s_L-1}]\varepsilon^{(2)}_{s_L-1}\right| \\ & +\left|\int[\nabla\phi_A.\nabla(\varepsilon^{(1)}_{s_L-1}) + \frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}]\tilde{\psi}^{(2)}_{b,s_L-1}\right| \\ \leq & C(A)\sqrt{\mathcal{E}_{s_L}}(\| \tilde{\psi}^{(1)}_{b,s_L} \|_{L^2(\leq A)} + \| \tilde{\psi}^{(2)}_{b,s_L-1} \|_{L^2(\leq A)} \leq C(A)\sqrt{\mathcal{E}_\sigma}b_1^{L+3}. \end{array} \tag{3.3.141}$$

The small linear term is also estimated easily. Indeed, we have that:

$$L(\varepsilon)=p(Q^{p-1}-\tilde{Q}_b^{p-1})\varepsilon^{(1)}=b_1\varepsilon^{(1)}O(1)$$

for $y\leq A$ for b_1 small enough. This gives using Cauchy-Schwarz:

$$\left|\int[\nabla\phi_A.\nabla(\varepsilon^{(1)}_{s_L-1})+\frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}]L(\varepsilon)_{s_L-1}\right|\leq C(A)b_1\mathcal{E}_{s_L}. \tag{3.3.142}$$

For the nonlinear term we use what we already showed during the proof of the monotonicity formula for the high Sobolev norm, see (*3.3.96*):

$$\begin{array}{rcl}\left|\int[\nabla\phi_A.\nabla(\varepsilon^{(1)}_{s_L-1})+\frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}]NL(\varepsilon)_{s_L-1}\right| & \leq & C(A)\sqrt{\mathcal{E}_{s_L}}\parallel NL(\varepsilon)_{s_L-1}\parallel_{L^2}\\ & \leq & C(A)b_1\mathcal{E}_{s_L},\end{array} \tag{3.3.143}$$

which is negligeable for b_1 small enough as we said before. Finally it just remains to control the modulation terms. We just compute for the second coordinate, a similar estimate holding for the first one. Let i be odd, $1\leq i\leq L$. As $A\ll B_1$ for s_0 large enough, we do not see the the cut χ_{B_1} in the integral: $\chi_{B_1}\equiv 1$ for $y\leq 2A$. Because $\boldsymbol{H}^i\boldsymbol{T}_i=(\boldsymbol{T}_i)_{i-1}(-1)^{\frac{i+1}{2}}=(-1)^i\boldsymbol{\Lambda Q}$ this term cancels in the integral because $(\boldsymbol{T}_i)_{s_L-1}=((\boldsymbol{T}_i)_{i-1})_{s_L-i}=0$ as $s_L-i=L+k_0-i\geq 1$.

$$\int[\nabla\phi_A\nabla(\varepsilon^{(1)}_{s_L-1})+\frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}](b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})(\chi_{B_1}T_i)^{(2)}_{s_L-1}=0.$$

For the terms of the form $\frac{\partial S_j}{\partial b_i}$ we always have at least one parameter b_i involved in this expression, which gives that for $y\leq A$ there holds: $\left|\frac{\partial S_j}{\partial b_i}(y)\right|\leq C(A)b_1$. We then use the modulation equation proven in Lemma 3.3.3 to estimate:

$$\begin{array}{rc} & \left|\int[\nabla\phi_A.\nabla(\varepsilon^{(1)}_{s_L-1})+\frac{(1-\delta)\Delta\phi_A}{2}\varepsilon^{(1)}_{s_L-1}](b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})(\chi_{B_1}\frac{\partial S_j}{\partial b_i})^{(2)}_{s_L-1}\right|\\ \leq & C(A,M)\mathcal{E}_{s_L}b_1+C(A,M)\sqrt{\mathcal{E}_{s_L}}b_1^{L+3}.\end{array}$$

As we said, the same reasoning applies to treat the first coordinate. Consequently we have the following bound for the modulation terms:

$$\begin{array}{rc} & \left|\int\nabla\phi_A.\nabla[\tilde{Mod}(t)^{(1)})]_{s_L-1}\varepsilon^{(2)}_{s_L-1}+\int\frac{(1-\delta)\Delta\phi_A}{2}(\tilde{Mod}(t)^{(1)})_{s_L-1}\varepsilon^{(2)}_{s_L-1}\right.\\ & \left.\int\nabla\phi_A.\nabla\varepsilon^{(1)}_{s_L-1}[\tilde{Mod}(t)^{(2)}]_{s_L-1}+\int\frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)})_{s_L-1}[\tilde{Mod}(t)^{(2)}]_{s_L-1}\right|\\ \leq & C(A,M)\mathcal{E}_{s_L}b_1+C(A,M)\sqrt{\mathcal{E}_{s_L}}b_1^{L+3}.\end{array} \tag{3.3.144}$$

We now come back to our initial decomposition (*3.3.138*). We have the expected control from the linear term in (*3.3.139*), and have estimated all the other terms in (*3.3.140*), (*3.3.141*), (*3.3.142*), (*3.3.143*) and (*3.3.144*). It gives the desired result. □

3.4 End of the proof

3.4.1 End of the Proof of Proposition 3.3.2

We now end the proof of the proposition 3.3.2. We will reintagrate in time the equations giving the time evolution of the parameters and the norms for the error term to obtain improved bounds. The definition of the minimal time s^* for which the bootstrap assumptions are violated implies that at time s^* at least one of the following three facts is true:

(i) *The error term has grown too big:*

$$\mathcal{E}_{s_L}(s^*) = K_1 b_1(s^*)^{2L+2(1-\delta_0)(1+\eta)} \text{ or } \mathcal{E}_\sigma = K_2 b_1(s^*)^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}},$$

(ii) *Exit of the stable modes*

$$V_1(s^*) = \frac{1}{(s^*)^{\tilde\eta}} \text{ or } |b_k(s^*)| = \frac{\epsilon_k}{(s^*)^{k+\tilde\eta}},$$

(iii) *Exit of the instable modes:*

$$(V_2(s^*), ..., V_\ell(s^*)) \in \mathcal{S}^{\ell-1}\left(\frac{1}{(s^*)^{\tilde\eta}}\right).$$

We will show in this section that the cases (i) and (ii) never happen for any initial solution. Indeed, the estimates of the error term can be improved using all the preceding monotonicity formulas, and are in fact smaller than what we asked for. The exit of the stable modes is impossible because their evolution is governed by a linear equation for which 0 is an attractor, plus a force term whose size is under control.

There are initial data leading to the exit of the unstable modes because they are driven by unstable dynamics. Indeed from the study of the linearized equation for the parameters we have seen that 0 is a repulsive equilibrium for these modes. However this equilibrium must persist[20] when we add the perturbative term to the equation, because the contrary would go against Brouwer fixed point theorem. This part will be made clearer in our precise case later on.

We begin with integrating the scaling equations.

Lemma 3.4.1 (law for the scaling in the trapped regime). *Up to time s^* there holds the following estimations for the scaling:*

$$\lambda(s) = \left(\frac{s_0}{s}\right)^{\frac{\ell}{\ell-\alpha}}\left[1+O\left(\frac{1}{s_0^{\tilde\eta}}\right)\right]. \tag{3.4.1}$$

Proof of Lemma 3.4.1 Untill s^*, we have under the bootstrap assumptions (*3.3.26*) and (*3.3.25*) for the parameters that $b_i(s) = b_i^e + \frac{U_i}{s^{i+1}}$ with $U_i \leq \frac{1}{s^{\tilde\eta}}$. So we use the modulation equation proved in Lemma 3.3.3:

$$-\frac{\lambda_s}{\lambda} = b_1 + O\left(b_1\mathcal{E}_{s_L} + b_1^{L+3}\right) = \frac{\ell}{(\ell-\alpha)s} + O\left(\frac{1}{s^{1+\tilde\eta}}\right).$$

We rewrite this equation as:

$$\left|\frac{d}{ds}(\log(s^{\frac{\ell}{\ell-\alpha}}\lambda))\right| \lesssim \frac{1}{s^{1+\tilde\eta}}.$$

After integration this gives:

$$\lambda(s) = \left(\frac{s_0}{s}\right)^{\frac{\ell}{\ell-\alpha}}\left[1+O\left(\frac{1}{s_0^{\tilde\eta}}\right)\right].$$

$\square$

We now rule out the case (i). We recall that K_1 and K_2 are used to quantify the control of the error term ε in the trapped regime of proposition 3.3.2.

[20] this is a way of speaking, there is no fixed point but one trajectory staying bounded.

Lemma 3.4.2 (Integrating the evolution equations for the norms). *Assume all the other constants of Proposition 3.3.2 are fixed in their range. There exist $K_1, K_2 > 0$, $N > 0$, $\nu > 0$ and ϵ such that for s_0 big enough, η small enough, under the bootstrap assumptions untill time s^* the norms enjoy a better estimation. There holds in fact:*

$$\mathcal{E}_{s_L} \leq \frac{K_1}{2} b_1^{2L+2(1-\delta_0)(1+\eta)}, \tag{3.4.2}$$

and:

$$\mathcal{E}_\sigma \leq \frac{K_2}{2} b_1^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}}. \tag{3.4.3}$$

Remark 3.4.3. The constant $\frac{1}{2}$ is not really important, we could have stated it for any constant.

Proof of Lemma 3.4.2 The low Sobolev norm: We recall the bound on the time evolution of the low Sobolev norm from Proposition 3.3.6:

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} \leq \frac{b_1\sqrt{\mathcal{E}_\sigma} b_1^{(\sigma-s_c)(1+\nu)}}{\lambda^{2(\sigma-s_c)+1}}\left[b_1^{\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c}{L}\right)} + b_1^{\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c}{L}\right)} \sum_{k=2}^p \left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-1}\right]$$

with $\nu = \frac{\alpha}{\ell-\alpha}$. One has $\sum_{k=2}^p \left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-1} \ll 1$ under the bootstrap conditions (*3.3.27*). Therefore, we see that there exists a small constant $0 < \delta \ll 1$, such that if one chooses s_0 large enough, this equation can be rewriten as:

$$\frac{d}{ds}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(s_L-s_c)}}\right\} \leq \frac{1}{\lambda^{2(\sigma-s_c)}}\frac{b_1}{\lambda}\sqrt{\mathcal{E}_\sigma} b_1^{(\sigma-s_c)\frac{\ell}{\ell-\alpha}+\delta}.$$

Still under the bootstrap assumption we can integrate this equation:

$$\mathcal{E}_\sigma(s) \leq \mathcal{E}_\sigma(0)\lambda^{2(\sigma-s_c)} + \lambda^{2(\sigma-s_c)}\int_{s_0}^s \frac{b_1}{\lambda^{2(\sigma-s_c)}}\sqrt{K_2} b_1^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}+\delta}. \tag{3.4.4}$$

We recall that $\lambda(0) = 1$ and from (*3.4.1*) and the bootstrap assumptions (*3.3.26*) and (*3.3.25*) on b_1:

$$\left|\lambda(s) - \left(\frac{s_0}{s}\right)^{\frac{\ell}{\ell-\alpha}}\right| \leq \frac{1}{s_0^{c\tilde{\eta}}}\left(\frac{s_0}{s}\right)^{\frac{\ell}{\ell-\alpha}} \text{ and } \left|b_1 - \frac{c_1}{s}\right| \leq \frac{1}{s^{1+\tilde{\eta}}}.$$

It implies: $\lambda(s) \leq \frac{C}{s^{\frac{\ell}{\ell-\alpha}}}$ and $b_1 \sim \frac{c_1}{s}$. Consequently:

$$\mathcal{E}_\sigma(0)\lambda^{2(\sigma-s_c)} \leq C\mathcal{E}_\sigma(0) b_1^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}}.$$

Given the initial condition (*3.3.21*) on $\mathcal{E}_\sigma(0)$ it yields:

$$\mathcal{E}_\sigma(0)\lambda^{2(\sigma-s_c)} \leq b_1^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}}. \tag{3.4.5}$$

For the integral term one has:

$$\lambda^{2(\sigma-s_c)}\int_{s_0}^s \frac{b_1}{\lambda^{(2(\sigma-s_c)}} b_1^{2\frac{\ell}{\ell-\alpha}(\sigma-s_c)+\delta} \leq C\lambda^{2(\sigma-s_c)} \leq C b_1^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}}$$

because the integral is convergent ($\frac{b_1}{\lambda^{(2(\sigma-s_c)}} b_1^{2\frac{\ell}{\ell-\alpha}(\sigma-s_c)} \leq s^{-1-\delta}$). Therefore:

$$\lambda^{2(\sigma-s_c)}\int_{s_0}^s \frac{b_1}{\lambda^{(2(\sigma-s_c)}} b_1^{2\frac{\ell}{\ell-\alpha}(\sigma-s_c)}\sqrt{K_2} \leq C\sqrt{K_2} b_1^{\frac{\ell}{\ell-\alpha}(\sigma-s_c)}. \tag{3.4.6}$$

Injecting the two estimates (*3.4.5*) and (*3.4.6*) we found in (*3.4.4*) gives:

$$\mathcal{E}_\sigma(s) \leq b_1^{2(\sigma-s_c)\frac{\ell}{\ell-\alpha}}\left(1+C\sqrt{K_2}\right),$$

and $(1+C\sqrt{K_2}) \leq \frac{K_2}{2}$ for K_2 large enough.

The high Sobolev norm: We recall the estimate of Proposition 3.3.7:

$$\begin{array}{rcl} \frac{d}{dt}\left\{\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}+O\left(\frac{\mathcal{E}_{s_L}b_1^{\eta(1-\delta_0)}}{\lambda^{2(s_L-s_c)}}\right)\right\} & \leq & \frac{C(M)}{\lambda^{2(s_L-s_c)}}\frac{b_1}{\lambda}\left[\mathcal{E}_{s_L}b_1^{\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c}{L}\right)}\sum_{k=2}^{p}\left[\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right]^{k-1}\right. \\ & & \left.+C(N)\mathcal{E}_{s_L,\text{loc}}+\frac{\mathcal{E}_{s_L}}{N^{\frac{\delta_0}{2}}}+\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+\eta)}\right] \end{array}$$

with $C(M)$ independent of N. In the trapped regime (*3.3.27*), by taking s_0 large enough one has:

$$\mathcal{E}_{s_L}b_1^{\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c}{L}\right)}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{b_1^{\sigma-s_c}}\right)^{k-1} \leq \frac{C\mathcal{E}_{s_L}}{N^{\frac{\delta_0}{2}}}.$$

So the previous equation becomes:

$$\begin{array}{rl} & \frac{d}{ds}\left\{\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}+O\left(\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}b_1^{\eta(1-\delta_0)}\right)\right\} \\ \leq & \frac{Cb_1}{\lambda^{2(s_L-s_c)}}\times\left[\frac{\mathcal{E}_{s_L}}{N^{\frac{\delta_0}{2}}}+\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+\eta)}+C(N)\mathcal{E}_{s_L,loc}\right], \end{array}$$

(by multipliying the constant C by 2). We also have by the Proposition 3.3.9:

$$\frac{d}{ds}\left(\frac{\mathcal{M}}{\lambda^{2(s_L-s_c}}\right) \geq \frac{\delta}{2N^\delta\lambda^{2(s_L-s_c)}}\mathcal{E}_{s_L,loc}-\frac{C}{A^\delta\lambda^{2(s_L-s_c)}}\mathcal{E}_{s_L}-\frac{C(A,N)\sqrt{\mathcal{E}_{s_L}}}{\lambda^{2(s_L-s_c)}}b_1^{L+3}.$$

Let $a>0$. Once N, K_1 and A are chosen, for s_0 small enough we have:

$$\frac{CC(N)b_1}{\lambda^{2(s_L-s_c)}}\mathcal{E}_{s_L,\text{loc}} \leq \frac{C(N)}{a}\left(\frac{d}{ds}\left(\frac{\mathcal{M}}{\lambda^{2(s_L-s_c)}}\right)\right)+\frac{C(N,M)b_1}{A^\delta\lambda^{2(s_L-s_c)}}\mathcal{E}_{s_L}+\frac{C(A,N)}{\lambda^{2(s_L-s_c)}}\sqrt{\mathcal{E}_{s_L}},$$

which gives for the evolution of the high Sobolev norm the following monotonicity formula:

$$\begin{array}{rl} & \frac{d}{ds}\left\{\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}+O\left(\frac{\mathcal{E}_\sigma}{\lambda^{2(s_L-s_c)}}b_1^{\eta(1-\delta_0)}\right)\right\} \\ \leq & \frac{Cb_1}{\lambda^{2(s_L-s_c)}}\left[\frac{\mathcal{E}_\sigma}{N^{\frac{\delta_0}{2}}}+C(A,N)\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+\eta)}+\frac{C(N)}{A^\delta}\mathcal{E}_{s_L}\right]+\frac{C(N)}{a}\frac{d}{ds}\left(\frac{\mathcal{M}}{\lambda^{2(s_L-s_c)}}\right). \end{array}$$

Let $a'>0$ be a large constant. By letting N be large enough, then by letting A and a be large enough we can reformulate it as:

$$\begin{array}{rcl} \frac{d}{ds}\left\{\frac{\mathcal{E}_{s_L}}{\lambda^{2(s_L-s_c)}}+O\left(\frac{\mathcal{E}_\sigma}{\lambda^{2(s_L-s_c)}}b_1^{\eta(1-\delta_0)}\right)\right\} & \leq & \frac{b_1}{\lambda^{2(s_L-s_c)}}\left[\frac{\mathcal{E}_\sigma}{a'}+C\sqrt{\mathcal{E}_{s_L}}b_1^{L+(1-\delta_0)(1+\eta)}\right] \\ & & +\frac{1}{a'}\frac{d}{ds}\left(\frac{\mathcal{M}}{\lambda^{2(s_L-s_c)}}\right), \end{array}$$

with C independent of a'. We will now integrate it in time as we did for the low Sobolev norm, using the bootstrap assumption (*3.3.27*):

$$\begin{array}{rcl} \mathcal{E}_{s_L}(s) & \leq & C(s_0)(\mathcal{E}_{s_L}(s_0)+|\mathcal{M}(s_0|)\lambda^{2(s_L-s_c)}+\frac{1}{a'}|\mathcal{M}(s)| \\ & + & \lambda^{2(s_L-s_c)}\int_{s_0}^{s}\frac{b_1}{\lambda^{(2(s_L-s_c)}}\left(\frac{K_1}{a'}+C\sqrt{K_1}\right)b_1^{2(L+(1-\delta_0)(1+\eta))}. \end{array}$$

We recall that: $|\mathcal{M}| \leq C(A)\mathcal{E}_{s_L}$, so:

$$\left|\frac{\mathcal{M}}{a'}\right| \leq \frac{C(M)}{a'}\mathcal{E}_{s_L}.$$

We then compare using the equivalents for b_1 and λ:

$$b_1^{2L+2(1-\delta_0)(1+\eta)} \approx \frac{1}{s^{2L+2(1-\delta_0)(1+\eta)}}.$$

$$\lambda^{2(s_L-s_c)} \sim \frac{1}{s^{2\frac{\ell}{\ell-\alpha}(L+k_0-\frac{d}{2}+\frac{2}{p-1})}} \lesssim \frac{1}{s^{2L+\frac{\alpha}{\ell}L+O\left(\frac{1}{L^2}\right)}}.$$

This implies $\lambda^{2(s_L-s_c)}=o(b_1^{2(L+(1-\delta_0)(1+\eta)})$ (remember that $\ell \ll L$). Because of the initial bound (*3.3.21*) on $\mathcal{E}_{s_L}(0)$ there holds for all $s_0\leq s\leq s^*$:

$$C(s_0)(\mathcal{E}_{s_L}(0)+|\mathcal{M}(s_0)|)\lambda^{2(s_L-s_c)}\leq b_1^{2L+2(1-\delta_0)(1+\eta)}.$$

We now treat the integral term using the equivalents for $\lambda(s)$ and $b_1(s)$:

$$\begin{array}{rl} & \lambda^{2(s_L-s_c)}\int_{s_0}^s \frac{b_1}{\lambda^{(2(s_L-s_c)}} b_1^{2(L+(1-\delta_0)(1+\eta))} \\ \leq & Cs^{-2(s_L-s_c)\frac{\ell}{\ell-\alpha}}\int_{s_0}^s s^{-1-2(L+(1-\delta_0)(1+\eta)+2(s_L-s_c)(\frac{\ell}{\ell-\alpha})} \\ \leq & Cs^{-2L-2(1-\delta_0)(1+\eta)} \leq Cb_1^{2L+2(1-\delta_0)(1+\eta)}, \end{array}$$

with the constant C just depending on c_1 and s_0. The integral is indeed divergent from $-2(L+(1-\delta_0)(1+\eta))+2(s_L-s_c)\frac{\ell}{\ell-\alpha}>0$ (as $\ell\ll L$). Eventually the three estimations we have shown allow us to conclude:

$$\left(1-\frac{C(N)}{a'}\right)\mathcal{E}_{s_L}(s)\leq b_1^{2L+2(1-\delta_0)(1+\eta)}\left(\frac{C}{a'}K_1+C\sqrt{K_1}+C\right).$$

For a' and K_1 big enough one has:

$$\frac{\frac{C}{a'}K_1+C\sqrt{K_1}+C}{1-\frac{C(N)}{a'}}\leq\frac{K_1}{2}$$

(remember that here $\frac{C(N)}{a'}=\frac{C(N)}{a}$ and since we choose a after M this term can be arbitrarily small. $\square$

We now rule out case (ii) in the possible exit scenarios. We recall that the small coefficients $(\epsilon_i)_{\ell+1\leq i\leq L}$ are used to quantify the control over the stable modes in the trapped regime of Proposition 3.3.2.

Lemma 3.4.4 (control of the stable modes). *After having choosen the other constants correctly, there exists small enough constants $\tilde{\eta}$, and $(\epsilon_i)_{\ell+1\leq i\leq L}$ such that for s_0 big enough, untill time s^* there holds:*

$$|V_1|\leq\frac{1}{2s^{\tilde{\eta}}},\ \textit{and}\ |b_k(s)|\leq\frac{\epsilon_i}{2s^{k+\tilde{\eta}}}\ \textit{for}\ \ell+1\leq k\leq L. \tag{3.4.7}$$

Proof of Lemma 3.4.4 The stable modes for $\ell+1\leq i\leq L-1$: Let i be an integer, $\ell+1\leq i\leq L-1$. We recall that the evolution of b_i is given by:

$$\begin{array}{rcl} b_{i,s} & = & -(i-\alpha)b_1b_i+b_{i+1}+O(b_1\sqrt{\mathcal{E}_{s_L}}+b_1^{L+3}) \\ & = & -\frac{c_1(i-\alpha)}{s}b_i-(i-\alpha)\frac{U_1b_i}{s}+b_{i+1}+O(s^{-L-1-(1-\delta_0)}) \\ & = & -\frac{c_1(i-\alpha)}{s}b_i+b_{i+1}+O(s^{-1-i-2\tilde{\eta}}), \end{array}$$

for $\tilde{\eta}$ small enough, because $U_1 b_i = O(s^{-2\tilde{\eta}})$ under the bootstrap assumptions. Hence for s_0 large enough it gives:

$$|b_{i,s} + (i-\alpha)c_1 \frac{b_i}{s}| \leq \frac{2\epsilon_{i+1}}{s^{i+1+\tilde{\eta}}},$$

which we rewrite as:

$$|\frac{d}{ds}(s^{(i-\alpha)c_1} b_i)| \leq 2\epsilon_{i+1} s^{(i-\alpha)c_1-(i+1+\tilde{\eta})}. \tag{3.4.8}$$

We notice that $(i-\alpha)c_1 = \frac{l(i-\alpha)}{l-\alpha} > i$. So for $\tilde{\eta}$ small enough one has $(i-\alpha)c_1 \geq i+\tilde{\eta}$. With these two facts in mind we integrate the last equation and estimate using the initial condition (*3.3.20*):

$$\begin{array}{rcl} |b_i(s)| & \leq & b_i(0)\frac{s^{(i-\alpha)c_1}}{s^{(i-\alpha)c_1}} + \frac{2\epsilon_{i+1}}{s^{(i-\alpha)c_1}} \int_{s_0}^s \tau^{(i-\alpha)c_1-(i+1+\tilde{\eta})} d\tau \\ & \leq & \frac{\epsilon_i}{10 s^{i+\tilde{\eta}}} + \frac{2\epsilon_{i+1}}{((i-\alpha)c_1 - i)s^{i+\tilde{\eta}}}, \end{array}$$

the integral that appeared being divergent. We therefore see here that we can choose the constants of initial smallness $(\epsilon_i)_{\ell+1\leq i\leq L}$ one after each other: once ϵ_i is choosen we can take ϵ_{i+1} small enough to produce $\frac{\epsilon_i}{10} + \frac{2\epsilon_{i+1}}{(i-\alpha)c_1} < \frac{\epsilon_i}{2}$. This, of course, makes only sense if one is able to bootstrap the estimate on the last parameter b_L.

<u>The stable mode $i = L$:</u> We recall the improved modulation equation for b_L:

$$\begin{array}{rl} & \left| b_{L,s} + (L-\alpha)b_1 b_L - \frac{d}{ds}\left[\frac{\langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, \chi_{B_0} \boldsymbol{\Lambda Q}\rangle}{\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}} \left(\frac{\partial S_{L+2}}{\partial b_L}\right)_{L-1} \right\rangle} \right] \right| \\ \leq & \frac{1}{B_0^{\delta_0}} C(M)\left[\sqrt{\mathcal{E}_{s_L}} + b_1^{L+(1-\delta_0)(1+\eta)}\right]. \end{array} \tag{3.4.9}$$

We have seen in (*3.3.106*) that:

$$\left| \frac{\langle \boldsymbol{H}^L \boldsymbol{\varepsilon}, \chi_{B_0} \boldsymbol{\Lambda Q}\rangle}{\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}} \left(\frac{\partial S_{L+2}}{\partial b_L}\right)_{L-1} \right\rangle} \right| \leq C\sqrt{\mathcal{E}_{s_L}} b_1^{\delta_0 - 1} \lesssim s^{-L-1-\eta(1-\delta_0)},$$

We integrate the time evolution of b_L the same way we did for the other stable modes. This time, however, the force term comes from the error ε. We reformulate (*3.4.9*)

$$\frac{d}{ds}(s^{(L-\alpha)c_1} b_L) = s^{(L-\alpha)c_1}\frac{d}{ds} O\left(\frac{1}{s^{L+\eta(1-\delta_0)}}\right) + s^{(L-\alpha)c_1} O\left(\frac{1}{s^{L+1+\eta(1-\delta_0)}}\right). \tag{3.4.10}$$

Then as for the b_i's for $\ell+1 \leq i \leq L-1$, we integrate and use integration by parts to find, under the initial smallness assumption on b_L and for $\tilde{\eta}$ small enough:

$$|b_L(s)| \leq \frac{\epsilon_L}{10 s^{L+\tilde{\eta}}} + \frac{C}{s^{L+\eta(1-\delta_0)}},$$

where C is just some integration constant. Hence by choosing s_0 large enough and $\tilde{\eta} < \eta(1-\delta_0)$ we have: $|b_L(s)| \leq \frac{\epsilon_L}{2 s^{L+\tilde{\eta}}}$.

<u>control of V_1.</u> We recall that V_1 is the eigenvector associated to the eigenvalue -1 of the linearized operator A_ℓ, defined by (*3.3.18*): $V_1 = (P_\ell U)_1 = \sum_1^\ell p_{1,i} U_i$. We first calculate the time evolution of the U_i's for for $1 \leq i \leq \ell$ thanks to the modulation equation (3.3.3):

$$\begin{array}{rcl} U_{i,s} & = & \frac{(AU)_i}{s} + \frac{O(|U^2|)}{s} + s^i O(b_1 C(M)\sqrt{\mathcal{E}_{s_L}} + C(M) b_1^{L+3}) \\ & := & \frac{(AU)_i}{s} + \frac{O(|U^2|)}{s} + s^i g_i(s), \end{array}$$

where $g_i(s)$ stands for the terms added in the full equation. It leads to the following expression for the time evolution of V_1:

$$V_{1,s} = -\frac{1}{s}V_1 + \frac{O(|V|^2)}{s} + \sum_{j=1}^{L} p_{1,j}s^j g_j(s) + q_1 s^\ell b_{\ell+1}, \tag{3.4.11}$$

where q_1 is a constant defined by (*3.2.79*). We reformulate it under the bootstrap assumptions as:

$$\frac{d}{ds}(sV_1) = sO\left(\frac{1}{s^{1+2\tilde{\eta}}} + \frac{1}{s^{L-l}}\right) + sq_1 s^\ell b_{\ell+1}.$$

As $|b_{\ell+1}| \leq \epsilon_{\ell+1}s^{-\tilde{\eta}}$ under the bootstrap assumptions, for s_0 large enough the time integration gives:

$$|V_1(s)| \leq \frac{s_0|V_1(s_0)|}{s} + O\left(\frac{\epsilon_{\ell+1}}{s^{\tilde{\eta}}}\right).$$

We recall the initial assumption $V_1(s_0) \leq \frac{1}{10s_0}$. For $\epsilon_{\ell+1}$ small enough the last equation becomes:

$$|V_1(s)| \leq \frac{1}{2s^{\tilde{\eta}}}.$$

□

We now fix all the constants of the analysis, and the constants of smallness, so that the last two lemmas hold. We just allow us to increase the initial time s_0 if necessary, as it still make these two lemmas hold.

Remark 3.4.5. we now know that s^* is characterized by:

$$(V_2(s^*), ..., V_\ell(s^*)) \in \mathcal{S}^{\ell-1}\left(\frac{1}{s^{\tilde{\eta}}}\right).$$

We fix $\varepsilon(s_0)$, $V_1(s_0)$ and $b_i(s_0)$ satisfying the smallness assumptions (*3.3.21*) and (*3.3.20*). we define the following application:

$$\begin{array}{llll} f: & \mathcal{D}(f) \subset \mathcal{B}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right) & \to & \mathcal{S}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right) \\ & (V_2(s_0), ...V_\ell(s_0)) & \mapsto & \frac{(s^*)^{\tilde{\eta}}}{s_0^{\tilde{\eta}}}(V_2(s^*), ..., V_\ell(s^*)), \end{array} \tag{3.4.12}$$

With domain:

$$\mathcal{D}(f) = \left\{(V_2(s_0), ..., V_\ell(s_0)) \in \mathcal{B}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right), \text{ such that } s^* < +\infty\right\}. \tag{3.4.13}$$

We prove in the following lemma that $\mathcal{D}$ is non empty, open in $\mathcal{B}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right)$, that f is continuous and equivalent to the identity on the sphere $\mathcal{S}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right)$.

Lemma 3.4.6. (Topological properties of f) *The following properties hold:*

(i) $\mathcal{D}(f)$ *is non empty and open, satisfying* $\mathcal{S}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right) \subset \mathcal{D}(f)$.

(ii) f *is continuous and is the identity on the sphere* $\mathcal{S}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right)$.

Proof of Lemma 3.4.6 We recall that V_i is the projection of U on the unstable direction v_i associated to the eigenvalue $\frac{i\alpha}{\ell-\alpha}$ of the matrix A_ℓ, see Lemma (3.2.17). To ease notation we will write $\mu_i := \frac{i\alpha}{\ell-\alpha}$ the eigenvalues. From the time evolution of U_i for $1\leq i\leq \ell$ computed in (3.4.1) we get that the time evolution of V_i is:

$$\begin{array}{rcl} V_{i,s} & = & \frac{\mu_i}{s}V_i + O(s^{-1-2\tilde{\eta}}) + O(s^{L-\ell}) + O(\epsilon_{\ell+1}s^{-1-\tilde{\eta}}) \\ & = & \frac{\mu_i}{s}V_i + O(\epsilon_{\ell+1}s^{-1-\tilde{\eta}}). \end{array}$$

Let $(V_2(s_0),...,V_\ell(s_0))\in \mathcal{S}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right)$ be an initial data on the sphere. We claim that $s^*=0$ which implies of course:

$$f((V_2(s_0)),...V_l(s_0))) = (V_2(s_0)),...V_l(s_0)).$$

This will prove that $\mathcal{D}(f)$ is non empty and that f is equivalent to the identity on $\mathcal{S}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right)$. To prove that, we just compute the scalar product between the time derivative of $(V_2(s),...V_\ell(s))$ and an outgoing normal vector to the sphere at the point $(V_2(s_0),...V_\ell(s_0))$:

$$(V_2(s_0),...V_\ell(s_0)).(V_{2,s}(s_0),...V_{\ell,s}(s_0)) = \sum_{i=2}^{\ell}\frac{\mu_i}{s_0}|V_i|^2 + O(\epsilon_{\ell+1}s_0^{-1-2\tilde{\eta}}) > 0$$

for $\epsilon_{\ell+1}$ small enough. In addition, this inequality uniformly holds on the sphere. For any small time s', we have that $(V_2(s_0+s'),...V_\ell(s_0+s'))$ is outside the ball, which implies $s^*=s_0$.

At $s=s_0$, this says that close to the border of the ball $\mathcal{B}^{\ell-1}(\frac{1}{s_0^{\tilde{\eta}}})$ the force term whose size is $O(\epsilon_{\ell+1}s_0^{-1-\tilde{\eta}})$ is overtaken by the linear repulsive dynamics. We are going to show that this is also true for $s_0\leq s\leq s^*$.

We now prove that f is continuous. Let s be such that $s_0\leq s\leq s^*$ and let $(V_2(s_0),...,V_\ell(s_0))$ be an initial data such that at time s, $\frac{1}{2s^{\tilde{\eta}}}\leq (V_2(s),...,V_\ell(s))$. The same computation gives:

$$\begin{array}{rcl} \frac{d}{ds}|V|^2 & = & (V_2(s),...V_\ell(s)).(V_{2,s}(s),...V_{\ell,s}(s)) \\ & \geq & \min((\mu_i)_{2\leq i\leq \ell})\frac{1}{4s^{1+2\tilde{\eta}}} + O(\frac{\epsilon_{\ell+1}}{s^{1+2\tilde{\eta}}}) \\ & > & 0, \end{array}$$

once again provided one has taken $\epsilon_{\ell+1}$ small enough. It implies that at time s fixed, there exists a small enough time $s^+>0$ and a small enough distance $r>0$ such that:

$$\frac{1}{s^{\tilde{\eta}}} - r \leq |V(s)| \leq \frac{1}{\tilde{\eta}} \text{ implies } s\leq s^*\leq s^+,$$

ie the orbit leaves the ball $\mathcal{B}^{\ell-1}(\frac{1}{s^{\tilde{\eta}}})$ in finite time. Let now $(V_2(s_0),...,V_\ell(s_0))$ be an initial data such that $s^*<+\infty$. Since the time evolution of V is a lipischitz continuous function of our problem, there is local continuity of the trajectories. Take $s^-<s^*$ close enough to s^* so that $1/s^{\tilde{\eta}}-\frac{r}{2}\leq |V(s^-)|$, there exists a small enough distance $r_0>0$ such that if $|V'(s_0)-V(s_0)|<r_0$ then $|V'(s)-V(s)|<\frac{r}{4}$ for $s_0\leq s\leq s^-$. The exit result we just stated implies that $s^-<s^*(V')$ and that $1/s^{\tilde{\eta}}-\frac{3r}{4}\leq V'(s^-)$. So that $s^-\leq s^*(V')\leq s^-+s^+$. We have proven that $\mathcal{D}(f)$ is open.

From direct inspection, with the use of continuity properties, it is easy to prove in the same spirit that the function s^* is continuous on $\mathcal{D}$, and that f is continuous too on $\mathcal{D}(f)$. $\square$

We have reached the end of the proof. Indeed, if for all choices of initial data $(V_2(s_0), ..., V_\ell(s_0))$ we had $s^* < +\infty$, ie that no solution stayed in the trapped regime for all time, then f would be a continuous function from the ball $\mathcal{B}^{\ell-1}(\frac{1}{s_0^{\tilde{\eta}}})$ onto the sphere $\mathcal{S}^{\ell-1}(\frac{1}{s_0^{\tilde{\eta}}})$ being equal to the identity at the border. This would be a contradiction to Brouwer's fixed point theorem. It implies the existence of at least one initial data $(V_2(s_0), ..., V_\ell(s_0)) \in \mathcal{B}^{\ell-1}\left(\frac{1}{s_0^{\tilde{\eta}}}\right)$ such that the solution of (NLW) stays in the trapped regime described by Proposition 3.3.2.

We now end the proof of the main theorem. We know from Proposition 3.3.2 that there exists an orbit satisfying the assumptions of the trapped regime. We have computed that in that case there exists a constant $c > 0$ such that:

$$\frac{1}{c} s^{-\frac{\ell}{\ell-\alpha}} \leq \lambda \leq c s^{-\frac{\ell}{\ell-\alpha}}.$$

Since $\frac{ds}{dt} = \frac{1}{\lambda}$ it gives:

$$\frac{1}{c'} s^{\frac{\ell}{\ell-\alpha}} \leq \frac{ds}{dt} \leq c' s^{\frac{\ell}{\ell-\alpha}}.$$

This is an explosive ODE, we have that there exists a maximal time T with:

$$s \sim C(\boldsymbol{u}(0))(T-t)^{-\frac{\ell-\alpha}{\alpha}} \text{ as } t \to T.$$

This implies:

$$\frac{1}{c}(T-t)^{\frac{\ell}{\alpha}} \leq \lambda(t) \leq c(T-t)^{\frac{\ell}{\alpha}} \text{ as } t \to T.$$

3.4.2 Behavior of Sobolev norms near blow-up time

We now prove the convergence of the norms (*2.2.10*), (*2.2.11*), (*2.2.12*) and (*2.2.13*). First note that our analysis relies only on the study of supercritical Sobolev norms $(\dot{H}^\sigma \cap \dot{H}^{s_L}) \times (\dot{H}^{\sigma-1} \cap \dot{H}^{s_L-1})$ for the perturbative term $\tilde{\boldsymbol{\alpha}}_{b,\frac{1}{\lambda}} + \boldsymbol{w}$. For this reason, the finiteness of the $\dot{H}^1 \times L^2$ norm of the initial data is not a requirement. Still, it is worth studying the behavior of lower order Sobolev norms because it applies when taking "nice" initial data, say smooth and with compact support, and because their asymptotic really corresponds to the concentration of a critical object. We still consider a solution described by Proposition 3.3.2 but now under the following decompositions:

$$\boldsymbol{u} = \boldsymbol{Q}_{\frac{1}{\lambda}} + \tilde{\boldsymbol{w}} = (\boldsymbol{Q} + \tilde{\boldsymbol{\varepsilon}})_{\frac{1}{\lambda}}, \ \text{ ie } \tilde{\boldsymbol{w}} = \boldsymbol{w} + \tilde{\boldsymbol{\alpha}}_{b,\frac{1}{\lambda}}, \ \text{ and } \tilde{\boldsymbol{\varepsilon}} = \boldsymbol{\varepsilon} + \tilde{\boldsymbol{\alpha}}_b, \tag{3.4.14}$$

$$\boldsymbol{u} = \chi \boldsymbol{Q}_{\frac{1}{\lambda}} + \boldsymbol{w}' = (\chi_{\frac{1}{\lambda}} \boldsymbol{Q} + \boldsymbol{\varepsilon}')_{\frac{1}{\lambda}}, \ \text{ ie } \boldsymbol{w}' = \tilde{\boldsymbol{w}} + ((1-\chi_{\frac{1}{\lambda}})\boldsymbol{Q})_{\frac{1}{\lambda}}, \ \text{ and } \boldsymbol{\varepsilon}' = \tilde{\boldsymbol{\varepsilon}} + (1-\chi_{\frac{1}{\lambda}})\boldsymbol{Q}. \tag{3.4.15}$$

We recall that the subscript $\frac{1}{\lambda}$ has a different meaning when it applies to χ, see (*3.1.12*). First note that because of (*3.3.27*) and because $\mathcal{E}_{s_L}$ controls the usual Sobolev norms, see (*3.D.25*), one has by interpolation:

$$\parallel \boldsymbol{\varepsilon} \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \underset{t \to T}{\to} 0 \text{ for all } \sigma \leq s \leq s_L. \tag{3.4.16}$$

Moreover, this convergence is also true for the perturbation on the manifold of approximate blow-up solutions:

$$\parallel \tilde{\boldsymbol{\alpha}}_b \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \underset{t \to T}{\to} 0 \text{ for all } \sigma \leq s \leq s_L.$$

so we get for the perturbation:

$$\| \tilde{\varepsilon} \|_{\dot{H}^s\times\dot{H}^{s-1}} \underset{t\to T}{\to} 0 \text{ for all } \sigma\leq s\leq s_L. \tag{3.4.17}$$

We suppose from now on that $\| \boldsymbol{u}(0) \|_{\dot{H}^1\times L^2}$ is finite. This implies the boundedness of the perturbation at initial time: $\| \boldsymbol{\varepsilon}'(0) \|_{\dot{H}^1\times L^2}=\| \boldsymbol{w}'(0) \|_{\dot{H}^1\times L^2}\leq C(\boldsymbol{u}(0))$. We show first that this last quantity stays bounded.

Lemma 3.4.7 (Boundedness in $\dot{H}^1\times L^2$). *Suppose $\boldsymbol{u}$ is a solution described by Proposition 3.3.2, such that $\boldsymbol{u}(0)\in\dot{H}^1\times L^2$. Then there exists a constant $C(\boldsymbol{u}(0))$ such that for all $0\leq t<T$:*

$$\| \boldsymbol{u} \|_{\dot{H}^1\times L^2}\leq C(\boldsymbol{u}(0)) \tag{3.4.18}$$

Proof of Lemma 3.4.7 We first compute that under the decomposition (*3.4.15*), the soliton's contribution to the $\dot{H}^1$ norm is finite:

$$\| \chi Q_{\frac{1}{\lambda}} \|_{\dot{H}^1}=\frac{1}{\lambda^{1-s_c}}\| \chi_{\frac{1}{\lambda}}Q \|_{\dot{H}^1}\leq \frac{1}{\lambda^{1-s_c}}C\left(\int_1^{\frac{1}{\lambda}} y^{d-\frac{4}{p-1}-2}\right)^{\frac{1}{2}}\leq C. \tag{3.4.19}$$

Therefore, the lemma is proven once we show that the $\dot{H}^1\times L^2$ norm of $\boldsymbol{w}'$ stays finite. We are going to prove this by computing its time evolution under the bootstrap regime. We claim that:

$$\frac{d}{dt}\| \boldsymbol{w}' \|^2_{\dot{H}^1\times L^2}\leq C\| \boldsymbol{w}' \|_{\dot{H}^1\times L^2}+C\sum_{k=1}^p \| \boldsymbol{w} \|^{2-c_k}_{\dot{H}^1\times L^2}\| \boldsymbol{w} \|^{c_k'}_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}, \tag{3.4.20}$$

where for each k, $0<c_k\leq 2$. We start by proving this bound. The time evolution of $\boldsymbol{w}'$ is:

$$\boldsymbol{w}'_t=\boldsymbol{L}+\frac{1}{\lambda}\boldsymbol{\mathfrak{F}}_{\frac{1}{\lambda}}+\frac{1}{\lambda}\boldsymbol{I}_{\frac{1}{\lambda}} \tag{3.4.21}$$

where $\boldsymbol{L}$ is the linear part, $\boldsymbol{L}:=\begin{pmatrix} w^{(2)} \\ \Delta w^{(1)}\end{pmatrix}$, $\boldsymbol{\mathfrak{F}}$ is the force term:

$$\boldsymbol{\mathfrak{F}}=\begin{pmatrix} \lambda_t\chi_{\frac{1}{\lambda}}\Lambda^{(1)}Q \\ \chi_{\frac{1}{\lambda}}Q^p(\chi_{\frac{1}{\lambda}}^{p-1}-1)+(\lambda^2(\partial_{rr}\chi)_{\frac{1}{\lambda}}+\frac{d-1}{r}\lambda(\partial_r\chi)_{\frac{1}{\lambda}})Q+2\lambda(\partial_r\chi)_{\frac{1}{\lambda}}\partial_r Q\end{pmatrix},$$

and $\boldsymbol{I}$ is the interaction term: $\boldsymbol{I}=\begin{pmatrix} 0 \\ \sum_{k=1}^p C_k(\chi_{\frac{1}{\lambda}}Q)^{p-k}(\varepsilon'^{(1)})^k\end{pmatrix}$. It leads to the following expression for the time derivative of the norm:

$$\frac{d}{dt}\| \boldsymbol{w}' \|^2_{\dot{H}^1\times L^2}=2\int\nabla w'^{(1)}.\nabla(L^{(1)}+\frac{1}{\lambda}\mathfrak{F}^{(1)})+2\int w'^{(2)}(L^{(2)}+\frac{1}{\lambda}\mathfrak{F}^{(2)}+\frac{1}{\lambda}I^{(2)}). \tag{3.4.22}$$

We now want to estimate everything in the right hand side of (*3.4.22*). The linear term's contribution is null:

$$\int\nabla w'^{(1)}.\nabla w'^{(2)}+w'^{(2)}\Delta w'^{(1)}=0. \tag{3.4.23}$$

We then compute the size of the force term. For the first coordinate:

$$\begin{aligned}\int\frac{1}{\lambda^2}|\nabla\mathfrak{F}^{(1)}|^2 &= \frac{1}{\lambda^2}\frac{1}{\lambda^{2(1-s_c)}}\int\lambda_t^2|\nabla(\chi_{\frac{1}{\lambda}}\Lambda^{(1)}Q)|^2\leq C\frac{1}{\lambda^{2(2-s_c)}}\int_1^{\frac{1}{\lambda}} y^{d-2\gamma-2-1}dy \\ &\leq C\frac{\lambda_t^2}{\lambda^{2(2-s_c)+d-2\gamma-2}}\leq C\lambda_t^2\lambda^{2\alpha-2}\leq C,\end{aligned} \tag{3.4.24}$$

because $\alpha > 2$ and $\lambda_t = b_1 \to 0$ as $t \to T$. For the second coordinate:

$$\begin{array}{rcl} \int \frac{1}{\lambda^2}|\mathcal{F}^{(2)}|^2 & = & \frac{1}{\lambda^2}\frac{1}{\lambda^{2(1-s_c)}}\Big(\int |\chi_{\frac{1}{\lambda}}Q^p(\chi_{\frac{1}{\lambda}}^{p-1}-1) \\ & & +(\lambda^2(\partial_{rr}\chi)_{\frac{1}{\lambda}}+\frac{d-1}{r}\lambda(\partial_r\chi)_{\frac{1}{\lambda}})Q+2\lambda(\partial_r\chi)_{\frac{1}{\lambda}}\partial_r Q|^2\Big) \\ & \leq & C\frac{1}{\lambda^{2(2-s_c)}}\int_1^{\frac{1}{\lambda}} y^{d-4-\frac{4}{p-1}-1}dy \leq C\frac{1}{\lambda^{2(2-s_c)}}\frac{1}{\lambda^{d-\frac{4}{p-1}-4}} = C. \end{array} \tag{3.4.25}$$

The bounds (*3.4.24*) and (*3.4.25*) imply the bound for the force term's contribution:

$$\left|\int \frac{1}{\lambda}\nabla w'^{(1)}.\nabla\mathcal{F}^{(1)} + \frac{1}{\lambda}w'^{(2)}\mathcal{F}^{(2)}\right| \leq C \parallel \boldsymbol{w'} \parallel_{\dot{H}^1\times L^2} . \tag{3.4.26}$$

We now turn to the L^2 norm of the interaction term. First we rescale:

$$\left|\frac{1}{\lambda}\int w'^{(2)}I^{(2)}\right| \leq \frac{C}{\lambda^{1+2(1-s_c)}}\sum_{k=1}^{p}\int |\varepsilon'^{(2)}|(\chi_{\frac{1}{\lambda}}Q)^{(p-k)}|\varepsilon'^{(1)}|^k. \tag{3.4.27}$$

We first take $k=1$. Because of the asymptotic $Q^{p-1} \sim \frac{c}{y^2}$ we use Hardy inequality and interpolation:

$$\begin{array}{rcl} \int |\varepsilon'^{(2)}|(\chi_{\frac{1}{\lambda}}Q)^{(p-1)}|\varepsilon'^{(1)}| & \leq & C \parallel \varepsilon'^{(2)} \parallel_{L^2}\parallel \nabla^2\varepsilon'^{(1)} \parallel_{L^2} \\ & \leq & C \parallel \varepsilon'^{(2)} \parallel_{L^2}\parallel \varepsilon'^{(1)} \parallel_{\dot{H}^1}^{\frac{\sigma-2}{\sigma-1}}\parallel \varepsilon'^{(1)} \parallel_{\dot{H}^\sigma}^{\frac{1}{\sigma-1}} . \end{array}$$

As $\frac{\sigma-2}{\sigma-1}(1-s_c)+\frac{\sigma-s_c}{\sigma-1} = 2-s_c$ this gives the the estimate when applying the scale change:

$$\frac{1}{\lambda^{1+2(1-s_c)}}\int |\varepsilon'^{(2)}|(\chi_{\frac{1}{\lambda}}Q)^{(p-1)}|\varepsilon'^{(1)}| \leq C \parallel w'^{(2)} \parallel_{L^2}\parallel w'^{(1)} \parallel_{\dot{H}^1}^{\frac{\sigma-2}{\sigma-1}}\parallel w'^{(1)} \parallel_{\dot{H}^\sigma}^{\frac{1}{\sigma-1}} . \tag{3.4.28}$$

Now let k be an integer, $2 \leq k \leq p$. We have the asymptotic: $Q^{p-k} \sim \frac{c}{y^{2\frac{p-k}{p-1}}}$. We put this weighted decay on $\varepsilon'^{(2)}$, use Hardy inequality and interpolation:

$$\parallel (\chi_{\frac{1}{\lambda}}Q)^{p-k}\varepsilon'^{(2)} \parallel_{L^2} \leq C \parallel \nabla^{\frac{2(p-k)}{p-1}}\varepsilon'^{(2)} \parallel_{L^2} \leq C \parallel \varepsilon'^{(2)} \parallel_{L^2}^{1-\theta}\parallel \nabla^{\sigma-1}\varepsilon'^{(2)} \parallel_{L^2}^{\theta} \tag{3.4.29}$$

for $\theta = \frac{2(p-k)}{(p-1)(\sigma-1)}$. From Sobolev injection $|\varepsilon'^{(1)}|^k \in L^q$ for $q \in [\frac{2d}{k(d-2)}, \frac{2d}{k(d-2\sigma)}]$. Because we work in a high dimension $d \geq 11$ and p is an integer ≥ 2 one has:

$$\frac{2d}{k(d-2)} \leq 2 \leq \frac{2d}{k(d-2\sigma)} = \frac{(p-1)d}{2k} + O(\sigma - s_c).$$

This implies that $\varepsilon'^{(1)k} \in L^2$ with the estimate:

$$\parallel \varepsilon'^{(1)k} \parallel_{L^2} = \parallel \varepsilon'^{(1)} \parallel_{L^{2k}}^k \leq C \parallel \varepsilon'^{(1)} \parallel_{\dot{H}^1}^{k(1-\theta')}\parallel \varepsilon'^{(1)} \parallel_{\dot{H}^\sigma}^{k\theta'}, \tag{3.4.30}$$

for $\frac{(1-\theta')(d-2)}{2d}+\frac{\theta'(d-2\sigma)}{2d} = \frac{1}{2k}$. The estimates (*3.4.29*) and (*3.4.30*) allow us to apply Cauchy Schwarz and find:

$$\int |\varepsilon'^{(2)}|(\chi_{\frac{1}{\lambda}}Q)^{(p-k)}|\varepsilon'^{(1)}|^k \leq C \parallel \varepsilon'^{(2)} \parallel_{L^2}^{1-\theta}\parallel \varepsilon'^{(2)} \parallel_{\dot{H}^{\sigma-1}}^{\theta}\parallel \varepsilon'^{(1)} \parallel_{\dot{H}^1}^{k(1-\theta')}\parallel \varepsilon'^{(1)} \parallel_{\dot{H}^\sigma}^{k\theta'} .$$

We now compute:

$$(1-\theta)(1-s_c)+\theta(\sigma-s_c)+k(1-\theta')(1-s_c)+k\theta'(\sigma-s_c = 1+2(1-s_c)).$$

Hence when applying the scale change the last estimate gives:

$$\frac{1}{\lambda^{1+2(1-s_c)}}\int |\varepsilon'^{(2)}|(\chi_{\frac{1}{\lambda}}Q)^{(p-k)}|\varepsilon'^{(1)}|^k \leq C \parallel \boldsymbol{w} \parallel_{\dot{H}^1\times L^2}^{1-\theta+k(1-\theta')}\parallel \boldsymbol{w} \parallel_{\dot{H}^\sigma\times \dot{H}^{\sigma-1}}^{\theta+k\theta'} \qquad (3.4.31)$$

we compute the power involved for the $\parallel \boldsymbol{w} \parallel_{\dot{H}^1\times L^2}$ term:

$$1-\theta+k(1-\theta')=2-\frac{1-(k-1)(\sigma-s_c)}{\sigma-1}=2-c_k.$$

We now go back to the expression (*3.4.27*). We have computed the right hand side for the linear case in (*3.4.28*), and in the non linear case in (*3.4.31*). We have computed the coefficient condition for the non linear case in the last equation (it is straightforward in the linear case). Therefore we have the following estimate for the interaction term:

$$\left|\frac{1}{\lambda}\int w'^{(2)}I^{(2)}\right| \leq C\sum_{k=1}^{p}\parallel \boldsymbol{w} \parallel_{\dot{H}^1\times L^2}^{2-c_k}\parallel \boldsymbol{w} \parallel_{\dot{H}^\sigma\times \dot{H}^{\sigma-1}}^{c_k'}. \qquad (3.4.32)$$

We now come back to the identity (*3.4.22*). We estimated the right hand side in (*3.4.23*), (*3.4.26*) and (*3.4.32*), proving the bound (*3.4.20*) we claimed. We now integrate this equation in time. We recall that $\boldsymbol{w}'=\boldsymbol{w}+\tilde{\boldsymbol{\alpha}}_{b,\frac{1}{\lambda}}+(1-\chi_{\frac{1}{\lambda}}\boldsymbol{Q})_{\frac{1}{\lambda}}$. We take s slightly supercritical: $s_c<s\leq\sigma$. The profile $\tilde{\boldsymbol{\alpha}}_{b,\frac{1}{\lambda}}$ has finite supercritical norm:

$$\parallel \tilde{\boldsymbol{\alpha}}_{b,\frac{1}{\lambda}} \parallel_{\dot{H}^s\times\dot{H}^{s-1}} \underset{t\to T}{\to} 0. \qquad (3.4.33)$$

The tail of the soliton has also a bounded size:

$$\parallel ((1-\chi_{\frac{1}{\lambda}})\boldsymbol{Q})_{\frac{1}{\lambda}} \parallel_{\dot{H}^s\times\dot{H}^{s-1}}\leq C. \qquad (3.4.34)$$

From the bound (*3.3.27*), the same property holds for $\boldsymbol{w}$ for $s=\sigma$: $\parallel \boldsymbol{w} \parallel_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\leq C$. Consequently, we have the boundedness of the σ Sobolev norm for $\boldsymbol{w}'$:

$$\parallel \boldsymbol{w}' \parallel_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\leq C.$$

Coming back to the identity (*3.4.20*) it gives:

$$\frac{d}{dt}\parallel \boldsymbol{w}' \parallel_{\dot{H}^1\times L^2}^2\leq C\parallel \boldsymbol{w}' \parallel_{\dot{H}^1\times L^2}+C\sum_{k=1}^{p}\parallel \boldsymbol{w} \parallel_{\dot{H}^1\times L^2}^{2-c_k}.$$

The growth of this quantity is sub linear: it stays bounded until time T. □

We now know from the previous Lemma 3.4.7 that our solution stays bounded in L^2 until blow-up time. Using (*3.4.19*) we have that:

$$\parallel \boldsymbol{w}' \parallel_{\dot{H}^1\times L^2}\leq C.$$

This implies for the renormalized error:

$$\parallel \boldsymbol{\varepsilon}' \parallel_{\dot{H}^1\times L^2}\leq \lambda^{1-s_c}C.$$

On the other hand, the bootstrap bound (*3.3.27*) gives:

$$\parallel \boldsymbol{\varepsilon}' \parallel_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\leq \lambda^{\sigma-s_c}C.$$

By interpolation, we get that for any $1 \leq s \leq \sigma$:

$$\parallel \boldsymbol{\varepsilon}' \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \leq \lambda^{s-s_c} C. \tag{3.4.35}$$

We now come back to the decomposition: $\boldsymbol{\varepsilon}' = \boldsymbol{\varepsilon} + \tilde{\boldsymbol{\alpha}}_b + (1 - \chi_{\frac{1}{\lambda}} \boldsymbol{Q})$. From (*3.4.33*) and (*3.4.34*) the perturbation $\tilde{\boldsymbol{\alpha}}_b$ and the tail of the solitary waves enjoy the bound:

$$\parallel \tilde{\boldsymbol{\alpha}}_b + (1 - \chi_{\frac{1}{\lambda}}) \boldsymbol{Q} \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \leq \lambda^{s-s_c} C.$$

Combined with the previous bound (*3.4.35*), it gives for the original error term:

$$\parallel \boldsymbol{\varepsilon} \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \leq \lambda^{s-s_c} C \to 0 \text{ as } t \to T.$$

This proves the convergence to 0 of the renormalized perturbation in slightly supercritical norms:

$$\parallel \tilde{\boldsymbol{\varepsilon}} \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \to 0 \quad \text{as} \quad t \to T, \quad \text{for} \quad s_c < s \leq \sigma. \tag{3.4.36}$$

We now put (*3.4.17*) and (*3.4.36*) together: for any $s_c < s \leq s_L$,

$$\parallel \tilde{\boldsymbol{\varepsilon}} \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \to 0 \quad \text{as} \quad t \to T. \tag{3.4.37}$$

Now we turn to subcritical Sobolev norms. Let s be such that $1 \leq s < s_c$. From (*3.4.35*), the perturbation has finite subcritical norms:

$$\parallel \boldsymbol{w}' \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \leq C.$$

As the localized soliton also has finite subcritical norms:

$$\parallel (\chi_{\frac{1}{\lambda}} \boldsymbol{Q})_{\frac{1}{\lambda}} \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \leq C,$$

this means that the full solution stays bounded in subcritical norms:

$$\parallel \boldsymbol{u} \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \leq C(\boldsymbol{u}(0)). \tag{3.4.38}$$

We now turn the the critical norm. From (*3.4.35*), the perturbation has finite critical and slightly supercritical norms:

$$\parallel \boldsymbol{w}' \parallel_{\dot{H}^s \times \dot{H}^{s-1}} \leq C(\boldsymbol{u}(0)) \text{ for } s_c \leq s \leq \sigma$$

As the soliton is located on the first coordinate, this implies the boundedness of the time derivative in the critical and slightly critical spaces:

$$\parallel u_t^{(1)} \parallel_{\dot{H}^{s-1}} = \parallel u^{(2)} \parallel_{\dot{H}^{s-1}} \leq C(\boldsymbol{u}(0)) \text{ for } s_c \leq s \leq \sigma \tag{3.4.39}$$

The critical norm for the first coordinate comes then from the soliton cut in a fixed zone:

$$\parallel u^{(1)} \parallel_{\dot{H}^{s_c}} \sim \parallel \chi Q_{\frac{1}{\lambda}} \parallel_{\dot{H}^{s_c}} = C(d,p) \sqrt{\ell} \sqrt{|\log(T-t)|}(1 + o(1)) \quad \text{as } t \to T. \tag{3.4.40}$$

We have therefore proved all the bounds at the different levels of regularity that we claimed in Theorem 2.2.4.

3.5 Lipschitz aspect and codimension of the set of solutions described by Proposition 3.3.2

We first recall the main arguments taken from the previous sections that highlight why there should be a manifold structure for the blow up solutions we constructed there. In Proposition 3.2.14, we have constructed an approximate blow up profile described by a set of $L+1$ parameters $\lambda, b_1, ..., b_L$: $\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}}$. We studied the approximate dynamics of (NLW) for such profiles, and found in Lemma 3.2.16 that for each integer $\ell > \alpha$, the time dependent profile $\tilde{\boldsymbol{Q}}_{b^e,\frac{1}{\lambda^e}}$ was a good approximate blow up solution. In Proposition 3.3.2, we showed the existence of a real solution of (NLW), under the form $\tilde{\boldsymbol{Q}}_{b^e+(\frac{U_1}{s},...,\frac{U_L}{s^L}),\frac{1}{\lambda}} + \boldsymbol{w}$, that stayed close to this approximate blow up solution.

To prove it, we studied the parameters $V_1, ..., V_\ell, U_{\ell+1}, .., U_L$ (obtained from the U_i's by a linear change of variables). We showed that at leading order, $V_1, U_{\ell+1}, ..., U_L$ were evolving according to a stable linear dynamics, whereas $V_2, ..., V_\ell$ were evolving via a unstable linear one. The error $\boldsymbol{w}$ was shown to be a stable perturbation. For each initial values of the stable parameters $V_1(s_0), U_{\ell+1}(s_0), ..., U_L(s_0)$ and error $\boldsymbol{w}(s_0)$, we proved in Lemma 3.4.6 that we could apply Brouwer's continuity argument to find the existence of at least one initial perturbation $V_2(s_0), ..., V_\ell(s_0)$ such that the orbit $V_2, ..., V_\ell$ stayed small., giving the existence of the real blow up solution.

Now one could wonder: is the choice $V_2(s_0), ...V_\ell(s_0)$ unique? If yes, how does it depend on the initial perturbation along the stable directions $V_1(s_0), U_{\ell+1}(s_0)$,..., $U_L(s_0)$ and $\boldsymbol{w}(s_0)$? We show in this section the uniqueness and the Lipschitz dependence. It will imply that the set of type II blow up solutions described by Proposition 3.3.2 is a Lipschitz manifold of codimension $\ell - 1$.

Theorem 3.5.1. *We keep the assumptions and notations of Proposition 3.3.2, and recall that σ and s_L are defined in (3.3.13) and (3.1.8). There exists a choice of constants implied in this proposition such that its result still holds, and that moreover the set of initial data leading to such solutions is a locally Lipschitz manifold of codimension $\ell - 1$ in the space $\left(\dot{H}^\sigma \cap \dot{H}^{s_L}\right) \times \left(\dot{H}^{\sigma-1} \cap \dot{H}^{s_L-1}\right)$.*

Roughly speaking, the proof of Theorem 3.5.1 is the adaptation of everything we did in the proof of Proposition 3.3.2, this time to study the difference of two solutions and to see what informations we can get. For this reason, some technical points in the proofs to come will be treated in a faster way whenever we already treated them in Section 3.

Our strategy of the proof is the following:

(i) *Lipschitz aspect of the unstable modes under extra assumptions.* We first prove that for initial data starting at the same scale and having extra regularity assumptions, the coefficients of the unstable modes $V_2(s_0), ..., V_\ell(s_0)$ have Lipschitz dependence on the stable modes $V_1(s_0)$, $b_{\ell+1}(s_0), ..., b_L(s_0)$ and $\boldsymbol{w}(s_0)$.

(ii) *removal of the extra assumptions.* We then show how to remove the extra assumptions we needed in the first step: it just consists in performing a lower order decomposition at initial time. Instead of studying the decomposition $\boldsymbol{U} = (\tilde{\boldsymbol{Q}}_b + \boldsymbol{\varepsilon})_{\frac{1}{\lambda}}$ for b a L-tuple $b = (b_1, ..., b_L)$, we study the decomposition $\boldsymbol{U} = (\tilde{\boldsymbol{Q}}_{\bar{b}} + \bar{\boldsymbol{\varepsilon}})_{\frac{1}{\lambda}}$ for $\bar{b}$ a $L-1$-tuple. We apply the result of the first step to this new decomposition. As b_L is small because it represents a small perturbation along the last stable mode, it imply the result for the original decomposition.

3.5.1 Lipschitz dependence of the unstable modes under extra assumptions

We now perform the first part of the analysis. Let $\boldsymbol{U}$ and $\boldsymbol{U}'$ be two solutions described by Proposition (3.3.2). For $\boldsymbol{U}$ we keep the notations introduced in the analysis throughout the previous section. For $\boldsymbol{U}'$ we adapt them:

$$\boldsymbol{U'} := (\boldsymbol{Q}_{b'} + \boldsymbol{\varepsilon'})_{\frac{1}{\lambda'}} = \boldsymbol{Q}_{b', \frac{1}{\lambda'}} + \boldsymbol{w}',$$

with ε' satisfying the orthogonality conditions (*3.3.9*). Its renormalized time is s' (defined by (*3.3.15*)), its associated scale λ', and associated parameters U' and V'. We use the same notation for the norms of the error we already used and introduce a higher derivative adapted norm:

$$\mathcal{E}'_\sigma := \int |\nabla^\sigma \varepsilon'^{(1)}|^2 + |\nabla^{\sigma-1} \varepsilon'^{(2)}|^2, \ \mathcal{E}_{s_L} := \int |\varepsilon'^{(1)}_{s_L}|^2 + |\varepsilon'^{(2)}_{s_L - 1}|^2,$$

$$\mathcal{E}'_{s_L+1} := \int |\varepsilon'^{(1)}_{s_L+1}|^2 + |\varepsilon'^{(2)}_{s_L}|^2$$

where f_k, the k-th adapted derivative of f, is defined by (*3.2.21*). We introduce the following notations for the differences:

$$\triangle U'_i := U_i - U'_i, \ \triangle V_i := V_i - V'_i, \ \triangle V_{uns} := (\triangle V_2, ..., \triangle V_\ell), \tag{3.5.1}$$

$$\triangle \mathcal{E}_{s_L} := \int |(\varepsilon^{(1)} - \varepsilon'^{(1)})_{s_L}|^2 + |(\varepsilon^{(2)} - \varepsilon'^{(2)})_{s_L-1}|^2, \tag{3.5.2}$$

$$\triangle \mathcal{E}_\sigma := \int |\nabla^\sigma (\varepsilon^{(1)} - \varepsilon'^{(1)})|^2 + |\nabla^{\sigma-1} (\varepsilon^{(2)} - \varepsilon'^{(2)})|^2. \tag{3.5.3}$$

In the analysis, it will be easier to use the following renormalized differences:

$$\triangle_r \mathcal{E}_{s_L} := b_1^{-2L-2(1-\delta_0)(1+\frac{\eta}{2})} \triangle \mathcal{E}_{s_L}, \ \triangle_r \mathcal{E}_\sigma := b_1^{-2(\sigma - s_c)(1+\nu)} \triangle \mathcal{E}_\sigma. \tag{3.5.4}$$

The presence of $\frac{\eta}{2}$ instead of the usual η is just technical. Here is the main proposition of this subsection, the Lipschitz dependence of the unstable coefficients under some extra assumptions: the two solutions start at the same scale and have some additional regularity.

Proposition 3.5.2. *Suppose that $\boldsymbol{U}_0 = \boldsymbol{Q}_{b_0} + \boldsymbol{\varepsilon}_0$ and $\boldsymbol{U'}_0 = \boldsymbol{Q}_{b'_0} + \boldsymbol{\varepsilon'}_0$ are two initial data of solutions described by Proposition* (3.3.2)*, starting at the same scale*[21]. *Suppose that they are close initially:*

$$b_0 = b^e(s_0) + \left(\frac{U_1(s_0)}{s_0}, ..., \frac{U_L(s_0)}{s_0^L} \right), b'_0 = b^e(s_0) + \left(\frac{U'_1(s_0)}{s_0}, ..., \frac{U'_L(s_0)}{s_0^L} \right), \tag{3.5.5}$$

which means $s_0 = s'_0$. Suppose moreover that we have the following additional regularity for $\boldsymbol{\varepsilon}'$:

$$\mathcal{E}'_{s_L+1}(s') \leq K_3 (b'_1)^{(2L+2+2(1-\delta_0)(1+\eta)}, \ for \ all \ s_0 \leq s', \tag{3.5.6}$$

[21]Indeed here one has $\lambda(0) = \lambda'(0) = 1$

for some constant $K_3 = K_3(K_1, K_2)$. Then there exist a constant $C > 0$ such that for s_0 small enough the following bound at initial time holds:

$$|\triangle V_{uns}(s_0)| \leq C\left(|\triangle V_1(s_0)| + \sum_{\ell+1}^{L}|\triangle U_i(s_0)| + \sqrt{\triangle_r \mathcal{E}_\sigma(s_0)} + \sqrt{\triangle_r \mathcal{E}_{s_L}(s_0)}\right). \quad (3.5.7)$$

The next subsubsections are devoted to the proof of this Proposition. We first introduce an adapted time for comparison $\hat{s}'$, and associate to $\boldsymbol{U'}$ the adapted variables for the analysis $\hat{\boldsymbol{\varepsilon}}'$, $\hat{U}'$ and $\hat{V}'$. We then write the time evolution equation for the differences of the parameters and error, yielding a system of coupled equations. We study this system and we show that if the initial size of the difference of the unstable parameters is too big compared to the initial size of the differences of the stable parameters and error, then repellency wins and it cannot converge to zero, preventing one of the two solutions to stay forever in the trapped regime.

3.5.1.1 Adapted time for comparison, notations and identities

The two solutions $\boldsymbol{U}$ and $\boldsymbol{U'}$ might blow up at different times. In addition, we have seen that the values of λ, s and the parameters b are correlated, see the equivalences in the trapped regime (*3.2.73*), (*3.3.25*), (*3.3.26*) and (*3.4.1*). Thus, we do not compare $\boldsymbol{U'}$ to $\boldsymbol{U}$ at the same time t, but at the times for which their scale coincide: $\lambda = \lambda'$.

Definition 3.5.3 (adapted time and variables for comparison). We define

$$\hat{s}'(s) = (\lambda')^{-1}(\lambda(s)), \quad (3.5.8)$$

as the adapted time for comparison, where $\lambda' : [s_0; +\infty[\to]0;1]$ is seen as C^1 diffeomorphism (remember that $\lambda'_{s'} \sim -\lambda' \frac{c_1}{s'} < 0$ from (*3.3.36*)). We associate to $\boldsymbol{U'}$ the variables $\hat{\varepsilon}'$, $\hat{b}'$, $\hat{U}'$, $\hat{V}'$ defined by (P_ℓ being defined in (*3.2.78*)):

$$\hat{\boldsymbol{w}}'(t) = \boldsymbol{w}'(t(\hat{s}'(s))), \quad \hat{\boldsymbol{\varepsilon}}'(s) = \boldsymbol{\varepsilon}'(\hat{s}'(s)), \quad \hat{b}'(s) = b'(\hat{s}'(s)), \quad (3.5.9)$$

$$\hat{U}_i'(s) = \left(\frac{s}{\hat{s}'(s)}\right)^i U_i'(\hat{s}'(s)), \text{ for } 1 \leq i \leq L, \text{ and } \hat{V}' = P_\ell(\hat{U}'). \quad (3.5.10)$$

We use the following notations for the norms of $\hat{\varepsilon}'$:

$$\hat{\mathcal{E}}'_\sigma := \int |\nabla^\sigma \hat{\varepsilon}'^{(1)}|^2 + |\nabla^{\sigma-1}\hat{\varepsilon}'^{(2)}|^2, \ \hat{\mathcal{E}}_{s_L+i} := \int |\hat{\varepsilon}'^{(1)}_{s_L+i}|^2 + |\hat{\varepsilon}'^{(2)}_{s_L-1+i}|^2, \ i = 0, 1. \quad (3.5.11)$$

We now prove that the times s and $\hat{s}'$ are close. All the analysis bounds of the trapped regime for $\boldsymbol{U}'$, expressed in function of $\hat{b}_1'$, then still hold interchanging $\hat{b}_1'$ with b_1.

Lemma 3.5.4 (Bounds on the change of variables). *The following bound holds:*

$$\hat{s}' = s(1 + O(s_0^{-\tilde{\eta}})). \quad (3.5.12)$$

The bounds of the trapped regime (*3.3.27*) *and the bound* (*3.5.6*) *can be written as:*

$$\hat{\mathcal{E}}'_{s_L+i} \leq 2K_2 b_1^{2L+2i+2(1-\delta_0)(1+\eta)}, \ i = 0, 1 \textit{ and } \hat{\mathcal{E}}'_\sigma \leq 2K_1 b_1^{2(\sigma - s_L)(1+\nu)}. \quad (3.5.13)$$

The parameters also enjoy the same estimates:

$$|\hat{V}_1'| \leq \frac{C}{s^{\tilde{\eta}}}, \ |\hat{V}_i'| \leq \frac{C}{s^{\tilde{\eta}}} \text{ for } 2 \leq i \leq \ell, \ |\hat{b}_i'| \leq \frac{C}{s^{i+\tilde{\eta}}} \text{ for } \ell+1 \leq i \leq L, \tag{3.5.14}$$

the constant C being independent of the other parameters. Moreover, $\hat{\varepsilon}'$ still enjoys the orthogonality conditions:

$$\langle \hat{\varepsilon}', \boldsymbol{H}^* \boldsymbol{\Phi}_M \rangle = 0, \text{ for } 1 \leq i \leq L. \tag{3.5.15}$$

Proof of Lemma 3.5.4. The orthogonality conditions are a straightforward consequence of those for ε', see (*3.3.9*). We use the formula (*3.4.1*) relating λ and s:

$$\lambda(t) = \left(\frac{s_0}{s(t)}\right)^{\frac{l}{l-\alpha}} (1 + O(s_0^{-\tilde{\eta}})).$$

This implies:

$$s(t) = \frac{s_0}{\lambda(t)^{\frac{l-\alpha}{l}}}(1+O(s_0^{-\tilde{\eta}})), \quad \text{and} \quad \hat{s}'(s) = \frac{s_0}{\lambda(t)^{\frac{l-\alpha}{l}}}(1+O(s_0^{-\tilde{\eta}})).$$

From that we get the first bound of the lemma: $\frac{s(t)}{s'(t')} = 1 + O(s_0^{-\tilde{\eta}})$. Now we recall that in the trapped regime: $b_1(s) = \frac{c_1}{s} + \frac{U_1}{s} = \frac{c_1}{s} + O(s^{-1-\tilde{\eta}})$ and $\hat{b}_1'(s) = \frac{c_1}{\hat{s}'} + \frac{U_1'(\hat{s}')}{\hat{s}'} = \frac{c_1}{\hat{s}'} + O((\hat{s}')^{-1-\tilde{\eta}})$. Hence, (*3.5.12*) implies $\frac{b_1(s)}{\hat{b}_1'(s)} = 1 + O(s_0^{-\tilde{\eta}})$. The bounds (*3.5.13*) and (*3.5.14*) are just a rewriting of the bootstrap bounds (*3.3.27*) and (*3.5.6*) knowing this equivalence.

□

We use the following notation for the differences (all terms taken at time s):

$$\triangle \hat{b}_i := b_i - \hat{b}_i', \ \triangle \hat{U}_i := U_i - \hat{U}_i', \ \triangle \hat{V}_i := V_i - \hat{V}_i', \ \triangle \hat{V}_{\text{uns}} := (\triangle \hat{V}_2, ..., \triangle \hat{V}_\ell) \tag{3.5.16}$$

$$\triangle \hat{\mathcal{E}}_{s_L} := \int |(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L}|^2 + |(\varepsilon^{(2)} - \hat{\varepsilon}'^{(2)})_{s_L - 1}|^2, \tag{3.5.17}$$

$$\triangle \hat{\mathcal{E}}_\sigma := \int |\nabla^\sigma (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})|^2 + |\nabla^{\sigma-1}(\varepsilon^{(2)} - \hat{\varepsilon}'^{(2)})|^2, \tag{3.5.18}$$

In the analysis, it will be easier to use the following renormalized differences:

$$\triangle_r \hat{\mathcal{E}}_{s_L} := b_1^{-2L-2(1-\delta_0)(1+\frac{\eta}{2})} \triangle \hat{\mathcal{E}}_{s_L}, \quad \triangle_r \hat{\mathcal{E}}_\sigma := b_1^{-2(\sigma - s_c)(1+\nu)} \triangle \hat{\mathcal{E}}_\sigma, \tag{3.5.19}$$

The coefficient $\frac{\eta}{2}$ instead of the η we had previously is because we will loose a bit in the analysis later on. We adapt the notation for the terms involved the analysis[22]:

$$\boldsymbol{\psi}_{\hat{b}'}(s) := \tilde{\boldsymbol{\psi}}_{b'}(\hat{s}'), \ \hat{\boldsymbol{L}}'(s) := \boldsymbol{L}'(\boldsymbol{\varepsilon}')(\hat{s}'(s)), \ \hat{\boldsymbol{NL}}'(s) := \boldsymbol{NL}'(\boldsymbol{\varepsilon}'(\hat{s}'(s))). \tag{3.5.20}$$

$$\hat{\boldsymbol{Mod}}'(s) := \frac{d\hat{s}'}{ds} \tilde{\boldsymbol{Mod}}'(\hat{s}'(s)), \ \hat{B}_1' := (\hat{b}_1')^{-(1+\eta)}, \text{ and } \hat{\boldsymbol{S}}_i' := \boldsymbol{S}_i(\hat{b}'). \tag{3.5.21}$$

The change of variables of Definition (3.5.3) produces the following identities:

$$\hat{b}_i'(s) = \frac{c_i}{\hat{s}'^i} + \frac{\hat{U}_i'}{s^i}, \tag{3.5.22}$$

[22] We do not mention the dependance of $\boldsymbol{L}$ and $\boldsymbol{NL}$ in ε and $\boldsymbol{w}$ anymore to ease notations, as it will be clear to which variable we are refering to in future computations.

$$\begin{array}{rcl} \hat{\boldsymbol{Mod}}'(s) & = & \chi_{\hat{B}_1'} \sum_{i=1}^{L} (\hat{b}_{i,s}' + \frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}_1'\hat{b}_i' - \hat{b}_{i+1}')) \left(\boldsymbol{T}_i + \sum_{j=i+1}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i}(\hat{b}') \right) \\ & & -(\frac{\lambda_s}{\lambda} + \frac{d\hat{s}'}{ds}\hat{b}_1')\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_{\hat{b}'}. \end{array} \tag{3.5.23}$$

We introduce the following notation for $1 \leq i \leq L$:

$$\begin{array}{rcl} \triangle\hat{\boldsymbol{Mod}}_i & := & (b_{i,s} + (i-\alpha)b_1 b_i - b_{i+1})\chi_{B_1}(\boldsymbol{T}_i + \sum\limits_{j=i+1}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i}) \\ & & -(\hat{b}_{i,s}' + \frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}_1'\hat{b}_i' - \hat{b}_{i+1}'))\chi_{\hat{B}_1'}(\boldsymbol{T}_i + \sum\limits_{j=i+1}^{L+2} \frac{\partial \hat{\boldsymbol{S}}_j'}{\partial b_i}), \end{array} \tag{3.5.24}$$

and define:

$$\triangle\hat{\boldsymbol{Mod}}_0 := -(\frac{\lambda_s}{\lambda} + b_1)\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b + (\frac{\lambda_s}{\lambda} + \frac{d\hat{s}'}{ds}\hat{b}_1')\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_{\hat{b}'} \tag{3.5.25}$$

So that $\tilde{\boldsymbol{Mod}} - \hat{\boldsymbol{Mod}}' = \sum\limits_{i=0}^{L} \triangle\hat{\boldsymbol{Mod}}_i$. With these new notations the time evolution of the difference of errors in renormalized variables is given by:

$$\begin{array}{rl} & \frac{d}{ds}(\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}') - \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda}(\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}') + \boldsymbol{H}(\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}') + (1 - \frac{d\hat{s}'}{ds})\boldsymbol{H}(\hat{\boldsymbol{\varepsilon}}') \\ = & -\tilde{\boldsymbol{Mod}} + \hat{\boldsymbol{Mod}}' - \tilde{\boldsymbol{\psi}}_b + \frac{d\hat{s}'}{ds}\hat{\boldsymbol{\psi}}_{\hat{b}'} + \boldsymbol{NL} - \frac{d\hat{s}'}{ds}\hat{\boldsymbol{NL}}' + \boldsymbol{L} - \frac{d\hat{s}'}{ds}\hat{\boldsymbol{L}}'. \end{array} \tag{3.5.26}$$

The time evolution of the original variables $\boldsymbol{w} - \hat{\boldsymbol{w}}'$ is:

$$\begin{array}{rl} & \frac{d}{dt}(\boldsymbol{w} - \hat{\boldsymbol{w}}') + \boldsymbol{H}_{\frac{1}{\lambda}}(\boldsymbol{w} - \hat{\boldsymbol{w}}') + (1 - \frac{d\hat{s}'}{ds})\boldsymbol{H}_{\frac{1}{\lambda}}(\hat{\boldsymbol{w}}') \\ = & -\frac{1}{\lambda}\tilde{\boldsymbol{Mod}}_{\frac{1}{\lambda}} + \frac{1}{\lambda}\hat{\boldsymbol{Mod}}'_{\frac{1}{\lambda}} - \frac{1}{\lambda}\tilde{\boldsymbol{\psi}}_{b,\frac{1}{\lambda}} + \frac{d\hat{s}'}{ds}\frac{1}{\lambda}\hat{\boldsymbol{\psi}}_{\hat{b}'} + \boldsymbol{NL} - \frac{d\hat{s}'}{ds}\hat{\boldsymbol{NL}}' + \boldsymbol{L} - \frac{d\hat{s}'}{ds}\hat{\boldsymbol{L}}'. \end{array} \tag{3.5.27}$$

3.5.1.2 Modulation equations for the difference

In this subsection we compute the time evolution of the difference of parameters between the first solution and the modified second solution defined in Definition (3.5.3). We relate it to the difference $\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}'$ and itself. We start with a technical lemma linking the differences of some profiles to the differences of the parameters.

Lemma 3.5.5 (Asymptotic for some differences of profiles for $y \leq 2B_0$:). *The following bounds hold, k denoting an integer $k \in \mathbb{N}$.*

(i) Differences of potentials: *For $1 \leq j \leq p-1$:*

$$|\partial_y^k((\tilde{Q}_{\hat{b}'}^{(1)})^{p-j} - (\tilde{Q}_b^{(1)})^{p-j})| \leq \frac{Cb_1}{1 + y^{\frac{2(p-j)}{p-1} - 1 + \alpha + k - C_k\eta}} \underset{1\leq i\leq L}{\textit{sup}} (b_1^{-i}|\triangle\hat{b}_i|). \tag{3.5.28}$$

(ii) Difference of the errors in the central zone: *For $y \leq 2B_0$, one has that $\tilde{\boldsymbol{\psi}}_b - \tilde{\boldsymbol{\psi}}_{\hat{b}'} = \begin{pmatrix} 0 \\ \psi_b - \psi_{\hat{b}'} \end{pmatrix}$ is on the second coordinate and there holds:*

$$|\partial_y^k(\psi_b - \psi_{\hat{b}'})| \leq \frac{Cb_1^{L+3}}{1 + y^{\gamma + g' - L - 1 + k}} \underset{1\leq i\leq L}{\textit{sup}} (b_1^{-i}|\triangle\hat{b}_i|). \tag{3.5.29}$$

Proof of Lemma 3.5.5 Step 1: *Differences of polynomials of parameters.* We claim that for any L-tuple J there holds:

$$|b^J - \hat{b}'^J| \leq C b_1^{|J|_2} \sup_{1\leq i\leq L} (b_1^{-i}|\triangle \hat{b}_i|). \tag{3.5.30}$$

We recall the notations $|J|_1 = \sum_{i=1}^{L} J_i$ and $|J|_2 = \sum_{i=1}^{L} iJ_i$. We show this bound by iteration. It is true for the trivial case $|J|_1 = 0$. Take now $i \geq 1$ and suppose that it is true for all J' satisfying $|J'|_1 \leq i-1$. Take J satisfying $|J|_1 = i$. Let j be the first coordinate for which J is non null. We have then:

$$b^J - \hat{b}'^J = b_j b^{J'} - \hat{b}'_j \hat{b}'^{J'} = (b_j - \hat{b}'_j) b^{J'} + \hat{b}'_j (b^{J'} - \hat{b}'^{J'})$$

for some L tuple J' satisfying $|J'|_1 = i-1$ and $|J'|_2 = |J|_2 - j$. The bound (*3.5.14*) imply that the parameters of the two solutions have the same size: $|b_j|, |\hat{b}'_j| \lesssim b_1^j$. For the first term of the previous identity one then has:

$$|(b_j - \hat{b}'_j) b^{J'}| \leq C |\triangle \hat{b}_j| b_1^{|J'|_2} \leq C b_1^{|J|_2} (b_1^{-j} |\triangle \hat{b}_j|).$$

For the second term, from the induction hypothesis for J' one has:

$$|\hat{b}'_j (b^{J'} - \hat{b}'^{J'})| \leq C b_1^j b_1^{|J'|_2} \sup_{1\leq i\leq L} (b_1^{-i}|\triangle \hat{b}_i|) = C b_1^{|J|_2} \sup_{1\leq i\leq L} (b_1^{-i}|\triangle \hat{b}_i|).$$

This implies that the property is true for i.

Proof of (i): The difference of the two potentials is:

$$(\tilde{Q}_{\hat{b}'}^{(1)})^{p-j} - (\tilde{Q}_b^{(1)})^{p-j} = \sum_{i=1}^{p-1} C_i Q^{p-j-i} \left(\chi^i_{\hat{B}'_1} ((\alpha_{\hat{b}'}^{(1)})^i - (\alpha_b^{(1)})^i) + (\alpha_b^{(1)})^i (\chi^i_{\hat{B}'_1} - \chi^i_{B_1}) \right) \tag{3.5.31}$$

for some constants $(C_i)_{1\leq i\leq p-j}$. Let i be fixed, with $1 \leq i \leq p-1$. We first study the first term in the right hand side of (*3.5.31*). There holds:

$$(\alpha_{\hat{b}'}^{(1)})^i - (\alpha_b^{(1)})^i = \sum_{|J|_1 = i} C_J \Big(\prod_{n=2 \text{ even}}^{L-1} b_n^{J_n} T_n^{J_n} \prod_{n=2 \text{ even}}^{L+1} S_n^{\tilde{J}_n} - \prod_{n=2 \text{ even}}^{L-1} \hat{b}_n'^{J_n} T_n^{J_n} \prod_{n=2 \text{ even}}^{L+1} \hat{S}_n'^{\tilde{J}_n} \Big)$$

and the profiles $\boldsymbol{S}_n$ are homogeneous of degree $(n, n-g', n \text{ mod} 2, n)$ in the sense of Definition 3.2.10. This means that for n even:

$$S_n(b) = \sum_{J' \in \mathcal{J},\ |J'|_2 = n} b^{J'} f_{J'},$$

the sum being finite $\#\mathcal{J} < +\infty$, and the profiles satisfying $\partial_y^k f_{J'} = O\left(\frac{1}{1+y^{\gamma-n+g'+k}}\right)$. Therefore one has the identity:

$$\prod_{n=2,\text{ even}}^{L-1} b_n^{J_n} T_n^{J_n} \prod_{n=2,\text{ even}}^{L+1} S_n^{\tilde{J}_n} - \prod_{n=2,\text{ even}}^{L-1} \hat{b}_n'^{J_n} T_n^{J_n} \prod_{n=2,\text{ even}}^{L+1} \hat{S}_n'^{\tilde{J}_n} = \sum_{G\in\mathcal{G}} [b^G - (\hat{b}')^G] g_G,$$

the sum being finite $\#\mathcal{G} < +\infty$, for some determined profiles g_G having the following asymptotic behavior: $\partial_y^k g_G = O\left(\frac{1}{1+y^{i\gamma+g'\sum_2^{L+2}\tilde{J}_n - |G|_2 + k}}\right)$. Using the bound (*3.5.30*) on $b^G - \hat{b}'^G$, one has for $y \leq 2\hat{B}'_1$:

$$\begin{aligned} |\partial_y^k [b^G - (\hat{b}')^G] g_G| &\leq C(\sup_{1\leq k\leq L} b_1^{-k} |\triangle \hat{b}_k|) \frac{1}{1+y^{i\gamma+k}} \frac{b_1^{|G|_2}}{1+y^{-|G|_2}} \\ &\leq C(\sup_{1\leq k\leq L} b_1^{-k} |\triangle \hat{b}_k|) \frac{b_1}{1+y^{i\gamma-1+k+O(\eta)}}. \end{aligned}$$

With (*3.2.1*) one obtains the desired bound (i) for the first term in (*3.5.31*):

$$\partial_y^k\left(\sum_{i=1}^{p-1}C_iQ^{p-j-i}\chi^i_{\hat{B}_1'}(\alpha^i_{\hat{b}'}-\alpha^i_b)\right)=O\left(\frac{b_1\sup\limits_{1\leq k\leq L}(b_1^{-k}|\triangle\hat{b}_k|)}{1+y^{\frac{2(p-j)}{p-1}-1+\alpha+k+O(\eta)}}\right). \qquad (3.5.32)$$

We now turn to the second term in (*3.5.31*). First we factorize:

$$\chi^i_{\hat{B}_1'}-\chi^i_{B_1}=(\chi_{\hat{B}_1'}-\chi_{B_1})\sum_{n=0}^{i-1}C_n\chi^n_{\hat{B}_1'}\chi^{i-1-n}_{B_1},$$

for some constants $(C_n)_{0\leq n\leq i-1}$ and then we use the integral formulation:

$$\chi_{B_1}(y)-\chi_{\hat{B}_1'}(y)=y(b_1^{1+\eta}-\hat{b}_1^{'(1+\eta)})\int_0^1\partial_y\chi(y((1-\theta)\hat{b}_1^{'(1+\eta)}+\theta b_1^{1+\eta})d\theta, \qquad (3.5.33)$$

to find that: $\partial_y^k(\chi^i_{\hat{B}_1'}-\chi_{B_1}^i)=O\left(\frac{1}{1+y^k}b_1^{-1}|\triangle\hat{b}_1|\right)$. We know from the asymptotic of the T_i's and S_i's that for $y\leq 2\max(B_1,\hat{B}_1')$:

$$\partial_y^k(Q^{p-j-i}\alpha_b^i)=O\left(\frac{b_1}{1+y^{\frac{2(p-j)}{p-1}-1+i\alpha+O(\eta)}}\right).$$

The two last asymptotics give the desired bound for the second term in (*3.5.31*):

$$\partial_y^k\left(\sum_{i=1}^{p-1}C_iQ^{p-j-i}(\alpha_b^i(\chi^i_{\hat{B}_1'}-\chi^i_{B_1}))\right)=O\left(\frac{b_1b_1^{-1}|\triangle\hat{b}_1|}{1+y^{\frac{2(p-j)}{p-1}-1+\alpha+k+O(\eta)}}\right). \qquad (3.5.34)$$

Injecting (*3.5.32*) and (*3.5.34*) in (*3.5.31*) gives the desired result (*3.5.28*).

Proof of (ii): As we are in the zone $y\leq 2B_0$, from the localization property of Proposition 3.2.14 the error is given by:

$$\psi_b=\sum_{J\in\mathfrak{J},\ |J|_2\geq L+3}b^Jf_J,$$

the sum being finite $\#\mathfrak{J}<+\infty$ and the profiles satisfying $\partial_y^kf_J=O\left(\frac{1}{1+y^{\gamma+g'+1-|J|_2}}\right)$. The difference of the primary errors then writes: $\psi_b-\psi_{\hat{b}'}=\sum\limits_{J\in\mathfrak{J},\ |J|_2\geq L+3}(b^J-\hat{b}'^J)f_J$. Therefore, the bound (*3.5.29*) of the lemma is a consequence of the asymptotic of the f_J's and of the bound (*3.5.30*) on $b^J-\hat{b}'^J$. □

We can now relate the time evolution of the difference of the parameters to the difference of the errors $\varepsilon-\hat{\varepsilon}'$ and to itself.

Lemma 3.5.6 (Modulation estimates for the difference). *There holds the following identities. The difference of the two times obeys to:*

$$\frac{d\hat{s}'}{ds}-1=\frac{\triangle\hat{b}_1}{\hat{b}_1'}+O\left(b_1^{L+(1-\delta_0)(1+\frac{\eta}{2})}(b_1^{\frac{\eta}{2}(1-\delta_0)}\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}})\right). \qquad (3.5.35)$$

For the parameters, for $1\leq i\leq L-1$ *one has:*

$$\begin{aligned}&\left|b_{i,s}+(i-\alpha)b_1b_i-b_{i+1}-[\hat{b}'_{i,s}+\tfrac{d\hat{s}'}{ds}((i-\alpha)\hat{b}_1'\hat{b}_i'-\hat{b}'_{i+1})]\right|\\ \leq\ & Cb_1^{L+1+(1-\delta_0)(1+\frac{\eta}{2})}\left(b_1^{\frac{\eta}{2}(1-\delta_0)}\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}\right),\end{aligned} \qquad (3.5.36)$$

and for the last one we have the primary bound:

$$\begin{array}{l} \left| b_{L,s} + (L-\alpha) b_1 b_L - [\hat{b}'_{L,s} + \frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1 \hat{b}'_L] \right| \\ \leq \ C b_1^{L+(1-\delta_0)(1+\frac{\eta}{2})} \left(b_1^{\frac{\eta}{2}(1-\delta_0)} \underset{1\leq i\leq L}{\textit{sup}}\, b_1^{-i} |\triangle \hat{b}_i| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} \right). \end{array} \quad (3.5.37)$$

Proof of Lemma 3.5.6 We take the scalar product of (*3.5.26*) with the profile $\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M$ for $i=0,...,L$. It gives, because of the orthogonality conditions (*3.3.9*) and (*3.5.15*):

$$\begin{array}{rl} & \langle \tilde{\boldsymbol{Mod}} - \hat{\boldsymbol{Mod}}', \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle - \langle \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda}(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle + \langle \boldsymbol{H}(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle \\ = & \langle \frac{d\hat{s}'}{ds}\hat{\boldsymbol{\psi}}_{\hat{b}'} - \tilde{\boldsymbol{\psi}}_b + (\frac{d\hat{s}'}{ds}-1)\boldsymbol{H}(\hat{\boldsymbol{\varepsilon}}') + \boldsymbol{NL} - \frac{d\hat{s}'}{ds}\hat{\boldsymbol{NL}}' + \boldsymbol{L}(\boldsymbol{\varepsilon}) - \frac{d\hat{s}'}{ds}\hat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle. \end{array} \quad (3.5.38)$$

To simplify the analysis we introduce the following intermediate quantity:

$$\begin{array}{rcl} \triangle D(t) & = & \left| b_1 - \frac{d\hat{s}'}{ds}\hat{b}'_1 \right| \\ & & + \sum_{i=1}^L \left| b_{i,s} + (i-\alpha) b_1 b_i - b_{i+1} - [\hat{b}'_{i,s} + \frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1\hat{b}'_i - \hat{b}'_{i+1})] \right|. \end{array}$$

We notice that as $\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'$ still satisfy the orthogonality conditions (*3.3.9*) we can still use the coercivity of $\triangle\hat{\mathcal{E}}_{s_L}$ given by Corollary (3.D.3).

Step 1: Law for $\frac{d\hat{s}'}{ds}$. We take $i=0$ in the previous equation (*3.5.38*). The linear terms disappear because of the orthogonality conditions (*3.3.9*) and (*3.5.15*):

$$\langle \boldsymbol{H}(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{\Phi}_M\rangle - \langle (\frac{d\hat{s}'}{ds}-1)\boldsymbol{H}(\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{\Phi}_M\rangle = 0 \quad (3.5.39)$$

The non linear, small linear and error terms are not on the first coordinate, so:

$$\left\langle -\tilde{\boldsymbol{\psi}}_b + \frac{d\hat{s}'}{ds}\hat{\boldsymbol{\psi}}_{\hat{b}'} + \boldsymbol{NL} - \frac{d\hat{s}'}{ds}\hat{\boldsymbol{NL}}' + \boldsymbol{L}(\boldsymbol{\varepsilon}) - \frac{d\hat{s}'}{ds}\hat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \right\rangle = 0. \quad (3.5.40)$$

For the the scale changing term, the coercivity and the fact that $\frac{\lambda_s}{\lambda}\sim b_1$ give:

$$\left| \langle \frac{\lambda_s}{\lambda}\boldsymbol{\Lambda}(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle \right| \leq C(M) b_1^{L+1+(1-\delta_0)(1+\frac{\eta}{2})} \sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}. \quad (3.5.41)$$

The $\tilde{\boldsymbol{Mod}}$ term catches the dynamics on the manifold $(\tilde{\boldsymbol{Q}}_{b,\lambda})_{\lambda,b}$. Taking $i=0$ in (*3.5.38*) means that we are computing the law for the scaling. But by the very definition (*3.5.8*) of the time $\hat{s}'$, the two solutions have the same scale. This property induces the law for $\hat{s}'$ as we are going to see. Using the notations (*3.5.24*) and (*3.5.25*) one writes:

$$\langle \tilde{\boldsymbol{Mod}} - \hat{\boldsymbol{Mod}}', \boldsymbol{\Phi}_M\rangle = \sum_0^L \langle \triangle\hat{\boldsymbol{Mod}}_i, \boldsymbol{\Phi}_M\rangle. \quad (3.5.42)$$

Using the orthogonality conditions (*3.3.7*) and the fact that $M \ll B_1, B'_1$ one decomposes for $1\leq i\leq L$:

$$\begin{array}{rl} & \langle \triangle\hat{\boldsymbol{Mod}}_i, \boldsymbol{\Phi}_M\rangle \\ = & \langle \sum_{i=1}^L (b_{i,s} + (i-\alpha)b_1 b_i - b_{i+1})(\sum_{j=i+1}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i} - \frac{\partial \hat{\boldsymbol{S}}'_j}{\partial b_i}), \boldsymbol{\Phi}_M\rangle \\ & + \sum\limits_{i=1}^L (b_{i,s} + (i-\alpha)b_1 b_i - b_{i+1} - (\hat{b}'_{i,s} + \frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1\hat{b}'_i - \hat{b}'_{i+1})))\langle \sum\limits_{j=i+1}^{L+2} \frac{\partial \hat{\boldsymbol{S}}'_j}{\partial b_i}, \boldsymbol{\Phi}_M\rangle \end{array} \quad (3.5.43)$$

Now we recall that $\boldsymbol{S}_j$ is an homogeneous profile of degree $(j, j-g', j \bmod 2, j)$. It implies that for $1\leq i<j\leq L+2$, one has the bound: $\left|\frac{\partial \boldsymbol{S}_j}{\partial b_i}\right|\leq C(L,M)b_1$ on $y\leq 2M$ (and similarly for $\hat{\boldsymbol{S}}'$). Hence the bound for the second term in (*3.5.43*):

$$\begin{array}{rc} & \left|\sum\limits_{i=1}^{L}(b_{i,s}+(i-\alpha)b_1b_i-b_{i+1}-(\hat{b}'_{i,s}+\frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1\hat{b}'_i-\hat{b}'_{i+1})))\langle\sum\limits_{j=i+1}^{L+2}\frac{\partial\hat{\boldsymbol{S}}'_j}{\partial b_i},\boldsymbol{\Phi}_M\rangle\right| \\ \leq & C(L,M)b_1\triangle D(t). \end{array} \quad (3.5.44)$$

The homogeneity of the $\boldsymbol{S}_j$'s means that: $\frac{\partial \boldsymbol{S}_j}{\partial b_i}=\sum_{J\in\mathfrak{J}}b^Jf_J$ and $\frac{\partial \hat{\boldsymbol{S}}'_j}{\partial b_i}=\sum_{J\in\mathfrak{J}}\hat{b}'^Jf_J$ where the J's are non null: $J\neq(0,...,0)$. Using the bound (*3.5.30*) on $b^J-\hat{b}'^J$ we obtain that for $y\leq 2M$, $\left|\frac{\partial \boldsymbol{S}_j}{\partial b_i}-\frac{\partial \hat{\boldsymbol{S}}'_j}{\partial b_i}\right|\leq b_1C(L,M)\sup\limits_{1\leq i\leq \min(|J|_2-1,L)}|b_1^{-i}\triangle\hat{b}_i|$. Moreover, we know from the modulation equations (*3.3.36*) and (*3.3.37*) that $|b_{i,s}+(i-\alpha)b_1b_i-b_{i+1}|\leq C(L,M)b_1^{L+(1-\delta_0)(1+\eta)}$. Hence we get the following bound for the second term in the right hand side of (*3.5.43*):

$$\begin{array}{rc} & \left|\left\langle\sum_{i=1}^{L}(b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})(\sum_{j=i+1}^{L+2}\frac{\partial \boldsymbol{S}_j}{\partial b_i}-\frac{\partial\hat{\boldsymbol{S}}'_j}{\partial b_i}),\boldsymbol{\Phi}_M\right\rangle\right| \\ \leq & Cb_1^{L+1+(1-\delta_0)(1+\eta)}\sup\limits_{1\leq i\leq L}|b_1^{-i}\triangle\hat{b}_i|. \end{array} \quad (3.5.45)$$

The identity (*3.5.43*) and the bounds (*3.5.44*) and (*3.5.45*) give for $1\leq i\leq L$:

$$\left|\langle\triangle\hat{\boldsymbol{Mod}}_i,\boldsymbol{\Phi}_M\rangle\right|\leq C(L,M)[b_1^{L+(1-\delta_0)(1+\eta)+1}\sup_{1\leq i\leq L}|\triangle\hat{b}_i|+b_1\triangle D(t)]. \quad (3.5.46)$$

We now look at the first term in the sum in the right hand side of (*3.5.42*). Using the same ideas we just used for the others, one gets:

$$\begin{array}{rl} & \langle\triangle\hat{\boldsymbol{Mod}}_0,\boldsymbol{\Phi}_M\rangle=\langle(\frac{\lambda_s}{\lambda}+b_1)(\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b-\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_{\hat{b}'})+(b_1-\frac{d\hat{s}'}{ds}\hat{b}'_1))\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_{\hat{b}'},\boldsymbol{\Phi}_M\rangle \\ = & O(b_1^{L+1+(1-\delta_0)(1+\eta)}\sup\limits_{1\leq i\leq L}|b_1^{-i}\triangle\hat{b}_i|)+O(b_1\triangle D(t))+(b_1-\frac{d\hat{s}'}{ds}\hat{b}'_1)\langle\boldsymbol{\Lambda}\boldsymbol{Q},\boldsymbol{\Phi}_M\rangle. \end{array} \quad (3.5.47)$$

We have estimated all terms involved in the identity (*3.5.42*) for the modulation term in (*3.5.46*) and (*3.5.47*), giving:

$$\begin{array}{rcl} \langle\tilde{\boldsymbol{Mod}}-\hat{\boldsymbol{Mod}}',\boldsymbol{\Phi}_M\rangle & = & (b_1-\frac{d\hat{s}'}{ds}\hat{b}'_1)\langle\boldsymbol{\Lambda}\boldsymbol{Q},\boldsymbol{\Phi}_M\rangle \\ & & +O(b_1^{L+1+(1-\delta_0)(1+\eta)}\sup\limits_{1\leq i\leq L}|\triangle\hat{b}_i|+b_1\triangle D(t)). \end{array} \quad (3.5.48)$$

We can now come back to the modulation equation (*3.5.38*) for $i=0$. We have calculated all terms in the right hand side in (*3.5.39*), (*3.5.40*), (*3.5.41*) and (*3.5.48*), so it now writes (because $\langle\boldsymbol{\Lambda}\boldsymbol{Q},\boldsymbol{\Phi}_M\rangle\sim cM^{2k_0+2\delta_0}>0$ for $c>0$):

$$\left|b_1-\frac{d\hat{s}'}{ds}\hat{b}'_1\right|\leq C[b_1\triangle D(t)+b_1^{L+1+(1-\delta_0)(1+\frac{\eta}{2})}(b_1^{\frac{\eta}{2}(1-\delta_0)}\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}})]. \quad (3.5.49)$$

This identity gives a first bound for the law of $\hat{s}'$:

$$1-\frac{d\hat{s}'}{ds}=\frac{-\triangle\hat{b}_1}{\hat{b}'_1}+O[\triangle D(t)+b_1^{L+(1-\delta_0)(1+\frac{\eta}{2})}(b_1^{\frac{\eta}{2}(1-\delta_0)}\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}})]. \quad (3.5.50)$$

Step 2: Law for $\triangle \hat{b}_i$ for $1 \leq i \leq L-1$. We look at (3.5.38) for $1 \leq i \leq L-1$. The linear term disappear because of orthogonality conditions:

$$\left\langle \boldsymbol{H}(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}') - (\frac{d\hat{s}'}{ds}-1)\boldsymbol{H}(\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \right\rangle = 0. \tag{3.5.51}$$

The scale changing term is estimated as before:

$$\left| \langle \frac{\lambda_s}{\lambda} \boldsymbol{\Lambda}(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle \right| \leq C(L,M) b_1^{L+1+(1-\delta_0)(1+\frac{\eta}{2})} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}. \tag{3.5.52}$$

The bounds (3.5.29), (3.5.50) and (3.2.41) on on $\tilde{\boldsymbol{\psi}}_b - \tilde{\boldsymbol{\psi}}_{\hat{b}'}$, $|\frac{d\hat{s}'}{ds}-1|$ and $\hat{\boldsymbol{\psi}}_{\hat{b}'}$ imply:

$$\begin{aligned} \left| \langle \tilde{\boldsymbol{\psi}}_b - \tfrac{d\hat{s}'}{ds} \hat{\boldsymbol{\psi}}_{\hat{b}'}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle \right| &= \left| \langle \tilde{\boldsymbol{\psi}}_b - \hat{\boldsymbol{\psi}}_{\hat{b}'} + (1-\tfrac{d\hat{s}'}{ds})\hat{\boldsymbol{\psi}}_{\hat{b}'}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle \right| \\ &\leq C b_1^{L+3} (\sup_{1\leq i\leq L} b_1^{-i} |\triangle b_i| + \triangle D(t) + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}). \end{aligned} \tag{3.5.53}$$

For the nonlinear terms, we have that $NL = \sum_{j=0}^{p-2} C_j \tilde{Q}_b^{(1)j} \varepsilon^{(1)(p-j)}$ and similarly for $\hat{NL}'$. Fix j, $1 \leq j \leq p-2$. We estimate, using the bound (3.5.28) on $\tilde{Q}_b^{(1)j} - \tilde{Q}_{\hat{b}'}^{(1)j}$:

$$\begin{aligned} & \| \tilde{Q}_b^{(1)j} \varepsilon^{(1)(p-j)} - \tilde{Q}_{\hat{b}'}^{(1)j} (\hat{\varepsilon}'^{(1)})^{(p-j)} \|_{L^1, y\leq 2M} \\ \leq & \| \tilde{Q}_b^{(1)j} - \tilde{Q}_{\hat{b}'}^{(1)j} \|_{L^\infty, y\leq 2M} \| \| \varepsilon^{(1)(p-j)} \|_{L^1, y\leq 2M} \\ & + C \| \varepsilon^{(1)(p-j)} - (\hat{\varepsilon}'^{(1)})^{(p-j)} \|_{L^1, y\leq 2M} \\ \leq & C b_1^{2L+(1-\delta_0)(2+\eta)} \sup_{1\leq i\leq L} |\triangle \hat{b}_i| + C b_1^{2L+(1-\delta_0)(2+\eta)} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}. \end{aligned}$$

For $j=0$ one has: $\| \varepsilon^{(1)p} - (\hat{\varepsilon}'^{(1)})^p \|_{L^1, y\leq 2M} \leq C b_1^{2L+(1-\delta_0)(2+\eta)} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}$. The previous bounds and the bound (3.5.50) on $\frac{d\hat{s}'}{ds}-1$ finally imply:

$$\begin{aligned} & \left| \langle \boldsymbol{NL} - \tfrac{d\hat{s}'}{ds} \hat{\boldsymbol{NL}}', \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle \right| = \left| \langle \boldsymbol{NL} - \hat{\boldsymbol{NL}}' + (1-\tfrac{d\hat{s}'}{ds}) \hat{\boldsymbol{NL}}', \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle \right| \\ \leq & C \| NL - \hat{NL}' \|_{L^1, y\leq 2M} + C|1-\tfrac{d\hat{s}'}{ds}| \| \hat{NL}' \|_{L^1, y\leq 2M} \\ \leq & C b_1^{2L+(1-\delta_0)(2+\eta)-1} (\sup_{1\leq i\leq L} |\triangle \hat{b}_i| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} + \triangle D(t)). \end{aligned} \tag{3.5.54}$$

We treat the same way the small linear term:

$$\begin{aligned} & \left| \langle \boldsymbol{L}(\boldsymbol{\varepsilon}) - \tfrac{d\hat{s}'}{ds} \hat{\boldsymbol{L}}(\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle \right| \\ \leq & C \| (\tilde{Q}_b^{(1)(p-1)} - Q^{p-1}) \varepsilon^{(1)} - (\tilde{Q}_{\hat{b}'}^{(1)(p-1)} - Q^{p-1}) \hat{\varepsilon}'^{(1)} \|_{L^1, y\leq 2M} \\ & + C|1-\tfrac{d\hat{s}'}{ds}| \| \hat{L}(\hat{\varepsilon}') \|_{L^1} \\ \leq & C b_1^{L+1+(1-\delta_0)(1+\frac{\eta}{2})} [b_1^{\frac{\eta}{2}(1-\delta_0)} \sup_{1\leq i\leq L} |\triangle \hat{b}_i| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} + b_1^{\frac{\eta}{2}(1-\delta_0)} \triangle D(t)]. \end{aligned} \tag{3.5.55}$$

Finally, for the modulation term, using the same tools employed for $i=0$ we obtain:

$$\begin{aligned} & \langle \tilde{\boldsymbol{Mod}} - \hat{\boldsymbol{Mod}}', \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M \rangle = O(b_1 \triangle D(t) + b_1^{L+1+(1-\delta_0)(1+\eta)} \sup_{1\leq i\leq L} |\triangle \hat{b}_i|). \\ & + (b_{i,s} + (i-\alpha) b_1 b_i - b_{i+1} - (\hat{b}'_{i,s} + \tfrac{d\hat{s}'}{ds} ((i-\alpha) \hat{b}'_1 \hat{b}'_i - \hat{b}'_{i+1}))) \langle \boldsymbol{\Lambda Q}, \boldsymbol{\Phi}_M \rangle. \end{aligned} \tag{3.5.56}$$

We now collect all the estimates we have showed, (3.5.51), (3.5.52), (3.5.53), (3.5.54), (3.5.55) and (3.5.56) and inject them in (3.5.38). This gives:

$$\begin{aligned} & \left| b_{i,s} + (i-\alpha) b_1 b_i - b_{i+1} - (\hat{b}'_{i,s} + \tfrac{d\hat{s}'}{ds} ((i-\alpha) \hat{b}'_1 \hat{b}'_i - \hat{b}'_{i+1})) \right| \\ \leq & C [b_1 \triangle D(t) + b_1^{L+1+(1-\delta_0)(1+\frac{\eta}{2})} (b_1^{\frac{\eta}{2}(1-\delta_0)} \sup_{1\leq i\leq L} b_1^{-i} |\triangle \hat{b}_i| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}})]. \end{aligned} \tag{3.5.57}$$

Step 3: Law for $\triangle\hat{b}_L$. The computations we made in the previous step, to find the estimates (*3.5.53*), (*3.5.52*), (*3.5.54*), (*3.5.55*) and (*3.5.56*) still work when taking $i = L$. The difference is that the linear term does not cancel anymore. Namely, using the bound (*3.5.50*) on $\frac{d\hat{s}'}{ds} - 1$:

$$\begin{array}{rl} & \left|\langle \boldsymbol{H}(\boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}') - (\frac{d\hat{s}'}{ds} - 1)\boldsymbol{H}(\hat{\boldsymbol{\varepsilon}}'), \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle\right| \\ \leq & C \parallel \boldsymbol{\varepsilon} - \hat{\boldsymbol{\varepsilon}}' \parallel_{L^2, y\leq 2M} + |\frac{d\hat{s}'}{ds} - 1| \parallel \hat{\boldsymbol{\varepsilon}}' \parallel_{L^2, y\leq 2M} \\ \leq & C b_1^{L+(1-\delta_0)(1+\frac{\eta}{2})}\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} + C b_1^{L+(1-\delta_0)(1+\eta)}(\triangle D(t) + \sup\limits_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|). \end{array}$$

So for $i = L$ in (*3.5.38*) one obtains:

$$\begin{array}{rl} & \left|b_{L,s} + (L-\alpha)b_1 b_L - (\hat{b}'_{L,s} + \frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L)\right| \\ \leq & C\left(b_1\triangle D(t) + b_1^{L+(1-\delta_0)(1+\frac{\eta}{2})}(b_1^{\frac{\eta}{2}(1-\delta_0)}\sup\limits_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i| + \sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}})\right). \end{array} \tag{3.5.58}$$

Step 4: Gathering the bounds. We now put together the primary bounds we found so far for the scaling (*3.5.49*), for the parameters (*3.5.57*) and (*3.5.58*) to find that:

$$|D(t)| \leq C b_1^{L+(1-\delta_0)(1+\frac{\eta}{2})}[b_1^{\frac{\eta}{2}(1-\delta_0)}\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i| + \sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}})].$$

We reinject it in the previous primary bounds (*3.5.49*), (*3.5.57*) and (*3.5.58*) to obtain the bounds (*3.5.35*), (*3.5.36*) and (*3.5.37*) claimed in the lemma.

□

We are now going to improve our control over $\triangle b_L$ by the same technique we used in Lemma (3.3.5). After an integration by parts in time, the time evolution of $\triangle\hat{b}_L$ enjoys a sufficiently good estimate for our purpose, as the ones we just proved for $\triangle\hat{b}_i$ for $1 \leq i \leq L-1$ in Lemma 3.5.6.

Lemma 3.5.7 (Improved modulation equation for $\triangle\hat{b}_L$). *There holds*[23] *:*

$$\begin{array}{rl} & (b_{L,s} + (L-\alpha)b_1 b_L - (\hat{b}'_{L,s} + \frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L)) \\ = & \frac{d}{ds}\left[\frac{\langle \boldsymbol{H}^L(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'), \chi_{B_0}\boldsymbol{\Lambda Q}\rangle - \hat{b}'_L\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L} - \frac{\partial\hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}}{\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\right] \\ & +O[b_1^{L+1+\frac{\eta}{2}(1-\delta_0)}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}} + b_1^{\frac{\eta}{2}(1-\delta_0)}\textit{sup}\, b_1^{-i}|\triangle\hat{b}_i|)]. \end{array} \tag{3.5.59}$$

The quantity appearing via its derivative in time has the following size:

$$\begin{array}{rl} & \left|\frac{\langle \boldsymbol{H}^L(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'), \chi_{B_0}\boldsymbol{\Lambda Q}\rangle - \hat{b}'_L\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L} - \frac{\partial\hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}}{\left\langle \chi_{B_0}\Lambda^{(1)}Q, \Lambda^{(1)}Q + (-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\right| \\ \leq & C b_1^{L+\frac{\eta}{2}(1-\delta_0)}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}} + b_1^{g'+O(\eta)}\textit{sup}_{1\leq i\leq 2}\, b_1^{-i}|\triangle\hat{b}_i|). \end{array} \tag{3.5.60}$$

[23] the denominator being strictly positive from (*3.3.69*).

Proof of Lemma 3.5.7 We will do the same computations we did to prove Lemma (3.3.5), this time expressing everything in function of the differences $\triangle\hat{b}_i$ and $\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'$.

Step 1: Time derivative of the numerator in (*3.5.59*). We compute for the first term:

$$\frac{d}{ds}\langle \boldsymbol{H}^L(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'),\chi_{B_0}\boldsymbol{\Lambda Q}\rangle = \langle \boldsymbol{H}^L(\boldsymbol{\varepsilon}_s-\hat{\boldsymbol{\varepsilon}}'_s),\chi_{B_0}\boldsymbol{\Lambda Q}\rangle + \langle \boldsymbol{H}^L(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'),b_{1,s}y\partial_y\chi(\frac{y}{B_0})\boldsymbol{\Lambda Q}\rangle. \qquad (3.5.61)$$

We now calculate everything in the right hand side. For the second term:

$$\left|\langle \boldsymbol{H}^L(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'),b_{1,s}\partial_y\chi(\frac{y}{B_0})\boldsymbol{\Lambda Q}\rangle\right| \leq C(M)\sqrt{\triangle\hat{\mathcal{E}}_{s_L}}b_1^{-(2k_0+\delta_0)}. \qquad (3.5.62)$$

We will now estimate the first term in the right hand side of (*3.5.61*). From the time evolution of the difference (*3.5.26*), one gets:

$$\begin{array}{rl} & (-1)^{\frac{L+1}{2}}\langle \boldsymbol{H}^L(\boldsymbol{\varepsilon}_s-\hat{\boldsymbol{\varepsilon}}_s),\chi_{B_0}\boldsymbol{\Lambda Q}\rangle \\ = & \int \chi_{B_0}\Lambda^{(1)}Q\times(\mathcal{L}(\hat{\varepsilon}'^{(1)}-\varepsilon^{(1)})+\frac{\lambda_s}{\lambda}\Lambda^{(2)}(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})-(\tilde{Mod}(t)^{(2)}-\hat{Mod}'(2)) \\ & -\tilde{\psi}_b^{(2)}+\tilde{\psi}_{\hat{b}'}^{(2)}+NL-\hat{NL}'+L-\hat{L}'+(\frac{d\hat{s}'}{ds}-1)(\mathcal{L}\hat{\varepsilon}'^{(1)}+\tilde{\psi}_{\hat{b}'}^{(2)}-\hat{NL}'-\hat{L}'))_{L-1} \end{array} \qquad (3.5.63)$$

and we now consider each term in the right hand side.

• *Linear term:* One has the bound from coercivity:

$$\left|\int \chi_{B_0}\Lambda^{(1)}Q(\mathcal{L}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}))_{L-1}\right| \leq C(M)\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}b_1^{-2(k_0+\delta_0)+L+1+\frac{\eta}{2}(1-\delta_0)}. \qquad (3.5.64)$$

•*Scale changing term:* One has the same bound:

$$\left|\int \chi_{B_0}\Lambda^{(1)}Q\frac{\lambda_s}{\lambda}(\Lambda^{(2)}(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)}))_{L-1}\right| \leq C(M)\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}b_1^{-2(k_0+\delta_0)+L+1+\frac{\eta}{2}(1-\delta_0)}. \qquad (3.5.65)$$

• *Error term:* As we are in the zone $y\leq 2B_0$ we can use the asymptotic (*3.5.29*):

$$\left|\int \chi_{B_0}\Lambda^{(1)}Q(\tilde{\psi}_b^{(2)}-\tilde{\psi}_{\hat{b}'}^{(2)})_{L-1}\right| \leq Cb_1^{-2(k_0+\delta_0)+L+1+g'}\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|. \qquad (3.5.66)$$

• NL *term:* We start by puting all the adapted derivatives on $\chi_{B_0}\Lambda^{(1)}Q$, localizing the integral in $B_0\leq y\leq 2B_0$ as $A\Lambda^{(1)}Q=0$:

$$\left|\int \chi_{B_0}\Lambda^{(1)}Q(NL-\hat{NL}')_{L-1}\right| \leq C\int_{B_0}^{2B_0}\frac{1}{y^{\gamma+L-1}}|NL-\hat{NL}'|.$$

We know that NL is a sum of terms of the form[24]: $\tilde{Q}_b^{p-k}\varepsilon^{(1)k}$ for $k>2$, and similarly for $\hat{NL}'$. Suppose that $k=p$, then:

$$\begin{array}{rcl} \int_{B_0}^{2B_0}\frac{|\varepsilon^{(1)p}-\hat{\varepsilon}'^{(1)p}|}{y^{\gamma+L-1}} & \leq & C\max(\|\varepsilon^{(1)}\|_{L^\infty}^{p-1},\|\varepsilon'^{(1)}\|_{L^\infty}^{p-1})\int_{B_0}^{2B_0}\frac{1}{y^{\gamma+L-1}}|\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}| \\ & \leq & C\left(\frac{\sqrt{\mathcal{E}_\sigma}+\sqrt{\hat{\mathcal{E}}'_\sigma}}{b_1^{\sigma-s_c}}\right)^{p-1}b_1^{2+\frac{2\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sqrt{\triangle\hat{\mathcal{E}}_{s_L}}b_1^{-2(k_0+\delta_0)-2} \\ & \leq & Cb_1^{-2(k_0+\delta_0)+L+1+\frac{\alpha}{2}+O\left(\eta,\frac{\sigma-s_c}{L}\right)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}, \end{array} \qquad (3.5.67)$$

where we used the estimates (*3.5.13*) of the trapped regime (we recall that they hold for both $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}'$ as $b_1\sim\hat{b}_1'$ from (*3.5.12*)). Suppose now $2\leq k\leq p-1$. We start by splitting in two parts:

$$\left|\int_{B_0}^{2B_0}\frac{\tilde{Q}_b^{p-k}\varepsilon^{(1)k}-\tilde{Q}_{\hat{b}'}^{p-k}\hat{\varepsilon}'^{(1)k}}{y^{\gamma+L-1}}\right| = \left|\int_{B_0}^{2B_0}\frac{(\tilde{Q}_b^{p-k}-\tilde{Q}_{\hat{b}'}^{p-k})\varepsilon^{(1)k}}{y^{\gamma+L-1}}+\frac{\tilde{Q}_{\hat{b}'}^{p-k}(\varepsilon^{(1)k}-\hat{\varepsilon}'^{(1)k})}{y^{\gamma+L-1}}\right|.$$

[24] we write $\tilde{Q}_b$ instead of $\tilde{Q}_b^{(1)}$ to ease notations.

For the first part, from the bound (*3.5.28*) for $\tilde{Q}_b^{p-k}-\tilde{Q}_{\hat{b}'}^{p-k}$, one gets:

$$\begin{array}{rcl} \left|\int_{B_0}^{2B_0} \frac{(\tilde{Q}_b^{p-k}-\tilde{Q}_{\hat{b}'}^{p-k})\varepsilon^{(1)k}}{y^{\gamma+L-1}}\right| & \leq & C b_1^{\frac{2(p-k)}{p-1}+\alpha+O(\eta)} \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i| \parallel \varepsilon^{(1)} \parallel_{L^\infty}^{k-1} \int_{B_0}^{2B_0} \frac{|\varepsilon^{(1)}|}{y^{\gamma+L-1}} \\ & \leq & C b_1^{-2(k_0+\delta_0)+L+1+\alpha+O\left(\eta,\sigma-s_c,\frac{1}{L}\right)} \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|. \end{array}$$

For the second part, as $|\tilde{Q}_{\hat{b}'}^{(p-k)}|\leq C b_1^{\frac{2(p-k)}{p-1}}$ for $B_0\leq y\leq 2B_0$ one gets using again the L^∞ estimate and coercivity:

$$\begin{array}{rcl} \left|\int_{B_0}^{2B_0} \frac{(\tilde{Q}_{\hat{b}'}^{p-k}(\varepsilon^{(1)k}-\hat{\varepsilon}'^{(1)k}))}{y^{\gamma+L-1}}\right| & \leq & \max(\parallel \varepsilon^{(1)} \parallel_{L^\infty}^{k-1}, \parallel \hat{\varepsilon}'^{(1)} \parallel_{L^\infty}^{k-1}) b_1^{-\frac{2(p-k)}{p-1}-(2k_0+\delta_0)} \sqrt{\triangle \hat{\mathcal{E}}_{s_L}} \\ & \leq & b_1^{-2(k_0+\delta_0)+L+1+\frac{2(k-1)\alpha}{(p-1)L}+O\left(\eta,\frac{\sigma-s_c}{L}\right)} \sqrt{\triangle_r \hat{\mathcal{E}_{s_L}}}. \end{array}$$

As $\eta\ll 1$ the last bounds give the following estimate for the non linear term:

$$\begin{array}{rl} & \left|\int \chi_{B_0}\Lambda^{(1)}Q(NL-\hat{NL}')_{L-1}\right| \\ \leq & C b_1^{-2(k_0+\delta_0)+L+1+\frac{\eta}{2}(1-\delta_0)}(\sqrt{\triangle_r \hat{\mathcal{E}_{s_L}}}+b_1^{\frac{\eta}{2}(1-\delta_0)} \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|). \end{array} \quad (3.5.68)$$

• *Small linear term:* One has: $L=(\tilde{Q}_b^{p-1}-Q^{p-1})\varepsilon^{(1)}$ and similarly for $\hat{L}'$. As for the non-linear we start by decomposing:

$$\left|\int \chi_{B_0}\Lambda^{(1)}Q(L-\hat{L}')_{L-1}\right| \leq C\int_{B_0}^{2B_0} \frac{|\tilde{Q}_b^{p-1}-\tilde{Q}_{\hat{b}'}^{p-1}||\varepsilon^{(1)}|+|\tilde{Q}_{\hat{b}'}^{p-1}||\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})|}{y^{\gamma+L-1}}.$$

For the first term we use the asymptotic (*3.5.28*) for $\tilde{Q}_b^{p-1}-\tilde{Q}_{\hat{b}'}^{p-1}$, yielding:

$$\int_{B_0}^{2B_0} \frac{1}{y^{\gamma+L-1}}|\tilde{Q}_b^{p-1}-\tilde{Q}_{\hat{b}'}^{p-1}||\varepsilon^{(1)}| \leq b_1^{-2(k_0+\delta_0)+L+1+\alpha+O(\eta)}(\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|).$$

For the second term, from $|\tilde{Q}_{\hat{b}'}^{p-1}|\leq C b_1^2$ for $B_0\leq y\leq 2B_0$ one gets:

$$\int_{B_0}^{2B_0} \frac{1}{y^{\gamma+L-1}}(|\tilde{Q}_{\hat{b}'}^{p-1}||\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})|) \leq C b_1^{-2(k_0+\delta_0)+L+\frac{\eta}{2}(1-\delta_0)} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}.$$

The last two bounds show that for the small linear term:

$$\left|\int \chi_{B_0}\Lambda^{(1)}Q(L-\hat{L}')_{L-1}\right| \leq C b_1^{-2(k_0+\delta_0)+L+\frac{\eta(1-\delta_0)}{2}}(\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}+b_1^{\frac{\eta(1-\delta_0)}{2}} \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|) \quad (3.5.69)$$

• *The modulation term:* From the localization of the T_i' and S_i's ((*3.2.28*) and (*3.2.43*)), and because $(T_i)_{L-1}=0$ for $i<L-1$:

$$\begin{array}{rl} & \int (\tilde{Mod}^{(2)}-\hat{Mod}'^{(2)})_{L-1}\chi_{B_0}\Lambda^{(1)}Q \\ = & \int \chi_{B_0}\Lambda^{(1)}Q(b_{L,s}+(L-\alpha)b_1 b_L)(T_L+\frac{\partial S_{L+2}}{\partial b_L})_{L-1} \\ & -\int \chi_{B_0}\Lambda^{(1)}Q(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L)(T_L+\frac{\partial \hat{S}'_{L+2}}{\partial b_L}))_{L-1} \\ & +\sum_{i=1}^{L-1}\int \chi_{B_0}\Lambda^{(1)}Q((b_{i,s}+(i-\alpha)b_1 b_i-b_{i+1})\left(\sum_{j=i+1,\ j \text{ odd}}^{L+2}\frac{\partial S_j}{\partial b_i}\right)_{L-1} \\ & -\sum_{i=1}^{L-1}\int \chi_{B_0}\Lambda^{(1)}Q((\hat{b}'_{i,s}+\frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1\hat{b}'_i-\hat{b}'_{i+1}))\left(\sum_{j=i+1,\ j \text{ odd}}^{L+2}\frac{\partial \hat{S}'_j}{\partial b_i}\right)_{L-1} \\ & -\int \chi_{B_0}\Lambda^{(1)}Q((\frac{\lambda_s}{\lambda}+b_1)\Lambda^{(2)}\tilde{Q}_b^{(2)}-(\frac{\lambda_s}{\lambda}+\frac{d\hat{s}'}{ds}\hat{b}'_1)\Lambda^{(2)}\tilde{Q}_{\hat{b}'}^{(2)})_{L-1}. \end{array} \quad (3.5.70)$$

We start by studying the first term in (*3.5.70*). Since $\boldsymbol{H}(\boldsymbol{T}_L)=(-1)^L\boldsymbol{\Lambda Q}$:

$$\begin{array}{rl} & \int \chi_{B_0}\Lambda^{(1)}Q(b_{L,s}+(L-\alpha)b_1b_L)(T_L+\frac{\partial S_{L+2}}{\partial b_L})_{L-1} \\ & -\int \chi_{B_0}\Lambda^{(1)}Q(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L)(T_L+\frac{\partial \hat{S}'_{L+2}}{\partial b_L}))_{L-1} \\ = & (-1)^{\frac{L-1}{2}}(b_{L,s}+(L-\alpha)b_1b_L-(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L)) \\ & \times\int \chi_{B_0}\Lambda^{(1)}Q\left(\Lambda^{(1)}Q+\left(\frac{\partial S_{L+2}}{\partial b_L}\right)_{L-1}\right) \\ & +(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L)\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1} \\ = & (-1)^{\frac{L-1}{2}}(b_{L,s}+(L-\alpha)b_1b_L-(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L)) \\ & \times\int \chi_{B_0}\Lambda^{(1)}Q\left(\Lambda^{(1)}Q+\left(\frac{\partial S_{L+2}}{\partial b_L}\right)_{L-1}\right) \\ & +\hat{b}'_{L,s}\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}+O(b_1^{-2(k_0+\delta_0)+L+1+g'}\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|). \end{array}$$

For the second, third and fourth terms in (*3.5.70*), using the modulation bounds (*3.5.35*) and (*3.5.36*) from the proof of the last Lemma and splitting as we did before:

$$\begin{array}{rl} & |\sum_{i=1}^{L-1}\int \chi_{B_0}\Lambda^{(1)}Q((b_{i,s}+(i-\alpha)b_1b_i-b_{i+1})\left(\sum_{j=i+1,\ j\text{ odd}}^{L+2}\frac{\partial S_j}{\partial b_i}\right)_{L-1} \\ & -\sum_{i=1}^{L-1}\int \chi_{B_0}\Lambda^{(1)}Q((\hat{b}'_{i,s}+\frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1\hat{b}'_i-\hat{b}'_{i+1}))\left(\sum_{j=i+1,\ j\text{ odd}}^{L+2}\frac{\partial \hat{S}'_j}{\partial b_i}\right)_{L-1} \\ & -\int \chi_{B_0}\Lambda^{(1)}Q((\frac{\lambda_s}{\lambda}+b_1)\Lambda^{(2)}\tilde{Q}_b^{(2)}-(\frac{\lambda_s}{\lambda}+\frac{d\hat{s}'}{ds}\hat{b}'_1)\Lambda^{(2)}\tilde{Q}_{\hat{b}'}^{(2)})_{L-1}| \\ \leq & b_1^{-2(k_0+\delta_0)+L+1+g'+(1-\delta_0)+O(\eta)}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+\sup b_1^{-i}|\triangle\hat{b}_i|)). \end{array}$$

With the previous computations, (*3.5.70*) becomes eventually:

$$\begin{array}{rl} & (-1)^{\frac{L-1}{2}}\int(M\tilde{od}(t)^{(2)}-M\hat{od}^{'(2)})_{L-1}\chi_{B_0}\Lambda^{(1)}Q \\ = & (b_{L,s}+(L-\alpha)b_1b_L-(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L)) \\ & \times\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}(\frac{\partial S_{L+2}}{\partial b_L})_{L-1}\rangle \\ & +O(b_1^{-2(k_0+\delta_0)+g'+L+1}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}})+\hat{b}'_{L,s}\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1} \\ & +O[\sup b_1^{-i}|\triangle\hat{b}_i|(b_1^{-2(k_0+\delta_0)+g'+L+1})]. \end{array}\tag{3.5.71}$$

• *The time error term:* Using the upper bound (*3.5.35*) for $\left|\frac{d\hat{s}'}{ds}-1\right|$ and the previous bounds(*3.3.60*), (*3.3.62*), (*3.3.64*) and (*3.3.63*) from the original Lemma about the improved modulation:

$$\begin{array}{rl} & \left|\int \chi_{B_0}\Lambda^{(1)}Q(\frac{d\hat{s}'}{ds}-1)(\mathcal{L}\hat{\varepsilon}'^{(1)}+\tilde{\psi}_{\hat{b}'}^{(2)}-\hat{NL}'-\hat{L}')_{L-1}\right| \\ \leq & b_1^{-2(k_0+\delta_0)+L+1}(b_1^{L+1-\delta_0+O(\eta)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+b_1^{\eta(1-\delta_0)}\sup b_1^{-i}|\triangle\hat{b}_i|). \end{array}\tag{3.5.72}$$

We can now gather all the bounds (*3.5.64*), (*3.5.65*), (*3.5.66*), (*3.5.68*), (*3.5.69*), (*3.5.70*) and (*3.5.72*), inject them in (*3.5.63*) to find that the first term in the rhs of (*3.5.61*) is:

$$\begin{array}{rl} & \langle\boldsymbol{H}^L(\boldsymbol{\varepsilon}_s-\hat{\boldsymbol{\varepsilon}}'_s),\chi_{B_0}\boldsymbol{\Lambda Q}\rangle-\hat{b}'_{L,s}\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1} \\ = & (b_{L,s}+(L-\alpha)b_1b_L-(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}((L-\alpha)\hat{b}'_1\hat{b}'_L)) \\ & \times\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S_{L+2}^{(2)}}{\partial b_L}\right)_{L-1}\right\rangle \\ & +O(b_1^{-2(k_0+\delta_0)+L+1+\frac{\eta}{2}(1-\delta_0)}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+b_1^{\frac{\eta}{2}(1-\delta_0)}\sup b_1^{-i}|\triangle\hat{b}_i|)). \end{array}\tag{3.5.73}$$

Combining the two computations we made, (*3.5.73*) and (*3.5.62*), the time evolution of the first term of the numerator in (*3.5.61*) is now:

$$\begin{array}{rl} & \frac{d}{ds}\langle \boldsymbol{H}^L(\varepsilon-\hat{\varepsilon}'),\chi_{B_0}\boldsymbol{\Lambda Q}\rangle - \hat{b}'_{L,s}\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1} \\ = & (b_{L,s}+(L-\alpha)b_1b_L-(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}((L-\alpha)\hat{b}'_1\hat{b}'_L)) \\ & \times\left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle \\ & +O(b_1^{-2(k_0+\delta_0)+L+1+\frac{\eta}{2}(1-\delta_0)}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+b_1^{\frac{\eta}{2}(1-\delta_0)}\sup b_1^{-i}|\triangle\hat{b}_i|)). \end{array} \tag{3.5.74}$$

<u>Step 2</u>: End of the computation. We can now end the proof of the Lemma. We recall that the denominator in (*3.5.59*) and its time derivative have the following size:

$$\begin{array}{l} \left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle \sim cb_1^{-2k_0-2\delta_0}, \quad (c \text{ a constant}, \ c>0) \\ \left|\frac{d}{ds}\left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle\right| \leq Cb_1^{-2(k_0+\delta_0)+1}. \end{array}$$

We get by coercivity of the adapted norm:

$$|\langle \boldsymbol{H}^L(\varepsilon-\hat{\varepsilon}'),\chi_{B_0}\boldsymbol{\Lambda Q}\rangle| \leq Cb_1^{-2(k_0+\delta_0)-1+L+\frac{\eta}{2}(1-\delta_0)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}. \tag{3.5.75}$$

The last three bounds, together with the identity (*3.5.74*) we established in Step 1, give:

$$\begin{array}{rl} & \frac{d}{ds}\left[\frac{\langle \boldsymbol{H}^L(\varepsilon-\hat{\varepsilon}'),\chi_{B_0}\boldsymbol{\Lambda Q}\rangle}{\left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\right]-\hat{b}'_{L,s}\frac{\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}}{\left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle} \\ = & (b_{L,s}+(L-\alpha)b_1b_L-(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_L))+O(b_1^{L+1+\frac{\eta}{2}(1-\delta_0)}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}} \\ & +b_1^{\frac{\eta}{2}(1-\delta_0)}\sup b_1^{-i}|\triangle\hat{b}_i|)). \end{array} \tag{3.5.76}$$

As $\frac{\partial \boldsymbol{S}_{L+2}}{\partial b_L}$ is homogeneous of degree $(L+2,L+2,1,2)$ and does not depend on b_L, we have using the modulation bounds (*3.3.36*) and (*3.5.36*):

$$\left|\frac{d}{ds}\left[\frac{\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}}{\left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\right]\right| \leq Cb_1^{g'+1}\left(\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}\right).$$

Integrating by parts then yields:

$$\begin{array}{rl} & \frac{\hat{b}'_{L,s}\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}}{\left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}+O[b_1^{L+g'+1}(\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}})] \\ = & \frac{d}{ds}\left(\hat{b}'_L\frac{\int \chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}}{\left\langle \chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\right). \end{array}$$

Injecting this last identity in (*3.5.76*) give the identity (*3.5.59*) we had to prove. To finish, the gain when integrating is a consequence of (*3.5.75*), of the size of the denominator (*3.3.69*), and of the asymptotic:

$$\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial \hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}=O(y^{-\gamma-g'+2}b_1^2\sup_{1\leq i\leq 2}b_1^{-i}|\triangle\hat{b}|_i)$$

□

3.5.1.3 Energy identities for the difference of errors

In the previous section, the key norm of ε we had to control was the adapted high Sobolev norm $\mathcal{E}_{s_L}$. We recall the non linear tools we used to find a sufficient estimate: we control ε at another level of regularity to close the non linear term, integrate in time the modulation part that is not controlled directly, and establish a Morawetz type identity to manage a local term. Here we want to know how the time evolution of the adapted high Sobolev norm of the difference of the errors, $\varepsilon-\hat{\varepsilon}'$ depends on the differences of the parameters and itself, and will do it using the same non linear tools.

We start with a technical lemma linking the difference of the profiles to the difference of the parameters.

Lemma 3.5.8 (Bounds on the differences of profiles:). *The following bounds hold:*

$$\parallel \tilde{\boldsymbol{\psi}}_b-\tilde{\boldsymbol{\psi}}_{\hat{b}'}\parallel_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\leq Cb_1^{\alpha+1+O(\sigma-s_c,\eta)}\left(\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}\right), \tag{3.5.77}$$

$$\begin{array}{ll} & \parallel(\tilde{\psi}_b^{(1)}-\tilde{\psi}_{\hat{b}'}^{(1)})_{s_L}\parallel_{L^2}+\parallel(\tilde{\psi}_b^{(2)}-\tilde{\psi}_{\hat{b}'}^{(2)})_{s_L-1}\parallel_{L^2}\\ \leq & Cb_1^{L+1+(1-\delta_0)(1+\eta)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)+Cb_1^{2L+2-2\delta_0+O(\eta)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}.\end{array} \tag{3.5.78}$$

Proof of Lemma (3.5.8) We recall from (*3.2.58*) the expression of the differences of the errors:

$$\begin{array}{rcl}\tilde{\boldsymbol{\psi}}_b-\tilde{\boldsymbol{\psi}}_{\hat{b}'} & = & \chi_{B_1}\boldsymbol{\psi}_b-\chi_{\hat{B}_1'}\boldsymbol{\psi}_{\hat{b}'}+\chi_{B_1,s}\boldsymbol{\alpha}_b-\partial_{\hat{s}'}(\chi_{B_1})\boldsymbol{\alpha}_{\hat{b}'}\\ & & +b_1(\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_b-\chi_{B_1}\boldsymbol{\Lambda}\boldsymbol{Q}_b)-\hat{b}_1'(\boldsymbol{\Lambda}\tilde{\boldsymbol{Q}}_{\hat{b}'}-\chi_{B_1}\boldsymbol{\Lambda}\boldsymbol{Q}_{\hat{b}'})\\ & & -(\boldsymbol{F}(\tilde{\boldsymbol{Q}}_b)-\boldsymbol{F}(\boldsymbol{Q})-\chi_{B_1}(\boldsymbol{F}(\boldsymbol{Q}_b)-\boldsymbol{F}(\boldsymbol{Q}))\\ & & +(\boldsymbol{F}(\tilde{\boldsymbol{Q}}_{\hat{b}'})-\boldsymbol{F}(\boldsymbol{Q})-\chi_{B_1}(\boldsymbol{F}(\boldsymbol{Q}_{\hat{b}'})-\boldsymbol{F}(\boldsymbol{Q})),\end{array} \tag{3.5.79}$$

We have to estimate everything in the right hand side. It always rely on finding the asymptotic of the profiles and relating it to the difference of the parameters. We will just do it for the first two terms: the same methodology giving the same results for the others. The first one is on the second coordinate and we decompose:

$$\chi_{B_1}\boldsymbol{\psi}_b-\chi_{\hat{B}_1'}\boldsymbol{\psi}_{\hat{b}'}=\begin{pmatrix}0\\ \chi_{B_1}(\psi_b-\psi_{\hat{b}'})+\psi_{\hat{b}'}(\chi_{B_1}-\chi_{\hat{B}_1'})\end{pmatrix}. \tag{3.5.80}$$

For the first term in (*3.5.80*), from the asymptotic (*3.5.29*) of $\psi_b-\psi_{\hat{b}'}$ we obtain:

$$\parallel\chi_{B_1}(\psi_b-\psi_{\hat{b}'})\parallel_{\dot{H}^{\sigma-1}}\leq Cb_1^{\alpha+1+g'+O(\eta,\sigma-s_c)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|), \tag{3.5.81}$$

$$\| (\chi_{B_1}(\psi_b-\psi_{\hat{b}'}))_{s_L-1} \|_{L^2} \leq C b_1^{L+1+(1-\delta_0)+g'+O(\eta)} \sup_{1\leq i\leq L} (b_1^{-i}|\triangle \hat{b}_i|), \tag{3.5.82}$$

We now turn to the second term in (*3.5.80*). The integral formula (*3.5.33*) for $\chi_{B_1}-\chi_{\hat{B}_1'}$ implies that $\chi_{B_1}-\chi_{\hat{B}_1'}=(b_1^{1+\eta}-\hat{b}_1^{'(1+\eta)})f(y)$ with the function f having its support in $[\min(B_1,\hat{B}_1'), 2\max(B_1,\hat{B}_1')]$, and satisfying: $\partial_y^k f=O(y^{1-k})$. As one has $|b_1^{1+\eta}-\hat{b}^{'(1+\eta)}|\leq C|b_1-\hat{b}_1'|$, using the previous result (*3.2.55*) we get:

$$\| \psi_{\hat{b}'}(\chi_{B_1}-\chi_{\hat{B}_1'}) \|_{\dot{H}^{\sigma-1}} \leq C b_1^{\alpha+1+g'+O(\eta,\sigma-s_c)} b_1^{-1}|\triangle \hat{b}_1|. \tag{3.5.83}$$

$$\| (\psi_{\hat{b}'}(\chi_{B_1}-\chi_{\hat{B}_1'}))_{s_L-1} \|_{L^2} \leq C b_1^{L+1+(1-\delta_0)+g'+O(\eta,\sigma-s_c)} b_1^{-1}|\triangle \hat{b}_1|. \tag{3.5.84}$$

The decomposition (*3.5.80*) and the bounds (*3.5.81*), (*3.5.83*), (*3.5.82*) and (*3.5.84*) imply for the following bounds for the first term in (*3.5.79*):

$$\| \chi_{B_1}\boldsymbol{\psi}_b-\chi_{\hat{B}_1'}\boldsymbol{\psi}_{\hat{b}'} \|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \leq C b_1^{\alpha+1+g'+O(\eta,\sigma-s_c)} \sup_{1\leq i\leq L} (b_1^{-i}|\triangle \hat{b}_i|). \tag{3.5.85}$$

$$\| (\chi_{B_1}\psi_b-\chi_{\hat{B}_1'}\psi_{\hat{b}'})_{s_L-1} \|_{L^2} \leq C b_1^{L+1+(1-\delta_0)+g'+O(\eta,\sigma-s_c)} \sup_{1\leq i\leq L} (b_1^{-i}|\triangle \hat{b}_i|). \tag{3.5.86}$$

We now turn to the second difference of terms in (*3.5.79*). We compute:

$$\begin{array}{rl} & \chi_{B_1,s}\boldsymbol{\alpha}_b-\partial_{\hat{s}'}(\chi_{B_1})\boldsymbol{\alpha}_{\hat{b}'} := y(1+\eta)(\boldsymbol{A}_1+\boldsymbol{A}_2+\boldsymbol{A}_3+\boldsymbol{A}_4) \\ = & y(1+\eta)\Big[(b_{1,s}-\hat{b}_{1,s}')b_1^\eta\partial_y\chi(yb_1^{1+\eta})\boldsymbol{\alpha}_b+\hat{b}_{1,s}'(b_1^\eta-\hat{b}_1^{'\eta})\partial_y\chi(yb_1^{1+\eta})\boldsymbol{\alpha}_b \\ & +\hat{b}_{1,s}'\hat{b}_1^{'\eta}(\partial_y\chi(yb_1^{1+\eta})-\partial_y\chi(y\hat{b}_1^{'(1+\eta)}))\boldsymbol{\alpha}_b+\hat{b}_{1,s}'\hat{b}_1^{'\eta}\partial_y\chi(y\hat{b}_1^{'(1+\eta)})(\boldsymbol{\alpha}_b-\boldsymbol{\alpha}_{\hat{b}'})\Big]. \end{array} \tag{3.5.87}$$

and will estimate everything in the right hand side. From the expressions (*3.5.36*) and (*3.5.35*) for $b_{1,s}-\hat{b}_{1,s}'$ and $\frac{d\hat{s}'}{ds}-1$ we deduce that for the first term:

$$|b_{1,s}-\hat{b}_{1,s}'|\leq Cb_1^2\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|+b_1^{L+1+(1-\delta_0)(1+\frac{\eta}{2})}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}.$$

For the second term one has $|b_{1,s}(b_1^\eta-\hat{b}_1^{'\eta})|\leq b_1^{2+\eta}|b_1-\hat{b}_1'|$. For the third term an integral formula similar to (*3.5.33*) holds, giving:

$$\partial_y^k(\partial_y\chi(yb_1^{1+\eta})-\partial_y\chi(y\hat{b}_1^{'(1+\eta)}))=O\left(\frac{b_1^\eta|\triangle\hat{b}_1|}{1+y^{-1+k}}\right).$$

Therefore we get for the first three terms in (*3.5.87*):

$$\begin{array}{rcl} \| y(1+\eta)(\boldsymbol{A}_1+\boldsymbol{A}_2+\boldsymbol{A}_3) \|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} & \leq & Cb_1^{\alpha+1+O(\eta,\sigma-s_c)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|) \\ & & +Cb_1^{L+1+\alpha+O(\eta,\sigma-s_c)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}, \end{array} \tag{3.5.88}$$

$$\begin{array}{rl} & \| (y(A_1^{(1)}+A_2^{(1)}+A_3^{(1)}))_{s_L} \|_{L^2}+\| (y(A_1^{(2)}+A_2^{(2)}+A_3^{(2)}))_{s_L-1} \|_{L^2} \\ \leq & Cb_1^{L+1+(1-\delta_0)(1+\eta)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)+Cb_1^{2L+2-2\delta_0+O(\eta)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}. \end{array} \tag{3.5.89}$$

We turn to the fourth term in (*3.5.87*). One has:

$$\boldsymbol{\alpha}_b-\boldsymbol{\alpha}_{\hat{b}'}=\sum_1^L(b_i-\hat{b}_i')\boldsymbol{T}_i+\sum_2^{L+2}\boldsymbol{S}_i-\hat{\boldsymbol{S}}_i'.$$

The bound (*3.5.30*), the fact that the $\boldsymbol{S}_i$'s are homogeneous, using their asymptotic and the one of the $\boldsymbol{T}_i$'s yield:

$$\| y\hat{b}'_{1,s}\hat{b}_1'^{\eta}\partial_y\chi(\tfrac{y}{\hat{B}_1'})(\boldsymbol{\alpha}_b-\boldsymbol{\alpha}_{\hat{b}'}) \|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\leq \ Cb_1^{\alpha+1+O(\eta,\sigma-s_c)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|) \tag{3.5.90}$$

$$\begin{aligned} &\| (y\hat{b}'_{1,s}\hat{b}_1'^{\eta}\partial_y\chi(\tfrac{y}{\hat{B}_1'})(\alpha_b^{(1)}-\alpha_{\hat{b}'}^{(1)})_{s_L} \|_{L^2}+\| (y\hat{b}'_{1,s}\hat{b}_1'^{\eta}\partial_y\chi(\tfrac{y}{\hat{B}_1'})(\alpha_b^{(2)}-\alpha_{\hat{b}'}^{(2)})_{s_L-1} \|_{L^2} \\ \leq\ & Cb_1^{L+1+(1-\delta_0)(1+\eta)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|). \end{aligned}\tag{3.5.91}$$

because $|\hat{b}'_{1,s}\hat{b}_1'^{\eta}|\leq Cb_1^{2+\eta}$. We collect the bounds (*3.5.88*), (*3.5.90*), (*3.5.89*) and (*3.5.91*) to find that for the second term in (*3.5.79*):

$$\begin{aligned} \| \chi_{B_1,s}\boldsymbol{\alpha}_b-\partial_{\hat{s}'}(\chi_{B_1})\boldsymbol{\alpha}_{\hat{b}'} \|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \ \leq\ & Cb_1^{\alpha+1+O(\eta,\sigma-s_c)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|) \\ &+b_1^{L+1+\alpha+O(\eta,\sigma-s_c)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}, \end{aligned}\tag{3.5.92}$$

$$\begin{aligned} &\| (\chi_{B_1,s}\boldsymbol{\alpha}_b^{(1)}-\partial_{\hat{s}'}(\chi_{B_1})\boldsymbol{\alpha}_{\hat{b}'}^{(1)})_{s_L} \|_{L^2}+\| \chi_{B_1,s}\boldsymbol{\alpha}_b^{(2)}-\partial_{\hat{s}'}(\chi_{B_1})\boldsymbol{\alpha}_{\hat{b}'}^{(2)})_{s_L-1} \|_{L^2} \\ \leq\ & Cb_1^{L+1+(1-\delta_0)(1+\eta)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)+Cb_1^{2L+2-2\delta_0+O(\eta)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}. \end{aligned}\tag{3.5.93}$$

We claim that the bounds (*3.5.92*) and (*3.5.93*) also holds for the last two differences of profiles in (*3.5.79*) and that they can be proven using verbatim the same tools we employed so far. This fact give us the bounds for the remaining terms in (*3.5.79*), which combined with the previous estimates for the first two terms (*3.5.85*), (*3.5.86*), (*3.5.92*) and (*3.5.93*) proves the two estimates (*3.5.77*) and (*3.5.78*) of the lemma.

□

We state now how the time evolution of the low Sobolev norm of the difference of the errors $\varepsilon-\hat{\varepsilon}'$ is influenced by itself and the difference between the parameters and the renormalized times. It is the analogue of Proposition (3.3.6).

Lemma 3.5.9. (Time evolution of the low Sobolev norm of $\varepsilon-\hat{\varepsilon}'$). *We keep the assumptions and notations of Proposition 3.5.2. There holds:*

$$\frac{d}{dt}\left\{\frac{\triangle\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\}\leq\frac{Cb_1^{1+2(\sigma-s_c)(1+\nu)+\frac{\alpha}{2L}}}{\lambda^{2(\sigma-s_c)+1}}\left(\triangle_r\hat{\mathcal{E}}_\sigma+\triangle_r\hat{\mathcal{E}}_{s_L}+(\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|)^2\right)\tag{3.5.94}$$

(the norm $\triangle\mathcal{E}_\sigma$ is defined in (3.5.18), the renormalized norms $\triangle_r\hat{\mathcal{E}}_\sigma$ and $\triangle_r\hat{\mathcal{E}}_{s_L}$ are defined in (3.5.19)).

Proof of Lemma 3.5.9 We start by computing the following identity:

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{\triangle\hat{\mathcal{E}}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} \\ =\ & \int\nabla^\sigma(w^{(1)}-\hat{w}'^{(1)}).\nabla^\sigma(w^{(2)}-\hat{w}'^{(2)}+\frac{(\hat{Mod}(t)'^{(1)}-\tilde{Mod}(t)^{(1)}+\tilde{\psi}_{\hat{b}'}^{(1)}-\tilde{\psi}_b^{(1)})_{\frac{1}{\lambda}}}{\lambda} \\ &+(1-\tfrac{d\hat{s}'}{ds})\int\nabla^\sigma(w^{(1)}-\hat{w}'^{(1)}).\nabla^\sigma(-\hat{w}'^{(1)}-\tfrac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}) \\ &+\int\nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}(\mathcal{L}(\hat{w}'^{(1)}-w^{(1)})+\frac{(\hat{Mod}'(t)^{(2)}-\tilde{Mod}(t)^{(2)}+\tilde{\psi}_{\hat{b}'}^{(2)}-\tilde{\psi}_b{}^{(2)})_{\frac{1}{\lambda}}}{\lambda} \\ &+NL-\hat{NL}'+L-\hat{L}'+(1-\tfrac{d\hat{s}'}{ds})(-\mathcal{L}\hat{w}'^{(1)}-\tfrac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}+\hat{NL}'+\hat{L}')). \end{aligned}\tag{3.5.95}$$

We now compute the size of every term in the right hand side of equation (*3.5.95*).

• ***Linear terms:*** The norm studied here being adapted to a wave equation:

$$\begin{array}{rc} & \int \nabla^\sigma(w^{(1)}-\hat{w}'^{(1)}).\nabla^\sigma(w^{(2)}-\hat{w}'^{(2)})+\nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}\mathcal{L}(w^{(1)}-\hat{w}'^{(1)}) \\ = & \int \nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}(pQ_{\frac{1}{\lambda}}^{p-1}(w^{(1)}-\hat{w}'^{(1)})) \\ \leq & O(\parallel \nabla^\sigma(w^{(2)}-\hat{w}'^{(2)})\parallel_{L^2}\parallel \nabla^{\sigma-2}(Q_{\frac{1}{\lambda}}^{p-1}(w^{(1)}-\hat{w}'^{(1)}))\parallel_{L^2}). \end{array}$$

We recall the asymptotic $Q^{p-1}\sim \frac{c}{x^2}$ $(c>0)$. Using the weighted Hardy estimate from Lemma 3.C.2 one has for the second term:

$$\parallel \nabla^{\sigma-2}(Q_{\frac{1}{\lambda}}^{p-1}(w^{(1)}-\hat{w}'^{(1)}))\parallel_{L^2}\leq \frac{C}{\lambda^{\sigma-s_c}}\parallel \nabla^\sigma(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})\parallel_{L^2}=C\frac{\sqrt{\triangle\mathcal{E}_\sigma}}{\lambda^{\sigma-s_c}}.$$

By interpolation, we get for the other term:

$$\parallel \nabla^\sigma(w^{(2)}-\hat{w}'^{(2)})\parallel_{L^2}\leq \frac{C}{\lambda^{\sigma-s_c+1}}\sqrt{\triangle\mathcal{E}_\sigma}^{1-\frac{1}{s_L-\sigma}}\sqrt{\triangle\mathcal{E}_{s_L}}^{\frac{1}{s_L-\sigma}}.$$

Using the definition of the renormalized norms of the difference (*3.5.19*) and the fact that one has the identity $\frac{L+(1-\delta_0)(1+\eta)-(\sigma-s_c)(1+\nu)}{s_L-\sigma}=1+\frac{\alpha}{L}+O((\sigma-s_c)L^{-1},\eta L^{-1},L^{-2})$ we conclude:

$$\begin{array}{rc} & \left|\int \nabla^\sigma(w^{(1)}-\hat{w}'^{(1)}.\nabla^\sigma(w^{(2)}-\hat{w}'^{(2)})-\nabla^{\sigma-1}w^{(2)}.\nabla^{\sigma-1}\mathcal{L}(w^{(1)}-\hat{w}'^{(1)})\right| \\ \leq & \frac{Cb_1^{2(\sigma-s_c)(1+\nu)+1+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L},\frac{\eta}{L},\frac{1}{L^2}\right)}}{\lambda^{2(\sigma-s_c)+1}}\sqrt{\triangle_r\mathcal{E}_\sigma}^{2-\frac{1}{s_L-\sigma}}\sqrt{\triangle_r\mathcal{E}_{s_L}}^{\frac{1}{s_L-\sigma}}. \end{array} \quad (3.5.96)$$

• $\tilde{\boldsymbol{Mod}}(t)$ ***terms:*** We only compute for the $\tilde{Mod}^{(2)}$ terms, the calculation being the same for the first coordinate. Rescaling, using Cauchy-Schwarz and the notations (*3.5.24*) and (*3.5.25*):

$$\begin{array}{rc} & \left|\frac{1}{\lambda}\int \nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}(\tilde{Mod}^{(2)}-\hat{Mod}'^{(2)})_{\frac{1}{\lambda}}\right| \\ \leq & \frac{1}{\lambda^{2(\sigma-s_c)+1}}\sqrt{\triangle\mathcal{E}_\sigma}\left(\sum_{i=0}^L\parallel \nabla^{\sigma-1}\triangle\hat{Mod}_i^{(2)}\parallel_{L^2}\right). \end{array} \quad (3.5.97)$$

We will just compute a bound for the last term: $\triangle\hat{Mod}_L^{(2)}$. Indeed it is for this one that we have the worst bound, see Lemma 3.5.6. We first split:

$$\begin{array}{rcl} \triangle\hat{Mod}_L^{(2)} & = & (b_{L,s}+(L-\alpha)b_1b_i-(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_i))\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L}) \\ & & +(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_i)(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L})-\chi_{\hat{B}'_1}(T_L+\frac{\partial \hat{S}'_{L+2}}{\partial b_L})). \end{array} \quad (3.5.98)$$

For the first term, the bound (*3.5.37*) established in the previous Lemma 3.5.6 implies:

$$\begin{array}{rc} & \parallel \nabla^{\sigma-1}(b_{L,s}+(L-\alpha)b_1b_i-(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_i))\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L})\parallel_{L^2} \\ \leq & Cb_1^{\alpha+1-\delta_0+O(\eta,\sigma-s_c)}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)). \end{array} \quad (3.5.99)$$

We now want to estimate the second term in (*3.5.98*). We decompose:

$$\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L})-\chi_{\hat{B}'_1}(T_L+\frac{\partial\hat{S}'_{L+2}}{\partial b_L}) \quad = \quad (\chi_{B_1}-\chi_{\hat{B}'_1})(T_L+\frac{\partial S_{L+2}}{\partial b_L})+\chi_{B_1}(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial S_{L+2}}{\partial b_L}).$$

The identity (*3.5.33*) gives that $\chi_{B_1}(y)-\chi_{\hat{B}'_1}(y)=(b_1^{1+\eta}-\hat{b}_1'^{(1+\eta)})f_{b_1,\hat{b}'_1}(y)$ with $f_{b_1,\hat{b}'_1}(y)$ being a C^∞ function with support in $[\text{min}(B_1,\hat{B}'_1),2\text{max}(B_1,\hat{B}'_1)]$ satisfying: $|\partial_y^k f_{b_1,\hat{b}'_1}|\leq Cb_1^{k-1+C_k\eta}$. We recall the meaning of our notation:

$$\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_i=\frac{d\hat{s}'}{ds}\left(\frac{d}{d\hat{s}'}b'_L(\hat{s}')+(L-\alpha)b'_1(\hat{s}')b'_L(\hat{s}')\right).$$

From the bound on the modulation (*3.3.37*), and from $|b_1^{1+\eta}-\hat{b}_1^{\prime(1+\eta)}|\leq C|b_1-\hat{b}_1'|$ one gets using the asymptotic of T_L and $\frac{\partial S_{L+2}}{\partial b_L}$:

$$\| \nabla^{\sigma-1}(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_i)(\chi_{B_1}-\chi_{\hat{B}'_1})(T_L+\frac{\partial S_{L+2}}{\partial b_L}) \|_{L^2}\leq Cb_1^{\alpha+1-\delta_0+O(\eta,\sigma-s_c)-1}|b_1-\hat{b}'_1|.$$

For the second part, using again the bound (*3.3.37*), the fact that $\frac{S_{L+2}}{\partial b_L}$ is homogeneous and that its degree is $(L+2,L+2-g',1,2)$ and the bound (*3.5.30*):

$$\| \nabla^{\sigma-1}(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_i)\chi_{B_1}(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial S_{L+2}}{\partial b_L}) \|_{L^2}\ \leq Cb_1^{\alpha+1-\delta_0+O(\eta,\sigma-s_c)}\sup_{1\leq i\leq L}(b_1^{-i}|b_i-\hat{b}'_i)|.$$

Eventually we have found, gathering the two previous bounds:

$$\begin{array}{ll} & \| \nabla^{\sigma-1}(\hat{b}'_{L,s}+\frac{d\hat{s}'}{ds}(L-\alpha)\hat{b}'_1\hat{b}'_i)(\chi_{B_1}(T_L+\frac{\partial S_{L+2}}{\partial b_L})-\chi_{\hat{B}'_1}(T_L+\frac{\partial \hat{S}'_{L+2}}{\partial b_L})) \|_{L^2} \\ \leq & Cb_1^{\alpha+1-\delta_0+O(\eta,\sigma-s_c)}\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)). \end{array} \quad (3.5.100)$$

We can now go back to (*3.5.98*) and inject the bounds (*3.5.99*) and (*3.5.100*) for the terms in the right hand side. This gives for the L-th modulation term:

$$\| \nabla^{\sigma-1}\triangle\hat{Mod}_L^{(2)} \|_{L^2}\leq Cb_1^{\alpha+1-\delta_0+O(\eta,\sigma-s_c)}(\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}). \quad (3.5.101)$$

The primary modulation bounds for the evolution of b_L and $\triangle\hat{b}_L$ being worst than the ones for b_i and $\triangle\hat{b}_i$ (compare (*3.3.36*) and (*3.3.37*), (*3.5.57*) and (*3.5.58*)) we claim that a better estimate than (*3.5.101*) also holds for the other terms in (*3.5.97*) and that it also work for the first coordinate, yielding when injected in (*3.5.97*):

$$\begin{array}{ll} & |\frac{1}{\lambda}\int\nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}(\tilde{Mod}^{(2)}-\hat{Mod}'^{(2)})_{\frac{1}{\lambda}} \\ & +\frac{1}{\lambda}\int\nabla^{\sigma}(w^{(1)}-\hat{w}'^{(1)}).\nabla^{\sigma}(\tilde{Mod}^{(1)}-\hat{Mod}'^{(1)})_{\frac{1}{\lambda}}| \\ \leq & C\frac{b_1^{1+\alpha-\delta_0+O(\eta,\sigma-s_c)}}{\lambda^{2(\sigma-s_c)+1}}\sqrt{\triangle_r\mathcal{E}_\sigma}(\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}), \end{array} \quad (3.5.102)$$

and we recall that $\alpha-\delta_0>2-\delta_0>1$.

- $\tilde{\psi}_b$ ***term:*** We use the bound (*3.5.77*) on $\tilde{\psi}_b-\tilde{\psi}_{\hat{b}'}$:

$$\begin{array}{ll} & \left|\frac{1}{\lambda}\int\nabla^{\sigma}(w^{(1)}-\hat{w}'^{(1)}).\nabla^{\sigma}(\tilde{\psi}_b-\tilde{\psi}_{\hat{b}'})^{(1)}_{\frac{1}{\lambda}}+\nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}(\tilde{\psi}_b-\tilde{\psi}_{\hat{b}'})^{(2)}_{\frac{1}{\lambda}}\right| \\ \leq & C\frac{b_1^{1+\alpha+O(\eta,\sigma-s_c)}}{\lambda^{2(\sigma-s_c)+1}}\sqrt{\triangle_r\mathcal{E}_\sigma}(\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}), \end{array} \quad (3.5.103)$$

and we recall that $\alpha>2$.

- $L(w)$ ***term:*** We compute the following identity:

$$\begin{array}{lll} L-\hat{L}' & = & (p-1)(Q^{p-1}-\tilde{Q}_b^{p-1})\varepsilon^{(1)}-(p-1)(Q^{p-1}-\tilde{Q}_{\hat{b}'}^{p-1})\hat{\varepsilon}'^{(1)} \\ & = & (p-1)(Q-\tilde{Q}_b^{p-1})(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})+(p-1)(\tilde{Q}_{\hat{b}'}^{p-1}-\tilde{Q}_b^{p-1})\hat{\varepsilon}'^{(1)}. \end{array}$$

We recall that thanks to the asymptotic (*3.3.80*) and to the fractional Hardy inequality one has for the first term:

$$\begin{array}{ll} & \| \nabla^{\sigma-1}(Q-\tilde{Q}_b^{p-1})(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}) \|_{L^2}\leq Cb_1 \| \nabla^{\sigma+\frac{1}{p-1}}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}) \|_{L^2} \\ \leq & Cb_1^{1+\frac{1}{p-1}+O(L^{-1},\eta,\sigma-s_c)}\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}^{1-\frac{1}{(p-1)(s_L-\sigma)}}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}^{\frac{1}{(p-1)(s_L-\sigma)}}. \end{array}$$

For the second one, the bound (*3.5.28*) on the asymptotic of $\tilde{Q}_{\hat{b}'}^{p-1}-\tilde{Q}_b^{p-1}$ and the Hardy inequality yield:

$$\parallel \nabla^{\sigma-1}(\tilde{Q}_{\hat{b}'}^{p-1}-\tilde{Q}_b^{p-1})\hat{\varepsilon}'^{(1)} \parallel_{L^2}\leq Cb_1^{1+\frac{1}{p-1}+O(L^{-1},\eta,\sigma-s_c)} \sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|).$$

Therefore we end up with the following bound on the small linear term:

$$\begin{array}{ll} & \left|\int \nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}(L-\hat{L}')\right| \leq \frac{C\|w^{(2)}-\hat{w}'^{(2)}\|_{\dot{H}^{\sigma-1}}\|L-\hat{L}'\|_{\dot{H}^{\sigma-1}}}{\lambda^{2(\sigma-s_c)+1}} \\ \leq & C\frac{b_1^{1+\frac{1}{p-1}+O(L^{-1},\sigma-s_c,\eta)}\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}(\sup\limits_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}^{1-\frac{1}{(p-1)(s_L-\sigma)}}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}^{\frac{1}{(p-1)(s_L-\sigma)}})}{\lambda^{2(\sigma-s_c)+1}}. \end{array} \quad (3.5.104)$$

• NL *term:* The difference of the non linear terms is[25]:

$$\begin{array}{lll} NL-\hat{NL}' & = & \sum_{j=2}^p C_j\tilde{Q}_b^{p-j}\varepsilon^{(1)j}-\sum_{j=2}^p C_j\tilde{Q}_{\hat{b}'}^{p-j}\hat{\varepsilon}'^{(1)j} \\ & = & \sum_{j=2}^p C_j\tilde{Q}_b^{p-j}(\varepsilon^{(1)j}-\hat{\varepsilon}'^{(1)j})+\sum_{j=2}^p C_j(\tilde{Q}_b^{p-j}-\tilde{Q}_{\hat{b}'}^{p-j})\hat{\varepsilon}'^{(1)j}. \end{array} \quad (3.5.105)$$

for some coefficients C_j appearing when developing the polynomial $(X+Y)^p$. We start with the second term of this identity, assuming $j\neq p$. We now recall the bound (*3.3.84*) we found for the non-linear term in the proof of Proposition (3.3.6):

$$\left\|\nabla^{\sigma-2+(j-1)(\sigma-s_c)}(v(y)\varepsilon'^{(1)j})\right\|_{L^2}\leq C\sqrt{\hat{\mathcal{E}}'_\sigma}^j,$$

for potentials v satisfying $\partial_y^k v=O\left(\frac{1}{1+y^{2\frac{p-j}{p-1}+k}}\right)$. Here, thanks to the asymptotic (*3.5.28*), the potential is even better because of an extra gain $y^{-\alpha}$, therefore:

$$\left\|\nabla^{\sigma-2+(j-1)(\sigma-s_c)}((\tilde{Q}_{\hat{b}'}^{p-j}-\tilde{Q}_b^{p-j})\hat{\varepsilon}'^{(1)j})\right\|_{L^2}\leq C\sqrt{\hat{\mathcal{E}}'_\sigma}^j\sup_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|).$$

This last bound imply that, integrating by part, for the second term in (*3.5.105*):

$$\begin{array}{ll} & \left|\int\nabla^{\sigma-1}(w^{(1)}-\hat{w}'^{(1)}).\nabla^{\sigma-1}((\tilde{Q}_b^{p-j}-\tilde{Q}_{\hat{b}'}^{p-j})\hat{\varepsilon}'^{(1)j})\right| \\ \leq & \frac{C}{\lambda^{2(\sigma-s_c)+1}}\parallel\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}\parallel_{\dot{H}^{\sigma-(j-1)(\sigma-s_c)}}\left\|(\tilde{Q}_{\hat{b}'}^{p-j}-\tilde{Q}_b^{p-j})\hat{\varepsilon}'^{(1)j}\right\|_{\dot{H}^{\sigma-2+(j-1)(\sigma-s_c)}} \\ \leq & \frac{Cb_1^{2(\sigma-s_c)(1+\nu)+1+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sup\limits_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}^{1-\frac{1-(j-1)(\sigma-s_c)}{s_L-\sigma}}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}^{\frac{1-(j-1)(\sigma-s_c)}{s_L-\sigma}}}{\lambda^{2(\sigma-s_c)+1}}. \end{array} \quad (3.5.106)$$

We now turn to the first term in (*3.5.105*). We factorize the non linear term:

$$(\varepsilon^{(1)j}-\hat{\varepsilon}'^{(1)j})=(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})\sum_{i=0}^{j-1}C_i\varepsilon^{(1)i}(\hat{\varepsilon}'^{(1)})^{j-1-i},$$

for some coefficients $(C_i)_{0\leq i\leq j-1}$. We can then apply the same reasoning we used in the proof of the bound (*3.3.84*), giving this time:

$$\left\|\nabla^{\sigma-2+(j-1)(\sigma-s_c)}(\tilde{Q}_b^{p-j}(\varepsilon^{(1)j}-\hat{\varepsilon}'^{(1)j}))\right\|_{L^2}\leq C\sqrt{\triangle\hat{\mathcal{E}}_\sigma}\sum_{i=0}^{j-1}\sqrt{\mathcal{E}_\sigma}^i\sqrt{\hat{\mathcal{E}}'_\sigma}^{j-1-i}.$$

[25] we make here the abuse of notation $\tilde{Q}_b^{p-j}=\tilde{Q}_b^{(1)(p-j)}$ to ease notations.

As we did previously for the second term in (*3.5.105*), we now use interpolation and inject the bootstrap bounds (*3.3.27*) to find:

$$
\begin{aligned}
& \left| \int \nabla^{\sigma-1}(w^{(1)} - \hat{w}'^{(1)}).\nabla^{\sigma-1}(\tilde{Q}_b^{p-j}(\varepsilon^{(1)j} - \hat{\varepsilon}'^{(1)j})) \right| \\
\leq \ & \frac{C}{\lambda^{2(\sigma-s_c)+1}} \parallel \varepsilon^{(1)} - \hat{\varepsilon}'^{(1)} \parallel_{\dot{H}^{\sigma-(j-1)(\sigma-s_c)}} \left\| \tilde{Q}_b^{p-j}(\varepsilon^{(1)j} - \hat{\varepsilon}'^{(1)j}) \right\|_{\dot{H}^{\sigma-2+(j-1)(\sigma-s_c)}} \\
\leq \ & \frac{C b_1^{2(\sigma-s_c)(1+\nu)+1+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}}{\lambda^{2(\sigma-s_c)+1}} \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma}^{2-\frac{1-(j-1)(\sigma-s_c)}{s_L-\sigma}} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}^{\frac{1-(j-1)(\sigma-s_c)}{s_L-\sigma}} .
\end{aligned}
\tag{3.5.107}
$$

In (*3.5.106*) and (*3.5.107*) we have found an estimate for the two terms in the right hand side of (*3.5.105*), giving the following bound for the non linear terms contribution:

$$
\begin{aligned}
& \left| \int \nabla^{\sigma-1}(w^{(1)} - \hat{w}'^{(1)}).\nabla^{\sigma-1}(NL - \hat{NL}') \right| \\
\leq \ & \frac{C b_1^{2(\sigma-s_c)(1+\nu)+1+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}}{\lambda^{2(\sigma-s_c)+1}} \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma}^{1-\frac{1-(j-1)(\sigma-s_c)}{s_L-\sigma}} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}^{\frac{1-(j-1)(\sigma-s_c)}{s_L-\sigma}} \\
& \times \left(\sqrt{\triangle_r \hat{\mathcal{E}}_\sigma} + \sup_{1\leq i\leq L} (b_1^{-i}|\triangle \hat{b}_i|) \right) .
\end{aligned}
\tag{3.5.108}
$$

• *The time difference terms:* We now look for a bound for the terms involving $\frac{d\hat{s}'}{ds} - 1$ in (*3.5.95*). We have already computed the size of most of the terms in (*3.3.79*), (*3.3.81*) and (*3.3.85*), yielding:

$$
\begin{aligned}
& \left|\int \nabla^\sigma (w^{(1)} - \hat{w}'^{(1)}).\nabla^\sigma(-\tfrac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}) + \int \nabla^{\sigma-1}(w^{(2)} - \hat{w}'^{(2)}).\nabla^{\sigma-1}(-\tfrac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}} + \hat{NL}' + \hat{L}')\right| \\
\leq \ & \frac{C b_1^{2(\sigma-s_c)(1+\nu)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)} (\sqrt{\triangle_r \hat{\mathcal{E}}_\sigma} + \sum_{k=2}^p \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma}^{1-\frac{1-(k-1)(\sigma-s_c)}{s_L-\sigma}} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}^{\frac{1-(k-1)(\sigma-s_c)}{s_L-\sigma}})}{\lambda^{2(\sigma-s_c)+1}} .
\end{aligned}
$$

With the bound (*3.5.35*) on $|\frac{d\hat{s}'}{ds} - 1|$ we obtain:

$$
\begin{aligned}
& \left|(1 - \tfrac{d\hat{s}'}{ds}) \int \nabla^\sigma (w^{(1)} - \hat{w}'^{(1)}).\nabla^\sigma(-\tfrac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}) \right. \\
& \left. +(1 - \tfrac{d\hat{s}'}{ds}) \int \nabla^{\sigma-1}(w^{(2)} - \hat{w}'^{(2)}).\nabla^{\sigma-1}(-\tfrac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}} + \hat{NL}' + \hat{L}')\right| \\
\leq \ & \frac{C b_1^{2(\sigma-s_c)(1+\nu)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}}{\lambda^{2(\sigma-s_c)+1}} (\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} + \sup_{1\leq i\leq L} (b_1^{-i}|\triangle \hat{b}_i|)) \\
& \times \left(\sqrt{\triangle_r \hat{\mathcal{E}}_\sigma} + \sum_{k=2}^p \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma}^{1-\frac{1-(k-1)(\sigma-s_c)}{s_L-\sigma}} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}^{\frac{1-(k-1)(\sigma-s_c)}{s_L-\sigma}} \right) .
\end{aligned}
\tag{3.5.109}
$$

The only term we did not really estimate in the proof of Proposition 3.3.6 is the linear one, because we had a natural cancellation, the norm being adapted to a wave equation. We start with the terms involving derivatives:

$$
\begin{aligned}
& \left| \int \nabla^\sigma (w^{(1)} - \hat{w}'^{(1)}).\nabla^\sigma(-\hat{w}'^{(2)}) + \nabla^{\sigma-1}(w^{(2)} - \hat{w}'^{(2)}).\nabla^{\sigma-1}(\Delta \hat{w}'^{(1)}) \right| \\
\leq \ & \frac{C}{\lambda^{2(\sigma-s_c)+1}} \sqrt{\triangle \hat{\mathcal{E}_\sigma}} \parallel \hat{\varepsilon}' \parallel_{\dot{H}^{\sigma+1}\times \dot{H}^\sigma} \leq \frac{C b_1^{1+2(\sigma-s_c)(1+\nu)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}}{\lambda^{2(\sigma-s_c)+1}} \sqrt{\triangle_r \hat{\mathcal{E}_\sigma}} .
\end{aligned}
\tag{3.5.110}
$$

For the term involving the potential, integrating by parts, using Hardy inequality (as $Q^{p-1} = O(y^{-2})$) and interpolation yields:

$$
\begin{aligned}
& \left| \int \nabla^{\sigma-1}(w^{(2)} - \hat{w}'^{(2)}).\nabla^{\sigma-1}((p-1)Q^{p-1}_{\frac{1}{\lambda}} \hat{w}'^{(1)}) \right| \\
\leq \ & \frac{C}{\lambda^{2(\sigma-s_c)+1}} \parallel \nabla^\sigma (w^{(2)} - \hat{w}'^{(2)}) \parallel_{L^2} \parallel \nabla^{\sigma-2} (Q^{p-1}_{\frac{1}{\lambda}} \hat{w}'^{(1)}) \parallel_{L^2} \\
\leq \ & \frac{C b_1^{1+2(\sigma-s_c)(1+\nu)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}}{\lambda^{2(\sigma-s_c)+1}} \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma}^{1-\frac{1}{s_L-\sigma}} \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma}^{\frac{1}{s_L-\sigma}} .
\end{aligned}
\tag{3.5.111}
$$

The two previous bounds (*3.5.110*) and (*3.5.111*), combined with the bound (*3.5.35*) on $|\frac{d\hat{s}'}{ds}-1|$ give for the linear term:

$$
\begin{array}{l}
\left|(\frac{d\hat{s}'}{ds}-1)\int \nabla^\sigma(w^{(1)}-\hat{w}'^{(1)}).\nabla^\sigma \hat{w}'^{(2)}+\nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}\mathcal{L}\hat{w}'^{(1)}\right| \\
\leq \dfrac{Cb_1^{1+2(\sigma-s_c)(1+\nu)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}^{1-\frac{1}{s_L-\sigma}}\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}^{\frac{1}{s_L-\sigma}}(\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}+\sup\limits_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|))}{\lambda^{2(\sigma-s_c)+1}}.
\end{array}
\tag{3.5.112}
$$

The bounds (*3.5.109*) and (*3.5.112*) imply that for the terms in (*3.5.95*) involving $\frac{d\hat{s}'}{ds}-1$:

$$
\begin{array}{l}
\big|(1-\frac{d\hat{s}'}{ds})\int\nabla^\sigma(w^{(1)}-\hat{w}'^{(1)}).\nabla^\sigma(-\hat{w}'^{(1)}-\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}) \\
+(1-\frac{d\hat{s}'}{ds})\int\nabla^{\sigma-1}(w^{(2)}-\hat{w}'^{(2)}).\nabla^{\sigma-1}(-\mathcal{L}\hat{w}'^{(1)}-\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}+\hat{N}L'+\hat{L}')\big| \\
\leq \dfrac{Cb_1^{1+2(\sigma-s_c)(1+\nu)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}^{1-\frac{1}{s_L-\sigma}}\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}^{\frac{1}{s_L-\sigma}}(\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}+\sup\limits_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|))}{\lambda^{2(\sigma-s_c)+1}}.
\end{array}
\tag{3.5.113}
$$

Step 2: Gathering the bounds. We have made the decomposition (*3.5.95*) and have computed an upper bound for all terms in the right hand side in (*3.5.96*), (*3.5.102*), (*3.5.103*), (*3.5.104*), (*3.5.108*) and (*3.5.113*). Consequently:

$$
\frac{d}{dt}\left\{\frac{\triangle\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\}\leq\frac{Cb_1^{1+2(\sigma-s_c)(1+\nu)+\frac{\alpha}{2L}}}{\lambda^{2(\sigma-s_c)+1}}\left(\triangle_r\hat{\mathcal{E}}_\sigma+\triangle_r\hat{\mathcal{E}}_{s_L}+(\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|)^2\right),
$$

which is the bound we had to prove. □

We now turn to the control of the most important of the two norms of the difference of errors $\varepsilon-\hat{\varepsilon}'$: the adapted one at a high level of regularity. We state a similar result as the one in Proposition 3.3.7, this time relating the time evolution to the differences of the parameters and errors. Again, we will not be able to control directly a local norm, relegating it to the next lemma.

Lemma 3.5.10. (Lyapunov monotonicity for the high Sobolev norm:) *We recall that $\triangle\mathcal{E}_{s_L}$ and $\triangle\mathcal{E}_{s_L,loc}$ are defined in (3.3.11) and (3.3.12)). There holds for $s_0\leq s$:*

$$
\begin{array}{l}
\dfrac{d}{dt}\left\{\dfrac{\triangle\hat{\mathcal{E}}_{s_L}}{\lambda^{2(s_L-s_c)}}+O\left(\dfrac{b_1^{2L+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)}}(\triangle_r\hat{\mathcal{E}}_{s_L}+|\sup\limits_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i||^2)\right)\right\} \\
\leq \dfrac{Cb_1^{2L+2(1-\delta_0)(1+\frac{\eta}{2})+1}}{\lambda^{2(s_L-s_c)+1}}\Big(b_1^{\frac{\eta}{2}(1-\delta_0)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}\sup\limits_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|+C(N)\triangle_r\hat{\mathcal{E}}_{s_L,loc} \\
\qquad +\dfrac{C}{N^{\frac{\delta_0}{2}}}(\triangle_r\hat{\mathcal{E}}_{s_L}+\triangle_r\hat{\mathcal{E}}_\sigma)+b_1^{\frac{\eta}{2}(1-\delta_0)}\sup\limits_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|)^2\Big),
\end{array}
\tag{3.5.114}
$$

for some universal constant C that does not depend on N.

Proof of Lemma 3.5.10 The strategy of the proof of Lemma 3.5.10 is similar to the one of the proof of Proposition (3.3.7). We start by computing the following identity:

$$\begin{array}{rl} & \frac{d}{dt}\left(\frac{\triangle \mathcal{E}_{s_L}}{2\lambda^{2(s_L-s_c)}}\right) \\ = & \int (w^{(1)}-\hat{w}'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}\big[w^{(2)}-\hat{w}'^{(2)}-\frac{1}{\lambda}\tilde{\psi}^{(1)}_{b,\frac{1}{\lambda}}+\frac{1}{\lambda}\tilde{\psi}^{(1)}_{\hat{b}',\frac{1}{\lambda}}-\frac{1}{\lambda}\tilde{Mod}^{(1)}_{\frac{1}{\lambda}} \\ & +\frac{1}{\lambda}\hat{Mod}'^{(1)}_{\frac{1}{\lambda}}+(\frac{d\hat{s}'}{ds}-1)(\hat{w}'^{(2)}-\frac{1}{\lambda}\tilde{\psi}^{(1)}_{\hat{b}',\frac{1}{\lambda}})\big] \\ & +\int (w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}\Big[-\mathcal{L}_{\frac{1}{\lambda}}(w^{(1)}-\hat{w}'^{(1)})-\frac{1}{\lambda}(\tilde{\psi}^{(2)}_{b,\frac{1}{\lambda}}-\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}})-\frac{1}{\lambda}\tilde{Mod}^{(2)}_{\frac{1}{\lambda}} \\ & +\frac{1}{\lambda}\hat{Mod}'^{(2)}_{\frac{1}{\lambda}}+L-\hat{L}'+NL-\hat{NL}'+(1-\frac{d\hat{s}'}{ds})(-\mathcal{L}\hat{w}'^{(1)}-\frac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}+\hat{NL}'+\hat{L}')\Big] \\ & +\frac{1}{2}\sum_{i=1}^{k_0+L+1}\int (w^{(1)}-\hat{w}'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1-i}(w^{(1)}-\hat{w}'^{(1)}) \\ & +\frac{1}{2}\sum_{i=1}^{k_0+L}\int (w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L-i}(w^{(2)}-\hat{w}'^{(2)}). \end{array} \tag{3.5.115}$$

We now manage all terms in the right hand side.

Step 1: Direct bounds. The linear, non linear, error, and time error terms can be estimated via a direct bound. We claim the following identity:

$$\begin{array}{rl} & \frac{d}{dt}\left(\frac{\triangle \mathcal{E}_{s_L}}{2\lambda^{2(s_L-s_c)}}\right) \\ = & \int (w^{(1)}-\hat{w}'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}\big[-\frac{1}{\lambda}\tilde{Mod}^{(1)}_{\frac{1}{\lambda}}+\frac{1}{\lambda}\hat{Mod}'^{(1)}_{\frac{1}{\lambda}}\big] \\ & +\int (w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}\Big[-\frac{1}{\lambda}\tilde{Mod}^{(2)}_{\frac{1}{\lambda}}+\frac{1}{\lambda}\hat{Mod}'^{(2)}_{\frac{1}{\lambda}}+L-\hat{L}'\Big] \\ & +\frac{1}{2}\sum_{i=1}^{k_0+L+1}\int (w^{(1)}-\hat{w}'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1-i}(w^{(1)}-\hat{w}'^{(1)}) \\ & +\frac{1}{2}\sum_{i=1}^{k_0+L}\int (w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{i-1}\frac{d}{dt}\left(\mathcal{L}_{\frac{1}{\lambda}}\right)\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L-i}(w^{(2)}-\hat{w}'^{(2)}) \\ & +O\Big(\frac{Cb_1^{2L+(1-\delta_0)(2+\frac{3}{2}\eta)+1}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}\sup\limits_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|+b_1^{\frac{\alpha+O(\sigma-s_c,\eta)}{L}}(\triangle_r\hat{\mathcal{E}}_{s_L}+\triangle_r\hat{\mathcal{E}}_\sigma))}{\lambda^{2(s_L-s_c)+1}}\Big), \end{array} \tag{3.5.116}$$

which we are now going to prove by finding upper bounds for each term in the right hand side of (*3.5.115*).

• *Linear terms:* The fact that the form of the norm is adapted to the linear wave equation with operator $\mathcal{L}$ induces:

$$\int (w^{(1)}-\hat{w}'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}(w^{(2)}-\hat{w}'^{(2)})+(w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}(-\mathcal{L}_{\frac{1}{\lambda}}(w^{(1)}-\hat{w}'^{(1)}))=0. \tag{3.5.117}$$

• *Error terms:* Using the bound (*3.5.78*) on $\tilde{\boldsymbol{\psi}}_b-\tilde{\boldsymbol{\psi}}_{\hat{b}'}$:

$$\begin{array}{rl} & \left|\frac{1}{\lambda}\int (w^{(1)}-\hat{w}'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1}(\tilde{\psi}^{(1)}_{b,\frac{1}{\lambda}}-\tilde{\psi}^{(1)}_{\hat{b}',\frac{1}{\lambda}})+(w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{k_0+L}(\tilde{\psi}^{(2)}_{b,\frac{1}{\lambda}}-\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}})\right| \\ \leq & \frac{C\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}}{\lambda^{2(s_L-s_c)+1}}(b_1^{2L+(1-\delta_0)(2+\frac{3}{2}\eta)+1}\sup\limits_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|)+b_1^{3L+3-3\delta_0+O(\eta)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}). \end{array} \tag{3.5.118}$$

• *Non linear terms:* We know that NL is a sum of terms of the form $\tilde{Q}_b^{p-k}\varepsilon^{(1)k}$ for $2\leq k\leq p$. Therefore we start by decomposing:

$$\begin{array}{rcl} \|(NL-\hat{NL}')_{k_0+L}\|_{L^2} & \leq & C\sum_2^p\|(\tilde{Q}_b^{p-k}(\varepsilon^{(1)k}-\hat{\varepsilon}'^{(1)k}))_{k_0+L}\|_{L^2} \\ & & +\|(\hat{\varepsilon}'^{(1)k}(\tilde{Q}_b^{(1)(p-k)}-\tilde{Q}_{\hat{b}'}^{(1)(p-k)}))_{k_0+L}\|_{L^2} \end{array} \tag{3.5.119}$$

For the first term of this identity, we can do the same reasoning we used in the proof of the direct bound (*3.3.96*) in the proof of Proposition 3.3.7. What changes here is that we do not have to treat $\varepsilon^{(1)k}$, but $(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})\varepsilon^{(1)i}\hat{\varepsilon}'^{(1)(k-1-i)}$ because of the factorization:

$$\varepsilon^{(1)k}-\hat{\varepsilon}'^{(1)k}=\left(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}\right)\sum_{i=0}^{k-1}C_i\varepsilon^{(1)i}\hat{\varepsilon}'^{(1)(k-1-i)}$$

for some constants $(C_0)_{1\leq i\leq k-1}$. We recall that using various decompositions, Hardy inequalities and Sobolev injections, in (*3.3.96*) we proved:

$$\| (NL)_{s_L-1} \|_{L^2}^2\leq C(K_1,K_2)b_1^{2L+2+2(1-\delta_0)(1+\eta)+\frac{2\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}. \tag{3.5.120}$$

Whenever interpolating between $\triangle\hat{\mathcal{E}}_\sigma$ and $\triangle\hat{\mathcal{E}}_{s_L}$ one has for $0\leq\theta\leq 1$:

$$\begin{array}{rcl}\triangle\hat{\mathcal{E}}_\sigma^\theta\triangle\hat{\mathcal{E}}_{s_L}^{1-\theta} & \leq & b_1^{2\theta(\sigma-s_c)(1+\nu)+2(1-\theta)(L+(1-\delta_0)(1+\eta))}\triangle_r\hat{\mathcal{E}}_\sigma^\theta\triangle_r\hat{\mathcal{E}}_{s_L}^{1-\theta} \\ & \leq & b_1^{2\theta(\sigma-s_c)(1+\nu)+2(1-\theta)(L+(1-\delta_0)(1+\eta))}(\triangle_r\hat{\mathcal{E}}_\sigma+\triangle_r\hat{\mathcal{E}}_{s_L}).\end{array}$$

This is why in this case, (*3.5.120*) transforms into:

$$\| (\tilde{Q}_b^{p-k}(\varepsilon^{(1)k}-\hat{\varepsilon}'^{(1)k}))_{s_L-1} \|_{L^2}^2\leq Cb_1^{2L+2+2(1-\delta_0)(1+\eta)+\frac{2\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}(\triangle_r\hat{\mathcal{E}}_\sigma+\triangle_r\hat{\mathcal{E}}_{s_L}). \tag{3.5.121}$$

We now turn to the second term in (*3.5.119*). Using the bound (*3.5.28*) and again the same reasoning that proved (*3.3.96*) one gets:

$$\begin{array}{rl} & \| ((\tilde{Q}_b^{(1)(p-k)}-\tilde{Q}_{\hat{b}'}^{(1)(p-k)})\hat{\varepsilon}'^{(1)k})_{s_L-1} \|_{L^2}^2 \\ \leq & Cb_1^{2L+2+2(1-\delta_0)(1+\eta)+\frac{2\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sup\limits_{1\leq i\leq L}(b_1^{-i}|\triangle\hat{b}_i|).\end{array} \tag{3.5.122}$$

We can now come back to the identity (*3.5.119*), inject the bounds (*3.5.121*) and (*3.5.122*) to find that the size of the nonlinear term is:

$$\begin{array}{rl} & \| (NL-\hat{NL}')_{k_0+L} \|_{L^2} \\ \leq & Cb_1^{L+1+(1-\delta_0)(1+\eta)+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\left(\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+\sup\limits_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|\right).\end{array} \tag{3.5.123}$$

After rescaling and applying Cauchy-Schwarz, this gives the following bound on the nonlinear term's contribution:

$$\begin{array}{rl} & \left|\int(w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}(NL-\hat{NL}')\right| \\ \leq & \frac{Cb_1^{2L+2(1-\delta_0)(1+\eta)+1+\frac{\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}}{\lambda^{2(s_L-s_c)+1}}\left(\sup\limits_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_\sigma}+\sqrt{\triangle\hat{\mathcal{E}}_{s_L}}\right).\end{array} \tag{3.5.124}$$

- *Time difference terms:* For the small linear term involving $\hat{w}'$ we recall (*3.3.98*):

$$\| \hat{L}'_{s_L-1} \|_{L^2}\leq Cb_1\left\|\frac{\hat{\varepsilon}_{s_L}'^{(1)}}{1+y^\delta}\right\|_{L^2}\leq Cb_1\| \varepsilon_{s_L}'^{(1)} \|_{L^2}\leq Cb_1^{L+(1-\delta_0)(1+\eta)+1}.$$

For the linear term, we need the extra assumption (*3.5.6*) on the higher derivative of $\hat{\varepsilon}'$, it produces:

$$\| \hat{\varepsilon}_{s_L}'^{(2)} \|_{L^2}+\| \hat{\varepsilon}_{s_L+1}'^{(1)} \|_{L^2}\leq Cb_1^{L+(1-\delta_0)(1+\eta)+1}.$$

The two previous inequalities, with the estimates (*3.3.92*) and (*3.3.96*) we already established for the non linear and error terms, plus the bound (*3.5.35*) on $|\frac{d\hat{s}'}{ds}-1|$ yield:

$$\begin{array}{ll} & |\frac{d\hat{s}'}{ds}-1|\left[\| (\hat{w}'^{(2)}-\frac{1}{\lambda}\tilde{\psi}^{(1)}_{\hat{b}',\frac{1}{\lambda}})_{s_L}\|_{L^2}+\|(\mathcal{L}\hat{w}'^{(1)}+\frac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}-\hat{NL}'-\hat{L}')_{s_L-1}\|_{L^2}\right] \\ \leq & \frac{Cb_1^{L+(1-\delta_0)(1+\eta)+1}}{\lambda^{(s_L-s_c)+1}}(\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|+b_1^{L+(1-\delta_0)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}). \end{array} \quad (3.5.125)$$

This implies that the contribution of the terms involving the difference of the evolution of the renormalized times $\frac{d\hat{s}'}{ds}-1$ in (*3.5.115*) is:

$$\begin{array}{ll} & \Big|(\frac{d\hat{s}'}{ds}-1)\int(w^{(1)}-\hat{w}'^{(1)})\mathcal{L}^{s_L}(\hat{w}'^{(2)}-\frac{1}{\lambda}\tilde{\psi}^{(1)}_{\hat{b}',\frac{1}{\lambda}} \\ & +\int(w^{(2)}-\hat{w}'^{(2)})\mathcal{L}^{s_L-1}(-\mathcal{L}\hat{w}'^{(1)}-\frac{1}{\lambda}\tilde{\psi}^{(2)}_{\hat{b}',\frac{1}{\lambda}}+\hat{NL}'+\hat{L}')\Big| \\ \leq & \frac{Cb_1^{2L+(1-\delta_0)(2+\frac{3}{2}\eta)+1}}{\lambda^{2(s_L-s_c)+1}}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}(\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|+b_1^{L+1-\delta_0}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}). \end{array} \quad (3.5.126)$$

We reach the end of the proof of the first step. We now inject the bounds (*3.5.117*) for the linear terms, (*3.5.118*) for the error terms, (*3.5.124*) for the non linear terms and (*3.5.126*) for the time error term in (*3.5.115*), yielding the intermediate equation (*3.5.116*) claimed in this step 1.

Step 2: Terms making appear a local part that cannot be estimated directly. The small linear terms and the scale changing terms cannot be estimated directly. The aim of this step is to decompose their contribution into two parts: one that can be bounded directly and the other that requires the study of a Morawetz type quantity, see next Lemma 3.5.11. We claim that (*3.5.116*) can be transformed into:

$$\begin{array}{lll} \frac{d}{dt}\left(\frac{\triangle\mathcal{E}_{s_L}}{2\lambda^{2(s_L-s_c)}}\right) & = & \int(w^{(1)}-\hat{w}'^{(1)})\mathcal{L}^{k_0+L+1}_{\frac{1}{\lambda}}\left[-\frac{1}{\lambda}\tilde{Mod}^{(1)}_{\frac{1}{\lambda}}+\frac{1}{\lambda}\hat{Mod}'^{(1)}_{\frac{1}{\lambda}}\right] \\ & & +\int(w^{(2)}-\hat{w}'^{(2)})\mathcal{L}^{k_0+L}_{\frac{1}{\lambda}}\left[-\frac{1}{\lambda}\tilde{Mod}^{(2)}_{\frac{1}{\lambda}}+\frac{1}{\lambda}\hat{Mod}'^{(2)}_{\frac{1}{\lambda}}\right] \\ & & +O\Big(\frac{Cb_1^{2L+2(1-\delta_0)(1+\frac{\eta}{2})+1}}{\lambda^{2(s_L-s_c)+1}}(b_1^{\frac{\eta}{2}(1-\delta_0)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i| \\ & & +C(N)\triangle_r\hat{\mathcal{E}}_{s_L,\text{loc}}+\frac{C}{N^{\frac{\delta_0}{2}}}(\triangle_r\hat{\mathcal{E}}_{s_L}+\triangle_r\hat{\mathcal{E}}_\sigma)\Big), \end{array} \quad (3.5.127)$$

the constant C being independent of N. We now prove this identity by establishing bounds on the small linear terms and the scale changing terms in (*3.5.116*).

• *The small linear terms:* We start by decomposing:

$$\begin{array}{lll} L-\hat{L}' & = & p(\tilde{Q}_b^{(1)(p-1)}-Q^{p-1})\varepsilon^{(1)}-p(\tilde{Q}_{\hat{b}}^{(1)(p-1)}-Q^{p-1})\hat{\varepsilon}'^{(1)} \\ & = & p(\tilde{Q}_b^{(1)(p-1)}-\tilde{Q}_{\hat{b}'}^{(1)(p-1)})\varepsilon^{(1)}+p(\tilde{Q}_{\hat{b}}^{(1)(p-1)}-Q^{p-1})(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}) \end{array} \quad (3.5.128)$$

and we now estimate each term. For the first term in (*3.5.128*), from the bound (*3.5.28*) on $\tilde{Q}_b^{(1)(p-1)}-\tilde{Q}_{\hat{b}'}^{(1)(p-1)}$ one gets:

$$\|((\tilde{Q}_b^{(1)(p-1)}-\tilde{Q}_{\hat{b}'}^{(1)(p-1)})\varepsilon^{(1)})_{s_L-1}\|_{L^2}\leq b_1^{L+1+(1-\delta_0)(1+\eta)}\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|. \quad (3.5.129)$$

Now for the second term in (*3.5.128*), using the same reasoning we used to prove (*3.3.98*) we obtain:

$$\|((\tilde{Q}_{\hat{b}}^{(1)(p-1)}-Q^{p-1})(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}))_{s_L-1}\|_{L^2}^2\leq Cb_1^2\left\|\frac{(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L}}{1+y^{\frac{\delta_0}{2}}}\right\|_{L^2}^2.$$

By cutting at a distance N from the origin one gets:

$$\left\| ((\tilde{Q}_{\hat{b}}^{(1)(p-1)} - Q^{p-1})(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)}))_{s_L-1} \right\|_{L^2}^2 \leq \frac{b_1^2 C}{N^{\delta_0}} \triangle \hat{\mathcal{E}}_{s_L} + C(N) b_1^2 \triangle \hat{\mathcal{E}}_{s_L,\mathrm{loc}}. \tag{3.5.130}$$

We now come back to the expression (*3.5.128*) for which we have found bounds in (*3.5.129*) and (*3.5.130*), yielding the following size for the small linear terms:

$$\begin{array}{rl} & \| \, (L - \hat{L}')_{s_L-1} \, \|_{L^2} \\ \leq & C b_1^{L+(1-\delta_0)(1+\frac{\eta}{2}\eta)+1} \left(\frac{\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}}{N^{\frac{\delta_0}{2}}} + C(N) \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L,\mathrm{loc}}} + b_1^{\frac{\eta}{2}(1-\delta_0)} \sup_{1\leq i\leq L} b_1^{-i} |\triangle \hat{b}_i| \right). \end{array} \tag{3.5.131}$$

After rescaling, applying Cauchy-Schwarz inequality the contribution of the small linear terms can be split into:

$$\begin{array}{rl} & \left| \int (w^{(2)} - \hat{w}'^{(2)}) \mathcal{L}_{\frac{1}{\lambda}}^{k_0+L} (L - \hat{L}') \right| \\ \leq & \frac{C b_1^{2L+2(1-\delta_0)(1+\frac{1}{2}\eta)+1} \left(b_1^{\frac{\eta}{2}(1-\delta_0)} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} \sup_{1\leq i\leq L} b_1^{-i} |\triangle \hat{b}_i| + \frac{1}{N^{\frac{\delta_0}{2}}} \triangle_r \hat{\mathcal{E}}_{s_L} + C(N) \triangle_r \hat{\mathcal{E}}_{s_L,\mathrm{loc}} \right)}{\lambda^{2(s_L-s_c)+1}}. \end{array} \tag{3.5.132}$$

- *The scale changing term:* Using verbatim the same methodology we used to prove (*3.3.100*) we get:

$$\begin{array}{rl} & \Big| \sum_{i=1}^{k_0+L+1} \int (w^{(1)} - \hat{w}'^{(1)}) \mathcal{L}_{\frac{1}{\lambda}}^{i-1} \frac{d}{dt} \left(\mathcal{L}_{\frac{1}{\lambda}} \right) \mathcal{L}_{\frac{1}{\lambda}}^{k_0+L+1-i} (w^{(1)} - \hat{w}'^{(1)}) \\ & + \sum_{i=1}^{k_0+L} \int (w^{(2)} - \hat{w}'^{(2)}) \mathcal{L}_{\frac{1}{\lambda}}^{i-1} \frac{d}{dt} \left(\mathcal{L}_{\frac{1}{\lambda}} \right) \mathcal{L}_{\frac{1}{\lambda}}^{k_0+L-i} (w^{(2)} - \hat{w}'^{(2)}) \Big| \\ \leq & \frac{C(M) b_1^{2L+2(1-\delta_0)(1+\frac{\eta}{2})+1}}{\lambda^{2(s_L-s_c)+1}} \left(\frac{\triangle_r \hat{\mathcal{E}}_{s_L}}{N^{\frac{\delta_0}{2}}} + C(N) \triangle_r \hat{\mathcal{E}}_{s_L,\mathrm{loc}} \right), \end{array} \tag{3.5.133}$$

Coming back to the identity (*3.5.116*) we showed in step 1, and injecting the bounds (*3.5.132*) on the small linear terms and (*3.5.133*) on the scale changing terms gives the identity (*3.5.127*) that we had to prove in this step 2.

Step 3: The modulation term. We need to find a proper integration by parts in time to deal with the modulation terms. We claim that:

$$\begin{array}{rl} & \int (w^{(1)} - \hat{w}'^{(1)}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L} \frac{(\tilde{Mod}^{(1)} - \hat{Mod}'^{(1)})_{\frac{1}{\lambda}}}{\lambda} + (w^{(2)} - \hat{w}'^{(2)}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L-1} \frac{(\tilde{Mod}^{(2)} - \hat{Mod}'^{(2)})_{\frac{1}{\lambda}}}{\lambda} \\ = & \partial_t \left[O\left(\frac{b_1^{2L+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)}} (\triangle_r \hat{\mathcal{E}}_{s_L} + | \sup_{1\leq i\leq L} b_1^{-i} |\triangle \hat{b}_i||^2) \right) \right] \\ & + O\left(\frac{b_1^{2L+1+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)+1}} (\triangle_r \hat{\mathcal{E}}_{s_L} + | \sup_{1\leq i\leq L} b_1^{-i} |\triangle \hat{b}_i||^2) \right). \end{array} \tag{3.5.134}$$

Once this bound is proven, we finish the proof of the proposition by injecting it in (*3.5.127*). Therefore to finish to proof we now prove (*3.5.134*). We recall that $\triangle \boldsymbol{\hat{Mod}}_i$ is defined by (*3.5.24*) and (*3.5.25*), and that $\boldsymbol{\tilde{Mod}} - \boldsymbol{\hat{Mod}}' = \sum_{i=0}^L \triangle \boldsymbol{\hat{Mod}}_i$. First we find a direct bound for the all the modulation terms other than the L-th. Let i denote an even integer, $1 \leq i \leq L-1$. The fact that we assume i even is just to have a precise location for the profiles. In that case one decompose:

$$\begin{array}{rl} & \triangle \boldsymbol{\hat{Mod}}_i := \boldsymbol{A}_1 + \boldsymbol{A}_2 \\ = & (b_{i,s} + (i-\alpha) b_1 b_i - b_{i+1} - (\hat{b}'_{i,s} + \frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1 \hat{b}'_i - \hat{b}'_{i+1}))) \chi_{B_1} (\boldsymbol{T}_i + \sum_{j=i+1}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i}) \\ & + (\hat{b}'_{i,s} + \frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1 \hat{b}'_i - \hat{b}'_{i+1}))(\chi_{B_1} (\boldsymbol{T}_i + \sum_{j=i+1}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i}) - \chi_{\hat{B}'_1} (\boldsymbol{T}_i + \sum_{j=i+1}^{L+2} \frac{\partial \hat{\boldsymbol{S}}'_j}{\partial b_i})). \end{array} \tag{3.5.135}$$

For the first term of the previous equation, we employ the bound (*3.5.36*) on the modulation of the parameters b_i for $1 \leq i \leq L-1$, yielding:

$$\parallel (A_1^{(1)})_{s_L} \parallel_{L^2} + \parallel (A_1^{(2)})_{s_L-1} \parallel_{L^2} \leq C b_1^{L+3-\delta_0+O(\eta)} (\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}). \quad (3.5.136)$$

For the second term (*3.5.30*) and (*3.5.33*) imply that:

$$\begin{aligned} & \left\| \left(\chi_{B_1}(\boldsymbol{T}_i + \sum_{j=i+1 \text{ even}}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i}) - \chi_{\hat{B}_1'}(\boldsymbol{T}_i + \sum_{j=i+1 \text{ even}}^{L+2} \frac{\partial \hat{\boldsymbol{S}}_j'}{\partial b_i}) \right)_{s_L} \right\|_{L^2} \\ & + \left\| \left(\chi_{B_1} \sum_{j=i+1 \text{ odd}}^{L+2} \frac{\partial \boldsymbol{S}_j}{\partial b_i} - \chi_{\hat{B}_1'} \sum_{j=i+1 \text{ odd}}^{L+2} \frac{\partial \hat{\boldsymbol{S}}_j'}{\partial b_i} \right)_{s_L-1} \right\| \\ \leq & \quad C b_1 \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|. \end{aligned}$$

We then use the primary bound (*3.3.36*) on the modulation to find that:

$$\parallel (A_2^{(1)})_{s_L} \parallel_{L^2} + \parallel (A_2^{(2)})_{s_L-1} \parallel_{L^2} \leq C b_1^{L+3-\delta_0+O(\eta)} \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|. \quad (3.5.137)$$

We come back to the decomposition (*3.5.135*) for which we have found bounds for the terms in the right hand side in (*3.5.136*) and (*3.5.137*), in the case where i is even. Now if i is odd or $i=0$ the very same computations show that they still hold, yielding:

$$\begin{aligned} & \parallel \textstyle\sum_{i=0}^{L-1} (\triangle \hat{Mod}_i^{(1)})_{s_L} \parallel_{L^2} + \parallel \sum_{i=0}^{L-1} (\triangle \hat{Mod}_i^{(2)})_{s_L-1} \parallel_{L^2} \\ \leq & \quad C b_1^{L+3-\delta_0+O(\eta)} (\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}). \end{aligned} \quad (3.5.138)$$

The previous bound (*3.5.138*) then imply the intermediate identity:

$$\begin{aligned} & \int (w^{(1)} - \hat{w}'^{(1)}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L} \frac{(\tilde{Mod}^{(1)} - \hat{Mod}'^{(1)})^{(1)}_{\frac{1}{\lambda}}}{\lambda} + (w^{(2)} - \hat{w}'^{(2)}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L-1} \frac{(\tilde{Mod}^{(2)} - \hat{Mod}'^{(2)})_{\frac{1}{\lambda}}}{\lambda} \\ = & \quad \frac{1}{\lambda} \int (w^{(1)} - \hat{w}'^{(1)}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L} \triangle \hat{Mod}^{(1)}_{L,\frac{1}{\lambda}} + (w^{(2)} - \hat{w}'^{(2)}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L-1} \triangle \hat{Mod}^{(2)}_{L,\frac{1}{\lambda}} \\ & + O\left(\frac{b_1^{2L+2(1-\delta_0)+2+O(\eta)}}{\lambda^{2(s_L-s_c)+1}} (\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i| + \triangle_r \hat{\mathcal{E}}_{s_L}) \right). \end{aligned} \quad (3.5.139)$$

We now have to deal with the last modulation term. We know by the improved bound for the evolution of $\triangle \hat{b}_L$, see Lemma 3.5.7 that

$$b_{L,s} + (L-\alpha) b_1 b_L - (\hat{b}'_{L,s} + (L-\alpha) \hat{b}'_1 \hat{b}'_L)$$

is small enough up to the derivative in time of the projection of $\varepsilon - \hat{\varepsilon}'$ onto $\boldsymbol{H}^{*L} \chi_{B_1} \boldsymbol{\Lambda Q}$. We claim the following identity:

$$\begin{aligned} & \frac{1}{\lambda} \int (w^{(1)} - \hat{w}'^{(1)}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L} \triangle \hat{Mod}^{(1)}_{L,\frac{1}{\lambda}} + \frac{1}{\lambda} \int (w^{(2)} - \hat{w}'^{(2)}) \mathcal{L}_{\frac{1}{\lambda}}^{s_L-1} \triangle \hat{Mod}^{(2)}_{L,\frac{1}{\lambda}} \\ = & \quad \partial_t \left[O\left(\frac{b_1^{2L+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)}} (\triangle_r \hat{\mathcal{E}}_{s_L} + |\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i||^2 \right) \right] \\ & + O\left(\frac{b_1^{2L+1+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)+1}} (\triangle_r \hat{\mathcal{E}}_{s_L} + |\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i||^2 \right). \end{aligned} \quad (3.5.140)$$

Once this identity is proven, we can combine it with (*3.5.139*) to obtain the identity (*3.5.134*) we claimed in this step 3. The rest of the proof is now devoted to the proof of (*3.5.140*). We define two radiations:

$$\begin{aligned}\boldsymbol{\xi} \quad :=& \quad \frac{\langle \boldsymbol{H}^L(\boldsymbol{\varepsilon}-\hat{\boldsymbol{\varepsilon}}'),\chi_{B_0}\boldsymbol{\Lambda Q}\rangle-\hat{b}'_L\int\chi_{B_0}\Lambda^{(1)}Q\left(\frac{\partial S_{L+2}}{\partial b_L}-\frac{\partial\hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}}{\left\langle\chi_{B_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial S^{(2)}_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\\ &\times\left[\chi_{B_1}\left(\boldsymbol{T}_L+\frac{\partial\boldsymbol{S}_{L+1}}{\partial b_L}+\frac{\partial\boldsymbol{S}_{L+2}}{\partial b_L}\right)\right]_{\frac{1}{\lambda}},\\ \boldsymbol{\xi}' \quad :=& \quad \frac{\langle\boldsymbol{H}^L(\hat{\varepsilon}',\chi_{\hat{B}'_0}\boldsymbol{\Lambda Q}\rangle}{\left\langle\chi_{\hat{B}'_0}\Lambda^{(1)}Q,\Lambda^{(1)}Q+(-1)^{\frac{L-1}{2}}\left(\frac{\partial\hat{S}'_{L+2}}{\partial b_L}\right)_{L-1}\right\rangle}\left[\chi_{B_1}\left(\boldsymbol{T}_L+\frac{\partial\boldsymbol{S}_{L+1}}{\partial b_L}+\frac{\partial\boldsymbol{S}_{L+2}}{\partial b_L}\right)\right.\\ &\left.-\chi_{\hat{B}'_1}\left(\boldsymbol{T}_L+\frac{\partial\hat{\boldsymbol{S}}'_{L+1}}{\partial b_L}+\frac{\partial\hat{\boldsymbol{S}}'_{L+2}}{\partial b_L}\right)\right]_{\frac{1}{\lambda}}.\end{aligned}$$

They enjoy the bound for $i=0,1$:

$$\parallel(\xi^{(1)}+\xi'^{(1)})_{s_L+i}\parallel_{L^2}+\parallel(\xi^{(2)}+\xi'^{(2)})_{s_L-1+i}\parallel_{L^2}\leq C\frac{b_1^{L+(1-\delta_0)(1+\frac{3}{2}\eta)+i}}{\lambda^{s_L-s_c+i}}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|). \tag{3.5.141}$$

From (*3.5.59*) and (*3.3.56*) one has:

$$\partial_t(\boldsymbol{\xi}+\boldsymbol{\xi}')=\frac{1}{\lambda}\triangle\hat{\boldsymbol{Mod}}_{L,\frac{1}{\lambda}}+\boldsymbol{R}, \tag{3.5.142}$$

where $\boldsymbol{R}$ is a remainder satisfying:

$$\parallel R^{(1)}_{s_L}\parallel_{L^2}+\parallel R^{(2)}_{s_L-1}\parallel_{L^2}\leq\frac{Cb_1^{L+(1+\frac{3}{2}\eta)(1-\delta_0)+1}}{\lambda^{s_L-s_c+1}}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|) \tag{3.5.143}$$

In the time evolution of $\boldsymbol{w}-\hat{\boldsymbol{w}}'$, (*3.5.27*), we found a bound for almost all the terms in the right hand side in (*3.5.78*), (*3.5.120*), (*3.5.125*), (*3.5.131*) and (*3.5.138*). With the identity (*3.5.142*) and the bound (*3.5.143*) it gives the following identity:

$$\partial_t(\boldsymbol{w}-\hat{\boldsymbol{w}}')+\boldsymbol{H}_{\frac{1}{\lambda}}(\boldsymbol{w}-\hat{\boldsymbol{w}}')=-\frac{1}{\lambda}\triangle\hat{\boldsymbol{Mod}}_{L,\frac{1}{\lambda}}+\boldsymbol{R}'=-\partial_t(\boldsymbol{\xi}+\boldsymbol{\xi}')+\boldsymbol{R}'-\boldsymbol{R},$$

$\boldsymbol{R}'$ being a remainder with the following size:

$$\parallel R'^{(1)}_{s_L}\parallel_{L^2}+\parallel R'^{(2)}_{s_L-1}\parallel_{L^2}\leq C\frac{b_1^{L+1+(1-\delta_0)(1+\frac{\eta}{2})}}{\lambda^{s_L-s_c+1}}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|).$$

With the previous relations, we perform the following integration by parts in time:

$$\begin{aligned}&\tfrac{1}{\lambda}\textstyle\int(w^{(1)}-\hat{w}'^{(1)})\mathcal{L}^{s_L}_{\frac{1}{\lambda}}\triangle\hat{Mod}^{(1)}_{L,\frac{1}{\lambda}}+\tfrac{1}{\lambda}\int(w^{(2)}-\hat{w}'^{(2)})\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}\triangle\hat{Mod}^{(2)}_{L,\frac{1}{\lambda}}\\ =\quad&\partial_t\Big[\textstyle\int(w^{(1)}-\hat{w}'^{(1)})\mathcal{L}^{s_L}_{\frac{1}{\lambda}}(\xi^{(1)}+\xi'^{(1)})+\int(w^{(2)}-\hat{w}'^{(2)})\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}(\xi^{(2)}+\xi'^{(2)})\\ &+\tfrac{1}{2}\textstyle\int(\xi^{(1)}+\xi'^{(1)})\mathcal{L}^{s_L}_{\frac{1}{\lambda}}(\xi^{(1)}+\xi'^{(1)})+\tfrac{1}{2}\int(\xi^{(2)}+\xi'^{(2)})\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}}(\xi^{(2)}+\xi'^{(2)})\Big]\\ &-\textstyle\int(w^{(1)}-\hat{w}'^{(1)})\partial_t(\mathcal{L}^{s_L}_{\frac{1}{\lambda}})(\xi^{(1)}+\xi'^{(1)})+\int(w^{(2)}-\hat{w}'^{(2)})\partial_t(\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}})(\xi^{(2)}+\xi'^{(2)})\\ &-\tfrac{1}{2}\textstyle\int(\xi^{(1)}+\xi'^{(1)})\partial_t(\mathcal{L}^{s_L}_{\frac{1}{\lambda}})(\xi^{(1)}+\xi'^{(1)})+\tfrac{1}{2}\int(\xi^{(2)}+\xi'^{(2)})\partial_t(\mathcal{L}^{s_L-1}_{\frac{1}{\lambda}})(\xi^{(2)}+\xi'^{(2)})\\ &+O\left(\frac{b_1^{2L+1+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)+1}}(\triangle_r\hat{\mathcal{E}}_{s_L}+|\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i||^2)\right).\end{aligned}$$

Using the degeneracy of the derivative in time of the potential (*3.2.12*) one has the bound for the third and fourth terms in the previous identity:

$$\begin{array}{ll} & \Big| \int (w^{(1)}-\hat{w}'^{(1)})\partial_t(\mathcal{L}_{\frac{1}{\lambda}}^{s_L})(\xi^{(1)}+\xi'^{(1)})+\int (w^{(2)}-\hat{w}'^{(2)})\partial_t(\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1})(\xi^{(2)}+\xi'^{(2)}) \\ & -\frac{1}{2}\int (\xi^{(1)}+\xi'^{(1)})\partial_t(\mathcal{L}_{\frac{1}{\lambda}}^{s_L})(\xi^{(1)}+\xi'^{(1)})+\frac{1}{2}\int (\xi^{(2)}+\xi'^{(2)})\partial_t(\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1})(\xi^{(2)}+\xi'^{(2)}) \Big| \\ \leq & C\frac{b_1^{2L+1+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)+1}}(\triangle_r\hat{\mathcal{E}}_{s_L}+|\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i||^2). \end{array}$$

Hence we can write:

$$\begin{array}{ll} & \frac{1}{\lambda}\int (w^{(1)}-\hat{w}'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{s_L}\triangle \hat{Mod}^{(1)}_{L,\frac{1}{\lambda}}+\frac{1}{\lambda}\int (w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}\triangle \hat{Mod}^{(2)}_{L,\frac{1}{\lambda}} \\ = & \partial_t \Big[\int (w^{(1)}-\hat{w}'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{s_L}(\xi^{(1)}+\xi'^{(1)})+\int (w^{(2)}-\hat{w}'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}(\xi^{(2)}+\xi'^{(2)}) \\ & +\frac{1}{2}\int (\xi^{(1)}+\xi'^{(1)})\mathcal{L}_{\frac{1}{\lambda}}^{s_L}(\xi^{(1)}+\xi'^{(1)})+\frac{1}{2}\int (\xi^{(2)}+\xi'^{(2)})\mathcal{L}_{\frac{1}{\lambda}}^{s_L-1}(\xi^{(2)}+\xi'^{(2)}) \Big] \\ & +O\left(\frac{b_1^{2L+1+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)+1}}(\triangle_r\hat{\mathcal{E}}_{s_L}+|\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i||^2 \right). \end{array}$$

We now take the previous equation, inject the bound (*3.5.141*) for the terms integrated in time, it gives the intermediate identity (*3.5.140*) we had to prove.

□

To control the local term in (*3.5.114*), we study a Morawetz type quantity localized near the origin. We recall that ϕ_A is defined by (*3.3.122*). We define the following quantity:

$$\triangle \mathcal{M}=-\int \left[\nabla \phi_A.\nabla(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}+\frac{(1-\delta)\Delta \phi_A}{2}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}\right](\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1}. \quad (3.5.144)$$

$\triangle \mathcal{M}$ is controlled by the high Sobolev norm of the difference:

$$|\triangle \mathcal{M}|\leq C(A,M)\triangle \mathcal{E}_{s_L} \quad (3.5.145)$$

At the linear level of the dynamics (*3.5.26*) of $\varepsilon-\hat{\varepsilon}'$, this quantity controls the local term $\triangle \hat{\mathcal{E}}_{s_L,\text{loc}}$. Indeed, from Lemma 3.3.8 one has:

$$\begin{array}{ll} & \int [\nabla \phi_A.\nabla(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}+\frac{(1-\delta)\Delta \phi_A}{2}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}](\mathcal{L}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}) \\ & -\int [\nabla \phi_A.\nabla(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1}+\frac{(1-\delta)\Delta \phi_A}{2}(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1}](\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1} \\ \geq & \frac{\delta}{2N^{\delta}}\triangle \mathcal{E}_{s_L,loc}-\frac{C(M)}{A^{\delta}}\triangle \mathcal{E}_{s_L}. \end{array} \quad (3.5.146)$$

This control remains in the full non linear equation. We have the following result:

Lemma 3.5.11 (Control of the local term by a Morawetz type identity). *One has the following lower bound on the evolution of* $\triangle \mathcal{M}$*:*

$$\frac{d}{ds}\triangle \mathcal{M}\geq \frac{\delta}{2N^{\delta}}\triangle \mathcal{E}_{s_L,loc}-\frac{C(M)}{A^{\delta}}\triangle \mathcal{E}_{s_L}-C(A)\sqrt{\triangle \mathcal{E}_{s_L}}b_1^{L+1+(1-\delta_0)+O(\eta)}(\sqrt{\triangle_r\mathcal{E}_{\sigma}}+\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|). \quad (3.5.147)$$

Proof of Lemma 3.5.11

To prove the identity of the lemma, we first compute the time evolution of $\triangle\mathcal{M}$, use the control (*3.5.146*) obtained at the linear level, and show that the other terms are negligible. The time evolution of $\triangle\mathcal{M}$ is:

$$
\begin{array}{rl}
 & \frac{d}{ds}\triangle\mathcal{M} \\
= & -\int \nabla\phi_A.\nabla\Big[(\frac{\lambda_s}{\lambda}\Lambda^{(1)}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})+\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)}-\tilde{\psi}_b^{(1)}+\tilde{\psi}_{\hat{b}'}^{(1)}-\tilde{Mod}^{(1)} \\
 & +\hat{Mod}'^{(1)}+(\frac{d\hat{s}'}{ds}-1)(\tilde{\psi}_{\hat{b}'}^{(1)}-\hat{\varepsilon}'^{(2)})\Big]_{s_L-1}(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1} \\
 & -\int \frac{(1-\delta)\Delta\phi_A}{2}\Big[(\frac{\lambda_s}{\lambda}\Lambda^{(1)}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})+\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)}-\tilde{\psi}_b^{(1)}+\tilde{\psi}_{\hat{b}'}^{(1)}-\tilde{Mod}^{(1)} \\
 & +\hat{Mod}'^{(1)}+(\frac{d\hat{s}'}{ds}-1)(\tilde{\psi}_{\hat{b}'}-\hat{\varepsilon}'^{(2)})\Big]_{s_L-1}(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1} \\
 & -\int \nabla\phi_A.\nabla(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}\Big[-\mathcal{L}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})-\frac{\lambda_s}{\lambda}\Lambda^{(2)}(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)}) \\
 & -\tilde{\psi}_b^{(2)}+\tilde{\psi}_{\hat{b}'}^{(2)}-\tilde{Mod}^{(2)}+\hat{Mod}'^{(2)} \\
 & +L-\hat{L}'+NL-\hat{NL}'+(\frac{d\hat{s}'}{ds}-1)(\tilde{\psi}_{\hat{b}'}^{(2)}+\mathcal{L}\hat{\varepsilon}'^{(1)}-\hat{L}'-\hat{NL}')\Big]_{s_L-1} \\
 & -\int \frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}\Big[-\mathcal{L}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})-\frac{\lambda_s}{\lambda}\Lambda^{(2)}(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)}) \\
 & -\tilde{\psi}_b^{(2)}+\tilde{\psi}_{\hat{b}'}^{(2)}-\tilde{Mod}^{(2)}+\hat{Mod}'^{(2)} \\
 & +L-\hat{L}'+NL-\hat{NL}'+(\frac{d\hat{s}'}{ds}-1)(\tilde{\psi}_{\hat{b}'}^{(2)}+\mathcal{L}\hat{\varepsilon}'^{(1)}-\hat{L}'-\hat{NL}')\Big]_{s_L-1}.
\end{array}
\tag{3.5.148}
$$

We now compute everything in the right hand side. The linear part produces exactly the control we want thanks to the identity (*3.5.146*):

$$
\begin{array}{rl}
 & \int[\nabla\phi_A.\nabla(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}+\frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}](\mathcal{L}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}) \\
 & -\int[\nabla\phi_A.\nabla(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1}+\frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1}](\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1} \\
\geq & \frac{\delta}{2N^\delta}\triangle\mathcal{E}_{s_L,loc}-\frac{C(M)}{A^\delta}\triangle\mathcal{E}_{s_L}.
\end{array}
\tag{3.5.149}
$$

Now ϕ_A is of compact support. Hence by integrating by parts and using coercivity we can control the scale changing term:

$$
\begin{array}{rl}
 & \int[\nabla\phi_A.\nabla(\frac{\lambda_s\Lambda^{(1)}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}}{\lambda})+\frac{(1-\delta)\Delta\phi_A\lambda_s\Lambda^{(1)}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}}{2\lambda}](\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1} \\
 & +\int[\nabla\phi_A.\nabla(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1})+\frac{(1-\delta)\Delta\phi_A(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}}{2}]\frac{\lambda_s(\Lambda^{(2)}\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1}}{\lambda} \\
= & O(b_1C(A)\triangle\mathcal{E}_{s_L}),
\end{array}
\tag{3.5.150}
$$

As we work on a compact set, we do not see the bad tail of the error terms. Hence their contribution is:

$$
\begin{array}{rl}
 & \Big|\int[\nabla\phi_A.\nabla(\tilde{\psi}_b^{(1)}-\tilde{\psi}_{\hat{b}'})_{s_L-1})+\frac{(1-\delta)\Delta\phi_A}{2}(\tilde{\psi}_b^{(1)}-\tilde{\psi}_{\hat{b}'}^{(1)})_{s_L-1})](\varepsilon^{(2)}-\hat{\varepsilon}'^{(2)})_{s_L-1}\Big| \\
 & +\Big|\int[\nabla\phi_A.\nabla(\varepsilon^{(1)}-\hat{\varepsilon'}^{(1)}_{s_L-1})+\frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)})_{s_L-1}](\tilde{\psi}_b^{(2)}-\tilde{\psi}_{\hat{b}'}^{(2)})_{s_L-1}\Big| \\
\leq & C(A)\sqrt{\triangle\mathcal{E}_\sigma}b_1^{L+3}\sup_{1\leq i\leq L}b_1^{-i}|\triangle\hat{b}_i|.
\end{array}
\tag{3.5.151}
$$

For the small linear terms we use the decomposition:

$$
L-\hat{L}'=p(\tilde{Q}_b^{(1)(p-1)}-\tilde{Q}_{\hat{b}'}^{(1)(p-1)})\varepsilon^{(1)}+p(\tilde{Q}_{\hat{b}}^{(1)(p-1)}-Q^{p-1})(\varepsilon^{(1)}-\hat{\varepsilon}'^{(1)}).
$$

From (*3.5.28*) one has for the first term in this decomposition:

$$\begin{array}{rl} & \left| \int [\nabla \phi_A . \nabla (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1}]((\tilde{Q}_b^{(1)(p-1)} - \tilde{Q}_{\hat{b}'}^{(1)(p-1)})\varepsilon^{(1)})_{s_L-1} \right| \\ \leq & C(A)\sqrt{\triangle \hat{\mathcal{E}}_{s_L}} b_1^{L+2-\delta_0+O(\eta)} \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|. \end{array}$$

For the second term, as $|\partial_y^k(\tilde{Q}_{\hat{b}'}^{(1)(p-1)} - Q^{p-1})| \leq C(A,k)b_1$ because of the compactness of the support, one gets:

$$\begin{array}{rl} & \left| \int [\nabla \phi_A . \nabla (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1}]((\tilde{Q}_{\hat{b}}^{(1)(p-1)} - Q^{p-1})(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1} \right| \\ \leq & C(A)\triangle \hat{\mathcal{E}}_{s_L} b_1. \end{array}$$

Hence the contribution of the small linear terms is:

$$\begin{array}{rl} & \left| \int [\nabla \phi_A . \nabla (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1}) + \frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1}](L - \hat{L}')_{s_L-1} \right| \\ \leq & C(A) b_1^{2L+3-2\delta_0+O(\eta)} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} (\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} + \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|). \end{array} \tag{3.5.152}$$

For the nonlinear terms we use the bound (*3.5.123*) we showed in the proof of the monotonicity of the adapted high Sobolev norm to find:

$$\begin{array}{rl} & \left| \int [\nabla \phi_A . \nabla (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1}](NL - \hat{NL}')_{s_L-1} \right| \\ \leq & C(A)\sqrt{\mathcal{E}_{s_L}} b_1^{L+2-\delta_0+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c}{L},\eta\right)} (\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}} + \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma} + \sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i|). \end{array} \tag{3.5.153}$$

Using the bounds (*3.3.92*), (*3.3.99*), (*3.3.96*) we established in the proof of Proposition 3.3.7 plus the assumption (*3.5.6*) and the bound (*3.5.35*) on $\frac{d\hat{s}'}{ds} - 1$ one gets for the terms involving the evolution of the time difference:

$$\begin{array}{rl} & \left| (\frac{d\hat{s}'}{ds} - 1) \int [\nabla \phi_A . \nabla (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1}](\tilde{\psi}_{\hat{b}'}^{(2)} + \mathcal{L}\hat{\varepsilon}'^{(1)} - \hat{L}' + \hat{NL}')_{s_L-1} \right| \\ & + \left| (\frac{d\hat{s}'}{ds} - 1) \int [\nabla \phi_A . \nabla (\tilde{\psi}_{\hat{b}'}^{(1)} - \hat{\varepsilon}'^{(2)})_{s_L-1} + \frac{\Delta\phi_A (\tilde{\psi}_{\hat{b}'}^{(1)} - \hat{\varepsilon}'^{(2)})_{s_L-1}}{2}](\varepsilon^{(2)} - \hat{\varepsilon}'^{(2)})_{s_L-1} \right| \\ \leq & C(A)\sqrt{\triangle \mathcal{E}_{s_L}} b_1^{L+1+(1-\delta_0)(1+\frac{3}{2}\eta)} (\sup_{1\leq i\leq L} b_1^{-i}|\triangle \hat{b}_i| + b_1^{L+(1-\delta_0)(1+\eta)}\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}). \end{array} \tag{3.5.154}$$

To finish the proof it remains to estimate the modulation terms. We just compute for one difference of modulation terms located in the second coordinate, that is to say a term of the form:

$$\int [\nabla \phi_A . \nabla (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}(\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1}]\triangle \hat{Mod}_i^{(2)})_{s_L-1}$$

where we recall that $\triangle \hat{Mod}_i$ is defined by (*3.5.24*). We suppose also that i is odd. We claim that the same computations yield the same result for the other modulation terms. As we work on a compact support, we do not see the two cut off χ_{B_1} and $\chi_{\hat{B}_1'}$. So the profile T_i cancels as $(T_i)_{s_L-1} = 0$. Therefore the quantity we have to estimate simplifies into:

$$\begin{array}{rl} (\triangle \hat{Mod}_i^{(2)})_{s_L-1} = & (b_{i,s} + (i-\alpha)b_1 b_i - b_{i+1} - (\hat{b}'_{i,s} + \frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1\hat{b}'_i + \hat{b}'_{i+1}))) \sum_{j=i+1,\ j \text{ odd}}^{L+2} \frac{\partial S_j}{\partial b_i} \\ & +(\hat{b}'_{i,s} + \frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1\hat{b}'_i + \hat{b}'_{i+1})) \sum_{j=i+1,\ j \text{ odd}}^{L+2} \frac{\partial S_j}{\partial b_i} - \frac{\partial \hat{S}'_j}{\partial b_i} \end{array}$$

for $y \leq 2A$. Therefore, using the modulation bounds (*3.5.36*) and (*3.5.37*) one gets that the contribution of this term is:

$$\begin{array}{ll} & \left| \int [\nabla \phi_A . \nabla (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2} (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1}] (\triangle \hat{Mod}_i^{(2)})_{s_L-1} \right| \\ \leq & C(A) \sqrt{\triangle \hat{\mathcal{E}}_{s_L}} b_1^{L+2-\delta_0+O(\eta)} (\sup_{1\leq i\leq L} b_1^{-i} |\triangle \hat{b}_i| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}). \end{array}$$

For the other terms involved in the modulation terms, the same reasoning yield the same estimate, hence:

$$\begin{array}{ll} & \left| (\int [\nabla \phi_A . \nabla (\varepsilon^{(1)} - \hat{\varepsilon}'^{(1)})_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2}] (\tilde{Mod}^{(2)} - \hat{Mod}'^{(2)})_{s_L-1} \right| \\ & + \left| \int [\nabla \phi_A . \nabla (\tilde{Mod}^{(1)} - \hat{Mod}'^{(1)})_{s_L-1} + \frac{(1-\delta)\Delta\phi_A}{2} (\tilde{Mod}^{(1)} - \hat{Mod}'^{(1)})_{s_L-1}] (\varepsilon^{(2)} - \hat{\varepsilon}'^{(2)})_{s_L-1} \right| \\ \leq & C(A) \sqrt{\triangle \hat{\mathcal{E}}_{s_L}} b_1^{L+2-\delta_0+O(\eta)} (\sup_{1\leq i\leq L} b_1^{-i} |\triangle \hat{b}_i| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}). \end{array} \tag{3.5.155}$$

Now, gathering together all the bounds we have proven: the control on the linear terms (*3.5.155*), the bounds on the error terms (*3.5.151*), on the scale changing terms (*3.5.150*), on the small linear and non linear terms (*3.5.152*) and (*3.5.153*), and the time difference terms (*3.5.154*) and on the modulation terms (*3.5.155*) one gets the bound (*3.5.147*) claimed in the lemma. □

3.5.1.4 Study of the coupled dynamical system, end of the proof of Proposition (3.5.2)

So far in this section, we introduced new variables $(\hat{\varepsilon}', \hat{b}')$ that we could compare with the other solution (ε, b). We then computed the time evolution of the difference of relevant quantities. In the Lemmas 3.5.6 and 3.5.7 we calculated the time evolution of the difference of the parameters, and in Lemma 3.5.10 we related the time evolution of the adapted high Sobolev norm of the difference of errors to the difference of parameters. The two other Lemmas 3.5.9 and 3.5.11 for the low Sobolev norm and for the Morawetz quantity are just additional tools to close an estimate for the previous norm.

Thus, at this point we found a quite complicated coupled dynamical system for the differences of the variables of the two solutions $b_i - \hat{b}_i'$ for $1 \leq i \leq L$, $s - \hat{s}'$ and $\varepsilon - \hat{\varepsilon}'$. In the following lemma we analyse this dynamical system, and find that it is only weakly coupled. Namely: the difference of the unstable parameters evolves according to an repelling linear dynamic plus a smaller feedback from the difference of the stable parameters and errors, the difference of stable parameters evolves according to an attractive linear dynamic plus a smaller feedback from the difference of the unstable parameters and errors and the dynamics of the difference of the errors is also stable.

Lemma 3.5.12. *For any* $0 < \kappa \ll 1$, *there exists universal constants* $\tilde{C}$, $(C_i)_{\ell+1\leq i\leq L}$, C_1, $C_{\triangle \hat{s}}$, $0 < \kappa_1 < \kappa$, $0 < \kappa_i < \kappa$ *for* $\ell + 1 \leq i \leq L$ *and* $\bar{s}$ *such that if* $s_0 \geq \bar{s}$ *the following holds for* $s_0 \leq s$:

(i) Estimates on the stable parameters: *for* $\ell + 1 \leq i \leq L$ *one has*

$$\begin{array}{rcl} |\triangle \hat{V}_1(s)| & \leq & C_1 \left(\sup_{\ell+1\leq i\leq L} |\triangle \hat{U}_i(s_0)| + |\triangle \hat{V}_1(s_0)| + \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma(s_0)} + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}(s_0)} \right) \\ & & + \kappa_1 \sup_{s_0\leq s'\leq s,\ 2\leq i\leq \ell} |\triangle \hat{V}_i|, \end{array} \tag{3.5.156}$$

$$\begin{array}{rcl} |\triangle \hat{U}_i(s)| & \leq & C_i \left(\underset{\ell+1\leq i\leq L}{\textit{sup}} |\triangle \hat{U}_i(s_0)| + |\triangle \hat{V}_1(s_0)| + \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma(s_0)} + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}(s_0)} \right) \\ & & +\kappa_i \underset{s_0\leq s'\leq s,\ 2\leq i\leq \ell}{\textit{sup}} |\triangle \hat{V}_i|, \quad \textit{for } \ell+1\leq i\leq L. \end{array} \quad (3.5.157)$$

For the difference of renormalized times there holds:

$$\begin{array}{rcl} \frac{|s-\hat{s}'(s)|}{\textit{slog}(s)} & \leq & C_{\triangle \hat{s}} \Big(\underset{\ell+1\leq i\leq L}{\textit{sup}} |\triangle \hat{U}_i(s_0)| + |\triangle \hat{V}_1(s_0)| + \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma(s_0)} + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}(s_0)} \\ & & + \underset{s_0\leq s'\leq s,\ 2\leq i\leq \ell}{\textit{sup}} |\triangle \hat{V}_i| \Big). \end{array} \quad (3.5.158)$$

(ii) Estimates on the difference of errors: *One has the bounds:*

$$\begin{array}{rcl} \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma(s)} & \leq & \tilde{C} \Big(\underset{\ell+1\leq i\leq L}{\textit{sup}} |\triangle \hat{U}_i(s_0)| + |\triangle \hat{V}_1(s_0)| + \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma(s_0)} + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}(s_0)} \\ & & + \underset{s_0\leq s'\leq s,\ 2\leq i\leq \ell}{\textit{sup}} |\triangle \hat{V}_i| \Big), \end{array} \quad (3.5.159)$$

$$\begin{array}{rcl} \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}(s)} & \leq & \tilde{C} \Big(\underset{\ell+1\leq i\leq L}{\textit{sup}} |\triangle \hat{U}_i(s_0)| + |\triangle \hat{V}_1(s_0)| + \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma(s_0)} + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}(s_0)} \\ & & + \underset{s_0\leq s'\leq s,\ 2\leq i\leq \ell}{\textit{sup}} |\triangle \hat{V}_i| \Big). \end{array} \quad (3.5.160)$$

Proof of Lemma (3.5.12) The proof is based on a bootstrap technique: we inject the bounds of the lemma in the evolution equations, and find that they can be bootstrapped. From now on we fix the constants of the Lemma (3.5.12): κ is small $\kappa \ll 1$, and the C's are large. We just allow us to increase $\bar{s}$ if necessary. The bounds of the lemma are verified at least on a small interval of time $[s_0, s']$, so we define s_1 as the supremum of times s' such that all the bounds of the Lemma are verified on $[s_0, s_1[$. If $s_1 = +\infty$ the lemma is proven. So we now assume $s_1 < +\infty$ and look for a contradiction.

We recall that we have the following relation: $\tilde{\eta} \ll \eta \ll 1$. We first state the following identity:

$$\triangle \hat{b}_i = b_i - \hat{b}_i' = \frac{c_i}{s^i} + \frac{U_i}{s^i} - \frac{c_i}{(\hat{s}')^i} - \frac{\hat{U}_i'}{s^i} = c_i \frac{(\hat{s}')^i - s^i}{s^i(\hat{s}')^i} + \frac{\triangle \hat{U}_i}{s^i}. \quad (3.5.161)$$

To ease notations, we let:

$$D_{\text{stab}}(s_0) = \sup_{\ell+1\leq i\leq L} |\triangle \hat{U}_i(s_0)| + |\triangle \hat{V}_1(s_0)| + \sqrt{\triangle_r \hat{\mathcal{E}}_\sigma(s_0)} + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}(s_0)}.$$

Step 1: *the time difference.* We recall that because the two solutions we are studying are in the trapped regime one has: $b_1 \sim s^{-1}$ and $|U_i| + |\hat{U}_i| \lesssim s^{-\tilde{\eta}}$. We inject the identity (*3.5.161*) in the time evolution of $s - \hat{s}'$ given by (*3.5.35*):

$$\frac{d}{ds}\left(\frac{s-\hat{s}'}{s}\right) = O\left(\frac{|s-\hat{s}'|}{s^{2+\tilde{\eta}}} + \frac{|\triangle \hat{U}_1|}{s} + \frac{|\triangle U| + \sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}}{s^{L+(1-\delta_0)(1+\frac{\eta}{2})+1}}\right), \quad (3.5.162)$$

the constant in the $O()$ being independent on the constants of the Lemma we are proving. We integrate till s_1. As $\triangle \hat{U}_1$ is a linear combination of the $\triangle \hat{V}_i$ for $1 \leq i \leq \ell$, injecting the bounds (*3.5.157*), (*3.5.156*), (*3.5.158*) and (*3.5.160*) gives:

$$\begin{array}{rl} & \left|\int_{s_0}^{s_1} O\left(\frac{|s-\hat{s}'|}{s^{2+\tilde\eta}}+\frac{|\triangle \hat U_1|}{s}+\frac{|\triangle U|+\sqrt{\triangle_r \hat{\mathcal{E}}_{s_L}}}{s^{L+(1-\delta_0)(1+\frac{\eta}{2})+1}}\right)\right| \\ \leq & C\log(s_1)\left(\frac{\log(s_0)}{s_0^{\tilde\eta}}C_{\triangle\hat s}+C_1+\frac{\tilde C+\sum_{\ell+1}^L C_i}{s_0^{L+(1-\delta_0)(1+\frac{\eta}{2})}}\right)D_{\text{stab}}(s_0) \\ & +C\log(s_1)\left(1+\frac{\tilde C}{s_0^{L+(1-\delta_0)(1+\frac{\eta}{2})}}+\frac{\log(s_0)}{s_0^{\tilde\eta}}C_{\triangle\hat s}\right)\sup_{s_0\leq s'\leq s,\ 2\leq i\leq \ell}|\triangle \hat V_i|, \end{array}$$

for some constant C independent of the bootstrap constants (the κ_i's do not appear as they are small, $\kappa_i \ll 1$). Now we recall that at initial time $\hat s'(s_0)=s_0$. Hence when integrating (*3.5.162*):

$$\begin{array}{rl} |s_1-\hat s'(s_1))| \leq & s_1\log(s_1)\left(CC_1+O(s_0^{-\frac{\tilde\eta}{2}})(C_{\triangle\hat s}+\sum_{\ell+1}^L C_i+\tilde C)\right)D_{\text{stab}}(s_0) \\ & +s_1\log(s_1)\left(C+O(s_0^{-\frac{\tilde\eta}{2}})(\tilde C+C_{\triangle\hat s})\right)\sup_{s_0\leq s'\leq s,\ 2\leq i\leq \ell}|\triangle \hat V_i|. \end{array}$$

It means, as C_1 is a big constant and $1\ll s_0$, that the inequality (*3.5.158*) is strict at time s_1 provided:

$$C_{\triangle\hat s} > CC_1+O(s_0^{-\frac{\tilde\eta}{2}})(\tilde C+\sum_{\ell+1}^L C_i), \tag{3.5.163}$$

where the constant C and the constants hidden in the $O()$ are independent of the other constants of the Lemma we are proving.

<u>Step 2</u>: ***the parameter*** V_1. The identity (*3.5.22*) implies that for $1\leq i\leq \ell$:

$$\begin{array}{rl} & b_{i,s}+(i-\alpha)b_1b_i-b_{i+1}-(\hat b'_{i,s}+\frac{d\hat s'}{ds}((i-\alpha)\hat b'_1\hat b'_i-\hat b'_{i+1})) \\ = & \frac{1}{s^i}(\triangle \hat U_{i,s}-\frac{(A_\ell \triangle \hat U)_i}{s}+O(s^{-1-\tilde\eta}(\frac{|\hat s'-s|}{s}+|\triangle \hat U|+|\frac{d\hat s'}{ds}-1|))). \end{array} \tag{3.5.164}$$

We now inject it in (*3.5.36*) using the bound (*3.5.35*) on $\frac{d\hat s'}{ds}-1$ to find:

$$\triangle \hat U_{i,s}=\frac{(A_\ell\triangle \hat U)_i}{s}+O(s^{-1-\tilde\eta}(|\frac{\hat s'-s}{s}|+|\triangle\hat U|))+O(s^{-L-(1-\delta_0)(1+\frac{\eta}{2})+i}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}), \tag{3.5.165}$$

the constants in the $O()$ being independent of the constants of the Lemma we are proving. As $\triangle \hat V_1$ is a linear combination of the $\triangle \hat U_i$ for $1\leq i\leq \ell$ only, see (*3.3.18*), and because of the shape of the matrix A_ℓ, see (*3.2.79*), the previous identity yields:

$$\triangle \hat V_{1,s}=\frac{-\triangle \hat V_1}{s}+\frac{q_1\triangle U_{\ell+1}}{s}+O(|\frac{\hat s'-s}{s^{2+\tilde\eta}}|+\frac{|\triangle \hat U|}{s^{1+\tilde\eta}}))+O(s^{-L-(1-\delta_0)(1+\frac{\eta}{2})+\ell}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}),$$

for some coefficient q_1 coming from the change of variable. This can be rewritten the following way:

$$\frac{d}{ds}(s\triangle \hat V_1)=q_1\triangle U_{\ell+1}+O(s^{-\tilde\eta}(|\frac{\hat s'-s}{s}|+|\triangle \hat U|))+O(s^{-L-(1-\delta_0)(1+\frac{\eta}{2})+\ell+1}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}). \tag{3.5.166}$$

We now integrate till s_1 this identity. Injecting the bootstrap bounds (*3.5.157*), (*3.5.156*), (*3.5.158*) and (*3.5.160*) one finds:

$$\begin{array}{rl} & \frac{1}{(s_1-s_0)}\left|\int_{s_0}^{s_1} q_1\triangle U_{\ell+1}+O(|\frac{\hat s'-s}{s^{1+\tilde\eta}}|+\frac{|\triangle\hat U|}{s^{\tilde\eta}})+O(s^{-L-(1-\delta_0)(1+\frac{\eta}{2})+\ell+1}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}})\right| \\ \leq & (q_1C_{\ell+1}+C(\frac{\log(s_0)}{s_0^{\tilde\eta}}C_{\triangle\hat s}+\frac{C_1+\sum_{i=\ell+1}^L C_i}{s_0^{\tilde\eta}}+\frac{\tilde C}{s_0^{L+(1-\delta_0)(1+\frac{\eta}{2})-\ell-1}}))D_{\text{stab}}(s_0) \\ & +(q_1\kappa_{\ell+1}+C(\frac{\log(s_0)C_{\triangle\hat s}+\kappa_1+\sum_{i=\ell+1}^L\kappa_i}{s_0^{\tilde\eta}}+\frac{\tilde C}{s_0^{L+(1-\delta_0)(1+\frac{\eta}{2})-\ell-1}}))\sup_{s_0\leq s'\leq s,\ 2\leq i\leq \ell}|\triangle \hat V_i|. \end{array}$$

So after integrating (*3.5.166*) one obtains:

$$\begin{array}{rl} & |\triangle \hat{V}_1(s_1)| \\ \leq & (1+q_1C_{\ell+1}+C(\frac{\log(s_0)}{s_0^{\tilde{\eta}}}C_{\triangle \hat{s}}+\frac{C_1+\sum_{i=\ell+1}^L C_i}{s_0^{\tilde{\eta}}}+\frac{\tilde{C}}{s_0^{L+(1-\delta_0)(1+\frac{\eta}{2})-i-1}}))D_{\text{stab}}(s_0) \\ & +(q_1\kappa_{\ell+1}+C(\frac{\log(s_0)C_{\triangle \hat{s}}+\kappa_1+\sum_{i=\ell+1}^L \kappa_i}{s_0^{\tilde{\eta}}}+\frac{\tilde{C}}{s_0^{L+(1-\delta_0)(1+\frac{\eta}{2})-i-1}})) \sup_{s_0\leq s'\leq s,\ 2\leq i\leq \ell} |\triangle \hat{V}_i|. \end{array}$$

As $\ell \ll L$ and $1 \ll s_0$ the inequality (*3.5.156*) is thus strict at time s_1 provided:

$$\begin{array}{l} C_1 > 2+2q_1C_{\ell+1}+O(s_0^{-\frac{\tilde{\eta}}{2}})(C_{\triangle \hat{s}}+\sum_{i=\ell+1}^L C_i+\tilde{C}), \\ \kappa_1 > 2q_1\kappa_{\ell+1}+O(s_0^{-\frac{\tilde{\eta}}{2}})(C_{\triangle \hat{s}}+\sum_{i=\ell+1}^L \kappa_i+\tilde{C}). \end{array} \tag{3.5.167}$$

Step 3: *the parameters U_i for $\ell+1\leq i\leq L-1$.* Pick i satisfying $\ell+1\leq i\leq L-1$. One has the identity:

$$\begin{array}{rl} & b_{i,s}+(i-\alpha)b_1b_i-b_{i+1}-(\hat{b}'_{i,s}+\frac{d\hat{s}'}{ds}((i-\alpha)\hat{b}'_1\hat{b}'_i-\hat{b}'_{i+1})) \\ = & \frac{1}{s^i}(\triangle \hat{U}_{i,s}-\frac{(i-(i-\alpha)c_1)\triangle \hat{U}_i+\triangle \hat{U}_{i+1}}{s}+O(s^{-1-\tilde{\eta}}(\frac{|\hat{s}'-s|}{s}+|\triangle \hat{U}|+|\frac{d\hat{s}'}{ds}-1|))). \end{array}$$

Hence, using the bound (*3.5.35*), the modulation equation (*3.5.36*) can be rewritten as:

$$\begin{array}{rl} \triangle \hat{U}_{i,s} = & \frac{(i-(i-\alpha)c_1)\triangle \hat{U}_i}{s}+\frac{\triangle \hat{U}_{i+1}}{s} \\ & +O(s^{-1-\tilde{\eta}}(\frac{|s-\hat{s}'|}{s}+|\triangle \hat{U}|)+s^{-L-1-(1-\delta_0)(1+\frac{\eta}{2})+i}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}). \end{array}$$

As $i-(i-\alpha)c_1<0$, we can inject the bootstrap bounds (*3.5.157*), (*3.5.156*), (*3.5.158*) and (*3.5.160*) in the previous equation, and integrate till time s_1 as we did in the previous steps to find that:

$$\begin{array}{rl} |\triangle \hat{U}_i(s_1)| \leq & (1+CC_{i+1}+O(s_0^{\frac{-\tilde{\eta}}{2}})(C_{\triangle \hat{s}}+C_1+\sum_{j=\ell+1}^L C_j+\tilde{C}))D_{\text{stab}}(s_0) \\ & +(C\kappa_{i+1}+O(s_0^{\frac{-\tilde{\eta}}{2}})(C_{\triangle \hat{s}}+\kappa_1+\sum_{j=\ell+1}^L \kappa_i+\tilde{C})) \sup_{s_0\leq s'\leq s,\ 2\leq i\leq \ell} |\triangle \hat{V}_i|. \end{array}$$

Thus the inequality (*3.5.157*) is strict at time s_1 provided:

$$\begin{array}{l} C_i > 2+CC_{i+1}+O(s_0^{-\frac{\tilde{\eta}}{2}})(C_1+C_{\triangle \hat{s}}+\sum_{j=\ell+1,\ j\neq i}^L C_j+\tilde{C}), \\ \kappa_i > C\kappa_{i+1}+O(s_0^{-\frac{\tilde{\eta}}{2}})(\kappa_1+C_{\triangle \hat{s}}+\sum_{j=\ell+1,\ j\neq i}^L \kappa_j+\tilde{C}), \end{array} \tag{3.5.168}$$

the constant C being independent on the constants of the Lemma.

Step 4: *the last parameter U_L.* Similarly, we rewrite (*3.5.59*) as:

$$\begin{array}{rl} & \left|\frac{d}{ds}\left(s^{(L-\alpha)c_1-L)}\triangle \hat{U}_L+O(s^{(L-\alpha)c_1-L-\frac{\eta}{2}(1-\delta_0)}(\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+\frac{|\hat{s}'-s|}{s}+|\triangle \hat{U}|))\right)\right| \\ \leq & Cs^{(L-\alpha)c_1-L-1}(s^{-\frac{\eta}{2}(1-\delta_0)}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}+s^{-\tilde{\eta}}(\frac{|\hat{s}'-s|}{s}+|\triangle \hat{U}|)) \end{array}$$

because of the bound (*3.5.60*) (the constant in the $O()$ being independent on the other constants of the lemma we are proving). Because $(L-\alpha)c_1-L>0$, when integrating this equation till time s_1 one gets:

$$\begin{array}{rl} |\triangle \hat{U}_L(s_1)| \leq & (1+O(s_0^{\frac{-\tilde{\eta}}{2}})(C_{\triangle \hat{s}}+C_1+\sum_{j=\ell+1}^L C_j+\tilde{C}))D_{\text{stab}}(s_0) \\ & +O(s_0^{\frac{-\tilde{\eta}}{2}})(C_{\triangle \hat{s}}+\kappa_1+\sum_{i=\ell+1}^L \kappa_i+\tilde{C}) \sup_{s_0\leq s'\leq s,\ 2\leq i\leq \ell} |\triangle \hat{V}_i|. \end{array}$$

Thus the inequality (*3.5.157*) is strict at time s_1 provided:

$$\begin{array}{l} C_L > 2 + O(s_0^{-\frac{\tilde{\eta}}{2}})(C_1 + C_{\triangle\hat{s}} + \sum_{j=\ell+1}^{L-1} C_j + \tilde{C}), \\ \kappa_L > O(s_0^{-\frac{\tilde{\eta}}{2}})(\kappa_1 + C_{\triangle\hat{s}} + \sum_{j=\ell+1}^{L-1} \kappa_j + \tilde{C}), \end{array} \tag{3.5.169}$$

the constants in the $O()$ being independent on the constants of the Lemma.

Step 5: *the low Sobolev norm.* We consider the time evolution of the low Sobolev norm of the difference of the errors given by (*3.5.94*). Because $\lambda^{2(\sigma-s_c)} \sim cb_1^{2(\sigma-s_c)(1+\nu)}$ for some constant $c>0$ one can rewrite it as:

$$\frac{d}{ds}\left\{\frac{\triangle\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} \leq Cb_1^{1+\frac{\alpha}{2L}}\left(\triangle_r\hat{\mathcal{E}}_\sigma + \triangle_r\hat{\mathcal{E}}_{s_L} + (\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|)^2\right).$$

Now, in a similar way as we did in all the previous step, we inject the bootstrap bounds, and integrate this identity till time s_1, to find that the bound (*3.5.159*) is strict at time s_1 provided:

$$\tilde{C} > 2 + O(s_0^{-\frac{\alpha}{4L}})(C_1 + C_{\triangle\hat{s}} + \sum_{j=\ell+1}^{L-1} C_j), \tag{3.5.170}$$

the constants in the $O()$ being independent on the constants of the Lemma.

Step 6: *the high Sobolev norm.* We consider the time evolution of the adapted high Sobolev norm of the difference of the errors given by (*3.5.114*). We inject the control on the local term given by the Morawetz estimate (*3.5.147*), knowing $|\mathcal{M}| \lesssim \triangle\hat{\mathcal{E}}_{s_L}$, and rewrite it as (taking s_0 large enough and using Young's inequality):

$$\begin{array}{rc} & \left|\frac{d}{ds}\left\{\frac{\triangle\hat{\mathcal{E}}_{s_L}}{\lambda^{2(s_L-s_c)}} + O\left(\frac{b_1^{2L+2(1-\delta_0)(1+\eta)}}{\lambda^{2(s_L-s_c)}}(\triangle_r\hat{\mathcal{E}}_{s_L} + |\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i||^2\right)\right\}\right| \\ \leq & \frac{Cb_1^{2L+2(1-\delta_0)(1+\frac{\eta}{2})+1}}{\lambda^{2(s_L-s_c)}}\Big[\frac{\triangle\hat{\mathcal{E}}_{s_L}}{A^\delta}C(N) \\ & +\frac{C}{N^{\frac{\delta_0}{2}}}(\triangle_r\hat{\mathcal{E}}_{s_L} + \triangle_r\hat{\mathcal{E}}_\sigma) + C(N,A))b_1^{\frac{\eta}{2}(1-\delta_0)}(\sup_{1\leq i\leq L} b_1^{-i}|\triangle\hat{b}_i|)^2\Big]. \end{array}$$

We inject the bootstrap bounds (*3.5.157*), (*3.5.156*), (*3.5.158*), (*3.5.160*) and (*3.5.159*) in the previous identity and integrate this identity till time s_1 (we recall that $b_1 \sim \frac{c}{s}$ and $\lambda \sim \frac{c}{s^{\frac{\ell}{\ell-\alpha}}}$):

$$\begin{array}{rcl} \triangle_r\hat{\mathcal{E}}_{s_L}(s_1) & \leq & C(D_{\text{stab}}(s_0) + \sup_{s_0\leq s'\leq s,\ 2\leq i\leq \ell}|\triangle\hat{V}_i|)\Big[1 + \left(\frac{1}{N^{\frac{\delta_0}{2}}} + \frac{C(N)}{A^\delta}\right)\tilde{C}^2 \\ & & +O\left(\frac{\log(s_0)}{s_0^{\frac{\eta}{2}(1-\delta_0)}}\right)(C_1^2 + \sum_{\ell+1}^L C_i^2 + C_{\triangle\hat{s}}^2)\Big]. \end{array}$$

The κ's do not appear as they are small. The constant C is independent on the other constants. Thus, the bound (*3.5.160*) is strict at time s_1 provided:

$$\tilde{C}^2 > C\left[1 + \left(\frac{1}{N^{\frac{\delta_0}{2}}} + \frac{C(N)}{A^\delta}\right)\tilde{C}^2 + O\left(\frac{\log(s_0)}{s_0^{\frac{\eta}{2}(1-\delta_0)}}\right)(C_1^2 + \sum_{\ell+1}^L C_i^2 + C_{\triangle\hat{s}}^2)\right],$$

the constants in the $O()$ being independent on the other constants. Taking s_0, N, then A large enough, the previous inequality is met if:

$$\tilde{C}^2 > C\left[1+O\left(\frac{\log(s_0)}{s_0^{\frac{\eta}{2}(1-\delta_0)}}\right)(C_1^2+\sum_{\ell+1}^{L}C_i^2+C_{\triangle\hat{s}}^2)\right], \qquad (3.5.171)$$

for some constant C independent on the other constants.

Step 7: ***end of the proof.*** We have seen that the bootstrap inequalities (*3.5.157*), (*3.5.156*), (*3.5.158*), (*3.5.159*) and (*3.5.160*) are strict at time s_1 provided that the conditions (*3.5.163*), (*3.5.167*), (*3.5.168*), (*3.5.169*), (*3.5.170*) and (*3.5.171*) are met. Now, if one takes s_0 large enough, one can see that there exists constants C_1, $\tilde{C}$, $C_{\triangle\hat{s}}$, $(C_i)_{\ell+1\leq i\leq L}$, $\kappa_1\leq\kappa$, $(\kappa_i)_{1\leq i\leq L}$ with $\kappa_i\leq\kappa$ that satisfies all these conditions. Thus, if the time s_1 were finite, all the bootstrap bounds would be strict at this time, which is impossible from a continuity argument. □

Thanks to the previous Lemma we can now end the proof of Proposition (3.5.2).

Proof of Proposition (3.5.2) Let $\boldsymbol{U}$ and $\boldsymbol{U}'$ be two solutions satisfying the assumptions of Proposition (3.5.2). We recall that $\triangle\hat{V}_{\text{uns}}$ is defined by (*3.5.16*). At time s_0 one has: $\triangle\hat{V}_{\text{uns}}=\triangle V_{\text{uns}}$. Let i be an integer, $2\leq i\leq\ell$. As $\triangle\hat{V}_i$ is a linear combination of the $\triangle\hat{U}_j$ for $1\leq i\leq\ell$ only, see (*3.3.18*), and because of the shape of the matrix A_ℓ, see (*3.2.79*), the identity (*3.5.165*) gives that the time evolution of $\triangle\hat{V}_i$ is:

$$\triangle\hat{V}_{i,s}=\mu_i\frac{\triangle\hat{V}_1}{s}+q_i\frac{\triangle U_{\ell+1}}{s}+O\left(\left|\frac{\hat{s}'-s}{s^{2+\tilde{\eta}}}\right|+\frac{|\triangle\hat{U}|}{s^{1+\tilde{\eta}}}\right)+O\left(s^{-L-(1-\delta_0)(1+\frac{\eta}{2})+i}\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}\right), \qquad (3.5.172)$$

where $\mu_i>0$ denotes the i-th eigenvalue of the matrix A_ℓ, see Lemma 3.2.17, and q_i is some constant coefficient coming from the change of variables from $\triangle\hat{U}$ to $\triangle\hat{V}$. Now let $\mu:=\min_{2\leq i\leq\ell}\mu_i$ and $q:=\max_{2\leq i\leq\ell}|q_i|$. Using Cauchy-Schwarz inequality, the identity (*3.5.172*) gives for the evolution of the unstable parameters:

$$\begin{array}{rl} & \frac{d}{ds}|\triangle\hat{V}_{\text{uns}}|^2 \\ \geq & \frac{|\triangle\hat{V}_{\text{uns}}|}{s}\left(\frac{\mu}{2}|\triangle\hat{V}_{\text{uns}}|-q|\triangle\hat{U}_{\ell+1}|-\frac{1}{s^{\frac{\eta}{2}}}\left(\left|\frac{\hat{s}'-s}{s}\right|+|\triangle\hat{V}_1+|\sum_{\ell+1}^{L}|\triangle\hat{U}_i|+\sqrt{\triangle_r\hat{\mathcal{E}}_{s_L}}\right)\right), \end{array} \qquad (3.5.173)$$

if one has chosen s_0 big enough. Now, as q and μ are fixed constants of the problem, one can ask that:

$$q\kappa<\frac{\mu}{10}. \qquad (3.5.174)$$

Let the constants $\tilde{C}$, C_1, $(C_i)_{\ell+1\leq i\leq L}$, $C_{\triangle\hat{s}}$, $0<\kappa_1<\kappa$, $0<\kappa_i<\kappa$ for $\ell+1\leq i\leq L$ and $\bar{s}$ be such that the previous Lemma (3.5.12) holds. In particular, one can take s_0 big enough such that:

$$\frac{1}{s^{\frac{\eta}{2}}}(\log(s)C_{\triangle\hat{s}}+\tilde{C}+\kappa_1+\sum_{i=\ell+1}^{L}\kappa_i)\leq\frac{\mu}{10} \qquad (3.5.175)$$

We now argue by contradiction. Suppose one has at initial time:

$$|\triangle V_{\text{uns}}(s_0)|>\frac{10}{\mu}(C_1+\tilde{C}+C_{\triangle\hat{s}}+qC_{\ell+1}+\sum_{\ell+2}^{L}|\triangle\hat{U}_i|)D_{\text{stab}}(s_0). \qquad (3.5.176)$$

We are going to show that this leads to a contradiction. Indeed, (*3.5.173*) implies that at initial time the differences of unstable modes are growing:

$$\frac{d}{ds}|\triangle\hat{V}_{\text{uns}}|^2>0. \qquad (3.5.177)$$

Let s_1 denote the supremum of all times s with $s_0 \leq s$ such that (*3.5.177*) holds on $[s_0, s_1]$. We are going to prove that $s_1 = +\infty$. Indeed, suppose s_1 were finite. Then at time s_1 one has:

$$\sup_{s_0 \leq s' \leq s_1,\ 2 \leq i \leq \ell} |\triangle \hat{V}_i| \leq |\triangle \hat{V}_{\text{uns}}(s_1)|$$

because of the monotonicity (*3.5.177*) on $[s_0, s_1]$. Injecting the bounds (*3.5.160*), (*3.5.158*), (*3.5.156*) and (*3.5.157*) in (*3.5.173*) give, because of the inequalities (*3.5.174*) and (*3.5.175*) between the constants:

$$\begin{array}{rc} & \frac{d}{ds}|\triangle \hat{V}_{\text{uns}}|^2 \\ \geq & \frac{\mu|\triangle \hat{V}_{\text{uns}}|}{2s}\Big(|\triangle \hat{V}_{\text{uns}}|(1 - \frac{2q\kappa_{\ell+1}}{\mu} - \frac{2}{\mu s^{\frac{\tilde{\eta}}{2}}}(\log(s)C_{\triangle \hat{s}} + \tilde{C} + \kappa_1 + \sum_{i=\ell+1}^{L} \kappa_i)) \\ & -\frac{2}{\mu}(qC_{\ell+1} + \frac{1}{s^{\frac{\tilde{\eta}}{2}}}(\log(s)C_{\triangle \hat{s}} + \tilde{C} + C_1 + \sum_{i=\ell+1}^{L} C_i))D_{\text{stab}}(s_0)\Big) \\ \geq & \frac{\mu|\triangle \hat{V}_{\text{uns}}|}{2s}\Big(|\triangle \hat{V}_{\text{uns}}|\frac{1}{2} \\ & -\frac{2}{\mu}(qC_{\ell+1} + \frac{1}{s^{\frac{\tilde{\eta}}{2}}}(\log(s)C_{\triangle \hat{s}} + \tilde{C} + C_1 + \sum_{i=\ell+1}^{L} C_i))D_{\text{stab}}(s_0)\Big). \end{array}$$

But because $|\triangle \hat{V}_{\text{uns}}|$ is increasing on $[s_0, s_1]$, and because at initial time (*3.5.176*) holds, one has:

$$|\triangle \hat{V}_{\text{uns}}(s_1)|\frac{1}{2} - \frac{2}{\mu}(qC_{\ell+1} + \frac{1}{s^{\frac{\tilde{\eta}}{2}}}(\log(s)C_{\triangle \hat{s}} + \tilde{C} + C_1 + \sum_{i=\ell+1}^{L} C_i))D_{\text{stab}}(s_0) > 0$$

which in turn implies that at time s_1: $\frac{d}{ds}|\triangle \hat{V}_{\text{uns}}|^2 > 0$, contradicting the definition of s_1. Hence $s_1 = +\infty$. But if $s_1 = +\infty$, that means that $|\triangle \hat{V}_{\text{uns}}|$ does not converge toward 0. This is the desired contradiction, because as $\boldsymbol{U}$ and $\boldsymbol{U}'$ stay in the trapped regime, this should be true. $\square$

3.5.2 Removal of extra assumptions, end of the proof of Theorem 3.5.1

In the proof of Proposition 3.3.2, we have seen that in order to control the projection of a solution on the first L iterates of the kernel of $\boldsymbol{H}$, one needs to control the $k_0 + 1 + L$ adapted derivative of $\boldsymbol{\varepsilon}$. Therefore, we will decompose only on the first $L - 1$ modes, which will allow us to work with the $k_0 + L$-th adapted derivative, while keeping the bound (*3.5.6*) for the $k_0 + 1 + L$-th one. It will allow us to remove the regularity assumption (*3.5.6*) in Proposition 3.5.2. An other extra assumption in this proposition was the fact that the two solutions started with the same scale, what we will also remove. Our main result is the following improvement of Proposition (3.5.2):

Proposition 3.5.13. *Suppose* $\boldsymbol{U}(s_0) = \Big(\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}} + \boldsymbol{w}\Big)(s_0)$, $\boldsymbol{U}'(s_0) = \Big(\tilde{\boldsymbol{Q}}_{b',\frac{1}{\lambda'}} + \boldsymbol{w}'\Big)(s_0)$ *are two initial data whose solutions stay in the trapped regime described by Proposition 3.3.2. Suppose that they are close initially, that is to say that:*

$$b(s_0) = b^e(s_0) + \left(\frac{U_1(s_0)}{s_0}, ..., \frac{U_L(s_0)}{s_0^L}\right), b'(0) = b^e(s_0) + \left(\frac{U_1'(s_0)}{s_0}, ..., \frac{U_L'(s_0)}{s_0^L}\right). \tag{3.5.178}$$

Suppose that the scales are close to one:

$$|\lambda(s_0) - 1| + |\lambda'(s_0) - 1| \leq s_0^{-L} \tag{3.5.179}$$

Then there exists $C > 0$ such that for s_0 small enough the following bound holds:

$$\begin{array}{rcl} |\triangle V_{uns}(s_0)| & \leq & C\Big(|\triangle V_1(s_0)| + \sum_{\ell+1}^{L} |\triangle U_i(s_0)| + |\lambda'(s_0) - \lambda(s_0)| \\ & & C(s_0) \parallel \boldsymbol{w}(s_0) - \boldsymbol{w}'(s_0) \parallel_{\dot{H}^{\sigma} \cap \dot{H}^{s_L} \times \dot{H}^{\sigma-1} \cap \dot{H}^{s_L - 1}}\Big). \end{array} \tag{3.5.180}$$

3.5.2.1 Lower order decomposition

We start by lowering the number of modes on which we project on the manifold of approximate solutions $(\tilde{\boldsymbol{Q}}_{b,\lambda})_{b,\lambda}$. We let:

$$\bar{L} = L - 1. \tag{3.5.181}$$

Definition 3.5.14 (Lower order decomposition). Suppose $\boldsymbol{U} = \tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}} + \boldsymbol{w} = (\tilde{\boldsymbol{Q}}_b + \boldsymbol{w})_{\frac{1}{\lambda}}$ is a solution satisfying the assumptions of Proposition 3.5.13. We define the $\bar{L}$-tuple of real numbers $\bar{b}$, the scale $\bar{\lambda}$, and the error terms $\bar{\varepsilon}$ and $\overline{\boldsymbol{w}}$ by:

$$\boldsymbol{U}(t) = \tilde{\boldsymbol{Q}}_{\bar{b},\frac{1}{\bar{\lambda}}} + \overline{\boldsymbol{w}}(t) = (\tilde{\boldsymbol{Q}}_{\bar{b}} + \bar{\boldsymbol{\varepsilon}}(\bar{s}))_{\frac{1}{\bar{\lambda}}}, \tag{3.5.182}$$

where $\bar{\varepsilon}$ satisfies the L orthogonality conditions:

$$\langle \bar{\boldsymbol{\varepsilon}}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle = 0, \ \ for\ 0 \leq i \leq L-1. \tag{3.5.183}$$

The renormalized time is given by:

$$\bar{s} := s_0 + \int_0^t \frac{1}{\bar{\lambda}(\tau)} d\tau. \tag{3.5.184}$$

This decomposition is possible for $\boldsymbol{U}$ because as it is a solution given by Proposition 3.3.2, the result of subsubsection 3.3.1.2 applies for the integer $\bar{L}$. We then define the tuples of parameters $\bar{U}$ and $\bar{V}$ as ($\bar{P}_\ell$ being the analogue of P_ℓ defined by (*3.2.78*)):

$$\bar{U}_i := \bar{s}^i(\bar{b}_i - \frac{c_i}{\bar{s}^i}), \ \text{for } 1 \leq i \leq L, \ \text{and } \bar{V} := \bar{P}_\ell(\bar{U}). \tag{3.5.185}$$

We introduce the following notation for the norms of $\bar{\boldsymbol{\varepsilon}}$:

$$\bar{\mathcal{E}}_\sigma := \int |\nabla^\sigma \bar{\varepsilon}^{(1)}|^2 + |\nabla^{\sigma-1}\bar{\varepsilon}^{(2)}|^2, \ \bar{\mathcal{E}}_i := \int |\bar{\varepsilon}_i^{(1)}|^2 + |\nabla^{\sigma-1}\bar{\varepsilon}_{i-1}^{(2)}|^2, \ i = s_{\bar{L}}, s_{\bar{L}}+1. \tag{3.5.186}$$

This decomposition works as follows: we have approximately $\bar{b} \sim (b_1, ..., b_{L-1})$ and $\bar{\boldsymbol{\varepsilon}} \sim \boldsymbol{\varepsilon} + b_L \boldsymbol{T}_L$. The bounds of the trapped regime for the original decomposition transform into bounds for the lower order decomposition. This way we obtain a solution of the trapped regime (with respect to the integer $\bar{L}$ instead of L) with the extra higher regularity bound (*3.5.6*): this is the type of solution for which we proved a primary Lipschitz bound in Proposition 3.5.2.

Lemma 3.5.15 (Bounds for the lower order decomposition). *We keep the assumptions and notations of Definition 3.5.14. The following estimates for* $0 \leq t < T$ *hold:*

(i) Global closeness for the parameters: *The renormalized time satisfies:*

$$\bar{s} = s + O\left(\frac{1}{s_0^L}\right). \tag{3.5.187}$$

For all $1 \leq i \leq L-1$ *there holds:*

$$|U_i - \bar{U}_i| = O(\bar{s}^{-1}). \tag{3.5.188}$$

These two bounds imply in particular that:

$$\bar{b}_1 \sim b_1. \tag{3.5.189}$$

(ii) Bounds for the high adapted derivatives: *for* $i=0,1$ *one has*

$$\overline{\mathcal{E}}_{s_{\bar{L}}+i} \leq C(L,M)K_2 \bar{b}_1^{\bar{L}+i+(1-\delta_0)(1+\eta)}. \tag{3.5.190}$$

(iii) Bound at σ level of regularity:

$$\overline{\mathcal{E}}_{\sigma} \leq C(L,M)K_1 \bar{b}_1^{2(\sigma-s_c)(1+\nu)}. \tag{3.5.191}$$

We denote the canonical projection from $\mathbb{R}^L$ to $\mathbb{R}^{\bar{L}}$ by:

$$\pi \ : \ (b_1,...,b_L) \mapsto (b_1,...,b_{L-1}). \tag{3.5.192}$$

The difference between $\tilde{\boldsymbol{Q}}_b$ and $\tilde{\boldsymbol{Q}}_{\pi(b)}$ is denoted by:

$$\tilde{\boldsymbol{Q}}_b = \tilde{\boldsymbol{Q}}_{\pi(b)} + \chi_{B_1}(b_L \boldsymbol{T}_L + \boldsymbol{S}_{L+2} + \boldsymbol{S}_{L+1} - \overline{\boldsymbol{S}}_{L+1}) \tag{3.5.193}$$

where $\boldsymbol{S}_{L+1}$ is given by (*3.2.48*) for the L-tuple b and $\overline{\boldsymbol{S}}_{L+1}$ is the profile given by the same proposition, but for the $L-1$ tuple $\pi(b)$.

Proof of Lemma 3.5.15 Proof of (i):

• *Step 1:* primary bound. We claim that for all $0 \leq t < T$:

$$\left| \frac{\bar{\lambda}(t)}{\lambda(t)} - 1 \right| \leq C(L,M) b_1(t)^{L+1}, \tag{3.5.194}$$

$$\left| b_i(t) - \bar{b}_i(t) \right| \leq C(L,M) b_1(t)^{L+1}. \tag{3.5.195}$$

We start by proving these two estimates. $\overline{\varepsilon}$ is given by:

$$\begin{array}{rcl} \overline{\varepsilon} & = & \boldsymbol{\varepsilon}_{\frac{\bar{\lambda}}{\lambda}} + (\tilde{\boldsymbol{Q}}_{\frac{\bar{\lambda}}{\lambda}} - \boldsymbol{Q}) + (\tilde{\boldsymbol{\alpha}}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\boldsymbol{\alpha}}_{\bar{b}}) \\ & & + (\chi_{B_1}(b_L \boldsymbol{T}_L + \boldsymbol{S}_{L+2} + \boldsymbol{S}_{L+1} - \overline{\boldsymbol{S}}_{L+1}))_{\frac{\bar{\lambda}}{\lambda}}. \end{array} \tag{3.5.196}$$

We take the scalar product between $\overline{\varepsilon}$ and $\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M$ for $0 \leq i \leq \bar{L}$. For $i=0$ we obtain a bound for the scaling.

$$-\langle \boldsymbol{Q}_{\frac{\bar{\lambda}}{\lambda}} - \boldsymbol{Q}, \boldsymbol{\Phi}_M \rangle = \langle \boldsymbol{\varepsilon}_{\frac{\bar{\lambda}}{\lambda}} + \tilde{\boldsymbol{\alpha}}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\boldsymbol{\alpha}}_{\bar{b}} + (b_L \boldsymbol{T}_L + \boldsymbol{S}_{L+2} + \boldsymbol{S}_{L+1} - \overline{\boldsymbol{S}}_{L+1})_{\frac{\bar{\lambda}}{\lambda}}, \boldsymbol{\Phi}_M \rangle.$$

The left hand side is:

$$\langle \boldsymbol{Q}_{\frac{\bar{\lambda}}{\lambda}} - \boldsymbol{Q}, \boldsymbol{\Phi}_M \rangle = (\frac{\bar{\lambda}}{\lambda} - 1)\langle \boldsymbol{\Lambda Q}, \boldsymbol{\Phi}_M \rangle + O(|\frac{\bar{\lambda}}{\lambda} - 1|^2)$$

We now look at the terms in the right hand side. Performing a change of variables:

$$\langle \boldsymbol{\varepsilon}_{\frac{\bar{\lambda}}{\lambda}}, \boldsymbol{\Phi}_M \rangle = \left(\frac{\lambda}{\bar{\lambda}} \right)^{d-\frac{4}{p-1}} \langle \boldsymbol{\varepsilon}, \boldsymbol{\Phi}_{M,\frac{\lambda}{\bar{\lambda}}} \rangle = O\left(b_1^{L+(1-\delta_0)(1+\eta))} (\frac{\bar{\lambda}}{\lambda} - 1) \right).$$

For the second term we decompose:

$$\langle \tilde{\boldsymbol{\alpha}}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\boldsymbol{\alpha}}_{\bar{b}}, \boldsymbol{\Phi}_M \rangle = \langle \tilde{\boldsymbol{\alpha}}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\boldsymbol{\alpha}}_{\pi(b)} + \tilde{\boldsymbol{\alpha}}_{\pi(b)} - \tilde{\boldsymbol{\alpha}}_{\bar{b}}, \boldsymbol{\Phi}_M \rangle.$$

There holds for the first part:

$$\langle \tilde{\boldsymbol{\alpha}}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\boldsymbol{\alpha}}_{\pi(b)}, \boldsymbol{\Phi}_M \rangle = O\left(b_1^2 (\frac{\bar{\lambda}}{\lambda} - 1) \right).$$

For the second part, because of the orthogonality property (*3.3.7*):

$$\langle \tilde{\boldsymbol{\alpha}}_{\pi(b)} - \tilde{\boldsymbol{\alpha}}_{\bar{b}}, \boldsymbol{\Phi}_M \rangle = \left\langle \sum_{i=2}^{L-2} \boldsymbol{S}_i(\pi(b)) - \boldsymbol{S}_i(\bar{b}) + \overline{\boldsymbol{S}}_{L+1}(\pi(b)) - \overline{\boldsymbol{S}}_{L+1}(\bar{b}), \boldsymbol{\Phi}_M \right\rangle,$$

where we recall that $\overline{\boldsymbol{S}}_{L+1}$ is defined in (*3.5.193*). All these terms are of the form:

$$\int \Phi_M f(\prod_1^{L-1} b_i^{J_i} - \prod_1^{L-1} \bar{b}_i^{J_i})$$

where $|J|_2 \geq 2$ (the notation for the tupples are defined in (*3.2.32*)) and f is bounded. The bound (*3.5.30*) on the difference of polynomials of the b_i's then gives:

$$\langle \tilde{\boldsymbol{\alpha}}_{\pi(b)} - \tilde{\boldsymbol{\alpha}}_{\bar{b}}, \boldsymbol{\Phi}_M \rangle = O(b_1 \mathrm{sup}(|b_i - \bar{b}_i|)).$$

The last term gives:

$$\langle (\chi_{B_1}(b_L \boldsymbol{T}_L + \boldsymbol{S}_{L+2} + \boldsymbol{S}_{L+1} - \overline{\boldsymbol{S}}_{L+1}))_{\frac{\bar{\lambda}}{\lambda}}, \boldsymbol{\Phi}_M \rangle = O(b_1^{L+1}).$$

Put together, all the previous computations yield:

$$(\frac{\bar{\lambda}}{\lambda} - 1) = O(b_1^{L+1}) + O(b_1 sup(|b_i - \bar{b}_i|)). \qquad (3.5.197)$$

Similarly, taking the scalar product of (*3.5.196*) with $\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M$ for $1 \leq i \leq \bar{L}$ yields:

$$(b_i - \bar{b}_i) = O(b_1^{L+1}) + O(b_1 sup(|b_i - \bar{b}_i|)) + O\left(\left(b_1 + |\frac{\bar{\lambda}}{\lambda} - 1| \right) |\frac{\bar{\lambda}}{\lambda} - 1| \right). \qquad (3.5.198)$$

By summing (*3.5.198*) and (*3.5.197*) one obtains the primary bounds we claimed: (*3.5.195*) and (*3.5.194*).
• *Step 2:* integration of the primary bounds. Equation (*3.5.194*) gives a control on the renormalized time difference:

$$\frac{d\bar{s}}{ds} = \frac{d\bar{s}}{dt}\frac{dt}{ds} = \frac{\lambda}{\bar{\lambda}} = 1 + O(b_1^{L+1}).$$

As $b_1 \lesssim s^{-1}$ an integration in time yields:

$$\bar{s} = s + O\left(\frac{1}{s_0^L} \right).$$

This implies in particular that for $1 \leq i \leq L$:

$$b_i^e = \bar{b}_i^e + O(s^{-(i+1)}),$$

which, combined with the primary bound (*3.5.195*) ends the proof of (i).

Proof of (ii): We proved in the previous step that $s \sim \bar{s}$ and $b_1 \sim \bar{b}_1$. We first prove the bound at the level of regularity $s_{\bar{L}} + 1 = s_L$. We have to compute the adapted norm of the right hand side of (*3.5.196*).

We just show here the computations for the second coordinate $\bar{\varepsilon}^{(2)}$, because the estimate for the first one can be proven using the very same calculations. As ε satisfies the result of Proposition 3.3.2, and as $\bar{\lambda} \sim \lambda$, see (*3.5.194*), there holds:

$$\int |(\varepsilon^{(2)}_{\frac{\bar{\lambda}}{\lambda}})_{s_L-1}|^2 \leq CK_2 \bar{b}_1^{2L+2(1-\delta_0)(1+\eta)}$$

with C independent of the other constants. For the second term we decompose:

$$\int |(\tilde{\alpha}^{(2)}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\alpha}^{(2)}_{\bar{b}})_{s_L-1}|^2 \lesssim \int |(\tilde{\alpha}^{(2)}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\alpha}^{(2)}_{\pi(b)})_{s_L-1}|^2 + |(\tilde{\alpha}^{(2)}_{\pi(b)} - \tilde{\alpha}^{(2)}_{\bar{b}})_{s_L-1}|^2.$$

The first term of the right hand side satisfies:

$$(\tilde{\alpha}^{(2)}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\alpha}^{(2)}_{\pi(b)}) = (\frac{\bar{\lambda}}{\lambda} - 1) \int_0^1 \frac{1}{1-\theta+\theta\frac{\bar{\lambda}}{\lambda}} (\Lambda^{(2)} \tilde{\alpha}^{(2)}_{\pi(b)})_{1-\theta+\theta\frac{\bar{\lambda}}{\lambda}} d\theta.$$

And as:

$$\int |(\Lambda^{(2)} \tilde{\alpha}^{(2)}_{\pi(b')})_{1-\theta+\theta\frac{\bar{\lambda}}{\lambda'}, s_L-1}|^2 < +\infty,$$

we conclude using (*3.5.194*) that:

$$\int |(\tilde{\alpha}^{(2)}_{\pi(b),\frac{\bar{\lambda}}{\lambda}} - \tilde{\alpha}^{(2)}_{\pi(b)})_{s_L-1}|^2 \leq |\frac{\bar{\lambda}}{\lambda} - 1|^2 \lesssim \bar{b}_1^{2L+2} \lesssim \bar{b}_1^{2L+2(1-\delta_0)(1+\eta)}.$$

For the other term we compute:

$$\begin{array}{rcl} |(\tilde{\alpha}^{(2)}_{\pi(b)} - \tilde{\alpha}^{(2)}_{\bar{b}})_{s_L-1}|^2 & \lesssim & \sum_{i=2, \text{ odd}}^{L-1} |((b_i\chi_{B_1}T_i - \bar{b}_i\chi_{\bar{B}_1}T_i)_{s_L-1}|^2 \\ & & + \sum_{i=2, \text{ odd}}^{L+1} |(\chi_{B_1}S_i(\pi(b) - \chi_{\bar{B}_1}S_i(\bar{b}))_{s_L-1}|^2. \end{array}$$

We have:

$$\int |((b_i\chi_{B_1}T_i - \bar{b}_i\chi_{\bar{B}_1}T_i)_{s_L-1}|^2 \lesssim \int |(b_i'(\chi_{B_1} - \chi_{\bar{B}_1})T_i)_{s_L-1}|^2 + |((b_i - \bar{b}_i)\chi_{\bar{B}_1}T_i)_{s_L-1}|^2$$

and we estimate the two parts:

$$\int |(b_i(\chi_{B_1} - \chi_{\bar{B}_1})T_i)_{s_L-1}|^2 \lesssim \bar{b}_1^{2L+2(1-\delta_0)(1+\eta)},$$

$$\int |((b_i - \bar{b}_i)\chi_{\bar{B}_1}T_i)_{s_L-1}|^2 \lesssim \bar{b}_1^{2L+2(1-\delta_0)(1+\eta)+2(L+1-i)},$$

where we used (*3.5.195*) for the second inequality. A similar argument gives a similar control for the S_i's contribution, hence yielding to:

$$\int |(\tilde{\alpha}^{(2)}_{\pi(b)} - \tilde{\alpha}^{(2)}_{\bar{b}})_{s_L-1}|^2 \lesssim \bar{b}_1^{2L+2(1-\delta_0)(1+\eta)}.$$

We go on, estimating the next term. From the asymptotic of T_L, see Lemma 3.2.9:

$$\int b_L^2 |((\chi_{B_1}T_L)_{\frac{\bar{\lambda}}{\lambda}})_{s_L-1}|^2 \leq |b_L|^2 b_1^{2(1+\eta)(1-\delta_0)} \leq C\bar{b}_1^{2L+2(1-\delta_0)(1+\eta)},$$

with a constant C that just depends on the bootstrap constant[26] ϵ_L and on L, but which, if L is fixed, is uniformly bounded in ϵ_L. Similarly, from (*3.2.43*):

$$\begin{array}{rcl} \int |((\chi_{B_1}S_{L+2})_{\frac{\bar{\lambda}}{\lambda}})_{s_L-1}|^2 & \leq & \left\{ \begin{array}{l} C\bar{b}_1^{2L+2(1-\delta_0)-C'\eta+2g'} \text{ if } 2\delta_0 + 2 - 2g' > 0 \\ C\bar{b}_1^{2L+4} \text{ if } 2\delta_0 + 2 - 2g' < 0 \end{array} \right. \\ & \leq & C\bar{b}_1^{2L+2(1-\delta_0)(1+\eta)}, \end{array}$$

[26]remember that ϵ_L quantify the size of b_L, see (*3.3.26*).

for η small enough, (we recall the assumption $0<\delta_0$). All the previous estimates show the bound (ii) for the second coordinate:

$$\int |\bar{\varepsilon}^{(2)}_{s_L-1}|^2 \leq C(K_1,\epsilon_L,L,M)\bar{b}_1^{2L+2(1-\delta_0)(1+\eta)}.$$

We claim that the estimate for the first coordinate can be shown making verbatim the same computations. For the sake of completeness, we just show how to deal with the term involving the soliton. We compute first:

$$Q_{\frac{\bar{\lambda}}{\lambda}}-Q=(\frac{\bar{\lambda}}{\lambda}-1)\int_0^1\frac{1}{1-\theta+\theta(\frac{\bar{\lambda}}{\lambda})}(\Lambda^{(1)}Q)_{1-\theta+\theta(\frac{\bar{\lambda}}{\lambda'})}d\theta. \qquad (3.5.199)$$

As for all θ, $\int|((\Lambda Q)_{1-\theta+\theta(\frac{\bar{\lambda}}{\lambda'})})_{s_L}|^2<+\infty$, using (*3.5.194*) and because $0<\delta_0$ we get:

$$\int |(Q_{\frac{\bar{\lambda}}{\lambda}}-Q)_{s_L}|^2\leq C\bar{b}_1^{L+1}\leq \bar{b}_1^{2L+2(1-\delta_0)(1+\eta))}$$

for η small enough. This way we get the bound (ii) for $i=1$. To prove (ii) for $i=0$ we need to use the energy estimate we used to control the error in the proof of Proposition 3.3.2. In the proof of this proposition, we saw (see Section 3.4) that if a solution started in the trapped regime, the only way to escape it was by having unstable mode growing too big. Here the unstable modes are under control from the previous bounds (*3.5.188*). So if it starts in the trapped regime described by proposition 3.3.2 associated to the integer $\bar{L}$, it will imply the control (*3.5.190*) for $i=0$. We compute the adapted $s_{\bar{L}}$ norm of the right hand side of (*3.5.196*) at initial time s_0. One has for the error by interpolation of (*3.3.21*):

$$\parallel \varepsilon^{(1)}(s_0)_{\frac{\bar{\lambda}}{\lambda},s_{\bar{L}}}\parallel_{L^2}+\parallel \varepsilon^{(2)}(s_0)_{\frac{\bar{\lambda}}{\lambda},s_{\bar{L}-1}}\parallel_{L^2}\leq Cb_1(s_0)^{\bar{L}+2+(1-\delta_0)(1+\eta)}.$$

For the L-th mode one has using the bound (*3.3.20*):

$$\parallel b_L(s_0)(\chi_{B_1}T_L)_{s_{\bar{L}-1}}\parallel_{L^2}\leq C|b_L(s_0)|b_1^{-\delta_0(1+\eta)}\leq Cb_1^{\bar{L}+1-\delta_0+\alpha\frac{L-\ell}{\ell-\alpha}+O(\eta))}.$$

We claim that for all the other terms in the right hand side of (*3.5.196*), the same computations we did for the proof of (ii) in the case $i=1$ yield similar results. Hence at initial time one has:

$$\parallel \bar{\varepsilon}^{(1)}(s_0)_{s_{\bar{L}}}\parallel_{L^2}+\parallel \bar{\varepsilon}^{(2)}(s_0)_{s_{\bar{L}-1}}\parallel_{L^2}\leq Cb_1(s_0)^{\bar{L}+(1-\delta_0)(1+\eta)}.$$

Hence we use the result Remark 3.4.1: as the unstable modes are under control from (*3.5.188*), we get the desired bound for all time:

$$\parallel \bar{\varepsilon}^{(1)}_{s_{\bar{L}}}\parallel_{L^2}+\parallel \bar{\varepsilon}^{(2)}_{s_{\bar{L}-1}}\parallel_{L^2}\leq Cb_1^{\bar{L}+(1-\delta_0)(1+\eta)}.$$

Proof of (iii): The estimate for $\overline{\mathcal{E}}_\sigma$ can be done by direct computation as we did for (ii) using similar computations. We estimate again all the terms in the right hand side of (*3.5.196*). We only show the estimate for the first coordinate, as the proof for the second one relies on similar computations. From $\bar{\lambda}\sim\lambda$, and as ε satisfies the bound (*3.3.27*) one gets:

$$\int |\nabla^\sigma\varepsilon^{(1)}_{\frac{\bar{\lambda}}{\lambda}}|^2|\nabla^{\sigma-1}\varepsilon^{(2)}_{\frac{\bar{\lambda}}{\lambda}}|^2\leq=\left|\frac{\bar{\lambda}}{\lambda}\right|^{2(\sigma-s_c)}\mathcal{E}_\sigma\leq CK_1\bar{b}_1^{2(\sigma-s_c)(1+\nu)}$$

for a constant C independent of the other constants. For the soliton term, we use the expression (*3.5.199*) and Fubini to estimate:

$$\int|\nabla^\sigma(Q_{\frac{\bar{\lambda}}{\lambda}}-Q)|^2\leq\left|\frac{\bar{\lambda}}{\lambda}-1\right|^2\underset{\theta\in[0,1]}{sup}\left|1-\theta+\theta\frac{\bar{\lambda}}{\lambda}\right|^{2(\sigma-s_c)-2}\int|\nabla^\sigma(\Lambda^{(1)}Q)|^2\leq C\bar{b}_1^{2L+2}.$$

We used the bound $|\frac{\bar\lambda}{\lambda}-1|\lesssim \bar b_1^{L+1}$ and the fact that $\int|\nabla^\sigma(\Lambda^{(1)}Q)|^2<+\infty$ from the asymptotic (*3.2.3*). For the following term, we decompose:

$$\tilde\alpha^{(1)}_{\pi(b),\frac{\bar\lambda}{\lambda}}-\tilde\alpha^{(1)}_{\bar b})=\tilde\alpha^{(1)}_{\pi(b),\frac{\bar\lambda}{\lambda}}-\tilde\alpha^{(1)}_{\pi(b)}+\tilde\alpha^{(1)}_{\pi(b)}-\tilde\alpha^{(1)}_{\bar b}.$$

For the first part, using the analogue of formula (*3.5.199*):

$$\begin{aligned}\int|\nabla^\sigma\tilde\alpha^{(1)}_{\pi(b),\frac{\bar\lambda}{\lambda}}-\tilde\alpha^{(1)}_{\pi(b)}|^2 \quad\leq\quad & \left|\frac{\bar\lambda}{\lambda}-1\right|^2\underset{\theta\in[0,1]}{sup}\left|1-\theta+\theta\frac{\bar\lambda}{\lambda}\right|^{2(\sigma-s_c)-2}\\ &\times\int|\nabla^\sigma(\Lambda^{(1)}\tilde\alpha^{(1)}_{\pi(b)})|^2\leq C(L,M)\bar b_1^{2L+2}\end{aligned}$$

because $\int|\nabla^\sigma(\Lambda^{(1)}\tilde\alpha^{(1)}_{\pi(b)})|^2<+\infty$ from the asymptotic (*3.2.43*) and Lemma 3.2.9. For the other part, (*3.5.30*), (*3.5.195*) and again the same asymptotics yield:

$$\int|\nabla^\sigma(\tilde\alpha^{(1)}_{\pi(b)}-\tilde\alpha^{(1)}_{\bar b})|^2\leq\bar b_1^4.$$

Putting together the last two estimates gives:

$$\int|\nabla^\sigma(\tilde\alpha^{(1)}_{\pi(b),\frac{\bar\lambda}{\lambda}}-\tilde\alpha^{(1)}_{\bar b})|^2\lesssim\bar b_1^4.$$

The last remaining term is estimated similarily:

$$\int|\nabla^\sigma((S_{L+1}-\overline{S}_{L+1})_{\frac{\bar\lambda}{\lambda}})|^2\lesssim\bar b_1^4$$

The estimate we have done for each term of the right hand side of (*3.5.196*) give:

$$\int|\nabla^\sigma\bar\varepsilon^{(1)}|^2\leq C(K_2)\bar b_1^{2(\sigma-s_c)(1+\nu)}.$$

Using the very same method, one finds the same estimation for the second coordinate, leading to the result (iii). □

The same lower order decomposition also applies for the other solution $\boldsymbol{U}'$, and we have the analogue of the previous lemma. What we want to do now is to apply the Proposition 3.5.2 associated to the integer $\bar L$ to these two new solutions in the trapped regime associated to the integer $\bar L$. There remains two steps: we have to check that the differences between the parameters and errors under the lower order decomposition can be related to the original decomposition, and we have to deal with a possible scale difference at initial time. We use the following notations for the lower order decomposition associated to $\boldsymbol{U}'$ by the Definition (3.5.14) :

$$\boldsymbol{U}'(t)=\tilde{\boldsymbol{Q}}_{\bar b',\frac{1}{\bar\lambda'}}+\overline{\boldsymbol{w}}'(t)=(\tilde{\boldsymbol{Q}}_{\bar b'}+\overline{\boldsymbol{\varepsilon}}(\bar s'))_{\frac{1}{\bar\lambda'}},\qquad(3.5.200)$$

where $\overline{\boldsymbol{\varepsilon}}'$ satisfies the L orthogonality conditions:

$$\langle\overline{\boldsymbol{\varepsilon}}',\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle=0,\ \ for\ 0\leq i\leq\bar L.\qquad(3.5.201)$$

Similarly we define ($\bar P_\ell$ being the analogue of P_ℓ defined by (*3.2.78*)):

$$\bar s':=s_0+\int_0^t\frac{1}{\bar\lambda'(\tau)}d\tau,\qquad(3.5.202)$$

$$\bar{U}_i' := (\bar{s}')^i(\bar{b}_i' - \frac{c_i}{(\bar{s}')^i}), \text{ for } 1 \leq i \leq L, \text{ and } \bar{V} := \bar{P}_\ell(\bar{U}). \tag{3.5.203}$$

We use the following notations for the differences under lower order decomposition:

$$\triangle \bar{b} := \bar{b}_i - \bar{b}_i', \ \triangle \bar{U}_i := \bar{U}_i - \bar{U}_i', \ \triangle \bar{\lambda} := \bar{\lambda} - \bar{\lambda}'.$$

We make now a slight change regarding the former norm notations. They now concern $\boldsymbol{w}$ instead of ε:

$$\triangle \overline{\mathcal{E}}_\sigma :=\| \overline{\boldsymbol{w}} - \overline{\boldsymbol{w}}' \|^2_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}}, \ \triangle_r \overline{\mathcal{E}}_\sigma := b_1^{-2(\sigma-s_c)(1+\nu)} \triangle \overline{\mathcal{E}}_\sigma,$$

$$\triangle \overline{\mathcal{E}}_{s_{\bar{L}}} := \int (\bar{w}^{(1)} - \bar{w}'^{(1)})^2_{s_{\bar{L}}} + (\bar{w}^{(2)} - \bar{w}'^{(2)})^2_{s_{\bar{L}}}, \ \triangle_r \overline{\mathcal{E}}_{s_{\bar{L}}} := b_1^{-2\bar{L}-(1-\delta_0)(2+\eta)} \triangle \overline{\mathcal{E}}_{s_{\bar{L}}}.$$

In the following lemma we relate the differences between the lower order decomposition and the original decomposition at initial time. Basically, the differences of the two solutions in lower or higher order are almost the same.

Lemma 3.5.16 (Bounds for the differences at initial time). *We keep the assumptions and notations from Definitions 3.5.14 and Proposition 3.5.13. There holds initially:*

(i) bounds on the parameters: *For* $1 \leq i \leq \bar{L}$*:*

$$\begin{array}{rcl} \triangle \bar{U}_i(s_0) & = & \triangle U_i(s_0) + O[b_1 |\triangle U(s_0)| + b_1^{(1-\delta_0)(1+\eta)} |\triangle \lambda|] \\ & & +O(C(s_0) \| \boldsymbol{w} - \boldsymbol{w}' \|_{\dot{H}^\sigma \cap \dot{H}^{s_L} \times \dot{H}^{\sigma-1} \cap \dot{H}^{s_L-1}}), \end{array} \tag{3.5.204}$$

(ii) bounds on the errors:

$$\sqrt{\triangle_r \overline{\mathcal{E}}_{s_{\bar{L}}}} \leq C(s_0) \| \boldsymbol{w} - \boldsymbol{w}' \|_{\dot{H}^\sigma \cap \dot{H}^{s_L} \times \dot{H}^{\sigma-1} \cap \dot{H}^{s_L-1}} + C(b_1 |\triangle U| + b_1^{\frac{\eta}{2}} |\triangle \lambda|), \tag{3.5.205}$$

$$\sqrt{\triangle_r \overline{\mathcal{E}}_\sigma} \leq C(s_0) \| \boldsymbol{w} - \boldsymbol{w}' \|_{\dot{H}^\sigma \cap \dot{H}^{s_L} \times \dot{H}^{\sigma-1} \cap \dot{H}^{s_L-1}} + C b_1^{\bar{L}} (|\triangle U| + |\triangle \lambda|). \tag{3.5.206}$$

(iii) bound on the scales:

$$\begin{array}{rcl} \triangle \bar{\lambda}(s_0) & = & \triangle \lambda(s_0) + O[b_1 |\triangle U(s_0)| + b_1^{(1-\delta_0)(1+\eta)} |\triangle \lambda|] \\ & & +O(C(s_0) \| \boldsymbol{w} - \boldsymbol{w}' \|_{\dot{H}^\sigma \cap \dot{H}^{s_L} \times \dot{H}^{\sigma-1} \cap \dot{H}^{s_L-1}}), \end{array} \tag{3.5.207}$$

for some constant C *independent of the other constants.*

Remark 3.5.17. In all the previous computations, $\boldsymbol{w}$ and $\boldsymbol{w}'$, or $\overline{\boldsymbol{w}}$ and $\overline{\boldsymbol{w}}'$ were always at the same scale: there was no confusion regarding orthogonality conditions or adapted norms. Now, in the case of Lemma (3.5.16), each error has a different scale: λ, λ', $\bar{\lambda}$ and $\bar{\lambda}'$. From (*3.5.179*) and (*3.5.194*) they are all close to one:

$$|\lambda - 1| + |\lambda' - 1| + |\bar{\lambda} - 1| + |\bar{\lambda}' - 1| \lesssim b_1^L.$$

From coercivity (see (*3.3.100*)) we obtain that for $\boldsymbol{f} \in \dot{H}^\sigma \cap \dot{H}^{s_L} \times \dot{H}^{\sigma-1} \cap \dot{H}^{s_L-1}$ satisfying the orthogonality conditions (*3.3.9*) and $\tilde{\lambda}$ close enough to 1:

$$\| f^{(1)}_{s_L} - ((f^{(1)}_{\tilde{\lambda}})_{s_L})_{\frac{1}{\tilde{\lambda}}} \|_{L^2} + \| f^{(2)}_{s_L} - ((f^{(2)}_{\tilde{\lambda}})_{s_L})_{\frac{1}{\tilde{\lambda}}} \|_{L^2} \lesssim |\tilde{\lambda} - 1| (\| f^{(1)}_{s_L} \|_{L^2} + \| f^{(2)}_{s_L-1} \|_{L^2}),$$

from what we deduce that the scale does not matter for this adapted norm:

$$\| ((f^{(1)}_{\tilde{\lambda}})_{s_L})_{\frac{1}{\tilde{\lambda}}} \|_{L^2} + \| ((f^{(2)}_{\tilde{\lambda}})_{s_L})_{\frac{1}{\tilde{\lambda}}} \|_{L^2} \sim \| f^{(1)}_{s_L} \|_{L^2} + \| f^{(2)}_{s_L-1} \|_{L^2} .$$

Proof of Lemma 3.5.16. To ease notations, we do not mention the dependence with respect to time: all objects are taken at time s_0. At this initial time, one has:

$$\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}}-\tilde{\boldsymbol{Q}}_{b',\frac{1}{\lambda'}}-(\tilde{\boldsymbol{Q}}_{\bar{b},\frac{1}{\bar{\lambda}}}-\tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\bar{\lambda}'}})+\boldsymbol{w}-\boldsymbol{w}'-(\overline{\boldsymbol{w}}-\overline{\boldsymbol{w}}')=0. \tag{3.5.208}$$

We introduce the following the notation:

$$D=\sum_{i}^{\bar{L}}|\triangle b_i-\triangle\bar{b}_i|.$$

Throughout the proof we will use Remark 3.5.17 and the fact that from (*3.5.188*) the parameters have the same size:

$$b_i\approx b_i'\approx\bar{b}_i\approx\bar{b}_i' \text{ for } 1\leq i\leq\bar{L}.$$

$\overline{\boldsymbol{w}}'$ satisfies the orthogonality conditions (*3.3.9*), but at the scale $\frac{1}{\bar{\lambda}'}$. To deal with the problem of the scale in orthogonality conditions and adapted norms, we introduce:

$$\overline{\boldsymbol{v}}':=\overline{\boldsymbol{w}}'-\sum_{0}^{\bar{L}}\frac{\langle\overline{\boldsymbol{w}}',(\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M)_{\frac{1}{\bar{\lambda}}}\rangle}{\langle\Lambda^{(1)}Q,\chi_M\Lambda^{(1)}Q\rangle}\chi_{B_1}\boldsymbol{T}_i. \tag{3.5.209}$$

Thus, $\overline{\boldsymbol{v}}'$ satisfies the orthogonality conditions (*3.3.9*) at the scale $\bar{\lambda}$. It is very close to $\overline{\boldsymbol{w}}'$ and one has the estimates:

$$\parallel\overline{\boldsymbol{w}}'-\overline{\boldsymbol{v}}'\parallel_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}}\leq|\triangle\bar{\lambda}|b_1^{\alpha+1-\delta_0+O(\eta,\sigma-s_c)}, \tag{3.5.210}$$

$$\parallel(\overline{w}'^{(1)}-\overline{v}'^{(1)})_{s_{\bar{L}}}\parallel_{L^2}+\parallel(\overline{w}'^{(2)}-\overline{v}'^{(2)})_{s_{\bar{L}}-1}\parallel_{L^2}\leq|\triangle\bar{\lambda}|b_1^{\bar{L}+2(1-\delta_0)(1+\eta)}, \tag{3.5.211}$$

$$\parallel\overline{\boldsymbol{w}}'-\overline{\boldsymbol{v}}'\parallel_{L^\infty\times L^\infty(y\leq 2M)}\leq C|\triangle\bar{\lambda}|b_1^{\bar{L}+(1-\delta_0)(1+\eta)}, \tag{3.5.212}$$

Step 1: *Difference of differences of polynomials of parameters.* We claim that for any L-tuple J there holds:

$$|b^J-\hat{b}'^J-(\bar{b}^J-\bar{b}'^J)|\leq C(b_1^{|J|_2}|\triangle U-\triangle\bar{U}|+b_1^{L+1}|\triangle b|). \tag{3.5.213}$$

We show this bound by iteration on $|J|_1=i$. It is obviously true for $i=0$. We take $i\geq 1$ and J satisfying $|J|_1=i$ and suppose it is true for all J' with $|J'|_1\leq i-1$. Let j be the first coordinate for which J is non null and write $b^J=b_jb^{J'}$ with $|J'|=i-1$. We decompose:

$$\begin{aligned} b^J-\hat{b}'^J-(\bar{b}^J-\bar{b}'^J) &= \bar{b}_j(b^{J'}-b'^{J'}-(\bar{b}^{J'}-\bar{b}'^{J'}))+\bar{b}'^{J'}(b_j-b_j'-(\bar{b}_j-\bar{b}_j'))\\ &\quad +(b_j-\bar{b}_j)(b^{J'}-b'^{J'})+(b'^{J'}-\bar{b}'^{J'})(b_j-b_j'). \end{aligned}$$

From (*3.5.188*) one gets for the last two terms $|(b_j-\bar{b}_j)(b^{J'}-b'^{J'})+(b'^{J'}-\bar{b}'^{J'})(b_j-b_j')|\leq b_1^{L+1}|\triangle b|$. For the first two terms we apply the iteration hypothesis for J' and conclude.

Step 2: *The scale.* We claim the first bound:

$$\triangle\bar{\lambda}=\triangle\lambda+O(b_1D)+O[b_1^{L+1}|\triangle U|+b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|)]+O(b_1^L\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}}). \tag{3.5.214}$$

We prove it by taking the scalar product of (*3.5.208*) with $(\boldsymbol{\Phi}_M)_{\frac{1}{\lambda}}$. For the part on the manifold of approximate solutions one has the following decomposition:

$$\begin{array}{rcl} \tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{b',\frac{1}{\lambda'}} - (\tilde{\boldsymbol{Q}}_{\bar{b},\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda'}}) & = & (\tilde{\boldsymbol{Q}}_b - \tilde{\boldsymbol{Q}}_{b'} - (\tilde{\boldsymbol{Q}}_{\bar{b}} - \tilde{\boldsymbol{Q}}_{\bar{b}'}))_{\frac{1}{\lambda}} \\ & & +((\tilde{\boldsymbol{Q}}_b - \tilde{\boldsymbol{Q}}_{b'})_{\frac{1}{\lambda}} - (\tilde{\boldsymbol{Q}}_b - \tilde{\boldsymbol{Q}}_{b'})_{\frac{1}{\lambda}}) \\ & & +((\tilde{\boldsymbol{Q}}_{b'} - \tilde{\boldsymbol{Q}}_{\bar{b}'})_{\frac{1}{\lambda}} - (\tilde{\boldsymbol{Q}}_{b'} - \tilde{\boldsymbol{Q}}_{\bar{b}'})_{\frac{1}{\lambda'}}) \\ & & +(\tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda'}} - (\tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda'}})). \end{array} \tag{3.5.215}$$

We aim at estimating the contribution of each term in the right hand side. For the first term, from the orthogonality conditions (*3.3.6*) and the localization (*3.3.5*):

$$\begin{array}{rl} & \langle (\tilde{\boldsymbol{Q}}_b - \tilde{\boldsymbol{Q}}_{b'} - (\tilde{\boldsymbol{Q}}_{\bar{b}} - \tilde{\boldsymbol{Q}}_{\bar{b}'}))_{\frac{1}{\lambda}}, \boldsymbol{\Phi}_{M,\frac{1}{\lambda}} \rangle \\ = & \langle (\tilde{Q}_b^{(1)} - \tilde{Q}_{b'}^{(1)} - (\tilde{Q}_{\bar{b}}^{(1)} - \tilde{Q}_{\bar{b}'}^{(1)})), \Phi_M^{(1)} \rangle \\ & +O(b_1^L \parallel \tilde{Q}_b^{(1)} - \tilde{Q}_{b'}^{(1)} - (\tilde{Q}_{\bar{b}}^{(1)} - \tilde{Q}_{\bar{b}'}^{(1)}) \parallel_{L^2(\leq 2M)}) \\ = & \langle (S_{L+2} - S'_{L+2} + S_{L+1} - \bar{S}_{L+1} - (S'_{L+1} - \bar{S}'_{L+1}), \Phi_M^{(1)} \rangle \\ & +O(\parallel \sum_{i=1,\text{ even}}^{L} S_i - S'_i - (\overline{S}_i - \overline{S}'_i) \parallel_{L^2(\leq 2M)}) \\ & +O(b_1^L \parallel \tilde{Q}_b^{(1)} - \tilde{Q}_{b'}^{(1)} - (\tilde{Q}_{\bar{b}}^{(1)} - \tilde{Q}_{\bar{b}'}^{(1)}) \parallel_{L^2(\leq 2M)}) \end{array}$$

Now, one decomposes the profiles S_i's for $1 \leq i \leq L$ as a finite sum $S_i = \sum b^J f$ with $|J|_2 = i$ and f a C^∞ function. Applying (*3.5.213*) gives (we recall that D is defined at the begining of the proof):

$$\parallel S_i - S'_i - (\overline{S}_i - \overline{S}'_i) \parallel_{L^2(\leq 2M)} = O(b_1 D) + O(b_1^{L+1}|\triangle b|).$$

So for the first term in (*3.5.215*) we obtain:

$$\langle \tilde{\boldsymbol{Q}}_b - \tilde{\boldsymbol{Q}}_{b'} - (\tilde{\boldsymbol{Q}}_{\bar{b}} - \tilde{\boldsymbol{Q}}_{\bar{b}'})_{\frac{1}{\lambda}}, \boldsymbol{\Phi}_{M,\frac{1}{\lambda}} \rangle = O(b_1 D) + O(b_1^{L+1}|\triangle U|).$$

From (*3.5.194*) we get for the second:

$$\langle (\tilde{\boldsymbol{Q}}_b - \tilde{\boldsymbol{Q}}_{b'})_{\frac{1}{\lambda}} - (\tilde{\boldsymbol{Q}}_b - \tilde{\boldsymbol{Q}}_{b'})_{\frac{1}{\lambda}}, \boldsymbol{\Phi}_{M,\frac{1}{\lambda}} \rangle = O(b_1^{L+1}|\triangle b|).$$

For the third from (*3.5.188*) one has:

$$\langle (\tilde{\boldsymbol{Q}}_{b'} - \tilde{\boldsymbol{Q}}_{\bar{b}'})_{\frac{1}{\lambda}} - (\tilde{\boldsymbol{Q}}_{b'} - \tilde{\boldsymbol{Q}}_{\bar{b}'})_{\frac{1}{\lambda'}}, \boldsymbol{\Phi}_{M,\frac{1}{\lambda}} \rangle = O(b_1^{L+1}|\triangle \lambda|).$$

For the fourth we decompose, use (*3.5.188*) and (*3.5.194*) to find:

$$\begin{array}{rl} & \langle \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda'}} - (\tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda'}}), \boldsymbol{\Phi}_{M,\frac{1}{\lambda}} \rangle \\ = & \langle (\tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda'}}) - (\tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda'}})_{\frac{\lambda}{\bar{\lambda}}}, \boldsymbol{\Phi}_{M,\frac{1}{\lambda}} \rangle + \langle \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\lambda'}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{\lambda}{\bar{\lambda}\lambda'}}, \boldsymbol{\Phi}_M \rangle \\ = & O(b_1^{L+1}|\triangle \lambda|) - (\bar{\lambda}' - \frac{\lambda' \bar{\lambda}}{\lambda})(\langle \chi_M \Lambda^{(1)} Q, \Lambda^{(1)} Q \rangle + O(b_1)) \\ = & (\triangle \lambda - \triangle \bar{\lambda})(\langle \chi_M \Lambda^{(1)} Q, \Lambda^{(1)} Q \rangle + O(b_1)) + O(b_1^{L+1}|\triangle \lambda|) \end{array}$$

from the identity $\bar{\lambda}' - \frac{\lambda'\bar{\lambda}}{\lambda} = \triangle\bar{\lambda} - \triangle\lambda + \frac{1}{\lambda}(\lambda - \bar{\lambda})(\lambda' - \lambda)$. The decomposition (*3.5.215*) and the four previous equations give for the contribution of the difference of differences of approximate profiles in (*3.5.208*):

$$\begin{array}{rl} & \langle \tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{b',\frac{1}{\lambda'}} - (\tilde{\boldsymbol{Q}}_{\bar{b},\frac{1}{\lambda}} - \tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\bar{\lambda}'}}), \boldsymbol{\Phi}_{M,\frac{1}{\lambda}} \rangle \\ = & (\triangle \lambda - \triangle \bar{\lambda})(\langle \chi_M \Lambda^{(1)} Q, \Lambda^{(1)} Q \rangle + O(b_1)) + O[b_1^{L+1}(|\triangle U| + |\triangle \lambda|)] + O(b_1 D). \end{array} \tag{3.5.216}$$

We now turn to the contribution of the difference of differences of errors in (*3.5.208*). We compute using the orthogonality conditions (*3.3.9*) and the hypothesis $|1-\lambda|\leq b_1^L$:

$$\langle \boldsymbol{w}-\boldsymbol{w}',\boldsymbol{\Phi}_{M,\frac{1}{\lambda}}\rangle=-\langle \boldsymbol{w}',\boldsymbol{\Phi}_{M,\frac{1}{\lambda}}-\boldsymbol{\Phi}_{M,\frac{1}{\lambda'}}\rangle=O(b_1^{L+(1-\delta_0)(1+\eta)}|\triangle\lambda|).$$

Using the variable $\overline{\boldsymbol{v}}'$ introduced at the begining of the proof, one can use coercivity thanks to Remark (3.5.17):

$$\begin{array}{rcl}\langle \overline{\boldsymbol{w}}-\overline{\boldsymbol{w}}',\boldsymbol{\Phi}_{M,\frac{1}{\lambda}}\rangle&=&\langle \overline{\boldsymbol{w}}-\overline{\boldsymbol{v}}',\boldsymbol{\Phi}_{M,\frac{1}{\lambda}}\rangle+\langle \overline{\boldsymbol{v}}'-\overline{\boldsymbol{w}}',\boldsymbol{\Phi}_{M,\frac{1}{\lambda}}\rangle\\ &=&\langle \overline{\boldsymbol{w}}-\overline{\boldsymbol{v}}',\boldsymbol{\Phi}_{M,\frac{1}{\lambda}}-\boldsymbol{\Phi}_{M,\frac{1}{\overline{\lambda}}}\rangle+\langle \overline{\boldsymbol{v}}'-\overline{\boldsymbol{w}}',\boldsymbol{\Phi}_{M,\frac{1}{\lambda}}\rangle\\ &=&O(b_1^L\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}})+O(b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\bar{\lambda}|).\end{array}$$

where we used the estimates (*3.5.211*) and (*3.5.212*). We put the two previous estimates for the contribution of the errors and (*3.5.216*) in (*3.5.208*), it gives the estimate (*3.5.214*) we claimed in this step 2.

Step 3: *The parameters.* We claim that the techniques employed in the previous step adapts when we consider the scalar product between (*3.5.208*) and $(\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M)_{\frac{1}{\lambda}}$ for $1\leq i\leq L$, yielding:

$$\begin{array}{rcl}\triangle\bar{b}_i&=&\triangle b_i+O(b_1D)+O[b_1^{L+1}(|\triangle U|+|\triangle\lambda|)]+O(b_1^L\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}})\\ &&+O(b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\bar{\lambda}|).\end{array}$$

Injecting the bound (*3.5.217*), the previous equation simplifies into:

$$\triangle\bar{b}_i=\triangle b_i+O(b_1D)+O[b_1^{L+1}(|\triangle U|)+b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|)]+O(b_1^L\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}})\qquad(3.5.217)$$

Step 4: *Improving the bounds.* We sum the previous identity (*3.5.217*) from $i=1$ to $\bar{L}$, it gives:

$$D=O[b_1^{L+1}(|\triangle U|)+b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|)]+O(b_1^L\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}})+O(b_1^L\sqrt{\triangle\mathcal{E}_{s_L}}).$$

Putting back this bound in (*3.5.214*) and (*3.5.217*) yield:

$$\triangle\bar{\lambda}=\triangle\lambda+O[b_1^{L+1}(|\triangle U|)+b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|)]+O(b_1^L\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}}),\qquad(3.5.218)$$

$$\triangle\bar{b}_i=\triangle b_i+O[b_1^{L+1}(|\triangle U|)+b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|)]+O(b_1^L\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}}).\qquad(3.5.219)$$

Step 5 *The error terms.* The difference between the two error terms in lower order decomposition is:

$$\overline{\boldsymbol{w}}-\overline{\boldsymbol{w}}'=\tilde{\boldsymbol{Q}}_{b,\frac{1}{\lambda}}-\tilde{\boldsymbol{Q}}_{b',\frac{1}{\lambda'}}-(\tilde{\boldsymbol{Q}}_{\bar{b},\frac{1}{\bar{\lambda}}}-\tilde{\boldsymbol{Q}}_{\bar{b}',\frac{1}{\bar{\lambda}'}})+\boldsymbol{w}-\boldsymbol{w}'.$$

Injecting the bounds (*3.5.218*) and (*3.5.219*) in the decomposition (*3.5.215*) gives:

$$\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}}=\sqrt{\triangle\mathcal{E}}_{s_{\bar{L}}}+O[b_1^{L+1}(|\triangle U|)+b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|)]+O(b_1^L\sqrt{\triangle\overline{\mathcal{E}}_{s_{\bar{L}}}}),$$

Now, as:

$$\sqrt{\triangle\mathcal{E}}_{s_{\bar{L}}}\lesssim\parallel\boldsymbol{w}-\boldsymbol{w}'\parallel_{\dot{H}^\sigma\cap\dot{H}^{s_L}\times\dot{H}^{\sigma-1}\cap\dot{H}^{s_L-1}}$$

it gives:

$$\sqrt{\triangle_r\overline{\mathcal{E}}_{s_{\bar{L}}}}\leq C(s_0)\parallel\boldsymbol{w}-\boldsymbol{w}'\parallel_{\dot{H}^\sigma\cap\dot{H}^{s_L}\times\dot{H}^{\sigma-1}\cap\dot{H}^{s_L-1}}+\sqrt{\triangle_r\mathcal{E}_\sigma})+C(b_1|\triangle U|+b_1^{\frac{\eta}{2}}|\triangle\lambda|).$$

We turn back to the previous identities (*3.5.218*) and (*3.5.219*), inject the bound we just found to obtain:

$$\triangle\bar{\lambda} = \triangle\lambda + O[b_1^{L+1}|\triangle U| + b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|] + O(\| \boldsymbol{w} - \boldsymbol{w}' \|_{\dot{H}^\sigma\cap\dot{H}^{s_L}\times\dot{H}^{\sigma-1}\cap\dot{H}^{s_L-1}}),$$

$$\triangle\bar{b}_i = \triangle b_i + O[b_1^{L+1}|\triangle U| + b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|] + O(\| \boldsymbol{w} - \boldsymbol{w}' \|_{\dot{H}^\sigma\cap\dot{H}^{s_L}\times\dot{H}^{\sigma-1}\cap\dot{H}^{s_L-1}}),$$

where the constant in the $O()$ depends on s_0. These two bounds allow us to compute the last norm of $\overline{\boldsymbol{w}} - \overline{\boldsymbol{w}}'$:

$$\sqrt{\triangle\bar{\mathcal{E}}_\sigma} \leq C(b_1^{L+1}|\triangle U| + b_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\triangle\lambda|) + C(s_0) \| \boldsymbol{w} - \boldsymbol{w}' \|_{\dot{H}^\sigma\cap\dot{H}^{s_L}\times\dot{H}^{\sigma-1}\cap\dot{H}^{s_L-1}} .$$

The four last bounds directly imply the bounds of the lemma we had to prove.

□

We are now ready to end the proof of Proposition 3.5.13.

Proof of Proposition 3.5.13 Let $\boldsymbol{U}$ and $\boldsymbol{U}'$ be two solutions described by the Proposition 3.5.13. We associate to the two solutions their lower order decomposition described by Definition 3.5.14. Without loss of generality, we can assume $\bar{\lambda}(s_0) \leq \bar{\lambda}'(s_0)$, which means that in lower order decomposition, the second solution starts at a higher scale than the first one.

We let the second solution evolve with time and define $\bar{s}_1'$ as the time at which its scale is the same as the initial scale of the first solution in lower order decomposition: $\bar{\lambda}'(\bar{s}_1') = \bar{\lambda}(s_0)$. We now estimate the difference between the second solution, in lower order decomposition, taken at these two times. From the equation (*3.3.36*) governing the time evolution of the scale one has:

$$|\bar{s}_1' - s_0| \leq Cb_1^{-1}|\bar{\lambda}'(s_0) - \bar{\lambda}(s_0)|.$$

We can then estimate, from (*3.3.36*) and (*3.3.56*) the time variation of the parameters:

$$|\bar{b}_i'(\bar{s}_1') - \bar{b}_i'(s_0)| \leq Cb_1^i|\bar{\lambda}'(s_0) - \bar{\lambda}(s_0)|.$$

Let us now quantify how the error changed. In the proof of the energy estimate for the high adapted Sobolev norm (*3.3.88*), we computed the size of everything in the right hand side of (*3.3.32*). We computed also the influence of the scale changing in (*3.3.100*). The form of this energy estimate was meant to cancel the linear part, see (*3.3.91*). But we have here the additional regularity (*3.5.6*) for the second solution under the lower order decomposition. Thus all these estimates yield:

$$\| \frac{d}{d\bar{s}'}[(((\bar{w}_{\bar{\lambda}'}^{'(1)})_{s_{\bar{L}}})_{\frac{1}{\bar{\lambda}'}}], \frac{d}{d\bar{s}'}[(((\bar{w}_{\bar{\lambda}'}^{'(2)})_{s_{\bar{L}-1}})_{\frac{1}{\bar{\lambda}'}}] \|_{L^2\times L^2} \leq Cb_1^{\bar{L}+1+(1-\delta_0)(1+\eta)}.$$

From that we deduce (using Remark 3.5.17):

$$\begin{aligned} &\| \bar{w}_{s_{\bar{L}}}^{'(1)}(\bar{s}_1') - ((\bar{w}_{\frac{\bar{\lambda}'}{\bar{\lambda}}}^{'(1)}(s_0))_{s_{\bar{L}}})_{\frac{\bar{\lambda}}{\bar{\lambda}'}}, \bar{w}_{s_{\bar{L}}-1}^{'(2)}(\bar{s}_1') - ((\bar{w}_{\frac{\bar{\lambda}'}{\bar{\lambda}}}^{'(2)}(s_0))_{s_{\bar{L}}-1})_{\frac{\bar{\lambda}}{\bar{\lambda}'}} \|_{L^2\times L^2} \\ \leq\ & Cb_1^{\bar{L}+(1-\delta_0)(1+\eta)}|\bar{\lambda}'(s_0) - \bar{\lambda}(s_0)|. \end{aligned}$$

A similar result holds for the low regularity Sobolev norm:

$$\| \bar{w}^{'(1)}(\bar{s}_1') - \bar{w}^{'(1)}(s_0), \bar{w}^{'(2)}(\bar{s}_1') - \bar{w}^{'(2)}(s_0) \|_{\dot{H}^\sigma\times\dot{H}^{\sigma-1}} \leq Cb_1^{(\sigma-s_c)(1+\nu)}|\bar{\lambda}'(s_0) - \bar{\lambda}(s_0)|.$$

We now apply the result of Proposition 3.5.2 to $\overline{\boldsymbol{U}}(s_0)$ and $\overline{\boldsymbol{U}}'(\bar{s}_1')$. It gives the primary Lipschitz bound (using Remark 3.5.17):

$$\begin{array}{rl} & |\bar{V}_{\text{uns}}(s_0) - \bar{V}'_{\text{uns}}(\bar{s}_1')| \\ \leq & C\Big(|\bar{V}_1(s_0) - \bar{V}_1'(\bar{s}_1')| + \sum_{\ell+1}^L |\bar{U}_i(s_0) - \bar{U}_i'(\bar{s}_1')| \\ & + b_1^{-(\sigma-s_c)(1+\nu)} \parallel \overline{\boldsymbol{w}}(s_0) - \overline{\boldsymbol{w}}'(\bar{s}_1') \parallel_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \\ & + b_1^{-\bar{L}-(1-\delta_0)(1+\eta)} \parallel \bar{w}^{(1)}_{s_{\bar{L}}}(s_0) - \bar{w}'^{(1)}_{s_{\bar{L}}}(\bar{s}_1'), \bar{w}^{(2)}_{s_{\bar{L}}-1}(s_0) - \bar{w}'^{(1)}_{s_{\bar{L}}-1}(\bar{s}_1') \parallel_{L^2\times L^2} \Big) \end{array}$$

for the variables under lower order decomposition and at different times s_0 and $\bar{s}_1'$. We now use the four previous bounds that link the variables for the second solution under lower order decomposition between the times s_0 and $\bar{s}_1'$ to obtain from the previous equation:

$$\begin{array}{rl} & |\bar{V}_{\text{uns}}(s_0) - \bar{V}'_{\text{uns}}(s_0)| \\ \leq & C\Big(|\bar{V}_1(s_0) - \bar{V}_1'(s_0)| + \sum_{\ell+1}^L |\bar{U}_i(s_0) - \bar{U}_i'(s_0)| + |\bar{\lambda}(s_0) - \bar{\lambda}'(s_0)| \\ & + b_1^{-(\sigma-s_c)(1+\nu)} \parallel \overline{\boldsymbol{w}}(s_0) - \overline{\boldsymbol{w}}'(s_0) \parallel_{\dot{H}^\sigma \times \dot{H}^{\sigma-1}} \\ & + b_1^{-\bar{L}-(1-\delta_0)(1+\eta)} \parallel \bar{w}^{(1)}_{s_{\bar{L}}}(s_0) - \bar{w}'^{(1)}_{s_{\bar{L}}}(s_0), \bar{w}^{(2)}_{s_{\bar{L}}-1}(s_0) - \bar{w}'^{(1)}_{s_{\bar{L}}-1}(s_0) \parallel_{L^2\times L^2} \Big), \end{array}$$

where we used again Remark 3.5.17. The previous identity is the Lipschitz aspect under lower order decomposition. To relate it to the original higher order decomposition, we use the bounds (*3.5.204*), (*3.5.205*), (*3.5.206*) and (*3.5.207*) of the previous Lemma (3.5.16), and we obtain the result of the Proposition.

□

We can now end the proof of the main Theorem 3.5.1 of this section.

Proof of Theorem (3.5.1) We let $\boldsymbol{X} := \left(\dot{H}^\sigma \cap \dot{H}^{s_L}\right) \times \left(\dot{H}^{\sigma-1} \cap \dot{H}^{s_L-1}\right)$ and $\boldsymbol{U}_0 \in \boldsymbol{X}$ be a solution leading to a type II blow up as described by Proposition 3.3.2. Without loss of generality we can assume that its scale is 1. We then write:

$$\boldsymbol{U}_0 = \tilde{\boldsymbol{Q}}_{b_0} + \boldsymbol{w}_0,$$

with $b_0 = b^e(s_0) + (\frac{U_1(s_0)}{s_0}, ..., \frac{U_L(s_0)}{s_0^L})$ according to the decomposition explained in Subsubsection 3.3.1.2.

Step 1: Flattening the non linear coordinates. Let $\boldsymbol{U}_0' \in \boldsymbol{X}$ be another initial datum. It can be written as:

$$\boldsymbol{U}_0' = \boldsymbol{U}_0 + \delta\lambda \frac{\partial}{\partial\lambda}(\tilde{\boldsymbol{Q}}_{b_0,\frac{1}{\lambda}})_{|\lambda=1} + \sum_1^L \frac{\delta U_i}{s_0^i}\frac{\partial}{\partial b_i}(\tilde{\boldsymbol{Q}}_b)_{|b=b_0} + \delta\boldsymbol{w}, \tag{3.5.220}$$

where $\delta\boldsymbol{w} \in X$ satisfies fixed orthogonality conditions at scale 1: $\langle \delta\boldsymbol{w}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle = 0$ for $0 \leq i \leq M$. We have seen that for the parameters one had stable directions of perturbation $V_1, U_{\ell+1}, ..., U_L$, unstable ones $V_2, ...V_\ell$ and that the error $\boldsymbol{w}$ was a stable perturbation. We recall the notation $V_i = \sum_1^\ell p_{i,j}U_j$. With the decomposition we just stated we can define the stable and unstable spaces of linearized directions of perturbation:

$$\begin{array}{rll} \boldsymbol{X}_s & := \Big\{ & \delta\lambda \frac{\partial}{\partial\lambda}(\tilde{\boldsymbol{Q}}_{b_0,\frac{1}{\lambda}})_{|\lambda=1} + \delta V_1 (\sum_1^\ell \frac{p_{1,j}}{s_0^j}\frac{\partial}{\partial b_j}(\tilde{\boldsymbol{Q}}_b)_{|b=b_0}) + \sum_{\ell+1}^L \frac{\delta U_i}{s_0^i}\frac{\partial}{\partial b_i}(\tilde{\boldsymbol{Q}}_b)_{|b=b_0} + \boldsymbol{w}, \\ & & (\delta\lambda, \delta V_1, \delta U_{\ell+1}, ..., \delta U_L) \in \mathbb{R}^{L-\ell+2}, \\ & & \delta\boldsymbol{w} \in \boldsymbol{X},\ \langle \delta\boldsymbol{w}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle = 0 \text{ for } 0 \leq i \leq M \Big\}, \\ \boldsymbol{X}_u & := \Big\{ & \sum_2^\ell \delta V_i (\sum_1^\ell \frac{p_{i,j}}{s_0^j}\frac{\partial}{\partial b_j}(\tilde{\boldsymbol{Q}}_b)_{|b=b_0}),\ (\delta V_2, ..., \delta V_\ell) \in \mathbb{R}^{\ell-1} \Big\}. \end{array}$$

So that we decompose in an affine way $\boldsymbol{X}=\boldsymbol{U}_0+(\boldsymbol{X}_u\oplus\boldsymbol{X}_s)$.

Step 2: From linear to adapted coordinates. To be able to use the results of Proposition 3.3.2 and Proposition 3.5.13 we consider the following mapping:

$$\begin{array}{rccc}\boldsymbol{\phi}: & \boldsymbol{X} & \to & \boldsymbol{X} \\ & \boldsymbol{U} & \mapsto & \tilde{\boldsymbol{Q}}_{b+\delta b,\frac{1}{1+\delta\lambda}}+\tilde{\boldsymbol{w}}\end{array}$$

where, using the decomposition (*3.5.220*), we define δb as $\delta b:=(\frac{\delta U_1}{s_0},...,\frac{\delta U_L}{s_0^L})$ and:

$$\tilde{\boldsymbol{w}}:=\boldsymbol{w}+\delta\boldsymbol{w}-\sum_0^L\frac{\langle\boldsymbol{w}+\delta\boldsymbol{w},(\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M)_{\frac{1}{1+\delta\lambda}}\rangle}{\langle\boldsymbol{T}_{i,\frac{1}{1+\delta\lambda}},(\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M)_{\frac{1}{1+\delta\lambda}}\rangle}\boldsymbol{T}_{i,\frac{1}{1+\delta\lambda}} \tag{3.5.221}$$

satisfies the orthogonality conditions (*3.3.9*) at the scale $\frac{1}{1+\delta\lambda}$:

$$\langle\delta\boldsymbol{w},(\boldsymbol{H}^{*i}\boldsymbol{\Phi}_M)_{\frac{1}{1+\delta\lambda}}\rangle=0\text{ for }0\leq i\leq M.$$

$\boldsymbol{\phi}$ is a C^∞ diffeomorphism that preserves $\boldsymbol{U}_0$: $\boldsymbol{\phi}(\boldsymbol{U}_0)=\boldsymbol{U}_0$.

Step 3: the Lipschitz manifold properties. Let

$$\delta\lambda\frac{\partial}{\partial\lambda}(\tilde{\boldsymbol{Q}}_{b_0,\frac{1}{\lambda}})_{|\lambda=1}+\delta V_1(\sum_1^\ell\frac{p_{1,j}}{s_0^j}\frac{\partial}{\partial b_j}(\tilde{\boldsymbol{Q}}_b)_{|b=b_0})+\sum_{\ell+1}^L\frac{\delta U_i}{s_0^i}\frac{\partial}{\partial b_i}(\tilde{\boldsymbol{Q}}_b)_{|b=b_0}+\boldsymbol{w}:=\delta\boldsymbol{U}_s\in X_s$$

be small enough. We apply the result of Proposition 3.3.2 to $\boldsymbol{\phi}(\boldsymbol{U}_0+\delta\boldsymbol{U}_s)$. There exists a choice of unstable modes $\delta V_2,...,\delta V_\ell$ such that $\tilde{\boldsymbol{U}}:=\tilde{\boldsymbol{Q}}_{b+\delta b,\frac{1}{1+\delta\lambda}}+\tilde{\boldsymbol{w}}$ is an initial datum leading to a blow up as described in this Proposition, where $\delta b:=(\frac{\delta U_1}{s_0},...,\frac{\delta U_L}{s_0^L})$, and $\delta V_i:=\sum_1^\ell p_{i,j}U_j$ for $1\leq i\leq\ell$. Moreover, from Proposition (3.5.13) the $\ell-1$-tuple $\delta V_2,...,\delta V_\ell$ is unique. We then have:

$$\boldsymbol{\phi}^{-1}(\tilde{\boldsymbol{U}})=\boldsymbol{U}_0+\delta\boldsymbol{U}_s+\delta\boldsymbol{U}_u,$$

with $\delta\boldsymbol{U}_u:=\sum_2^\ell\delta V_i(\sum_1^\ell\frac{p_{i,j}}{s_0^j}\frac{\partial}{\partial b_j}(\tilde{\boldsymbol{Q}}_b)_{|b=b_0})\in\boldsymbol{X}_u$. Let $\mathcal{O}$ be a small enough open set of $\boldsymbol{X}$ with $\boldsymbol{0}\in\mathcal{O}$. We define the application $\boldsymbol{f}$ as:

$$\begin{array}{rccc}\boldsymbol{f}: & \delta\mathcal{O}\cap\boldsymbol{X}_s & \to & \delta\boldsymbol{X}_u \\ & \boldsymbol{U}_s & \mapsto & \boldsymbol{U}_u\end{array}$$

with $\boldsymbol{X}_u$ being defined by the previous construction. For $\boldsymbol{U}_s\in\boldsymbol{X}_s\cap\mathcal{O}$, the function $\boldsymbol{\phi}(\boldsymbol{U}_0+\delta\boldsymbol{U}_s+\boldsymbol{f}(\delta\boldsymbol{U}_s))$ yields a type II blow up as described by Proposition 3.3.2. Moreover, Proposition 3.5.13 implies that $\boldsymbol{f}$ is a Lipsichitz mapping. Let $\mathcal{M}$ denote the set of initial data described by Proposition 3.3.2. We just have proved that $\boldsymbol{\phi}^{-1}(\mathcal{M}\cap(\boldsymbol{U}_0+\mathcal{O}))$ is the graph of the Lipschitz mapping $\boldsymbol{f}:\boldsymbol{X}_s\cap\mathcal{O}\to\boldsymbol{X}_u$ with $\boldsymbol{X}=\boldsymbol{X}_u\oplus\boldsymbol{X}_s$ and $\boldsymbol{X}_u$ of dimension $\ell-1$. This means that $\boldsymbol{\phi}^{-1}(\mathcal{M}\cap(\boldsymbol{U}_0+\mathcal{O}))$ is a Lipschitz manifold of codimension $\ell-1$. As $\boldsymbol{\phi}$ is a C^∞ diffeomorphism, it implies that $\mathcal{M}\cap(\boldsymbol{U}_0+\mathcal{O}))$ is a Lipschitz manifold of codimension $\ell-1$. Hence $\mathcal{M}$ is a locally Lipschitz manifold of codimension $\ell-1$ in $\boldsymbol{X}$.

□

3.A Properties of the stationnary state

We state here the fundamental decomposition for the asymptotic of the stationary state Q. These results are now standard, see [95] [67] for exemple, and see also [114] for its role in type II blow-up involving Q for the Schrödinger equation. An important fact, the non nullity of the second term in the expansion, is however not proven in these works. We therefore prove it hereafter.

Lemma 3.A.1. *(*Asymptotic expansion for the stationary state:*) We have the expansion:*

$$\partial_y^k Q(y) = \partial_y^k \left(\frac{c_\infty}{y^{\frac{2}{p-1}}} + \frac{a_1}{y^\gamma} \right) + O\left(\frac{1}{y^{\gamma+g+k}} \right) \quad \text{as } y \text{ goes to } +\infty,$$

with a_1 being a strictly negative (in particular $a_1 \neq 0$) coefficient:

$$a_1 < 0 \tag{3.A.1}$$

In [67] and references therein, the authors show the expansion, but they do not show that $a_1 \neq 0$. This appendix is devoted to prove this fact. In the chapter the authors show the following result:

Lemma 3.A.2 (Gui Ni Wang, [67], Theorem 2.5). *We recall that $0 < \alpha < \alpha_2$ are the roots of the polynomial:*

$$X^2 - \left(d - 2 - \frac{4}{p-1}\right) X + 2\left(d - 2 - \frac{2}{p-1}\right). \tag{3.A.2}$$

Then the following expansion is true.

(i) If $\frac{\alpha_2}{\alpha} \notin \mathbb{N}$, *then for all $k_1,\ k_2 \in \mathbb{N}$, as $y \to +\infty$ one has:*

$$Q(y) = \frac{c_\infty}{y^{\frac{2}{p-1}}} + \sum_{i,j=1}^{k_1,k_2} \frac{a_{i,j}}{y^{\frac{2}{p-1}+k_1\alpha+k_2\alpha_2}} + O\left(\frac{1}{y^{\frac{2}{p-1}+(k_1+1)\alpha}} \right). \tag{3.A.3}$$

(ii) If $\frac{\alpha_2}{\alpha} = k+1 \in \mathbb{N}$: *then as $y \to +\infty$ one has::*

$$Q(y) = \frac{c_\infty}{y^{\frac{2}{p-1}}} + \sum_{i=1}^{k+1} \frac{a_i}{y^{\frac{2}{p-1}+i\alpha}} + \frac{a_k \log(y) + a_k'}{y^{\frac{2}{p-1}+k\alpha}} + O\left(\frac{1}{y^{\frac{2}{p-1}+(k+1)\alpha}} \right). \tag{3.A.4}$$

As in the previous case the expansion can be continued to higher terms, but it does not matter for the analysis of the present chapter.

(iii) This expansion adapts for higher derivatives of Q.

This proves the expansion of Lemma 3.A.1. The rest of this section is devoted to the proof that a_1 is strictly negative.

Proof of the assertion (*3.A.1*)

As a consequence of the previous lemma we get that, noting $k := E[\frac{\lambda_2}{\lambda_1}]$ if $\frac{\alpha_2}{\alpha} \notin \mathbb{N}$, and $k := \frac{\alpha_2}{\alpha} - 1$ if $\frac{\alpha_2}{\alpha} \in \mathbb{N}$ we have in both cases:

$$\Lambda^{(1)} Q = \sum_{i=1}^{k} -i\alpha \frac{a_1}{y^{\frac{2}{p-1}+i\alpha}} + O\left(\frac{\log(y)}{y^{\frac{2}{p-1}+\alpha_2}} \right), \tag{3.A.5}$$

and:

$$\partial_y \Lambda^{(1)} Q = \sum_{i=1}^{k} (i\alpha) \left(\frac{2}{p-1} + i\alpha \right) \frac{a_1}{y^{\frac{2}{p-1}+i\alpha+1}} + O\left(\frac{\log(y)}{y^{\frac{2}{p-1}+\alpha_2+1}} \right). \tag{3.A.6}$$

The key point is that the coefficient a_i are linked with a reccurence relation:

Lemma 3.A.3. *For $1 \leq i \leq k$, a_i is given by $a_i = P_i(a_1)$ where P_i is a polynomial such that $P_i(0) = 0$ for all $1 \leq i \leq k$.*

This lemma is proved later. Hence we have the following alternative:

$$either\ a_1 \neq 0\ or\ \partial_y \Lambda^{(1)} Q = O\left(\frac{log(y)}{y^{\frac{2}{p-1}+\alpha_2+1}} \right). \tag{3.A.7}$$

The remainder term of (*3.A.6*) is in L^2. Indeed, we compute:

$$d - 2\frac{2}{p-1} - 2\alpha_2 - 2 = -\sqrt{\triangle} < 0.$$

So If $a_1 = 0$ then $\Lambda^{(1)} Q \in \dot{H}^1$. The term associated to a_1 is not in L^2 because $d - 2\frac{2}{p-1} - 2\alpha - 2 = \sqrt{\triangle} > 0$, see (*2.2.5*).
But we know from [78] that $\mathcal{L}$ is positive definite on $\dot{H}^1$, and that $\mathcal{L}\Lambda^{(1)} Q = 0$. We then must have $\Lambda^{(1)} Q \notin \dot{H}^1$. Considering what was said previously, this implies $a_1 \neq 0$.

We also know from [78] that $\Lambda^{(1)} Q > 0$. From the expansion (*3.A.5*) This implies that a_1 is strictly negative. □

We now give the proof of the recurrence relation between the a_i's stated in Lemma 3.A.3.

Proof of Lemma 3.A.3 We use here the ideas developped in [95]. In this chapter or in references therin, the following facts are proven:

Lemma 3.A.4 ([95] Lemmas 4.3 and 4.4). *The following holds:*

(i) the solitary wave exists and has C^∞ regularity.

(ii) $y^{\frac{2}{p-1}} Q(y)$ has a limit as $y \to +\infty$, denoted c_∞.

(iii) If we renormalise the space variable by $y = e^t$ and define:

$$Z(t) = y^{\frac{2}{p-1}} Q(y) - c_\infty. \tag{3.A.8}$$

Z then satisfies the differential equation for t large:

$$Z_{tt} + \left(d - 2 - \frac{4}{p-1} \right) + 2\left(d - 2 - \frac{2}{p-1} \right) Z + P(Z) = 0, \tag{3.A.9}$$

where P denotes the polynomial:

$$(X + c_\infty)^p - c_\infty^p - pc_\infty^{p-1} X. \tag{3.A.10}$$

(iv) Z has the following begining of expansion at infinity:

$$Z(t) = \begin{cases} a_1 e^{-\alpha t} + O(e^{-\alpha_2 t}) \text{ if } \alpha_2 < 2\alpha \\ a_1 e^{-\alpha t} + O(te^{-\alpha_2 t}) \text{ if } \alpha_2 = 2\alpha \\ a_1 e^{-\alpha t} + O(e^{-2\alpha t}) \text{ if } \alpha_2 > 2\alpha. \end{cases} \tag{3.A.11}$$

We will now compute the other coefficients of the expansion. As Z is a solution of (*3.A.9*), basic ODE theory states that there exists two coefficients a and b such that:

$$Z(t) = ae^{-\alpha t} + be^{-\alpha_2 t} + \frac{1}{\alpha_2 - \alpha}\int_{T_0}^t (e^{\alpha_2(s-t} - e^{\alpha(s-t)})P(Z)ds. \tag{3.A.12}$$

We now prove lemma 3.A.3 by iteration. Our iteration hypothesis is the following for $1 \le j \le k-1$:

$$\mathcal{H}(j): \quad \begin{array}{l} Z(t) = \sum_{i=1}^j a_i e^{-i\alpha t} + O(e^{-(j+1)\alpha t}), \text{ with } a_i = P_i(a_1), \\ P_i \text{ being a polynomial such that } P_i(0) = 0. \end{array} \tag{3.A.13}$$

<u>Initialization:</u> For $i = 1$, $a_1 = P_1(a_1)$ with $P_1 = X$ and of course $P_1(0) = 0$. Because of the preliminary expansion (iv), the property is true for $j = 1$.

<u>Heredity:</u> We now suppose it is true for $1 \le j \le k-1$. We then plug the expansion (*3.A.13*) into (*3.A.12*). It gives the following expression for Z:

$$Z(t) = ae^{-\alpha t} + \frac{1}{\alpha_2 - \alpha} + \int_{T_0}^t (e^{\alpha_2(s-t} - e^{\alpha(s-t)})P(Z)ds + O(e^{-(j+1)\alpha t}), \tag{3.A.14}$$

since $(j+1)\alpha < \alpha_2$ (because $1 \le j \le k-1$). But with the definition (*3.A.10*) of P and the hypothesis (*3.A.13*) on the a_i for $i \le j$ we have that:

$$P(Z(t)) = \sum_{i=2}^{j+1} \tilde{a}_i e^{-i\alpha t} + O(e^{-(j+2)\alpha t}),$$

where $\tilde{a_i} = \tilde{P_i}(a_1)$ with $\tilde{P_i}$ being a polynomial such that $\tilde{P_i}(0) = 0$. We now put this expression in (*3.A.14*) and compute the integral of the right hand side. For $2 \le i \le j+1$:

$$\begin{aligned} \textstyle\int_{T_0}^t (e^{\alpha_2(s-t} - e^{\alpha(s-t)})e^{-i\alpha s}ds &= \tfrac{1}{\alpha_2 - i\alpha}e^{-\alpha_2 t} - \tfrac{1}{(i-1)\alpha}e^{-\alpha t} \\ &\quad + \left(\tfrac{1}{\alpha_2 - i\alpha} + \tfrac{1}{(i-1)\alpha}\right)e^{-i\alpha t}, \end{aligned}$$

and:

$$\begin{aligned} \textstyle\int_{T_0}^t (e^{\alpha_2(s-t} - e^{\alpha(s-t)})O(e^{-(j+2)\alpha s})ds &= e^{-\alpha_2 t}\textstyle\int_{T_0}^t O(e^{(\alpha_2-(j+2)\alpha)s})ds \\ &\quad - e^{-\alpha t}\textstyle\int_{T_0}^t O(e^{-(j+1)\alpha s})ds. \end{aligned} \tag{3.A.15}$$

Since $\alpha_2 > (j+2)\alpha$ the first integral diverges, the second term is integrable. Hence:

$$\begin{aligned} \textstyle\int_{T_0}^t (e^{\alpha_2(s-t} - e^{\alpha(s-t)})O(e^{-(j+2)\alpha s})ds &= e^{-\alpha_2 t}O(\textstyle\int_{T_0}^t e^{(\alpha_2-(j+2)\alpha)s}ds) \\ &\quad - e^{-\alpha t}(\textstyle\int_{T_0}^{+\infty} O(e^{-(j+1)\alpha}) \\ &\quad - \textstyle\int_t^{+\infty} O(e^{-(j+1)\alpha}) \\ &= Ce^{-\alpha t} + O(e^{-(j+2)\alpha t}). \end{aligned}$$

So we finally get for a constant C:

$$\begin{aligned} Z(t) &= Ce^{-\alpha t} + \sum_{i=2}^{j+1} \tilde{a}_i \frac{1}{\alpha_2 - \alpha_2}\left(\frac{1}{\alpha_2 - i\alpha} - \frac{1}{-(i-1)\alpha}\right) e^{-i\alpha t} \\ &+ O(e^{-(i+2)\alpha t}). \end{aligned} \tag{3.A.16}$$

By identifying this last identity with the expansion (*3.A.13*) given by the induction hypothesis, one finds that in fact $C = a_1$ and $a_i = \tilde{a}_i$ for $i \leq j$. Therefore the property $\mathcal{H}(j)$ is true for $j+1$.

By induction, we have proved that (*3.A.13*) is valid for $j = k-1$. To finish the proof one needs to do the same computation that we did before for the case $j = k-1$

(i) If $\frac{\alpha_2}{\alpha} \neq \mathbb{N}$. Then the only things that changes is that we do not have $e^{-\alpha_2 t} = O(e^{-(k+1)\alpha t})$, so we cannot throw away the terms involving $e^{-\alpha_2 t}$ and we get:

$$Z(t) = Ce^{-\alpha t} + \sum_{i=2}^{k} \tilde{a}_i e^{-i\alpha t} + O(e^{-(k+1)\alpha t}).$$

(ii) If $\frac{\alpha_2}{\alpha}$ is an integer, and $k = \frac{\alpha_2}{\alpha} - 1$ to go from $k-1$ to k we also do the same computations as before. Now what changes is that we have a t corrective term in (*3.A.15*):

$$\int_{T_0}^{t} e^{\alpha_2(s-t)} O(e^{-(k+1)\alpha t}) = O(te^{-\alpha_2 t}).$$

which is what produces the log term in the expansion of Q in that case.

□

3.B Equivalence of norms

In this subsection we show that the notion of degree for admissible functions (see Definition 3.2.7) is equivalent for usual derivatives and adapted ones. We also show that the weighted usual Sobolev norms are equivalent, to some extent, to the weighted adapted ones.

Lemma 3.B.1. (equivalence of the degree)

Let p_2 be a real number and f a C^∞ radial function. We recall that f_k is the k-th adapted derivative defined in (3.2.21). The two following proposition are equivalents:

(i) $\forall k \geq 0$, $\partial_y^k f = O\left(\frac{1}{y^{p_2+k}}\right)$ *as* $y \to +\infty$.

(ii) $\forall k \geq 0$, $f_k = O\left(\frac{1}{y^{p_2+k}}\right)$ *as* $y \to +\infty$.

Let $a \in \mathbb{R}$. For any $u \in C^\infty_{rad}$ there holds.[27]

$$\sum_{i=0}^{k} \int_{y\geq 1} \frac{|\partial_y^i u|^2}{1+y^{2k-2i+2a}} \sim \sum_{i=0}^{k} \int_{y\geq 1} \frac{|u_i|^2}{1+y^{2k-2i+2a}}. \tag{3.B.1}$$

[27] the quantity need not be finite. By $x \sim y$ we mean here $\frac{x}{c} \leq y \leq cx$ for $c > 0$.

Proof of Lemma 3.B.1

We just show that (i) implies (ii), the other implication being similar. So we suppose:

$$f \in C^\infty_{rad}, \text{ with } \forall k \geq 0, \ f_k = O\left(\frac{1}{y^{p_2+k}}\right) \text{ as } y \to +\infty.$$

We are going to show to following property by induction: for i an integer:, for all $0 \leq j \leq i$ and $k \in \mathbb{N}$ there holds:

$$\mathcal{H}(i) \quad \partial_y^k f_j = O\left(\frac{1}{y^{p_2+j+k}}\right) \text{ for all } 0 \leq j \leq i \text{ and } k \in \mathbb{N}.$$

The property $\mathcal{H}(0)$ is obviously true from the supposition on f. Suppose now $\mathcal{H}(i)$ is true for i, and let $k \in \mathbb{N}$, suppose in addition that i is odd. Then:

$$\partial_y^k f_{i+1} = \partial_y^k (\mathcal{A}^* f_i) = \partial_y^k \left[\partial_y f_i + \left(\frac{d-1}{y} + W\right) f_i\right].$$

As $\partial_y^l \left(\frac{d-1}{y} + W\right) = O\left(\frac{1}{y^{l+1}}\right)$ the property $\mathcal{H}(i+1)$ is then true. If i is even, then replacing A^* by A leads to the same result as they have the same structure (they divide or multiply by a potential similar to y^{-1}) at infinity. We have proven that if $\mathcal{H}(i)$ is true then so is $\mathcal{H}(i+1)$. Hence we have showed the first proposition of the lemma by induction.

For the equivalence of the weighted norms away from the origin, we note that what we have just proven is the fact that for any integer i:

$$\partial_y^i f = \sum_{j=0}^{i} a_{i,j} f_j \quad \text{and} \quad f_i = \sum_{j=0}^{i} \tilde{a}_{i,j} \partial_y^j f,$$

the functions $a_{i,j}$ and $\tilde{a}_{i,j}$ being radial and C^∞ outside the origin, with $a_{i,j} = O(y^{-(i-j)})$ and $\tilde{a}_{i,j} = O(y^{-(i-j)})$ as $y \to +\infty$. This implies (*3.B.1*). $\square$

We recall that the Laplace based derivatives of a C^∞ functions are:

$$D^{2k} u := \Delta^k u, \text{ and } D^{2k+1} u := \partial_y \Delta^k u.$$

Lemma 3.B.2. (Equivalence of weighted adapted norms) *There holds for all $u \in C^\infty$ radial function and integer k:*

$$\sum_{i=0}^{k} \int \frac{u_i^2}{1+y^{2k-2i}} \sim \sum_{i=0}^{k} \int \frac{|D^i u|^2}{1+y^{2k-2i}}. \tag{3.B.2}$$

Proof of Lemma 3.B.2 step 1: *Leibnitz rule.* Let f and u be C^∞ radial, with:

$$\partial_y^k f = O\left(y^{a-k}\right) \text{ as } y \to +\infty,$$

for some real number a. We will show the following property by induction: for any integer i:

$$\mathcal{H}(i): \quad (fu)_i = \sum_{j=0}^{i} V_{i,j}(f) u_j,$$

$V_{i,j}(f) \in C^\infty$ depending just on f, with $\partial_y^k V_{i,j}(f) \sim y^{a-(j-i)-k}$, and with the regularity $\frac{V_{i,j}(f)}{y} \in C^\infty$ for $i-j$ odd.

he property $\mathcal{H}(0)$ is obviously true. Suppose now it is true for i odd:

$$\begin{array}{rcl} (fu)_{i+1} & = & A^*((fu)_i) = \sum_{j=0,\ j \text{ even}}^i A^*(V_{i,j}u_j) + \sum_{j=0,\ j \text{ odd}}^i A^*(V_{i,j}u_j) \\ & = & \sum_{j=0,\ j \text{ even}}^i \left(-A + 2W + \frac{d-1}{y}\right)(V_{i,j}u_{j+1}) \\ & & + \sum_{j=0,\ j \text{ odd}}^i \partial_y V_{i,j}u_j + V_{i,j}u_{j+1} \\ & = & \sum_{j=0,\ j \text{ even}}^i V_{i,j}u_{j+1} + \left(\partial_y V_{i,j} + 2WV_{i,j} + \frac{(d-1)V_{i,j}}{y}\right)u_j \\ & & + \sum_{j=0,\ j \text{ odd}}^i \partial_y V_{i,j}u_j + V_{i,j}u_{j+1} \\ & = & \sum_{j=0,\ (i+1-j)\text{even}} \left(\partial_y V_{i,j} + 2WV_{i,j} + \frac{d-1}{y}V_{i,j} + V_{i,j-1}\right)u_j \\ & & + \sum_{j=0,\ (i+1-j)\text{odd}} \partial_y V_{i,j}u_j + V_{i,j-1}u_j. \end{array}$$

For the terms in the first sum we have: $\partial_y V_{i,j} + 2WV_{i,j} + \frac{d-1}{y}V_{i,j} + V_{i,j-1} \in C^\infty$ because of the property for i, and it satisfies the decay propriety:

$$\partial_y^k \left(\partial_y V_{i,j} + 2WV_{i,j} + \frac{d-1}{y}V_{i,j} + V_{i,j-1}\right) = O\left(y^{a-(i+1-j)-k}\right).$$

For the second one the asymptotic property is also true from the induction hypothesis $\mathcal{H}(i)$, and we have indeed: $\frac{1}{y}(\partial_y(V_{i,j}) + V_{i,j-1}) \in C^\infty$. We have showed that if $\mathcal{H}(i)$ is true for i odd, then $\mathcal{H}(i+1)$ is true. For i even a similar reasoning gives also that $\mathcal{H}(i)$ implies $\mathcal{H}(i+1)$. Consequently, the proprerty $\mathcal{H}(i)$ holds for all $i \in \mathbb{N}$.

Step 2: ***passing from one derivation to the other:*** We now claim that for any integer i another property holds:

$$\mathcal{H}'(i) \quad D^i u = \sum_{j=0}^i \tilde{V_{i,j}}u_j,$$

with $V_{i,j} \in C^\infty$ satisfaying $\partial_y^k V_{i,j} \sim y^{-(i-j)-k}$, and for $j-i$ odd $\frac{1}{y}\tilde{V}_{i,j} \in C^\infty$. We show this property also by induction. It is true for $i = 0, 1, 2$. Suppose now it is true for $i \geq 2$. Suppose i even, then:

$$\begin{array}{rcl} D^{i+1}u & = & \partial_y(D^i u) = \sum_{j=0,\ j \text{ even}}^i (-A + W)(V_{i,j}u_j) \\ & & + \sum_{j=0,\ j \text{ odd}}^i (A^* - W - \frac{d-1}{y})(V_{i,j}u_j) \\ & = & \sum_{j=0,\ j \text{ even}}^i -V_{i,j}u_{j+1} + \partial_y V_{i,j}u_j \\ & & + \sum_{j=0,\ j \text{ odd}}^i V_{i,j}u_{j+1} + (\partial_y V_{i,j} - WV_{i,j} - \frac{d-1}{y}V_{i,j})u_j. \end{array}$$

The asymptotic behavior of the potentials is easily checked from the induction hypothesis. For $i+1-j$ odd we have: $\tilde{V}_{i+1,j} = \partial_y V_{i,j} + V_{i,j-1}$, which verifies indeed $\frac{1}{y}\tilde{V}_{i+1,j} \in C^\infty$ from the induction hypothesis $\mathcal{H}'(i)$. Hence $\mathcal{H}'(i+1)$ is true. We have shown $\mathcal{H}(i)$ implies $\mathcal{H}'(i+1)$ for i even and claim that for i odd a very similar proof shows the heredity. Therefore, the propriety $\mathcal{H}'(i)$ is true for any integer i

This implies:

$$\int |D^i u|^2 \leq C \sum_{j=0}^i \int \frac{u_j^2}{1+y^{2(i-j)}},$$

which implies the control of the Laplace derivatives by adapted derivatives in the Lemma. The other inequality of the equivalence can be proved exactly the same way. The opposite formula holds indeed also:

$$u_i=\sum_{j=0}^{i}\tilde{V}'_{i,j}D^ju,$$

with $\tilde{V}'_{i,j}\in C^\infty$, $\partial_y^k\tilde{V}'_{i,j}\sim y^{-(i-j)-k}$ and $\frac{1}{y}\tilde{V}'_{i,j}\in C^\infty$ if $i-j$ odd. The proof is left to the reader.

$\square$

3.C Hardy inequalities

In this subsection we recall the standard Hardy estimates we used in the chapter, in order to make this chapter self contained. We use them to prove Hardy type estimates for the adapted norms, see next subsection. These analysis results, used to relate a norm that is adapted to a linear flow to the standard L^2 norms for usual derivatives, is now used in a canonical way in some works about blow-up, see for exemple [138] in a more subtle critical setting, [114] in a supercritical setting.

Lemma 3.C.1. (Hardy inequality with best constant)

(i) Hardy near the origin: *Let* $u\in\cap_{0<r<1}H^1(\mathcal{C}(r,1))$, *then:*[28]

$$\int_{y\leq 1}|\partial_y u|^2y^{d-1}dy\geq\frac{(d-2)^2}{4}\int_{y\leq 1}\frac{u^2}{y^2}y^{d-1}dy-C(d)u^2(1). \quad (3.C.1)$$

(ii) Hardy away from the origin, non critical exponent: *Let* $p>0$, $p\neq\frac{d-2}{2}$, *and* $u\in\cap_{1<R}H^1(\mathcal{C}(1,R))$. *If* p *is supercritical,* $p>\frac{d-2}{2}$ *then:*[29]

$$\int_{y\geq 1}\frac{|\partial_y u|^2}{y^{2p}}y^{d-1}dy\geq\left(\frac{d-(2p+2)}{2}\right)^2\int_{y\geq 1}\frac{u^2}{y^{2p+2}}y^{d-1}dy-C(d,p)u^2(1), \quad (3.C.2)$$

$$\frac{(2p+2-d)^2}{4}\int_1^R\frac{u^2}{y^{2p+2}}y^{d-1}dy\leq\int_1^R\frac{|\partial_y u|^2}{y^{2p}}y^{d-1}dy+C(d,p)u^2(1). \quad (3.C.3)$$

If p *is subcritical,* $0<p<\frac{d-2}{2}$, *if:*[30]

$$\int_{y\geq 1}\frac{|u|^2}{y^{2p+2}}y^{d-1}dy<+\infty, \quad (3.C.4)$$

then:

$$\int_{y\geq 1}\frac{|\partial_y u|^2}{y^{2p}}y^{d-1}dy\geq\left(\frac{d-(2p+2)}{2}\right)^2\int_{y\geq 1}\frac{u^2}{y^{2p+2}}y^{d-1}dy. \quad (3.C.5)$$

Proof of Lemma 3.C.1

A proof of this lemma can be found in [114].

$\square$

[28] Note that the quantities can be infinite.
[29] Note that the quantities can be infinite.
[30] we need integrability this time, a constant function violates this rule for example.

We now state a useful refined version of Hardy inequality for arbitrary weight function and number of derivatives. We denote by $x := (x_1, ..., x_d)$ an element $x \in \mathbb{R}^d$. We introduce a notation for the partial derivatives of a function:

$$\partial^\kappa f = \frac{\partial f}{\partial_{x_1}^{\kappa_1} ... \partial_{x_d}^{\kappa_d}} \tag{3.C.6}$$

for a d-tuple $\kappa := (\kappa_1, ..., \kappa_d)$ with $|\kappa|_1 = \sum_{i=1}^d \kappa_i$.

Lemma 3.C.2. (Weighted Fractional Hardy :) *Let:*

$$0 < \nu < 1, \ k \in \mathbb{N} \text{ and } 0 < \alpha \text{ satisfying } \alpha + \nu + k < \frac{d}{2},$$

and let f be a smooth function with decay estimates:

$$|\partial^\kappa f(x)| \leq \frac{C(f)}{1 + |x|^{\alpha+i}}, \text{ for } |\kappa|_1 = i, \ i = 0, 1, ..., k+1, \tag{3.C.7}$$

then for $\varepsilon \in \dot{H}^{\alpha+k+\nu}$, there holds $\varepsilon f \in \dot{H}^{\nu+k}$ with:

$$\| \nabla^{\nu+k}(\varepsilon f) \|_{L^2} \leq C(C(f), \nu, k, \alpha, d) \| \nabla^{\alpha+k+\nu} \varepsilon \|_{L^2} . \tag{3.C.8}$$

If f is a smooth radial function satisfying:

$$|\partial_{|x|}^i f(|x|)| \leq \frac{C(f)}{1 + |x|^{\alpha+i}}, \ i = 0, 1, ..., k+1, \tag{3.C.9}$$

then (3.C.8) holds.

Proof of Lemma 3.C.2 We first proove for f satisfying the non radial condition (*3.C.7*), and show after that for a radial function, this condition is equivalent to (*3.C.9*) the radial condition mentionned in the Lemma.

Step 1: case for $k = 0$. A proof of the case $k = 0$ can be found in [114] for example.

Step 2: Proof for $k \geq 1$. Let f, ε, α, ν and k satisfying the conditions of the lemma, with $k \geq 1$. Using Liebnitz rule for the integer part of the derivation:

$$\| \nabla^{\nu+k}(\varepsilon f) \|_{L^2}^2 \leq C \sum_{(\kappa, \tilde{\kappa}), |\kappa||_1 + |\tilde{\kappa}|_1 = k} \| \nabla^\nu (\partial^{\kappa_k} \varepsilon \partial^{\tilde{\kappa}_k} f \|_{L^2}^2 \tag{3.C.10}$$

We can now apply the result obtained for $k = 0$ to the norms $\| \nabla^\nu (\partial^{\kappa_k} \varepsilon \partial^{\tilde{\kappa}_k} f \|_{L^2}^2$ in (*3.C.10*). We have indeed that $\partial^{\kappa_k} \varepsilon \in \dot{H}^{\alpha+k_2+\nu}$, and that $\partial^{\tilde{\kappa}}$ satisfyies the decay property from (*3.C.7*). It implies that for all $\kappa, \tilde{\kappa}$:

$$\| \nabla^\nu (\partial^{\kappa_k} \varepsilon \partial^{\tilde{\kappa}_k} f \|_{L^2}^2 \leq C \| \nabla^{\nu+\alpha+k} \varepsilon \|_{L^2}^2$$

which implies the result: $\| \nabla^{\nu+k}(\varepsilon f) \|_{L^2}^2 \leq C(C(f), \nu, d, k, \alpha) \| \nabla^{\nu+\alpha+k} \varepsilon \|_{L^2}^2$.

Step 3: equivalence between the decay properties. We want to show that (*3.C.7*) and (*3.C.9*) are equivalents for radial smooth functions, therefore implying the last assertion of the lemma. Suppose that f is smooth, radial, and satisfies (*3.C.7*). Then one has:

$$\partial_y^i f(y) = \frac{\partial f}{\partial_{x_1}^i}(|y| e_1)$$

where e_1 stands for the unit vector $(1,...,0)$ of $\mathbb{R}^d$. From this formula, we see that the condition (*3.C.7*) on $\frac{\partial f}{\partial^i_{x_1}}(|y|e_1)$ implies the radial condition (*3.C.9*). We now suppose that f is a smooth radial function satisfying the radial condition (*3.C.9*). Then there exists a smooth radial function ϕ such that:

$$f(y)=\phi(y^2).$$

With a proof by iteration left to the reader one has that the decay property (*3.C.9*) for f implies the following decay property for ϕ:

$$|\partial_y^i\phi(y)|\leq \frac{C(f)}{1+y^{\frac{\alpha}{2}+i}},\ i=0,1,...,k+1,$$

Now the standard derivatives of f are easier to compute with ϕ. We claim that for all d-tuple κ there exists a finite number of polynomials $P_i(x):=C_ix_1^{i_1}...x_d^{i_d}$, for $1\leq i\leq l(\kappa)$, such that:

$$\partial^\kappa f(x)=\sum_{i=1}^{l(\kappa)}P_i(x)\partial_{|x|}^{q(i)}\phi(|x|^2)$$

with for all i, $2q(i)-\sum_{j=1}^d i_j=|\kappa|_1$. This fact is also left to the reader. The decay property for ϕ then implies:

$$|P_i(x)\partial_{|x|}^{q(i)}\phi(|x|^2)|\leq \frac{C}{1+y^{\alpha+2q(i)-\sum_{j=1}^d i_j}}=\frac{C}{1+y^{\alpha+|\kappa|_1}},$$

which implies the property (*3.C.7*). $\square$

3.D Coercivity of the adapted norms

Here we prove Hardy type inequalities for the operators A, A^* and $\mathcal{L}$. Such quantities are easier to manipulate for the linear flow of the operator $\boldsymbol{H}$ (defined in (*3.1.13*)). As for the previous section of the Appendix, this kind of bounds is now standard and we refer to the papers quoted therein for the use of similar techniques. We start with A^*, then A, and after that we are able to deal with the coercivity of the adapted norms.

We recall that the profile $\boldsymbol{\Phi}_M$ is defined by equation (*3.3.3*). Its main properties that we will use in this section are its localization on the first coordinate and its non-orthogonality with respect to $\boldsymbol{\Lambda Q}$ (from (*3.3.5*) and (*3.3.6*)):

$$\boldsymbol{\Phi}_M=\begin{pmatrix}\Phi_M\\0\end{pmatrix},\ \langle\boldsymbol{\Phi}_M,\boldsymbol{\Lambda Q}\rangle=\langle\Phi_m,\Lambda^{(1)}Q\rangle\sim CM^{2k_0+2\delta_0}>0\ (C>0). \tag{3.D.1}$$

We also recall the structure of the two first order differential operators on radial functions A and A^*:

$$A^*=\partial_y+\left(\frac{d-1}{y}+W\right),\ A=-\partial_y+W, \tag{3.D.2}$$

where W is a smooth radial function with the asymptotic at infinity from (*3.2.11*):

$$W=\frac{-\gamma}{y}+O\left(\frac{1}{y^{1+g}}\right)\ \text{as}\ y\ \to+\infty \tag{3.D.3}$$

Lemma 3.D.1. (Weighted coercivity for A^*). *Let p be a non negative real number. Then there exists a constant $c_p > 0$ such that for all radial $u \in H^1_{loc}(\mathbb{R}^d)$ there holds*[31]*:*

$$\int \frac{|A^*u|^2}{1+y^{2p}} \geq c_p \left[\int \frac{u^2}{y^2(1+y^{2p})} + \int \frac{|\partial_y u|^2}{1+y^{2p}} \right]. \tag{3.D.4}$$

Proof of Lemma 3.D.1 We take u satisfying the conditions of the lemma.

Step 1: *Subcoercivity for A^**. We claim the subcoercivity lower bound:

$$\begin{array}{rcl} \int \frac{|A^*u|^2}{1+y^{2p}} & \geq & c\left[\int \frac{u^2}{y^2(1+y^{2p})} + \int \frac{|\partial_y u|^2}{1+y^{2p}} \right] \\ & & -\frac{1}{c}\left[u^2(1) + \int \frac{u^2}{1+y^{2p+g}} \right], \end{array} \tag{3.D.5}$$

for a universal constant $c = c(d,p) > 0$. We introduce the operator: $\tilde{W} := W + \frac{d-1}{y}$. First we estimate close to the origin:

$$\begin{array}{rcl} \int_{y\leq 1} |A^*u|^2 & = & \int_{y\leq 1} (|\partial_y u|^2 + \tilde{W}^2 u^2 + 2\tilde{W} u \partial_y u) \\ & = & \int_{y\leq 1} |\partial_y u|^2 + \int_{y\leq 1} u^2 \left(\tilde{W}^2 - \frac{1}{y^{d-1}}(y^{d-1}\tilde{W}) \right) + W(1)^2 u(1)^2 \\ & \geq & \int_{y\leq 1} |\partial_y u|^2 + \int_{y\leq 1} u^2 (\frac{(d-1)^2-(d-1)(d-2)}{y^2} + O(1)) \\ & = & \int_{y\leq 1} |\partial_y u|^2 + (d-1) \int_{y\leq 1} \frac{u^2}{y^2} + O(\int_{y\leq 1} u^2). \end{array} \tag{3.D.6}$$

Away from the origin, from the asymptotic (*3.D.3*):

$$\begin{array}{rcl} \int_1^R \frac{|A^*u|^2}{y^{2p}} & = & \int_1^R \frac{1}{y^{2p}} (\partial_y u + \frac{d-1-\gamma}{y} u + O(\frac{1}{y^{1+g}}) u)^2 \\ & = & \int_1^R \frac{1}{y^{2p}} [\partial_y u + \frac{d-1-\gamma}{y} u]^2 + \int_1^R uO\left(\frac{1}{y^{2p+1+g}}\right)\left(\partial_y u + uO\left(\frac{1}{y}\right)\right) \\ & = & \int_1^R \frac{1}{y^{2p+2(d-1-\gamma)}} |\partial_y (y^{d-1-\gamma} u)|^2 + \int_1^R uO\left(\frac{1}{y^{2p+1+g}}\right)\left(\partial_y u + uO\left(\frac{1}{y}\right)\right). \end{array} \tag{3.D.7}$$

Let $v = y^{d-1-\gamma}u$, and $p' = p+d-1-\gamma$. We have: $2p'-(d-2) = 2p+d-2\gamma > 0$. Hence we can apply the identity (*3.C.3*):

$$\begin{array}{rcl} \int_1^R \frac{1}{y^{2p+2(d-1-\gamma)}} |\partial_y (y^{d-1-\gamma} u)|^2 & = & \int_1^R \frac{1}{y^{2p'}} |\partial_y v|^2 \geq C(d,p) \int_1^R \frac{v^2}{y^{2p'+2}} - C' v^2(1) \\ & = & \int_1^R \frac{u^2}{y^{2p+2}} - C' u^2(1). \end{array}$$

We have by developing the expression, using Cauchy Schwarz and Young's inequality:

$$\begin{array}{rcl} \int_1^R \frac{1}{y^{2p+2(d-1-\gamma)}} |\partial_y (y^{d-1-\gamma} u)|^2 & \geq & \int_1^R \frac{|\partial_y u|^2}{y^{2p}} + C \frac{u^2}{y^{2p+2}} - C' \left(\int_1^R \frac{|\partial_y u|^2}{y^{2p}} \right)^{\frac{1}{2}} \left(\int_1^R \frac{u^2}{y^{2p+2}} \right)^{\frac{1}{2}} \\ & \geq & (1 - \frac{\epsilon}{2} C') \int_1^R \frac{|\partial_y u|^2}{y^{2p}} + (C - \frac{C'}{2\epsilon}) \int_1^R \frac{u^2}{y^{2p+2}}. \end{array}$$

Combining the last two estimates gives:

$$\int_1^R \frac{1}{y^{2p+2(d-1-\gamma)}} |\partial_y (y^{d-1-\gamma} u|^2 \geq c \left(\int_1^R \frac{u^2}{y^{2p+2}} + \int_1^R \frac{|\partial_y u|^2}{y^{2p}} \right) - C' u^2(1), \tag{3.D.8}$$

for a constant $c > 0$. We come back to (*3.D.7*) and inject the bound (*3.D.8*), it yields:

$$\begin{array}{rcl} \int_1^R \frac{|A^*u|^2}{y^{2p}} & \geq & c \left(\int_1^R \frac{u^2}{y^{2p+2}} + \int_1^R \frac{|\partial_y u|^2}{y^{2p}} \right) - \frac{1}{c} u^2(1) \\ & & + \int_1^R uO\left(\frac{1}{y^{2p+1+g}}\right)\left(\partial_y u + uO\left(\frac{1}{y}\right)\right). \end{array} \tag{3.D.9}$$

[31] The quantities need not be finite.

We now use Cauchy-Schwarz and Young inequalities on better decaying term:

$$\begin{array}{rl} & \left| \int_1^R uO\left(\frac{1}{y^{2p+1+g}}\right)\left(\partial_y u + uO\left(\frac{1}{y}\right)\right)\right| \\ \leq & C\epsilon \int_1^R \frac{|\partial_y u|^2}{y^{2p}} + \frac{C}{\epsilon}\int_1^R \frac{|u|^2}{y^{2p+2+2g}} + C\int_1^R \frac{|u|^2}{y^{2p+2+g}}. \end{array}$$

Taking ϵ small enough and combining this bound with (*3.D.9*) gives for a constant $c>0$:

$$\int_1^R \frac{|A^*u|^2}{y^{2p}} \geq c\left(\int_1^R \frac{u^2}{y^{2p+2}} + \int_1^R \frac{|\partial_y u|^2}{y^{2p}}\right) - \frac{1}{c}\left(u^2(1) + \int_1^R \frac{u^2}{y^{2p+2+g}}\right)$$

Because of the additional decay in the last term we have that if $\frac{u^2}{y^{2p+2}}$ or $\frac{|\partial_y u|^2}{y^{2p}}$ is non integrable at infinity, then going to the limit $R\to 0$ gives that $\frac{|A^*u|^2}{y^{2p}}$ is non integrable. Therefore in that case all quantities in (*3.D.4*) are infinite and the inequality is proven. Now, if they are integrable, then going to the limit $R\to+\infty$ in the last inequality and combining it with the estimate close to the origin (*3.D.6*)we proved earlier gives the subcoercivity bound (*3.D.5*).

Step 2: Coercivity. We argue by contradiction. We suppose that there exists a sequence of functions $(u_n)_{n\in\mathbb{N}}$ such that, up to a renormalization:

$$\int \frac{|A^*u|^2}{1+y^{2p}} \leq \frac{1}{n}, \quad \text{and} \int \frac{u^2}{y^2(1+y^{2p})} + \int \frac{|\partial_y u|^2}{1+y^{2p}} = 1 \tag{3.D.10}$$

From the subcoercivity estimate (*3.D.5*) it implies that:

$$u_n(1)^2 + \int \frac{u_n^2}{1+y^{2p+2+g}} \gtrsim 1.$$

And by (*3.D.10*) we have that u_n is uniformly bounded in $H^1[r,R]$. Hence by compacity and by an extraction argument there exists a limit profile $u_\infty\in H^1_{loc}$ such that up to a subsequence,

$$u_n \rightharpoonup u_\infty \text{ in } H^1_{\text{loc}}.$$

From continuity of functions in H^1 in one dimension, and from compactness of the injection $H^1\hookrightarrow L^2$ on compact sets we have also:

$$u_n\to u_\infty \text{ in } L^2_{\text{loc}},\ u_n(1)\to u_\infty(1).$$

We now show that $u_\infty\neq 0$. We have that $u_n^2(1)\to u_\infty^2(1)$. Indeed the continuity of the H^1_{loc} functions in 1 dimension, the strong convergence L^2 and of the equi-continuity of the family $\{u_n\}$ implies the convergence in L^∞. If $u_\infty^2(1)\neq 0$, then $u_\infty\neq 0$. If $u_\infty(1)=0$ then the subcoercivity bound implies that $\int \frac{u_n^2}{1+y^{2p+2+g}}\gtrsim 1$. The local L^2 convergence, and the fact that $\int \frac{u_n^2}{y^2(1+y^{2p})}$ is uniformly bounded implies that:

$$\int \frac{u_n^2}{1+y^{2p+2+g}} \to \int \frac{u_\infty^2}{1+y^{2p+2+g}}.$$

Hence $\int \frac{u_\infty^2}{1+y^{2p+2+g}}>0$ so $u_\infty\neq 0$. In any cases we have found: $u_\infty\neq 0$. On the other hand from semi-continuity again we have that:

$$A^*u_\infty = 0.$$

This equation has for unique solution in H^1 the function Γ up to multiplication by a scalar. Hence:

$$u_\infty = c\Gamma.$$

c is non zero because u_∞ is non zero. But:

$$\int_{y\leq 1}\frac{\Gamma^2}{y^2} \gtrsim \int_{y\leq 1}\frac{y^{d-1}}{y^{2(d-2)+2}}dy = +\infty,$$

which contradicts (*3.D.11*). □

We now focus on the coercivity of the operator A.

Lemma 3.D.2. (Weighted coercivity for A:) *Let p be a non negative real number. Let k_0 and δ_0 be defined by* (*3.1.1*) *($\delta_0>0$)). Then:*

(i) case p small: *if $0\leq p<k_0+\delta_0-1$, then there exists a constant $c_p>0$ such that for all $u\in H^1_{rad,loc}(\mathbb{R}^d)$ satisfying:*

$$\int_{y\geq 1}\frac{u^2}{y^{2p+2}}<+\infty, \tag{3.D.11}$$

there holds the coercivity:[32]

$$\int\frac{|Au|^2}{1+y^{2p}} \geq c_k\left[\int\frac{|\partial_y u|^2}{1+y^{2p}}+\frac{u^2}{y^2(1+y^{2p})}\right]. \tag{3.D.12}$$

(ii) case p large: *let $p>k_0+\delta_0-1$, let M be large enough (depending on d and p only), then there exists $c_{M,p}>0$ such that if $u\in H^1_{rad,loc}$ satisfies:*

$$\langle u,\Phi_M\rangle = 0. \tag{3.D.13}$$

then:[33]

$$\int\frac{|Au|^2}{1+y^{2p}} \geq c_{M,p}\left[\int\frac{|\partial_y u|^2}{1+y^{2p}}+\frac{u^2}{y^2(1+y^{2p})}\right]. \tag{3.D.14}$$

Proof of Lemma 3.D.2 As for A^* we first show a subcoercivity bound and then show that if we want to violate the Hardy type inequality, one must get closer and closer to the zero of A which is $\Lambda^{(1)}Q$, but this is impossible due to integrability conditions in the case p small and due to the orthogonality condition for the case p large.

Step 1: *subcoercivity*. Let $p\geq 0$. Then we claim that if u satisfies (*3.D.11*):

$$\int\frac{|Au|^2}{1+y^{2p}} \geq c\left[\int\frac{|\partial_y u|^2}{1+y^{2p}}+\frac{u^2}{y^2(1+y^{2p})}\right]-\frac{1}{c}\left[u^2(1)+\int\frac{u^2}{1+y^{2p+2+g}}\right], \tag{3.D.15}$$

for a universal constant $c>0$. We start by computing close to the origin using (*3.D.2*), with the help of the Hardy inequality close to the origin (3.C.1):

$$\begin{array}{rcl}\int_{y\leq 1}|Au|^2 & = & \int_{y\leq 1}|\partial_y u|^2+\int_{y\leq 1}O(u^2)+\int u\partial_y uO(1)\\ & \geq & c\left(\int_{y\leq 1}|\partial_y u|^2+\frac{u^2}{y^2}\right)-\frac{1}{c}\left(u^2(1)+\int_{y\leq 1}u^2\right)+\int u\partial_y uO(1).\end{array}$$

[32] the quantities in the coercivity estimate need not be finite.

[33] idem.

We apply Cauchy-Schwarz and Young inequality to control the last term:

$$\Big|\int u\partial_y u O(1)| \leq \epsilon C \int_{y\leq 1}\Big|\partial_y u|^2 + \frac{C}{\epsilon}\int_{y\leq 1} u^2.$$

Taking ϵ small enough gives close to the origin:

$$\int_{y\leq 1} |Au|^2 \geq c\left(\int_{y\leq 1} |\partial_y u|^2 + \frac{u^2}{y^2}\right) - \frac{1}{c}\left(u^2(1) + \int_{y\leq 1} u^2\right). \tag{3.D.16}$$

Away from the origin, we use the asymptotics (*3.D.3*) of the potential W to compute:

$$\begin{array}{rcl} \int_1^R \frac{|Au|^2}{y^{2p}} & = & \int_2^R \frac{1}{y^{2p}}\left[\partial_y u + \frac{\gamma}{y}u + O\left(\frac{u^2}{y^{1+g}}\right)\right]^2 \\ & = & \int_1^R \frac{1}{y^{2p}}\left[\partial_y u + \frac{\gamma}{y}u\right]^2 + \int_1^R O\left(\frac{u}{y^{2p+1+g}}\right)\left(\partial_y u + uO\left(\frac{1}{y}\right)\right). \end{array} \tag{3.D.17}$$

This time we let $v = y^\gamma u$, and $2p' = 2p + 2\gamma$. We observe: $2p' - (d-2) = 2p - 2k_0 + 2 - 2\delta_0 < 0$ in the case p small and > 0 in the case p large. For p small we have from (*3.C.5*):

$$\begin{array}{rcl} \int_1^R \frac{1}{y^{2p}}\left[\partial_y u + \frac{\gamma}{y}u\right]^2 = \int_1^R \frac{|\partial_y v|^2}{y^{2p'}} & \geq & c\int_1^R \frac{v^2}{y^{2p'+2}} - \frac{R^{d-2p'-2}}{d-2-2p'}v^2(R) \\ & = & c\int_1^R \frac{u^2}{y^{2p+2}} - \frac{R^{d-2-2k}}{d-2-2p}u^2(R). \end{array} \tag{3.D.18}$$

As we did in the proof of the sub-coercivity estimate for A^*, the identity (*3.D.17*) and the control (*3.D.18*) imply using Cauchy-Schwarz and Young inequality:

$$\int_1^R \frac{|Au|^2}{y^{2p}} \geq c'\left(\int_1^R \frac{u^2}{y^{2p+2}} + \frac{|\partial_y u|^2}{y^{2p}}\right) - \frac{1}{c}\left(\frac{R^{d-2-2p}}{d-2-2p}u^2(R) + \int_1^R \frac{u^2}{y^{2p+2+g}}\right).$$

The integrability condition (*3.D.11*) gives that along a sequence R_n the $u(R_n)$ term goes to zero. This allow us to conclude that if $\frac{|\partial_y u|^2}{y^{2p}}$ is not integrable, then $\frac{|A^* u|^2}{y^{2p}}$ is not integrable neither. This gives the Hardy inequality in the case the quantities are infinite. We can now suppose that the involved quantities are finite. We go to the limit in the previous equation along R_n and combine it with (*3.D.16*) to obtain the subcoercivity estimate.

For p large we are in the supercritical case in the standard Hardy inequality for v. We can do verbatim the same reasoning we did for the proof of the subcoercivity estimate for A^*.

Step 2: *Coercivity.* We argue by contradiction. If the hardy inequality we want to show was wrong, there would exist a sequence $(u_n)_{n\in\mathbb{N}}$, such that:

$$\int \frac{|\partial_y u_n|^2}{1+y^{2p}} + \frac{u_n^2}{y^2(1+y^{2p})} = 1, \ \int \frac{|Au|^2}{1+y^{2p}} \to 0.$$

From the subcoercivity estimate implies:

$$u_n^2(1) + \int \frac{u_n^2}{1+y^{2p+2+g}} \gtrsim 1,$$

and $u_n \rightharpoonup u_\infty$ in $H^1_{\text{loc}}(]0,+\infty[)$. The quantities go the same way to the limit and we find that u_∞ is not zero and must satisfy:

$$Au = 0.$$

This implies $u_\infty = c\Lambda^{(1)}Q$, $c \neq 0$.

If $k \geq k_0$ then the orthogonality condition goes to the limit with the weak topology and we find $\langle u_\infty, \Phi_M\rangle = 0$ which violates (*3.D.1*). If $k \leq k_0 - 1$, we have from lower semi continuity that:

$$\int \frac{u_\infty^2}{1+y^{2p+2}} < +\infty,$$

but $\Lambda^{(1)}Q$ does not satisfy this inequality because as $-2\gamma - 2p - 2 + d = 2(k_0 - p) - 2(1-\delta_0) > 0$ we have:

$$\int \frac{\Lambda^{(1)}Q^2}{1+y^{2p+2}} = +\infty.$$

In both cases there is a contradiction. Hence the lemma are proven. □

Once the coercivity properties of A and A^* have been established, we can turn to the core of this part: the coercivity estimates for the adapted norms provided some orthogonality conditions are satisfied.

Lemma 3.D.3 (Coercivity of $\mathcal{E}_k$). *We still assume $\delta_0 \neq 0$. k denotes an integer. We recall that u_j, the j-th adapted derivative of u, is defined in* (*3.2.21*).

(i) case k small *Let $0 \leq k \leq k_0$ and $0 \leq \delta < \delta_0$. Then there exists a constant $c_{k,\delta} > 0$ such that for all $u \in H^k_{rad,loc}(\mathbb{R}^d)$ satisfying:*

$$\sum_{p=0}^{k} \int \frac{u_p^2}{1+y^{2k-2p}} < +\infty, \tag{3.D.19}$$

there holds:

$$\int \frac{u_k^2}{1+y^{2\delta}} \geq c_k \sum_{p=0}^{k-1} \int \frac{u_p^2}{1+y^{2k-2p+2\delta}}. \tag{3.D.20}$$

(ii) case k large *Let $k \geq k_0 + 1$ and $0 \leq \delta < \delta_0$, let $j = E(\frac{k-k_0}{2})$. Then for $M = M(k)$ large enough, there exists $c_{M,k} > 0$ such that for all $H^k_{loc,rad}(\mathbb{R}^d)$ satisfying:*

$$\sum_{p=0}^{k} \int \frac{u_p^2}{1+y^{2k-2}} < +\infty \text{ and } \langle u, \mathcal{L}^p \Phi_M\rangle = 0, \text{ for } 0 \leq p \leq j-1, \tag{3.D.21}$$

there holds:

$$\int \frac{u_k^2}{1+y^{2\delta}} \geq c_{M,k} \sum_{p=0}^{k-1} \int \frac{u_p^2}{1+y^{2k-2p+2\delta}}. \tag{3.D.22}$$

Corollary 3.D.4 (Coercivity of $\mathcal{E}_{s_L}$). *Let L and σ be defined by* (*3.2.37*) *and* (*3.3.13*) *(L is odd) and $0 \leq \delta < \delta_0$. Then there exists a constant $c > 0$ such that for all radial $\boldsymbol{\varepsilon} \in \dot{H}^{s_L} \times \dot{H}^{s_L - 1} \cap \dot{H}^\sigma \times \dot{H}^{\sigma-1}$ satisfying:*

$$\langle \boldsymbol{\varepsilon}, \boldsymbol{H}^{*i}\boldsymbol{\Phi}_M\rangle = 0 \text{ for } 0 \leq i \leq L, \tag{3.D.23}$$

there holds:

$$\sum_{p=0}^{s_L-1} \int \frac{|\varepsilon_p^{(1)}|^2}{1+y^{2s_L-2p+2\delta}} + \sum_{p=0}^{s_L-2} \int \frac{|\varepsilon_p^{(2)}|^2}{1+y^{2s_L-2-2p+2\delta}} \leq c\left(\int \frac{|\varepsilon_{s_L}^{(1)}|^2}{1+y^{2\delta}} + \int \frac{|\varepsilon_{s_L-1}^{(2)}|^2}{1+y^{2\delta}}\right) \tag{3.D.24}$$

$$\| \boldsymbol{\varepsilon} \|^2_{\dot{H}^{s_L} \times \dot{H}^{s_L-1}} \leq c\mathcal{E}_{s_L} < +\infty, \tag{3.D.25}$$

the adapted derivatives u_k being defined by (*3.2.21*) *and $\mathcal{E}_{s_L}$ being defined by* (*3.3.11*).

Proof of Corollary (3.D.4) Step 1: Proof that $\mathcal{E}_{s_L} < +\infty$. From the equivalence between Laplace derivatives and adapted ones, (*3.B.2*), one has:

$$\int |\varepsilon^{(1)}_{s_L}|^2 \leq C \sum_{i=0}^{s_L} \int \frac{|D^i \varepsilon^{(1)}|^2}{1+y^{2s_L-2i}}.$$

For $\sigma \leq i \leq s_L$ one has by interpolation $\int |D^i \varepsilon^{(1)}|^2 < +\infty$, hence $\int \frac{|D^i \varepsilon^{(1)}|^2}{1+y^{2s_L-2i}} < +\infty$. For $0 \leq i \leq \sigma$ one has $\frac{D^i \varepsilon^{(1)}}{1+y^{\sigma-i}} \in L^2$ from the Hardy inequality (*3.C.8*). Consequently in that case we also have $\frac{D^i \varepsilon^{(1)}}{1+y^{s_L-i}} \in L^2$. This proves:

$$\int |\varepsilon^{(1)}_{s_L}|^2 < +\infty.$$

Similarly one has $\int |\varepsilon^{(2)}_{s_L-1}|^2 < +\infty$, implying $\mathcal{E}_{s_L} < +\infty$. Step 2: Proof of the coercivity estimate. We want to apply the previous Lemma 3.D.3 for $k = s_L$. We have seen in the previous step 1 that the integrability condition (*3.D.21*) is met. Now from the formula (*3.2.25*) giving the powers of H^* we compute that the orthogonality condition (*3.D.23*) implies:

$$\langle \varepsilon^{(1)}, \mathcal{L}^i \Phi_M \rangle = \langle \varepsilon^{(2)}, \mathcal{L}^i \Phi_M \rangle = 0 \text{ for } 0 \leq i \leq \frac{L-1}{2}.$$

We compute: $E\left[\frac{k-k_0}{2}\right] = E\left[\frac{L+k_0+1-k_0}{2}\right] = \frac{L+1}{2}$. Therefore the Lemma 3.D.3 applies and gives the bound (*3.D.24*). Now we use the equivalence between Laplace and adapted derivatives (*3.B.2*), with the bound we just proved for (*3.D.24*) for $\delta = 0$ and it yields (*3.D.25*). □

Proof of Lemma 3.D.3 Case k small: We suppose $1 \leq k \leq k_0$, and that u is a function satisfying the conditions of the lemma. We have, depending on the parity of k:

$$u_k = A u_{k-1} \text{ or } u_k = A^* u_{k-1}.$$

In both cases, the conditions required to apply to u_{k-1} Lemma 3.D.2 or Lemma 3.D.1 are fulfilled. Consequently:

$$\int \frac{u_k^2}{1+y^{2\delta}} \gtrsim \int \frac{u_{k-1}^2}{1+y^{2+2\delta}}.$$

If $k-1=0$ we have finished. If not, then again, $u_{k-1} = A u_{k-2}$ or $u_{k-1} = A^* u_{k-2}$ and in both cases we can apply Lemma 3.D.2 or Lemma 3.D.1 which gives:

$$\int \frac{u_k^2}{1+y^{2\delta}} \gtrsim \int \frac{u_{k-1}^2}{1+y^{2+2\delta}} \gtrsim \int \frac{u_{k-2}^2}{1+y^{4+2\delta}}.$$

We can iterate k times what we did previously to obtain:

$$\int \frac{u_k^2}{1+y^{2\delta}} \gtrsim \int \frac{u_{k-1}^2}{1+y^{2+2\delta}} \gtrsim ... \gtrsim \int \frac{u_1^2}{1+y^{2(k-2)+2\delta}} \gtrsim \int \frac{u^2}{1+y^{2k+2\delta}},$$

which gives the result in that case.

Case k large: Suppose first that $k \geq k_0+1$ and that $j = \frac{k-k_0}{2} \in \mathbb{N}^*$, so $k = k_0 + 2j$. We can apply the result for k small we just showed to compute:

$$\int \frac{u_k^2}{1+y^{2\delta}} \gtrsim \int \frac{u_{k-k_0}^2}{1+y^{2k_0+2\delta}} = \int \frac{u_{2j}^2}{1+y^{2k_0+2\delta}}.$$

Since $2j$ is even we know that: $u_{2j} = A^*A...A^*Au = A^*u_{2j-1}$ and we can apply Lemma 3.D.1 to find:

$$\int \frac{u_{2j}^2}{1+y^{2k_0+2\delta}} \gtrsim \int \frac{u_{2j-1}^2}{1+y^{2k_0+2+2\delta}} = \frac{Au_{2j-2}^2}{1+y^{2k_0+2+2\delta}}.$$

We need an orthogonality condition for u_{2j-2} in order to go on. This is given by the orthogonality condition on u. Indeed:

$$\langle u_{2j-2}, \Phi_M\rangle = \langle u, \mathcal{L}^{j-1}\Phi_M\rangle = 0.$$

Hence:

$$\int \frac{u_{2j-1}^2}{1+y^{2(k_0+1+\delta)}} \gtrsim \int \frac{u_{2j-2}^2}{1+y^{2(k_0+2+\delta)}}.$$

We need exactly the j orthogonality conditions to iterate like that till we reach 0.

Suppose now that $k = k_0 + 2j + 1$. Then it works the same, indeed without use of orthogonality conditions:

$$\int \frac{u_k^2}{1+y^{2\delta}} \gtrsim \int \frac{u_{k-1}^2}{1+y^{2+2\delta}} \gtrsim ... \gtrsim \frac{u_{k-k_0}^2}{1+y^{2k_0+2\delta}} = \int \frac{|Au_{2j}|^2}{1+y^{2k_0+2\delta}}.$$

We have exactly j orthogonality conditions to go down to zero as we did before:

$$\int \frac{|Au_{2j}|^2}{1+y^{2k_0+2\delta}} \gtrsim \int \frac{u_{2j}^2}{1+y^{2k_0+2\delta}} \gtrsim ... \gtrsim \int \frac{u^2}{1+y^{2k+2\delta}}.$$

This ends the proof. □

3.E Specific bounds for the analysis

We make use here of the tools established in the last subsection to control ε. Again, the use of such estimate is standard in blow-up issues, and we refer to the papers quoted in Appendix C. Although their proofs are not very hard to write once one has the previous results, we put it here for the reader's convenience. aAs the non-linearity just acts on $\varepsilon^{(1)}$ we just state results for this coordinate.

Lemma 3.E.1. *Under the bootstrap conditions* (*3.3.27*) *of Proposition 3.3.2 and provided that ε satisfies the orthogonality conditions* (*3.3.9*) *there holds ($\mathcal{E}_{s_L}$ and $\mathcal{E}_\sigma$ being defined in* (*3.3.11*) *and* (*3.3.14*)*):*

(i) Improved Hardy inequality: *For $j \in \mathbb{N}$ and $p > 0$ satisfying $\sigma \leq j + p \leq s_L$:*

$$\int_{y\geq 1} \frac{|\partial_y^j \varepsilon^{(1)}|^2}{1+y^{2p}} \leq C(M)\mathcal{E}_\sigma^{\frac{s_L-(j+p)}{s_L-\sigma}} \mathcal{E}_{s_L}^{\frac{j+p-\sigma}{s_L-\sigma}}, \tag{3.E.1}$$

(ii) L^∞ control:

$$\| \epsilon^{(1)} \|_{L^\infty} \leq C(K_1, K_2, M)\sqrt{\mathcal{E}_\sigma} b_1^{\left(\frac{d}{2}-\sigma\right)+\frac{2\alpha}{(p-1)L}+O\left(\frac{\sigma-s_c}{L}\right)}, \tag{3.E.2}$$

(iii) Weighted L^∞ bound: *for* $0 < a < \frac{d}{2}$

$$\left\| \frac{\epsilon^{(1)}}{1+x^a} \right\|_{L^\infty} \leq C(K_1, K_2, M)\sqrt{\mathcal{E}_\sigma} b_1^{a+\left(\frac{d}{2}-\sigma\right)+\frac{\left(\frac{2}{p-1}+a\right)\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}. \tag{3.E.3}$$

Proof of Lemma 3.E.1

Proof of (i): Let $j \in \mathbb{N}$ and p satisfying $\sigma \leq j+p \leq s_L$. For a slow decaying potential, ie if p satisfies in addition $p < \frac{d}{2}$ then the equivalence between Laplace derivatives and ∂_y ones away from the origin, together with the weighted Hardy inequality (Lemma 3.C.2) gives:

$$\int \frac{|\partial_y^j \varepsilon^{(1)}|^2}{1+y^{2p}} \leq C \int |\nabla^{j+p} \varepsilon^{(1)}|^2,$$

and we conclude by interpolation. We claim now that:

$$\sum_{i=0}^{s_L} \int_{y \geq 1} \frac{|\partial_y^i \varepsilon^{(1)}|^2}{1+y^{2(s_L-i)}} \leq C(M) \mathcal{E}_{s_L}.$$

Indeed, from the equivalence between ∂_y and adapted derivatives (Lemma 3.B.1), and from coercivity we have:

$$\sum_{i=0}^{s_L} \int_{y \geq 1} \frac{|\partial_y^i \varepsilon^{(1)}|^2}{1+y^{2(s_L-i)}} \sim \sum_{i=0}^{s_L} \int_{y \geq 1} \frac{|\varepsilon_i^{(1)}|^2}{1+y^{2(s_L-i)}} \leq C(M) \mathcal{E}_{s_L}.$$

This claim implies that for a fast decaying potential, ie $p = s_L - j$:

$$\int \frac{|\partial_y^j \varepsilon^{(1)}|^2}{1+y^{2p}} \leq \mathcal{E}_{s_L}.$$

Now, for $\frac{d}{2} \leq p \leq s_l - j$ we interpolate the last two results, as for $a \leq b \leq c$:

$$\frac{|\varepsilon^{(1)}|^2}{1+y^{2b}} \sim \left(\frac{|\varepsilon|^2}{1+y^{2a}} \right)^{\frac{c-b}{c-a}} \left(\frac{|\varepsilon|^2}{1+y^{2b}} \right)^{\frac{b-a}{c-a}}$$

and this gives (i).

Proof of (ii). We prove it for $\varepsilon^{(1)}$, the proof for the second coordinate being similar. By the coercivity bound (*3.D.25*) we have that:

$$\| \nabla^{s_L} \varepsilon^{(1)} \|_{L^2}^2 \leq C(M) \mathcal{E}_{s_L}.$$

We have by interpolation that for all $\sigma \leq k \leq s_L$, $\nabla^k \varepsilon^{(1)} \in L^2$ with the control

$$\| \nabla^k \varepsilon^{(1)} \|_{L^2}^2 \leq C(M) \mathcal{E}_\sigma^{\frac{s_L-k}{s_L-\sigma}} \mathcal{E}_{s_L}^{\frac{k-\sigma}{s_L-\sigma}}.$$

Denoting by $\hat{\varepsilon^{(1)}}$ the Fourier transform of $\varepsilon^{(1)}$ we have:

$$|\varepsilon^{(1)}(y)| \leq \int_{|\xi| \leq 1} \frac{|\hat{\varepsilon^{(1)}}||\xi|^{\frac{k_1}{2}}}{|\xi|^{\frac{k_1}{2}}} + \int_{|\xi| \geq 1} \frac{|\hat{\varepsilon^{(1)}}||\xi|^{\frac{k_2}{2}}}{|\xi|^{\frac{k_2}{2}}} \lesssim \| \nabla^{k_1} \varepsilon^{(1)} \|_{L^2} + \| \nabla^{k_2} \varepsilon^{(1)}, \|_{L^2}$$

with $\sigma < k_1 < \frac{d}{2} < k_2 < s_L$. Using the interpolation bound previously established and taking $k_1, k_2 \to \frac{d}{2}$ gives:

$$\begin{aligned} |\varepsilon^{(1)}(y)|^2 &\leq C \mathcal{E}_\sigma^{\frac{s_L - \frac{d}{2}}{s_L-\sigma}} \mathcal{E}_{s_L}^{\frac{\frac{d}{2}-\sigma}{s_L-\sigma}} \leq C \mathcal{E}_\sigma b_1^{2\left(L+(1-\delta_0)(1+\eta)-(\sigma-s_c)\frac{\ell}{\ell-\alpha}\right)(\frac{d}{2}-\sigma)\left(\frac{1}{s_L}+\frac{\sigma}{s_L^2}+O(\frac{1}{L^3})\right)} \\ &= C \mathcal{E}_\sigma b_1^{(\frac{d}{2}-\sigma)2+\frac{\frac{2}{p-1}(2\eta(1-\delta_0)+2\alpha)}{L}+O\left(\frac{(\sigma-s_c)}{L}\right)} \leq C \mathcal{E}_\sigma b_1^{(\frac{d}{2}-\sigma)2+\frac{\frac{2}{p-1}\alpha}{L}+O\left(\frac{(\sigma-s_c)}{L}\right)} \end{aligned}$$

which gives the result.

Proof of (iii) Take $a \geq 1$, $\alpha \leq a \ll s_L$. Then from (i):

$$\| \nabla^{E\left[\frac{d}{2}+1\right]} \frac{\varepsilon^{(1)}}{1+y^a} \|_{L^2}^2 \quad \sim \quad \int \left| D^{E\left[\frac{d}{2}+1\right]} \left(\frac{\varepsilon^{(1)}}{1+a}\right)\right|^2 \leq C(M) \mathcal{E}_\sigma^{\frac{s_L - E\left[\frac{d}{2}+1\right]-a}{s_L-\sigma}} \mathcal{E}_\sigma^{\frac{E\left[\frac{d}{2}+1\right]+a-\sigma}{s_L-\sigma}}.$$

And we estimate the same way $\| \nabla^{E\left[\frac{d}{2}-1\right]} \frac{\varepsilon^{(1)}}{1+y^a} \|_{L^2}^2$. We can the interpolate this two estimations to have an estimate for $\| \frac{\varepsilon^{(1)}}{1+y^a} \|_{L^\infty}$. By calculating the exponents the same way we did for the proof of (ii) we get the result of the lemma for a. Now we can interpolate this result with (ii) to conclude for any exponent $0 \leq a \leq s_L$. $\square$

4

Concentration of the ground state for the energy supercritical semilinear heat equation in the non-radial setting

4.1 Introduction, organization and notations

In this chapter we prove Theorem 2.2.9. This work is to appear in Analysis and PDE [26]. We gave a detailed sketch of the proof of this Theorem in Subsection 2.2.3.

We recall that we motivated the result of of Theorem 2.2.9 and presented related works in the previous Chapter 2, in Section 2.2. We turn here to its rigorous proof. As in the previous Chapter 3, certain objects here and in other chapters are different but play a similar role in the analysis, and we consequently use the same notation. For this reason, we start by giving all the specific notations used in this chapter, and invite the reader to come back here whenever he or she has some doubts.

The Chapter is organized in two parts. The first one is devoted to the construction of an approximate blow-up profile in the non-radial setting, but with domain $\mathbb{R}^d$ instead of a smooth bounded one. In Section 4.2 we describe the kernel of the linearized operator H near Q in Lemma 4.2.1. This provides a formula to invert elliptic equations of the form $Hu = f$, stated in Lemma 4.2.4 and allows to describe the generalized kernel of H in Lemma 4.2.8. The blow-up profile is built on functions depending polynomially on some parameters and with explicit asymptotic at infinity, and we introduce the concept of homogeneous functions in Definition 4.2.12 and Lemma 4.2.13 to track these informations easily. With these tools, in Section 4.3 we construct a first approximate blow-up profile for which the error is localized at infinity in Proposition 4.3.1 and we cut it in the self-similar zone in Proposition 4.3.3. The evolution of the parameters describing the approximate blow-up profile is an explicit dynamical system with special solutions given in Lemma 4.3.4 for which the linear stability is investigated in Lemma 4.3.5.

Then, we turn to the existence of a true solution of (NLH) on a bounded domain with Dirichlet boundary conditions, being a perturbation of the approximate blow-up profile. In Section 4.4 we define a bootstrap regime for solutions of the full equation close to the approximate blow-up profile. We give a suitable decomposition for such solutions, using orthogonality conditions that are provided by Definition 4.4.1 and Lemma 4.4.2, in Lemma 4.4.3. They must satisfy in addition some size assumption, and all the conditions describing the bootstrap regime are given in Definition 4.4.4. The main result of the chapter is Proposition 4.4.6, stating the existence of a solution staying for all times in the boostrap regime, whose proof is relegated to the next Section. With this result we end the proof of Theorem 2.2.9 in Subsection 4.4.2. To to this, the modulation equations are computed in Lemma 4.4.7, yielding that solutions staying in the bootstrap regime must concentrate in Lemma 4.4.8 with an explicit asymptotic for Sobolev norm in Lemma 4.4.9. In Section 4.5 we prove the main Proposition 4.4.6. For solutions in the boostrap regime, an improved modulation equation is established in Lemma 4.5.1, and Lyapunov type monotonicity formulas are established in Propositions 4.5.3 and 4.5.5 for the low regularity Sobolev norms of the remainder, and in Propositions 4.5.6 and 4.5.8 for the high regularity norms. With this analysis one can characterize the conditions under which a solution leaves the boostrap regime in Lemma 4.5.9, and with a topological argument provided in Lemma 4.5.10 one ends the proof of Proposition 4.4.6 in Proof 4.5.4.

Some results are relegated to the appendix, which is organized as follows. In Section 4.A we give the proof of Lemma 4.2.1 describing the kernel of H. In Section 4.B we recall some Hardy and Rellich type estimates. An important version of these inequalies that we use here, Lemma 3.C.2, was already in the appendix of the previous Chapter 3. In Section 4.C we investigate the coercivity of H in Lemmas 4.C.2 and 4.C.3. In Section 4.D we prove some bounds for solutions in the bootstrap regime. In Section 4.E we give the proof of the decomposition Lemma 4.4.3.

Notations

We collect here the main notations. In the analysis the notation C will stand for a constant whose value just depends on d and p which may vary from one line to another. The notation $a \lesssim b$ means that $a \leq Cb$ for such a constant C, and $a = O(b)$ means $|a| \lesssim b$.

Supercritical numerology. for $d \geq 11$ the condition $p > p_{JL}$ where p_{JL} is defined by (*1.4.1*) is equivalent to $2 + \sqrt{d-1} < s_c < \frac{d}{2}$. We define the sequences of numbers describing the asymptotic of particular zeros of H for $n \in \mathbb{N}$:

$$-\gamma_n := \frac{-(d-2)+\sqrt{\triangle_n}}{2}, \quad \triangle_n := (d-2)^2 - 4cp_\infty + 4n(d+n-2), \tag{4.1.1}$$

$$\alpha_n := \gamma_n - \frac{2}{p-1} \tag{4.1.2}$$

where $\triangle_n > 0$ for $p > p_{JL}$. We will use the following facts in the sequel:

$$\gamma_0 = \gamma, \ \gamma_1 = \frac{2}{p-1} + 1, \ \gamma_n < \frac{2}{p-1} \text{ for } n \geq 2 \text{ and } \gamma_n \sim -n, \tag{4.1.3}$$

see Lemma 4.A.1 (where γ is defined in (*2.2.5*)). In particular $\alpha_0 = \alpha$, $\alpha_1 = 1$ and $\alpha_n < 0$ for $n \geq 2$. A computation yields the bound:

$$2 < \alpha < \frac{d}{2} - 1$$

(see [114]). We let:

$$g := \min(\alpha, \triangle) - \epsilon, \quad g' := \frac{1}{2}\min(g, 1, \delta_0 - \epsilon) \tag{4.1.4}$$

where $0 < \epsilon \ll 1$ is a very small constant just here to avoid to track some logarithmic terms later on. For $n \in \mathbb{N}$ we define[1]:

$$m_n := E\left[\frac{1}{2}(\frac{d}{2} - \gamma_n)\right] \tag{4.1.5}$$

and denote by δ_n the positive real number $0 \leq \delta_n < 1$ such that:

$$d = 2\gamma_n + 4m_n + 4\delta_n. \tag{4.1.6}$$

[1] $E[x]$ stands for the entire part: $x - 1 < E[x] \leq x$.

For $1 \ll L$ a very large integer we define the Sobolev exponent:

$$s_L := m_0 + L + 1 \tag{4.1.7}$$

In this chapter we assume the technical condition (*2.2.8*) for $s_+ = s_L$ which means:

$$0 < \delta_n < 1 \tag{4.1.8}$$

for all integer n such that $d - 2\gamma_n \leq 4s_L$ (there is only a finite number of such integers from (*4.1.3*)). We let n_0 be the last integer to satisfy this condition:

$$n_0 \in \mathbb{N}, \quad d - 2\gamma_{n_0} \leq 4s_L \quad \text{and} \quad d - 2\gamma_{n_0+1} > 4s_L \tag{4.1.9}$$

and we define:

$$\delta_0' := \max_{0 \leq n \leq n_0} \delta_n \in (0,1). \tag{4.1.10}$$

For all integer $n \leq n_0$ we define the integer:

$$L_n := s_L - m_n - 1 \tag{4.1.11}$$

and in particular $L_0 = L$. Given an integer $\ell > \frac{\alpha}{2}$ (that will be fixed in the analysis later on), for $0 \leq n \leq n_0$ we define the real numbers:

$$i_n = \ell - \frac{\gamma - \gamma_n}{2}. \tag{4.1.12}$$

Notations for the analysis. For $R \geq 0$ the euclidian sphere and ball are denoted by:

$$\begin{array}{l} \mathcal{S}^{d-1}(R) := \left\{ x \in \mathbb{R}^d, \ \sum_1^d x_i^2 = R^2 \right\}, \\ \mathcal{B}^d(R) := \left\{ x \in \mathbb{R}^d, \ \sum_1^d x_i^2 \leq R^2 \right\}. \end{array}$$

We use the Kronecker delta-notation:

$$\delta_{i,j} := \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j, \end{cases}$$

for $i,j \in \mathbb{N}$. We let:

$$F(u) := \Delta u + f(u), \quad f(u) := |u|^{p-1}u$$

so that (NLH) writes:

$$\partial_t u = F(u).$$

When using the binomial expansion for the nonlinearity we use the constants

$$f(u+v) = \sum_{l=0}^{p} C_l^p u^l v^{p-l}, \quad C_l^p := \binom{p}{l}.$$

The linearized operator close to Q (defined in (*2.2.3*)) is:

$$Hu := -\Delta u - pQ^{p-1}u \tag{4.1.13}$$

so that $F(Q+\varepsilon)\sim -H\varepsilon$. We introduce the potential

$$V := -pQ^{p-1} \tag{4.1.14}$$

so that $H=-\Delta+V$. Given a strictly positive real number $\lambda>0$ and function $u:\mathbb{R}^d\to\mathbb{R}$, we define the rescaled function:

$$u_\lambda(x)=\lambda^{\frac{2}{p-1}}u(\lambda x). \tag{4.1.15}$$

This semi-group has the infinitesimal generator:

$$\Lambda u := \frac{\partial}{\partial\lambda}(u_\lambda)_{|\lambda=1}=\frac{2}{p-1}u+x.\nabla u.$$

The action of the scaling on (NLH) is given by the formula:

$$F(u_\lambda):=\lambda^2(F(u))_\lambda.$$

For $z\in\mathbb{R}^d$ and $u:\mathbb{R}^d\to\mathbb{R}$, the translation of vector z of u is denoted by:

$$\tau_z u(x):=u(x-z). \tag{4.1.16}$$

This group has the infinitesimal generator:

$$\left[\frac{\partial}{\partial z}(\tau_z u)\right]_{|z=0}=-\nabla u.$$

The original space variable will be denoted by $x\in\Omega$ and the renormalized one by y, related through $x=z+\lambda y$. The number of spherical harmonics of degree n is:

$$k(0):=1,\ k(1):=d,\ k(n):=\frac{2n+p-2}{n}\binom{n+p-3}{n-1}\ \text{ for } n\geq 2$$

The Laplace-Beltrami operator on the sphere $\mathbb{S}^{d-1}(1)$ is self-adjoint with compact resolvent and its spectrum is $\{n(d+n-2),\ n\in\mathbb{N}\}$. For $n\in\mathbb{N}$ the eigenvalue $n(d+2-n)$ has geometric multiplicity $k(n)$, and we denote by $(Y^{(n,k)})_{n\in\mathbb{N},\ 1\leq k\leq k(n)}$ an associated orthonormal Hilbert basis of $L^2(\mathbb{S}^d)$:

$$L^2(\mathbb{S}^{d-1}(1))=\overset{+\infty\perp}{\underset{n=0}{\oplus}}\text{Span}\left(Y^{(n,k)},\ 1\leq k\leq k(n)\right),$$

$$\Delta_{S^{d-1}(1)}Y^{(n,k)}=n(d+n-2)Y^{(n,k)},\quad \int_{S^{d-1}(1)}Y^{(n,k)}Y^{(n',k')}=\delta_{(n,k),(n',k')}, \tag{4.1.17}$$

with the special choices:

$$Y^{(0,1)}(x)=C_0,\quad Y^{1,k}(x)=-C_1x_k \tag{4.1.18}$$

where C_0 and C_1 are two renormalization constants. The action of H on each spherical harmonics is described by the family of operators on radial functions

$$H^{(n)}:=-\partial_{rr}-\frac{d-1}{r}\partial_r+\frac{n(d+n-2)}{r^2}-pQ^{p-1} \tag{4.1.19}$$

for $n\in\mathbb{N}$ as for any radial function f they produce the identity

$$H\left(x\mapsto f(|x|)Y^{(n,k)}\left(\frac{x}{|x|}\right)\right)=x\mapsto (H^{(n)}(f))(|x|)Y^{(n,k)}\left(\frac{x}{|x|}\right). \tag{4.1.20}$$

For two strictly positive real number $b_1^{(0,1)}>0$ and $\eta>0$ we define the scales:

$$M\gg 1 \quad B_0=|b_1^{(0,1)}|^{-\frac12}, \quad B_1=B_0^{1+\eta}, \tag{4.1.21}$$

The blow-up profile of this chapter will is an excitation of several direction of stability and instability around the soliton Q. Each one of these directions of perturbation, denoted by $T_i^{(n,k)}$ will be associated to a triple (n,k,i), meaning that it is the i-th perturbation located on the spherical harmonics of degree (n,k). For each (n,k) with $n\leq n_0$, there will be L_n+1 such perturbations for $i=0,...,L_n$ except for the cases $n=0$, $k=1$, and $n=1$, $k=1,...,d$, where there will be L_n perturbations for $i=1,...,L_n$ $(n=1,2)$. Hence the set of triple (n,k,i) used in the analysis is:

$$\begin{aligned}\mathcal{I} \ := \ & \{(n,k,i)\in\mathbb{N}^3, \ 0\leq n\leq n_0, \ 1\leq k\leq k(n), \ 0\leq i\leq L_n\} \\ & \backslash(\{(0,1,0)\}\cup\{(1,1,0),...,(1,d,0)\})\end{aligned} \tag{4.1.22}$$

with cardinal

$$\#\mathcal{I}:=\sum_{n=0}^{n_0}k(n)(L_n+1)-d-1. \tag{4.1.23}$$

For $j\in\mathbb{N}$ and a n-tuple of integers $\mu=(\mu_i)_{1\leq i\leq j}$ the usual length is denoted by:

$$|\mu|:=\sum_{i=1}^{j}\mu_i.$$

If $j=d$ and h is a smooth function on $\mathbb{R}^d$ then we use the following notation for the differentiation:

$$\partial^\mu h:=\frac{\partial^{|\mu|}}{\partial_{x_1}^{\mu_1}...\partial_{x_d}^{\mu_d}}h.$$

For J is a $\#\mathcal{I}$-tuple of integers we introduce two others weighted lengths:

$$|J|_2=\sum_{n,k,i}(\frac{\gamma-\gamma_n}{2}+i)J_i^{(n,k)}, \tag{4.1.24}$$

$$|J|_3=\sum_{i=1}^{L}iJ_i^{(0,1)}+\sum_{1\leq i\leq L_1, \ 1\leq k\leq d}iJ_i^{(1,k)}+\sum_{(n,k,i)\in\mathcal{I}, \ 2\leq n}(i+1)J_i^{(n,k)}. \tag{4.1.25}$$

To localize some objects we will use a radial cut-off function $\chi\in C^\infty(\mathbb{R}^d)$:

$$0\leq\chi\leq 1, \quad \chi(|x|)=1 \text{ for } |x|\leq 1, \quad \chi(|x|)=0 \text{ for } |x|\geq 2 \tag{4.1.26}$$

and for $B>0$, χ_B will denote the cut-off around $\mathcal{B}^d(0,B)$:

$$\chi_B(x):=\chi\left(\frac{x}{B}\right).$$

4.2 Preliminaries on Q and H

We first summarize the content and ideas of this section. The instabilities near Q underlying the blow up that we study result from the excitement of modes in the generalized kernel of H. We first describe this set. H being radial, we use a decomposition into spherical harmonics: restricted to spherical harmonics of degree n, see (*4.1.20*), it becomes the operator $H^{(n)}$ on radial functions defined by (*4.1.20*). Using ODE techniques, the kernel is described in Lemma 4.2.1 and the inversion of $H^{(n)}$ is given by Definition 4.2.4 and (4.2.11). By inverting successively the elements in the kernel of $H^{(n)}$ one obtains the generators of the generalized kernel $\cup_j \mathrm{Ker}((H^{(n)})^j)$ of this operator in Lemma 4.2.8.

To track the asymptotic behavior and the dependance in some parameters of various profiles during the construction of the approximate blow up profile in the next section, we introduce the framework of "homogeneous" functions in Definition 4.2.12 and Lemma 4.2.13.

4.2.1 Properties of the ground state and of the potential

We recall that all the properties of the ground state that we will use are contained in Lemma 3.2.1. The standard Hardy inequality $\int_{\mathbb{R}^d} |\nabla u|^2 \geq \frac{(d-2)^2}{4} \int_{\mathbb{R}^d} \frac{u^2}{|y|^2} dy$ and (*3.2.2*) then imply the positivity of H on $\dot{H}^1(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^d} uHudy \geq \int_{\mathbb{R}^d} \frac{\delta(p)u^2}{|y|^2} dy. \tag{4.2.1}$$

It is worth mentioning that the aforementioned expansion (*3.2.1*) is false for $p \leq p_{JL}$. This asymptotics at infinity of Q is decisive for type II blow up via perturbation of it, as from [97, 128] it cannot occur for $\frac{d+2}{d-2} < p < p_{JL}$.

4.2.2 Kernel of H

Lemma 4.2.1 (Kernel of $H^{(n)}$). *We recall that the numbers $(\gamma_n)_{n\in\mathbb{N}}$ and g are defined in (4.1.1). Let $n \in \mathbb{N}$. There exist $T_0^{(n)}, \Gamma^{(n)} : (0,+\infty) \to \mathbb{R}$ two smooth functions such that if $f : (0,+\infty) \to \mathbb{R}$ is smooth and satisfies $H^{(n)} f = 0$, then $f \in Span(T_0^{(n)}, \Gamma^{(n)})$. They enjoy the asymptotics:*

$$\begin{cases} T_0^{(n)}(r) \underset{r\to 0}{=} \sum_{j=0}^{l} c_j^{(n)} r^{n+2j} + O(r^{n+2+2l}), \quad \forall l \in \mathbb{N}, \quad c_0^{(n)} \neq 0, \\ T_0^{(n)} \underset{r\to+\infty}{\sim} C_n r^{-\gamma_n} + O(r^{-\gamma_n - g}), \quad C_n \neq 0, \\ \Gamma^{(n)} \underset{r\to 0}{\sim} \frac{c_n'}{r^{d-2+n}} \text{ and } \Gamma^{(n)} \underset{r\to+\infty}{\sim} \tilde{c}_n' r^{-\gamma_n}, \quad c_n', \tilde{c}_n' \neq 0. \end{cases} \tag{4.2.2}$$

Moreover, $T_0^{(n)}$ is strictly positive, and for $1 \leq k \leq k(n)$ the functions $y \mapsto T_0^{(n)}(|y|) Y_{n,k}\left(\frac{|y|}{y}\right)$ are smooth on $\mathbb{R}^d$. The first two regular and strictly positive zeros are explicit:

$$T_0^{(0)} = \frac{1}{C_0} \Lambda Q \text{ and } T_0^{(1)} = -\frac{1}{C_1} \partial_y Q. \tag{4.2.3}$$

where C_0 and C_1 are the renormalized constants defined by (4.1.18).

Proof.

The proof of this lemma is done in Appendix 4.A.

□

Remark 4.2.2. The presence of the renormalized constants in (*4.2.3*) is here to produce the identities $T_0^{(0)} Y^{(0,0)} = \Lambda Q$ and $T_0^{(1)} Y^{(1,k)} = \partial_{x_k} Q$ from (*4.1.18*). For each $n \in \mathbb{N}$, only one zero, $T_0^{(n)}$, is regular at the origin. We insist on the fact that $-\gamma_n > 0$ is a positive number[2] for n large from (*4.1.3*) making these profile grow as $r \to +\infty$.

[2]This notation seems unnatural but matches the standard notation in the literature.

4.2.3 Inversion of $H^{(n)}$

We start by a useful factorization formula for $H^{(n)}$. Let $n\in\mathbb{N}$ and $W^{(n)}$ denote the potential:

$$W^{(n)}:=\partial_r(\log(T_0^{(n)})), \tag{4.2.4}$$

where $T_0^{(n)}$ is defined in (*4.2.2*) and define the first order operators on radial functions:

$$A^{(n)}:u\mapsto -\partial_r u+W^{(n)}u,\ A^{(n)*}:u\mapsto \frac{1}{r^{d-1}}\partial_r(r^{d-1}u)+W^{(n)}u. \tag{4.2.5}$$

Lemma 4.2.3 (Factorization of $H^{(n)}$). *There holds the factorization:*

$$H^{(n)}=A^{(n)*}A^{(n)}. \tag{4.2.6}$$

Moreover one has the adjunction formula for smooth functions with enough decay:

$$\int_0^{+\infty}(A^{(n)}u)vr^{d-1}dr=\int_0^{+\infty}u(A^{(n)*}v)r^{d-1}dr.$$

Proof of Lemma 4.2.3

As $T_0^{(n)}>0$ from (*4.2.2*), $W^{(n)}$ is well defined. This factorization is a standard property of Schrödinger operators with a non-vanishing zero. We start by computing:

$$A^{(n)*}A^{(n)}u=-\partial_{rr}u-\frac{d-1}{r}\partial_r u+\left(\frac{d-1}{r}W^{(n)}+\partial_r W^{(n)}+(W^{(n)})^2\right)u.$$

As $W^{(n)}=\frac{\partial_r T_0^{(n)}}{T_0^{(n)}}$, the potential that appears is nothing but:

$$\begin{aligned}\tfrac{d-1}{r}W^{(n)}+\partial_r W^{(n)}+(W^{(n)})^2 &= \frac{\partial_{rr}T_0^{(n)}+\frac{d-1}{r}T_0^{(n)}}{T_0^{(n)}}=\frac{-H^{(n)}T_0^{(n)}+(\frac{n(d+n-2)}{r^2}+V)T_0^{(n)}}{T_0^{(n)}}\\ &= \tfrac{n(d+n-2)}{r^2}+V,\end{aligned}$$

as $H^{(n)}T_0^{(n)}=0$, which proves the factorization formula (*4.2.6*). The adjunction formula comes from a direct computation using integration by parts.

□

From the asymptotic behavior (*4.2.2*) of $T_0^{(n)}$ at the origin and at infinity, we deduce the asymptotic behavior of $W^{(n)}$:

$$W^{(n)}=\begin{cases}\frac{n}{r}+O(1)\text{ as }r\to 0,\\ \frac{-\gamma_n}{r}+O\left(\frac{1}{r^{1+g+j}}\right)\text{ as }r\to+\infty,\end{cases} \tag{4.2.7}$$

which propagates for the derivatives. Using the factorization (*4.2.6*), to define the inverse of $H^{(n)}$ we proceed in two times, first we invert $A^{(n)*}$, then $A^{(n)}$.

Definition 4.2.4 (Inverse of $H^{(n)}$). Let $f:(0,+\infty)\to\mathbb{R}$ be smooth with $f(r)=O(r^n)$ as $r\to 0$. We define[3] the inverses $(A^{(n)*})^{-1}f$ and $(H^{(n)})^{-1}f$ by:

$$(A^{(n)*})^{-1}f(r)=\frac{1}{r^{d-1}T_0^{(n)}}\int_0^r fT_0^{(n)}s^{d-1}ds, \tag{4.2.8}$$

[3] u is well defined because from the decay of f at the origin one deduces $(A^{(n)*})^{-1}f=O(r^{n+1})$ as $y\to 0$ and so $\frac{u'}{T_0^n}$ is integrable at the origin from the asymptotic behavior (*4.2.2*).

$$(H^{(n)})^{-1}f(r)=\begin{cases} T_0^{(n)}\int_r^{+\infty}\frac{(A^{(n)*})^{-1}f}{T_0^{(n)}}ds \ \text{ if } \frac{(A^{(n)*})^{-1}f}{T_0^{(n)}} \text{ is integrable on } (0,+\infty), \\ -T_0^{(n)}\int_0^r\frac{(A^{(n)*})^{-1}f}{T_0^{(n)}}ds \ \text{ if } \frac{(A^{(n)*})^{-1}f}{T_0^{(n)}} \text{ is not integrable on } (0,+\infty). \end{cases} \tag{4.2.9}$$

Direct computations give indeed $H^{(n)}\circ(H^{(n)})^{-1}=A^{(n)*}\circ(A^{(n)*})^{-1}=\mathrm{Id}$, and $A^{(n)}\circ(H^{(n)})^{-1}=(A^{(n)*})^{-1}$. As we do not have uniqueness for the equation $Hu=f$, one may wonder if this definition is the "right" one. The answer is yes because this inverse has the good asymptotic behavior, namely, if $f\underset{r\to+\infty}{\approx}r^q$ one would expect $u\underset{r\to+\infty}{\approx}r^{q+2}$, which will be proven in Lemma 4.2.7. To keep track of the asymptotic behaviors at the origin and at infinity, we now introduce the notion of admissible functions.

Definition 4.2.5 (Simple admissible functions). Let n be an integer, q be a real number and $f:(0,+\infty)\to\mathbb{R}$ be smooth. We say that f is a simple admissible function of degree (n,q) if it enjoys the asymptotic behaviors:

$$\forall l\in\mathbb{N},\ f=\sum_{j=0}^{l}c_jr^{n+2j}+O(r^{n+2l+2}) \tag{4.2.10}$$

at the origin for a sequence of numbers $(c_l)_{l\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, and at infinity:

$$f=O(r^q)\ \text{ as } r\to+\infty, \tag{4.2.11}$$

and if the two asymptotics propagate for the derivatives of f.

Remark 4.2.6. Let $f:(0,+\infty)$ be smooth, we define the sequence of n-adapted derivatives of f by induction:

$$f_{[n,0]}:=f \text{ and for } j\in\mathbb{N},\ f_{[n,j+1]}:=\begin{cases} A^{(n)}f_{[n,j]} \text{ for } j \text{ even}, \\ A^{(n)*}f_{[n,j]} \text{ for } j \text{ odd}. \end{cases} \tag{4.2.12}$$

From the definition (*4.2.5*) of $A^{(n)}$ and $A^{(n)*}$, and the asymptotic behavior (*4.2.7*) of the potential $W^{(n)}$, one notices that the condition (*4.2.11*) on the asymptotic at infinity for a simple admissible function of degree (n,q) and its derivatives is equivalent to the following condition for all $j\in\mathbb{N}$:

$$f_{[n,j]}=O(r^{q-j})\ \text{ as } r\to+\infty \tag{4.2.13}$$

where the adapted derivatives $(f_{[n,j]})_{j\in\mathbb{N}}$ are defined by (*4.2.12*). We will use this fact many times in the rest of this subsection, as it is more adapted to our problem.

The operators $H^{(n)}$ and $(H^{(n)})^{-1}$ leave this class of functions invariant, and the asymptotic at infinity is increased by -2 and 2 under some conditions (that will always hold in the sequel) on the coefficient q to avoid logarithmic corrections.

Lemma 4.2.7 (Action of $H^{(n)}$ and $(H^{(n)})^{-1}$ on simple admissible functions). *Let $n\in\mathbb{N}$ and f be a simple admissible function of degree (n,q) in the sense of Definition 4.2.5, with $q>\gamma_n-d$ and $-\gamma_n-2-q\notin 2\mathbb{N}$. Then for all integer $i\in\mathbb{N}$:*

(i) $(H^{(n)})^if$ *is simple admissible of degree* $(n,q-2i)$.

(ii) $(H^{(n)})^{-i}f$ *is simple admissible of degree* $(n,q+2i)$.

Proof of Lemma 4.2.7

step 1 Action of $H^{(n)}$. For each integer i and j one has from (*4.2.12*) and (*4.2.6*): $((H^{(n)})^i f)_{[n,j]} = f_{[n,j+2i]}$. Using the equivalent formulation (*4.2.13*), the asymptotic at infinity (*4.2.11*) for $H^i f$ is then a straightforward consequence of the asymptotic at infinity (*4.2.11*) for f. Close to the origin, one notices that $H^{(n)} = -\Delta^{(n)} + V$ with $\Delta^{(n)} = \partial_{rr} + \frac{d-1}{r}\partial_r - n(d+n-2)$. If f satisfies (*4.2.10*) at the origin, then so does $(\Delta^{(n)})^i f$ by a direction computation. As V is smooth at the origin, $(H^{(n)})^i f$ satisfies also (*4.2.10*). Hence $(H^{(n)})^i f$ is a simple admissible function of degree $q-2i$.

step 2 Action of $(H^{(n)})^{-1}$. We will prove the property for $(H^{(n)})^{-1} f$, and the general result will follow by induction on i. Let u denote the inverse by $H^{(n)}$: $u = (H^{(n)})^{-1} f$.

- *Asymptotic at infinity.* We will prove the equivalent formulation (*4.2.13*) of the asymptotic at infinity (*4.2.11*). From (*4.2.12*), (*4.2.8*), (*4.2.9*) and (*4.2.6*), $u_{[n,j]} = f_{[n,j-2]}$ for $j \geq 2$ so the asymptotic behavior (*4.2.13*) at infinity for the n-adapted derivatives of u are true for $j \geq 2$. Therefore it remains to prove them for $j = 0, 1$.

Case $j = 1$. From the definition of the inverse (*4.2.9*) and of the adapted derivatives (*4.2.12*), one has:

$$u_{[n,1]} = \frac{1}{r^{d-1} T_0^{(n)}} \int_0^r f T_0^{(n)} s^{d-1} ds.$$

From the asymptotic behaviors (*4.2.11*) and (*4.2.2*) for f and $T_0^{(n)}$ at infinity and the condition $q > \gamma_n - d$, the integral diverges and we get

$$u_{[n,1]}(r) = O(r^{q+1}) \text{ as } r \to +\infty \tag{4.2.14}$$

which is the desired asymptotic (*4.2.13*) for $u_{[n,1]}$.

Case $j = 0$. Suppose $\frac{(A^{(n)*})^{-1} f}{T_0^{(n)}} = \frac{u_{[n,1]}}{T_0^{(n)}}$ is integrable on $(0, +\infty)$. In that case:

$$u = T_0^{(n)} \int_r^{+\infty} \frac{u_{[n,1]}}{T_0^{(n)}} ds.$$

If $q > -\gamma_n - 2$, then from the integrability of the integrand and (*4.2.2*) one gets the asymptotic we desire $u_{[n,0]} = u = O(r^{-\gamma_n}) = O(r^{q+2})$. If $q < -\gamma_n - 2$ then from (*4.2.14*) one has $\frac{u_{[n,1]}}{T_0^{(n)}} = O(r^{q+1+\gamma_n})$ and then $\int_r^{+\infty} \frac{u_{[n,1]}}{T_0^{(n)}} ds = O(r^{q+2+\gamma_n})$, from what we get the desired asymptotic $u = O(r^{q+2})$. Suppose now $\frac{u_{[n,1]}}{T_0^{(n)}}$ is not integrable, then we must have $q > -\gamma_n + 2$ from (*4.2.14*). u is then given by:

$$u = -T_0^{(n)} \int_0^r \frac{u_{[n,1]}}{T_0^{(n)}} ds$$

and the integral has asymptotic $O(r^{q+2+\gamma_n})$. We hence get $u = O(r^{q+2})$ at infinity using (*4.2.2*).

Conclusion. In both cases, we have proven that the asymptotic at infinity (*4.2.13*) holds for u.

- *Asymptotic at the origin.* We have:

$$u = -T_0^{(n)} \int_0^r \frac{u_{[n,1]}}{T_0^{(n)}} ds + a T_0^{(n)}$$

where $a=0$ if $\frac{u_{[n,1]}}{T_0^{(n)}}$ is not integrable, and $a=\int_0^{+\infty}\frac{u_{[n,1]}}{T_0^{(n)}}ds$ if it is. From (*4.2.2*), $T_0^{(n)}$ satisfies (*4.2.10*). So it remains to prove (*4.2.10*) for $-T_0^{(n)}\int_0^r\frac{u_{[n,1]}}{T_0^{(n)}}ds$. We proceed in two steps. First, from (*4.2.10*) for f we obtain that for every integers j,p:

$$u_{[n,1]}=\frac{1}{r^{d-1}T_0^{(n)}}\int_0^r fT_0^{(n)}s^{d-1}ds=\sum_{j=0}^{l}\tilde{c}_j r^{n+1+2j}+\tilde{R}_l,$$

where $\partial_r^k\tilde{R}_l \underset{r\to 0}{=} O(r^{\max(n+2l+3-k,0)})$ for some coefficients $\tilde{c}_j$ depending on the c_j's and the asymptotic at the origin of T_0^n. It then follows that

$$-T_0^{(n)}\int_0^r\frac{u_{[n,1]}}{T_0^{(n)}}ds=\sum_{j=0}^{l}\hat{c}_j r^{n+2+2j}+\hat{R}_l,\ \text{where } \partial_r^k\hat{R}_l\underset{r\to 0}{=}O(r^{\max(n+2l+4-k,0)})$$

for some coefficients $\hat{c}_l$. This implies that u satisfies (*4.2.10*) at the origin.

□

We can now invert the elements in the kernel of $H^{(n)}$ and construct the generalized kernel of this operator.

Lemma 4.2.8 (Generators of the generalized kernel of $H^{(n)}$). *Let $n\in\mathbb{N}$, γ_n, g', $(H^{(n)})^{-1}$ and $T_0^{(n)}$ be defined by (4.1.1), (4.1.4), Definition 4.2.4 and* (4.2.1). *We denote by $(T_i^{(n)})_{i\in\mathbb{N}}$ the sequence of profiles given by:*

$$T_{i+1}^{(n)}:=-(H^{(n)})^{-1}T_i^{(n)},\ i\in\mathbb{N}. \tag{4.2.15}$$

Let $(\Theta_i^{(n)})_{i\in\mathbb{N}}$ be the associated sequence of profiles defined by:

$$\Theta_i^{(n)}:=\Lambda T_i^{(n)}-\left(2i+\frac{2}{p-1}-\gamma_n\right)T_i^{(n)},\ i\in\mathbb{N}. \tag{4.2.16}$$

Then for each $i\in\mathbb{N}$:

$$(i)\quad T_i^{(n)} \text{ is simple admissible of degree } (n,-\gamma_n+2i), \tag{4.2.17}$$
$$(ii)\quad \Theta_i^{(n)} \text{ is simple admissible of degree } (n,-\gamma_n+2i-g'), \tag{4.2.18}$$

where simple admissibility is defined in Definition (4.2.5).

Proof of Lemma 4.2.8

step 1 Admissibility of $T_i^{(n)}$. From the asymptotic behaviors (*4.2.2*) at infinity and at the origin, $T_0^{(n)}$ is simple admissible of degree $(n,-\gamma_n)$ in the sense of Definition (4.2.5). $-\gamma_n>\gamma_n-d$ since $-2\gamma_n+d\geq -2\gamma_0+d=2+\sqrt{\triangle}>0$ from (*2.2.5*) and since $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing from (*4.1.1*). One has also $-\gamma_n-2-(-\gamma_n)=-2\notin 2\mathbb{N}$. Therefore one can apply Lemma 4.2.7: for all $i\in\mathbb{N}$, $T_i^{(n)}$ given by (*4.2.15*) is an admissible profile of degree $(n,-\gamma_n+2i)$.

Step 2 Admissibility of $\Theta_i^{(n)}$. We start by computing the following commutator relations from (*4.1.20*) (*4.2.4*) and (*4.2.5*):

$$\begin{aligned} A^{(n)}\Lambda&=\Lambda A^{(n)}+A^{(n)}-(W^{(n)}+y\partial_y W^{(n)}),\\ H^{(n)}\Lambda&=\Lambda H^{(n)}+2H^{(n)}-(2V+y.\nabla V). \end{aligned} \tag{4.2.19}$$

We now proceed by induction. From the previous equation, and the asymptotic behaviors (*4.2.2*), (*3.2.10*) and (*4.2.7*) of the functions $T_0^{(n)}$, V and $W^{(n)}$, we get that $\Theta_0^{(n)}$ is simple admissible of degree $(n,-\gamma_n-g')$. Now let $i\geq 1$ and suppose that the property (ii) is true for $i-1$. Using the previous formula and (*4.2.16*) we obtain:

$$H^{(n)}\Theta_i^n=-\Theta_{i-1}^{(n)}-(2V+y.\nabla V)T_i^{(n)}.$$

The asymptotic at infinity (*3.2.10*) of V yields the decay $2V+y.\nabla V=(y^{-2-\alpha})$. This, as $T_i^{(n)}$ is simple admissible of degree $(n,2i-\gamma_n)$ and from the induction hypothesis, gives that $H^{(n)}\Theta_i^{(n)}$ is simple admissible of degree $(n,2i-2-\gamma_n-g')$ because $g'<\alpha$ from (*4.1.4*). One has $2i-2-\gamma_n-g'>\gamma_n-d$ because

$$2i-2-2\gamma_n-g'+d\geq -2\gamma_0-g'+d=2+\sqrt{\triangle}-g'>0$$

as $0<g'<1$, $i\geq 1$, $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing from (*4.1.1*) and from (*2.2.5*). Similarly $-\gamma_n-2-(2i-2-\gamma_n-g')=-2i+g'\notin 2\mathbb{N}$. Therefore we can apply Lemma (4.2.7) and obtain that $(H^{(n)})^{-1}H^{(n)}\Theta_i^{(n)}$ is of degree $(n,2i-\gamma_n-g')$. From Lemma (4.2.1) one has $(H^{(n)})^{-1}H^{(n)}\Theta_i^{(n)}=\Theta_i^{(n)}+aT_0^{(n)}+b\Gamma^{(n)}$, for two integration constants $a,b\in\mathbb{R}$. At the origin $\Gamma^{(n)}$ is singular from (*4.2.2*), hence $b=0$. As $T_0^{(n)}$ is of degree $(n,-\gamma_n)$ with $-\gamma_n+2i-g'>-\gamma_n$ (because $i\geq 1$) we get that $\Theta_i^{(n)}$ is of degree $(n,2i-\gamma_n-g')$. □

4.2.4 Inversion of H on non radial functions

The Definition 4.2.4 of the inverse of $H^{(n)}$ naturally extends to give an inverse of H by inverting separately the components onto each spherical harmonics. There will be no problem when summing as for the purpose of the present chapter one can restrict to the following class of functions that are located on a finite number of spherical harmonics.

Definition 4.2.9 (Admissible functions). Let $f:\mathbb{R}^d\to\mathbb{R}$ be a smooth function and denote its decomposition by $f(y)=\sum_{n,k}f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)$, and q be a real number. We say that f is admissible of degree q if there is only a finite number of couples (n,k) such that $f^{(n,k)}\neq 0$, and that for every such couple $f^{(n,k)}$ is a simple admissible function of degree (n,q) in the sense of Definition 4.2.5.

For $f=\sum_{n,k}f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)$ an admissible function we define its inverse by H by (the sum being finite):

$$(H^{(-1)}f)(y):=\sum_{n,k}[(H^{(n)})^{-1}f^{(n,k)}(|y|)]Y^{(n,k)}\left(\frac{y}{|y|}\right) \qquad (4.2.20)$$

where $(H^{(n)})^{-1}$ is defined by Definition 4.2.4. For n, k and i three integers with $1\leq k\leq k(n)$, we define the profile $T_i^{(n,k)}:\mathbb{R}^d\to\mathbb{R}$ as:

$$T_i^{(n,k)}(y)=T_i^{(n)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right) \qquad (4.2.21)$$

where the radial function $T_i^{(n)}$ is defined by (*4.2.15*). From Lemma 4.2.8, $T_i^{(n,k)}$ is an admissible function of degree $(-\gamma_n+2i)$ in the sense of Definition 4.2.9. The class of admissible functions has some structural properties: it is stable under summation, multiplication and differentiation, and its elements are smooth with an explicit decay at infinity. This is the subject of the next lemma.

Lemma 4.2.10 (Properties of admissible functions). *Let f and g be two admissible functions of degree q and q' in the sense of Definition 4.2.9, and $\mu \in \mathbb{N}^d$. Then:*

(i) f is smooth.

(ii) fg is admissible of degree $q+q'$.

(iii) $\partial^\mu f$ is admissible of degree $q-|\mu|$.

(iv) There exists a constant $C(f,\mu)$ such that for all y with $|y|\geq 1$:

$$|\partial^\mu f(y)| \leq C(f,\mu)|y|^{q-|\mu|}.$$

Proof of Lemma 4.2.10

From the Definition 4.2.9, $f=\sum_{n,k} f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)$ and $g=\sum_{n,k} g^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)$ and both sums involve finitely many non zero terms. Therefore, without loss of generality, we will assume that f and g are located on only one spherical harmonics: $f=f^{(n,k)}Y^{(n,k)}$ and $g=g^{(n',k')}Y^{(n',k')}$, for $f^{(n,k)}$ and $g^{(n',k')}$ simple admissible of degree (n,q) and (n',q') in the sense of Definition 4.2.5. The general result will follow by a finite summation.

Proof of (i). $y\mapsto f^{(n,k)}(|y|)$ is smooth outside the origin since f is smooth, and $y\mapsto Y^{(n,k)}\left(\frac{y}{|y|}\right)$ is also smooth outside the origin, hence f is smooth outside the origin. The Laplacian on spherical harmonics is:

$$(-\Delta)^i f = (-\Delta)^i\left(f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right) = ((-\Delta^{(n)})^i f^{(n,k)})(|y|)Y^{(n,k)}$$

where $-\Delta^{(n)}=-\partial_{rr}-\frac{d-1}{r}\partial r+n(d+n-2)$. From the expansion of $f^{(n,k)}$ (*4.2.10*), $(-\Delta^{(n)})^i f^{(n,k)}$ is bounded at the origin for each $i\in\mathbb{N}$. Therefore $(-\Delta)^i f$ is bounded at the origin for each i and f is smooth at the origin from elliptic regularity.

Proof of (ii). We treat the case where $n+n'$ is even, and the case $n+n'$ odd can be treated with verbatim the same arguments. As the product of the two spherical harmonics $Y^{(n,k)}Y^{n,k'}$ decomposes onto spherical harmonics of degree less than $n+n'$ with the same parity than $n+n'$, the product fg can be written:

$$fg = \sum_{0\leq \tilde n\leq n+n',\ \tilde n \text{ even},\ 1\leq \tilde k\leq k(\tilde n)} a_{n,k,n',k',\tilde n,\tilde k} f^{(n,k)}g^{(n',k')}Y^{(\tilde n,\tilde k)}$$

with $a_{n,k,n',k',\tilde n,\tilde k}$ some fixed coefficients. Now fix $\tilde n$ and $\tilde k$ in the sum, one has $n+n'=\tilde n+2i$ for some $i\in\mathbb{N}$. Using the Leibniz rule, as $\partial_r^j f^{(n,k)}=O(r^{q-j})$ and $\partial_r^j g^{(n,k)}=O(r^{q'-j})$ at infinity, we get that $\partial_r^j(f^{(n,k)}g^{(n',k')})=O(r^{q+q'-j})$ as $y\to+\infty$, which proves that $f^{(n,k)}g^{(n',k')}$ satisfies the asymptotic at infinity (*4.2.11*) of a simple admissible function of degree $(\tilde n, q+q')$. Close to the origin, the two expansions (*4.2.10*) for $f^{(n,k)}$ and $g^{(n',k')}$, starting at r^n and $r^{n'}$ respectively, imply the same expansion (*4.2.10*) starting at $y^{n+n'}$ for the product $f^{(n,k)}g^{(n',k')}$. As $n+n'=\tilde n+2i$, $f^{(n,k)}g^{(n,k)}$ satisfies the expansion at the origin (*4.2.10*) of a simple admissible function of degree $(\tilde n,q+q')$. Therefore $f^{(n,k)}g^{(n,k)}$ is simple admissible of degree $(\tilde n,q+q')$ and thus fg is simple admissible of degree $q+q'$.

Proof of (iii). We treat the case where n is even, and the case n odd can be treated with exactly the same reasoning. Let $1\leq i\leq d$, we just have to prove that $\partial_{y_i}f$ is admissible of degree $q-1$ and the result for higher order derivatives will follow by induction. We recall that $Y^{(n,k)}$ is the restriction of an homogenous harmonic polynomial of degree n to the sphere. We will still denote by $Y^{(n,k)}(y)$ this polynomial extended to the whole space $\mathbb{R}^d$ and they are related by $Y^{(n,k)}(y)=|y|^nY^{(n,k)}\left(\frac{y}{|y|}\right)$. This homogeneity implies $y.\nabla(Y^{(n,k)})(y)=nY^{(n,k)}(y)$ and leads to the identity:

$$\begin{array}{rcl}\partial_{y_i}\left[f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right] & = & \left(\partial_r f^{(n,k)}(|y|)-n\frac{f(|y|)}{|y|}\right)\frac{y_i}{|y|}Y^{(n,k)}\left(\frac{y}{|y|}\right)\\ & & +\frac{f(|y|)}{|y|}\partial_{y_i}Y^{(n,k)}\left(\frac{y}{|y|}\right).\end{array} \quad (4.2.22)$$

One has now to prove that the two terms in the right hand side are admissible of degree $q-1$. We only show it for the last term, the proof being the same for the first one. As $\partial_{y_i}Y^{(n,k)}\left(\frac{y}{|y|}\right)$ is an homogeneous polynomial of degree $n-1$ restricted to the sphere, it can be written as a finite sum of spherical harmonics of odd degrees (because n is even) less than $n-1$ and this gives:

$$\frac{f}{|y|}\partial_{y_i}Y^{(n,k)}\left(\frac{y}{|y|}\right)=\sum_{1\leq n'\leq n-1,\ n'\text{ odd},\ 1\leq k\leq k(n')}a_{i,n,k,n',k'}\frac{f}{|y|}Y^{(n',k')}\left(\frac{y}{|y|}\right)$$

for some coefficients $a_{i,n,k,n',k'}$. Now fix n',k' in the sum. At infinity $a_{i,n,k,n',k'}\frac{f(|y|)}{|y|}$ satisfies the asymptotic behavior (*4.2.11*) of a simple admissible function of degree $(n',q-1)$. Close to the origin, one has from (*4.2.10*), the fact that $n'+2j=n-1$ for some $j\in\mathbb{N}$, that for any $i\in\mathbb{N}$:

$$a_{i,n,k,n',k'}\frac{f(r)}{r}=\sum_{l=0}^{i}\tilde{c}_l r^{n-1+2l}+O(r^{n-1+2i+2})=\sum_{l=0}^{i}\hat{c}_l r^{n'+2j+2l}+O(r^{n'+2j+2i+2}),$$

which is the asymptotic behavior (*4.2.10*) of a simple admissible function of degree $(n',q-1)$ close to the origin. Therefore, $a_{i,n,k,n',k'}\frac{f(r)}{r}$ is a simple admissible function of degree $(n',q-1)$. Thus $\frac{f}{|y|}\partial_{y_i}Y^{(n,k)}\left(\frac{y}{|y|}\right)$ is an admissible function of degree $(q-1)$. The same reasoning works for the first term in the right hand side of (*4.2.22*), and therefore $\partial_{y_i}\left[f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right]$ is admissible of degree $q-1$.

Proof of (iv). We just showed in the last step that $\partial^\mu f$ is admissible of degree $q-|\mu|$ for all $\mu\in\mathbb{N}^d$, we then only have to prove (iv) for the case $\mu=(0,...,0)$. This can be showed via the following brute force bound for $|y|\geq 1$:

$$|f(y)|=\left|f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)\right|\leq\|Y^{(n,k)}\|_{L^\infty}|f^{(n,k)}(|y|)|\leq C|y|^q$$

from (*4.2.11*) since f is a simple admissible function of degree (n,q).

□

The next Lemma extends Lemma 4.2.7 to admissible functions. We do not give a proof, as it is a direct consequence of the latter.

Lemma 4.2.11 (Action of H on admissible functions). *Let f be an admissible function in the sense of Definition 4.2.9 written as $f(y)=\sum_{n,k}f^{(n,k)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right)$, of degree q, with $q>\gamma_n-d$. Assume that for all $n\in\mathbb{N}$ such that there exists k, $1\leq k\leq k(n)$ with $f^{(n,k)}\neq 0$ q satisfies $-q-\gamma_n-2\notin 2\mathbb{N}$. Then for all integer $i\in\mathbb{N}$, recalling that $H^{-1}f$ is defined by (4.2.20):*

(i) H^if is admissible of degree $q-2i$.

(ii) $H^{-i}f$ is admissible of degree $q+2i$.

4.2.5 Homogeneous functions

The approximate blow up profile we will build in the following subsection will look like $Q+\sum b_i^{(n,k)}T_i^{(n,k)}$ for some coefficients $b_i^{(n,k)}$ ($T_i^{(n,k)}$ being defined in (*4.2.21*)). The nonlinearity in the semilinear heat equation (NLH) will then produce terms that will be products of the profiles $T_i^{(n,k)}$ and coefficients $b_i^{(n,k)}$. Such non-linear terms are admissible functions multiplied by monomials of the coefficients $b_i^{(n,k)}$. The set of triples (n,k,i) for wich we will make a perturbation along $T_i^{(n,k)}$ is $\mathcal{I}$, defined in (*4.1.22*). Hence the vector b representing the perturbation will be:

$$b=(b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}=(b_1^{(0,1)},...,b_L^{(0,1)},b_1^{(1,1)},...,b_{L_1}^{(1,1)},...,b_0^{(n_0,k(n_0))},...,b_{L_{n_0}}^{(n_0,k(n_0))}) \tag{4.2.23}$$

We will then represent a monomial in the coefficients $b_i^{(n,k)}$ by a tuple of $\#\mathcal{I}$ integers:

$$J=(J_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}=(J_1^{(0,1)},...,J_L^{(0,1)},J_1^{(1,1)},...,J_{L_1}^{(1,1)},...,J_0^{(n_0,k(n_0))},...,J_{L_{n_0}}^{(n_0,k(n_0))})$$

through the formula:

$$b^J:=(b_1^{(0,1)})^{J_1^{(0,1)}}\times...\times(b_{L_{n_0}}^{(n_0,k(n_0))})^{J_{L_{n_0}}^{(n_0,k(n_0))}} \tag{4.2.24}$$

We associate three different lengths to J for the analysis. The first one, $|J|:=\sum J_i^{(n,k)}$, represents the number of parameters $b_i^{(n,k)}$ that are multiplied in the above formula, counted with multiplicity, i.e. the standard degree of b^J. In the analysis the coefficients $b_i^{(nk)}$ will have the size $|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$. The second length, $|J|_2:=\sum_{n,k,i}(\frac{\gamma-\gamma_n}{2}+i)J_i^{(n,k)}$ is tailor made to produce the following identity if these latter bounds hold:

$$|b^J|\lesssim(b_1^{(0,1)})^{|J|_2},$$

i.e. $|J|_2$ encodes the "size" of the real number b^J. For the construction of the approximate blow up profile, we will invert several times some elliptic equations, and the i-th inversion will be related to the following third length, $|J|_3:=\sum_{i=1}^L iJ_i^{(0,1)}+\sum_{1\leq i\leq L_1,\ 1\leq k\leq d}iJ_i^{(1,k)}+\sum_{(n,k,i)\in\mathcal{I},\ 2\leq n}(i+1)J_i^{(n,k)}$. To track information about of the non-linear terms generated by the semilinear heat equation (NLH) we eventually introduce the class of homogeneous functions.

Definition 4.2.12 (Homogeneous functions). Let b denote a $\#\mathcal{I}$-tuple under the form (*4.2.23*), $m\in\mathbb{N}$ and $q\in\mathbb{R}$. We recall that $|J|_2$ and $|J|_3$ are defined by (*4.1.24*) (*4.1.25*) and b^J is given by (*4.2.24*). We say that a function $S:\mathbb{R}^{\mathcal{I}}\times\mathbb{R}^d\to\mathbb{R}$ is homogeneous of degree (m,q) if it can be written as a finite sum:

$$S(b,y)=\sum_{J\in\mathcal{J}}b^JS_J(y),$$

$\#\mathcal{J}<+\infty$, where for each tuple $J\in\mathcal{J}$, one has that $|J|_3=m$ and that the function S_J is admissible of degree $2|J|_2+q$ in the sense of Definition 4.2.9.

As a direct consequence of the Lemma 4.2.10, and so we do not write here the proof, we obtain the following properties for homogeneous functions.

Lemma 4.2.13 (Calculus on homogeneous functions). *Let S and S' be two homogeneous functions of degree (m,q) and (m',q') in the sense of Definition 4.2.12, and $\mu\in\mathbb{N}^d$. Then:*

(i) $\partial^\mu S$ is homogeneous of degree $(m,q-|\mu|)$.

(ii) SS' *is homogeneous of degree* $(m+m', q+q')$.

(iii) If, writing $S=\sum_{J\in\mathfrak{J}} b^J \sum_{n,k} S_J^{(n,k)} Y^{(n,k)}$, *one has that* $2|J|_2+q>\gamma_n-d$ *and* $-2|J|_2-q-\gamma_n-2\notin 2\mathbb{N}$ *for all* n,J *such that there exists* k, $1\leq k\leq k(n)$ *with* $S_J^{(n,k)}\neq 0$, *then for all* $i\in\mathbb{N}$, $H^{-i}(S)$ *(given by (4.2.20)) is homogeneous of degree* $(m,q+2i)$.

4.3 The approximate blow-up profile

4.3.1 Construction

We first summarize the content and ideas of this section. We construct an approximate blow-up profile relying on a finite number of parameters close to the set of functions $(\tau_z(Q_\lambda))_{\lambda>0,\ z\in\mathbb{R}^d}$. It is built on the generalized kernel of H, Span$((T_i^{(n,k)})_{n,i\in\mathbb{N},\ 1\leq k\leq k(n)})$ defined by (*4.2.21*), and can therefore be seen as a part of a center manifold. The profile is built on the whole space $\mathbb{R}^d$ for the moment and will be localized later.

In Proposition 4.3.1 we construct a first approximate blow up profile. The procedure generates an error terms ψ, and by inverting elliptic equations, i.e. adding the term $H^{-1}\psi$ to our approximate blow up profile, one can always convert this error term into a new error term that is localized far away from the origin. We apply several times this procedure to produce an error term that is very small close to the origin. Then, in Proposition 4.3.3 we localize the approximate blow-up profile to eliminate the error terms that are far away from the origin. We will cut in the zone $|y|\approx B_1=B_0^{1+\eta}$ where $\eta\ll 1$ is a very small parameter. In this zone, the perturbation in the approximate blow-up profile has the same size than ΛQ, being the reference function for scale change. It will correspond to the self-similar zone $|x|\sim\sqrt{T-t}$ for the true blow-up function, where T will be the blow-up time.

The blow-up profile is described by a finite number of parameters whose evolution is given by the explicit dynamical system (*4.3.58*). In Lemma 4.3.4 we show the existence of special solutions describing a type II blow up with explicit blow-up speed. The linear stability of these solutions is investigated in Lemma 4.3.5.

There is a natural renormalized flow linked to the invariances of the semilinear heat equations (NLH). For u a solution of (NLH), $\lambda:[0,T(u_0))\to\mathbb{R}_+^*$ and $z:[0,T(u_0))\to\mathbb{R}^d$ two C^1 functions, if one defines for $s_0\in\mathbb{R}$ the renormalized time:

$$s(t):=s_0+\int_0^t\frac{1}{\lambda(t')^2}dt' \qquad (4.3.1)$$

and the renormalized function:

$$v(s,\cdot):=(\tau_{-z}u(t,\cdot))_\lambda,$$

then from a direct computation v is a solution of the renormalized equation:

$$\partial_s v-\frac{\lambda_s}{\lambda}\Lambda v-\frac{z_s}{\lambda}.\nabla v-F(v)=0. \qquad (4.3.2)$$

Our first approximate blow up profile is adapted to this new flow and is a special perturbation of Q.

Proposition 4.3.1 (First approximate blow up profile). *Let $L\in\mathbb{N}$, $L\gg 1$, and let $b=(b_i^{(n,k)})_{(n,k,i)\in\mathfrak{I}}$ denote a $\#\mathfrak{I}$-tuple of real numbers with $b_1^{(0,1)}>0$. There exists a $\#\mathfrak{I}$-dimensional manifold of C^∞ functions $(Q_b)_{b\in\mathbb{R}_+^*\times\mathbb{R}^{\#\mathfrak{I}-1}}$ such that:*

$$F(Q_b)=b_1^{(0,1)}\Lambda Q_b+b_1^{(1,\cdot)}.\nabla Q_b+\sum_{(n,k,i)\in\mathfrak{I}}\left(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)}\right)\frac{\partial Q_b}{\partial b_i^{(n,k)}}-\psi_b, \tag{4.3.3}$$

where $b_1^{(1,\cdot)}$ denotes the d-tuple of real numbers $(b_1^{(1,1)},...,b_1^{(1,d)})$ and where we used the convention $b_{L_n+1}^{(n,k)}=0$. ψ_b is an error term. Let B_1 be defined by (4.1.21). If the parameters satisfy the size conditions[4] *$b_1^{(0,1)}\ll 1$ and $|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ for all $(n,k,i)\in\mathfrak{I}$, then ψ_b enjoys the following bounds:*

(i) Global[5] bounds: *For* $0\le j\le s_L$,

$$\|H^j\psi_b\|^2_{L^2(|y|\le 2B_1)}\le C(L)(b_1^{(0,1)})^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}, \tag{4.3.4}$$

$$\|\nabla^j\psi_b\|^2_{L^2(|y|\le 2B_1)}\le C(L)(b_1^{(0,1)})^{2(\frac{j}{2}-m_0)+2(1-\delta_0)+g'-C(L)\eta} \tag{4.3.5}$$

where $C(L)$ is a constant depending on L only.

(ii) Local bounds:

$$\forall j\ge 0,\ \forall B>1,\ \int_{|y|\le B}|\nabla^j\psi_b|^2dy\le C(j,L)B^{C(j,L)}(b_1^{(0,1)})^{2L+6}. \tag{4.3.6}$$

where $C(L,j)$ is a constant depending on L and j only.

The profile Q_b is of the form:

$$Q_b:=Q+\alpha_b,\quad \alpha_b:=\sum_{(n,k,i)\in\mathfrak{I}}b_i^{(n,k)}T_i^{(n,k)}+\sum_{i=2}^{L+2}S_i, \tag{4.3.7}$$

where $T_i^{(n,k)}$ is given by (4.2.21), and the profiles S_i are homogeneous functions in the sense of definition 4.2.12 with:

$$deg(S_i)=(i,-\gamma-g') \tag{4.3.8}$$

and with the property that for all $2\le j\le L+2$, $\frac{\partial S_j}{\partial b_i^{(n,k)}}=0$ if $j\le i$ for $n=0,1$ and if $j\le i+1$ for $n\ge 2$.

Remark 4.3.2. The previous proposition is to be understood the following way. We have a special function depending on some parameters b close to Q, that it to say at scale 1 and with concentration point 0 for the moment. (*4.3.3*) means that the force term (i.e. when applying F) generated by (NLH) makes it concentrate at speed $b_1^{(0,1)}$ and translate at speed $b_1^{(1,\cdot)}$, while the time evolution of the parameters is an explicit dynamical system given by the third term. These approximations involve an error for which we have some explicit bounds (*4.3.4*) and (*4.3.6*).

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[4]This means that under the bounds $|b_i^{(n,k)}|\le K|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ for some $K>0$, there exists $b^*(K)$ such that the estimates that follow hold if $b_1^{(0,1)}\le b^*(K)$ with constants depending on K. K will be fixed independently of the other important constants in what follows.

[5]The zone $y\le B_1$ is called global because in the next proposition we will cut the profile Q_b in the zone $|y|\sim B_1$.

The size of this approximate profile is directly related to the size of the perturbation along $T_1^{(0,1)}$, the first term in the generalized kernel of H responsible for scale variation. Indeed we ask for $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$, and the size of the error is measured via $b_1^{(0,1)}$, see (*4.3.4*), (*4.3.5*) and (*4.3.6*). $b_1^{(0,1)}$ will therefore be the the universal order of magnitude in our problem.

Because of the shape of this approximate blow up profile (*4.3.7*), when including the time evolution of the parameters in (*4.3.3*) we get:

$$\partial_s(Q_b) - F(Q_b) + b_1^{(0,1)}\Lambda Q_b + b_1^{(1,\cdot)}.\nabla Q_b = \text{Mod}(s) + \psi_b, \tag{4.3.9}$$

where[6]:

$$\text{Mod}(s) = \sum_{(n,k,i)\in\mathcal{I}} [b_{i,s}^{(n,k)} + (2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)} - b_{i+1}^{(n,k)}] \left[T_i^{(n,k)} + \sum_{j=i+1+\delta_{n\geq 2}}^{L+2} \frac{\partial S_j}{\partial b_i^{(n,k)}} \right]. \tag{4.3.10}$$

For all $2 \leq j \leq L+2$, as S_j is homogeneous of degree $(j, -\gamma - g')$ in the sense of Definition 4.2.12 from (*4.3.8*), and from the fact that $\frac{\partial S_j}{\partial b_i^{(n,k)}} = 0$ if $j \leq i$ for $n = 0,1$ and if $j \leq i+1$ for $n \geq 2$, one has that for all j,n,k,i, $\frac{\partial S_j}{\partial b_i^{(0,1)}}$ is either 0 or is homogeneous of degree (a,b) with $a \geq 1$, meaning that it never contains non trivial constant functions independent of the parameters b. Hence, if the bounds $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ hold, since $|b_1^{(0,1)}| \lesssim 1$ and $-\gamma_n \geq -\gamma$ from (*4.1.1*), one has in particular that on compact sets for any $2 \leq j \leq L+2$ and $(n,k,i) \in \mathcal{I}$:

$$\frac{\partial S_j}{\partial b_i^{(n,k)}} = O(|b_1^{(0,1)}|). \tag{4.3.11}$$

Proof of Proposition 4.3.1

step 1 Computation of ψ_b. We first find an appropriate reformulation for the error ψ_b given by (*4.3.3*) when Q_b has the form (*4.3.7*).

- *rewriting of* $F(Q_b)$ *in* (*4.3.3*). We start by computing:

$$\begin{array}{rcl} -F(Q_b) & = & H(\alpha_b) - (f(Q_b) - f(Q) - \alpha_b f'(Q)) \\ & = & \sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} H T_i^{(n,k)} + \sum_{i=2}^{L+2} H(S_i) - (f(Q_b) - f(Q) - \alpha_b f'(Q)) \\ & = & -b_1^{(0,1)}\Lambda Q - b_1^{(1,\cdot)}.\nabla Q - \sum_{(n,k,i)\in\mathcal{I}} b_{i+1}^{(n,k)} T_i^{(n,k)} \\ & & + \sum_{i=2}^{L+2} H(S_i) - (f(Q_b) - f(Q) - \alpha_b f'(Q)) \end{array} \tag{4.3.12}$$

where we used the definition of the profiles $T_i^{(n,k)}$ from (*4.2.21*), and the convention $b_{L_n+1}^{(n,k)} = 0$. Now, for $i = 2, ..., L$, we regroup the terms that involve the multiplication of i parameters $b_j^{(n,k)}$ in the non linear term $-(f(Q_b) - f(Q) - \alpha_b f'(Q))$. Since p is an odd integer:

$$\begin{array}{rl} & (f(Q_b) - f(Q) - \alpha_b f'(Q)) = \sum_{k=2}^p C_k^p Q^{p-k} \alpha_b^k \\ = & \sum_{k=2}^p C_k^p Q^{p-k} \left[\sum_{|J|_1 = k} C_J \prod_{(n,k,i)\in\mathcal{I}} (b_i^{(n,k)})^{J_i^{(n,k)}} (T_k^{(n,k)})^{J_i^{(n,k)}} \prod_{i=2}^{L+2} S_i^{J_i} \right], \end{array} \tag{4.3.13}$$

where $J = (J_1^{(0,1)}, ..., J_{L_{n_0}}^{(n_0,k(n_0))}, J_2, ..., J_{L+2})$ represents a $(\#\mathcal{I} + L + 1)$-tuple of integers. Anticipating that the profile S_i will be an homogeneous profile of degree $(i, \gamma - g')$, we define for such tuples J:

$$|J|_3 = \sum_{i=1}^{L} i J_i^{(0,1)} + \sum_{1\leq i\leq L_1,\ 1\leq k\leq d} i J_i^{(1,k)} + \sum_{(n,k,i)\in\mathcal{I},\ 2\leq n} (i+1) J_i^{(n,k)} + \sum_{i=2}^{L+2} i J_i. \tag{4.3.14}$$

[6]Here $\delta_{n\geq 2} = 1$ if $n \geq 2$, and is zero otherwise.

We reorder the sum in the previous equation (*4.3.13*), partitioning the $\#\mathcal{I}+L+1$-tuples J according to their length $|J|_3$ instead of their length J_1:

$$(f(Q_b)-f(Q)-\alpha_b f'(Q))=\sum_{j=2}^{L+2}P_j+R,$$

P_j captures the terms with polynomials of the parameters $b_i^{(n,k)}$ of length $|J|_3=j$:

$$P_j=\sum_{k=2}^{p}C_kQ^{p-k}\left(\sum_{|J|=k,|J|_3=j}C_J\prod_{(n,k,i)\in\mathcal{I}}(b_i^{(n,k)})^{J_i^{(n,k)}}(T_k^{(n,k)})^{J_i^{(n,k)}}\prod_{i=2}^{L+2}S_i^{J_i}\right) \tag{4.3.15}$$

and the remainder contains only terms involving polynomials of the parameters $b_i^{(n,k)}$ of length $|\cdot|_3$ greater or equal to $L+3$:

$$R=(f(Q_b)-f(Q)-\alpha_b f'(Q))-\sum_{i=2}^{L+2}P_i. \tag{4.3.16}$$

From (*4.3.12*) we end up with the final decomposition :

$$-F(Q_b)=-b_1^{(0,1)}\Lambda Q-b_1^{(1,\cdot)}.\nabla Q-\sum_{(n,k,i)\in\mathcal{I}}b_{i+1}^{(n,k)}T_i^{(n,k)}+\sum_{i=2}^{L}H(S_i)-\sum_{i=2}^{L+2}P_i-R. \tag{4.3.17}$$

- *rewriting of the other terms in* (*4.3.3*). One has from the form of Q_b (*4.3.7*):

$$b_1^{(0,1)}\Lambda Q_b=b_1^{(0,1)}\Lambda Q+\sum_{(n,k,i)\in\mathcal{I}}b_1^{(0,1)}b_i^{(n,k)}\Lambda T_i^{(n,k)}+\sum_{i=2}^{L+2}b_1^{(0,1)}\Lambda S_i, \tag{4.3.18}$$

$$b_1^{(1,\cdot)}.\nabla Q_b=b_1^{(1,\cdot)}.\nabla Q+\sum_{j=1}^{d}\left(\sum_{(n,k,i)\in\mathcal{I}}b_1^{(1,j)}b_i^{(n,k)}\partial_{x_j}T_i^{(n,k)}+\sum_{i=2}^{L+2}b_1^{(1,j)}\partial_{x_j}S_i\right), \tag{4.3.19}$$

$$\begin{aligned}&\textstyle\sum_{(n,k,i)\in\mathcal{I}}\left(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)}\right)\frac{\partial Q_b}{\partial b_i^{(n,k)}}\\ =\;&\textstyle\sum_{(n,k,i)\in\mathcal{I}}\left(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)}\right)\left(T_i^{(n,k)}+\sum_{j=2}^{L+2}\frac{\partial S_j}{\partial b_i^{(n,k)}}\right).\end{aligned} \tag{4.3.20}$$

- ***Expression of the error term*** ψ_b. We define from (*4.2.16*):

$$\Theta_i^{(n,k)}(y):=\Theta_i^{(n)}(|y|)Y^{(n,k)}\left(\frac{y}{|y|}\right).$$

From (*4.3.17*), (*4.3.18*), (*4.3.19*) and (*4.3.20*), ψ_b given by (*4.3.3*) is a sum of terms that are polynomials in b, and, denoting a monomial by b^J, we rearrange them according to the value $|J|_3$:

$$\begin{aligned}\psi_b\;=\;&\textstyle\sum_{i=2}^{L+2}[\Phi_i+H(S_i)]+b_1^{(0,1)}\Lambda S_{L+2}+\sum_{j=1}^{d}b_1^{(1,j)}\partial_{x_j}S_{L+2}\\&\textstyle+\sum_{(n,k,i)\in\mathcal{I}}(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)})\frac{\partial S_{L+2}}{\partial b_i^{(n,k)}}-R,\end{aligned} \tag{4.3.21}$$

where the profiles Φ_i are given by the following formulas:

$$\begin{aligned}\Phi_2\;:=\;&\textstyle(b_1^{(0,1)})^2\Theta_1^{(0,1)}+\sum_{k=1}^{d}b_1^{(0,1)}b_1^{(1,k)}\Theta_1^{(1,k)}\\&\textstyle+\sum_{j=1}^{d}\left(b_1^{(1,j)}b_1^{(0,1)}\partial_{x_j}T_1^{(0,1)}+\sum_{k=1}^{d}b_1^{(1,j)}b_1^{(1,k)}\partial_{x_j}T_1^{(1,k)}\right)\\&\textstyle+\sum_{(n,k,0)\in\mathcal{I},\ n\geq2}\left(b_1^{(0,1)}b_0^{(n,k)}\Theta_0^{(n,k)}+\sum_{j=1}^{d}b_1^{(1,j)}b_0^{(n,k)}\partial_{x_j}T_0^{(n,k)}\right)-P_2,\end{aligned} \tag{4.3.22}$$

for $i=3...L+1$:

$$\begin{array}{lll} \Phi_i & := & b_1^{(0,1)}b_{i-1}^{(0,1)}\Theta_{i-1}^{(0,1)}+\sum_{k=1,\ (1,k,i-1)\in\mathfrak{I}}^d b_1^{(0,1)}b_{i-1}^{(1,k)}\Theta_{i-1}^{(1,k)} \\ & & +\sum_{j=1}^d \left(b_1^{(1,j)}b_{i-1}^{(0,1)}\partial_{x_j}T_{i-1}^{(0,1)}+\sum_{k=1,\ (1,k,i-1)\in\mathfrak{I}}^d b_1^{(1,j)}b_{i-1}^{(1,k)}\partial_{x_j}T_1^{(1,k)}\right) \\ & & +\sum_{(n,k,i-2)\in\mathfrak{I},\ n\geq 2}\left(b_1^{(0,1)}b_{i-2}^{(n,k)}\Theta_{i-2}^{(n,k)}+\sum_{j=1}^d b_1^{(1,j)}b_{i-2}^{(n,k)}\partial_{x_j}T_{i-2}^{(n,k)}\right) \\ & & +b_1^{(0,1)}\Lambda S_{i-1}+\sum_{m=1}^d b_1^{(1,m)}\partial_{x_m}S_{i-1} \\ & & +\sum_{(n,k,j)\in\mathfrak{I}}(-(2j-\alpha_n)b_1^{(0,1)}b_j^{(n,k)}+b_{j+1}^{(n,k)})\frac{\partial S_{i-1}}{\partial b_j^{(n,k)}}-P_i, \end{array} \quad (4.3.23)$$

$$\begin{array}{lll} \Phi_{L+2} & := & b_1^{(0,1)}\Lambda S_{L+1}+\sum_{m=1}^d b_1^{(1,m)}\partial_{x_m}S_{L+1} \\ & & +\sum_{(n,k,j)\in\mathfrak{I}}(-(2j-\alpha_n)b_1^{(0,1)}b_j^{(n,k)}+b_{j+1}^{(n,k)})\frac{\partial S_{L+1}}{\partial b_j^{(n,k)}}-P_{L+2} \end{array} \quad (4.3.24)$$

step 2 Definition of the profiles $(S_i)_{2\leq i\leq L+2}$ and simplification of ψ_b. We define by induction a sequence of couples of profiles $(S_i)_{2\leq i\leq L+2}$ by:

$$\left\{\begin{array}{l} S_2 := -H^{-1}(\Phi_2) \\ S_i := -H^{-1}(\Phi_i)\ \text{ for } 3\leq i\leq L+2,\ \ \Phi_i \text{ being defined by } (4.3.22),\ (4.3.23),\ (4.3.24) \end{array}\right. \quad (4.3.25)$$

where H^{-1} is defined by (*4.2.20*). In the next step we prove that there is no problem in this construction. The S_i's being defined this way, from (*4.3.21*) we get the final expression for the error:

$$\psi_b=b_1^{(0,1)}\Lambda S_{L+2}+\sum_{j=1}^d b_1^{(1,j)}\partial_{x_j}S_{L+2}+\sum_{(n,k,i)\in\mathfrak{I}}(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)})\frac{\partial S_{L+2}}{\partial b_i^{(n,k)}}-R. \quad (4.3.26)$$

step 3 Properties of the profiles S_i. We prove by induction on $i=2,...,L+2$ that S_i is homogeneous of degree $(i,-\gamma-g')$ in the sense of Definition 4.2.12, and that for all $2\leq j\leq L+2$, $\frac{\partial S_j}{\partial b_i^{(n,k)}}=0$ if $j\leq i$ for $n=0,1$ and if $j\leq i+1$ for $n\geq 2$.

- ***Initialization.*** We now prove that S_2 is homogeneous of degree $(2,-\gamma-g')$, and that $\frac{\partial S_2}{\partial b_i^{(n,k)}}=0$ if $2\leq i$ for $n=0,1$ and if $1\leq i$ for $n\geq 2$. We claim that Φ_2 is homogeneous of degree $(2,-\gamma-g'-2)$ and that $\frac{\partial \Phi_2}{\partial b_i^{(n,k)}}=0$ if $2\leq i$ for $n=0,1$ and if $1\leq i$ for $n\geq 2$. To prove this, we prove that these two properties are true for every term in the right hand side of (*4.3.22*).

From Lemma 4.2.8, $\Theta_1^{(0,1)}$ is simple admissible of degree $(0,-\gamma+2-g')$ in the sense of Definition 4.2.9. $(b_1^{(0,1)})^2$ can be written under the form $J_1^{(0,1)}=2$ and $J_i^{(n,k)}=0$ otherwise and one has $|J|_2=2$ and $|J|_3=2$. Therefore, $(b_1^{(0,1)})^2\Theta_1^{(0,1)}$ is homogeneous of degree $(|J|_3,-\gamma+2-g'-2|J|_2)=(2,-\gamma-g'-2)$. The same reasoning applies for $b_1^{(0,1)}b_1^{(1,k)}\Theta_1^{(1,k)}$ for $1\leq k\leq d$.

For $1\leq j\leq d$, $T_1^{(0,1)}$ is admissible of degree $(0,-\gamma+2)$ from Lemma 4.2.10 so $\partial_{x_j}T_1^{(0,1)}$ is admissible of degree $(-\gamma+1)$ from Lemma 4.2.8. $b_1^{(1,j)}b_1^{(0,1)}$ can be written under the form b^J with $J_1^{(0,1)}=1$, $J_1^{(1,j)}=1$ and $J_i^{(n,k)}=0$ otherwise, therefore $|J|_3=2$ and $|J|_2=1+\frac{\gamma-\gamma_1}{2}+1=2+\frac{\alpha-1}{2}$ from (*4.1.1*). Thus $b_1^{(1,j)}b_1^{(0,1)}\partial_{x_j}T_1^{(0,1)}$ is homogeneous of degree $(|J|_3,-\gamma 1+1-2|J|_2)=(2,-\gamma-2-\alpha)$. As $g'<\alpha$, it is then homogeneous of degree $(2,-\gamma-g'-2)$. The same reasoning applies for $1\leq j,k\leq d$ to the term $b_1^{(1,j)}b_1^{(1,k)}\partial_{x_j}T_1^{(1,k)}$.

We now examine for $(n,k,0)\in\mathfrak{I}$ the profile:

$$b_1^{(0,1)}b_0^{(n,k)}\Theta_0^{(n,k)}+\sum_{j=1}^d b_1^{(1,j)}b_0^{(n,k)}\partial_{x_j}T_0^{(n,k)}.$$

$\Theta_0^{(n,k)}$ is simple admissible of degree $(n,-\gamma_n-g')$ from Lemma 4.2.8. $b_1^{(0,1)}b_0^{(n,k)}$ can be written under the form b^J for $J_1^{(0,1)}=1$, $J_0^{(n,k)}=1$ and $J_i^{(n',k')}=0$ otherwise, and one then has $|J|_3=2$ and $|J|_2=1+\frac{\gamma-\gamma_n}{2}$. Therefore, $b_1^{(0,1)}b_0^{(n,k)}\Theta_0^{(n,k)}$ is homogeneous of degree $(|J|_3,-\gamma_n-g'-2|J|_2)=(2,-\gamma-g'-2)$. Similarly the terms in the sum in the above identity are homogeneous of degree $(2,-\gamma-g'-2)$.

We now look at the non-linear term P_2. As for $2\leq i\leq L+2$ the profile S_i involves polynomials of b under the form b^J with $|J|_3=i$, from its definition (*4.3.15*) P_2 does not depend on the profiles S_i for $2\leq i\leq L+2$ and can be written as:

$$P_2=CQ^{p-2}\left(b_1^{(0,1)}T_1^{(0,1)}+\sum_{k=1}^d b_1^{(1,k)}T_1^{(1,k)}+\sum_{(n,k,0)\in\mathcal{I}} b_0^{(n,k)}T_0^{(n,k)}\right)^2$$

for a constant C. We have to prove that all the mixed terms that are produced by this formula are homogeneous of degree $(2,\gamma-g'-2)$. We write it only for one term, and apply the same reasonning to the others. For all $((n,k,0),(n',k',0))\in\mathcal{I}^2$, from Lemmas 4.2.8 and 4.2.13 and (*3.2.1*), the profile $b_0^{(n,k)}b_0^{(n',k')}Q^{p-2}T_0^{(n,k)}T_0^{(n',k')}$ is homogeneous of degree $(2,-\gamma-2-\alpha)$ and then of degree $(2,-\gamma-2-g')$. As we said, similar considerations yield that all the other terms are homogeneous of degree $(2,\gamma-g'-2)$. This implies that P_2 is homogeneous of degree $(2,-\gamma-g'-2)$.

We have examined all terms in (*4.3.22*) and consequently proved that Φ_2 is homogeneous of degree $(2,-\gamma-2-g')$. By a direct check at all the terms in the right hand side of (*4.3.22*), with P_2 given by the above identity, one has that $\frac{\partial\Phi_2}{\partial b_i^{(n,k)}}=0$ if $2\leq i$ for $n=0,1$ and if $1\leq i$ for $n\geq 2$. We now check that we can apply (iii) in Lemma 4.2.13 to invert Φ_2 and to propagate the homogeneity. For all $\#\mathcal{I}$-tuple J with $|J|_3=2$, one has indeed for all integer n that $2|J|_2-\gamma_n-2-g'>\gamma_n-d$ as the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing and $d-2\gamma-2>0$. For the second condition required by the Lemma, we notice that g' is not a "fixed" constant in our problem, as its definition (*4.1.4*) involves a parameter ϵ. The purpose of the parameter ϵ is the following: by choosing it appropriately, we can suppose that for every $0\leq n\leq n_0$ and $\#\mathcal{I}$-tuple J with $|J|_3=2$ there holds:

$$-2|J|_2+\gamma+g'-\gamma_n\notin 2\mathbb{N}.$$

This allows us to apply (iii) in Lemma 4.2.13: S_2 is homogeneous of degree $(2,-\gamma-g')$. We also get that $\frac{\partial S_2}{\partial b_i^{(n,k)}}=0$ if $2\leq i$ for $n=0,1$ and if $1\leq i$ for $n\geq 2$ as this is true for Φ_2. This proves the initialization of our induction.

- ***Heredity.*** Suppose $3\leq i\leq L+1$, and that for $2\leq i'\leq i$, $S_{i'}$ is homogeneous of degree $(i',-\gamma-g')$, and that $\frac{\partial S_i'}{\partial b_j^{(n,k)}}=0$ if $i'\leq j$ for $n=0,1$ and if $i'-1\leq j$ for $n\geq 2$. We claim that Φ_i is homogeneous of degree $(i,-\gamma-g'-2)$ and that $\frac{\partial\Phi_i}{\partial b_j^{(n,k)}}=0$ if $i\leq j$ for $n=0,1$ and if $i-1\leq j$ for $n\geq 2$. We prove it by looking at all the terms in the right hand side of (*4.3.23*). With the same reasoning we used for the initialization, we prove that

$$\begin{array}{cc} & b_1^{(0,1)}b_{i-1}^{(0,1)}\Theta_{i-1}^{(0,1)}+\sum_{k=1,\ (1,k,i-1)\in\mathcal{I}}^d b_1^{(0,1)}b_{i-1}^{(1,k)}\Theta_{i-1}^{(1,k)} \\ + & \sum_{j=1}^d\left(b_1^{(1,j)}b_{i-1}^{(0,1)}\partial_{x_j}T_{i-1}^{(0,1)}+\sum_{k=1,\ (1,k,i-1)\in\mathcal{I}}^d b_1^{(1,j)}b_{i-1}^{(1,k)}\partial_{x_j}T_1^{(1,k)}\right) \\ + & \sum_{(n,k,i-2)\in\mathcal{I},\ n\geq 2}\left(b_1^{(0,1)}b_{i-2}^{(n,k)}\Theta_{i-2}^{(n,k)}+\sum_{j=1}^d b_1^{(1,j)}b_{i-2}^{(n,k)}\partial_{x_j}T_{i-2}^{(n,k)}\right)\end{array}$$

is homogeneous of degree $(i,\gamma-g'-2)$. From the induction hypothesis, $b_1^{(0,1)}\Lambda S_{i-1}$ is homogeneous of degree $(i,-\gamma-g'-2)$. From Lemma 4.2.10, for $1\leq j\leq d$, $\partial_{x_j}S_{i-1}$ is homogeneous of degree

$(i-1,-\gamma-g'-1)$, so that $b_1^{(1,j)}\partial_{x_j}S_{i-1}$ is homogeneous of degree $(i,-\gamma-g'-2-\alpha)$, α being positive, it is then homogeneous of degree $(i,-\gamma-g'-2)$. Still from the induction hypothesis, for all $(n,k,i')\in\mathfrak{I}$, $(-(2i'-\alpha_n)b_1^{(0,1)}b_{i'}^{(n,k)}+b_{i'+1}^{(n,k)})\frac{\partial S_{i-1}}{\partial b_{i'}^{(n,k)}}$ is homogeneous of degree $(i,-\gamma-g'-2)$. The last term to be consider is P_i. As for $2\leq j\leq L+2$ the profile S_j involves polynomials of b under the form b^J with $|J|_3=i$, from its definition (*4.3.15*) P_i does not depend on the profiles S_j for $i\leq j\leq L+2$ and can be written as:

$$P_i=\sum_{k=2}^{p}C_kQ^{p-k}\left(\sum_{|J|=k,|J|_3=i}C_J\prod_{(n,k,i)\in\mathfrak{I}}(b_i^{(n,k)})^{J_i^{(n,k)}}(T_k^{(n,k)})^{J_i^{(n,k)}}\prod_{j=2}^{i-1}S_j^{J_j}\right)$$

Let k be an integer $2\leq k\leq p$, let J be a $\#\mathfrak{I}+L$-tuple with $|J|_3=i$. Then from the induction hypothesis,

$$Q^{p-k}\prod_{(n,k,i)\in\mathfrak{I}}(b_i^{(n,k)})^{J_i^{(n,k)}}(T_k^{(n,k)})^{J_i^{(n,k)}}\prod_{j=2}^{i-1}S_j^{J_j}$$

is homogeneous of degree $(i,-\gamma-2-(k-1)\alpha-g'\sum_{j=2}^{i-1}J_j)$. As $k\geq 2$ and $\alpha>g'$, it is homogeneous of degree $(i,\gamma-2-g')$.

We just proved that Φ_i is homogeneous of degree $(i,-\gamma-2-g')$. By a direct check at all the terms in the right hand side of (*4.3.23*), with P_i given by the above formula, one has that $\frac{\partial\Phi_i}{\partial b_j^{(n,k)}}=0$ if $i\leq j$ for $n=0,1$ and if $i-1\leq j$ for $n\geq 2$. We now check that we can apply (iii) from Lemma 4.2.13 to get the desired properties for $S_i=-H^{-1}\Phi_i$. For all $\#\mathfrak{I}$-tuple J with $|J|_3=i$ and integer n, the first condition $|J|_2-\gamma-2-g'>\gamma_n-d$ is fulfilled since $-2\gamma_n-d\geq-2\gamma-d>2$. For the second condition, again as in the initialization, as g' is not a "fixed" constant in our problem (its definition (*4.1.4*) involving a parameter ϵ), we can choose it such that for every $0\leq n\leq n_0$ and $\#\mathfrak{I}$-tuple J with $|J|_3=i$:

$$-2|J|_2+\gamma+g'-\gamma_n\notin 2\mathbb{N}.$$

We thus can apply (iii) in Lemma 4.2.13: S_i is homogeneous of degree $(i,-\gamma-g')$. One also obtains that $\frac{\partial S_i}{\partial b_j^{(n,k)}}=0$ if $i\leq j$ for $n=0,1$ and if $i-1\leq j$ for $n\geq 2$ as this is true for Φ_i. This proves the heredity in our induction.

The last step, that it is the heredity from $L+1$ to $L+2$, can be proved verbatim the same way and we do not write it here.

step 4 Bounds for the error term. In Step 2 we have computed the expression (*4.3.26*) of the error term ψ_b. In Step 3 we proved that the profiles S_i were well defined and homogeneous of degree $(i,-\gamma-g')$. We can now prove the bounds on ψ_b claimed in the Proposition. In the sequel we always assume the bounds $|b_i^{(n,k)}|\lesssim|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ and $|b_1^{(0,1)}|\ll 1$.

- ***Homogeneity of*** ψ_b. We claim that ψ_b is a finite sum of homogeneous functions of degree $(i,-\gamma-g'-2)$ for $i\geq L+3$. For this we consider all terms in the right hand side of (*4.3.26*). As S_{L+2} is homogeneous of degree $(L+2,-\gamma-g')$ from Step 3, the function $b_1^{(0,1)}\Lambda S_{L+2}$ is homogeneous of degree $(L+3,-\gamma-g'-2)$ from Lemma 4.2.13. Similarly for $1\leq j\leq d$, $b_1^{(1,j)}\partial_{x_j}S_{L+2}$ is homogeneous of degree $(L+3,-\gamma-g'-2-\alpha)$ (and then homogeneous of degree $(L+3,-\gamma-g'-2)$ as $\alpha>0$), and for $(n,k,i)\in\mathfrak{I}$, $(-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)})\frac{\partial S_{L+2}}{\partial b_i^{(n,k)}}$ is homogeneous of degree $(L+3,-\gamma-g'-2)$.

From its definition (*4.3.16*), and as for $2 \leq i \leq L+2$, S_i is homogeneos of degree $(i, -\gamma - g')$, R is a finite sum of homogeneous profiles of degree $(i, -\gamma - \alpha - 2)$ with $i \geq L+3$. All this implies that ψ_b is a finite sum of homogeneous functions of degree $(i, -\gamma - g' - 2)$ for $i \geq L+3$.

- ***Proof of an intermediate estimate.*** We claim that there exists an integer $A \geq L+3$ such that for μ a d-tuple of integers, $j \in \mathbb{N}$ and $B > 1$ there holds:

$$\int_{|y|\leq B} \frac{|\partial^\mu \psi_b|^2}{1+|y|^{2j}} dy \leq C(L) \sum_{i=L+3}^{A} |b_1^{(0,1)}|^{2i} B^{\max(4i+4(m_0-\frac{|\mu|+j}{2})+4(\delta_0-1)-2g',0)}. \qquad (4.3.27)$$

We now prove this bound. We proved earlier that ψ_b is a finite sum of homogeneous functions of degree $(i, -\gamma - g' - 2)$ for $i \geq L+3$. Consequently, it suffices to prove this bound for an homogeneous function $b^J f(y)$ of degree $(|J|_3, -\gamma - g' - 2)$ with $|J|_3 \geq L+3$. One then computes as f is admissible of degree $(2|J|_2 - \gamma - g' - 2)$:

$$\begin{array}{rcl} \int_{|y|\leq B} \frac{|b^J \partial^\mu f|^2}{1+|y|^{2j}} & \leq & C(f)|b_1^{(0,1)}|^{2|J|_2} \int_0^B (1+r)^{4|J|_2-2\gamma-2g'-4-2j-2|\mu|} r^{d-1} dr \\ & \leq & C(f)|b_1^{(0,1)}|^{2|J|_2} B^{\max(4|J|_2+4(m_0+\frac{j+|\mu|}{2})+4(\delta_0-1)-2g',0)} \end{array}$$

(we avoid the logarithmic case in the integral by changing a bit the value of g' defined in (*4.1.4*), by changin a bit the value of ϵ). This concludes the proof of (*4.3.27*).

- ***Proof of the local bounds for the error.*** Let j be an integer, and $\mu \in \mathbb{N}^d$ with $|\mu| = j$. From (*4.3.27*), $|b_1^{(0,1)}| \ll 1$ and $B > 1$ we obtain from (*4.3.27*):

$$\int_{|y|\leq B} |\partial^\mu \psi_b|^2 dy \leq C(L)|b_1^{(0,1)}|^{2L+6} B^{\max(4A+4(m_0-\frac{|\mu|+j}{2})+4(\delta_0-1)-2g',0)}$$

which gives the desired bound (*4.3.6*).

- ***Proof of the global bounds for the error.*** Let $j \leq 2s_L$, and $\mu \in \mathbb{N}^d$ with $|\mu| = j$. Using (*4.3.27*), we notice that for $L+3 \leq i \leq A$ one has

$$\max(4i + 4(m_0 - \frac{|\mu|+j}{2}) + 4(\delta_0 - 1) - 2g', 0) = 4i + 4(m_0 - \frac{|\mu|+j}{2}) + 4(\delta_0 - 1) - 2g'$$

This implies:

$$\begin{array}{rcl} \int_{|y|\leq B_1} \frac{|\partial^\mu \psi_b|^2}{1+|y|^{2j}} dy & \leq & C(L) \sum_{i=L+3}^{A} |b_1^{(0,1)}|^{2i} B_1^{4i+4(m_0-\frac{|\mu|+j}{2})+4(\delta_0-1)-2g'} \\ & \leq & C(L)|b_1^{(0,1)}|^{2(\frac{j}{2}-m_0)+2(1-\delta_0)+g'-C(L)\eta}. \end{array}$$

which is the desired bound (*4.3.5*). Let j be an integer, $j \leq s_L$. Now, as $H = -\Delta + V$ where V is a smooth potential satisfying $|\partial^\mu V| \leq C(\mu)(1+|y|)^{-2-|\mu|}$ from (*3.2.10*) one obtains using (*4.3.27*):

$$\begin{array}{rl} & \int_{|y|\leq B_1} |H^j \psi_b|^2 dy \leq C(L) \sum_{j'+|\mu|_1=2j} \int_{|y|\leq B_1} \frac{|\partial^\mu \psi_b|^2}{1+|y|^{2j'}} dy \\ \leq & C(L) \sum_{j'+|\mu|=2j} \sum_{i=L+3}^{A} |b_1^{(0,1)}|^{2i} B_1^{\max(4i+4(m_0-j)+4(\delta_0-1)-2g',0)} \\ \leq & C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta} \end{array}$$

(because again $4i + 4(m_0 - j) + 4(\delta_0 - 1) - 2g' > 0$ as $i \geq L+3$ and $j \leq s_L$). This proves the last estimate (*4.3.4*).

$\square$

We now localize the perturbation built in Proposition 4.3.1 in the zone $|y| \leq B_1$ and estimate error generated by the cut. We also include the time dependance of the parameters following Remark 4.3.2. We recall that s_L is defined by (*4.1.7*)

Proposition 4.3.3 (Localization of the perturbation). *χ is a cut-off defined by (4.1.26). We keep the notations from Proposition 4.3.1. $I=(s_0,s_1)$ is an interval, and*

$$\begin{array}{rcl} b:I & \to & \mathbb{R}^{\#\mathcal{I}} \\ s & \mapsto & (b_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I}} \end{array}$$

is a C^1 function with the following a priori bounds[7]*:*

$$|b_i^{(n,k)}|\lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}, \quad 0<b_1^{(0,1)}\ll 1, \quad |b_{1,s}^{(0,1)}|\lesssim |b_1^{(0,1)}|^2. \tag{4.3.28}$$

We define the profile $\tilde{Q}_b$ as:

$$\tilde{Q}_b := Q+\tilde{\alpha}_b = Q+\chi_{B_1}\alpha_b, \quad \tilde{\alpha}_b := \chi_{B_1}\alpha_b. \tag{4.3.29}$$

Then one has the following identity (Mod(s) being defined by (4.3.10)):

$$\partial_s\tilde{Q}_b - F(\tilde{Q}_b)+b_1^{(0,1)}\Lambda\tilde{Q}_b+b_1^{(1,\cdot)}.\nabla\tilde{Q}_b=\tilde{\psi}_b+\chi_{B_1}\textbf{\textit{Mod}}(s) \tag{4.3.30}$$

with, for $0<\eta\ll 1$ small enough, an error term $\tilde{\psi}_b$ satisfying the following bounds:

(i) Global bounds: *For any integer j with $1\leq j\leq s_L-1$ there holds:*

$$\int_{\mathbb{R}^d}|H^j\tilde{\psi}_b|^2dy\leq C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C_j\eta}. \tag{4.3.31}$$

For any real number $s_c\leq j<2s_L-2$:

$$\int_{\mathbb{R}^d}|\nabla^j\tilde{\psi}_b|^2dy\leq C(L)|b_1^{(0,1)}|^{2(\frac{j}{2}-m_0)+2(1-\delta_0)-C_j\eta}. \tag{4.3.32}$$

And for $j=s_L$ one has the improved bound:

$$\int_{\mathbb{R}^d}|H^{s_L}\tilde{\psi}_b|^2dy\leq C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)+2\eta(1-\delta_0')}. \tag{4.3.33}$$

(ii) Local bounds: *one has that (ψ_b being defined by (4.3.3)):*

$$\forall |y|<B_1, \ \ \tilde{\psi}_b(y)=\psi_b, \tag{4.3.34}$$

and for any $1\leq B\leq B_1$ and $j\in\mathbb{N}$:

$$\int_{|y|\leq B}|\nabla^j\tilde{\psi}_b|^2dy\leq C(L,j)B^{C(L,j)}|b_1^{(0,1)}|^{2L+6}. \tag{4.3.35}$$

Proof of Proposition 4.3.3

First, we compute the expression of the new error term by rewriting the left hand side of (*4.3.30*) using (*4.3.9*) and the fact that $F(Q)=0$:

$$\begin{array}{rcl} \tilde{\psi}_b & = & \chi_{B_1}\psi_b+\partial_s(\chi_{B_1})\tilde{\alpha}_b-[F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))] \\ & & +b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q)+b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b)-\chi_{B_1}\Lambda\alpha_b) \\ & & +b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q)+b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b). \end{array} \tag{4.3.36}$$

[7] This means that under the bounds $|b_i^{(n,k)}|\leq K|b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ for some $K>0$, there exists $b^*(K)$ such that the estimates that follow hold if $b_1^{(0,1)}\leq b^*(K)$ with constants depending on K. K will be fixed independently of the other important constants in what follows.

Local bounds. In the previous identity, one clearly sees that all the terms, except $\chi_{B_1}\psi_b$, have their support in $B_1 \leq |y|$. Thus, for $B \leq B_1$, the bound (*4.3.35*) is a direct consequence of the local bound (*4.3.6*) for ψ_b.

Global bounds. Let $m_1+1 \leq j \leq s_L$. We will prove the bounds (*4.3.31*) and (*4.3.33*) by proving that this estimate holds for all terms in the right hand side of (*4.3.36*). The reasoning to prove the estimates will be similar from one term to another. For this reason, we shall go quickly whenever an argument has already been used earlier.

- *The $\chi_{B_1}\psi_b$ term.* As $H=-\Delta+V$ for V a smooth potential with $\partial^\mu V \lesssim (1+|y|)^{-2-|\mu|}$ from (*3.2.10*), and as $(\partial_r^k(\chi_{B_1}))(r) = B_1^{-k}\partial_r^k\chi(\frac{r}{B_1})$ there holds the identity:

$$H^j(\chi_{B_1}\psi_b) = \chi_{B_1}H^j\psi_b + \sum_{\mu\in\mathbb{N}^d,\ 0\leq|\mu|\leq 2j-1}^{j} f_\mu \partial^\mu \psi_b$$

where for each $\mu \in \mathbb{N}^d$, $0 \leq |\mu| \leq j-1$, f_μ has its support in $B_1 \leq |x| \leq 2B_1$ and satisfies: $|f_\mu| \leq C(L)B_1^{-(2j-|\mu|)}$. Using (*4.3.4*) and (*4.3.5*) we obtain:

$$\begin{array}{rcl} \int_{\mathbb{R}^d} |H^j(\chi_{B_1}\psi_b)|^2 dy & \leq & C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta} \\ & & + \sum_{\mu\in\mathbb{N}^d,\ 0\leq|\mu|\leq 2j-1}^{j} B_1^{-(4j-2|\mu|)} b_1^{2(\frac{|\mu|}{2}-m_0+2(1-\delta_0)+g'-C(L)\eta)} \\ & \leq & C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}. \end{array} \quad (4.3.37)$$

Similarly, one obtains for any integer j' with $0 \leq j' \leq 2s_L - 2$:

$$\int_{\mathbb{R}^d} |\nabla^{j'}(\chi_{B_1}\psi_b)|^2 \leq C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)+g'-C(L)\eta}. \quad (4.3.38)$$

Using interpolation, this estimate remains true for any real number j' with $0 \leq j' \leq 2s_L - 2$.

- *The $\partial_s(\chi_{B_1})\alpha_b$ term.* We first split from (*4.3.7*):

$$\partial_s(\chi_{B_1})\alpha_b = \partial_s(\chi_{B_1})\left(\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)}T_i^{(n,k)} + \sum_{i=2}^{L+2} S_i\right) \quad (4.3.39)$$

We compute $\partial_s(\chi_{B_1}) = (b_1^{(0,1)})^{-1}b_{1,s}^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi_{B_1})(\frac{y}{B_1})$. One first treat the S_i terms. As we already explained in the study of the $\chi_{B_1}\psi_b$ term one has:

$$H^j(\partial_s(\chi_{B_1})S_i) = \sum_{\mu\in\mathbb{N}^d,\ |\mu|\leq 2j} f_\mu\partial^\mu S_i$$

with f_μ a smooth function, with support in $B_1 \leq |x| \leq 2B_1$ and satisfying $|f_\mu| \leq C(L)b_1^{(0,1)}B_1^{-(2j-|\mu|_1)}$ (because $|b_{1,s}^{(0,1)}| \lesssim |b_1^{(0,1)}|^2$ from (*4.3.28*)). As S_i is homogeneous of degree $(i, -\gamma-g')$ in the sense of Definition 4.2.12 from (*4.3.8*) and $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ we get using Lemma 4.2.13:

$$\int_{\mathbb{R}^d} |H^j(\partial_s(\chi_{B_1})S_i)|^2 dy \leq C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+g'-C(L)\eta}. \quad (4.3.40)$$

Now we treat the $T_i^{(n,k)}$ terms in the identity (*4.3.39*). Let $(i,n,k) \in \mathcal{I}$. Then again one has the decomposition:

$$H^j[\partial_s(\chi_{B_1})b_i^{(n,k)}T_i^{(n,k)}] = b_i^{(n,k)} \sum_{\mu\in\mathbb{N}^d,\ |\mu|\leq 2j} f_\mu\partial^\mu T_i^{n,k}$$

with f_μ a smooth function, with support in $B_1 \leq |y| \leq 2B_1$ and satisfying $|f_\mu| \leq C(L) b_1^{(0,1)} B_1^{-(2j-|\mu|)}$. As $T_i^{(n,k)}$ is an admissible profile of degree $(-\gamma_n + 2i)$ in the sense of Definition 4.2.9 from (*4.2.21*) and Lemma 4.2.8, $\partial^\mu T_i^{n,k}$ is admissible of degree $(-\gamma_n + 2i - |\mu|)$ from Lemma 4.2.10 and we compute:

$$
\begin{aligned}
\int_{\mathbb{R}^d} |b_i^{(n,k)} f_\mu \partial^\mu T_i^{n,k}|^2 dy &\leq \frac{C(L)|b_1^{(0,1)}|^{\gamma-\gamma_n+2i+2}}{B_1^{2(2j-|\mu|1)}} \int_{B_1}^{2B_1} r^{-2\gamma_n+4i-2|\mu|_1} r^{d-1} dr \\
&\leq C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\eta(2j-2i-2\delta_n-2m_n)}
\end{aligned}
$$

As $(i,n,k) \in \mathcal{I}$, $i \leq L_n$ so if $j = s_L$ one has: $2j - 2i - 2\delta_n - 2m_n \geq 2 - 2\delta_n$. Therefore we have proved the bound (we recall that $\delta_0' = \max\limits_{0 \leq n \leq n_0} \delta_n \in (0,1)$):

$$
\int_{\mathbb{R}^d} |H^j(\partial_s(\chi_{B_1}) b_i^{(n,k)} T_i^{(n,k)})|^2 dy \leq \left\{ \begin{array}{ll} C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } m_0+1 \leq j < s_L, \\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)+\eta(1-\delta_0')} & \text{if } j = s_L. \end{array} \right. \tag{4.3.41}
$$

From the decomposition (*4.3.39*), the bounds (*4.3.40*) and (*4.3.41*), we deduce the bound:

$$
\int_{\mathbb{R}^d} |H^j(\partial_s(\chi_{B_1})\alpha_b|^2 dy \leq \left\{ \begin{array}{ll} C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } 0 \leq j < s_L, \\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}(|b_1^{(0,1)}|^{2\eta(1-\delta_0')} + |b_1^{(0,1)}|^{g'-C(L)\eta}) & \text{if } j = s_L. \end{array} \right. \tag{4.3.42}
$$

Using verbatim the same arguments, one gets that for any integer $0 \leq j' \leq 2s_L - 2$:

$$
\int_{\mathbb{R}^d} |\nabla^{j'}(\partial_s(\chi_{B_1})\alpha_b|^2 dy \leq C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta}. \tag{4.3.43}
$$

which remains true for any real number j' with $0 \leq j' \leq 2s_L - 2$ from interpolation.
- ***The** $F(Q + \chi_{B_1}\alpha_b) - F(Q) - \chi_{B_1}(F(Q+\alpha_b) - F(Q))$ **term.*** It writes:

$$
\begin{aligned}
& F(Q + \chi_{B_1}\alpha_b) - F(Q) - \chi_{B_1}(F(Q+\alpha_b) - F(Q)) \\
= \ & \Delta(\chi_{B_1}\alpha_b) - \chi_{B_1}\Delta\alpha_b + (Q + \chi_{B_1}\alpha_b)^p - Q^p - \chi_{B_1}((Q+\alpha_b)^p - Q^p).
\end{aligned} \tag{4.3.44}
$$

We now prove the bound for the two terms that have appeared. From the identity:

$$
\Delta(\chi_{B_1}\alpha_b) - \chi_{B_1}\Delta\alpha_b = \Delta(\chi_{B_1})\alpha_b + 2\nabla\chi_{B_1}.\nabla\alpha_b,
$$

as χ is radial and as $(\partial_r^k(\chi_{B_1}))(r) = B_1^{-k}\partial_r^k\chi(\frac{r}{B_1})$, one sees that this term can be treated exactly the same we treated the previous term: $\partial_s(\chi_{B_1})\alpha_b$. This is why we claim the following estimates that can be proved using exactly the same arguments:

$$
\int_{\mathbb{R}^d} |H^j(\Delta(\chi_{B_1}\alpha_b) - \chi_{B_1}\Delta\alpha_b)|^2 dy \leq \left\{ \begin{array}{ll} C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } m_0+1 \leq j < s_L, \\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}(|b_1^{(0,1)}|^{2\eta(1-\delta_0')} + |b_1^{(0,1)}|^{g'-C(L)\eta}) & \text{if } j = s_L. \end{array} \right. \tag{4.3.45}
$$

We now turn to the other term in (*4.3.44*) that can be rewritten as:

$$
(Q + \chi_{B_1}\alpha_b)^p - Q^p - \chi_{B_1}((Q+\alpha_b)^p - Q^p) = \sum_{k=2}^{p} C_k^p Q^{p-k}\chi_{B_1}(\chi_{B_1}^{k-1} - 1)\alpha_b^k.
$$

All the terms are localized in the zone $B_1 \leq |y| \leq 2B_1$. From the definition (*4.3.7*) of α_b, (*4.3.8*), (*3.2.1*) and Lemma 4.2.13, for each $2 \leq k \leq p$ one has that $Q^{p-k}\alpha_b^k$ is a finite sum of homogeneous profiles of degree $(i, -\gamma - \alpha - 2)$ for $i \geq k$, yielding:

$$
\int_{\mathbb{R}^d} |H^j((Q+\chi_{B_1}\alpha_b)^p - Q^p - \chi_{B_1}((Q+\alpha_b)^p - Q^p))|^2 dy \leq C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{4.3.46}
$$

From the decomposition (*4.3.44*) and the estimates (*4.3.45*) and (*4.3.46*) one gets:

$$\begin{array}{ll} & \int_{\mathbb{R}^d} |H^j(F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q)))|^2 dy \\ \leq & C(L)\left\{ \begin{array}{ll} |b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } m_0+1\leq j<s_L, \\ |b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}(|b_1^{(0,1)}|^{2\eta(1-\delta_0')}+|b_1^{(0,1)}|^{\alpha-C(L)\eta}) & \text{if } j=s_L. \end{array}\right. \end{array} \quad (4.3.47)$$

As for the study of the two previous terms the same methods yield the analogue estimate for $\nabla^{j'}[F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q))]$ for any integer $0\leq j'\leq 2s_L-2$, and by interpolation, we obtain for any real number j' with $0\leq j'\leq 2s_L-2$:

$$\int_{\mathbb{R}^d} |\nabla^{j'}(F(Q+\chi_{B_1}\alpha_b)-F(Q)-\chi_{B_1}(F(Q+\alpha_b)-F(Q)))|^2 dy\leq C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta}. \quad (4.3.48)$$

- ***The*** $b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q)$ ***term.*** As $\partial^\mu(\Lambda Q)\leq C(\mu)(1+|y|)^{-\gamma-|\mu|}$ for all $\mu\in\mathbb{N}^d$ from (*4.2.2*) and $H\Lambda Q=0$ one computes:

$$\begin{array}{lll} \int_{\mathbb{R}^d} |H^j(b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q))|^2 dy & \leq & C(j)|b_1^{(0,1)}|^2\int_{B_1}^{2B_1} r^{-2\gamma-4j}r^{d-1}dr \\ & \leq & C(j)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+2\eta(j-m_0-\delta_0)} \end{array} \quad (4.3.49)$$

with for $j=s_L$, $s_L-m_0-\delta_0=L+1-\delta_0>1-\delta_0$. For any integer j' with $E[s_c]\leq j'\leq 2s_L-2$, similar reasonings yield the estimate:

$$\int_{\mathbb{R}^d} |\nabla^{j'}(b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q))|^2 dy\leq C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}.$$

By interpolation, one has for any real number j' with $E[s_c]\leq j'\leq 2s_L-2$:

$$\int_{\mathbb{R}^d} |\nabla^{j'}(b_1^{(0,1)}(\Lambda Q-\chi_{B_1}\Lambda Q))|^2 dy\leq C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}. \quad (4.3.50)$$

- ***The*** $b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b)-\chi_{B_1}\Lambda\alpha_b)$ ***term.*** First we write this term as:

$$b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b)-\chi_{B_1}\Lambda\alpha_b=b_1^{(0,1)}(y.\nabla\chi_{B_1})\alpha_b.$$

Now we notice that the term $b_1^{(0,1)}(y.\nabla\chi_{B_1})=b_1^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi)(\frac{|y|}{B_1})$ is very similar to the term we already studied $\partial_s(\chi_{B_1})=(b_1^{(0,1)})^{-1}b_{1,s}^{(0,1)}\frac{|y|}{B_1}(\partial_r\chi_{B_1})(\frac{y}{B_1})$, in the sense that it enjoys the same estimates, as $|b_{1,s}^{(0,1)}|\lesssim(b_1^{(0,1)})^2$ from (*4.3.28*). Thus, we can get exactly the same estimates for the term $b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b)-\chi_{B_1}\Lambda\alpha_b)$ that we obtained previously for the term $\partial_s(\chi_{B_1})\alpha_b$ with verbatim the same methodology, yielding:

$$\begin{array}{ll} & \int_{\mathbb{R}^d} |H^j(b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b)-\chi_{B_1}\Lambda\alpha_b))|^2 dy \\ \leq & \left\{ \begin{array}{ll} C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)-C(L)\eta} & \text{if } 0\leq j<s_L, \\ C(L)|b_1^{(0,1)}|^{2L+2+2(1-\delta_0)}(|b_1^{(0,1)}|^{2\eta(1-\delta_0')}+|b_1^{(0,1)}|^{g'-C(L)\eta}) & \text{if } j=s_L, \end{array}\right. \end{array} \quad (4.3.51)$$

and for any integer j' with $0\leq j'\leq 2s_L-2$:

$$\int_{\mathbb{R}^d} |\nabla^{j'}(b_1^{(0,1)}(\Lambda(\chi_{B_1}\alpha_b)-\chi_{B_1}\Lambda\alpha_b))|^2 dy\leq C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(L)\eta}. \quad (4.3.52)$$

- ***The*** $b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q)$ ***term.*** First we rewrite:

$$b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q)=\sum_{i=1}^d b_1^{(1,i)}(1-\chi_{B_1})\partial_{y_i}Q. \quad (4.3.53)$$

Now let i be an integer, $1\leq i\leq d$. From the asymptotic (*3.2.1*) of the ground state $|\partial^\mu Q|\leq C(\mu)(1+|y|)^{-\frac{2}{p-1}-|\mu|}$ and the fact that $H\partial_{x_i}Q=0$ we deduce:

$$\begin{array}{rcl} \int_{\mathbb{R}^d}|H^j(b_1^{(1,i)}((1-\chi_{B_1})\partial_{y_i}Q)|^2dy & \leq & C(j)|b_1^{(0,1)}|^{\gamma-\gamma_1+2}\int_{B_1}^{2B_1}r^{-2\gamma_1-4j}r^{d-1}dr \\ & \leq & C(j)|b_1^{(0,1)}|^{2(j-m_0)-2(1-\delta_0)+2\eta(j-m_1-\delta_1)}. \end{array}$$

with for $j=s_L$, $s_L-m_1-\delta_1=L+m_0-m_1+1-\delta_1>1-\delta_1$. So we finally get, putting together the two previous equations:

$$\begin{array}{rcl} \int_{\mathbb{R}^d}|H^j(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q))|^2dy & \leq & C(j)|b_1^{(0,1)}|^2\int_{B_1}^{+\infty}r^{-2\gamma-4j}r^{d-1}dr \\ & \leq & C(j)|b_1^{(0,1)}|^{2(j-m_0)-2(1-\delta_0)+2\eta(1-\delta_1)}. \end{array} \tag{4.3.54}$$

Now, for any integer j' with $E[s_c]\leq j'\leq 2s_L-2$, as $E[s_c]>s_c-1$, similar reasonings yield the estimate:

$$\int_{\mathbb{R}^d}|\nabla^{j'}(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q))|^2dy\leq C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}.$$

By interpolation, one has for any real number j' with $E[s_c]\leq j'\leq 2s_L-2$:

$$\int_{\mathbb{R}^d}|\nabla^{j'}(b_1^{(1,\cdot)}.(\nabla Q-\chi_{B_1}\nabla Q))|^2dy\leq C(j')|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)-C(j')\eta}. \tag{4.3.55}$$

- ***The*** $b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)$ ***term.*** We first rewrite:

$$b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b)=\sum_{i=1}^d b_1^{(1,i)}\partial_{y_i}(\chi_{B_1})\alpha_b.$$

Let i be an integer, $1\leq i\leq d$. For all $\mu\in\mathbb{N}^d$, $\partial^\mu(\chi_{B_1})\leq C(\mu)B_1^{-|\mu|}$. From (*4.3.7*) and (*4.3.8*), α_b is a sum of homogeneous profiles of degree $(i,-\gamma)$. Using Lemma 4.2.13 one computes:

$$\int_{\mathbb{R}^d}|H^j(b_1^{(1,i)}\partial_{y_i}(\chi_{B_1})\alpha_b)|^2dy\leq C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}.$$

With the two previous equations one has proved that:

$$\int_{\mathbb{R}^d}|H^j(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b))|^2dy\leq C(L)|b_1^{(0,1)}|^{2(j-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{4.3.56}$$

Using verbatim the same arguments, one can prove that for any integer $0\leq j'\leq 2s_L-2$, the analogue estimate for $\nabla^{j'}(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b))$ holds. By interpolation, it gives that for any real number $0\leq j'\leq 2s_L-2$ there holds:

$$\int_{\mathbb{R}^d}|\nabla^{j'}(b_1^{(0,1)}.(\nabla(\chi_{B_1}\alpha_b)-\chi_{B_1}\nabla\alpha_b))|^2dy\leq C(L)|b_1^{(0,1)}|^{2(\frac{j'}{2}-m_0)+2(1-\delta_0)+\alpha-C(L)\eta}. \tag{4.3.57}$$

- ***End of the proof.*** For the estimate concerning the operator H (resp. the operator ∇), we have estimated all terms in the right hand side of (*4.3.36*) in (*4.3.37*), (*4.3.42*), (*4.3.47*), (*4.3.49*), (*4.3.51*), (*4.3.54*) and (*4.3.56*) (resp. the right hand side of (*4.3.36*) in (*4.3.38*), (*4.3.43*), (*4.3.48*), (*4.3.50*), (*4.3.52*), (*4.3.55*) and (*4.3.57*)). Adding all these estimates, as $0<b_1^{(0,1)}\ll 1$ is a very small parameter, one sees that there exists $\eta_0:=\eta_0(L)$ such that for $0<\eta<\eta_0$, the bounds (*4.3.31*) and (*4.3.33*) hold (resp. the bound (*4.3.32*) holds).

□

4.3.2 Study of the approximate dynamics for the parameters

In Proposition 4.3.3 we have stated the existence of a profile $\tilde{Q}_b$ such that the force term $F(\tilde{Q}_b)$ generated by (NLH) has an almost explicit formulation in terms of the parameters $b=(b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ up to an error term $\tilde{\psi}_b$. Suppose that for some time, the solution that started at $\tilde{Q}_{b(0)}$ stays close to this family of approximate solutions, up to scaling and translation invariances, meaning that it can be written approximately as $\tau_{z(t)}\left(\tilde{Q}_{b(t),\frac{1}{\lambda(t)}}\right)$. Then $\tilde{Q}_{b(s)}$ is almost a solution of the renormalized flow (*4.3.2*) associated to the functions of time $\lambda(t)$ and $z(t)$, meaning that:

$$\partial_s(\tilde{Q}_b)-\frac{\lambda_s}{\lambda}\Lambda\tilde{Q}_b-\frac{z_s}{\lambda}.\nabla\tilde{Q}_b-F(\tilde{Q}_b)\approx 0.$$

Using the identity (*4.3.30*) this means:

$$-\left(b_1^{(0,1)}+\frac{\lambda_s}{\lambda}\right)\Lambda\tilde{Q}_b-(b_1^{1,\cdot)}+\frac{z_s}{\lambda}).\nabla\tilde{Q}_b+\chi_{B_1}\mathrm{Mod}(s)\approx 0.$$

From the very definition (*4.3.10*) of the modulation term $\mathrm{Mod}(s)$, projecting the previous relation onto the different modes that appeared[8] yields:

$$\begin{cases}\frac{\lambda_s}{\lambda}=-b_1^{(0,1)},\\ \frac{z_s}{\lambda}=-b_1^{(1,\cdot)},\\ b_{i,s}^{(n,k)}=-(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}+b_{i+1}^{(n,k)},\quad\forall(n,k,i)\in\mathcal{I}\end{cases}\tag{4.3.58}$$

with the convention $b_{L_n+1}^{(n,k)}=0$. The understanding of a solution starting at $\tilde{Q}_{b(0)}$ then relies on the understanding of the solutions of the finite dimensional dynamical system (*4.3.58*) driving the evolution of the parameters $b_i^{(n,k)}$. First we derive some explicit solutions such that $\lambda(t)$ touches 0 in finite time, signifying concentration in finite time.

Lemma 4.3.4 (Special solutions for the dynamical system of the parameters). *We recall that the renormalized time s is defined by (4.3.1). Let $\ell\leq L$ be an integer such that $2\alpha<\ell$. We define the functions:*

$$\begin{cases}\bar{b}_i^{(0,1)}(s)=\frac{c_i}{s^i}\ \textit{for}\ 1\leq i\leq\ell,\\ \bar{b}_i^{(0,1)}=0\ \textit{for}\ \ell<i\leq L,\\ \bar{b}_i^{(n,k)}=0\ \textit{for}\ \ (n,k,i)\in\mathcal{I}\ \textit{with}\ n\geq 1,\end{cases}\tag{4.3.59}$$

with $(c_i)_{1\leq i\leq\ell}$ being ℓ constants defined by induction as follows:

$$c_1=\frac{\ell}{2\ell-\alpha}\ \ \textit{and}\ \ c_{i+1}=-\frac{\alpha(\ell-i)}{2\ell-\alpha}c_i\ \ \textit{for}\ \ 1\leq i\leq\ell-1.\tag{4.3.60}$$

Then $\bar{b}=(\bar{b}_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ is a solution of the last equation in (4.3.58). Moreover, the solutions $\lambda(s)$ and $z(s)$ of the first two equations in (4.3.58) starting at $\lambda(0)=1$ and $z(0)=0$, taken in original time variable t are $z(t)=0$ and:

$$\lambda(t)=\left(\frac{\alpha}{(2\ell-\alpha)s_0}\right)^{\frac{\ell}{\alpha}}\left(\frac{(2\ell-\alpha)}{\alpha}s_0-t\right)^{\frac{\ell}{\alpha}}.\tag{4.3.61}$$

[8] This will be done rigorously in the next section.

Proof of Lemma 4.3.4

It is a direct computation that can safely be left to the reader.

□

As $s_0 > 0$ and $2\ell > \alpha$, (*4.3.61*) can be interpreted as: there exists $T > 0$ with $\lambda(t) \approx (T-t)^{\frac{\ell}{\alpha}}$ as $t \to T$. Now, given $\frac{\alpha}{2} < \ell \leq L$, we want to know the exact number of instabilities of the particular solution $\bar{b}$. In addition, in Propositions 4.3.1 and 4.3.3, we needed the a priori bounds $|b_i^{(n,k)}| \lesssim |b_1^{(0,1)}|^{\frac{\gamma-\gamma_n}{2}+i}$ to show sufficient estimates for the errors ψ_b and $\tilde{\psi}_b$. Around the solution $\bar{b}$ defined by (*4.3.59*), $b_1^{(0,1)}$ is of order s^{-1}, and so the a priori bounds we need become[9] $b_i^{(n,k)} \lesssim s^{\frac{\gamma_n-\gamma}{2}-i}$. Therefore, by "stability" of $\bar{b}$ we mean stability with respect to this size and introduce the following renormalization for a solution of (*4.3.58*) close to $\bar{b}$:

$$b_i^{(n,k)} = \bar{b}_i^{(n,k)} + \frac{U_i^{(n,k)}}{s^{\frac{\gamma-\gamma_n}{2}+i}}. \tag{4.3.62}$$

It defines a $\#\mathfrak{I}$-tuple of real numbers $U = (U_i^{(n,k)})_{(n,k,i)\in\mathfrak{I}}$, and we order the parameters as in (*4.2.23*) by

$$U = (U_1^{(0,1)}, ..., U_L^{(0,1)}, U_1^{(1,1)}, ..., U_{L_1}^{(1,1)}, ..., U_0^{(n_0,k(n_0))}, ..., U_{L_{n_0}}^{(n_0,k(n_0))}) \tag{4.3.63}$$

In the following lemma we state the linear stability result for the renormalized perturbation $(U_i^{(n,k)})_{(n,k,i)\in\mathfrak{I}}$.

Lemma 4.3.5. *(Linear stability of special solutions)*

Suppose b is a solution of the last equation in (4.3.58). Define $U = (U_i^{(n,k)})_{(n,k,i)\in\mathfrak{I}}$ by (4.3.62) and order it as in (4.3.63).

(i) Linearized dynamics*: the time evolution of U is given by:*

$$\partial_s U = \frac{1}{s} A U + O\left(\frac{|U|^2}{s}\right), \tag{4.3.64}$$

where A is the bloc diagonal matrix:

$$A = \begin{pmatrix} A_\ell & & & (0) \\ & \tilde{A}_1 & & \\ & & ... & \\ (0) & & & \tilde{A}_{n_0} \end{pmatrix}.$$

The matrix A_ℓ is defined by:

$$A_\ell = \begin{pmatrix}
-(2-\alpha)c_1 + \alpha\frac{\ell-1}{2\ell-\alpha} & 1 & & & & & & & \\
. & . & . & & & & & & \\
-(2i-\alpha)c_i & & \alpha\frac{\ell-i}{2\ell-\alpha} & 1 & & & & & \\
. & & & . & . & & (0) & & \\
-(2\ell-\alpha)c_\ell & & & & 0 & 1 & & & \\
0 & & & & & -\alpha\frac{1}{2\ell-\alpha} & . & & \\
. & & & & & & . & 1 & \\
0 & (0) & & & & & & -\alpha\frac{i-\ell}{2\ell-\alpha} & . & \\
. & & & & & & & & . & 1 \\
0 & & & & & & & & & -\alpha\frac{(L-\ell)}{2\ell-\alpha}
\end{pmatrix}, \tag{4.3.65}$$

[9] One notices that this bound holds for $\bar{b}_i^{(n,k)}$.

The matrix $\tilde{A}_1$ is a bloc diagonal matrix constituted of d matrices $\tilde{A}_1'$:

$$\tilde{A}_1=\begin{pmatrix}\tilde{A}_1' & & (0)\\ & . & \\ (0) & & \tilde{A}_1'\end{pmatrix},\quad \tilde{A}_1'=\begin{pmatrix}\alpha\frac{\ell-\frac{\alpha-1}{2}-1}{2\ell-\alpha} & 1 & & & & (0)\\ & . & . & & & \\ & & \alpha\frac{\ell-\frac{\alpha-1}{2}-i}{2\ell-\alpha} & 1 & & \\ & & & . & . & \\ & & & & . & 1\\ (0) & & & & & \alpha\frac{\ell-\frac{\alpha-1}{2}-L_1}{2\ell-\alpha}\end{pmatrix}, \tag{4.3.66}$$

and for $2\leq n\leq n_0$ the matrix $\tilde{A}_n$ is a bloc diagonal matrix constituted of $k(n)$ times the matrix $\tilde{A}_n'$:

$$\tilde{A}_n=\begin{pmatrix}\tilde{A}_n' & & (0)\\ & . & \\ (0) & & \tilde{A}_n'\end{pmatrix},\quad \tilde{A}_n'=\begin{pmatrix}\alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}}{2\ell-\alpha} & 1 & & & & (0)\\ & . & . & & & \\ & & \alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-i}{2\ell-\alpha} & 1 & & \\ & & & . & . & \\ & & & & . & 1\\ (0) & & & & & \alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-L_n}{2\ell-\alpha}\end{pmatrix}. \tag{4.3.67}$$

(ii) Diagonalization, stability and instability*: A is diagonalizable because A_ℓ and $\tilde{A}_n$ for $1\leq n\leq n_0$ are. A_ℓ is diagonalizable into the matrix*

diag$(-1,\frac{2\alpha}{2\ell-\alpha},..,\frac{i\alpha}{2\ell-\alpha},..,\frac{\ell\alpha}{2\ell-\alpha},\frac{-1}{2\ell-\alpha},..,\frac{\ell-L}{2\ell-\alpha})$. *We denote the eigenvector of A associated to the eigenvalue -1 by v_1 and the eigenvectors associated to the unstable modes $\frac{2\alpha}{\ell-\alpha}$, ..., $\frac{\ell\alpha}{\ell-\alpha}$ of A by $v_2,...,v_\ell$. They are a linear combination of the ℓ first components only. That is to say there exists a $\#\mathfrak{I}\times\#\mathfrak{I}$ matrix coding a change of variables:*

$$P_\ell:=\begin{pmatrix}P_\ell' & 0\\ 0 & Id_{\#\mathfrak{I}-\ell}\end{pmatrix}, \tag{4.3.68}$$

with P_ℓ' an invertible $\ell\times\ell$ matrix and $Id_{\#\mathfrak{I}-\ell}$ the $(\#\mathfrak{I}-\ell)\times(\#\mathfrak{I}-\ell)$ identity matrix such that:

$$P_\ell A P_\ell^{-1}=\begin{pmatrix}A_\ell' & & & (0)\\ & \tilde{A}_1 & & \\ & & ... & \\ (0) & & & \tilde{A}_{n_0}\end{pmatrix} \tag{4.3.69}$$

$$A_\ell'=\begin{pmatrix}-1 & & (0) & q_1 & & & & \\ & \frac{2\alpha}{2\ell-\alpha} & & q_2 & & & & \\ & & . & & & & & \\ & & & \frac{\ell\alpha}{2\ell-\alpha} & q_\ell & & (0) & \\ & & & & \frac{-\alpha}{2\ell-\alpha} & 1 & & \\ & & & & & . & . & \\ & & (0) & & & & . & 1\\ & & & & & & & \alpha\frac{\ell-L}{2\ell-\alpha}\end{pmatrix}. \tag{4.3.70}$$

with $(q_i)_{1\leq i\leq\ell}\in\mathbb{R}^\ell$ being some fixed coefficients. $\tilde{A}_1'$ has max$(E[i_1],0)$ non negative eigenvalues and L_1- max$(E[i_1],0)$ strictly negative eigenvalues (i_n being defined by (4.1.12)). For $2\leq n\leq n_0$, $\tilde{A}_n'$ has max$(E[i_n]+1,0)$ non negative eigenvalues and L_n+1- max$(E[i_n]+1,0)$ strictly negative eigenvalues.

Proof of Lemma 4.3.5

Proof of (i). as b and $\bar{b}$ are solutions of (*4.3.58*), we compute (with the convention $\bar{b}_{L_n+1}^{(n,k)}=0$ and $U_{L_n+1}^{(n,k)}=0$):

$$\begin{aligned} U_{i,s}^{(n,k)} &= \frac{1}{s}\Big[\Big(\frac{\gamma-\gamma_n}{2}+i-(2i-\alpha_n)\bar{b}_1^{(0,1)}s\Big)U_i^{(n,k)}-(2i-\alpha_n)\bar{b}_i^{(n,k)}s^{\frac{\gamma-\gamma_n}{2}+i}U_1^{(0,1)} \\ &\quad -(2k-\alpha_n)U_1^{(0,1)}U_i^{(n,k)}+U_{i+1}^{(n,k)}\Big]. \end{aligned}$$

As $\bar{b}_1^{(0,1)}=\frac{\ell}{2\ell-\alpha}$, we obtain $\frac{\gamma-\gamma_n}{2}+i-(2i-\alpha_n)\bar{b}_1^{(0,1)}=\alpha\frac{\ell-\frac{\gamma-\gamma_n}{2}-i}{2\ell-\alpha}$. We then get (*4.3.65*) by noticing that $\bar{b}_i^{(0,1)}=0$ for $i\geq \ell+1$ and because by definition $\gamma=\gamma_0$. We get (*4.3.66*) and (*4.3.67*) by noticing that $\bar{b}_i^{(n,k)}=0$ for $i\geq 1$.

Proof of (ii). $\tilde{A}_n$ for $1\leq n\leq n_0$ is diagonalizable because it is upper triangular. Their eigenvalues are then the values on the diagonal, and the last statement in (ii), about the stability and instability directions comes from the very definition (*4.1.12*) of the real number i_n for $1\leq n\leq n_0$. It remains to prove that A_ℓ is diagonalizable. We will do it by calculating its characteristic polynomial.

- *Computation of the characteristic polynomial for the top left corner matrix*: we let A_ℓ' be the $\ell\times\ell$ matrix:

$$A_\ell'=\begin{pmatrix} -(2-\alpha)c_1+\alpha\frac{\ell-1}{2\ell-\alpha} & 1 & & & (0) \\ . & . & . & & \\ -(2i-\alpha)c_i & & \alpha\frac{\ell-i}{2\ell-\alpha} & 1 & \\ . & (0) & & . & 1 \\ -(2\ell-\alpha)c_l & & & & 0 \end{pmatrix},$$

We recall that as $\alpha>2$, $\ell\geq 2$ so A_ℓ' has at least 2 rows and 2 lines. We let $\mathcal{P}_\ell(X)=\det(A_\ell'-XId)$. We compute this determinant by developing with respect to the last row and iterating by doing that again for the sub-determinant appearing in the process. Eventually we obtain an expression of the form:

$$\begin{aligned} \mathcal{P}_\ell=(-1)^\ell(2\ell-\alpha)c_\ell \quad &+ \quad (-X)\Big[(-1)^{\ell+1}(2\ell-2-\alpha)c_{\ell-1}+(\tfrac{\alpha}{2\ell-\alpha}-X) \\ &\times \quad \Big[(-1)^\ell(2\ell-4-\alpha)c_{\ell-2}+(\tfrac{2\alpha}{2\ell-\alpha}-X)[...]\Big]\Big]. \end{aligned} \tag{4.3.71}$$

We define the polynomials $(A_i)_{1\leq i\leq\ell}$ and $(B_i)_{1\leq i\leq \ell}$ and $(C_i)_{1\leq i\leq\ell-1}$ as:

$$A_i:=(-1)^{\ell-i+1}(2\ell+2-2i-\alpha)c_{\ell+1-i} \quad \text{and} \quad B_i:=(i-1)\frac{\alpha}{2\ell-\alpha}-X, \tag{4.3.72}$$

$$C_i:=(-1)^{\ell+1-i}(X(2\ell-2i-\alpha)c_{\ell-i}+\frac{2\ell-\alpha}{i}c_{\ell-i+1}). \tag{4.3.73}$$

This way, the determinant $\mathcal{P}_\ell$ given by (*4.3.71*) can be rewritten as:

$$\mathcal{P}_\ell=A_1+B_1\left(A_2+B_2\left[A_3+B_3\left[...\right]\right]\right). \tag{4.3.74}$$

We notice by a direct computation from (*4.3.72*) and (*4.3.73*) that:

$$A_1+B_1A_2=C_1.$$

Moreover, this identity propagates by induction and we claim that for $1\leq i\leq \ell-2$:

$$C_i+B_1B_2A_{i+2}=B_{i+2}C_{i+1}.$$

Indeed, from (*4.3.60*) one has $\frac{2\ell-\alpha}{i+1}c_{\ell-i}=-\alpha c_{\ell-i-1}$, and from (*4.3.72*) and (*4.3.73*):

$$\begin{array}{rc} & B_{i+2}C_{i+1}-C_i \\ = & ((i+1)\frac{\alpha}{2\ell-\alpha}-X)(-1)^{\ell-i}(X(2\ell-2i-2-\alpha)c_{\ell-i-1}+\frac{2\ell-\alpha}{i+1}c_{\ell-i}) \\ & -(-1)^{\ell+1-i}(X(2\ell-2i-\alpha)c_{\ell-i}+\frac{2\ell-\alpha}{i}c_{\ell-i+1}) \\ = & (-1)^{\ell-i}\Big(((i+1)\frac{\alpha}{2\ell-\alpha}-X)(X(2\ell-2i-2-\alpha)c_{\ell-i-1}-\alpha c_{\ell-i-1}) \\ & -X(2\ell-2i-\alpha)\alpha\frac{i+1}{2\ell-\alpha}c_{\ell-i-1}+\alpha^2\frac{i+1}{2\ell-\alpha}c_{\ell-i-1}\Big) \\ = & (-1)^{\ell-i}c_{\ell-i-1}X\Big(\alpha\frac{i+1}{2\ell-\alpha}(2\ell-2i-2-\alpha)+\alpha-X(2\ell-2i-2-\alpha)-\frac{2\ell-2i-\alpha}{2\ell-\alpha}\alpha(i+1)\Big) \\ = & (-1)^{\ell-i}c_{\ell-i-1}X(2\ell-2i-2-\alpha)(\frac{\alpha}{2\ell-\alpha}-X) \\ = & A_{i+2}B_1B_i \end{array}$$

From the above identity we can rewrite $\mathcal{P}_\ell$ given by (*4.3.74*) as:

$$\begin{aligned} \mathcal{P}_\ell &= A_1+B_1A_2+B_1B_2A_3+B_1B_2B_3(A_4+B_4(...)) \\ &= C_1+B_1B_2A_3+B_1B_2B_3(A_4+B_4(...)) \\ &= B_3(C_2+B_1B_2(A_4+B_4(...)) \\ &= B_3B_4(C_3+B_1B_2(A_5+B_5(...)) \\ &\dots \\ &= B_3...B_\ell(C_{\ell-1}+B_1B_2). \end{aligned} \tag{4.3.75}$$

The last polynomial that appeared is from (*4.3.72*) and (*4.3.73*):

$$C_{\ell-1}+B_1B_2=X(2-\alpha)c_1+\frac{2\ell-\alpha}{\ell-1}c_2-X\left(\frac{\alpha}{2\ell-\alpha}-X\right)=(X+1)\left(X-\frac{\alpha\ell}{2\ell-\alpha}\right)$$

and so we end up from (*4.3.75*) with the final identity for $\mathcal{P}_\ell$:

$$\mathcal{P}_\ell=(X+1)\prod_{i=2}^{\ell}\left(\frac{i\alpha}{2\ell-\alpha}-X\right).$$

This means that A_ℓ' is diagonalizable with eigenvalues $(1,-\frac{2\alpha}{2\ell-\alpha},...,\frac{\ell}{2\ell-\alpha})$: there exists an invertible $\ell\times\ell$ matrix $\tilde{P}_\ell$ such that $\tilde{P}_\ell A_\ell\tilde{P}_\ell^{-1}=\text{diag}(-1,\frac{2}{2\ell-\alpha},...,\frac{\ell}{2\ell-\alpha})$. We denote the by P_ℓ the matrix:

$$P_\ell':=\begin{pmatrix}\tilde{P}_\ell & \\ & \text{Id}_{L-\ell}\end{pmatrix}$$

Then, from (*4.3.65*), there exists ℓ real numbers $(q_i)_{1\leq i\leq n}\in\mathbb{R}^\ell$ such that:

$$P_\ell'A_\ell(P_\ell')^{-1}=\begin{pmatrix} -1 & & (0) & & q_1 & & & \\ & \frac{2\alpha}{2\ell-\alpha} & & & q_2 & & & \\ & & \cdot & & & & & \\ & & & \frac{\ell\alpha}{2\ell-\alpha} & q_\ell & & (0) & \\ & & & & \frac{-\alpha}{2\ell-\alpha} & 1 & & \\ & & & & & \cdot & \cdot & \\ & & (0) & & & & \cdot & 1 \\ & & & & & & & \alpha\frac{\ell-L}{2\ell-\alpha} \end{pmatrix}.$$

This implies that A_ℓ can be diagonalized and that its eigenvalues are of simple multiplicity and that they are given by $(-1, \frac{2\alpha}{2\ell-\alpha}, ..., \alpha\frac{\ell}{2\ell-\alpha}, -\frac{\alpha}{2\ell-\alpha}, .., -\alpha\frac{L-\ell}{2\ell-\alpha})$, and that the eigenvectors associated to the eigenvalues -1, and $\alpha\frac{2}{2\ell-\alpha}, ..., \alpha\frac{\ell}{2\ell-\alpha}$ are linear combinations of the ℓ first components only. This concludes the proof of Lemma.

□

4.4 Main proposition and proof of Theorem 2.2.9

We recall that the approximate blow up profile $\tau_z(\tilde{Q}_{\bar{b},\frac{1}{\lambda}})$ was designed for a blow up on the whole space $\mathbb{R}^d$. In this section, we state in the main Proposition 4.4.6 of this chapter the existence of solutions staying in a trapped regime (defined in Definition 4.4.4) close to the cut approximate blow up profile $\chi\tau_z(\tilde{Q}_{\bar{b},\frac{1}{\lambda}})$. We then end the proof of Theorem 2.2.9 by proving that such a solution will blow up as described in the theorem.

4.4.1 The trapped regime and the main proposition

4.4.1.1 Projection of the solution on the manifold of approximate blow up profiles

The following reasoning is made for a blow up on the whole space $\mathbb{R}^d$. As in this case our blow up solution should stay close to the manifold of approximate blow up profiles $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,z,\lambda}$ we want to decompose it as a sum $\tau_z(\tilde{Q}_{b,\lambda}+\varepsilon_\lambda)$ for some parameters b, z, λ such that ε has "minimal" size. The tangent space of $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,z,\lambda}$ at the point Q is $\text{Span}((T_i^{(n,k)})_{(n,k,i)\in\mathcal{I}\cup\{(0,1,0),(1,1,0),...,(1,d,0)\}}$. One could then think of an orthogonal projection at the linear level, i.e. $\langle T_i^{(n,k)}, \varepsilon\rangle = 0$. The profiles $T_i^{(n,k)}$'s are however not decaying quickly enough at infinity so that this duality bracket would make sense in the functional space where ε lies. For these grounds we will approximate such orthogonality conditions by smooth profiles that are compactly supported.

Definition 4.4.1 (Generators of orthogonality conditions). For a very large scale $M \gg 1$, for $n \leq n_0$ and $1 \leq k \leq k(n)$ we define:

$$\Phi_M^{(n,k)} = \sum_{i=0}^{L_n} c_{i,n,M}(-H)^i(\chi_M T_0^{(n,k)}) = \sum_{i=0}^{L_n} c_{i,n,M}(-H^{(n)})^i(\chi_M T_0^{(n)})Y^{(n,k)}, \tag{4.4.1}$$

(L_n and $T_0^{(n,k)}$ being defined by (*4.1.11*) and (*4.2.21*)) where:

$$c_{0,n,M} = 1 \ \text{ and } \ c_{i,n,M} = -\frac{\sum_{j=0}^{i-1} c_{j,n,M}\langle(-H)^j(\chi_M T_0^{(n,k)}), T_i^{(n,k)}\rangle}{\langle \chi_M T_0^{(n)}, T_0^{(n)}\rangle}. \tag{4.4.2}$$

Lemma 4.4.2 (Generation of orthogonality conditions). *For $n \leq n_0$, $1 \leq k \leq k(n)$, $0 \leq i \leq L_n$, $j \in \mathbb{N}$, $n' \in \mathbb{N}$ and $1 \leq k' \leq k(n')$ there holds for $c > 0$:*

$$\begin{array}{rcl} \langle(-H)^j\Phi_M^{(n,k)}, T_i^{(n',k')}\rangle & = & \delta_{(n,k,i),(n',k',j)} \int_0^{+\infty} \chi_M |T_0^{(n)}|^2 r^{d-1} \\ & \sim & cM^{4m_n+4\delta_n}\delta_{(n,k,i),(n',k',j)}, \quad c > 0. \end{array} \tag{4.4.3}$$

Proof of Lemma 4.4.2

The scalar product is zero if $(n,k)\neq(n',k')$ because by construction $\Phi_M^{(n,k)}$ (resp. $H^j(T_i^{(n',k')})$ lives on the spherical harmonic $Y^{(n,k)}$ (resp. $Y^{n',k'}$). We now suppose $(n,k)=(n',k')$ and compute from (*4.4.1*):

$$\langle(-H)^j\Phi_M^{(n,k)},T_i^{(n,k)}\rangle=\sum_{l=0}^{L_n}c_{l,n,M}\langle T_0^{(n)}\chi_M,(-H^{(n)})^{l+j}T_i^{(n)}\rangle.$$

If $j>i$ for all l, $(H^{(n)})^{l+j}T_i^{(n)}=0$ and then $\langle(-H)^j\Phi_M^{(n,k)},T_i^{(n,k)}\rangle=0$. If $j=i$ then only the first term in the sum is not zero since $(-H^{(n)})^iT_i^{(n)}=T_0^{(n,k)}$ and:

$$\sum_{l=0}^{L_n}c_{l,n,M}\langle T_0^{(n)}\chi_M,(-H^{(n)})^{l+j}T_i^{(n)}\rangle=\langle T_0^{(n)}\chi_M,T_0^{(n)}\rangle\sim cM^{4m_n+4\delta_n}$$

from the asymptotic behavior (*4.2.2*) of $T_0^{(n)}$. If $j<i$ then:

$$\begin{array}{rl}&\sum_{l=0}^{L_n}c_{l,n,M}\langle T_0^{(n)}\chi_M,(-H^{(n)})^{l+j}T_i^{n}\rangle\\=&c_{i-j,n,M}\langle T_0^{(n)}\chi_M,T_0^{(n)}\rangle+\sum_{l=0}^{i-j-1}c_{l,n,M}\langle T_0^{(n)}\chi_M,(-H^{(n)})^{l+j}T_i^{(n)}\rangle=0\end{array}$$

from the definition (*4.4.2*) of the constant $c_{i-j,n,M}$ which ends the proof.

$\square$

4.4.1.2 Geometrical decomposition

First we describe here how we decompose a solution of (NLH) on the unit ball $\mathcal{B}^d(1)$ onto the set $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,|z|\leq\frac18,0<\lambda<\frac{1}{8M}}$ of concentrated ground states, using the orthogonality conditions provided by Lemma 4.4.2. This provides a decomposition for any domain containing $\mathcal{B}^d(1)$. Let $0<\kappa\ll1$ to be fixed latter on. We study the set of functions close to $(\tau_z(\tilde{Q}_{b,\lambda}))_{b,|z|\leq\frac18,0<\lambda<\frac{1}{8M}}$ such that the projection onto the first element in the generalized kernel dominates[10]:

$$\exists(\tilde\lambda,\tilde z)\in\left(0,\frac{1}{8M}\right)\times\mathcal{B}^d\left(\frac18\right),\left|\begin{array}{l}\|u-Q_{\tilde z,\frac{1}{\tilde\lambda}}\|_{L^\infty(\mathcal{B}^d(1))}<\frac{\kappa}{\tilde\lambda^{\frac{2}{p-1}}}\text{ and}\\ \|(\tau_{-\tilde z}u)_{\tilde\lambda}-Q\|_{L^\infty(\mathcal{B}^d(3M))}<\langle(\tau_{-\tilde z}u)_{\tilde\lambda}-Q,H\Phi_M^{(0,1)}\rangle\end{array}\right.\qquad(4.4.4)$$

Lemma 4.4.3 (Decomposition). *There exist $\kappa,K>0$ such that for any solution $u\in\mathcal{C}^1([0,T),\times\mathcal{B}^d(1))$ of (NLH) satisfying (4.4.4) for all $t\in[0,T)$ there exist a unique choice of the parameters $\lambda:[0,T)\to(0,\frac{1}{4M})$, $z:[0,T)\to\mathcal{B}^d\left(\frac14\right)$ and $b:[0,T)\to\mathbb{R}^{\mathcal{I}}$ such that $b_1^{(0,1)}>0$ and*

$$u=(\tilde Q_b+v)_{z,\lambda}\ \text{ on }\mathcal{B}^d(1),\quad\sum_{(n,k,i)\in\mathcal{I}}|b_i^{(n,k)}|+\|v\|_{L^\infty\left(\frac1\lambda(\mathcal{B}^d(0,1)-\{z\})\right)}\leq K\kappa$$

with $v=(\tau_{-z}u)_\lambda-\tilde Q_b$ satisfying the orthogonality conditions:

$$\langle v,H^i\Phi_M^{(n,k)}\rangle=0,\quad\text{for }0\leq n\leq n_0,\ 1\leq k\leq k(n),\ 0\leq i\leq L_n$$

Moreover, λ, b and z are $\mathcal{C}^1$ functions.

Proof of Lemma 4.4.3

It is a direct consequence of Lemma 4.E.2 from the appendix.

$\square$

[10] Note that $(\tau_{-\tilde z}u)_{\tilde\lambda}$ is defined on $\frac{1}{\tilde\lambda}(\mathcal{B}^d(1)-\tilde z)$ which contains $\mathcal{B}^d(7M)$ as $|\tilde z|<\frac18$ and $0<|\tilde\lambda|<\frac{1}{8M}$, thus the second estimate makes sense.

Decomposition and adapted norms for the remainder inside a bounded domain

Let u be a solution of (NLH) in $C^1([0,T),\Omega)$ with Dirichlet boundary condition, such that the restriction[11] of u to $\mathcal{B}^d(1)$ satisfy the conditions of Lemma 4.4.3. Then from this Lemma, for all $t\in[0,T)$ we can decompose u according to:

$$u := \chi\tau_z\left(\tilde{Q}_{b,\frac{1}{\lambda}}\right)+w, \tag{4.4.5}$$

cutting the approximate blow-up profile in the zone $1\le|x|\le 2$, and w is a remainder term satisfying $w_{|\partial\Omega}=0$ as $\mathcal{B}^d(7)\subset\Omega$ and $u_{|\partial\Omega}=0$. To study w inside and outside the blow-up zone we decompose it according to:

$$w_{\text{int}} := \chi_3 w,\ w_{\text{ext}} := (1-\chi_3)w,\ \varepsilon := (\tau_{-z(t)}w_{\text{int}})_{\lambda(t)} \tag{4.4.6}$$

w_{int} and w_{ext} are the remainder cut in the zone $3\le|x|\le 6$, ε is the renormalized remainder at the blow up area, and is adapted to the renormalized flow. We notice that the support of w_{ext} does not intersect the support of the approximate blow up profile $\chi\tau_z\left(\tilde{Q}_{b,\frac{1}{\lambda}}\right)$, that the supports of w_{int} and w_{ext} overlap, and that $(w_{\text{ext}})_{|\partial\Omega}=0$. From Lemma 4.4.3 and its definition, ε is compactly supported and satisfies the orthogonality conditions (*4.4.11*). We measure ε through the following norms:

(i) *High order Sobolev norm adapted to the linearized flow:* We define

$$\mathcal{E}_{2s_L} := \int_{\mathbb{R}^d}|H^{s_L}\varepsilon|^2. \tag{4.4.7}$$

This norm controls the L^2 norms of all smaller order derivatives with appropriate weight from Lemma 4.C.3 since ε satisfy the orthogonality conditions (*4.4.11*), and the standard $\dot{H}^{2s_L}$ Sobolev norm:

$$\mathcal{E}_{2s_L}\ge C\sum_{|\mu|\le 2s_L}\int_{\mathbb{R}^d}\frac{|\partial^\mu\varepsilon|^2}{1+|x|^{4i-2\mu+}}+C\|\varepsilon\|^2_{\dot{H}^{2s_L}}$$

(ii) *Low order slightly supercritical Sobolev norm:* Let σ be a slightly supercritical regularity:

$$0<\sigma-s_c\ll 1. \tag{4.4.8}$$

We then define the following second norm for the remainder:

$$\mathcal{E}_\sigma := \|\varepsilon\|^2_{\dot{H}^\sigma}. \tag{4.4.9}$$

Existence of a solution staying in a trapped regime close to the approximate blow up solution

From now on we focus on solutions that are close to an approximate blow-up profile in the sense of the following definition.

Definition 4.4.4 (Solutions in the trapped regime). We say that a solution u of (NLH) in $C^1([0,T),\Omega)$ is trapped on $[0,T)$ if it satisfies all the following. First, it satisfies the condition (*4.4.4*) and then can be decomposed via Lemma 4.4.3 according to (*4.4.5*) and (*4.4.6*):

$$u := \chi\tau_z\left(\tilde{Q}_{b,\frac{1}{\lambda}}\right)+w,\ w_{\text{int}} := \chi_3 w,\ w_{\text{ext}} := (1-\chi_3)w,\ \varepsilon := (\tau_{-z(t)}w_{\text{int}})_{\lambda(t)} \tag{4.4.10}$$

[11] We recall that Ω contains $\mathcal{B}^d(7)$

with ε satisfying the orthogonality conditions:

$$\langle \varepsilon, H^i \Phi_M^{(n,k)} \rangle = 0, \quad \text{for } 0 \leq n \leq n_0, \ 1 \leq k \leq k(n), \ 0 \leq i \leq L_n \tag{4.4.11}$$

To the scale λ given by this decomposition we associate the renormalized time s defined by (*4.3.1*) with $s_0 > 0$. The $\#\mathfrak{I}$-tuple of parameters b is represented as a perturbation of the solution $\bar{b}$ of the dynamical system (*4.3.58*) given by (*4.3.59*):

$$b_i^{(n,k)}(s) = \bar{b}_i^{(n,k)}(s) + \frac{U_i^{(n,k)}(s)}{s^{\frac{\gamma-\gamma_n}{2}+i}} \tag{4.4.12}$$

and we let $U := (U_i^{(n,k)})_{(n,k,i)\in\mathfrak{I}}$. To use the eigenvectors of the linearized dynamics, Lemma (4.3.5), we define:

$$V_i := (P_\ell U)_i \ \text{ for } 1 \leq i \leq \ell \tag{4.4.13}$$

where P_ℓ is defined by (*4.3.68*). All these parameters must satisfy the following estimates, where $0 < \tilde{\eta} \ll 1$, $0 < \epsilon_i^{(n,k)} \ll 1$ for $(n,k,i) \in \mathfrak{I}$ with $(n,k,i) \notin \{1,...,\ell\} \times \{0\} \times \{1\}$, K_1 and K_2 will be fixed later on.

-*Initial conditions.* At time $t = 0$ (or equivalently $s = s_0$):

(i) Control of the unstable modes on the radial component:

$$|V_i(0)| \leq s_0^{-\tilde{\eta}} \ \text{ for } 2 \leq i \leq \ell \tag{4.4.14}$$

(ii) Control of the unstable modes on the other spherical harmonics:

$$|(U_i^{(n,k)}(0))| \leq \epsilon_i^{(n,k)} \ \text{ for } (n,k,i) \in \mathfrak{I} \text{ with } 1 \leq n, \ 0 \leq i < i_n \tag{4.4.15}$$

(ii) Control of the stable modes:

$$V_1(0) \leq \frac{1}{10 s_0^{\tilde{\eta}}}, \quad |U_i^{(0,1)}(0)| \leq \frac{\epsilon_i^{(0,1)}}{10 s_0^{\tilde{\eta}}} \ \text{for } \ell+1 \leq i \leq L, \tag{4.4.16}$$

$$|U_i^{(n,k)}(0)| \leq \frac{\epsilon_i^{(n,k)}}{10 s_0^{\tilde{\eta}}} \ \text{for } (n,k,i) \in \mathfrak{I}, \ \text{with } 1 \leq n \text{ and } i_n < i \leq L_n, \tag{4.4.17}$$

$$|U_i^{(n,k)}(0)| \leq \frac{\epsilon_i^{(n,k)}}{10} \ \text{for } (n,k,i) \in \mathfrak{I}, \ \text{with } 1 \leq n \text{ and } i = i_n. \tag{4.4.18}$$

(iii) Smallness of the remainder:

$$\|w\|_{H^{2s_L}}^2 < \frac{1}{s_0^{\frac{2\ell}{2\ell-\alpha}(2s_L - s_c)}}. \tag{4.4.19}$$

(iv) Compatibility conditions at the border[12]:

$$\begin{cases} \tilde{w}_0 := w(0) \in H_0^1(\Omega), \ \tilde{w}_1 := \partial_t w(0) = \Delta w(0) + w(0)^p \in H_0^1(\Omega), \\ \tilde{w}_2 := \partial_t^2 w(0) = \Delta^2 w(0) + \Delta(w(0)^p) + p w(0)^{p-1}(\Delta w(0) + w(0)^p) \in H_0^1(\Omega), ... \\ ..., \ \tilde{w}_{s_L - 1} := \partial_t^{s_L - 1} w(0) \in H_0^1(\Omega) \end{cases} \tag{4.4.20}$$

[12]We make an abuse of notations here. The identities given for the time derivatives of w are only true close to the border of Ω, but which is enough as the required conditions are trace type conditions, see [52].

(v) Initial scale and initial blow-up point:

$$\lambda(0)=s_0^{-\frac{\ell}{2\ell-\alpha}} \quad \text{and} \quad z(0)=0. \tag{4.4.21}$$

-*Pointwise in time estimates.* The following bounds hold on $(0,T)$:

(i) Parameters on the first spherical harmonics:

$$|V_i(s)|\le s^{-\tilde\eta} \ \text{ for } \ 1\le i\le \ell, \quad |U_i^{(0,1)}(s)|\le \epsilon_i^{(0,1)}s^{-\tilde\eta} \ \text{ for } \ell+1\le i\le L \tag{4.4.22}$$

(ii) Parameters on the other spherical harmonics: for $(n,k,i)\in\mathcal{I}$ with $n\ge 1$:

$$|(U_i^{(n,k)}(s))|\le 1 \ \text{ if } \ 0\le i<i_n, \tag{4.4.23}$$

$$|U_i^{(n,k)}(s)|\le \frac{\epsilon_i^{(n,k)}}{s^{\tilde\eta}}, \ \text{ if } \ i_n<i\le L_n \ \text{ and } \ |U_i^{(n,k)}(s)|\le \epsilon_i^{(n,k)}, \ \text{ if } i=i_n. \tag{4.4.24}$$

(iii) Control of the remainder:

$$\begin{array}{ll} \mathcal{E}_{s_L}(s)\le \frac{K_2}{s^{2L+2(1-\delta_0)+2(1-\delta_0')\eta}}, & \mathcal{E}_\sigma(s)\le \frac{K_1}{s^{2(\sigma-s_c)\frac{\ell}{2\ell-\alpha}}}, \\ \|w_{\text{ext}}\|_{H^{2s_L}}^2\le \frac{K_2}{\lambda^{2(2s_L-s_c)}s^{2L+2(1-\delta_0)+2(1-\delta_0')\eta}}, & \|w_{\text{ext}}\|_{H^\sigma}^2\le K_1. \end{array} \tag{4.4.25}$$

(iv) Estimates on the scale and the blow-up point:

$$\lambda\le 2s^{-\frac{\ell}{2\ell-\alpha}} \quad \text{and} \quad |z|\le \frac{1}{10}. \tag{4.4.26}$$

Remark 4.4.5. For a trapped solution one has the above estimates on the parameters from (*4.3.59*), (*4.4.12*), (*4.4.13*), (*4.4.22*), (*4.4.23*) and (*4.4.24*):

$$|b_i^{(n,k)}|\le \frac{C}{s^{\frac{\gamma-\gamma_n}{2}+i}}, \quad b_1^{(0,1)}=\frac{\ell}{2\ell-\alpha}\frac{1}{s}+O(s^{-1-\tilde\eta}) \tag{4.4.27}$$

for C independent independent of the other constants. The bounds (*4.4.25*) on the remainders for the solution described by Proposition (4.4.6), because of the the coercivity estimate (4.C.3) implies that

$$\|w\|_{H^\sigma(\Omega)}\le CK_1, \quad \|w\|_{H^{2s_L}}(\Omega)\le \frac{C(K_1,K_2,M)}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{4.4.28}$$

A trapped solution must first satisfy the condition (*4.4.4*) in order to apply the decomposition Lemma 4.E.1, and then the variables of this decomposition must satisfy suitable bounds. However, these additional bounds in turn provide a much stronger estimate than (*4.4.4*). Indeed, one has from (*4.4.10*), (*4.3.29*), (*4.3.7*), (*4.4.27*), (*4.D.2*):

$$\begin{array}{rc} & \inf_{(\tilde\lambda,\tilde z)\in\left(0,\frac{1}{8M}\right)\times\mathcal{B}^d\left(\frac18\right)} \tilde\lambda^{\frac{2}{p-1}}\|u-Q_{\tilde z,\frac{1}{\tilde\lambda}}\|_{L^\infty(\mathcal{B}^d(1))}\le \lambda^{\frac{2}{p-1}}\|u-Q_{z,\frac{1}{\lambda}}\|_{L^\infty(\mathcal{B}^d(1))} \\ = & \|\tilde Q_b+\varepsilon-Q\|_{L^\infty\left(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\})\right)}=\|\chi_{B_1}\alpha_b+\varepsilon\|_{L^\infty\left(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\})\right)} \\ \le & \|\chi_{B_1}\alpha_b\|_{L^\infty(\mathbb{R}^d)}+\|\varepsilon\|_{L^\infty(\mathbb{R}^d)}\le \frac{C}{s}+\frac{C}{s^{\frac d4-\frac\sigma2}}\ll\kappa, \end{array}$$

$$\|(\tau_{-z})u_\lambda-Q\|_{L^\infty(\mathcal{B}^d(3M))}\le\|\alpha_b\|_{L^\infty(\mathcal{B}^d(3M))}+\|\varepsilon\|_{L^\infty(\mathcal{B}^d(3M))}\le\frac{C}{s}+\frac{C}{s^2}. \tag{4.4.29}$$

Using (*4.4.10*), (*4.4.11*), (*4.3.29*), (*4.3.7*), (*4.4.27*), (*4.4.3*) and (*4.2.2*) one gets

$$\begin{aligned} &\langle (\tau_{-z})u_\lambda - Q, H\Phi_M^{(0,1)}\rangle = \langle \alpha_b, H\Phi_M^{(0,1)}\rangle \\ = \quad & b_1^{(0,1)}\langle T_0^{(0,1)}, \chi_M T_0^{(0,1)}\rangle + O(s^{-2}) \sim \tfrac{c}{s} = \tfrac{c_1}{s} c M^{d-2\gamma} + O(s^{-2}) \end{aligned}$$

for some $c>0$, which, combined with the above estimate gives:

$$\|(\tau_{-z})u_\lambda - Q\|_{L^\infty(\mathcal{B}^d(3M))} \ll \langle (\tau_{-z})u_\lambda - Q, H\Phi_M^{(0,1)}\rangle$$

for M large enough as $d-2\gamma>0$. Therefore, a solution cannot exit the trapped regime because the condition (*4.4.4*) fails: the estimates on the parameters and the remainder have to be violated first. We thus forget about this condition in the following.

The key result of this chapter is the existence of solutions that are trapped on their whole lifespan.

Proposition 4.4.6 (Existence of fully trapped solutions:). *There exists a choice of universal constants for the analysis*[13]*:*

$$\begin{array}{l} L = L(\ell,d,p) \gg 1, \;\; 0<\eta=\eta(d,p,L)\ll 1, \;\; M = M(d,p,L)\gg 1, \\ \sigma = \sigma(L,d,p), \; K_1 = K_1(d,p,L) \gg 1, \;\; K_2 = K_2(d,p,L)\gg 1, \\ 0<\epsilon_i^{(0,1)} = \epsilon_i^{(0,1)}(L,d) \ll 1 \textit{ for } \ell+1\leq i\leq L, \; 0<\epsilon_1=\epsilon_1(L,d)\ll 1, \\ 0<\epsilon_i^{(n,k)} = \epsilon_i^{(n,k)}(L,d)\ll 1 \textit{ for } (n,k,i)\in\mathcal{I} \textit{ with } 1\leq n, i_n+1\leq i\leq L_n \\ 0<\tilde\eta = \tilde\eta(\ell,L,d,p,\eta) \ll 1 \textit{ and } s_0 = s_0(\ell,d,p,L,M,K_1,K_2,\epsilon_i^{(n,k)},\tilde\eta)\gg 1, \end{array} \tag{4.4.30}$$

such that the following fact holds close to $\chi \tilde Q_{\bar b(s_0),\frac{1}{\lambda(s_0)}}$ *where* $\bar b$ *is given by* (*4.3.59*) *and* $\lambda(s_0)$ *satisfies* (*4.4.21*)*. Given a perturbation along the stable directions, represented by* $w(s_0)$*, decomposed in* (*4.4.5*)*, satisfying* (*4.4.19*) *and* (*4.4.11*)*, and* $V_1(s_0)$, $\left(U_{\ell+1}^{(0,1)}(s_0),...,U_L^{(0,1)}(s_0)\right)$, $\left(U_i^{(n,k)}(s_0)\right)_{(n,k,i)\in\mathcal{I},\, n\geq 1,\, i_n\leq i}$ *satisfying* (*4.4.16*)*,* (*4.4.17*) *and* (*4.4.18*)*, there exists a correction along the unstable directions represented by* $(V_2(s_0),...V_\ell(s_0))$ *and* $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\leq n,\, i<i_n}$ *satisfying* (*4.4.14*) *and* (*4.4.15*) *such that the solution* $u(t)$ *of* (NLH) *with initial datum* $u(0)=\chi\tilde Q_{b(s_0),\frac{1}{\lambda(s_0)}} + w(s_0)$ *with:*

$$b(s_0) = \left(\bar b_i^{(n,k)} + \frac{U_i^{(n,k)}(s_0)}{s_0^{\frac{\gamma-\gamma_n}{2}+i}}\right)_{(n,k,i)\in\mathcal{I}} \tag{4.4.31}$$

is trapped until its maximal time of existence in the sense of Definition 4.4.4.

Proof of Proposition 4.4.6

The proof is relegated to Section 4.5.

□

[13] The interdependence of the constants is written here so that the reader knows, for example, that s_0 is chosen after all the other constants.

4.4.2 End of the proof of Theorem 2.2.9 using Proposition 4.4.6

In this subsection we end the proof of the main Theorem 2.2.9 by proving that the solutions given by Proposition 4.4.6 lead to a finite time blow up with the properties described in Theorem 2.2.9. The proof of Theorem 2.2.9 is a direct consequence of Proposition 4.4.6, Lemmas 4.4.8 and 4.4.9. Until the end of this subsection, u will denote a solution that is trapped in the sense of Definition 4.4.4) on its maximal interval of existence. First, we describe the time evolution equation for ε. It then allows us to compute how the time evolution law for the parameters λ and z related to the decomposition (*4.4.5*) depends on the other parameters. The bounds on the parameters and the remainder for a trapped solution then imply that λ goes to zero with explicit asymptotic in finite time, that z converges, and that the solution undergoes blow up by concentration with a control on the asymptotic behavior for Sobolev norms.

4.4.2.1 Time evolution for the error

Let u be a trapped solution. From the decomposition (*4.4.5*) we compute that the time evolution of the remainder is:

$$\begin{aligned} w_t \ &= \ -\tfrac{1}{\lambda^2}\chi\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}}+\tilde{\psi}_{b,\frac{1}{\lambda}})+\Delta w+\textstyle\sum_{k=1}^p C_k^p(\chi\tau_z\tilde{Q}_{b,\frac{1}{\lambda}})^{p-k}w^k \\ &\quad +\Delta\chi\tau_z Q_{\frac{1}{\lambda}}+2\nabla\chi.\nabla\tau_z Q_{\frac{1}{\lambda}}+\chi\tau_z Q^p_{\frac{1}{\lambda}}(\chi^{p-1}-1). \end{aligned} \tag{4.4.32}$$

with the new modulation term being defined as:

$$\tilde{\text{Mod}}(t):=\chi_{B_1}\text{Mod}(t)-\left(\frac{\lambda_s}{\lambda}+b_1^{0,1)}\right)\Lambda\tilde{Q}_b-\left(\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right).\nabla\tilde{Q}_b, \tag{4.4.33}$$

From (*4.4.32*) and (*4.4.6*), as the support of w_{ext} is outside $\mathcal{B}^d(2)$ and as $\tau_z(\tilde{Q}_{b,\lambda})$ is cut in the zone $1\leq|x|\leq 2$, the time evolution of w_{ext} is:

$$\partial_t w_{\text{ext}}=\Delta w_{\text{ext}}+\Delta\chi_3 w+2\nabla\chi_3.\nabla w+(1-\chi_3)w^p. \tag{4.4.34}$$

The excitation of the solitary wave $\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})$ has support in the zone $|x-z|\leq 2\lambda B_1$ and from (*4.4.26*), $|z|+\lambda B_1\ll 1$, so it does not see the cut by χ of the approximate blow up profile. From this, (*4.4.32*) and (*4.4.6*) the time evolution of w_{int} is therefore given by:

$$\partial_t w_{\text{int}}+H_{z,\frac{1}{\lambda}}w_{\text{int}}=-\frac{1}{\lambda^2}\chi\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}}+\tilde{\psi}_{b,\frac{1}{\lambda}})+L(w_{\text{int}})+NL(w_{\text{int}})+\tilde{L}+\tilde{NL}+\tilde{R} \tag{4.4.35}$$

where $H_{z,\frac{1}{\lambda}}$, $\text{NL}(w_{\text{int}})$, $L(w_{\text{int}})$ are the linearized operator, the non linear term and the small linear terms resulting from the interaction between w_{int} and a non cut approximate blow up profile $\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})$:

$$H_{z,\frac{1}{\lambda}}:=-\Delta-p\left(\tau_z(\tilde{Q}_{\frac{1}{\lambda}})\right)^{p-1},\quad H_{b,z,\frac{1}{\lambda}}:=-\Delta-p\left(\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})\right)^{p-1} \tag{4.4.36}$$

$$\begin{aligned} \text{NL}(w_{\text{int}}) &:= F\left(\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})+w_{\text{int}}\right)-F\left(\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})\right)+H_{b,\frac{1}{\lambda}}(w_{\text{int}}), \\ L(w_{\text{int}}) &:= H_{z,\frac{1}{\lambda}}w_{\text{int}}-H_{b,z,\frac{1}{\lambda}}w_{\text{int}}=\tfrac{p}{\lambda^2}\tau_z(\chi_{B_1}^{p-1}\alpha_b^{p-1})_{\frac{1}{\lambda}}. \end{aligned} \tag{4.4.37}$$

The last terms in (*4.4.35*) are the corrective terms induced by the cut of the approximate blow up profile and the cut of the error term[14]:

$$\tilde{L}:=-\Delta\chi_3 w-2\nabla\chi_3.\nabla w+p\tau_z Q^{p-1}_{\frac{1}{\lambda}}(\chi^{p-1}-\chi_3)w, \tag{4.4.38}$$

[14] Again, the excitation of the solitary wave $\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})$ is not present here as its support is in the zone $|x|\ll 1$, see (*4.4.26*)

$$\tilde{NL} := \sum_{k=2}^{p} C_k^p \tau_z Q_{\frac{1}{\lambda}}^{p-k}(\chi^{p-k}-\chi_3^{k-1})\chi_3 w^k, \tag{4.4.39}$$

$$\tilde{R} := \Delta\chi\tau_z Q_{\frac{1}{\lambda}} + 2\nabla\chi\nabla\tau_z Q_{\frac{1}{\lambda}} + \chi\tau_z Q_{\frac{1}{\lambda}}^p(\chi^{p-1}-1), \tag{4.4.40}$$

and one notices that their support is in the zone $1 \leq |x| \leq 6$. Using the definition of the renormalized flow (*4.3.2*) and the decomposition (*4.4.5*) we compute from (*4.4.32*):

$$\begin{array}{rcl} \partial_s\varepsilon - \frac{\lambda_s}{\lambda}\Lambda\varepsilon - \frac{z_s}{\lambda}.\nabla\varepsilon + H\varepsilon & = & -\chi(\lambda y+z)(\tilde{\text{Mod}}(s)+\tilde{\psi}_b) \\ & & +\text{NL}(\varepsilon) + L(\varepsilon) + \lambda^2[\tau_{-z}(\tilde{L}+\tilde{R}+\tilde{NL})]_\lambda, \end{array} \tag{4.4.41}$$

with the the purely non linear term and the small linear term in adapted renormalized variables being defined as:

$$\text{NL}(\varepsilon) := F(\tilde{Q}_b+\varepsilon) - F(\tilde{Q}_b) + H_b(\varepsilon), \quad L(\varepsilon) := H\varepsilon - H_b\varepsilon, \tag{4.4.42}$$

where $H_b := -\Delta - p\tilde{Q}_b^{p-1}$ is the linearized operator near $\tilde{Q}_b$. One notices that the extra terms induced by the cut, $\lambda^2[\tau_{-z}(\tilde{L}+\tilde{R}+\tilde{NL})]_\lambda$, have support in the zone $\frac{1}{2\lambda} \leq |y| \leq \frac{7}{\lambda}$ (from (*4.4.26*)).

4.4.2.2 Modulation equations

We now quantify how the evolution of one of the parameters $b_i^{(n,k)}$, λ or z depends on all the parameters $(b_i^{(n,k)})_{(n,k,i)\in\mathcal{I}}$ and the remainder ε.

Lemma 4.4.7 (Modulation). *Let all the constants of the analysis described in Proposition 4.4.6 be fixed except s_0. Then for s_0 large enough, for any solution u that is trapped on $[s_0,s')$ in the sense of Definition 4.4.4 there holds for $s_0 \leq s < s'$:*

$$\begin{array}{rl} & \left|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right| + \left|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right| + \sum\limits_{(n,k,i)\in\mathcal{I},\ i\neq L_n} |b_{i,s}^{(n,k)} + (2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)} + b_{i+1}^{(n,k)}| \\ \leq & \frac{C(L,M)}{s^{L+3}} + \frac{C(L,M)}{s}\sqrt{\mathcal{E}_{2s_L}}, \end{array} \tag{4.4.43}$$

$$\sum_{(n,k,i)\in\mathcal{I},\ i=L_n} |b_{i,s}^{(n,k)} + (2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}| \leq \frac{C(M,L)}{s^{L+3}} + C(M,L)\sqrt{\mathcal{E}_{2s_L}}. \tag{4.4.44}$$

Proof of Lemma 4.4.7

We let:

$$D(s) = \left|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}\right| + \left|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}\right| + \sum_{(n,k,i)\in\mathcal{I}} |b_{i,s}^{(n,k)} + (2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)} - b_{i+1}^{(n,k)}|. \tag{4.4.45}$$

with the convention that $b_{L_n+1}^{(n,k)} = 0$. Taking the scalar product of (*4.4.41*) with $(-H)^i\Phi_M^{(n,k)}$, using (*4.4.3*), gives [15]:

$$\begin{array}{rcl} \langle\tilde{\text{Mod}}(s),(-H)^i\Phi_M^{(n,k)}\rangle & = & \langle -H\varepsilon,(-H)^i\Phi_M^{(n,k)}\rangle - \langle\tilde{\psi}_b,(-H)^i\Phi_M^{(n,k)}\rangle \\ & & +\langle\frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon + \text{NL}(\varepsilon) + L(\varepsilon),(-H)^i\Phi_M^{(n,k)}\rangle. \end{array} \tag{4.4.46}$$

Now we look closely at each one of the terms of this identity.

[15] We do not see the extra terms $\tilde{L}$, $\tilde{R}$ and $\tilde{NL}$ because their support is in the zone $\frac{1}{2\lambda} \leq |y|$ (from (*4.4.26*)) which is very far away from the support of $\Phi_M^{(n,k)}$, in the zone $|y| \leq 2M$ (s_0 being chosen large enough so that this statement holds).

- *The modulation term.* From the expression (*4.3.29*) of $\tilde{Q}_b$, the bound (*4.3.11*) on $\frac{\partial S_j}{\partial b_i^{(n,k)}}$, the bounds (*4.4.27*) on the parameters, one has:

$$\tilde{Q}_b = Q + \chi_{B_1}\alpha_b = Q + O(s^{-1}), \quad \text{and} \quad \frac{\partial S_j}{\partial b_i^{(n,k)}} = O(s^{-1}) \text{ on } \mathcal{B}^d(0,2M).$$

From (*4.3.10*), (*4.4.33*) and (*4.4.45*) the modulation term can then be rewritten as:

$$\begin{array}{rcl} \text{Mod}(s) & = & \chi_{B_1} \sum\limits_{(n,k,i)\in\mathcal{I}} [b_{i,s}^{(n,k)} + (2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)} - b_{i+1}^{(n,k)}] \left[T_i^{(n,k)} + \sum_{j=i+1+\delta_{n\geq 2}}^{L+2} \frac{\partial S_j}{\partial b_i^{(n,k)}}\right] \\ & & - \left(\frac{\lambda_s}{\lambda} + b_1^{0,1)}\right)\Lambda\tilde{Q}_b - \left(\frac{z_s}{\lambda} + b_1^{(1,\cdot)}\right).\nabla\tilde{Q}_b \\ & = & \chi_{B_1}\sum_{(n,k,i)\in\mathcal{I}}[b_{i,s}^{(n,k)} + (2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)} - b_{i+1}^{(n,k)}]T_i^{(n,k)} \\ & & - \left(\frac{\lambda_s}{\lambda} + b_1^{0,1)}\right)\Lambda Q - \left(\frac{z_s}{\lambda} + b_1^{(1,\cdot)}\right).\nabla Q + O(\frac{|D(s)|}{s}) \end{array}$$

where the $O(\frac{|D(s)|}{s})$ is valid in the zone $|y| \leq 2M$. From the orthogonality relations (*4.4.3*) we then get:

$$\begin{array}{c} \langle\tilde{\text{Mod}}(s), (-H)^i\Phi_M^{(n,k)}\rangle + O\left(\frac{|D(s)|}{s}\right) \\ = \left\{ \begin{array}{l} -C\langle\chi_M\Lambda Q, \Lambda Q\rangle\left(\frac{\lambda_s}{\lambda} + b_1^{(0,1)}\right) \text{ for } (n,k,i) = (0,1,0) \\ -C'\langle\chi_M\nabla Q, \nabla Q\rangle\left(\frac{z_{j,s}}{\lambda} + b_1^{(1,k)}\right) \text{ for } (n,i) = (1,0), \ 1\leq k\leq d \\ \langle\chi_M T_0^{(n,k)}, T_0^{(n,k)}\rangle\left(b_{i,s}^{(n,k)} + (2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)} - b_{i+1}^{(n,k)}\right) \text{ otherwise} \end{array}\right. \end{array} \quad (4.4.47)$$

where C and C' are two positive renormalization constants.

- *The main linear term.* The coercivity estimate (*4.C.16*) and Hölder inequality imply:

$$\int_{|y|\leq 2M} |\varepsilon| dy \lesssim C(M)\sqrt{\mathcal{E}_{2s_L}}.$$

Hence, from the orthogonality (*4.4.11*) for ε we obtain for $0\leq n\leq n_0$, $1\leq k\leq k(n)$:

$$\left|\langle H\varepsilon, H^i\Phi_M^{(n,k)}\rangle\right| = \left\{ \begin{array}{l} 0 \ \text{ for } i < L_n \\ \left|\langle\varepsilon, (-H)^{i+1}\Phi_M^{(n,k)}\rangle\right| = O(\sqrt{\mathcal{E}_{2s_L}}) \ \text{ for } i = L_n. \end{array}\right. \quad (4.4.48)$$

- *The error term.* Using the local bound (*4.3.35*) for $\tilde{\psi}_b$ and (*4.4.27*):

$$\left|\langle\tilde{\psi}_b, H^i\Phi_M^{(n,k)}\rangle\right| \leq \frac{C(L,M)}{s^{L+3}}. \quad (4.4.49)$$

- *The extra terms.* From (*4.4.27*), the coercivity estimate (*4.C.16*), the bound (*4.4.25*) on $\mathcal{E}_{2s_L}$ and (*4.4.45*) one obtains:

$$\left|\left\langle\frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon, H^i\Phi_M^{(n,k)}\right\rangle\right| \leq \frac{C(L,M)}{s}\sqrt{\mathcal{E}_{2s_L}} + \frac{|D(s)|}{s^{L+1-\delta_0+\eta(1-\delta_0')}}.$$

Now, as $Q^{p-1} - \tilde{Q}_b^{p-1} = O(s^{-1})$ on the set $|y|\leq 2M$ from (*4.3.7*) and (*4.4.27*), using the estimate (*4.D.2*) on $\|\varepsilon\|_{L^\infty}$, from the definition (*4.4.42*) of $NL(\varepsilon)$ and $L(\varepsilon)$ and the coercivity (*4.C.16*) one gets for s_0 large enough:

$$\left|\langle\text{NL}(\varepsilon) + L(\varepsilon), H^i\Phi_M^{(n,k)}\rangle\right| \leq C(L,M)\mathcal{E}_{2s_L} + C(L,M)\frac{\sqrt{\mathcal{E}_{2s_L}}}{s} \leq C(L,M)\frac{\sqrt{\mathcal{E}_{2s_L}}}{s}.$$

Putting together the last two estimates yields:

$$\left|\left\langle\frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon + \text{NL}(\varepsilon) + L(\varepsilon), H^i\Phi_M^{(n,k)}\right\rangle\right| \leq \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s} + \frac{C(L,M)|D(s)|}{s^{L+1-\delta_0+\eta(1-\delta_0')}}. \quad (4.4.50)$$

- *Final bound on* $|D(s)|$. Summing the previous estimates we performed on each term of (*4.4.46*) in (*4.4.47*), (*4.4.48*), (*4.4.49*) and (*4.4.50*) yields:

$$|D(s)| \leq C(L,M)\sqrt{\mathcal{E}_{s_L}} + \frac{C(L,M)}{s^{L+3}}.$$

We now come back to (*4.4.46*), inject again (*4.4.47*) with the above bound on $|D|$, (*4.4.48*), (*4.4.49*) and (*4.4.50*), yielding the desired bounds (*4.4.43*) and (*4.4.44*) of the lemma.

□

4.4.2.3 Finite time blow up

We now reintegrate in time the time evolution of λ and z we found in Lemma 4.4.7 to obtain their behavior and show the blow up.

Lemma 4.4.8 (Concentration and asymptotic of the blow up point). *Let u be a solution that is trapped on its maximal interval of existence. Then it blows up in finite time $T>0$ with $s(t)\to+\infty$ as $t\to T$ and:*

(i) Concentration speed*: $\lambda \underset{t\to T}{\sim} C(u(0))(T-t)^{\frac{\ell}{\alpha}}$, C(u(0))>0.*

(ii) Behavior of the blow up point*: there exists z_0 such that $\lim_{t\to T} z(t)=z_0$ and for all times $s\geq s_0$:*

$$|z(s)| = O(s_0^{-\tilde{\eta}}) \tag{4.4.51}$$

Proof of Lemma 4.4.8

From the Cauchy theory in L^∞, (*4.3.1*) and (*4.4.26*), if $T\in(0,+\infty]$ denotes the maximal time of existence of u, one necessarily have $\lim_{s\to T} s(t)=+\infty$. From the estimate (*4.4.27*) on $b_1^{(0,1)}$, the modulation (*4.4.43*) and (*4.4.25*) one has:

$$\frac{\lambda_s}{\lambda} = -\frac{c_1}{s} + O(s^{-1-\tilde{\eta}}).$$

We reintegrate using (*4.4.21*) (we recall that $c_1=\frac{\ell}{2\ell-\alpha}$ from (*4.3.59*)):

$$\lambda = \frac{(1+O(s_0^{-\tilde{\eta}}))}{s^{\frac{\ell}{2\ell-\alpha}}} \tag{4.4.52}$$

which is valid as long as the solution u is trapped. In addition, if the solution is trapped on its maximal interval of existence, then the function represented by the $O()$ that admits a limit as $s\to+\infty$. In turn, from $\frac{ds}{dt}=\frac{1}{\lambda^2}$ we obtain:

$$s = \frac{s_0}{\left(1-\frac{\alpha s_0^{\frac{\alpha}{2\ell-\alpha}}}{2\ell-\alpha}\int_0^t(1+O(s_0^{-\tilde{\eta}}))dt'\right)^{\frac{2\ell-\alpha}{\alpha}}}$$

Hence there exists $T>0$ with:

$$s \underset{t\to T}{\sim} C(u(0))(T-t)^{-\frac{2\ell-\alpha}{\alpha}}. \tag{4.4.53}$$

Injecting this identity in (*4.4.52*) then gives $\lambda \underset{t\to T}{\sim} C(u(0))(T-t)^{\frac{\ell}{\alpha}}$. Now we turn to the asymptotic behavior of the point of concentration z. From (*4.4.43*), using $b_1^{(1,i)}=O(s^{-\frac{\alpha+1}{2}})$ from (*4.4.23*) for $1\leq i\leq d$, one gets:

$$|z_{i,s}| = O(s^{-c_1-\frac{\alpha+1}{2}}) = O(s^{-1-\frac{\alpha}{2}(1+\frac{1}{2\ell-\alpha})}). \tag{4.4.54}$$

As $\alpha > 0$ this implies the convergence and the estimate of z claimed in the lemma.

□

4.4.2.4 Behavior of Sobolev norms near blow up time

From Lemma 4.4.8, the L^∞ bound on the error (*4.D.2*) and the bounds on the parameters (*4.4.27*), any solution that is trapped on its maximal interval of existence indeed blows up at the time T given by Lemma 4.4.8 because $\lim_{t\to T} \|u\|_{L^\infty} = +\infty$. The behavior of the Sobolev norms is the following.

Lemma 4.4.9 (Asymptotic behavior for subcritical norms). *Let u be a solution that is trapped for all times $s \geq s_0$ and T be its finite maximal lifespan*[16]*. Then*

(i) Behavior of subcritical norms:

$$\textit{limsup}_{t\to T} \|u\|_{H^m(\Omega)} < +\infty, \quad \textit{for}\, 0 \leq m < s_c.$$

(ii) Behavior of the critical norm:

$$\|u\|_{H^{s_c}(\Omega)} \underset{t\to T}{=} C(d,p)\sqrt{\ell}\sqrt{|\textit{log}(T-t)|}(1+o(1)).$$

(iii) Boundedness of the perturbation in slightly supercritical norms

$$\textit{limsup}_{t\to T} \|u - \chi\tau_z(Q_{\frac{1}{\lambda}})\|_{H^m(\Omega)} < +\infty, \quad \textit{for}\ s_c < m \leq \sigma. \tag{4.4.55}$$

Proof of Lemma 4.4.9

The trapped solution u can be written as:

$$u = \chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}}) + w = \chi\tau_z(Q_{\frac{1}{\lambda}}) + \tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}}) + w$$

We first look at the second term $\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})$, being the excitation of the ground state. It has compact support in the zone $|x| \leq 2B_1\lambda$. From (*4.1.21*), (*4.4.52*), one gets $2B_1\lambda \ll 1$ as $s_0 \gg 1$, so that $\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})$ has compact support inside $\mathcal{B}^d(1)$. This implies that $\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{H^\sigma(\Omega)} \leq C\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{\dot{H}^\sigma(\mathbb{R}^d)}$, this later norm being easier to compute. Indeed by renormalizing one has:

$$\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{\dot{H}^\sigma(\mathbb{R}^d)} = \frac{1}{\lambda^{\sigma-s_c}}\|\tilde{\alpha}_b\|_{\dot{H}^\sigma(\mathbb{R}^d)}.$$

As $\tilde{\alpha}_b = \chi_{B_1}\left(\sum_{(n,k,i)\in\mathcal{I}} b_i^{(n,k)} T_i^{(n,k)} + \sum_{i=2}^{L+2} S_i\right)$ from (*4.3.29*) and (*4.3.7*), the bounds (*4.4.27*) on the parameters $b_i^{(n,k)}$, together with the asymptotic at infinity of the profiles $T_i^{(n,k)}$ and S_i described in Lemma 4.2.8 and Proposition 4.3.3 imply that $\|\tilde{\alpha}_b\|_{\dot{H}^\sigma} \leq \frac{C}{s}$. Hence $\|\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})\|_{H^\sigma} \leq \frac{C}{s^{1-\frac{\ell(\sigma-s_c)}{2\ell-\alpha}}} \to 0$ as $t \to T$ as $\sigma - s_c \ll 1$.

Now, following the second paragraph of Remark 4.4.5, we get that $\|w\|_{H^\sigma} \leq CK_1$ is uniformly bounded till the blow up time. Combined with what was just said about the boundedness of $\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})$, we get that (iii) holds for all $0 \leq m \leq \sigma$. This, together with the asymptotic of the ground state (*3.2.1*) then gives (i) and (ii).

□

[16] T is finite from Lemma 4.4.8.

4.4.2.5 The blow-up set

We recall that $x \in \Omega$ is a blow-up point of u if there exists $(t_n, x_n) \to (T, x)$ such that $|u(t_n, x_n)| \to +\infty$. For trapped solutions one has the following result.

Lemma 4.4.10 (Description of the blow-up set). *Let u be a solution that is trapped for all times $s \geq s_0$ and T be its finite maximal lifespan*[17]*. Then z_0 given by Lemma 4.4.8 is a blow-up point of u, and it is the only one.*

Proof of Lemma 4.4.10

From the L^∞ bound (*4.4.29*) and the fact that $\lim_{t\to T} s(t) = +\infty$ from Lemma 4.4.8, $u(s, z(s)) \sim \lambda(s)^{-\frac{2}{p-1}} Q(0)$ as $s \to +\infty$. From Lemma 4.4.8, this implies that $u(t, z(t)) \to +\infty$ as $t \to T$ and that $z_0 = \lim_{t\to T} z(t)$ is indeed a blow-up point.

Now take another point $x \in \Omega$, $x \neq z_0$. From (*4.4.55*), the asymptotic of Q (Lemma 3.2.1), and Lemma 4.4.8, there exists $R > 0$ such that

$$\sup_{0\leq t<T} \|u(t)\|_{H^\sigma(\mathcal{B}^d(x,R))} < +\infty.$$

This local boundedness, by Sobolev embedding and Hölder, implies that

$$\sup_{0\leq t<T} \|u(t)\|_{W^{1,q}(\mathcal{B}^d(x,R))} < +\infty, \quad q = \frac{2d}{d+2-2\sigma} > \frac{2d}{d+2-2s_c} = d\frac{p-1}{p+1}.$$

The above inequality, after applying several times Lemma 4.4.11 below and using Sobolev embedding, implies that there exists $r > 0$ such that

$$\sup_{0\leq t<T} \|u(t)\|_{L^\infty(\mathcal{B}^d(x,r))} < +\infty.$$

Therefore, x is not a blow-up point of u.

□

In the proof of the previous Lemma, we used the following result.

Lemma 4.4.11 (Parabolic bootstrap). *Let $R > 0$ and $x \in \Omega$ such that $B(x, R) \subset \Omega$. Let $q_0 > \frac{p-1}{p+1} d$. There exists $\kappa(q_0) > 0$ such that for any $q > q_0$, if $u \in C([0,T), W^{1,\infty}(\Omega))$ is a solution of (NLH) satisfying*

$$\sup_{0\leq t<T} \|u(t)\|_{W^{1,q}(\mathcal{B}^d(x,R))} < +\infty \qquad (4.4.56)$$

then

$$\sup_{0\leq t<T} \|u(t)\|_{W^{1,q(1+\kappa)}(\mathcal{B}^d(x,\frac{R}{2}))} < +\infty. \qquad (4.4.57)$$

Proof of Lemma 4.4.11

[17] T is finite from Lemma 4.4.8.

The proof relies on a classical use of estimates for the heat kernel. Without loss of generality we assume $q_0 < d$. If u solves (NLH) and satisfies (*4.4.56*) then the localisation $v = \chi_{\frac{R}{2}} u$ solves

$$v_t = \Delta v - 2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta\chi_{\frac{R}{2}} u + \chi_{\frac{R}{2}} |u|^{p-1}u$$

and using Duhamel formula can then be written as

$$v(t) = K_t * v(0) + \int_0^t K_{t-s} * [-2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta\chi_{\frac{R}{2}} u + \chi_{\frac{R}{2}} |u|^{p-1}u] ds$$

where the heat kernel is $K_t(x) = (4\pi t)^{-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}$. One then has the formula

$$\begin{array}{rcl} \nabla v(t) & = & \nabla K_t * v(0) + \int_0^t \nabla K_{t-s} * [-2\nabla.\chi_{\frac{R}{2}}.\nabla u - \Delta\chi_{\frac{R}{2}} u] ds \\ & & + \int_0^t K_{t-s} * [\nabla\chi_{\frac{R}{2}} |u|^{p-1}u + \chi_{\frac{R}{2}} \nabla u |u|^{p-1}] ds. \end{array} \quad (4.4.58)$$

We estimate the last term using Hölder, Sobolev and Young inequalities[18]:

$$\begin{array}{l} \| \int_0^t K_{t-s} * [\chi_{\frac{R}{2}} \nabla u |u|^{p-1}] ds \|_{L^{q(1+\kappa)}} \leq \int_0^t \| K_{t-s} * [\chi_{\frac{R}{2}} \nabla u |u|^{p-1}] \|_{L^{q(1+\kappa)}} ds \\ \lesssim \int_0^t \|K_{t-s}\|_{L^{\frac{1}{1+\frac{1}{q(1+\kappa)} - \left(\frac{(d-q_0)(p-1)}{dq_0} - \frac{1}{q}\right)}}} \|\nabla u |u|^{p-1}]\|_{L^{\frac{1}{\frac{(d-q_0)(p-1)}{dq_0} + \frac{1}{q}}}} ds \\ \lesssim \int_0^t \|K_{t-s}\|_{L^{\frac{1}{1-\frac{(d-q_0)(p-1)}{dq_0} - \frac{\kappa}{q(1+\kappa)}}}} \|\nabla u\|_{L^q} \||u|^{p-1}\|_{L^{\frac{dq_0}{(d-q_0)(p-1)}}} ds \\ \lesssim \int_0^t \frac{1}{(t-s)^{\theta(\kappa,q)}} \|\nabla u\|_{L^q} \|\nabla u\|_{L^{q_0}}^{p-1} ds \lesssim \int_0^T \frac{ds}{(t-s)^{\theta(\kappa,q)}}. \end{array}$$

for

$$\theta(\kappa, q) = \frac{(d-q_0)(p-1)}{2q_0} + \frac{\kappa d}{2q(1+\kappa)} \quad (\theta \geq 0 \text{ as } q_0 < d).$$

For $\kappa \geq 0$ and $\frac{p-1}{p+1} d \leq q \leq d$, if κ is fixed θ is strictly decreasing with respect to q, and if q is fixed θ is strictly increasing with respect to κ. As $\theta(0, q_0) < 1$ since $q_0 > \frac{p-1}{p+1} d$ this implies that there exists $\kappa(q_0) > 0$ such that for all $q_0 \leq q \leq d$, and $0 < \kappa \leq \kappa(q_0)$, $\theta(\kappa, q) < 1$. The above inequality then implies that in that range

$$\| \int_0^t K_{t-s} * [\chi_{\frac{R}{2}} \nabla u |u|^{p-1}] ds \|_{L^{q(1+\kappa)}} < +\infty.$$

We claim that this term was the "worst" to be estimated in (*4.4.58*) and that using the very same techniques one can estimate similarly all the other terms in the right hand side in the same range $0 < \kappa \leq \kappa(q_0)$ leading to

$$\sup_{0 \leq t < T} \|\nabla v(t)\|_{L^{(1+\kappa)q}} < +\infty$$

which implies that $\sup_{0 \leq t < T} \|v(t)\|_{W^{1,(1+\kappa)q}} < +\infty$ by Sobolev embedding and Hölder inequality. This concludes the proof as $v = u$ on $B(x, \frac{R}{2})$.

□

[18] As $q \geq q_0 > \frac{p-1}{p+1} d$, $p > \frac{d+2}{d-2}$ and $d \geq 11$ all the computations below are rigorous.

4.5 Proof of Proposition 4.4.6

This section is devoted to the proof of this latter proposition, which will then end the proof of the main theorem. For all trapped solution u in the sense of Definition 4.4.4 we let $s^*=s^*(u(0))$ be the exit time from the trapped regime:

$$s^*=\sup\{s\geq s_0 \text{ such that } (4.4.22),\ (4.4.23),\ (4.4.24),\ (4.4.25) \text{ and } (4.4.26) \text{ hold on}[s_0,s)\} \qquad (4.5.1)$$

If $s^*<+\infty$, after s^*, one of the bounds (*4.4.22*), (*4.4.23*), (*4.4.24*), (*4.4.25*) or (*4.4.26*) must then be violated. The result of the first part of this section is a refinement of this exit condition. In Lemma 4.5.1, Propositions 4.5.3, 4.5.5, 4.5.6 and 4.5.8 we quantify accurately the time evolution of the parameters and the remainder in the trapped regime. Combined with the modulation equations of Lemma 4.4.7, this allows us to show that in the trapped regime, all the components of the solution along the stable directions of perturbation are under control, see Lemma 4.5.9. Moreover, from (*4.4.52*), (*4.4.26*) is always fulfilled as long as the other bounds hold. As a consequence, the exit time of the trapped regime is in fact characterized by the following condition: just after s^*, one of the bounds in (*4.4.22*) and (*4.4.23*) regarding the unstable parameters is violated.

Proposition 4.4.6 is then proven by contradiction. Suppose that given a stable perturbation of $\chi\tilde{Q}_{\bar{b}(s_0),\frac{1}{\lambda(s_0)}}$ as described in Proposition 4.4.6, for all initial corrections along the unstable directions $(V_2(s_0),...V_\ell(s_0))$ and $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\leq n,\ i<i_n}$, the solution starting from $\chi\tilde{Q}_{b(s_0),\frac{1}{\lambda(s_0)}}+w(s_0)$ leaves the trapped regime. This means that the trajectory of $(V_2(s),...V_\ell(s),(U_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I},1\leq n,\ i<i_n})$ leaves the set[19] $\mathcal{B}_\infty^{\ell-1}(s^{-\tilde{\eta}})\times\mathcal{B}_\infty^K(1)$ in finite time, from the previous paragraph. But at the leading order, the dynamics of this trajectory is a linear repulsive one. In Lemma 4.5.10 we show how the fact that all the trajectories leave this ball is a contradiction to Brouwer's fixed point theorem.

4.5.1 Improved modulation for the last parameters $b_{L_n}^{(n,k)}$

In Lemma 4.4.7, the modulation estimates (*4.4.43*) for the first parameters are better than the ones for the last parameters $b_{L_n}^{(n,k)}$, (*4.4.44*). When looking at the proof of Lemma 4.4.7, we see that this is a consequence of the fact that the projection of the linearized dynamics onto the profile generating the orthogonality conditions, $\langle H\varepsilon,H^i\Phi_M^{(n,k)}\rangle$ cancels only for $i<L_n$. However, as we explained in the introduction of Lemma 4.4.2, $H^i\Phi_M^{(n,k)}$ has to be thought as an approximation of $T_i^{(n,k)}$, and in that case the previous term would cancel also for $i=L_n$. It is therefore natural to look for a better modulation estimate for $b_{L_n}^{(n,k)}$. In the next Lemma we find a better bound by, roughly speaking, integrating by part in time the projection of ε onto $T_{L_n}^{(n,k)}$ in the self similar zone.

Lemma 4.5.1 (Improved modulation equation for $b_{L_n}^{(n,k)}$). *Suppose all the constants in Proposition 4.4.6 are fixed except s_0. Then for s_0 large enough, for any solution that is trapped on $[s_0,s')$, for $0\leq n\leq n_0$,*

[19] here K is the number of directions of instabilities on the spherical harmonics of degree greater than 0, $K=d(E[i_1]-\delta_{i_1\in\mathbb{N}})+\sum_{2\leq n\leq n_0}k(n)(E[i_n]+1-\delta_{i_n\in\mathbb{N}})$, $\mathcal{B}_\infty^a(r)$ is the ball of radius r of $\mathbb{R}^a$ for the usual $|\cdot|_\infty$ norm.

$1\leq k\leq k(n)$ *there holds for* $s\in[s_0,s')$*:*

$$\begin{array}{ll} & \left| b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}-\frac{d}{ds}\left[\frac{\langle H^{L_n}(\varepsilon-\sum_2^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle}{\left\langle \chi_{B_0}T_0^{(n,k)},T_0^{n,k}\right\rangle}\right]\right| \\ \leq & \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{\delta_n}}+\frac{C(L,M)}{s^{L+\frac{g'}{2}+\delta_n-\delta_0+1}}. \end{array} \quad (4.5.2)$$

Remark 4.5.2. From (*4.5.19*), we see that the denominator is not zero. From (*4.5.19*) and (*4.5.20*) one has the following bound for the new quantity that appeared when comparing this new modulation estimate to the former one (*4.4.44*):

$$\left|\frac{\langle H^{L_n}(\varepsilon-\sum_2^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle}{\left\langle \chi_{B_0}T_0^{(n,k)},T_0^{n,k}\right\rangle}\right|\leq C(L,M)s^{-L-\frac{g'}{2}+\delta_0-\delta_n}+C(L,M,K_2)s^{-L+\delta_0-\delta_n+\eta(1-\delta_0')}. \quad (4.5.3)$$

This is a better bound compared to the required bound (*4.4.24*) on $b_{L_n}^{(n,k)}$ in the trapped regime because that one is: $|b_{L_n}^{(n,k)}|\leq Cs^{-\frac{\gamma-\gamma_n}{2}-L_n}=Cs^{-L-\delta_n+\delta_0}$.

Proof of Lemma 4.5.1

First, from the fact that $HT_0^{(n,k)}=0$, the asymptotic (*4.2.2*) of $T_0^{(n,k)}$ and (*4.4.27*) we obtain:

$$\text{supp}[H^{L_n}(\chi_{B_0}T_0^{(n,k)})]\subset\{B_0\leq|y|\leq 2B_0\},\ \text{and}\ |H^{L_n}(\chi_{B_0}T_0^{(n,k)})|\leq\frac{C(L)}{s^{\frac{\gamma_n}{2}+L_n}}. \quad (4.5.4)$$

step 1 Computation of a first identity. We claim the following identity:

$$\begin{array}{rcl}\frac{d}{ds}\left(\langle H^{L_n}\varepsilon,\chi_{B_0}T_0^{(n,k)}\rangle\right) & = & (b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)})\langle T_0^{(n,k)},\chi_{B_0}T_0^{(n,k)}\rangle \\ & & +\frac{d}{ds}\left(\sum_{j=2}^{L+2}\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right) \\ & & +O(\sqrt{\mathcal{E}_{2s_L}}B_0^{4m_n+2\delta_n})+O\left(\frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}\right)\end{array} \quad (4.5.5)$$

what we are going to prove now. From the evolution equation (*4.4.41*) and the fact that H is self adjoint we obtain:

$$\begin{array}{rcl}\frac{d}{ds}\left(\langle H^{L_n}\varepsilon,\chi_{B_0}T_0^{(n,k)}\rangle\right) & = & \langle\varepsilon,H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle+\Big\langle-\tilde{\text{Mod}}(s)-\tilde{\psi}_b+\frac{\lambda_s}{\lambda}\Lambda\varepsilon \\ & & +\frac{z_s}{\lambda}.\nabla\varepsilon-H\varepsilon+\text{NL}(\varepsilon)+L(\varepsilon),H^{L_n}(\chi_{B_0}T_0^{(n,k)})\Big\rangle.\end{array} \quad (4.5.6)$$

The terms created by the cut of the solitary wave $\lambda^2\tau_{-z}[(\tilde{L}+\tilde{R}+\tilde{NL})_\lambda]$ do not appear because they have their support in the zone $\frac{1}{2\lambda}\leq|y|$ which is far away from the zone $|y|\leq 2B_0$ as $B_0\ll\frac{1}{\lambda}$ in the trapped regime from (*4.4.52*). We now look at all the terms in the above equation.

- *The* $\partial_s(\chi_{B_0})$ *term.* From the modulation equation (*4.4.43*) and the bound (*4.4.25*) one has $|b_{1,s}^{(0,1)}|\leq Cs^{-2}$. Hence, using the asymptotic (*4.2.2*) of $T_0^{(n,k)}$ and the fact that $HT_0^{(n,k)}=0$ and (*4.4.27*) we get that $H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})$ has support in $B_0\leq|y|\leq 2B_0$ and satisfies the bound $|H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})|\leq\frac{C(L)}{s^{\frac{\gamma_n}{2}+L_n+1}}$. Using the coercivity estimate (*4.C.16*) we obtain:

$$\left|\langle\varepsilon,H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle\right|\leq C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \quad (4.5.7)$$

- *The error term.* For $|y|\leq 2B_0$ one has $\tilde{\psi}_b=\psi_b$ from (*4.3.34*). As ψ_b is a finite sum of homogeneous profiles of degree $(i,-\gamma-2-g')$ for some $i\in\mathbb{N}$ (what was proved in Step 4 of the proof of Proposition

4.3.1), the bounds on the parameters (*4.4.27*) imply that $|\psi_b(y)| \leq C(L)s^{-\frac{\gamma+2+g}{2}}$ for $B_0 \leq |y| \leq 2B_0$. Combined with (*4.5.4*) this yield:

$$\left|\left\langle \tilde{\psi}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right| \leq C(L)B_0^{d-\gamma_n-2L_n-\gamma-g'-2} \leq \frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}. \tag{4.5.8}$$

- *The remainder's contribution.* Using (*4.5.4*), the bounds $|\frac{\lambda_s}{\lambda}| \leq Cs^{-1}$ and $|\frac{z_s}{\lambda}| \leq Cs^{-\frac{\alpha+1}{2}}$ (which are consequences of the modulation estimate (*4.4.43*) and (*4.4.25*)) and the coercivity estimate (4.C.3) one gets:

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon - H\varepsilon, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right| \leq C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \tag{4.5.9}$$

The small linear term writes $L(\varepsilon) = (pQ^{p-1} - p\tilde{Q}_b^{p-1})$, hence from the form of $\tilde{Q}_b$, see (*4.3.29*), one has $|(pQ^{p-1} - p\tilde{Q}_b^{p-1})| \leq C(L)s^{-1-\frac{\alpha}{2}}$. It's contribution is then of smaller order using (*4.5.4*):

$$\left|\langle L(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right| \leq C(L)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n-\frac{\alpha}{2}}. \tag{4.5.10}$$

The nonlinear term writes: $\text{NL}(\varepsilon) = \sum_{k=2}^p C_k^p \varepsilon^k \tilde{Q}_b^{p-k}$. From the coercivity estimate (4.C.3) we get:

$$\int_{B_0\leq|y|\leq 2B_0} \frac{\varepsilon^2}{|y|^{\gamma_n+2L_n}}dy \leq C(L,M)\mathcal{E}_{2s_L}s^{2s_L-\frac{\gamma_n}{2}-L_n}.$$

One computes using the bootstrap bounds (*4.4.25*) and (*4.4.27*):

$$\sqrt{\mathcal{E}_{2s_L}}s^{2s_L-\frac{\gamma_n}{2}-L_n} \leq K_2 s^{\delta_n+2m_n-(\frac{\gamma-2}{4}+\frac{\eta(1-\delta_0')}{2})} \leq B_0^{\delta_n+2m_n}$$

for s_0 large enough (because $\gamma > 2$). For $2 \leq k \leq p$, $|\varepsilon^{k-2}\tilde{Q}_b^{p-k}| \leq C$ is bounded from (*4.D.2*), so one gets using the two previous equations and (*4.5.4*):

$$\left|\langle \text{NL}(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right| \leq \sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n} \tag{4.5.11}$$

for s_0 large enough. Gathering (*4.5.9*), (*4.5.10*) and (*4.5.11*) we have found the following upper bound for the remainder's contribution:

$$\left|\left\langle \frac{\lambda_s}{\lambda}\Lambda\varepsilon + \frac{z_s}{\lambda}.\nabla\varepsilon - H\varepsilon + \text{NL}(\varepsilon) + L(\varepsilon), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right| \leq C(L,M)\sqrt{\mathcal{E}_{2s_L}}s^{2m_n+\delta_n}. \tag{4.5.12}$$

- *The modulation term.* For $(n',k',i) \in \mathcal{I}$, one has

$$\langle T_i^{(n,k)}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle = \langle H^{L_n}T_i^{(n,k)}, \chi_{B_0}T_0^{(n,k)}\rangle = 0$$

if $(n',k',i) \neq (n,k,L_n)$. Indeed, if $(n',k') \neq (n,k)$ then the two functions are located on different spherical harmonics and their scalar product is 0. If $i \neq L_n$ then $i < L_n$ and $H^{L_n}T_i^{(n,k)} = 0$. This implies the identity from (*4.4.33*) since $B_1 \gg B_0$:

$$\begin{array}{rl} & \langle \tilde{\text{Mod}}(s), H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle \\ = & (b_{L_n,s}^{(n,k)} + (2L_n - \alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)})\langle T_0^{(n,k)}, \chi_{B_0}T_0^{(n,k)}\rangle \\ & + \sum_{j=2}^{L+2}\sum_{(n',k',i)\in\mathcal{I}\}}(b_{i,s}^{(n',k')} + (2i-\alpha_{n'})b_1^{(0,1)}b_i^{(n',k')})\langle \frac{\partial S_j}{\partial b_i^{(n',k')}}, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle \\ & -(\frac{\lambda_s}{\lambda} + b_1^{(1,0)})\langle \Lambda\tilde{Q}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle - \langle (\frac{z_s}{\lambda} + b_1^{(1,\cdot)}).\nabla\tilde{Q}_b, H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle \end{array} \tag{4.5.13}$$

For $2\leq j\leq L+2$, and $(n',k',i)\in \mathcal{I}$ there holds, as S_i is homogeneous of degree $(i,-\gamma-g')$, using (*4.4.27*) and (*4.5.4*):

$$\left|(2i-\alpha_{n'})b_1^{(0,1)}b_i^{(n',k')})\left\langle \frac{\partial S_j}{\partial b_i^{(n',k')}},H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle\right|\leq \frac{C(L,M)}{s^{L-\delta_0-\delta_n+2m_n+1+\frac{g'}{2}}}. \tag{4.5.14}$$

Using the modulation bound (*4.4.43*), the asymptotics (*3.2.1*) and (*4.2.2*) of Q and ΛQ, (*4.4.27*) and (*4.5.4*) we find that:

$$\begin{aligned}&\left|(\tfrac{\lambda_s}{\lambda}+b_1^{(1,0)})\langle\Lambda\tilde{Q}_b,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle-\langle(\tfrac{z_s}{\lambda}+b_1^{(1,\cdot)}).\nabla\tilde{Q}_b,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right|\\ \leq\ & \frac{C(L,M)}{s^{2L+\frac{3-\alpha}{2}-2m_n-\delta_n}}\end{aligned} \tag{4.5.15}$$

is very small as $L\gg 1$. Moreover for $2\leq j\leq L+2$ one has:

$$\begin{aligned}\sum_{(n',k',i)\in\mathcal{I}}b_{i,s}^{(n',k')}\left\langle\frac{\partial S_j}{\partial b_i^{(n',k')}},H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle &= \frac{d}{ds}\left(\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right)\\ &\quad -\langle S_j,H^{L_n}(\partial_s\chi_{B_0}T_0^{(n,k)})\rangle.\end{aligned}$$

From similar arguments we used to derive (*4.5.14*) one has the similar bound for the last term, yielding:

$$\begin{aligned}\sum_{(n',k',i)\in\mathcal{I}}b_{i,s}^{(n',k')}\left\langle\frac{\partial S_j}{\partial b_i^{(n',k')}},H^{L_n}(\chi_{B_0}T_0^{(n,k)})\right\rangle &= \frac{d}{ds}\left(\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right)\\ &\quad +O(s^{-L+\delta_0+\delta_n+2m_n-1-\frac{g'}{2}}).\end{aligned} \tag{4.5.16}$$

Coming back to the decomposition (*4.5.13*), and injecting (*4.5.14*) and (*4.5.16*) gives:

$$\begin{aligned}\langle\tilde{\text{Mod}}(s),H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle &= (b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)})\langle T_0^{(n,k)},\chi_{B_0}T_0^{(n,k)}\rangle\\ &\quad +\frac{d}{ds}\left(\sum_{j=2}^{L+2}\langle S_j,H^{L_n}(\chi_{B_0}T_0^{(n,k)})\rangle\right)\\ &\quad +O(s^{-L+\delta_0+\delta_n+2m_n-1-\frac{g'}{2}})\end{aligned} \tag{4.5.17}$$

In the decomposition (*4.5.6*) we examined each term in (*4.5.7*), (*4.5.8*), (*4.5.12*) and (*4.5.17*), yielding the identity (*4.5.5*) we claimed in this first step.

step 2 End of the proof. From (*4.5.5*) one obtains:

$$\begin{aligned}&\frac{d}{ds}\left(\frac{\left(\langle H^{L_n}(\varepsilon-\sum_2^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle\right)}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle}\right)\\ =\ & b_{L_n,s}^{(n,k)}+(2L_n-\alpha_n)b_1^{(0,1)}b_{L_n}^{(n,k)}+\frac{O(\sqrt{\mathcal{E}_{2s_L}}B_0^{4m_n+2\delta_n})+O\left(\frac{C(L)}{s^{L+1+\frac{g'}{2}-\delta_0-\delta_n-2m_n}}\right)}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle}\\ &+\langle H^{L_n}(\varepsilon-\textstyle\sum_2^{L+2}S_i),\chi_{B_0}T_0^{(n,k)}\rangle\frac{d}{ds}\left(\frac{1}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle}\right).\end{aligned} \tag{4.5.18}$$

The size of the denominator is, from the asymptotic (*4.2.2*) of $T_0^{(n,k)}$ and (*4.4.27*):

$$\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle\sim cs^{2m_n+2\delta_n} \tag{4.5.19}$$

for some constant $c>0$. As the denominator just depends on $b_1^{(0,1)}$, using the bound $|b_{1,s}^{(0,1)}|\leq Cs^{-2}$ and the asymptotics (*4.2.2*) of $T_0^{(n,k)}$ we obtain:

$$\left|\frac{d}{ds}\left(\frac{1}{\langle\chi_{B_0}T_0^{(n,k)},T_0^{(n,k)}\rangle}\right)\right|\leq\frac{C(L,M)}{s^{2m_n+2\delta_n+1}}.$$

Also, using again the coercivity estimate (4.C.3), (*4.5.4*) and the fact that for $2 \leq j \leq L+2$, S_j is homogeneous of degree $(j, -\gamma - g')$ we obtain:

$$\left| \langle H^{L_n}(\varepsilon - \sum_2^{L+2} S_i), \chi_{B_0} T_0^{(n,k)} \rangle \right| \leq C(L,M)(\sqrt{\mathcal{E}_{2s_L}} s^{2m_n+\delta_n} + s^{-L-\frac{g'}{2}+\delta_0+\delta_n+2m_n}). \qquad (4.5.20)$$

Hence, plugging the three previous identities in (*4.5.18*) gives the identity (*4.5.3*) claimed in the Lemma.

□

4.5.2 Lyapunov monotonicity for low regularity norms of the remainder

The key estimate concerning the remainder w is the bound on the high regularity adapted Sobolev norm at the blow up area: $\mathcal{E}_{2s_L}$. However, the nonlinearity can transfer energy from low to high frequencies, and consequently to control $\mathcal{E}_{2s_L}$ we need to control the low frequencies. This is the purpose of the following two propositions 4.5.3 and 4.5.5 where we find an upper bound for the time evolution of $\|w_{\text{int}}\|_{\dot{H}^\sigma(\mathbb{R}^d)}$ and $\|w_{\text{ext}}\|_{H^\sigma(\Omega)}$.

Proposition 4.5.3 (Lyapunov monotonicity for the low Sobolev norm in the blow up zone). *Suppose all the constants involved in Proposition 4.4.6 are fixed except s_0 and η. Then for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0, s')$ there holds for $0 \leq t < t(s')$:*

$$\frac{d}{dt}\left\{ \frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}} \right\} \leq \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}} \frac{1}{s^{\frac{\alpha}{4L}}} \left[1 + \sum_{k=2}^{p} \left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}} \right)^{k-1} \right] \qquad (4.5.21)$$

where the norm $\mathcal{E}_\sigma$ is defined in (4.4.9).

Remark 4.5.4. (*4.5.21*) should be interpreted as follows. The term $\frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}$ is from (*4.4.25*) and (*4.4.52*) of order $\frac{1}{s}\frac{ds}{dt}$ (as $\frac{ds}{dt} = \lambda^{-2}$). The $\frac{1}{s^{\frac{\alpha}{4L}}}$ then represents a gain: it gives that the right hand side of (*4.5.21*) is of order $\frac{1}{s^{1+\frac{\alpha}{4L}}}\frac{ds}{dt}$, which when reintegrated in time is convergent and arbitrarily small for s_0 large enough. The third term shows that one needs to have $\sqrt{\mathcal{E}_\sigma} \lesssim s^{-\frac{\sigma-s_c}{2}}$ to control the non linear terms, which holds because of the bootstrap bound (*4.4.25*).

Proof of Proposition 4.5.3

To show this result, we compute the left hand side of (*4.5.21*) and we upper bound it using all the bounds that hold in the trapped regime. The time evolution w_{int} given by (*4.4.35*) yields:

$$\begin{aligned} \frac{d}{dt}\left\{ \frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}} \right\} &= \frac{d}{dt}\left\{ \int |\nabla^\sigma w_{\text{int}}|^2 \right\} \\ &= \int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(-H_{z,\frac{1}{\lambda}} w_{\text{int}} - \frac{1}{\lambda^2}\chi\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}} + \tilde{\psi}_{b\frac{1}{\lambda}}) \\ &\quad +\text{NL}(w_{\text{int}}) + L(w_{\text{int}}) + \tilde{L} + \tilde{NL} + \tilde{R}). \end{aligned} \qquad (4.5.22)$$

We now give an upper bound for each term in (*4.5.22*). As all the terms involve functions that are compactly supported in Ω since w_{int} is, all integrations by parts are legitimate and all computations and integrations are performed in $\mathbb{R}^d$ (e.g. L^2 denotes $L^2(\mathbb{R}^d)$).

step 1 Inside the blow-up zone (all terms except the three last ones in (*4.5.22*)).

- *The linear term:* We first compute from (*4.4.36*) using dissipation:

$$\begin{array}{rcl} \int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(-H_{z,\frac{1}{\lambda}}w_{\text{int}}) & = & \int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(\Delta w_{\text{int}}+p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w_{\text{int}}) \\ & \leq & \int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w_{\text{int}}) \end{array}$$

which becomes after an integration by parts and using Cauchy-Schwarz inequality:

$$\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w_{\text{int}}) \leq \|\nabla^{\sigma+2}w_{\text{int}}\|_{L^2}\|\nabla^{\sigma-2}(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w_{\text{int}})\|_{L^2}$$

Using interpolation, the coercivity estimate (*4.C.16*) and the bounds of the trapped regime (*4.4.25*) on ε, one has for the first term (performing a change of variables to go back to renormalized variables):

$$\begin{array}{rl} & \|\nabla^{\sigma+2}w_{\text{int}}\|_{L^2} = \frac{1}{\lambda^{\sigma+2-s_c}}\|\nabla^{\sigma+2}\varepsilon\|_{L^2} \leq \frac{C}{\lambda^{\sigma+2-s_c}}\|\nabla^\sigma\varepsilon\|_{L^2}^{1-\frac{2}{2s_L-\sigma}}\|\varepsilon\|_{\dot{H}^{2s_L}}^{\frac{2}{2s_L-\sigma}} \\ \leq & \frac{C(L,M)}{\lambda^{\sigma+2-s_c}}\sqrt{\mathcal{E}_\sigma}^{1-\frac{2}{2s_L-\sigma}}\sqrt{\mathcal{E}_{2s_L}}^{\frac{2}{2s_L-\sigma}} \leq \frac{C(L,M,K_1,K_2)}{\lambda^{\sigma+2-s_c}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+\frac{2}{2s_L-\sigma}(L+1-\delta_0+\eta(1-\delta_0'))-\frac{(\sigma-s_c)\ell}{2\ell-\alpha}}} \\ = & \frac{C(L,M,K_1,K_2)}{\lambda^{\sigma+2-s_c}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}} \end{array}$$

As $Q^{p-1}=O((1+|y|)^{-2})$ from (*3.2.10*), using the Hardy inequality (*3.C.8*) we get for the second term after a change of variables:

$$\begin{array}{rcl} \|\nabla^{\sigma-2}(p(\tau_z(Q_{\frac{1}{\lambda}}))^{p-1}w)\|_{L^2} & = & \frac{p}{\lambda^{\sigma-s_c}}\|\nabla^{\sigma-2}(Q^{p-1}\varepsilon)\|_{L^2} \leq \frac{C}{\lambda^{\sigma-s_c}}\|\nabla^\sigma\varepsilon\|_{L^2} \\ & = & \frac{C}{\lambda^{\sigma-s_c}}\sqrt{\mathcal{E}_\sigma}. \end{array}$$

Combining the four above identities we obtain:

$$\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(-H_{z,\frac{1}{\lambda}}w_{\text{int}}) \leq \frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}. \tag{4.5.23}$$

- *The modulation term:* To treat the error induced by the cut separately, we decompose as follows, going back to renormalized variables using Cauchy-Schwarz:

$$\begin{array}{rl} & \left|\int \nabla^\sigma w.\nabla^\sigma(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}}))\right| \\ \leq & \left|\int \nabla^\sigma w.\nabla^\sigma(\frac{1}{\lambda^2}(1+(\chi-1)))\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}}))\right| \\ \leq & \frac{1}{\lambda^{2(\sigma-s_c)+2}}\sqrt{\mathcal{E}_\sigma}\left[\|\nabla^\sigma\tilde{\text{Mod}}(s)\|_{L^2}+\|\nabla^\sigma(\frac{1}{\lambda^2}(\chi-1)\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}}))\|_{L^2}\right]. \end{array} \tag{4.5.24}$$

For the first term in the above equation, using (*4.4.33*) and the modulation estimates (*4.4.43*) and (*4.4.44*) we get:

$$\begin{array}{rl} & \|\nabla^\sigma\tilde{\text{Mod}}(s)\|_{L^2} \\ \leq & \sum\limits_{(n,k,i)\in\mathcal{I}}|b_{i,s}^{(n,k)}+(2i-\alpha_n)b_1^{(0,1)}b_i^{(n,k)}-b_{i+1}^{(n,k)}|\|\nabla^\sigma(\chi_{B_1}(T_i^{(n,k)}+\sum\limits_2^{L+2}\frac{\partial S_j}{\partial b_i^{(n,k)}}))\|_{L^2} \\ & +|\frac{\lambda_s}{\lambda}+b_1^{(0,1)}|\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}+|\frac{z_s}{\lambda}+b_1^{(1,\cdot)}|\|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2} \\ \leq & C(L,M)(\sqrt{\mathcal{E}_{2s_L}}+s^{-L-3})\Big[\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}+\|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2} \\ & +\sum_{(n,k,i)\in\mathcal{I}}\|\nabla^\sigma(\chi_{B_1}T_i^{(n,k)})\|_{L^2}+\sum_2^{L+2}\|\nabla^\sigma(\chi_{B_1}\frac{\partial S_j}{\partial b_i^{(n,k)}})\|_{L^2}\Big]. \end{array}$$

Under the trapped regime bound (*4.4.25*) one has $\sqrt{\mathcal{E}_{2s_L}}+s^{-L-3}\leq s^{-L-1+\delta_0-\eta(1-\delta_0')}$. Moreover, from the asymptotic of Q, ΛQ, $T_i^{(n,k)}$ and S_j ((*3.2.1*), (*4.2.2*), Lemma 4.2.8 and (*4.3.8*)), and the bounds on the parameters (*4.4.27*) one has:

$$\|\nabla^\sigma(\Lambda\tilde{Q}_b)\|_{L^2}\leq C, \quad \|\nabla^{\sigma+1}(\tilde{Q}_b)\|_{L^2}\leq C,$$

$$\begin{array}{ll} & \sum_{(n,k,i)\in\mathcal{J}} \|\nabla^\sigma(\chi_{B_1} T_i^{(n,k)})\|_{L^2} + \sum_2^{L+2} \|\nabla^\sigma(\chi_{B_1} \frac{\partial S_j}{\partial b_i^{(n,k)}})\|_{L^2} \leq C(L) \\ \leq & C(L) s^{L+\sup\limits_{0\leq n\leq n_0} \delta_n - \delta_0 - \frac{\alpha}{2} - \frac{(\sigma-s_c)}{2} + C(L)\eta} + C(L) s^{L+\sup\limits_{0\leq n\leq n_0} \delta_n - \delta_0 - \frac{\alpha}{2} - \frac{(\sigma-s_c)}{2} + C(L)\eta - \frac{g'}{2}} \end{array}$$

All these bounds then imply that for the modulation term that is located at the blow up zone in (*4.5.24*) there holds:

$$\begin{array}{lll} \frac{1}{\lambda^{2(\sigma-s_c)+2}} \sqrt{\mathcal{E}_\sigma} \|\nabla^\sigma \tilde{\text{Mod}}(s)\|_{L^2} & \leq & \frac{C(L,M)\sqrt{\mathcal{E}_\sigma} s^{L+\sup\limits_{0\leq n\leq n_0} \delta_n - \delta_0 - \frac{\alpha}{2} - \frac{(\sigma-s_c)}{2} + C(L)\eta}}{\lambda^{2(\sigma-s_L)+2} s^{L+1-\delta_0+(1-\delta_0')\eta}} \\ & \leq & \frac{C(L,M)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{1+\left(\frac{\alpha}{2} - \sup\limits_{0\leq n\leq n_0} \delta_n\right) + \frac{\sigma-s_c}{2} - C(L)\eta}} \end{array}$$

We now turn to the second term in (*4.5.24*). The blow up point z is arbitrarily close to 0 from (*4.4.51*) and from the expression of the modulation term (*4.4.33*), all the terms except $\tau_z([\frac{\lambda_s}{\lambda} + b_1^{(0,1)}]\Lambda Q + [b_1^{(1,\cdot} + \frac{z_s}{\lambda}].\nabla Q)_{\frac{1}{\lambda}}$ have support in the zone $\{|x-z| \leq 2B_1\lambda\} \subset B(0,\frac{1}{2})$ because $B_1\lambda \ll 1$. This means that from the modulation estimates (*4.4.43*):

$$\begin{array}{ll} & \|\nabla^\sigma(\frac{1}{\lambda^2}(\chi-1)\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}}))\|_{L^2} \\ = & \|\nabla^\sigma(\frac{1}{\lambda^2}(\chi-1)\tau_z([\frac{\lambda_s}{\lambda} + b_1^{(0,1)}]\Lambda Q + [b_1^{(1,\cdot} + \frac{z_s}{\lambda}].\nabla Q)_{\frac{1}{\lambda}}))\|_{L^2} \\ \leq & \frac{C[|\frac{\lambda_s}{\lambda} + b_1^{(0,1)}| + |\frac{z_s}{\lambda} + b_1^{(1,\cdot)}|]}{\lambda^2} \leq \frac{C}{\lambda^2 s^{L+1}} \end{array}$$

We inject the two previous equations in the expression (*4.5.24*), yielding:

$$\left|\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}}))\right| \leq \frac{C(L,M)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{1+\left(\frac{\alpha}{2} - \sup\limits_{0\leq n\leq n_0} \delta_n\right) + \frac{\sigma-s_c}{2} - C(L)\eta}} \quad (4.5.25)$$

- *The error term:* as $|z| \ll 1$ from (*4.4.51*) and $B_1\lambda \ll 1$ from (*4.4.27*) and (*4.4.52*), from the expression of the error term (*4.3.36*), all the terms except $\tau_z(b_1^{(0,1)}\Lambda Q + b_1^{(1,\cdot)}.\nabla Q)_{\frac{1}{\lambda}}$ have support in the zone $\{|x-z| \leq 2B_1\lambda\} \subset B(0,\frac{1}{2})$. Therefore, one computes, making the following decomposition and coming back to renormalized variables, using the estimates(*4.3.32*) and (*4.4.43*):

$$\begin{array}{ll} & \left|\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(\frac{1}{\lambda^2}\chi\tau_z(\tilde{\psi}_{b\frac{1}{\lambda}}))\right| \\ \leq & \frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{\sigma-s_c+2}}(\frac{\|\nabla^\sigma\tilde{\psi}_b\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}} + \|\nabla^\sigma((\chi-1)\tau_z(\tilde{\psi}_{b\frac{1}{\lambda}}))\|_{L^2}) \\ \leq & \frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}} + \frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{\sigma-s_c+2}}\|\nabla^\sigma(\chi-1)(\tau_z(b_1^{(0,1)}\Lambda Q + b_1^{(1,\cdot}.\nabla Q)_{\frac{1}{\lambda}})\|_{L^2} \\ \leq & \frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}} + C\frac{\|\nabla^\sigma\varepsilon\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}}(|b_1^{(0,1)}|\lambda^{\alpha+\sigma-s_c} + |b_1^{(1,\cdot)}|\lambda^{1+\sigma-s_c}) \\ \leq & \frac{C(L)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{1+\frac{\alpha}{2}+\frac{\sigma-s_c}{2}-C(L)\eta}} \end{array} \quad (4.5.26)$$

- *The non linear term:* First, coming back to renormalized variables, as $\text{NL}(\varepsilon) = \sum_{k=2}^p C_k^p \tilde{Q}_b^{p-k}\varepsilon^k$, and performing an integration by parts we write:

$$|\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(\text{NL}(w_{\text{int}}))| \leq C\sum_{k=2}^p \frac{\|\nabla^{\sigma+2-(k-1)(\sigma-s_c)}\varepsilon\|_{L^2}\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\tilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}}{\lambda^{2(\sigma-s_c)+2}} \quad (4.5.27)$$

We fix k, $2 \leq k \leq p$ and focus on the k-th term in the sum. The first term is estimated using interpolation, the coercivity estimate (*4.C.16*) and the bound (*4.4.25*):

$$\begin{array}{lll} \|\nabla^{\sigma+2-(k-1)(\sigma-s_c)}\varepsilon\|_{L^2} & \leq & C\|\nabla^\sigma\varepsilon\|_{L^2}^{1-\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\|\nabla^{2s_L}\varepsilon\|_{L^2}^{\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}} \\ & \leq & C(L,M)\sqrt{\mathcal{E}_\sigma}^{1-\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}}\sqrt{\mathcal{E}_{2s_L}}^{\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}} \\ & \leq & \frac{C(L,M,K_1,K_2)}{s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1-\frac{(k-1)(\sigma-s_c)}{2}+\frac{\alpha}{2L}+O\left(\frac{|\sigma-s_c|+|\eta|}{L}\right)}}. \end{array} \quad (4.5.28)$$

For the second term in (*4.5.27*), as $\tilde Q_b=O((1+|y|)^{-2})$ from (*4.3.29*) and (*4.4.27*) we first use the Hardy inequality (*3.C.8*):

$$\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\tilde Q_b^{p-k}\varepsilon^k)\|_{L^2}\leq C\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}}(\varepsilon^k)\|_{L^2}. \tag{4.5.29}$$

We write

$$\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}=\sigma(n,k)+\delta(n,k)$$

where $\sigma(n,k):=E[\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}]\in\mathbb{N}$ and $0\leq\delta(n,k)<1$. Developing the entire part of the derivative yields:

$$\begin{array}{rc} & \|\nabla^{\sigma-2+(k-1)(\sigma-s_c)+\frac{2(p-k)}{p-1}}(\varepsilon^k)\|_{L^2} \\ \leq & \displaystyle\sum_{(\mu_i)_{1\leq i\leq k}\in\mathbb{N}^{kd},\ \sum_i|\mu_i|=\sigma(n,k)}\|\nabla^{\delta(\sigma,k)}(\textstyle\prod_1^k\partial^{\mu_i}\varepsilon)\|_{L^2}.\end{array} \tag{4.5.30}$$

Fix $(\mu_i)_{1\leq i\leq k}\in\mathbb{N}^{kd}$ satisfying $\sum_{i=1}^k|\mu_i|=\sigma(n,k)$ in the above sum. We define the following family of Lebesgue exponents (that are well-defined since $\sigma<\frac d2$):

$$\frac{1}{p_i}:=\frac12-\frac{\sigma-|\mu_i|_1}{d},\quad \frac{1}{p_i'}:=\frac12-\frac{\sigma-|\mu_i|-\delta(\sigma,k)}{d}\quad\text{for } 1\leq i\leq k.$$

One has $p_i>2$ and a direct computation shows that

$$\frac{1}{p_j'}+\sum_{i\neq j}\frac{1}{p_i}=\frac12.$$

We now recall the commutator estimate:

$$\|\nabla^{\delta_\sigma}(uv)\|_{L^q}\leq C\|\nabla^{\delta_\sigma}u\|_{L^{p_1}}\|v\|_{L^{p_2}}+C\|\nabla^{\delta_\sigma}v\|_{L^{p_1'}}\|u\|_{L^{p_2'}},$$

for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_1'}+\frac{1}{p_2'}=\frac1q$, provided $1<q,p_1,p_1'<+\infty$ and $1\leq p_2,p_2'\leq+\infty$. This estimate, combined with the Hölder inequality allows us to compute by iteration:

$$\begin{array}{rl} & \|\nabla^{\delta(\sigma,k)}(\prod_{i=1}^k\partial^{\mu_i}\varepsilon)\|_{L^2} \\ \leq & C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\|\prod_2^k\partial^{\mu_i}\varepsilon\|_{L^{(\sum_2^k\frac{1}{p_i})^{-1}}} \\ & +C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\nabla^{\delta(\sigma,k)}(\prod_2^k\partial^{\mu_i}\varepsilon)\|_{L^{(\frac12-\frac{1}{p_1})^{-1}}} \\ \leq & C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\prod_2^k\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}} \\ & +C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2+\delta(\sigma,k)}\varepsilon\|_{L^{p_2'}}\|\prod_3^k\partial^{\mu_i}\varepsilon\|_{L^{(\sum_3^k\frac{1}{p_i})^{-1}}} \\ & +C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2}\varepsilon\|_{L^{p_2}}\|\nabla^{\delta(\sigma,k)}(\prod_3^k\partial^{\mu_i}\varepsilon)\|_{L^{(\frac12-\frac{1}{p_1}-\frac{1}{p_2})^{-1}}} \\ \leq & C\|\partial^{\mu_1+\delta(\sigma,k)}\varepsilon\|_{L^{p_1'}}\prod_2^k\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}}+C\|\partial^{\mu_2+\delta(\sigma,k)}\varepsilon\|_{L^{p_2'}}\prod_{i\neq2}\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}} \\ & +C\|\partial^{\mu_1}\varepsilon\|_{L^{p_1}}\|\partial^{\mu_2}\varepsilon\|_{L^{p_2}}\|\nabla^{\delta(\sigma,k)}(\prod_3^k\partial^{\mu_i}\varepsilon)\|_{L^{(\frac12-\frac{1}{p_1}-\frac{1}{p_2})^{-1}}} \\ \leq & \dots \\ \leq & C\sum_{i=1}^k\|\partial^{\mu_i+\delta(\sigma,k)}\varepsilon\|_{L^{p_i'}}\prod_{j=1,\ j\neq i}^k\|\partial^{\mu_i}\varepsilon\|_{L^{p_j}}.\end{array}$$

From Sobolev embedding, one has on the other hand that:

$$\|\partial^{\mu_i+\delta(\sigma,k)}\varepsilon\|_{L^{p_i'}}+\|\partial^{\mu_i}\varepsilon\|_{L^{p_i}}\leq C\|\nabla^\sigma\varepsilon\|_{L^2}=C\sqrt{\mathcal{E}_\sigma}.$$

Therefore (the strategy was designed to obtain this):

$$\left\| \nabla^{\delta(\sigma,k)} (\prod_{i=1}^k \partial^{\mu_i} \varepsilon) \right\|_{L^2} \leq \sqrt{\mathcal{E}_\sigma}^k.$$

Plugging this estimate in (*4.5.29*) using (*4.5.30*) gives:

$$\|\nabla^{\sigma-2+(k-1)(\sigma-s_c)}(\tilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2} \leq C\sqrt{\mathcal{E}_\sigma}^k.$$

Injecting this bound and the bound (*4.5.28*) in the decomposition (*4.5.27*) yields:

$$\begin{array}{ll} & |\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(\text{NL}(w_{\text{int}}))| \\ \leq & \frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{|\eta|+|\sigma-s_c|}{L}\right)}} \sum_{k=2}^p \left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}. \end{array} \quad (4.5.31)$$

- *The small linear term:* One has: $L(\varepsilon) = -p(Q^{p-1} - \tilde{Q}^{p-1})\varepsilon$. The potential here admits the asymptotic $Q^{p-1} - \tilde{Q}^{p-1} \lesssim |y|^{-2-\alpha}$ at infinity which is better than the asymptotic of the potential appearing in the linear term $Q^{p-1} \sim |y|^{-2}$ we used previously to estimate it. Hence using verbatim the same techniques one can prove the same estimate:

$$\left|\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(L(w_{\text{int}}))\right| \leq \frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1+\frac{\alpha}{2L}+O\left(\frac{|\eta|+|\sigma-s_c|}{L}\right)}}. \quad (4.5.32)$$

- *End of Step 1:* We come back to the first identity we derived (*4.5.22*) and inject the bounds we found for each term in (*4.5.23*), (*4.5.25*), (*4.5.26*), (*4.5.31*) and (*4.5.32*) to obtain:

$$\begin{array}{ll} & |\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(-H_{z,\frac{1}{\lambda}} w_{\text{int}} - \frac{1}{\lambda^2}\chi\tau_z(\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}} + \tilde{\psi}_{b\frac{1}{\lambda}}) + \text{NL}(w_{\text{int}}) + L(w_{\text{int}}))| \\ \leq & \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2} s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}} \Bigg[\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}} + \frac{C(L,M,K_2)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+(\frac{\alpha}{2}-\sup_{0\leq n\leq n_0}\delta_n)-C(L)\eta}} \\ & + \frac{C(L)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+\frac{\alpha}{2}-C(L)\eta}} + \frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}} \sum_{k=2}^p \left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1} \Bigg]. \end{array} \quad (4.5.33)$$

step 2 The last three terms outside the blow up zone in (*4.5.22*). By a change of variables we see that the extra error term (*4.4.40*) is bounded:

$$\|\nabla^\sigma \tilde{R}\|_{L^2(\mathbb{R}^d)} \leq C.$$

Then, the extra linear term in (*4.5.22*) is estimated directly via interpolation using the bound (*4.4.28*):

$$\begin{array}{ll} & \|\nabla^\sigma(-\Delta\chi_{B(0,3)}w - 2\nabla\chi_{B(0,3)}.\nabla w + p\tau_z Q_{\frac{1}{\lambda}}^{p-1}(\chi_{B(0,1)}^{p-1} - \chi_{B(0,3)})w)\|_{L^2(\mathbb{R}^d)} \\ \leq & \|w\|_{H^{\sigma+1}} \leq \|w\|_{H^\sigma}^{1-\frac{1}{2s_L-\sigma}} \|w\|_{H^{2s_L}}^{\frac{1}{2s_L-\sigma}} \leq C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-\sigma} s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{1}{2s_L-\sigma}} \\ \leq & C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-\sigma} s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2}{2s_L-\sigma}} = \frac{C(K_1,K_2)}{\lambda^2 s^{1+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}} \end{array}$$

because $\frac{1}{\lambda^{2s_L-\sigma} s^{L+1-\delta_0+\eta(1-\delta_0')}} \gg 1$ in the trapped regime. For the last non linear in (*4.5.22*) one has using (*4.D.4*) and (*4.4.28*):

$$\begin{array}{ll} & \|\tilde{NL}\|_{H^\sigma} \leq C\|w\|_{H^\sigma}\|w\|_{H^{\frac{d}{2}+\sigma-s_c}}^{p-1} \leq C(K_1)\|w\|_{H^{2s_L}}^{(p-1)\frac{\frac{d}{2}+\sigma-s_c-\sigma}{2s_L-\sigma}} \\ \leq & C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2}{2s_L-\sigma}} \leq C(K_1,K_2)\frac{1}{\lambda^2 s^{1+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}. \end{array}$$

The three previous estimates imply that for the terms created by the cut in (*4.5.22*) there holds the estimate (we recall that $\frac{\lambda^{\sigma-s_c}}{s^{\frac{\ell(\sigma-s_c)}{2\ell-\alpha}}}=1+O(s_0^{-\tilde\eta})$ from (*4.4.52*)):

$$\left|\int \nabla^\sigma w_{\text{int}}.\nabla^\sigma(\tilde L+\tilde R+\tilde{NL})\right|\leq \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}. \quad (4.5.34)$$

step 3 Conclusion. We now come back to the first identity we derived (*4.5.22*) and inject the bounds (*4.5.33*) and (*4.5.34*), yielding:

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\}\leq{}& \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\left[\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}+\frac{C(L,M,K_2)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+(\frac{\alpha}{2}-\sup\limits_{0\leq n\leq n_0}\delta_n)-C(L)\eta}}\right.\\ &\left.+\frac{C(L)}{s^{-\frac{(\sigma-s_c)\alpha}{2\ell-\alpha}+\frac{\alpha}{2}-C(L)\eta}}+\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\sum_{k=2}^p\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\right].\end{aligned}$$

As the constants never depends on s_0 or on η, as $L\gg 1$ is an arbitrary large integer, $0<\sigma-s_c\ll 1$, $\frac{\alpha}{2}-\sup\limits_{0\leq n\leq n_0}\delta_n>0$, we see that for s_0 sufficiently large and η sufficiently small, the terms in the right hand side of the previous equation can be as small as we want, and (*4.5.21*) is obtained.

□

Proposition 4.5.5 (Lyapunov monotonicity for the low Sobolev norm outside the blow up area). *Let all the constants involved in Proposition 4.4.6 be fixed except s_0 and η. Then for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0,s')$ there holds for $t\in[0,t(s'))$:*

$$\frac{d}{dt}\left[\|w_{ext}\|^2_{H^\sigma}\right]\leq\frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}\lambda^2}\|w_{ext}\|_{H^\sigma}. \quad (4.5.35)$$

Proof.

From the evolution equation of w_{ext} (*4.4.34*) we deduce:

$$\frac{d}{dt}\|w_{\text{ext}}\|^2_{H^\sigma(\Omega)}\leq C\|w_{\text{ext}}\|_{H^\sigma(\Omega)}\|\Delta w_{\text{ext}}+\Delta\chi_3 w+2\nabla\chi_3.\nabla w+(1-\chi_3)w^p\|_{H^\sigma(\Omega)}. \quad (4.5.36)$$

For the linear terms, using interpolation and the bounds (*4.4.25*) and (*4.4.28*) one finds:

$$\begin{aligned}&\|\Delta w_{\text{ext}}+\Delta\chi_3 w+2\nabla\chi_3.\nabla w\|_{H^\sigma(\Omega)}\leq C\|w_{\text{ext}}\|_{H^{\sigma+2}(\Omega)}+C\|w\|_{H^{\sigma+1}(\Omega)}\\ \leq{}& C\|w_{\text{ext}}\|^{1-\frac{2}{2s_L-\sigma}}_{H^\sigma(\Omega)}\|w_{\text{ext}}\|^{\frac{2}{2s_L-\sigma}}_{H^{2s_L}(\Omega)}+C\|\|w\|^{1-\frac{1}{2s_L-\sigma}}_{H^\sigma(\Omega)}\|w\|^{\frac{1}{2s_L-\sigma}}_{H^{2s_L}(\Omega)}\\ \leq{}& C(K_1,K_2)\left[\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{1}{2s_L-\sigma}}+\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2}{2s_L-\sigma}}\right]\\ \leq{}& C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2}{2s_L-\sigma}}\leq C(K_1,K_2)\frac{1}{\lambda^2 s^{1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\end{aligned}$$

because $\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\gg 1$ in the trapped regime from (*4.4.52*). For the nonlinear term, using, using (*4.D.4*), interpolation and then the bootstrap bound (*4.4.28*):

$$\begin{aligned}&\|(1-\chi_3)w^p\|_{H^\sigma}\leq C\|w^p\|_{H^\sigma(\Omega)}\leq C\|w\|_{H^\sigma(\Omega)}\|w\|^{p-1}_{H^{\frac{d}{2}+\sigma-s_c}(\Omega)}\\ \leq{}& C(K_1)\|w\|^{(p-1)\frac{\frac{d}{2}+\sigma-s_c-\sigma}{2s_L-\sigma}}_{H^{2s_L}(\Omega)}\leq C(K_1)\|w\|^{\frac{2}{2s_L-\sigma}}_{H^{2s_L}(\Omega)}\leq\frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}\lambda^2}\end{aligned}$$

Injecting the two above estimates in (*4.5.36*) yields the desired identity (*4.5.35*).

□

4.5.3 Lyapunov monotonicity for high regularity norms of the remainder

We derive Lyapunov type monotonicity formulas for the high regularity norms of the remainder inside and outside the blow-up zone, $\mathcal{E}_{2s_L}$ and $\|w_{\text{ext}}\|_{H^{2s_L}}$, in Propositions 4.5.6 and 4.5.8. In our general strategy, we have to find a way to say that w is of smaller order compared to the excitation $\chi\tau_z(\tilde{\alpha}_{b,\frac{1}{\lambda}})$ and does not affect the blow up dynamics induced by this latter. This is why we study the quantity $\mathcal{E}_{2s_L}$: it controls the usual Sobolev norm H^{2s_L} and any local norm of lower order derivative which is useful for estimates, and is adapted to the linear dynamics as it undergoes dissipation. Finally, for this norm one sees that the error $\tilde{\psi}_b$ is of smaller order compared to the main dynamics of $\chi\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})$ (this is the $\eta(1-\delta_0')$ gain in (*4.3.33*)).

Proposition 4.5.6 (Lyapunov monotonicity at high regularity inside the blow up area). *Let all the constants of Proposition 4.4.6 be fixed, except s_0 and η. Then there exists a constant $\delta>0$, such that for any constant $N\gg 1$, for s_0 large enough and η small enough, for any solution u that is trapped on $[s_0,s')$ there holds for $0\leq t<t(s')$:*

$$\begin{array}{rl} & \frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}+O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}})\right)\right\} \\ \leq & \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\left[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')}}+\frac{C(L,M)}{N^{2\delta}}\mathcal{E}_{2s_L}\right. \\ & \left.+\mathcal{E}_{2s_L}\sum_2^p\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O\left(\frac{1}{L}\right)}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}+\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}\right], \end{array} \quad (4.5.37)$$

where $O_{L,M}(f)$ denotes a function depending on time such that $|O_{L,M}(f)(t)|\leq C(L,M)f$ for a constant $C(L,M)>0$, and where $\mathcal{E}_\sigma$ and $\mathcal{E}_{2s_L}$ are defined in (4.4.9) and (4.4.7).

Remark 4.5.7. (*4.5.37*) has to be understood the following way. The $O()$ in the time derivative is a corrective term coming from the refinement of the last modulation equations, see (*4.4.44*) and (*4.5.2*), it is of smaller order for our purpose so one can "forget" it. In the right hand side of (*4.5.37*), the first two terms come from the error $\tilde{\psi}_b$ made in the approximate dynamics. The third one results from the competition of the dissipative linear dynamics and the lower order linear terms that are of smaller order (the motion of the potential in the operator $H_{z,\frac{1}{\lambda}}$ involved in $\mathcal{E}_{2s_L}$, and the difference between the potentials $\tau_z(\tilde{Q}_{b,\frac{1}{\lambda}})^{p-1}$ and $\tau_z(Q_{\frac{1}{\lambda}})^{p-1}$). The penultimate represents the effect of the main nonlinear term, and shows that one needs $\mathcal{E}_\sigma$ smaller than $s^{s_c-\sigma}$ to control the energy transfer from low to high frequencies. The last one results from the cut of w at the border of the blow up zone.

Proof of Proposition 4.5.6

From (*4.4.41*) one has the identity:

$$\begin{array}{rl} & \frac{d}{dt}\left(\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s_L-s_c)}}\right)=\frac{d}{dt}\left(\int|H_{z,\frac{1}{\lambda}}^{s_L}w_{\text{int}}|^2\right) \\ = & -2\int H_{z,\frac{1}{\lambda}}^{s_L}w_{\text{int}}H_{z,\frac{1}{\lambda}}^{s_L+1}w_{\text{int}}+\int H_{z,\frac{1}{\lambda}}^{s_L}w_{\text{int}}H_{z,\frac{1}{\lambda}}^{s_L}(\frac{1}{\lambda^2}\chi\tau_z(-\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}})) \\ & +2\int H_{z,\frac{1}{\lambda}}^{s_L}w_{\text{int}}\left[H_{z,\frac{1}{\lambda}}^{s_L}[\frac{1}{\lambda^2}\chi\tau_z(-\tilde{\psi}_{b\frac{1}{\lambda}})+\text{NL}(w_{\text{int}})+L(w_{\text{int}})]+\frac{d}{dt}(H_{z,\frac{1}{\lambda}}^{s_L})w_{\text{int}}\right] \\ & +2\int H_{z,\frac{1}{\lambda}}^{s_L}w_{\text{int}}H_{z,\frac{1}{\lambda}}(\tilde{L}+\tilde{NL}+\tilde{R}). \end{array} \quad (4.5.38)$$

The proof is organized as follows. For the terms appearing in this identity: for some (those on the second line) we find direct upper bounds (step 1), then we integrate by part in time some modulation terms that are problematic to treat the second term in the right hand side (step 2), and eventually we prove that the terms created by the cut of the solitary wave (the last line) are harmless and use some dissipation property at the linear level (produced by the first term in the right hand side) to improve the result (step 3). Throughout the proof, the estimates are performed on $\mathbb{R}^d$ as w_{int} has compact support inside Ω, and we omit it in the notations.

step 1 Brute force upper bounds. We claim that the non linear term, the error term, the small linear term and the term involving the time derivative of the linearized operator in (*4.5.38*) can be directly upper bounded, yielding:

$$\begin{array}{ll} & \| H^{s_L}_{z,\frac{1}{\lambda}}[\text{NL}(w_{\text{int}})-\frac{1}{\lambda^2}\chi\tau_z(\tilde{\psi}_{b,\frac{1}{\lambda}})+L(w_{\text{int}})]+\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\text{int}}\|_{L^2} \\ \leq & \frac{1}{\lambda^{(2s_L-s_c)+2}s}\Big[\sqrt{\mathcal{E}_{2s_L}}\sum_2^p\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c+L-1}{L}\right)}}+\frac{C(L)}{s^{L+1-\delta_0+\eta(1-\delta_0)'}} \\ & +C(L,M)\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}\Big] \end{array} \qquad (4.5.39)$$

for some constant $\delta>0$. We now analyse these four terms separately.

- *The error term.* We decompose between the main terms and the terms created by the cut. The cut induced by $\tilde{\chi}:=\chi(\lambda y+z)$ only sees the terms $b_1^{(0,1)}\Lambda Q+b_1^{(1,\cdot)}.\nabla Q$ because all the other terms in the expression (*4.3.36*) of $\tilde{\psi}_b$ have support inside $\mathcal{B}^d(2B_1)$, and that $|z|\ll 1$ (*4.4.51*) and $B_1\ll\frac{1}{\lambda}$ from (*4.4.52*). For the main term we use the estimate (*4.3.33*) and for the second the bound on the parameters (*4.4.27*) and the asymptotics (*4.2.2*) and (*3.2.1*) of ΛQ and ∂Q:

$$\begin{array}{ll} & \|H^{s_L}_{z,\frac{1}{\lambda}}(\frac{1}{\lambda^2}\chi\tau_z\tilde{\psi}_{b,\frac{1}{\lambda}})\|_{L^2} \\ \leq & C\|H^{s_L}_{z,\frac{1}{\lambda}}(\frac{1}{\lambda^2}\tau_z\tilde{\psi}_{b,\frac{1}{\lambda}})\|_{L^2}+C\|H^{s_L}_{z,\frac{1}{\lambda}}(\frac{1}{\lambda^2}(1-\chi)\tau_z\tilde{\psi}_{b,\frac{1}{\lambda}})\|_{L^2}. \\ \leq & \frac{\|H^{s_L}\tilde{\psi}_b\|_{L^2}}{\lambda^{2s_L-s_c}}+\frac{1}{\lambda^{2(2s_L-s_c)+4}}\int|H^{s_L}[(1-\tilde{\chi})(b_1^{(0,1)}\Lambda Q+b_1^{(1,\cdot)}.\nabla Q]|^2 \\ \leq & \frac{C(L)}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')}}+\frac{C\lambda^{2(\alpha-1)}}{s}+\frac{C}{s^{\frac{\alpha+1}{2}}}\leq\frac{C(L)}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')}} \end{array} \qquad (4.5.40)$$

since $\alpha>1$, hence $\frac{\lambda^{2(\alpha-1)}}{s}+\frac{1}{s^{\frac{\alpha+1}{2}}}\ll 1$, and since $\frac{1}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')}}\gg 1$ in the trapped regime from (*4.4.52*).

- *The non linear term:* We begin by coming back to renormalized variables:

$$\begin{array}{lll} \|H^{s_L}_{z,\frac{1}{\lambda}}(\text{NL}(w_{\text{int}}))\|_{L^2} & \leq & \frac{\|H^{s_L}(\text{NL}(\varepsilon))\|_{L^2}}{\lambda^{(2s_L-s_c)+2}} \\ & \leq & C\sum_{k=2}^p\frac{\|H^{s_L}(\tilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}}{\lambda^{(2s_L-s_c)+2}} \end{array} \qquad (4.5.41)$$

because $\text{NL}(\varepsilon)=\sum_{k=2}^pC_k^p\tilde{Q}_b^{p-k}\varepsilon^k$. We fix k with $2\leq k\leq p$ and study the corresponding term in the above sul. One has $H=-\Delta-pQ^{p-1}$, and Q is a smooth profile satisfying the estimate $Q=O((1+|y|)^{-\frac{2}{p-1}})$ which propagates to its derivatives from (*3.2.1*). Similarly, from (*4.4.27*) and (*4.3.29*) one has: $\tilde{Q}_b=O((1+|y|)^{-\frac{2}{p-1}})$ and it propagates for the derivatives. The Leibniz rule for derivation then yields:

$$\begin{array}{rcl} \|H^{s_L}(\tilde{Q}_b^{p-k}\varepsilon^k)\|_{L^2}^2 & \leq & C(L) \sum_{\mu\in\mathbb{N}^d,\ 0\leq|\mu|\leq 2s_L} \int \frac{|\partial^\mu(\varepsilon^k)|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu|}} \\ & \leq & C(L) \sum_{(\mu_i)_{1\leq i\leq k}\in\mathbb{N}^{kd},\ \sum_i^k|\mu_i|\leq 2s_L} \int \frac{\prod_1^k |\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_1^k|\mu_i|}}. \end{array} \tag{4.5.42}$$

We fix $\mu_i \in \mathbb{N}^{kd}$ with $\sum|\mu_i|_1 \leq 2s_L$ and focus on the corresponding term in the above equation. Without loss of generality we order by increasing length: $|\mu_1| \leq ... \leq |\mu_k|$. We now distinguish between two cases.

Case 1: if $|\mu_k| + \frac{2(p-k)}{p-1} + 2s_L - \sum_1^k|\mu_i| \leq 2s_L$. As one has $|\mu_k|_1 + \frac{(p-k)}{p-1} + 2s_L - \sum_1^k|\mu_i|_1 \geq \sigma$ because the $|\mu_i|_1$'s are increasing and $\sum|\mu_i|_1 \leq 2s_L$, using (*4.D.1*):

$$\int \frac{|\partial^{\mu_k}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_1^k|\mu_i|_1}} \leq C(M)\mathcal{E}_\sigma^{\frac{\sum|\mu_i|-|\mu_k|_1-\frac{2(p-k)}{p-1}}{2s_L-\sigma}} \mathcal{E}_{2s_L}^{\frac{2s_L-\sigma-\sum|\mu_i|+|\mu_k|_1+\frac{2(p-k)}{p-1}}{2s_L-\sigma}}.$$

As the coefficients are in increasing order and L is arbitrarily very large, for $1 \leq j < k$ there holds $|\mu_i| + \frac{d}{2} \leq 2s_L$. We then recall the L^∞ estimate (*4.D.3*):

$$\|\partial^{\mu_i}\varepsilon\|_{L^\infty} \leq \sqrt{\mathcal{E}_\sigma}^{\frac{2s_L-|\mu_i|_1-\frac{d}{2}}{2s_L-\sigma}+O\left(\frac{1}{L}^2\right)} \sqrt{\mathcal{E}_{2s_L}}^{\frac{|\mu_i|_1+\frac{d}{2}-\sigma}{2s_L-\sigma}+O\left(\frac{1}{L^2}\right)}.$$

The two previous estimates imply that:

$$\begin{array}{rcl} \int \frac{\prod_1^k|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_1^k|\mu_i|_1}} & \leq & \int \frac{|\partial^{\mu_k}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_1^k|\mu_i|_1}} \prod_1^{k-1}\|\partial^{\mu_i}\varepsilon\|_{L^\infty}^2 \\ \leq & & \mathcal{E}_\sigma^{\frac{2(k-1)s_L-(k-1)\frac{d}{2}-2\frac{p-k}{p-1}}{2s_L-\sigma}+O\left(\frac{1}{L^2}\right)} \mathcal{E}_{2s_L}^{\frac{(k-1)\frac{d}{2}-k\sigma+2s_L+2\frac{p-k}{p-1}}{2s_L-\sigma}+O\left(\frac{1}{L^2}\right)} \\ \leq & & \mathcal{E}_\sigma^{k-1+\frac{-2+(k-1)(\sigma-s_c)}{2s_L-\sigma}+O\left(\frac{1}{L^2}\right)} \mathcal{E}_{2s_L}^{1+\frac{2-(k-1)(\sigma-s_c)}{2s_L-\sigma}+O\left(\frac{1}{L^2}\right)} \\ \leq & & \mathcal{E}_{2s_L}\left(\frac{\mathcal{E}_\sigma^{1+O\left(\frac{1}{L}\right)}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1} \frac{C(L,M,K_1,K_2)}{s^{1+\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c+L^{-1}}{L}\right)}}. \end{array} \tag{4.5.43}$$

Case 2: if $|\mu_k| + \frac{2(p-k)}{p-1} + 2s_L - \sum_1^k|\mu_i| > 2s_L$. This means $\frac{2(p-k)}{p-1} - \sum_1^{k-1}|\mu_i| > 0$. Hence, there are two subcases: the subcase $|\mu_i| = 0$ for $1 \leq i \leq k-1$ and the subcase $|\mu_{k-1}| = 1$ (because the μ_i's are ordered by increasing size $|\mu_i|$). If $|\mu_i| = 0$ for $1 \leq i \leq k-1$, then, using the weighted L^∞ estimate (*4.D.2*), the coercivity estimate (*4.C.16*) and the bound (*4.4.25*) we obtain:

$$\begin{array}{rcl} \int \frac{\prod_1^k|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_1^k|\mu_i|}} & = & \int \frac{|\varepsilon|^{2(k-1)}}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu_k|}} \\ \leq \|\frac{\varepsilon}{1+|y|^{\frac{2(p-k)}{p-1}}}\|_{L^\infty}^2\|\varepsilon\|_{L^\infty}^{2(k-2)}\mathcal{E}_{s_L} & \leq & \left(\frac{\mathcal{E}_\sigma^{1+O\left(\frac{1}{L}\right)}}{s^{-(\sigma-s_c)}}\right)^{(k-1)} \frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c+L^{-1}}{L}\right)}}. \end{array}$$

If $|\mu_{k-1}| = 1$, then, using the weighted L^∞ estimate (*4.D.2*) for $\nabla\varepsilon$, the coercivity estimate (*4.C.16*) and the bound (*4.4.25*) we obtain:

$$\begin{array}{rcl} \int \frac{\prod_1^k|\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_1^k|\mu_i|}} & = & \int \frac{|\partial^{\mu_{k-1}}\varepsilon|^2|\varepsilon|^{2(k-2)}}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2|\mu_k|}-2} \\ \leq \|\frac{\partial^{\mu_{k-1}}\varepsilon}{1+|y|^{\frac{2(p-k)}{p-1}}-1}\|_{L^\infty}^2\|\varepsilon\|_{L^\infty}^{2(k-2)}\mathcal{E}_{s_L} & \leq & \left(\frac{\mathcal{E}_\sigma^{1+O\left(\frac{1}{L}\right)}}{s^{-(\sigma-s_c)}}\right)^{(k-1)} \frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c+L^{-1}}{L}\right)}}. \end{array}$$

In both subcases there holds:

$$\int \frac{\prod_1^k |\partial^{\mu_i}\varepsilon|^2}{1+|y|^{\frac{4(p-k)}{p-1}+4s_L-2\sum_1^k|\mu_i|}} \leq \left(\frac{\mathcal{E}_\sigma^{1+O\left(\frac{1}{L}\right)}}{s^{-(\sigma-s_c)}}\right)^{(k-1)} \frac{C(L,M,K_1,K_2)\mathcal{E}_{s_L}}{s^{1+\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c+L-1}{L}\right)}}. \tag{4.5.44}$$

Now we come back to (*4.5.41*), which we reformulated in (*4.5.42*) where we estimated the terms appearing in the sum in (*4.5.43*) and (*4.5.44*), obtaining the following bound for the nonlinear term's contribution in (*4.5.38*):

$$\|H^{s_L}_{z,\frac{1}{\lambda}}(\mathrm{NL}(w_{\text{int}}))\|_{L^2} \leq \frac{\sqrt{\mathcal{E}_{2s_L}}}{\lambda^{(2s_L-s_c)+2}}\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O\left(\frac{1}{L}\right)}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{1+\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c+L-1}{L}\right)}}. \tag{4.5.45}$$

- ***The small linear term and the term involving the time derivative of the linearized operator:*** we claim that there exists a constant $\delta := \delta(d,L,p) > 0$ such that:

$$\|H^{s_L}_{z,\frac{1}{\lambda}}(L(w_{\text{int}}))+\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\text{int}}\|_{L^2} \leq \frac{C(L,M)}{\lambda^{2s_L-s_c+2}s}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}. \tag{4.5.46}$$

We now prove this estimate. The small linear term is in renormalized variables from (*4.4.37*):

$$\int |H^{s_L}_{z,\frac{1}{\lambda}}(L(w_{\text{int}}))|^2 = \frac{p^2}{\lambda^{2(2s_L-s_c)+4}}\int (H^{s_L}((Q^{p-1}-\tilde{Q}_b^{p-1})\varepsilon))^2.$$

For $\mu\in\mathbb{N}^s$, one has the following asymptotic behavior for the potential that appeared, from the bounds on the parameters (*4.4.27*) and the expression of $\tilde{Q}_b$ (*4.3.29*):

$$|\partial^\mu(Q^{p-1}-\tilde{Q}_b^{p-1})| \leq \frac{1}{s}\frac{C(\mu)}{1+|y|^{\alpha-C(L)\eta+|\mu|}} \leq \frac{1}{s}\frac{C(\mu)}{1+|y|^{\delta+|\mu|}}$$

for η small enough, because $\alpha>2$, and for some constant δ that can be chosen small enough so that:

$$0<\delta\ll 1, \quad \text{with } \delta < \sup_{0\leq n\leq n_0}\delta_n \quad \text{and} \quad \delta < \frac{d}{4}-\frac{\gamma_{n_0+1}}{2}-s_L \tag{4.5.47}$$

(this technical condition is useful to apply a coercivity estimate for the next equation, all the terms appearing are indeed strictly positive from (*4.1.8*)). We recall that $H=-\Delta-pQ^{p-1}$ where Q is a smooth potential satisfying $|\partial^\mu Q|\leq\frac{C(\mu)}{1+|y|^{\frac{2}{p-1}+|\mu|}}$. Using the Leibniz rule this implies:

$$\begin{array}{rcl} \int (H^{s_L}((Q^{p-1}-\tilde{Q}_b^{p-1})\varepsilon))^2 & \leq & \frac{C(L)}{s^2}\sum_{\mu_i\in\mathbb{N}^d,\ |\mu_i|\leq 2s_L,\ i=1,2}\int\frac{|\partial^{\mu_1}\varepsilon||\partial^{\mu_2}\varepsilon|}{1+|y|^{4s_L+2\delta-2|\mu_1|-2|\mu_2|}} \\ & \leq & \frac{C(L)}{s^2}\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}} \end{array} \tag{4.5.48}$$

where we used for the last line the weighted coercivity estimate (*4.C.16*), which we could apply because δ satisfies the technical condition (*4.5.47*). We now turn to the term involving the time derivative of the linearized operator in (*4.5.38*). Going back to renormalized variables it can be written as:

$$\int |\frac{d}{dt}H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}|^2 = \frac{p^2(p-1)^2}{\lambda^{2(2s_L-s_c)+4}}\sum_{i=1}^{s_L}\int (H^{i-1}[(Q^{p-2}\frac{z_s}{\lambda}.\nabla Q+\frac{\lambda_s}{\lambda}Q^{p-2}\Lambda Q)H^{s_L-i}\varepsilon])^2.$$

For $\mu\in\mathbb{N}^d$, one has the following asymptotic behavior for the two potentials that appeared (from the asymptotic (*3.2.1*) and (*4.2.2*) of Q and ΛQ):

$$|\partial^{\mu}(Q^{p-2}\partial_{y_i}Q)|\leq \frac{C(\mu)}{1+|y|^{2+1+|\mu|}} \text{ for } 1\leq i\leq d, \text{ and } |\partial^{\mu}(Q^{p-2}\Lambda Q)|\leq \frac{C(\mu)}{1+|y|^{2+\alpha}}.$$

Therefore, as $H=-\Delta-pQ^{p-1}$ where Q is a smooth potential satisfying $|\partial^{\mu}Q|\leq \frac{C(\mu)}{1+|y|^{\frac{2}{p-1}+|\mu|_1}}$, using the Leibniz rule and the two above identities:

$$\begin{array}{rl} & \left|\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\text{int}}\right| \\ \leq & \frac{C(L)(|\frac{\lambda_s}{\lambda}|^2+|\frac{z_s}{\lambda}|^2)}{\lambda^{2(2s_L-s_c)+4}}\sum_{\mu_i\in\mathbb{N}^d,\ |\mu_i|_1\leq 2s_L,\ i=1,2}\int\frac{|\partial^{\mu_1}\varepsilon||\partial^{\mu_2}\varepsilon|}{1+|y|^{4s_L+2-2|\mu_1|-2|\mu_2|}}. \\ \leq & \frac{C(L)}{\lambda^{2(2s_L-s_c)+4}s^2}\sum_{\mu_i\in\mathbb{N}^d,\ |\mu_i|_1\leq 2s_L,\ i=1,2}\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}} \end{array} \quad (4.5.49)$$

for $\delta<\alpha,1$ being defined by (*4.5.47*), where we used the weighted coercivity estimate (*4.C.16*) and the fact that $|\frac{\lambda_s}{\lambda}|\sim s^{-1}$ and $|\frac{z_s}{\lambda}|\sim s^{-1-\frac{\alpha-1}{2}}$ from (*4.4.43*) and (*4.4.27*). We now combine the estimates we have proved, (*4.5.48*) and (*4.5.49*), to obtain the estimate (*4.5.46*) we claimed.

- *End of the proof of Step 1:* we now gather the brute force upper bounds we have found for the terms we had to treat in (*4.5.40*), (*4.5.45*) and (*4.5.46*), yielding the bound (*4.5.39*) we claimed in this first step.

step 2 Integration by part in time to treat the modulation term. We now focus on the modulation term in (*4.5.38*) which requires a careful treatment. Indeed, the brute force upper bounds on the modulation (*4.4.43*) are not sufficient and we need to make an integration by part in time to treat the problematic term $b^{(n,k)}_{L_n,s}$. We do this in two times. First we define a radiation term. Next we use it to prove a modified energy estimate.

- *Definition of the radiation.* We recall that $\alpha_b=\sum_{(n,k,i)\in\mathcal{I}}b_i^{(n,k)}T_i^{(n,k)}+\sum_2^{L+2}S_i$, where $T_i^{(n,k)}$ is defined by (*4.2.21*) and S_i is homogeneous of degree $(i,-\gamma-g')$ in the sense of Definition 4.2.12, see (*4.3.8*). We want to split α_b in two parts to distinguish the problematic terms involving the parameters $b^{(n,k)}_{L_n}$. For $i=2,...,L+2$, as S_i is homogeneous of degree $(i,-\gamma-g')$ it is a finite sum:

$$S_i=\sum_{J\in\mathfrak{J}(i)}b^Jf_J, \quad \text{with } b^J=\prod_{(n,k,i)\in\mathcal{I}}(b_i^{(n,k)})^{J_i^{(n,k)}} \quad (4.5.50)$$

where $\mathfrak{J}(i)$ is a finite subset of $\mathbb{N}^{\#\mathcal{I}}$ and for all $J\in\mathfrak{J}(i)$, $|J|_3=i$ and f_J is admissible of degree $(2|J|_2-\gamma-g')$ in the sense of Definition 4.2.9. We then define the following partition of $\mathfrak{J}(i)$:

$$\begin{array}{l} \mathfrak{J}_1(i):=\{J\in\mathfrak{J}(i),\ J_{L_n}^{(n,k)}=0 \text{ for all } 0\leq n\leq n_0,\ 1\leq k\leq k(n)\}, \\ \mathfrak{J}_2(i):=\{J\in\mathfrak{J}(i),\ |J|=2 \text{ and } \exists(n,k,L_n)\in\mathcal{I},\ J_{L_n}^{(n,k)}\geq 1\}, \\ \mathfrak{J}_3(i):=\mathfrak{J}(i)\backslash[\mathfrak{J}_1(i)\cup\mathfrak{J}_2(i)], \\ \bar{S}_i:=\sum_{J\in\mathfrak{J}_2(i)}b^Jf_J,\ \ \bar{S}_i':=\sum_{J\in\mathfrak{J}_3(i)}b^Jf_J, \end{array} \quad (4.5.51)$$

and the following radiation term:

$$\begin{array}{rcl} \xi & := & H^{s_L}\left(\chi_{B_1}\left[\sum_{0\leq n\leq n_0,\ 1\leq k\leq k(n)}b_{L_n}^{(n,k)}T_{L_n}^{(n,k)}+\sum_{i=2}^{L+2}\bar{S}_i'\right]\right) \\ & & +\sum_{i=2}^{L+2}H^{s_L}\left(\chi_{B_1}\bar{S}_i\right)-\chi_{B_1}H^{s_L}\bar{S}_i. \end{array} \quad (4.5.52)$$

From (*4.5.51*), for all $J \in \mathcal{J}_3(i)$ there exists n with $0 \leq n \leq n_0$ such that $J_{L_n}^{(n,k)} \geq 1$ and $|J| \geq 3$. As $\delta_{n'} > 0$ this implies:

$$\forall J \in \mathcal{J}_3(i), \quad |J|_2 > L + 2 - \delta_0. \tag{4.5.53}$$

Using this fact, (*4.2.2*), the fact that $H^{s_L} T_{L_n}^{(n,k)} = 0$ since $s_L > L_n$ for all $0 \leq n \leq n_0$, (*4.5.51*) and (*4.4.27*) the radiation satisfies:

$$\|\xi\|_{L^2} \leq \frac{C(L,M)}{s^{L+1-\delta_0+\eta(1-\delta_0')}}, \quad \|H\xi\|_{L^2} \leq \frac{C(L,M)}{s^{L+2-\delta_0+\eta(2-\delta_0')}}, \tag{4.5.54}$$

$$\|\nabla\xi\|_{L^2} \leq \frac{C(L,M)}{s^{L+\frac{3}{2}-\delta_0+\eta(\frac{3}{2}-\delta_0')}}, \quad \|\Lambda\xi\|_{L^2} \leq \frac{C(L,M)}{s^{L+1-\delta_0+\eta(1-\delta_0')}}. \tag{4.5.55}$$

We eventually introduce the following remainders:

$$\begin{array}{rcl} R_1 & := & H^{s_L}\Big(\chi_{B_1} \sum\limits_{(n,k,i)\in\mathcal{I},\ i\neq L_n} (b_{i,s}^{(n,k)} + (2i-\alpha_n) b_i^{(n,k)} b_1^{(0,1)} - b_{i+1}^{(n,k)}) \\ & & (T_i^{(n,k)} + \sum_2^{L+2} \frac{\partial S_j}{\partial b_i^{(n,k)}})\Big) - (\frac{\lambda_s}{\lambda} + b_1^{(0,1)}) H^{s_L}\Lambda\tilde{Q}_b - (\frac{z_s}{\lambda} + b_1^{(1,\cdot)}).H^{s_L}\nabla\tilde{Q}_b \\ & & +H^{s_L}\Big(\chi_{B_1} \sum\limits_{(n,k,L_n)\in\mathcal{I}} (2L_n - \alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)} (T_{L_n}^{(n,k)} + \sum_2^{L+2} \frac{\partial \bar{S}_j'}{\partial b_{L_n}^{(n,k)}})) \Big) \\ & & + \sum\limits_{(n,k,L_n)\in\mathcal{I}} (2L_n - \alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)} \Big(\sum\limits_{j=2}^{L+2} H^{s_L}(\chi_{B_1} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}}) - \chi_{B_1} H^{s_L} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}} \Big) \end{array}$$

$$R_2 := \sum_{(n,k,L_n)\in\mathcal{I}} (b_{L_n,s}^{(n,k)} + (2L_n - \alpha_n) b_{L_n}^{(n,k)} b_1^{(0,1)}) \Big(\sum_2^{L+2} \chi_{B_1} H^{s_L} \frac{\partial \bar{S}_j}{\partial b_{L_n}^{(n,k)}} \Big),$$

$$R_3 := \sum_{(n,k,i)\in\mathcal{I},\ i\neq L_n} b_{i,s}^{(n,k)} \frac{\partial}{\partial_{b_i^{(n,k)}}} \xi,$$

so that they produce from (*4.5.52*) and (*4.4.33*) the identity:

$$H^{s_L}(\tilde{\text{Mod}}(s)) = \partial_s \xi + R_1 + R_2 + R_3. \tag{4.5.56}$$

The remainder R_1 enjoys the following bounds from (*4.4.43*), (*4.2.17*), (*4.3.8*), (*4.5.51*), (*4.5.53*) and (*4.4.27*):

$$\|R_1\|_{L^2} \leq \frac{C(L,M)}{s^{L+2-\delta_0+(1-\delta_0')\eta}} + \frac{C(L,M)\mathcal{E}_{2s_L}}{s^2}. \tag{4.5.57}$$

From the definition (*4.5.51*) of $\mathcal{S}_j$ and the construction (*4.3.25*) of S_j one has:

$$\begin{array}{rcl} \sum_{j=2}^{L+2} H\bar{S}_j & = & -\sum_{(n,k,L_n)\in\mathcal{I}} b_1^{(0,1)} b_{L_n}^{(n,k)} \left(\Lambda T_{L_n}^{(n,k)} - (2L_n - \alpha_n) T_{L_n}^{(n,k)}\right) \\ & & -\sum_{(n,k,L_n)\in\mathcal{I}} b_{L_n}^{(n,k)} b_1^{(1,\cdot)}.\nabla\Lambda T_{L_n}^{(n,k)} \\ & & +p(p-1)Q^{p-2} \left(\sum\limits_{(n,k,L_n)\in\mathcal{I}} b_{L_n}^{(n,k)} T_{L_n}^{(n,k)} \right) \left(\sum\limits_{(n',k',i)\in\mathcal{I}} b_i^{(n',k')} T_i^{(n',k')} \right). \end{array}$$

As $H^{s_L} T_{L_n}^{(n,k)} = 0$ since $s_L > L_n$ for all $0 \leq n \leq n_0$, using the commutator identity (*4.2.19*), the asymptotic (*4.2.17*) of $T_i^{(n,k)}$, (*4.4.27*) and (*3.2.10*) (as $\alpha > 2$) one has:

$$\int (1+|y|^{4+2\delta}) \left(\chi_{B_1} H^{s_L+1} \sum_{j=2}^{L+2} \frac{\partial \bar{S}_j}{\partial_{b_{L_n}^{(n,k)}}} \right)^2 \leq \frac{C(L)}{s}$$

where δ is defined by (*4.5.47*) from what we deduce using (*4.4.44*):

$$\|(1+|y|)^{2+\delta}HR_2\|_{L^2}\leq \frac{C(L,M)}{s^{L+4}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s}. \tag{4.5.58}$$

Finally for the last remainder one has the estimate from (*4.5.52*), (*4.4.43*), (*4.4.27*), (*4.4.25*), (*4.2.17*) and (*4.5.51*) for s_0 large enough:

$$\|R_3\|_{L^2}\leq \frac{C(L,M)}{s^{L+2-\delta_0+\eta(1-\delta_0')}} \tag{4.5.59}$$

- *Modified energy estimate:* we claim that the following modified energy estimate (compared to (*4.5.38*)) holds:

$$\begin{array}{c} \frac{d}{dt}\left\{\int (H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}}))^2\right\} \\ \leq \ \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Big[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int \frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}} \\ +\mathcal{E}_{2s_L}\sum_{k=2}^p\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O\left(\frac{1}{L}\right)}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\Big]-2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L+1}_{z,\frac{1}{\lambda}}w_{\text{int}} \\ +2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L}_{z,\frac{1}{\lambda}}(\tilde{L}+\tilde{R}+\tilde{NL}), \end{array} \tag{4.5.60}$$

what we are going to prove now. From the time evolution (*4.5.56*), (*4.4.32*) of ξ and w and because the support of $\tau_z(\xi_{\frac{1}{\lambda}})$ is disjoint from the one of $\tilde{L}$, $\tilde{R}$, and $\tilde{NL}$ one gets the following expression for the left hand side of the previous equation (*4.5.60*):

$$\begin{array}{rl} & \frac{d}{dt}\left\{\int (H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}}))^2\right\} \\ = & -2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L+1}_{z,\frac{1}{\lambda}}w_{\text{int}}-\frac{2}{\lambda^{2s_L+2}}\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}\tau_z(R_{2,\frac{1}{\lambda}}) \\ & -\frac{2}{\lambda^{2s_L}}\int \tau_z(\xi_{\frac{1}{\lambda}})H^{s_L+1}_{z,\frac{1}{\lambda}}w_{\text{int}}+2\int\Big[H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}})\Big] \\ & \times\Big[H^{s_L}_{z,\frac{1}{\lambda}}(\text{NL}(w_{\text{int}})-\frac{1}{\lambda^2}\tau_z(\tilde{\psi}_{b,\frac{1}{\lambda}}+(\chi-1)\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}})+L(w_{\text{int}})) \\ & +\frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w_{\text{int}}-\frac{1}{\lambda^{2+2s_L}}\tau_z((R_1+R_3+\frac{\lambda_s}{\lambda}\Lambda\xi+2s_L\frac{\lambda_s}{\lambda}\xi-\frac{z_s}{\lambda}.\nabla\xi)_{\frac{1}{\lambda}})\Big] \\ & -\frac{2}{\lambda^{4s_L+2}}\int\tau_z(\xi_{\frac{1}{\lambda}})\tau_z(R_{2,\frac{1}{\lambda}})+2\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L}_{z,\frac{1}{\lambda}}(\tilde{L}+\tilde{NL}+\tilde{R}). \end{array} \tag{4.5.61}$$

We now analyse all the terms in this identity except the first one and the last one that we will study in the next step. Using the estimate (*4.5.58*) on the remainder R_2, going back in renormalized variables and using the coercivity (*4.C.16*) one gets for the second term in (*4.5.61*):

$$\begin{array}{rcl} \left|\frac{2}{\lambda^{2s_L+2}}\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}\tau_z(R_{2,\frac{1}{\lambda}})\right| & \leq & C\int\frac{|H^{s_L-1}\varepsilon|}{1+|y|^{2+\delta}}(1+|y|^{2+\delta})|HR_2| \\ & \leq & \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{\lambda^{2(2s_L-s_c)+2}s}\left(\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}+\frac{1}{s^{L+3}}\right). \end{array}$$

Going back to renormalized variables, integrating by parts and using the estimate (*4.5.54*) on $H\xi$ gives for the third term in (*4.5.61*):

$$\left|\frac{2}{\lambda^{2s_L}}\int\tau_z(\xi_{\frac{1}{\lambda}})H^{s_L+1}_{z,\frac{1}{\lambda}}w_{\text{int}}\right|\leq\frac{C(L,M)}{\lambda^{2(2s_L-s_c)+2}}\frac{\sqrt{\mathcal{E}_{2s_L}}}{s^{L+2-\delta_0+\eta(2-\delta_0')}}.$$

To upper bound the fourth and the fifth terms in (*4.5.61*), we go back to renormalized variables and use the bound (*4.5.39*) on the error, the nonlinear term, the small linear term and the term involving the time derivative of the linearized operator we derived in Step 1, together with the bounds (*4.5.54*) and (*4.5.55*) on ξ, $\Lambda\xi$, $\nabla\xi$ and the fact that $|\frac{\lambda_s}{\lambda}|\leq Cs^{-1}$ and $|\frac{z_s}{\lambda}|\leq Cs^{-1-\frac{\alpha-1}{2}}$ in the trapped regime, and the bound (*4.5.57*) and (*4.5.59*) on the remainders R_1 and R_3, yielding:

$$
\begin{array}{rl}
 & \Big| \int \Big[H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}} + \frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}})\Big] \Big[H^{s_L}_{z,\frac{1}{\lambda}}(\text{NL}(w_{\text{int}}) - \frac{1}{\lambda^2}\tau_z(\tilde{\psi}_{b,\frac{1}{\lambda}} + (\chi-1)\tilde{\text{Mod}}(t)_{\frac{1}{\lambda}}) \\
 & + L(w_{\text{int}})) + \frac{d}{dt}(H^{s_L}_{z,\frac{1}{\lambda}})w - \frac{1}{\lambda^{2+2s_L}}\tau_z((R_1+R_3+\frac{\lambda_s}{\lambda}\Lambda\xi + 2s_L\frac{\lambda_s}{\lambda}\xi - \frac{z_s}{\lambda}.\nabla\xi)_{\frac{1}{\lambda}})\Big] \\
 & - \frac{2}{\lambda^{4s_L+2}}\int \tau_z(\xi_{\frac{1}{\lambda}})\tau_z(R_{1,\frac{1}{\lambda}})\Big| \\
\leq & \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Big[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}} + \frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}} + C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int \frac{|H^{s_L}\varepsilon|^2}{1+|x|^{2\delta}}\right)^{\frac{1}{2}} \\
 & + \mathcal{E}_{2s_L}\sum_{k=2}^p \left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O\left(\frac{1}{L}\right)}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1} \frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}\Big].
\end{array}
$$

We finish the proof of the bound (*4.5.60*) by injecting in the identity (*4.5.61*) the three previous bounds we proved on the second, third, fourth and fifth terms.

step 3 Use of dissipation. We put an upper bound for the last terms in (*4.5.60*) and improve the energy estimate using the coercivity of the quantity $-\int H^{s_L+1}\varepsilon H^{s_L}\varepsilon$.

- ***The dissipation estimate:*** we recall that $H=-\Delta-pQ^{p-1}$, the potential $-pQ^{p-1}$ being below the Hardy potential, $pQ^{p-1}<\frac{(d-2)^2-4\delta(p)}{4|y|^2}$ for some constant $\delta(p)>0$ from (*3.2.5*). Hence, using the standard Hardy inequality one gets for the linear term:

$$
\begin{array}{rl}
 & -\int H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}} H_{z,\frac{1}{\lambda}} H^{s_L}_{z,\frac{1}{\lambda}} w_{\text{int}} = -\frac{1}{\lambda^{2(2s-L-s_c)+2}}\int H^{s_L}\varepsilon H H^{s_L}\varepsilon \\
= & \frac{1}{\lambda^{2(2s-L-s_c)+2}}\left(-\int |\nabla H^{s_L}\varepsilon|^2 + \int pQ^{p-1}|H^{s_L}\varepsilon|^2\right) \\
= & \frac{1}{\lambda^{2(2s-L-s_c)+2}}\left(\left[\frac{(d-2)^2-\frac{\delta(p)}{2}}{(d-2)^2} + \frac{\delta(p)}{2(d-2)^2}\right]\int |\nabla H^{s_L}\varepsilon|^2 + \int pQ^{p-1}|H^{s_L}\varepsilon|^2\right) \\
\leq & \frac{1}{\lambda^{2(2s-L-s_c)+2}}\Big(-\frac{(d-2)^2-\frac{\delta(p)}{2}}{4}\int \frac{|H^{s_L}\varepsilon|^2}{|y|^2} - \frac{\delta(p)}{2(d-2)^2}\int |\nabla H^{s_L}\varepsilon|^2 \\
 & + \frac{(d-2)^2-\delta(p)}{4}\int \frac{|H^{s_L}\varepsilon|^2}{|y|^2}\Big) \\
= & -\frac{\delta(p)}{8\lambda^{2(2s-L-s_c)+2}}\int \frac{|H^{s_L}\varepsilon|^2}{|y|^2} - \frac{\delta(p)}{2(d-2)^2\lambda^{2(2s-L-s_c)+2}}\int |\nabla H^{s_L}\varepsilon|^2.
\end{array}
\tag{4.5.62}
$$

- ***Bounds for the terms created by the cut.*** We study the last terms in (*4.5.60*). From its definition (*4.4.40*), and as $\lambda+|z|\ll 1$ from (*4.4.52*) and (*4.4.51*), the remainder $\tilde{R}$ is bounded by a constant independent of the others:

$$
\|H^{s_L}_{z,\frac{1}{\lambda}}\tilde{R}\|_{L^2}\leq C. \tag{4.5.63}
$$

For the non linear term, for any very small $\kappa>0$, from (*4.D.4*), (*4.4.39*) and (*4.4.28*):

$$
\begin{array}{rl}
 & \|H^{s_L}_{z,\frac{1}{\lambda}}\tilde{NL}\|_{L^2} \leq C\sum_{k=2}^p \|w^k\|_{H^{2s_L}} \leq C\|w\|_{H^{2s_L}}\sum_{k=2}^p \|w\|^{k-1}_{H^{\frac{d}{2}+\kappa}} \\
\leq & C\|w\|_{H^{2s_L}}\sum_{k=2}^p \|w\|_{H^\sigma}^{(k-1)(1-\frac{\frac{d}{2}+\kappa-\sigma}{2s_L-\sigma})}\|w\|_{H^{2s_L}}^{(k-1)(\frac{\frac{d}{2}+\kappa-\sigma}{2s_L-\sigma})} \\
\leq & C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{1+(p-1)\frac{\frac{d}{2}+\kappa-\sigma}{2s_L-\sigma}} \\
= & C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{1+(p-1)\frac{\frac{2}{p-1}-\sigma-s_c+\kappa}{2s_L-\sigma}} \\
\leq & C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{1+\frac{2}{2s_L-\sigma}} \\
= & \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}.
\end{array}
\tag{4.5.64}
$$

because $\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\gg 1$ from (*4.4.52*), if κ has been chosen small enough. For the extra linear term in (*4.5.60*), performing an integration by parts, using Young's inequality for any $\epsilon>0$, (*4.4.25*) and (*4.4.28*):

$$\begin{array}{ll} & \left|\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L}_{z,\frac{1}{\lambda}}\tilde{L}\right| \\ = & \left|\int H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}H^{s_L}_{z,\frac{1}{\lambda}}[-\Delta\chi_3 w-2\nabla\chi_3.\nabla w+p\tau_z Q^{p-1}_{\frac{1}{\lambda}}(\chi_1^{p-1}-\chi_3)w]\right| \\ \leq & C\|H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}\|_{L^2}\|w\|_{H^{2s_L}}+C\epsilon\|\nabla H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}\|^2_{L^2}+\frac{C}{\epsilon}\|w_{\text{int}}\|^2_{H^{2s_L}} \\ \leq & C\epsilon\|\nabla H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}\|^2_{L^2}+\frac{C(K_1,K_2,\epsilon)}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}} \\ \leq & \frac{C\epsilon}{\lambda^{2(2s-L-s_c)+2}}\int|\nabla H^{s_L}\varepsilon|^2+\frac{C(K_1,K_2,\epsilon)}{\lambda^{2(2s_L-s_c)+2}s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} \end{array} \quad (4.5.65)$$

because in the trapped regime $\lambda^2 s\sim s^{-\frac{\alpha}{2\ell-\alpha}}$ from (*4.4.52*).

- ***Conclusion*** we inject in the modified energy estimate (*4.5.60*) the bounds (*4.5.62*), (*4.5.63*), (*4.5.64*) and (*4.5.65*), yielding:

$$\begin{array}{ll} & \frac{d}{dt}\left\{\int(H^{s_L}_{z,\frac{1}{\lambda}}w_{\text{int}}+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}}))^2\right\} \\ \leq & \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Big[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}} \\ & +\mathcal{E}_{2s_L}\sum_2^p\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}-\frac{s\delta(p)}{8}\int\frac{|H^{s_L}\varepsilon|^2}{|y|^2}-\frac{s\delta(p)}{2(d-2)^2}\int|\nabla H^{s_L}\varepsilon|^2 \\ & +C\epsilon s\int|\nabla H^{s_L}\varepsilon|^2+\frac{C(K_1,K_2,M,L)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}\Big]. \end{array} \quad (4.5.66)$$

For any $N\gg 1$, using Young's inequality and splitting the weighted integrals in the zone $|y|\leq N^2$ and $|y|\geq N^2$ gives for ϵ small enough and s_0 large enough:

$$\begin{array}{ll} & C(L,M)\sqrt{\mathcal{E}_{2s_L}}\left(\int\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}\right)^{\frac{1}{2}}-\frac{s\delta(p)-sC\epsilon}{8}\int\frac{|H^{s_L}}{|y|^2} \\ \leq & \frac{C(L,M)\mathcal{E}_{2s_L}}{N^{2\delta}}+C(L,M)N^{2\delta}\int_{|y|\leq N^2}\frac{|H^{s_L}\varepsilon|^2}{1+|y|^{2\delta}}-\frac{s\delta(p)}{16}\int\frac{|H^{s_L}\varepsilon|^2}{|y|^2}\leq\frac{C(L,M)\mathcal{E}_{2s_L}}{N^{2\delta}} \end{array}$$

Finally, from the bound (*4.5.54*) on the size of ξ one has:

$$\begin{array}{ll} & \frac{d}{dt}\left\{\int(H^{s_L}_{z,\frac{1}{\lambda}}w+\frac{1}{\lambda^{2s_L}}\tau_z(\xi_{\frac{1}{\lambda}}))^2\right\} \\ = & \frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}\right\}+\frac{d}{dt}\left\{\int\frac{2}{\lambda^{2s_L}}H^{s_L}_{z,\frac{1}{\lambda}}w\tau_z(\xi_{\frac{1}{\lambda}})+\frac{1}{\lambda^{4s_L}}(\tau_z(\xi_{\frac{1}{\lambda}}))^2\right\} \\ = & \frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}\right\} \\ & +\frac{d}{dt}\left\{O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}})\right)\right\} \end{array}$$

where denotes $O_{L,M}(\cdot)$ the usual $O(\cdot)$ for a constant in the upper bound that depends only on L and M only. Plugging the two previous identities in the modified energy estimate (*4.5.66*) yields the bound (*4.5.37*) we claimed in this proposition.

□

Proposition 4.5.8 (Lyapunov monotonicity at high regularity Sobolev outside the blow up zone). *Let all the constants of Proposition 4.4.6 be fixed except* s_0. *Then for* s_0 *large enough, for any solution* u *that is* *trapped on* $[s_0, s')$ *there holds for* $0 \le t < t(s')$*:*

$$\begin{array}{rcl} \|w_{ext}\|^2_{H^{2s_L}} & \le & \|\partial_t^{s_L} w_{ext}(0)\|^2_{L^2} + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} dt' \\ & & + \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{ext}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}} dt' \\ & & + \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}. \end{array} \tag{4.5.67}$$

Proof.

From the time evolution (*4.4.34*) of w_{ext} we get that :

$$\partial_t^{k+1} w_{\text{ext}} = \Delta \partial_t^k w_{\text{ext}} + (1-\chi_3)\partial_t^k (w^p) + \Delta\chi_3 \partial_t^k w + 2\nabla\chi_3.\nabla\partial_t^k w. \tag{4.5.68}$$

We make an energy estimate for $\partial_t^{s_L} w_{\text{ext}}$ and propagate this bound via elliptic regularity by iterations, what is a standard in the study of parabolic problems. All computations, unless mentioned, are performed on Ω, and we forget about this in the notations to ease writing.

step 1 Estimate on the force terms. We first prove some estimates on the force terms in the right hand side of (*4.5.68*). From the decomposition (*4.4.10*) and the evolution (*4.4.32*) of w, in the exterior zone $\Omega\backslash\mathcal{B}^d(2)$, $\partial_t^k w$ can be written as:

$$\partial_t^k w = \sum_{j=0}^k \sum_{\mu=(\mu_i)_{1\le i\le 1+j(p-1)}\in\mathbb{N}^{dk(p-1)},\ \sum_{i=1}^{1+j(p-1)}|\mu_i|_1=2(k-j)} C(\mu) \prod_{i=1}^{1+j(p-1)} \partial^{\mu_i} w. \tag{4.5.69}$$

for some constants $C(\mu)$. Fix $k \le s_L$, an integer j, with $0 \le j \le k$ and a sequence of d-tuples $(\mu_i)_{1\le i\le 1+k(p-1)} \in \mathbb{N}^{dk(p-1)}$ satisfying $\sum_{i=1}^{1+j(p-1)}|\mu_i| = 2(k-j)$. One can assume that the d-tuples μ_i are order by decreasing length: $|\mu_1| \ge |\mu_2| \ge ....$

- *The case* $k = s_L$. We want to estimate the above term in the zone $\Omega\backslash\mathcal{B}^d(2)$.

Subcase 1: if $|\mu_1| \ge \sigma$. Using Hölder, Sobolev embedding (since in that case $\mu_i < 2s_L - \frac{d}{2}$ for $2 \le i \le 1 + j(p-1)$), interpolation and (*4.4.28*), for $\kappa > 0$ small enough:

$$\begin{array}{rl} & \|\prod_{i=1}^{1+j(p-1)} \partial^{\mu_i} w\|_{L^2} \le \|\partial^{\mu_1} w\|_{L^2} \prod_{i=2}^{1+j(p-1)} \|\partial^{\mu_i} w\|_{L^\infty} \\ \le & \|w\|_{H^{|\mu_1|}} \prod_{i=2}^{1+j(p-1)} \|w\|_{H^{\frac{d}{2}+\kappa+|\mu_i|}} \\ \le & C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{|\mu_1|-\sigma+\sum_{i=2}^{1+j(p-1)}|\mu_i|+\frac{d}{2}+\kappa-\sigma}{2s_L-\sigma}} \\ = & C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{1-\frac{(j(p-1)-1)(\sigma-s_c-\kappa)}{2s_L-\sigma}} \le \frac{C(K_1,K_2)}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}} \end{array} \tag{4.5.70}$$

as $\frac{1}{\lambda^{2s_L-s_c} s^{L+1-\delta_0+\eta(1-\delta_0')}} \gg 1$ from (*4.4.52*).

Subcase 2: if $|\mu_1| < \sigma$. Then $\mu_i < \sigma$ for all $1 \le i \le j(p-1)$ and $\partial^{\mu_i} w \in L^{p_i}$ with p_i given by $\frac{1}{p_i} = \frac{1}{2} - \frac{\sigma-|\mu_i|}{d}$ from Sobolev embedding. We define i_0 as the integer $2 \le i_0 \le 1 + j(p-1)$ such that $\sum_{i=1}^{i_0-1}\frac{1}{p_i} < \frac{1}{2}$ and $\sum_{i=1}^{i_0}\frac{1}{p_i} \ge \frac{1}{2}$. i_0 exists since $\frac{1}{p_1} < \frac{1}{2}$ and $\sum_{i=1}^{1+j(p-1)}\frac{1}{p_i} \gg \frac{1}{2}$. We define $\tilde{p}_{i_0} > 2$ by $\frac{1}{\tilde{p}_{i_0}} = \frac{1}{2} - \sum_{i=1}^{i_0-1}\frac{1}{p_i}$ and $\tilde{s} \ge \sigma$ as the regularity giving the Sobolev embedding $H^{\tilde{s}-|\mu_{i_0}|} \to L^{\tilde{p}_{i_0}}$:

$$\tilde{s} = \sum_{i=1}^{i_0}|\mu_i| + (i_0-1)\left(\frac{d}{2}-\sigma\right).$$

This implies that $\prod_{i=1}^{i_0}\partial^{\mu_i}w\in L^2$ with the estimate (from Hölder inequality):

$$\begin{array}{rcl}\|\prod_{i=1}^{i_0}\partial^{\mu_i}w\|_{L^2} & \leq & C\|\partial^{\mu_{i_0}}w\|_{L^{\tilde p_{i_0}}}\prod_{i=1}^{i_0-1}\|\partial^{\mu_i}w\|_{L^{p_i}}\leq \|w\|_{H^{\tilde s}}\prod_{i=1}^{i_0-1}\|w\|_{H^\sigma} \\ & \leq & C(K_1)\|w\|_{H^{2s_L}}^{\frac{\tilde s-\sigma}{2s_L-\sigma}}\end{array}$$

where we used interpolation and (*4.4.25*). Therefore, for $\kappa>0$ small enough, using Sobolev embedding, the above estimate, interpolation and (*4.4.25*):

$$\begin{array}{rcl}\|\prod_{i=1}^{1+j(p-1)}\partial^{\mu_i}w\|_{L^2} & \leq & \|\prod_{i=1}^{i_0}\partial^{\mu_i}w\|_{L^2}\prod_{i=i_0+1}^{1+j(p-1)}\|w\|_{H^{\frac d2+\kappa+|\mu_i|}} \\ \leq & & C(K_1)\|w\|_{H^{2s_L}}^{\frac{\tilde s-\sigma}{2s_L-\sigma}}\prod_{i=i_0+1}^{1+j(p-1)}\|w\|_{H^\sigma}^{1-\frac{\frac d2+\kappa+|\mu_i|-\sigma}{2s_L-\sigma}}\|w\|_{H^{2s_L}}^{\frac{\frac d2+\kappa+|\mu_i|-\sigma}{2s_L-\sigma}} \\ \leq & & C(K_1,K_2)\left(\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\right)^{\frac{2s_L-\sigma-j(p-1)(\sigma-s_c)+(j(p-1)-i_0+1)\kappa}{2s_L-\sigma}} \\ \leq & & C(K_1,K_2)\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\end{array}\qquad (4.5.71)$$

as $\frac{1}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}\gg 1$ from (*4.4.52*).

End of substep 1: injecting (*4.5.70*) and (*4.5.71*) in the identity we obtain:

$$\|\partial_t^{s_L}w\|_{L^2(\Omega\backslash\mathcal{B}^d(2))}\leq\frac{C(K_1,K_2)}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')}}.\qquad (4.5.72)$$

Estimate for the nonlinear term in (*4.5.68*). With the very same arguments used in the first substep one obtains the following bound:

$$\|\partial_t^{s_L}w^p\|_{L^2(\Omega\backslash\mathcal{B}^d(2))}\leq\frac{C(K_1,K_2)}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}.\qquad (4.5.73)$$

- *The case* $k<s_L$. Again, the verbatim same methods yields for $0\leq k<s_L$:

$$\|\partial_t^k w\|_{H^{2(s_L-1-k)}(\Omega\backslash\mathcal{B}^d(2))}\leq\frac{C(K_1,K_2)}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}+O\left(\frac1L\right)}}.\qquad (4.5.74)$$

$$\|\nabla\partial_t^k w\|_{H^{2(s_L-1-k)}(\Omega\backslash\mathcal{B}^d(2))}\leq\frac{C(K_1,K_2)}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2(2\ell-\alpha)}+O\left(\frac1L\right)}}.\qquad (4.5.75)$$

$$\|\partial_t^k w^p\|_{H^{2(s_L-1-k)}(\Omega\backslash\mathcal{B}^d(2))}\leq\frac{C(K_1,K_2)}{\lambda^{2s_L-s_c}s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}.\qquad (4.5.76)$$

step 2 Energy estimate for $\partial_t^{s_L}w_{\text{ext}}$. We claim that for $0\leq t<t'$:

$$\begin{array}{rcl}\|\partial_t^{s_L}w_{\text{ext}}\|_{L^2}^2 & \leq & \|\partial_t^{s_L}w_{\text{ext}}(0)\|_{L^2}^2+\int_0^t\frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2}s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}dt' \\ & & +\int_0^t\frac{C(K_1,K_2)\|\partial_t^{s_L}w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2}s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}dt'\end{array}\qquad (4.5.77)$$

and we now prove this estimate. From (*4.5.68*) one has the identity:

$$\begin{array}{rcl}\partial_t(\|\partial_t^{s_L}w_{\text{ext}}\|_{L^2}^2) & = & -2\int|\nabla\partial_t^{s_L}w_{\text{ext}}|^2+4\int\partial_t^{s_L}w_{\text{ext}}\nabla\chi_3.\nabla\partial_t^{s_L}w \\ & & +2\int\partial_t^{s_L}w_{\text{ext}}\partial_t^{s_L}((1-\chi_3)w^p+\Delta\chi_3 w)\end{array}\qquad (4.5.78)$$

and we are now going to study the right hand side of this equation.

- *Use of dissipation.* We study all the terms except the nonlinear one in (*4.5.78*). After an integration by parts, using Cauchy-Schwarz, Young's and Poincare's inequalities:

$$\begin{array}{rc} & |\int \partial_t^{s_L} w_{\text{ext}} \nabla\chi_3 . \nabla \partial_t^{s_L} w + \int \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L}(\Delta\chi_3 w)| \\ & = |-\int \Delta\chi_3 \partial_t^{s_L} w \partial_t^{s_L} w_{\text{ext}} - \nabla\chi_3 . \nabla \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L} w + \int \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L}(\Delta\chi_3 w)| \\ \leq & C[\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2} \|\partial_t^{s_L} w_{\text{ext}}\|_{L^2} + \|(1-\chi_2)\partial_t^{s_L} w\|_{L^2} \|\nabla \partial_t^{s_L} w_{\text{ext}}\|_{L^2}] \\ \leq & C(\epsilon)\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2} + \epsilon \|\nabla \partial_t^{s_L} w\|_{H^1}^2, \end{array}$$

for any $\epsilon > 0$. Adding the dissipation term in (*4.5.78*), taking ϵ small enough and using the bound (*4.5.72*) on the force term $\partial_t^{s_L} w$ gives:

$$\begin{array}{rl} & -\int |\nabla \partial_t^{s_L} w_{\text{ext}}|^2 + 4\int \nabla\chi_3 . \nabla \partial_t^{s_L} w \partial_t^{s_L} w_{\text{ext}} + \int \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L}(\Delta\chi_{B(0,3)} w) \\ \leq & C\|(1-\chi_2)\partial_t^{s_L} w\|_{L^2}^2 \leq C\|\partial_t^{s_L} w\|_{L^2}^2 \leq \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} \\ \leq & \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} \end{array} \quad (4.5.79)$$

because in the trapped regime, $\lambda^2 s \sim s^{-\frac{\alpha}{2\ell-\alpha}}$.

- *Estimate for the non linear term.* We now turn to the non linear term in (*4.5.78*), and use the estimate (*4.5.73*) for $\partial_t^{s_L} w^p$ we found in the first step, yielding:

$$\left| \int \partial_t^{s_L} w_{\text{ext}} \partial_t^{s_L}((1-\chi_3)w^p \right| \leq \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}. \quad (4.5.80)$$

- *End of Step 2*: we collect the estimates (*4.5.79*) and (*4.5.80*) found in the previous substeps, what gives the desired bound (*4.5.77*) we claimed in this Step.

step 3 Iteration of elliptic regularity. We claim that for $i = 0 ... s_L$:

$$\begin{array}{rl} \|\partial_t^i w_{\text{ext}}\|_{H^{2(s_L-i)}}^2 \leq & \|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2}^2 + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} dt' \\ & + \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}} dt' \\ & + \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}. \end{array} \quad (4.5.81)$$

We are going to show this estimate by induction. This is true for $i = s_L$ from the result (*4.5.77*) of the last step, and because of the compatibility conditions (*4.4.20*) at the border. Now suppose it is true for i, with $1 \leq i \leq s_L$. Then as $\partial_t^{i-1} w_{\text{ext}}$ solves (*4.5.68*), from elliptic regularity one gets (again because of the compatibility conditions (*4.4.20*) at the border), from the induction hypothesis and the bounds (*4.5.76*), (*4.5.76*) and (*4.5.76*) on the force terms:

$$\begin{array}{rl} & \|\partial_t^{i-1} w_{\text{ext}}\|_{H^{2(s_L-i)+2}}^2 \\ \leq & \|(1-\chi_{B(0,4)})\partial_t^{i-1}(w^p) + \Delta\chi_{B(0,4)} \partial_t^{i-1} w + 2\nabla\chi_{B(0,4)} . \nabla \partial_t^{i-1} w\|_{H^{2(s_L-i)}}^2 \\ & + \|\partial_t^i w_{\text{ext}}\|_{H^{2(s_L-i)}}^2 \\ \leq & \|\partial_t^{s_L} w_{\text{ext}}(0)\|_{L^2}^2 + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2} s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}} dt' \\ & + \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L} w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2} s^{L+2+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}} dt' \\ & + \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)} s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}} \end{array}$$

This shows that the inequality (*4.5.81*) is true for $i-1$. Hence, by iterations, the inequality (*4.5.81*) is true for $i = 0$, what gives the estimate (*4.5.67*) we had to prove.

□

4.5.4 End of the proof of Proposition 4.4.6

Proposition 4.4.6 states that, once the constants of involved in the analysis that are listed at its beginning are well chosen, given an initial data of (NLH) that is a perturbation of the approximate blow up profile along the stable directions of perturbation, there is a way to perturb it along the instable directions of perturbation to produce a solution that stays trapped for all time in the sense of Definition 4.4.4. The strategy of the proof is the following. We argue by contradiction and suppose that for all perturbations along the instable directions the corresponding solution will eventually escape from the trapped regime. First, we characterize the exit of the trapped regime through a condition on the size of the instable parameters, and then we show that arguing by contradiction would amount to go against Brouwer's fixed point theorem.

We fix $\lambda(s_0)$ satisfying (*4.4.21*), $w(s_0)$ decomposed in (*4.4.5*) satisfying (*4.4.19*) and (*4.4.11*), $V_1(s_0)$, $\left(U_{\ell+1}^{(0,1)}(s_0),...,U_L^{(0,1)}(s_0)\right)$ and $\left(U_i^{(n,k)}(s_0)\right)_{(n,k,i)\in\mathcal{I} \text{ with } 1\leq n,\ i_n\leq i}$ satisfying (*4.4.16*), (*4.4.17*) and (*4.4.18*). For any $(V_2(s_0),...V_\ell(s_0))$ and $(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\leq n,\ i<i_n}$ satisfying (*4.4.14*) and (*4.4.15*), let u denote the solution of (NLH) with initial datum $u(0)=\chi\tilde{Q}_{b(s_0),\frac{1}{\lambda(s_0)}}+w(s_0)$ with $b(s_0)$ given by (*4.4.31*). We define the renormalized exit time $s^*=s^*((V_2(s_0),...V_\ell(s_0)),(U_i^{(n,k)}(s_0))_{(n,k,i)\in\mathcal{I},1\leq n,\ i<i_n})$:

$$s^*:=\sup\{s\geq s_0,\ u \text{ is trapped in the sense of Definition } 4.4.4 \text{ on } [s_0,s)\} \tag{4.5.82}$$

From a continuity argument, one always have $s^*>s_0$.

Lemma 4.5.9 (Characterization of the exit of the trapped regime). *For L and M large enough and σ close enough to s_c, there exists a choice of the other constants in (4.4.30), except s_0 and η, such that for any s_0 large enough and η small enough, if $s^*<+\infty$, at least one of the following two scenarios hold:*

(i) Exit via instabilities on the first spherical harmonics:

$$V_i(s^*)=(s^*)^{-\tilde{\eta}} \ \textit{for some } 1\leq i\leq \ell.$$

(ii) Exit via instabilities on the other spherical harmonics:

$$U_i^{(n,k)}(s^*)=1 \ \textit{for some } (n,k,i)\in\mathcal{I},\ \textit{with } 1\leq n \textit{ and } i<i_n.$$

Proof of Lemma 4.5.9

A solution u is trapped if the parameters and the error involved in its decomposition (*4.4.10*) satisfy the bounds (*4.4.22*), (*4.4.23*), (*4.4.24*), (*4.4.25*) and (*4.4.52*). At time s^*, the bound (*4.4.52*) is strict at from (*4.4.51*) and (*4.4.52*), and we are going to prove that (*4.4.25*) is strict in step 1 and that (*4.4.24*) is strict in step 2. Thus, (*4.4.22*) or (*4.4.23*) must be violated at the time s^* and the Lemma is proved.

step 1 Improved bounds for the remainder w. We claim that:

$$\begin{aligned}&\mathcal{E}_\sigma(s^*)\leq \frac{K_1}{2(s^*)^{\frac{2(\sigma-s_c)\ell}{2\ell-\alpha}}}\text{ă},\quad \mathcal{E}_{2s_L}(s^*)\leq \frac{K_2}{2(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}},\\ &\|w_{\text{ext}}(s^*)\|_{H^\sigma}^2\leq \frac{K_1}{2} \ \text{ and } \ \|w_{\text{ext}}(s^*)\|_{H^{2s_L}}^2\leq \frac{K_2}{2\lambda^{2(2s_L-s_c)}s^{2L+2(1-\delta_0)+2\eta(1-\delta_0')}}\end{aligned} \tag{4.5.83}$$

and we now prove these estimates.

- ***Bound on*** $\mathcal{E}_\sigma$: Let K_1 and K_2 be any strictly positive real numbers. Then from Proposition 4.5.3 there holds for s_0 and η large enough:

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} \leq \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\frac{1}{s^{\frac{\alpha}{4L}}}\left[1+\sum_{k=2}^{p}\left(\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\right].$$

On $[s_0,s^*]$ one has $\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\leq K_1 s^{-\frac{\alpha(\sigma-s_c)}{4\ell-2\alpha}}$ from (*4.4.25*), hence for s_0 large enough:

$$\frac{d}{dt}\left\{\frac{\mathcal{E}_\sigma}{\lambda^{2(\sigma-s_c)}}\right\} \leq \frac{\sqrt{\mathcal{E}_\sigma}}{\lambda^{2(\sigma-s_c)+2}s^{\frac{(\sigma-s_c)\ell}{2\ell-\alpha}+1}}\frac{1}{s^{\frac{\alpha}{8L}}}.$$

One has $\lambda=\left(\frac{s_0}{s}\right)^{\frac{\ell}{2\ell-\alpha}}(1+O(s_0^{-\tilde\eta}))$ from (*4.4.52*) and we assume that $|O(s_0^{-\tilde\eta})|\leq\frac12$. We reintegrate the above equation using (*4.4.25*) and (*4.4.19*):

$$\mathcal{E}_\sigma(s^*)\leq\frac{1}{(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}\left(\left(\frac{3}{2}\right)^{2\sigma-s_c}+s_0^{\frac{2\ell(\sigma-s_c)}{2\ell-\alpha}}\frac{2^{2(\sigma-s_c)+3}L}{\alpha s_0^{\frac{\alpha}{8L}}}\sqrt{K_1}\right).$$

Therefore, once L is fixed we choose σ close enough to s_c so that $\frac{\alpha}{8L}>\frac{2\ell(\sigma-s_c)}{2\ell-\alpha}$ and then for s_0 large enough one has $s_0^{\frac{2\ell(\sigma-s_c)}{2\ell-\alpha}}\frac{2^{2(\sigma-s_c)+3}L}{\alpha s_0^{\frac{\alpha}{8L}}}\leq 1$. For any choice of the constants $K_1>10$ there then holds:

$$\mathcal{E}_\sigma(s^*)\leq\frac{1}{(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}\left(\left(\frac{3}{2}\right)^{2\sigma-s_c}+\sqrt{K_1}\right)\leq\frac{K_1}{2(s^*)^{\frac{2\ell(\sigma-s_c)}{2\ell-\sigma}}}. \tag{4.5.84}$$

- ***Bound on*** $\mathcal{E}_{2s_L}$: Let K_1 and K_2 be any strictly positive real numbers. From Proposition 4.5.6, for any $N\gg1$ there holds for s_0 and η large enough:

$$\begin{array}{ll} & \frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}+O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}})\right)\right\} \\ \leq & \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Big[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')}}+\frac{C(L,M)}{N^{2\delta}}\mathcal{E}_{2s_L} \\ & +\mathcal{E}_{2s_L}\sum_2^p\left(\frac{\sqrt{\mathcal{E}_\sigma}^{1+O\left(\frac{1}{L}\right)}}{s^{-\frac{\sigma-s_c}{2}}}\right)^{k-1}\frac{C(L,M,K_1,K_2)}{s^{\frac{\alpha}{L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}+\frac{C(L,M,K_1,K_2)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}\Big],\end{array}$$

In the trapped regime, from (*4.4.25*) one has: $\frac{\sqrt{\mathcal{E}_\sigma}}{s^{-\frac{\sigma-s_c}{2}}}\leq K_1 s^{-\frac{\alpha(\sigma-s_c)}{4\ell-2\alpha}}$. Consequently, for N and s_0 large enough the previous identity becomes:

$$\begin{array}{ll} & \frac{d}{dt}\left\{\frac{\mathcal{E}_{2s_L}}{\lambda^{2(2s-L-s_c)}}+O_{(L,M)}\left(\frac{1}{\lambda^{2(2s_L-s_c)}s^{L+1-\delta_0+\eta(1-\delta_0')}}(\sqrt{\mathcal{E}_{2s_L}}+\frac{1}{s^{L+1-\delta_0+\eta(1-\delta_0')}})\right)\right\} \\ \leq & \frac{1}{\lambda^{2(2s_L-s_c)+2}s}\Big[\frac{C(L,M)}{s^{2L+2-2\delta_0+2(1-\delta_0')}}+\frac{C(L,M)\sqrt{\mathcal{E}_{2s_L}}}{s^{L+1-\delta_0+\eta(1-\delta-0')}}+\frac{1}{N^{2\delta}}\mathcal{E}_{2s_L}\Big].\end{array}$$

As from (*4.4.52*), $\lambda=\left(\frac{s_0}{s}\right)^{\frac{\ell}{2\ell-\alpha}}(1+O(s_0^{-\tilde\eta}))$ one gets, when reintegrating in time the previous equation using the trapped regime bounds (*4.4.25*) and (*4.4.19*):

$$\begin{array}{lll}\mathcal{E}_{2s_L}(s^*) & \leq & \lambda(s^*)^{2(2s_L-s_c)}\Big[O_{(L,M)}\left(\frac{1}{\lambda(s^*)^{2(2s_L-s_c)}(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}(\sqrt{K_1}+1)\right) \\ & & +\mathcal{E}_{2s_L}(s_0)+O_{L,M}\left(\frac{1}{s_0^{L+1-\delta_0+\eta(1-\delta_0')}}(\sqrt{\mathcal{E}_{2s_L}(s_0)}+\frac{1}{s_0^{L+1-\delta_0+\eta(1-\delta_0')}})\right) \\ & & +\int_{s_0}^{s^*}\frac{1}{\lambda^{2(2s_L-s_c)}s^{2L+3-2\delta_0+\eta(1-\delta_0')}}\left(C(L,M)\sqrt{K_2}+C(L,M)+\frac{K_2}{N^{2\delta}}\right)\Big] \\ & \leq & \frac{1}{(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}[C(L,M)(1+\sqrt{K_2})+C(L)\frac{K_2}{N^{2\delta}}] \\ & \leq & \frac{1}{K_2(s^*)^{2L+2-2\delta_0+2\eta(1-\delta_0')}}\end{array} \tag{4.5.85}$$

if N and K_1 have been chosen large enough.

- ***Bound on*** $\|w_{ext}\|_{H^\sigma}$. We recall the estimate (*4.5.35*):

$$\frac{d}{dt}\left[\|w_{\text{ext}}\|_{H^\sigma}^2\right] \leq \frac{C(K_1,K_2)}{s^{1+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}\lambda^2}\|w_{\text{ext}}\|_{H^\sigma}.$$

For any choice of the constants of the analysis in Proposition 4.4.6 such that all the previous propositions and lemmas hold, then for s_0 large enough:

$$\frac{d}{dt}\left[\|w_{\text{ext}}\|_{H^\sigma}^2\right] \leq \frac{1}{s^{\frac{\alpha}{4L}}\lambda^2}\|w_{\text{ext}}\|_{H^\sigma}.$$

We reintegrate this equation in the bootstrap regime, by injecting the bounds (*4.4.25*) and (*4.4.19*) on $\|w_{\text{ext}}\|_{H^\sigma}$ (using the relation $\frac{ds}{dt}=\frac{1}{\lambda^2}$):

$$\|w_{\text{ext}}(s^*)\|_{H^\sigma} \leq \sqrt{K_2}\frac{C(L)}{s_0^{\frac{\alpha}{4L}}} + \frac{C}{s_0^{\frac{2\ell}{2\ell-\alpha}(2s_L-s_c)}} \leq \frac{K_2}{2} \tag{4.5.86}$$

For K_2 chosen large enough.

- ***Bound on*** $\|w_{ext}\|_{H^{2s_L}}$. We recall the estimate (*4.5.67*):

$$\begin{aligned}
\|w_{\text{ext}}\|_{H^{2s_L}}^2 \quad \leq \quad & \|\partial_t^{s_L}w_{\text{ext}}(0)\|_{L^2}^2 + \int_0^t \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)+2}s^{2L+3-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha}{2\ell-\alpha}}}dt' \\
& + \int_0^t \frac{C(K_1,K_2)\|\partial_t^{s_L}w_{\text{ext}}(t')\|_{L^2}}{\lambda^{2s_L-s_c+2}s^{L+2-\delta_0+\eta(1-\delta_0')+\frac{\alpha}{2L}+O\left(\frac{\eta+\sigma-s_c}{L}\right)}}dt' \\
& + \frac{C(K_1,K_2)}{\lambda^{2(2s_L-s_c)}s^{2L+2-2\delta_0+2\eta(1-\delta_0')+\frac{\alpha(p-1)(\sigma-s_c)}{2(2\ell-\alpha)}+O\left(\frac{\sigma-s_c+\eta}{L}\right)}}.
\end{aligned}$$

One has $w_{\text{ext}}=(1-\chi_3)w$, so $\partial_t^{s_L}w_{\text{ext}}=(1-\chi_3)\partial_t^{s_L}w$. We recall that we proved the bound (*4.5.72*) in the trapped regime for $\partial_t^{s_L}w(t)$ outside the blow up zone in the proof of Proposition 4.5.8. The same proof gives for s_0 large enough, taking in account the bound (*4.4.19*) on w at initial time:

$$\|\partial_t^{s_L}w_{\text{ext}}(0)\|_{L^2} \leq 1.$$

Injecting this estimate and (*4.5.72*) in the previous identity gives for s_0 large enough:

$$\begin{aligned}
\|w_{\text{ext}}\|_{H^{2s_L}}^2 \quad & \leq \quad 1 + \int_0^t \frac{dt'}{\lambda^{2(s_L-s_c)+2}s^{2L+3-2\delta_0+2\eta(1-\delta_0')}} + \frac{1}{\lambda^{2(2s_L-s_c)}s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} \\
& \leq \frac{2}{\lambda^{2(2s_L-s_c)}s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \int_0^t \frac{C dt'}{s^{-\frac{\ell[2(2s_L-s_c)+2]}{2\ell-\alpha}}s^{2L+3-2\delta_0+2\eta(1-\delta_0')}} \\
& \leq \frac{2}{\lambda^{2(2s_L-s_c)}s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \frac{C(L)}{s^{-\frac{\ell 2(2s_L-s_c)}{2\ell-\alpha}}s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} \\
& \leq \frac{2}{\lambda^{2(2s_L-s_c)}s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} + \frac{C(L)}{\lambda^{2(2s_L-s_c)}s^{2L+2-2\delta_0+2\eta(1-\delta_0')}} \\
& \leq \frac{K_2}{2\lambda^{2(2s_L-s_c)}s^{2L+2-2\delta_0+2\eta(1-\delta_0')}}
\end{aligned} \tag{4.5.87}$$

where we used the equivalence $\lambda \sim s^{-\frac{\ell}{2\ell-\alpha}}$ from (*4.4.52*), and where the last lines hold for K_2 large enough.

- *End of step 1:* we have proven (*4.5.84*), (*4.5.85*), (*4.5.86*) and (*4.5.87*), yielding the estimate we claimed (*4.5.83*).

step 2 Improved bounds for the stable parameters. We claim that once L, M, η, K_1 and K_2 have been chosen so that the result of step 1 hold, there exist $\tilde\eta>0$ and strictly positive constants $(\epsilon_i^{(0,1)})_{\ell+1\le i\le L}$, $(\epsilon_i^{(n,k)})_{(n,k,i)\in\mathfrak{I},\ 1\le n,\ i_n\le i}$ such that:

$$|V_1(s^*)|\le \frac{1}{2(s^*)^{-\tilde\eta}},\quad |U_i^{(0,1)}(s^*)|\le \frac{\epsilon_i^{(0,1)}}{2(s^*)^{\tilde\eta}}\ \text{ for } \ell+1\le i\le L, \tag{4.5.88}$$

$$\text{for } (n,k,i)\in\mathfrak{I},\ n\ge 1,\ \ |U_i^{(n,k)}(s^*)|\le \frac{\epsilon_i^{(n,k)}}{2(s^*)^{\tilde\eta}} \text{ if } i_n<i,\ |U_i^{(n,k)}(s^*)|\le \frac{\epsilon_i^{(n,k)}}{2} \text{ if } i_n=i. \tag{4.5.89}$$

We now prove all these improved bounds: first we prove the one for $b_{L_n}^{(n,k)}$, then the one for the $U_i^{(n,k)}$, $i\neq L_n$, and finally the one for V_1. For technical reasons, we introduce for $(n,k,i)\in\mathfrak{I}$ the function $g_i^{(n,k)}$ solution of the ODE:

$$\frac{\frac{d}{ds}g_i^{(n,k)}}{g_i^{(n,k)}}=(2i-\alpha_n)b_1^{(0,1)},\quad g(s_0)=s_0^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}}. \tag{4.5.90}$$

As $b_1^{(0,1)}=\frac{\ell}{s(2\ell-\alpha)}+O(s^{-1-\tilde\eta})$, for $\tilde\eta$ small enough and s_0 large enough one has:

$$g_i^{(n,k)}(s)=s^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}}(1+O(s_0^{-\tilde\eta}))\ \text{ with }\ |O(s_0^{-\tilde\eta})|\le\frac12. \tag{4.5.91}$$

- ***Improved bound for*** $b_{L_n}^{(n,k)}$***:*** first we notice that since L is chosen after ℓ one can assume that for all $0\le n\le n_0$, $i_n<L$. We rewrite the improved modulation equation (*4.5.2*) for $b_{L_n}^{(n,k)}$, using the estimate (*4.5.3*) for the extra term in the time derivative and the function $g_{L_n}^{(n,k)}$ (satisfying (*4.5.90*) and (*4.5.91*)), yielding:

$$\begin{array}{rl} & \left|\frac{d}{ds}\left[g_{L_n}^{(n,k)}b_{L_n}^{(n,k)}+O_{L,M,K_2}(s^{-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}})\right]\right| \\ \le & C(L,M,K_2)s^{-1-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}} \end{array}$$

as $\eta(1-\delta_0')<\frac{g'}{2}$ for η small enough (g' being fixed). The notation $O_{L,M,K_2}()$ is the usual $O()$ notation with a constant depending on L, M and K_2. One has $2L_n-\alpha_n=2L-\frac{d}{2}-2\delta_n+2m_0+\frac{2}{p-1}$. Hence for L large enough, the quantity $-L-\eta(1-\delta_0')+\delta_0-\delta_n+\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}$ is strictly positive for all $0\le n\le n_0$. Therefore, reintegrating in time the previous identity yields using (*4.4.16*) and (*4.4.17*):

$$\begin{array}{rcl} |b_{L_n}^{(n,k)}(s^*)| & \le & \frac{C(L,M,K_2)}{(s^*)^{L+\eta(1-\delta_0')+\delta_0-\delta_n}} \\ & & +\frac{1}{s^{L+\delta_0-\delta_n+\tilde\eta}}\frac{s_0^{\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}-L-\delta_0+\delta_n-\tilde\eta}}{(s^*)^{\frac{\ell(2L_n-\alpha_n)}{2\ell-\alpha}-L-\delta_0+\delta_n-\tilde\eta}}\frac32 s_0^{L+\delta_0-\delta_n+\tilde\eta}|b_{L_n}^{(n,k)}(s_0)| \\ & \le & \frac{C(L,M,K_2)}{(s^*)^{L+\eta(1-\delta_0')+\delta_0-\delta_n}}+\frac{3\epsilon_{L_n}^{(n,k)}}{20}\frac{1}{(s^*)^{L+\delta_0-\delta_n+\tilde\eta}} \end{array}$$

Therefore, if $\tilde\eta<\eta(1-\delta_0')$, for any $0<\epsilon_{L_n}^{(n,k)}<1$, for s_0 large enough there holds:

$$|b_{L_n}^{(n,k)}(s^*)|\le\frac{\epsilon_{L_n}^{(n,k)}}{2(s^*)^{L+\delta_0-\delta_n+\tilde\eta}}. \tag{4.5.92}$$

- ***Improved bound for*** $b_i^{(n,k)}$***,*** $i_n<i<L_n$***:*** using the same methodology we used to study the parameter $b_{L_n}^{(n,k)}$, we take the modulation equation (*4.4.43*), we integrate it in time injecting the bounds (*4.4.22*),

(*4.4.23*), (*4.4.24*) and (*4.4.25*), yielding:

$$\begin{array}{rcl} |\frac{d}{ds}(g_i^{(n,k)}b_i^{(n,k)})| & \leq & \frac{3\epsilon_{i+1}^{(n,k)}s^{\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)-\frac{\gamma-\gamma_n}{2}-i-\tilde{\eta}-1}}{2} \\ & & +C(L,M,K_1)s^{-L-1+\delta_0-\eta(1-\delta_0')+\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)}. \end{array}$$

The condition $i_n<i$ ensures that $\frac{\ell}{2\ell-\alpha}(2i-\alpha_n)-\frac{\gamma-\gamma_n}{2}-i>0$. For $\tilde{\eta}$ small enough, we can then integrate in time the previous equation, the first term in the right hand side giving then a divergent integral, and inject the bound (*4.5.91*) on $g_i^{(n,k)}$ and the initial bound (*4.4.17*) on $b_i^{(n,k)}$ one obtains:

$$\begin{array}{rcl} |b_i^{(n,k)}(s^*)| & \leq & \frac{1}{(s^*)^{\frac{\gamma-\gamma_n}{2}+i+\tilde{\eta}}}\Big(\frac{3\epsilon_i^{(n,k)}}{20}+C(L)\epsilon_{i+1}^{(n,k)} \\ & & +\frac{C(L,M)}{(s^*)^{\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}-\frac{\gamma-\gamma_n}{2}-i-\tilde{\eta}}}\int_{s_0}^{s^*}s^{-L-1+\delta_0-\eta(1-\delta_0')+\frac{\ell(2i-\alpha)}{2\ell-\alpha}}ds\Big) \\ & \leq & \frac{\epsilon_i^{(n,k)}}{2(s^*)^{\frac{\gamma-\gamma_n}{2}+i}} \end{array} \tag{4.5.93}$$

if s_0 is large enough and $\epsilon_{i+1}^{(n,k)}$ is small enough, because $L-\delta_0>\frac{\gamma-\gamma_n}{2}+i$.

- *Improved bound for* $b_i^{(n,k)}$ *if* $i_n=i$ *and* $1\leq n$*:* in that case, $\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}=\frac{\gamma-\gamma_n}{2}+i$, hence one has $\frac{1}{2}\leq\frac{g_i^{(n,k)}}{s^{\frac{\gamma-\gamma_n}{2}+i}}\leq\frac{3}{2}$. Integrating the modulation equation and making the same manipulations we made for $i_n<i$ then yields:

$$|b_i^{(n,k)}(s^*)|\leq\frac{1}{(s^*)^{\frac{\gamma-\gamma_n}{2}+i}}\left(\frac{3\epsilon_i^{(n,k)}}{20}+C(L)\epsilon_{i+1}^{(n,k)}+\frac{C(L,M)}{s_0^{L-\delta_0-\frac{\gamma-\gamma_n}{2}-i}}\right)\leq\frac{\epsilon_i^{(n,k)}}{2(s^*)^{\frac{\gamma-\gamma_n}{2}}+i} \tag{4.5.94}$$

if $\epsilon_{i+1}^{(n,k)}$ is small enough and s_0 is large enough.

- *Improved bound for* V_1*:* we recall that from (*4.4.13*), V_1 denotes the stable direction of perturbation for the dynamical system (*4.3.58*) contained in $\text{Span}((U_i^{(0,1)})_{1\leq i\leq\ell})$. From the quasi diagonalization (*4.3.69*) of the linearized matrix A_ℓ its time evolution is given by, under the bootstrap bounds (*4.4.22*), (*4.4.23*), (*4.4.24*) and (*4.4.25*):

$$\begin{array}{rcl} V_{1,s} & = & -\frac{V_1}{s}+O\left(\frac{|(V_i)_{1\leq i\leq\ell}|^2}{s}\right)+O(C(L,M,K_2)s^{-L-\ell})+\frac{q_1}{s}U_{i+1}^{(0,1)} \\ & = & -\frac{V_1}{s}+O\left(\frac{1}{s^{1+2\tilde{\eta}}}+s^{-L-\ell}+\frac{\epsilon_{\ell+1}^{(0,1)}}{s^{1+\tilde{\eta}}}\right) \end{array}$$

which when reintegrated in time gives, if $\epsilon_{\ell+1}^{(0,1)}$ is small enough, s_0 is large enough and using (*4.4.16*):

$$|V_1(s^*)|\leq\frac{s_0V_1(s_0)}{s^*}+\frac{C(L,M,K_1)}{(s^*)^{2\tilde{\eta}}}+\frac{C(L)\epsilon_{\ell+1}^{(0,1)}}{(s^*)^{\tilde{\eta}}}\leq\frac{1}{2s^{\tilde{\eta}}} \tag{4.5.95}$$

- *End of Step 2:* We choose the constants of smallness in the following order so that all the improved bounds we proved, (*4.5.92*), (*4.5.93*), (*4.5.94*), (*4.5.95*), hold together. For any choice of K_1, K_2, L, M, η in their range, there exists $\tilde{\eta}>0$ such that $\tilde{\eta}<\eta(1-\delta_0')$ and $\frac{\gamma-\gamma_n}{2}+i+\tilde{\eta}<\frac{\ell(2i-\alpha_n)}{2\ell-\alpha}$ for all $(n,k,i)\in\mathfrak{I}$ such that $i_n<i$. Then, we first choose the constant $\epsilon_{\ell+1}^{(0,1)}$ small enough so that the improved bounds (*4.5.95*) for V_1 holds for s_0 large enough. Next we choose $\epsilon_{\ell+2}^{(0,1)}$ such that the improved bound (*4.5.93*) for $U_{\ell+1}^{(0,1)}$ holds for s_0 large enough. By iteration we then choose $\epsilon_{\ell+3}^{(0,1)}$, ..., $\epsilon_L^{(0,1)}$ to make all the bounds (*4.5.93*) hold till the one for $U_{L-1}^{(0,1)}$. The last one, (*4.5.92*), for $U_L^{(0,1)}$, holds for s_0 large enough without

any conditions on $\epsilon_i^{(0,1)}$ for $\ell+1\leq i\leq L-1$. The same reasoning applies for the stable parameters on the spherical harmonics of higher degree ($1\leq n\leq n_0$). We have proved (*4.5.88*).

□

We fix all the constants of the analysis so that Lemma 4.5.9 holds, and we will just possibly increase the initial renormalized time s_0, which does not change its validity. The number of instability directions is:

$$m=\ell-1+d(E[i_1]-\delta_{i_1\in\mathbb{N}})+\sum_{2\leq n\leq n_0}k(n)(E[i_n]+1-\delta_{i_n\in\mathbb{N}}).$$

To prove Proposition 4.4.6, we have to prove that there exists an additional perturbation along the instable directions of perturbations such that the solution stays forever trapped. We prove it via a topological argument, by looking at all the solutions associated to the possible perturbations along the instable directions of perturbation. For this purpose we introduce the following set:

$$\begin{aligned}\mathcal{B}:=\{\quad &(V_2(s_0),...,V_\ell(s_0),(U^{(n,k)}(s_0)_i)_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n})\in\mathbb{R}^m,\ |V_i(s_0)|\leq s_0^{-\tilde\eta}\\ &\text{for } 2\leq i\leq\ell,\ |U^{(n,k)}(s_0)_i|\leq\epsilon_i^{(n,k)} \text{ for } (n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n\qquad\}\end{aligned}$$

which represents all the possible values of the instable parameters so that the solution to (NLH) with initial data given by (*4.4.5*) and (*4.4.31*) starts in the trapped regime. We then define the following application $f:\mathcal{D}(f)\subset\mathcal{B}\to\partial\mathcal{B}$ that gives the last value taken by the instable parameters before the solution leaves the trapped regime (when it does):

$$\begin{aligned}&f\left(V_2(s_0),...,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n}\right)\\ =&\left(\frac{(s^*)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s^*),...,\frac{(s^*)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s^*),(U_i^{(n,k)}(s^*))_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n}\right).\end{aligned}\tag{4.5.96}$$

The domain $\mathcal{D}(f)$ of the application f is the set of the m-tuples of real numbers under the form $(V_2(s_0),...,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n})$ in $\mathcal{B}$ such that the solution starting initially with a decomposition given by (*4.4.5*) and (*4.4.31*) leaves the trapped regime in finite time s^*. The following lemma describes the topological properties of f.

Lemma 4.5.10 (Topological properties of the exit application). *There exists a choice of smallness constants $(\epsilon_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n+1}$ such that the following properties hold for s_0 large enough:*

(i) $\mathcal{D}(f)$ is non empty and open, and there holds the inclusion $\partial\mathcal{B}\subset\mathcal{D}(f)$.

(ii) f is continuous and is the identity on the boundary $\partial\mathcal{B}$.

Proof of Lemma 4.5.10

step 1 The outgoing flux property. We prove in this step that one can choose the smallness constants $(\epsilon_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n+1}$ such that for any $(V_2(s_0),...,V_\ell(s_0),(U_i^{(n,k)})_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n})$ in $\mathcal{B}$ such that the solution starting initially with the decomposition given by (*4.4.5*) and (*4.4.31*) is in the trapped regime on $[s_0,s]$ and satisfies at time s:

$$\left(\frac{(s)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s),...,\frac{(s)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s),(U_i^{(n,k)}(s))_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n}\right)\in\partial\mathcal{B},$$

then the exit time from the trapped regime is s. To prove this we compute the time derivative of the instable parameters when they are on $\partial\mathcal{B}$, and show that it points toward the exterior. Indeed from the modulation equation (*4.4.43*) and (*4.3.69*) (where we injected the bounds of the trapped regime (*4.4.22*), (*4.4.23*), (*4.4.24*) and (*4.4.25*)):

$$\begin{aligned} V_{i,s} &= \tfrac{i\alpha}{2\ell-\alpha}\tfrac{V_i}{s}+O(\tfrac{|(V_1(s),...,V_\ell(s))|^2}{s})+\tfrac{q_iU^{(0,1)}_{\ell+1}}{s}+O(s^{-L+\ell}) \\ &= \tfrac{i\alpha}{2\ell-\alpha}\tfrac{V_i}{s}+O(s^{-1-2\tilde\eta}+\tfrac{\epsilon^{(0,1)}_{\ell+1}}{s^{1+\tilde\eta}}), \end{aligned}$$

$$\begin{aligned} U^{(n,k)}_{i,s} &= \alpha\tfrac{\ell-\frac{\gamma-\gamma_n}{2}-i}{(2\ell-\alpha)s}U^{(n,k)}_i+\tfrac{U^{(n,k)}_{i+1}}{s}+O(s^{-1-\tilde\eta}) \\ &= \alpha\tfrac{i_n-i}{(2\ell-\alpha)s}U^{(n,k)}_i+O(\tfrac{\epsilon^{(n,k)}_{i+1}}{s}+s^{-1-\tilde\eta}). \end{aligned}$$

Therefore, as $i<i_n$, by iterations (ie by choosing first $\epsilon^{(n,k)}_0$, then $\epsilon^{(n,k)}_1$, and so on till choosing $\epsilon^{(n,k)}_{\ell+1}$) we can choose all the smallness constants and s_0 large enough so that:

$$\frac{i\alpha}{2\ell-\alpha}\frac{(-1)^j}{s^{1+\tilde\eta}}+O(s^{-1-2\tilde\eta}+\frac{\epsilon^{(0,1)}_{\ell+1}}{s^{1+\tilde\eta}})>0 \text{ (resp. } <0) \text{ if } j=0 \text{ (resp. } j=1),$$

$$\alpha\frac{i_n-i}{(2\ell-\alpha)s}(-1)^j\epsilon^{(n,k)}_i+O(\frac{\epsilon^{(n,k)}_{i+1}}{s}+s^{-L+\ell})>0 \text{ (resp. } <0) \text{ if } j=0 \text{ (resp. } j=1).$$

Consequently, any solution that is trapped until s such that at time s,

$$\left(\frac{(s)^{\tilde\eta}}{s_0^{\tilde\eta}}V_2(s),...,\frac{(s)^{\tilde\eta}}{(s_0)^{\tilde\eta}}V_\ell(s),(U^{(n,k)}_i(s))_{(n,k,i)\in\mathcal{I},\ 1\leq n,\ i<i_n}\right)\in\partial\mathcal{B},$$

leaves the trapped regime after s.

step 2 End of the proof of the lemma. Step 1 directly implies that $\mathcal{D}(f)$ contains $\partial\mathcal{B}$, and that f is the identity on $\partial\mathcal{B}$. If a solution u leaves at time s^*, it also implies that it never hit the boundary before s^*. Consequently, as the trapped regime is characterized by non strict inequalities, and because everything in the dynamics of (NLH) is continuous with respect to variation on these instable parameters, we get that $\mathcal{D}(f)$ is open, and that the exit time s^* and f are continuous on $\mathcal{D}(f)$.

□

We can now end the proof of Proposition 4.4.6.

Proof of Proposition 4.4.6

We argue by contradiction. If for any choice of initial perturbation along the instable directions of perturbation, the solution leaves the trapped regime, then it means that the domain of the exit application f defined by (*4.5.96*) is $\mathcal{D}(f)=\mathcal{B}$. But then from Lemma 4.5.10, f would be a continuous application from $\mathcal{B}$ towards its boundary, being the identity on the boundary, which is impossible thanks to Brouwer's theorem, and the contradiction is obtained.

□

4.A Properties of the zeros of H

This section is devoted to the proof of Lemma 4.2.1.

Proof of Lemma 4.2.1

The proof relies solely on ODE techniques (in the same spirit as [67, 93]) and is as follows. First, we describe the asymptotics of the equation $H^{(n)}f=0$ at the origin and at infinity in Lemma 4.A.1. Then we construct the special zeroes $T_0^{(n)}$ and $\Gamma^{(n)}$ in these asymptotic regimes using a perturbative argument and obtain their asymptotic behavior in Lemma 4.A.2. Finally we show that they are not equal via global invariance properties of the ODE in the phase space $(f,\partial_r f)$ in Lemma 4.A.3, yielding that they form indeed a basis of the set of solutions.

Let $f:(0,+\infty)$ be smooth such that $H^{(n)}f=0$. First we make the change of variables $f(r)=w(t)$ with $t=\ln(r)\in(-\infty,+\infty)$. w then solves:

$$w''+(d-2)w'-[e^{2t}V(e^t)+n(d+n-2)]w=0 \tag{4.A.1}$$

where V is defined by (*4.1.14*) and satisfies $e^{2t}V(e^t)=O(e^{2t})\to 0$ as $t\to-\infty$, and $e^{2t}V(e^t)=-pc_\infty^{p-1}+O(e^{-t\alpha})$ as $t\to+\infty$, from (*3.2.10*). Hence (*4.A.1*) is similar to the following ODEs as $t\to\pm\infty$:

$$w''+(d-2)w'+(pc_\infty^{p-1}-n(d+n-2))w=0, \tag{4.A.2}$$

$$w''+(d-2)w'-n(d+n-2)w=0. \tag{4.A.3}$$

The first step in the proof of Lemma 4.2.1 is to describe their solutions.

Lemma 4.A.1. *The set of solutions of* (*4.A.2*) *(resp.* (*4.A.3*)*) is Span*$(e^{-\gamma_n t},e^{-\gamma_n' t})$ *(resp. Span*$(e^{nt},e^{(-n-d+2)t})$*), where* γ_n *is defined in* (*4.1.1*) *and*

$$\gamma_n':=\frac{d-2+\sqrt{\triangle_n}}{2}, \tag{4.A.4}$$

$\triangle_n>0$ *being defined in* (*4.1.1*)*. These numbers satisfy:*

$$\gamma_0=\gamma,\ \ \gamma_1=\frac{2}{p-1}+1 \ \ \textit{and}\ \forall n\geq 2,\ \gamma_n<\frac{2}{p-1}\ \textit{and}\ \gamma_n'>\frac{(d-2)}{2} \tag{4.A.5}$$

where γ *is defined in* (*2.2.5*)*.*

Proof.

From the standard theory of second order differential equations with constant coefficients, the set of solutions of (*4.A.2*) (resp. (*4.A.3*)) is Span$(e^{-\gamma_n t},e^{-\gamma_n' t})$ (resp. Span$(e^{nt},e^{(-n-d+2)t})$), where γ_n and γ_n' are defined by (*4.1.1*) and (*4.A.4*). For any $n\in\mathbb{N}$, one computes from its definition in (*4.1.1*) that the number $\triangle_n$ used in the definitions (*4.1.1*) and (*4.A.4*) of γ_n and γ_n' is strictly positive: $\triangle_n>0$. Indeed, $\triangle_n\geq\triangle_0$ from (*4.1.1*), and $\triangle_0>0$ if and only if $p>p_{JL}$ where p_{JL} is defined in (*1.4.1*), and the present chapter is concerned with the case $p>p_{JL}$.

From the formula (*4.1.1*) one computes that $\gamma_0=\gamma$ and $\gamma_1=\frac{2}{p-1}+1$ where γ is defined in (*2.2.5*). For all $n\in\mathbb{N}$, from the definition (*4.A.4*) of γ_n' and since $\triangle_n>0$, one gets that $\gamma_n'>\frac{d-2}{2}$. Eventually we compute from (*4.1.1*) that

$$\triangle_1=(d-4-\frac{4}{p-1})^2,\ \ \triangle_2=(d-4-\frac{4}{p-1})^2+4d+4$$

which implies in particular that

$$\triangle_2-\triangle_1-4\sqrt{\triangle_1}-4=4d+4-4(d-4-\tfrac{4}{p-1})-4=16+\tfrac{16}{p-1}>0.$$

giving $\sqrt{\triangle_2}>\sqrt{\triangle_1}+2$. This, from (*4.1.1*), implies:

$$\gamma_2=\frac{d-2-\sqrt{\triangle_2}}{2}<\frac{d-2-\sqrt{\triangle_1}-2}{2}=\gamma_1-1=\frac{2}{p-1}+1-1=\frac{2}{p-1}.$$

This implies that $\gamma_n<\frac{2}{p-1}$ for all $n\geq 2$ because the sequence $(\gamma_n)_{n\in\mathbb{N}}$ is decreasing from its definition (*4.1.1*).

□

Lemma 4.A.2. *There exist* $w_1^{(n)}$, $w_2^{(n)}$, $w_3^{(n)}$ *and* $w_4^{(n)}$ *solving* (*4.A.1*) *such that:*

$$w_1^{(n)} \underset{t\to-\infty}{=} \sum_{i=0}^{q} c_i e^{(n+2i)t}+O(e^{(n+2q+2)t}),\ w_2^{(n)} \underset{t\to-\infty}{\sim} \tilde{c}_1 e^{(-n-d+2)t}, \tag{4.A.6}$$

$$w_3^{(n)} \underset{t\to+\infty}{=} \tilde{c}_2 e^{-\gamma_n t}+O(e^{(-\gamma_n-g)t}) \text{ and } w_4^{(n)} \underset{t\to+\infty}{\sim} \tilde{c}_3 e^{-\gamma_n' t}=O(e^{(-\gamma_n-g)t}), \tag{4.A.7}$$

with constants $c_1,\tilde{c}_1,\tilde{c}_2,\tilde{c}_3\neq 0$. *Moreover the asymptotics hold for the derivatives.*

Proof of Lemma 4.A.2

step 1 Existence of $w_1^{(n)}$. For $n=0$, we take the explicit solution $w_1^{(0)}=\Lambda Q(e^t)$, which satisfies indeed (*4.A.6*) from (*3.2.1*). Let now $n\geq 1$. Using the Duhamel formula for solutions of (*4.A.1*), the fundamental set of solutions for the constant coefficient ODE (*4.A.3*) being provided by Lemma 4.A.1, a solution of (*4.A.1*) satisfying the condition on the left in (*4.A.6*) with $c_0=1$ can be written as:

$$w_1^{(n)}(t)=e^{nt}+\frac{1}{2n+d-2}\int_{-\infty}^{t}(e^{n(t-t')}-e^{(-n-d+2)(t-t')})w_1^{(n)}(t')e^{2t'}V(e^{t'})dt'. \tag{4.A.8}$$

We now use a standard contraction argument. For $t_0\in\mathbb{R}$ we endow the space

$$X:=\left\{u\in C((-\infty,t_0],\mathbb{R}),\ \sup_{t\leq t_0}|u(t)|e^{-t}<+\infty\right\}$$

with the norm:

$$\|u\|_X:=\sup_{t\leq t_0}|u(t)|e^{-(n+1)t}. \tag{4.A.9}$$

For $u\in X$ we define the function $\Phi u:(-\infty,t_0]\to\mathbb{R}$ by:

$$(\Phi u)(t):=\frac{1}{2n+d-2}\int_{-\infty}^{t}(e^{n(t-t')}-e^{(-n-d+2)(t-t')})[e^{nt'}+u(t')]e^{2t'}V(e^{t'})dt'. \tag{4.A.10}$$

Φ maps X into itself. Indeed as the potential V is bounded from (*3.2.10*) a brute force bound on the above equation yields that:

$$|(\Phi u)(t)|\leq C\|V\|_{L^\infty}(e^t+\|u\|_X e^{2t})e^{(n+1)t}.$$

and therefore $\|\Phi u\|_X\leq C\|V\|_{L^\infty}(e^{t_0}+\|u\|_X e^{2t_0})$. The same brute force bound for the difference of two images under Φ of two elements gives:

$$|(\Phi u)(t)-(\Phi v)(t)|\leq C\|V\|_{L^\infty}e^{2t}\|u-v\|_X e^{(n+1)t}.$$

Hence $\|\Phi u - \Phi v\|_X \leq C\|V\|_{L^\infty} e^{2t_0}\|u-v\|_X$ and Φ is a contraction for $t_0 \ll 0$ small enough. Therefore, Φ admits a fixed point in X, denoted by u_1. From the Duhamel formula (*4.A.8*) and the definition (*4.A.10*) of Φ, $w_1^{(n)} := e^{nt} + u_1(t)$ is then a solution of (*4.A.1*) on $(-\infty, t_0]$ which satisfies from the definition (*4.A.9*) of X:

$$w_1^{(n)} = e^{nt} + O(e^{(n+1)t}) \ \text{ as } t \to -\infty. \tag{4.A.11}$$

We extend it to a solution of (*4.A.1*) on $\mathbb{R}$ ((*4.A.1*) being linear with smooth coefficients), still naming it $w_0^{(n)}$.

step 2 Asymptotics of $w_1^{(n)}$. At present, we will refine the asymptotics (*4.A.11*). We reason by induction. We claim that if for $k \in \mathbb{N}$ and $(c_i)_{0\leq i\leq k} \in \mathbb{R}^{k+1}$ one has:

$$w_1^{(n)} = \sum_{i=0}^{k} c_i e^{(n+2i)t} + O(e^{(n+2k+2)t}) \ \text{ as } t \to -\infty \tag{4.A.12}$$

then there exists $c_{k+1} \in \mathbb{R}$ such that:

$$w_1^{(n)} = \sum_{i=0}^{k+1} c_i e^{(n+2i)t} + O(e^{(n+2k+4)t}) \ \text{ as } t \to -\infty. \tag{4.A.13}$$

We now prove this fact. Fix $k \geq 1$ and assume that $w_1^{(n)}$ satisfies (*4.A.12*). As V is a smooth radial profile, one has that $\partial_r^{2q+1} V(0) = 0$ for any $q \in \mathbb{N}$, implying that there exists $(d_i)_{i\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that

$$V(e^t) = \sum_{i=0}^{k} d_i e^{2it} + O(e^{(2k+2)t}) \ \text{ as } t \to -\infty. \tag{4.A.14}$$

We inject this and (*4.A.12*) in (*4.A.8*) and integrate to find:

$$\begin{array}{rcl} w_1^{(n)} & = & e^{nt} + \frac{1}{2n+d-2}\int_{-\infty}^{t} (e^{n(t-t')} - e^{(2-n-d)(t-t')}) \\ & & \times \left[\sum_{i=0}^{k}\sum_{j=0}^{i} c_j d_{i-j} e^{(n+2i+2)t'} + O(e^{(n+2k+4)t'})\right] dt' \\ & = & e^{nt} + \sum_{i=0}^{k} \frac{e^{(n+2i+2)t}}{2n+d-2}\left(\frac{1}{2i+2} - \frac{1}{2n+d+2i}\right)\sum_{j=0}^{i} c_j d_{i-j} + O(e^{(2+2k+4)t}). \end{array}$$

This asymptotics has to be coherent with the assumption (*4.A.12*), hence for all $0 \leq i \leq k-1$ one has $\left(\frac{1}{2i+2} - \frac{1}{2n+d+2i}\right)\sum_{j=0}^{i} \frac{c_j d_{i-j}}{2n+d-2} = c_{i+1}$. The above identity is then the formula (*4.A.13*) one has to prove. Thus, one has proven that the asymptotics in the left of (*4.A.6*) holds for $w_1^{(n)}$. It remains to show that it also holds for the derivatives. Differentiating (*4.A.8*) gives:

$$(w_1^{(n)})'(t) = ne^{nt} + \frac{1}{2n+d-2}\int_{-\infty}^{t} [ne^{n(t-t')} + (n+d-2)e^{(2-n-d)(t-t')}] w_1^{(n)} e^{2t'} V.$$

We make the same reasoning we did for $w_1^{(n)}$: we inject the asymptotics (*4.A.12*) at any order for $w_1^{(n)}$ we just showed and (*4.A.14*) in the above formula, integrate in time and match the coefficients we find with (*4.A.12*), yielding that:

$$(w_1^{(n)})'(t) = \sum_{i=0}^{k} (n+2i) c_i e^{(n+2i)t} + O(e^{(n+2k+2)t})$$

for any $k \in \mathbb{N}$. Therefore, one has proven that the asymptotics in the left of (*4.A.6*) holds for $w_1^{(n)}$ and $(w_1^{(n)})'$. As $w_1^{(n)}$ solves (*4.A.1*) its second derivatives is given by:

$$(w_1^{(n)})'' = -(d-2)(w_1^{(n)})' + [e^{2t}V(e^t) + n(d+n-2)] w_1^{(n)}$$

and therefore from (*4.A.14*) the expansion also holds for $(w_1^{(n)})''$. Differentiating the above equation, using again (*4.A.14*) and the expansions for $w_1^{(n)}$, $(w_1^{(n)})'$ and $(w_1^{(n)})''$, one obtains the expansion for $(w_1^{(n)})'''$. By iterating this procedure we obtain the expansion in the left of (*4.A.6*) for any derivatives of $w_1^{(n)}$.

step 3 Existence and asymptotics of $w_2^{(n)}$. Let $t_0 \in \mathbb{R}$. We use the Duhamel formula for (*4.A.1*), the solutions of the underlying constant coefficient ODE (*4.A.3*) being provided by Lemma 4.A.1. For $t \le t_0$ the solution of (*4.A.1*) starting from $w_2^{(n)}(t_0) = e^{(2-d-n)t_0}$, $(w_2^{(n)})'(t_0) = (2-d-n)e^{(2-d-n)t_0}$ can be written as:

$$w_2^{(n)} = e^{(2-d-n)t} - \frac{1}{2n+d-2}\int_t^{t_0} (e^{n(t-t')} - e^{(2-n-d)(t-t')})V(e^{t'})e^{2t'}w_2^{(n)}(t')dt'. \qquad (4.A.15)$$

We claim that for $t_0 \ll 0$ small enough, there holds

$$|w_2^{(n)} - e^{(2-d-n)t}| \le \frac{e^{(2-d-n)}}{2} \qquad (4.A.16)$$

for all $t \le t_0$. To show that, let $\mathcal{T}$ be the set of times $t \le t_0$ such that this inequality holds. $\mathcal{T}$ is closed via a continuity argument, and is non empty as it contains t_0. For $t \in \mathcal{T}$ we compute by brute force on the above identity:

$$|w_2^{(n)} - e^{(2-d-n)t}| \le C\|V\|_{L^\infty} e^{(2-n-d)t}e^{2t_0}.$$

Hence, for $t_0 \ll 0$ small enough, $|w_2^{(n)} - e^{(2-d-n)t}| \le \frac{e^{(2-n-d)t}}{3}$ implying that $\mathcal{T}$ is open. Therefore, $\mathcal{T} = (-\infty, t_0]$ from a connectedness argument and $w_2^{(n)}$ satisfies (*4.A.16*) for all $t \le t_0$. We inject (*4.A.16*) in (*4.A.15*) to refine the asymptotics (the constant in the $O()$ depends on $\|V\|_{L^\infty}$):

$$\begin{aligned}
w_2^{(n)} &= e^{(2-d-n)t} + \textstyle\int_t^{t_0} (e^{n(t-t')} - e^{(2-d-n)(t-t')})O(e^{(4-n-d)(t-t')})dt' \\
&= e^{(2-d-n)t} + e^{nt}\textstyle\int_t^{t_0} O(e^{(4-2n-d)t'})dt' + e^{(2-n-d)t}\int_t^{t_0} O(e^{2t'})dt' \\
&= e^{(2-d-n)t} + O(e^{(4-n-d)t}) + e^{(2-n-d)t}\left(\textstyle\int_{-\infty}^{t_0} O(e^{2t'})dt' - \int_{-\infty}^{t} O(e^{2t'})dt'\right) \\
&= e^{(2-d-n)t}\left(1 + \textstyle\int_{-\infty}^{t_0} O(e^{2t'})dt'\right) + O(e^{(4-n-d)t}) \\
&= \tilde{c}_1 e^{(2-d-n)t} + O(e^{(4-n-d)t})
\end{aligned}$$

with $\tilde{c}_1 \neq 0$ if $t_0 \ll 0$ is chosen small enough. We just showed the asymptotic on the right of (*4.A.6*).

step 4 Existence and asymptotics of $w_3^{(n)}$ and $w_4^{(n)}$. Using verbatim the same techniques we used at $-\infty$ to construct $w_1^{(n)}$ and $w_2^{(n)}$ as perturbations of the solutions described by Lemma 4.A.1 of the asymptotic constant coefficients ODE (*4.A.3*), we can construct two solutions of (*4.A.1*), $w_3^{(n)}$ and $w_4^{(n)}$, satisfying:

$$w_3^{(n)} \sim \tilde{c}_2 e^{-\gamma_n t}, \ w_4^{(n)} \sim \tilde{c}_3 e^{-\gamma_n' t} \ \text{ as } t \to +\infty \qquad (4.A.17)$$

with $\tilde{c}_2, \tilde{c}_3 \neq 0$, as perturbations of the solutions $e^{-\gamma_n t}$ and $e^{-\gamma_n' t}$ of the asymptotic ODE (*4.A.2*) at $+\infty$. We leave safely the proof of this fact to the reader. We now show why the second term in the asymptotic of $w_3^{(n)}$ is $O(e^{(-\gamma_n - g)t})$ where g is defined in (*4.1.4*). Using Duhamel's formula for (*4.A.1*), with the set of fundamental solutions of the asymptotic equation (*4.A.2*) described in Lemma 4.A.1, $w_3^{(n)}$ can be written as

$$\begin{aligned}
w_3^{(n)} &= a_1 e^{-\gamma_n t} + b_1 e^{-\gamma_n' t} \\
&\quad -\tfrac{1}{-\gamma_n+\gamma_n'}\textstyle\int_0^t (e^{-\gamma_n(t-t')} - e^{-\gamma_n'(t-t')})e^{2t'}(V(e^{t'}) + pc_\infty^{p-1}e^{-2t'})w_3^{(n)}(t')dt'.
\end{aligned}$$

for a_1 and b_1 two coefficients. We use the bounds $V(e^{t'})+pc_\infty^{p-1}e^{-2t'}=O(e^{-\alpha t'})$ from (*3.2.10*) and (*4.A.17*) to find:

$$w_3^{(n)}(t)=a_1e^{-\gamma_n t}+b_1e^{-\gamma_n' t}-\frac{1}{-\gamma_n+\gamma_n'}\int_0^t(e^{-\gamma_n(t-t')}-e^{-\gamma_n'(t-t')})O(e^{(-\gamma_n-\alpha)t'}).$$

After few computations we obtain two new coefficients $\tilde{a}_1$ and $\tilde{a}_2$ such that:

$$w_3^{(n)}(t)=\tilde{a}_1e^{-\gamma_n t}+\tilde{b}_1e^{-\gamma_n' t}+O(e^{(-\gamma_n-\alpha)t}).$$

The asymptotic (*4.A.17*), as $-\gamma_n'<-\gamma_n$ from (*4.1.1*) implies $\tilde{a}_1=\tilde{c}_2\neq 0$. From the definition (*4.1.4*) of g, this parameter is tailor made to produce $-\gamma_0-g>-\gamma_0'$ (from (*2.2.5*) and (*4.1.1*)). From (*4.1.1*) one then has: $-\gamma_n-g+\gamma_n'\geq -\gamma_0-g+\gamma_0'>0$. As g satisfies also $g<\alpha$, the above identity then yields:

$$w_3^{(n)}(t)=\tilde{c}_2e^{-\gamma_n t}+O(e^{(-\gamma_n-g)t}).$$

Using exactly the same methods we use to propagate the asymptotic of $w_1^{(n)}$ to its derivatives in Step 2, the above identity propagates to the derivatives of $w_3^{(n)}$.

□

Lemma 4.A.3. *The solutions $w_1^{(n)}$ and $w_4^{(n)}$ given by Lemma 4.A.2 are not collinear. Moreover, $w_1^{(n)}$ has constant sign.*

Proof of Lemma 4.A.3

We see (ODE_n) as a planar dynamical system:

$$\frac{d}{dt}\begin{pmatrix}w^1\\w^2\end{pmatrix}=\begin{pmatrix}0&1\\n(d+n-2)+e^{2t}V(e^t)&-(d-2)\end{pmatrix}\begin{pmatrix}w^1\\w^2\end{pmatrix}.$$

with $w^1=w$ and $w^2=w'$. From their asymptotics from Lemma 4.A.1:

$$\begin{pmatrix}w_1^{(n)}(t)\\(w_1^{(n)})'(t)\end{pmatrix}=c_1e^{nt}\begin{pmatrix}1\\n\end{pmatrix}+O(e^{(n+2)t})\text{ as }t\to-\infty,$$

$$\begin{pmatrix}w_4^{(n)}(t)\\(w_4^{(n)})'(t)\end{pmatrix}\sim\tilde{c}_3e^{-\gamma_n' t}\begin{pmatrix}1\\-\gamma_n'\end{pmatrix}\text{ as }t\to-\infty$$

and we may take $c_1,\tilde{c}_3>0$ without loss of generality. Therefore, close to $-\infty$, $\left(w_1^{(n)}(t),(w_1^{(n)})'(t)\right)$ is in the top right corner of the plane. It cannot cross the ray $\{0\}\times(0,+\infty)$ because there the vector field $\begin{pmatrix}w^2\\-(d-2)w^2\end{pmatrix}$ points toward the right. Neither can it go below the ray $(x,-\frac{d-2}{2}x)_{x\geq 0}$. To see that we compute the scalar product between the vector field and a vector that is orthogonal to this ray and that points toward north at any time $t\in\mathbb{R}$:

$$\begin{aligned}&\left(\begin{pmatrix}0&1\\n(d+n-2)+e^{2t}V(e^t)&-(d-2)\end{pmatrix}\begin{pmatrix}1\\-\frac{d-2}{2}\end{pmatrix}\right)\cdot\begin{pmatrix}\frac{d-2}{2}\\1\end{pmatrix}\\=&\frac{(d-2)^2}{4}+e^{2t}V(e^t)+n(d+n-2)>0\end{aligned}$$

because $e^{2t}V(e^t) > \frac{(d-2)^2}{4}$, the potential $-V$ being below the Hardy potential (see (*3.2.5*)). Hence $\left(w_1^{(n)}(t),(w_1^{(n)})'(t)\right)$ stays in the top right zone whose border is $\{0\}\times(0,+\infty)\cup(x,-\frac{d-2}{2}x)_{x\geq 0}$. In particular, $w_1^{(n)}>0$ for all times, which proves the positivity of $w_1^{(n)}$. As the trajectory $\left(w_4^{(n)}(t),(w_4^{(n)})'(t)\right)$ is asymptotically collinear to the vector $\begin{pmatrix}1\\-\gamma_n'\end{pmatrix}$ which does not belong to this zone (from Lemma 4.A.1) nor its opposite, one obtains that $w_1^{(n)}$ and $w_4^{(n)}$ are not collinear.

□

We now end the proof of Lemma 4.2.1. The fundamental set of solutions of (*4.A.1*) is provided by Lemma 4.A.2. As $w_1^{(n)}$ is not collinear to $w_4^{(n)}$, there exists $a_1\neq 0$ and a_2 such that $w_1^{(n)}=a_1w_3^{(n)}+a_2w_4^{(n)}$. From the asymptotics (*4.A.7*) and the positivity of $w_1^{(n)}$ shown in Lemma 4.A.3 one then has:

$$w_1^{(n)}=be^{-\gamma_n t}+O(e^{(-\gamma_n-g)t}) \ \text{ as } t\to+\infty,\ b>0.$$

We call T_0^n the profile associated to $w_1^{(n)}$ in the original space variable r: $T_0^n(r)=w_1^{(n)}(\ln(r))$ which solves $H^{(n)}T_0^{(n)}=0$. The above identity means $T_0^n=a_1r^{-\gamma_n}+O(r^{(-\gamma_n-g)})$ as $r\to+\infty$, and (*4.A.6*) implies $T_0^n(r)\underset{r\to 0}{=}\sum_{i=0}^q b_i^n r^{n+2l}+O(r^{n+2+2q})$ as $r\to 0$, for some coefficients $(b_i)_{i\in\mathbb{N}}\in\mathbb{R}^{\mathbb{N}}$, for any $q\in\mathbb{N}$. These asymptotics propagate for the derivatives. This is the identity (*4.2.2*) we had to prove.

Now let us denote by w another solution of (*4.A.1*) that is not collinear to $w_1^{(n)}$ and $w_4^{(n)}$. (*4.A.6*) and (*4.A.7*) imply that $w\sim ce^{(2-n-d)t}$ as $t\to-\infty$ and $w=de^{-\gamma_n t}+O(e^{(-\gamma_n-g)t})$ as $t\to+\infty$ with $c,d\neq 0$. These asymptotics propagate for higher derivatives. The solution of $H^{(n)}\Gamma^{(n)}=0$ given by $\Gamma^{(n)}(r)=w(\ln(r))$ then satisfies the desired asymptotics (*4.2.2*) we had to prove. Eventually, the laplacian on spherical harmonics of degree n is (for f radial):

$$\Delta(fY_{n,k})=\left((\partial_{rr}+\frac{d-1}{r}\partial_r-\frac{n(d+n-2)}{r^2})f\right)Y_{n,k}$$

meanings from the asymptotics (*4.2.2*) that for any $j\in\mathbb{N}$, $\Delta^j(T_0^n(|x|)Y_{n,k}(\frac{x}{|x|}))$ is a continuous function near the origin. Therefore, $T_0^nY_{n,k}$ is smooth close to the origin from elliptic regularity. It is also smooth outside as a product of smooth functions, and thus smooth everywhere, ending the proof Lemma 4.2.1.

□

4.B Hardy and Rellich type inequalities

We recall in this section the Hardy and Rellich estimates to make this chapter self contained. They are used throughout the chapter, and especially to derive a fundamental coercivity property of the adapted high Sobolev norm in Appendix 4.C. We now state a useful and very general Hardy inequality with possibly fractional weights and derivatives. A proof can be found in [114], Lemma B.2.

Lemma 4.B.1 (Hardy type inequalities). *Let $\delta>0$, $q\geq 0$ satisfy $\left|q-(\frac{d}{2}-1)\right|\geq\delta$ and $u:[1,+\infty)\to\mathbb{R}$ be smooth and satisfy*

$$\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q}}y^{d-1}dy+\int_1^{+\infty}\frac{u^2}{y^{2q+2}}y^{d-1}dy<+\infty.$$

(i) If $q>\frac d2-1+\delta$, then there holds:

$$C(d,\delta)\int_{y\geq 1}\frac{u^2}{y^{2q+2}}y^{d-1}dy-C'(d,\delta)u^2(1)\leq \int_{y\geq 1}\frac{|\partial_y u|^2}{y^{2q}}y^{d-1}dy \qquad (4.B.1)$$

(ii) If $q<\frac d2-1-\delta$, then there holds:

$$C(d,\delta)\int_{y\geq 1}\frac{u^2}{y^{2q+2}}y^{d-1}dy\leq \int_{y\geq 1}\frac{|\partial_y u|^2}{y^{2q}}y^{d-1}dy. \qquad (4.B.2)$$

Proof of Lemma 4.B.1

Let $R>1$, the fundamental theorem of calculus gives:

$$\frac{u^2(R)}{R^{2q+2-d}}-u^2(1)=2\int_1^R\frac{u\partial_y u}{y^{2q+2-d}}dy-(2q+2-d)\int_1^R\frac{u^2}{y^{2q+2-d}}dy.$$

The integrability of $\frac{u^2}{y^{2q+3-d}}$ over $[1,+\infty)$ implies that $\frac{u^2(R_n)}{R_n^{2q+2-d}}\to 0$ along a sequence of radiuses $R_n\to+\infty$. Passing to the limit through this sequence we get:

$$(2q+2-d)\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}dy-u^2(1)=2\int_1^{+\infty}\frac{u\partial_y u}{y^{2q+2-d}}dy.$$

We apply Cauchy-Schwarz and Young inequalities to find:

$$\begin{aligned}\left|2\int_1^{+\infty}\frac{u\partial_y u}{y^{2q+2-d}}dy\right| &\leq 2\left(\int_1^{+\infty}\frac{u^2}{y^{2q+3-d}}dy\right)^{\frac12}\left(\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+1-d}}dy\right)^{\frac12}\\ &\leq \epsilon\int_1^{+\infty}\frac{u^2}{y^{2q+3-d}}dy+\frac1\epsilon\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}dy\end{aligned}$$

for any $\epsilon>0$. If $q>\frac d2-1+\delta$, then the two above identities give:

$$\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}dy\leq\frac{u^2(1)}{2\delta}+\frac{\epsilon}{2\delta}\int_1^{+\infty}\frac{u^2}{y^{2q+3-d}}dy+\frac{1}{2\delta\epsilon}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}dy.$$

Taking $\epsilon=\delta$, one gets $\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}dy\leq\frac{u^2(1)}{\delta}+\frac{1}{\delta^2}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}dy$ which is precisely the identity (*4.B.1*) we had to prove. If $q<\frac d2-1-\delta$ then one obtains:

$$\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}dy\leq-\frac{u^2(1)}{2(\frac d2-1-q)}+\frac{\epsilon}{2\delta}\int_1^{+\infty}\frac{u^2}{y^{2q+3-d}}dy+\frac{1}{2\delta\epsilon}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}dy.$$

Taking $\epsilon=\delta$, one gets $\int_1^{+\infty}\frac{u^2}{y^{2q+2-d}}dy\leq\frac{1}{\delta^2}\int_1^{+\infty}\frac{|\partial_y u|^2}{y^{2q+3-d}}dy$ which is precisely the second identity (*4.B.2*) we had to prove.

□

Lemma 4.B.2 (Rellich type inequalities). *For any $u\in H^2(\mathbb{R}^d)$ there holds*

$$\left(\frac{(d-4)d}{4}\right)^2\int_{\mathbb{R}^d}\frac{u^2}{|x|^4}dx\leq\int_{\mathbb{R}^d}|\Delta u|^2dx,\qquad \frac{d^2}{4}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^2}dx\leq\int_{\mathbb{R}^d}|\Delta u|^2dx. \qquad (4.B.3)$$

If $q\geq 0$ and $u:\mathbb{R}^d\to\mathbb{R}$ is a smooth function satisfying

$$\int_{\mathbb{R}^d}\left(\frac{|\Delta u|^2}{1+|x|^{2q}}+\frac{|\nabla u|^2}{1+|x|^{2q+2}}+\frac{u^2}{1+|x|^{2q+4}}\right)dx<+\infty.$$

then there holds:

$$C(d,q)\sum_{1\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}dx-C'(d,q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}dx\leq\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}dx. \qquad (4.B.4)$$

Proof of Lemma 4.B.2

(*4.B.3*) is a standard inequality and we omit its proof. To prove We prove (*4.B.4*) we reason with smooth and compactly supported functions, and then conclude by a density argument.

step 1 Control of the first derivatives. Making integration by parts we compute

$$\int_{\mathbb{R}^d}\frac{u\Delta u}{1+|x|^{2q+2}}dx=-\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}dx+\frac{1}{2}\int_{\mathbb{R}^d}u^2\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)dx$$

We then use Cauchy-Schwarz and Young's inequalities to obtain:

$$\begin{array}{rc} & C\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}dx-C'\int_{\mathbb{R}^d}u^2\left(\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)-\frac{1}{(1+|x|^{2q+2})(1+|x|)^2}\right)dx \\ \leq & \int_{\mathbb{R}^d}\frac{|\Delta u|^2}{(1+|x|^{2q+2})(1+|x|)^{-2}}dx \end{array}$$

It leads to the following estimate by noticing that $(1+|x|^{2q+2})(1+|x|)^{-1}\sim(1+|x|^{2q})$ and that $\left|\Delta\left(\frac{1}{1+|x|^{2q+2}}\right)-\frac{1}{(1+|x|^{2q+2})(1+|x|)^2}\right|\leq\frac{C}{1+|x|^{2q+4}}$:

$$C(d,p)\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}dx-C'(d,q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}dx\leq\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}dx \tag{4.B.5}$$

step 2 Control of the second order derivatives. Making again integrations by parts one finds:

$$\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}=\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}}+\sum_1^n\partial_{x_i}u\nabla\partial_{x_i}u.\nabla\left(\frac{1}{1+|x|^{2q}}\right)-\Delta u\nabla u.\nabla\left(\frac{1}{1+|x|^{2q}}\right)$$

in which by using Cauchy-Schwarz and Young's inequalities for any $\epsilon>0$ we can control the last two terms by:

$$\begin{array}{rc} & \left|\int_{\mathbb{R}^d}\sum_1^n\partial_{x_i}u\nabla\partial_{x_i}u.\nabla\left(\frac{1}{1+|x|^{2q}}\right)-\Delta u\nabla u.\nabla\left(\frac{1}{1+|x|^{2q}}\right)\right| \\ \leq & C\epsilon\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}}dx+\frac{C}{\epsilon}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^{2q+2}}dx. \end{array}$$

Therefore for ϵ small enough the two above identities yield:

$$\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^{2q}}dx\leq C\left(\int_{\mathbb{R}^d}\left(\frac{|\Delta u|^2}{1+|x|^{2q}}+\frac{|\nabla u|^2}{1+|x|^{2q+2}}+\frac{u^2}{1+|x|^{2q+4}}\right)dx\right)$$

Combining this identity and (*4.B.5*) one obtains the desired identity (*4.B.4*).

□

4.C Coercivity of the adapted norms

Here we prove coercivity estimates for the operator H under suitable orthogonality conditions, following the techniques of [138]. We recall that the profiles used as orthogonality directions, $\Phi_M^{(n,k)}$, are defined by (*4.4.1*). To perform an analysis on each spherical harmonics and to be able to track the constants, we will not study directly $A^{(n)}$ and $A^{(n)*}$, but the following asymptotically equivalent operators:

$$\tilde{A}^{(n)}:u\mapsto-\partial_y u+\tilde{W}^{(n)}u,\ \ A^{(n)*}:u\mapsto\frac{1}{y^{d-1}}\partial_y(y^{d-1}u)+\tilde{W}^{(n)}u \tag{4.C.1}$$

where:

$$\tilde{W}^{(n)}=-\frac{\gamma_n}{y}. \tag{4.C.2}$$

From the definition (*4.1.1*) of γ_n they factorize the following operator:

$$\tilde{H}^{(n)} := -\partial_{yy} - \frac{d-1}{y}\partial_y - \frac{pc_\infty^{p-1}}{y^2} + \frac{n(d+n-2)}{y^2} = \tilde{A}^{(n)*}\tilde{A}^{(n)}, \tag{4.C.3}$$

The strategy is the following. First we derive subcoercivity estimates for $\tilde{A}^{(n)*}$, $\tilde{A}^{(n)}$ and $H^{(n)}$. A summation yields subcoercivity for $-\Delta - \frac{pc_\infty^{p-1}}{|x|^2}$, and hence for H as they are asymptotically equivalent. Roughly, this subcoercivity implies that minimizing sequences of the functional $I(u) = \int uH^s u$ are "almost compact" on the unit ball of $\dot{H}^s \cap \left(\text{Span}(\Phi_M^{(n,k)})\right)^\perp$. In particular if the infimum of I on this set were 0 it would be attained, which is impossible from the orthogonality conditions, yielding the coercivity $\int uH^s u \gtrsim \|u\|_{\dot{H}^s}^2$ via homogeneity.

Lemma 4.C.1. *Let n be an integer, $q \geq 0$ and $u : [1, +\infty) \to \mathbb{R}$ be smooth satisfying:*

$$\int_1^{+\infty} \frac{|\partial_y u|^2}{y^{2q}} y^{d-1} dy + \int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1} dy < +\infty. \tag{4.C.4}$$

(i) There exist two constants $c, c' > 0$ independent of n and q such that:

$$c\int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1} dy - c'u^2(1) \leq \int_1^{+\infty} \frac{|\tilde{A}^{(n)*}u|^2}{y^{2q}} y^{d-1} dy. \tag{4.C.5}$$

(ii) Let $\delta > 0$ and suppose $|q - (\frac{d}{2} - 1 - \gamma_n)| > \delta$. Then there exist two constants $c(\delta), c'(\delta) > 0$ depending only on δ such that:

$$c(\delta)\int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1} dy - c'(\delta)u^2(1) \leq \int_1^{+\infty} \frac{|\tilde{A}^{(n)}u|^2}{y^{2q}} y^{d-1} dy. \tag{4.C.6}$$

Proof of Lemma 4.C.1

Coercivity for $\tilde{A}^{(n)*}$. We first compute:

$$\int_1^{+\infty} \frac{|\tilde{A}^{(n)*}u|^2}{y^{2q}} y^{d-1} dy = \int_1^{+\infty} \frac{|\partial_y u + y^{-1}(d-1-\gamma_n)u|^2}{y^{2q}} y^{d-1} dy.$$

We make the change of variable $u = vy^{\gamma_n+1-d}$. From (*4.C.4*), $\frac{v^2}{y^{2q-2\gamma_n+d+1}}$ and $\frac{|\partial_y v|^2}{y^{2q-2\gamma_n+d-1}}$ are integrable on $[1, +\infty)$. As $q + \frac{d}{2} - \gamma_n \geq \frac{d}{2} - \gamma > 1$ from (*2.2.5*) and (*4.1.1*), we can apply apply (*4.B.2*) to the above identity and obtain (*4.C.5*) via:

$$\begin{aligned}
\int_1^{+\infty} \frac{|\tilde{A}^{(n)*}u|^2}{y^{2q}} y^{d-1} dy &= \int_1^{+\infty} \frac{|\partial_y v|^2}{y^{2q-2\gamma_n+2d-2}} y^{d-1} dy \\
\geq\ C\int_1^{+\infty} \frac{v^2}{y^{2q-2\gamma_n+2d-2}} y^{d-1} dy - C'v^2(1) &= C\int_1^{+\infty} \frac{u^2}{y^{2q+2}} y^{d-1} dy - C'u^2(1).
\end{aligned}$$

Coercivity for $\tilde{A}^{(n)}$. This time the integral we have to estimate is:

$$\int_1^{+\infty} \frac{|\tilde{A}^{(n)}u|^2}{y^{2q}} y^{d-1} dy = \int_1^{+\infty} \frac{|\partial_y u + y^{-1}\gamma_n u|}{y^{2p}} y^{d-1} dy.$$

We make the change of variable $u = vy^{-\gamma_n}$. From (*4.C.4*), $\frac{v^2}{y^{2p+2\gamma_n-d+1}}$ and $\frac{|\partial_y v|^2}{y^{2p+2\gamma_n+3-d}}$ are integrable on $[1, +\infty)$. As $|q - (\frac{d}{2} - 1 - \gamma_n)| > \delta$ one can apply (*4.B.1*) or (*4.B.2*) to the above identity: there exists $c = c(\delta)$ and $c' = c'(\delta)$ such that:

$$\begin{aligned}
&\int_1^{+\infty} \tfrac{|\tilde{A}^{(n)}u|^2}{y^{2q}} y^{d-1} dy = \int_1^{+\infty} \tfrac{|\partial_y v|^2}{y^{2q+2\gamma_n}} y^{d-1} \geq c\int_1^{+\infty} \tfrac{v^2}{y^{2q+2\gamma_n+2}} y^{d-1} dy - c'v^2(1) \\
=\ &c\int_1^{+\infty} \tfrac{u^2}{y^{2q+2}} y^{d-1} dy - c'u^2(1).
\end{aligned}$$

which is precisely the identity (*4.C.6*).

$\square$

Lemma 4.C.2 (Coercivity of H under suitable orthogonality conditions). *Let $\delta > 0$ and $q \geq 0$ such that*[20] $|q - (\frac{d}{2} - 2 - \gamma_n)| \geq \delta$ *for all $n \in \mathbb{N}$. Let $n_0 \in \mathbb{N} \cup \{-1\}$ be the lowest number such that $q - (\frac{d}{2} - 2 - \gamma_{n_0+1}) < 0$. Then there exists a constant $c(\delta) > 0$ such that for all $u \in H^2_{loc}(\mathbb{R}^d)$ satisfying the integrability condition:*

$$\int_{\mathbb{R}^d} \frac{|\Delta u|^2}{1+|x|^{2q}} + \frac{|\nabla u|^2}{1+|x|^{2q+2}} + \int \frac{u^2}{1+|x|^{2q+4}} < +\infty$$

and the orthogonality conditions[21] *($\Phi_M^{(n,k)}$ being defined in (4.4.1)):*

$$\langle u, \Phi_M^{(n,k)} \rangle = 0 \ \text{ for } \ 0 \leq n \leq n_0, \ 1 \leq k \leq k(n), \tag{4.C.7}$$

one has the inequality:

$$c(\delta) \left(\int_{\mathbb{R}^d} \frac{|\Delta u|^2}{1+|x|^{2q}} + \frac{|\nabla u|^2}{|x|^2(1+|x|^{2q})} + \frac{u^2}{|x|^4(1+|x|^{2q})} \right) \leq \int_{\mathbb{R}} \frac{|Hu|^2}{1+|x|^{2q}}. \tag{4.C.8}$$

Proof of Lemma 4.C.2

In what follows, $C(\delta)$ and $C'(\delta)$ denote strictly positive constants that may vary but only depends on δ, d and p.

step 1 We claim the following subcoercivity estimate for $\tilde{H} := -\Delta - \frac{pc_\infty^{p-1}}{|x|^2}$:

$$\begin{array}{rcl} \int_{\mathbb{R}^d \setminus \mathcal{B}^d(1)} \frac{|\tilde{H}u|^2}{|x|^{2q}} dx & \geq & C(\delta) \int_{\mathbb{R}^d \setminus \mathcal{B}^d(1)} \frac{u^2}{|x|^{2q+4}} dx \\ & & -C'(\delta) \left(\|u_{|\mathbb{S}^{d-1}(1)}\|_{L^2}^2 + \|(\nabla u)_{|\mathbb{S}^{d-1}(1)}\|_{L^2}^2 \right) \end{array} \tag{4.C.9}$$

where $f_{|\mathbb{S}^{d-1}(1)}$ denotes the restriction of f to the sphere. We now prove this inequality. We start by decomposing $u(x) = \sum_{n, 1 \leq k \leq k(n)} u^{(n,k)}(|x|) Y^{(n,k)}\left(\frac{x}{|x|}\right)$. We recall the link between u and its decomposition ($\tilde{H}^{(n)}$ being defined by (*4.C.3*)):

$$\int_{\mathbb{R}^d \setminus \mathcal{B}^d(1)} \frac{|\tilde{H}u|^2}{|x|^{2q}} dx = \sum_{n, 1 \leq k \leq k(n)} \int_1^{+\infty} \frac{|\tilde{H}^{(n)} u^{(n,k)}|^2}{y^{2q}} y^{d-1} dy, \tag{4.C.10}$$

$$\int_{\mathbb{R}^d \setminus \mathcal{B}^d(1)} \frac{u^2}{|x|^{2q+4}} dx = \sum_{n, 1 \leq k \leq k(n)} \int_1^{+\infty} \frac{|u^{(n,k)}|^2}{y^{2q+4}} y^{d-1} dy. \tag{4.C.11}$$

As $\tilde{H}^{(n)} = \tilde{A}^{(n)*}\tilde{A}^{(n)}$ and $|q - (\frac{d}{2} - 2 - \gamma_n)| > \delta$ for all $n \in \mathbb{N}$, we apply (*4.C.5*) and (*4.C.6*) to obtain for each $n \in \mathbb{N}$:

$$\begin{array}{rcl} \int_1^{+\infty} \frac{|\tilde{H}^{(n)} u^{(n,k)}|^2}{y^{2q}} y^{d-1} dy & \geq & C(\delta) \int_1^{+\infty} \frac{|u^{(n,k)}|^2}{y^{2q+4}} y^{d-1} dy \\ & & -C'(\delta) \left((u^{(n,k)})^2(1) + \tilde{A}^{(n)} (u^{(n,k)})^2(1) \right). \end{array} \tag{4.C.12}$$

We now sum on n and k this identity. The second term in the right hand side is:

$$\sum_{n, 1 \leq k \leq k(n)} (u^{(n,k)})^2(1) = \int_{\mathbb{S}^{d-1}} \left(\sum_{n, 1 \leq k \leq k(n)} u^{(n,k)}(1) Y^{(n,k)}(x) \right)^2 dx = \int_{\mathbb{S}^{d-1}} u^2(x) dx$$

[20] We recall that $\gamma_n \to -\infty$, hence for δ small enough many qs satisfy this condition.
[21] With the convention that there is no orthogonality conditions required if $n_0 = -1$.

because $(Y^{(n,k)})_{n,1\leq k\leq n}$ is an orthonormal basis of $L^2(\mathbb{S}^{d-1})$. From (*4.C.1*), and as $\gamma_n\sim -n$ as $n\to+\infty$ from (*4.1.1*), the last term in the right hand side of (*4.C.12*) is

$$\begin{array}{rcl}\sum_{n,1\leq k\leq n}|\tilde{A}^{(n)}u^{(n,k)}|^2(1) & \leq & C\sum_{n,1\leq k\leq k(n)}(1+n^2)|u^{(n,k)}|^2(1)+|\partial_y u^{(n,k)}|^2\\ & \leq & C(\|u_{|\mathbb{S}^{d-1}(1)}\|^2_{H^1}+\|\nabla u_{|\mathbb{S}^{d-1}(1)}.\vec{n}\|^2_{L^2})\\ & \leq & C\left(\|u_{|\mathbb{S}^{d-1}}\|^2_{L^2}+\|\nabla u_{|\mathbb{S}^{d-1}(1)}\|^2_{L^2}\right)\end{array}$$

We inject the two above equations in (*4.C.12*) and obtain:

$$\begin{array}{rcl}\sum_{n,1\leq k\leq n}\int_1^{+\infty}\frac{|\tilde{H}^{(n)}u^{(n,k)}|^2}{y^{2q}}y^{d-1}dy & \geq & C(\delta)\sum_{n,1\leq k\leq n}\int_1^{+\infty}\frac{|u^{(n,k)}|^2}{y^{2q+4}}y^{d-1}dy\\ & & -C'(\delta)\left(\|u_{|\mathbb{S}^{d-1}}\|^2_{L^2}+\|\nabla u_{|\mathbb{S}^{d-1}(1)}\|^2_{L^2}\right).\end{array}$$

In turn, we inject this identity in (*4.C.10*) using (*4.C.11*) to obtain the desired estimate (*4.C.9*).

step 2 Subcoercivity for H. We claim the following estimate:

$$\begin{array}{rcl}\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2q}}dx & \geq & C(\delta)\left(\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}dx+\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^2(1+|x|^{2q})}dx+\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}dx\right)\\ & & -C'(\delta)\Big(\|u_{|S^{d-1}(1)}\|^2_{L^2}+\|(\nabla u)_{|S^{d-1}(1)}\|^2_{L^2}\\ & & +\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+\alpha}}+\|u\|^2_{H^1(\mathcal{B}^{d-1}(1))}\Big),\end{array}\quad (4.C.13)$$

which we now prove. Away from the origin, Cauchy-Schwarz and Young's inequalities, the bound $V+pc_\infty^{p-1}|x|^{-2}=O(|x|^{-2-\alpha})$ from (*3.2.10*) and (*4.C.9*) give (for $C>0$):

$$\begin{array}{rcl}\int_{\mathbb{R}^d\backslash\mathcal{B}^d(1)}\frac{|Hu|^2}{|x|^{2q}}dx & = & \int_{\mathbb{R}^d\backslash\mathcal{B}^d(1)}\frac{|\tilde{H}u+(V+pc_\infty^{p-1}|x|^{-2})u|^2}{|x|^{2q}}dx\\ & \geq & C\int_{\mathbb{R}^d\backslash\mathcal{B}^d(1)}\frac{|\tilde{H}u|^2}{|x|^{2q}}dx-C'\int_{\mathbb{R}^d\backslash\mathcal{B}^d(1)}\frac{|u|^2}{|x|^{2q+4+2\alpha}}dx\\ & \geq & C(\delta)\int_{\mathbb{R}^d\backslash\mathcal{B}^d(1)}\frac{u^2}{1+|x|^{2q+4}}-C'(\delta)\Big(\|u_{|\mathbb{S}^{d-1}(1)}\|^2_{L^2}\\ & & +\|(\nabla u)_{|\mathbb{S}^{d-1}(1)}\|^2_{L^2}+\int_{\mathbb{R}^d\backslash\mathcal{B}^d(1)}\frac{|u|^2}{1+|x|^{2q+4+2\alpha}}\Big)\end{array}$$

Close to the origin, using Rellich's inequality (*4.B.3*):

$$\begin{array}{rcl}\int_{\mathcal{B}^d(1)}|Hu|^2dx & \geq & C\int_{\mathcal{B}^d(1)}|\Delta u|^2dx-\frac{1}{C}\int_{\mathcal{B}^d(1)}|u|^2dx\\ & \geq & C\int_{\mathcal{B}^d(1)}\frac{|u|^2}{|x|^4}dx-\frac{1}{C}\|u\|_{H^1(\mathcal{B}^{d-1}(1))}.\end{array}$$

Combining the two previous estimates we obtain the intermediate identity:

$$\begin{array}{rcl}\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2q}}dx & \geq & C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}dx-C'(\delta)\Big(\|u_{|\mathbb{S}^{d-1}(1)}\|^2_{L^2}\\ & & +\|(\nabla u)_{|S^{d-1}(1)}\|^2_{L^2}+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}dx+\|u\|^2_{H^1(\mathcal{B}^{d-1}(1))}\Big).\end{array}$$

Now, as $H=-\Delta+V$ with $V=O((1+|x|)^{-2})$, using Young's inequality, the above identity and (*4.B.4*), for $\epsilon>0$ small enough (depending on δ) one has:

$$\begin{array}{rl} & \int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}dx=(1-\epsilon)\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}dx|Hu|^2dx+\epsilon\int_{\mathbb{R}^d}\frac{|Hu|^2}{1+|x|^{2p}}dx\\ \geq & (1-\epsilon)C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}dx-C'(\delta)\Big(\|u_{|\mathbb{S}^{d-1}(1)}\|^2_{L^2}+\|(\nabla u)_{|S^{d-1}(1)}\|^2_{L^2}\\ & +\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\Big)+\frac{\epsilon}{2}\int_{\mathbb{R}^d}\frac{|\Delta u|^2}{1+|x|^{2q}}dx-\epsilon\int_{\mathbb{R}^d}\frac{|Vu|^2}{1+|x|^{2q}}dx\\ \geq & (1-\epsilon)C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}dx-C'(\delta)\Big(\|u_{|\mathbb{S}^{d-1}(1)}\|^2_{L^2}+\|(\nabla u)_{|S^{d-1}(1)}\|^2_{L^2}\\ & +\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\Big)+C(q)\frac{\epsilon}{2}\sum_{1\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}dx\\ & -\epsilon C'(q)\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4}}dx\\ \geq & C(\delta)\int_{\mathbb{R}^d}\frac{u^2}{|x|^4(1+|x|^{2q})}+\frac{C(q)\epsilon}{2}\sum_{1\leq|\mu|\leq 2}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{2q+4-2\mu}}-C'(\delta)\Big(\|u_{|\mathbb{S}^{d-1}(1)}\|^2_{L^2}\\ & +\|(\nabla u)_{|S^{d-1}(1)}\|^2_{L^2}+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^{2q+4+2\alpha}}dx+\|u\|_{H^1(\mathcal{B}^{d-1}(1))}\Big)\end{array}$$

which is the identity (*4.C.13*) we claimed.

step 3 Coercivity for H. We now argue by contradiction. Suppose that (*4.C.8*) does not hold. Up to a renormalization, this means that there exists a sequence of functions $(u_n)_{n\in\mathbb{N}}$ such that:

$$\int_{\mathbb{R}} \frac{|Hu_n|^2}{1+|x|^{2q}} \to 0, \quad \int_{\mathbb{R}^d} \frac{|\Delta u_n|^2}{1+|x|^{2q}} + \frac{|\nabla u_n|^2}{|x|^2(1+|x|^{2q})} + \frac{|u_n|^2}{|x|^4(1+|x|^{2q})} = 1 \ \forall n. \tag{4.C.14}$$

Up to a subsequence, we can suppose that $u_n \to u_\infty \in H^2_{\text{loc}}(\mathbb{R}^d)$, the local convergence in L^2 being strong for $(u_n)_{n\in\mathbb{N}}$ and $(\nabla u_n)_{n\in\mathbb{N}}$, and weak for $(\nabla^2 u_n)_{n\in\mathbb{N}}$. (*4.C.14*) then implies:

$$\|u_n\|^2_{H^1(\mathcal{B}^{d-1}(1))} + \int_{\mathbb{R}^d} \frac{|u_n|^2}{1+|x|^{2q+4+\alpha}} \to \|u_\infty\|^2_{H^1(\mathcal{B}^{d-1}(1))} + \int_{\mathbb{R}^d} \frac{|u_\infty|^2}{1+|x|^{2q+4+\alpha}}.$$

u_n converges strongly to u_∞ in $H^s(\mathcal{B}^d(0,1))$ for any $0\le s<2$. The trace theorem for Sobolev spaces ensures that:

$$\|(u_n)_{|S^{d-1}(1)}\|^2_{L^2} + \|(\nabla u_n)_{|S^{d-1}(1)}\|^2_{L^2} \to \|(u_\infty)_{|S^{d-1}(1)}\|^2_{L^2} + \|(\nabla u_\infty)_{|S^{d-1}(1)}\|^2_{L^2}.$$

We inject the three previous identities in the subcoercivity estimate (*4.C.13*) yielding:

$$\|(u_\infty)_{|S^{d-1}(1)}\|^2_{L^2} + \|(\nabla u_\infty)_{|S^{d-1}(1)}\|^2_{L^2} + \int_{\mathbb{R}^d} \frac{|u_\infty|^2}{1+|x|^{2q+4+\alpha}} + \|u_\infty\|^2_{H^1(\mathcal{B}^d(1))} \neq 0$$

which means that $u_\infty \neq 0$. On the other hand the lower semicontinuity of norms for the weak topology and (*4.C.14*) imply:

$$Hu_\infty = 0.$$

Hence u_∞ is a non trivial function in the kernel of H, hence smooth from elliptic regularity. It satisfies the integrability condition (still from lower semicontinuity):

$$\int_{\mathbb{R}^d} \frac{|\Delta u_\infty|^2}{1+|x|^{2q}} dx + \frac{|\nabla u_\infty|^2}{1+|x|^{2q+2}} dx + \int \frac{|u_\infty|^2}{1+|x|^{2q+4}} dx < +\infty.$$

We now decompose u_∞ in spherical harmonics: $u_\infty = \sum_{n,1\le k\le k(n)} u_\infty^{(n,k)} Y_{(n,k)}$ and will show that for each n,k one must have $u_\infty^{(n,k)} = 0$ which will give a contradiction. For each n,k the nullity $Hu_\infty = 0$ implies $H^{(n)} u_\infty^{(n,k)}$ where $H^{(n)}$ is defined in (*4.1.20*). From Lemma 4.2.1 this means $u_\infty = aT_0^{(n)} + b\Gamma^{(n)}$ for a and b two real numbers. The previous equation implies the following integrability for $u_\infty^{(n,k)}$:

$$\int \frac{|u_\infty^{(n,k)}|^2}{1+y^{2q+4}} y^{d-1} dy < +\infty.$$

From (*4.2.2*), as $\Gamma^{(n)} \sim y^{-d-n+2}$ does not satisfy this integrability at the origin whereas $T_0^{(n)}$ is regular, one must have $b=0$. Then, if $n\ge n_0+1$, $\frac{|T_0^{(n)}|^2}{1+y^{2q+4}} y^{d-1} \sim y^{-2\gamma_n-2q-5+d}$. From the assumption on n_0 and (*4.1.1*), one has:

$$-2\gamma_n - 2q - 5 + d = -1 - 2(q+2+\gamma_{n_0+1} - \frac{d}{2}) + 2(\gamma_{n_0+1} - \gamma_n) > -1$$

implying that $\frac{|T_0^{(n)}|^2}{1+y^{2q+4}} y^{d-1}$ is not integrable on $[0,+\infty)$, hence $a=0$. If $n\le n_0$ then the orthogonality condition (*4.C.7*) goes to the limit as $\Phi_M^{(n,k)}$ is compactly supported and implies:

$$\langle u^\infty, \Phi_M^{(n,k)} \rangle = 0$$

which, in spherical harmonics, can be rewritten as:

$$0=\langle u_\infty^{(n,k)},\Phi_M^{(n,k)}\rangle=a\langle T_0^{(n)},\Phi_M^{(n,k)}\rangle.$$

However, from (*4.4.3*) this in turn implies $a=0$. We have proven that for all n,k $u_\infty^{(n,k)}=0$, hence $u_\infty=0$ which is the desired contradiction as we proved earlier that u_∞ is non trivial. The coercivity (*4.C.8*) must then be true.

□

If one adds analogous orthogonality conditions for the derivatives of u and uses a bit more the structure of the Laplacian, one gets that the weighted norm $\|\frac{H^i}{1+|x|^p}u\|_{L^2}$ controls all derivatives of lower order with corresponding weights.

Lemma 4.C.3 (Coercivity of the iterates of H). *Let i be an integer with $2i>\sigma$, such that for all $n\in\mathbb{N}$ satisfying $m_n+\delta_n\leq i$ one has $\delta_n\neq 0$. Let n_0 be the lowest integer such that $m_{n_0+1}+\delta_{n_0+1}>i$. Let $u\in\dot H^{2i}\cap\dot H^\sigma(\mathbb{R}^d)$ satisfy (where $\Phi_M^{(n,,k)}$ is defined in (4.4.1))*

$$\langle u,H^j\Phi_M^{n,k}\rangle=0 \ \textit{for}\ 0\leq n\leq n_0,\ 0\leq j\leq i-m_n-1,\ 1\leq k\leq k(n). \tag{4.C.15}$$

Then there exists a constant $\delta>0$ such that for all $0\leq\delta'\leq\delta$ there holds:

$$C(\delta,i)\sum_{|\mu|\leq 2i}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{4i-2\mu+2\delta'}}dx\leq\int_{\mathbb{R}^d}\frac{|H^iu|^2}{1+|x|^{2\delta'}}dx \tag{4.C.16}$$

which in particular implies that:

$$\|u\|_{\dot H^{2i}}\leq C(\delta,i)\left(\int_{\mathbb{R}}|H^iu|^2dx\right)^{\frac12} \tag{4.C.17}$$

Proof of Lemma 4.C.3

step 1 Equivalence of weighted norms. We claim that for all integer j there holds:

$$H^ju=(-\Delta)^ju+\sum_{|\mu|\leq 2j-2}f_{j,\mu}\partial^\mu u \tag{4.C.18}$$

for some smooth functions f_μ having the decay $|\partial^{\mu'}f_{j,\mu}|\leq C(1+|x|^{2j-|\mu|+|\mu'|})^{-1}$. This identity is true for $j=1$ because $Hu=-\Delta u+Vu$ with the potential V being smooth and having the required decay from (*3.2.10*). If the aforementioned identity holds true for $j\geq 1$ then:

$$\begin{aligned}H^{j+1}u&=(-\Delta+V)\left((-\Delta)^ju+\textstyle\sum_{|\mu|\leq 2j-2}f_{j,\mu}\partial^\mu u\right)\\&=(-\Delta)^{j+1}u+V(-\Delta)^ju+\textstyle\sum_{|\mu|\leq 2j-2}(-\Delta+V)(f_{j,\mu}\partial^\mu u)\end{aligned}$$

and hence it is true for $j+1$ since V is smooth and satisfies the decay (*3.2.10*). By induction it is true for all $j\in\mathbb{N}$ and (*4.C.18*) is proven. (*4.C.18*) then implies that:

$$\int_{\mathbb{R}^d}\frac{|H^iu|^2}{1+|x|^{2\delta}}dx\leq C\sum_{|\mu|\leq 2i}\int_{\mathbb{R}^d}\frac{|\partial^\mu u|^2}{1+|x|^{4i-2|\mu|+2\delta'}}dx \tag{4.C.19}$$

step 2 Weighted integrability in $\dot H^{2i}\cap \dot H^{\sigma}$. We claim that for all functions $u\in \dot H^{2i}\cap \dot H^{\sigma}(\mathbb{R}^d)$ and $\delta'>0$ there holds:

$$\sum_{|\mu|\leq 2i}\int_{\mathbb{R}^d}\frac{|\partial^{\mu}u|^2}{1+|x|^{4i-2|\mu|+2\delta'}}dx<+\infty. \tag{4.C.20}$$

Indeed, let μ be a $|\mu|$-tuple with $|\mu|\leq 2i$. We split in two cases. First if $|\mu|\leq \sigma$, as $\sigma<\frac{d}{2}$ and $2i>\sigma$ the Hardy inequality (3.C.2) yields:

$$\int_{\mathbb{R}^d}\frac{|\partial^{\mu}u|^2}{1+|x|^{4i-2|\mu|+2\delta'}}dx\leq \int_{\mathbb{R}^d}\frac{|\partial^{\mu}u|^2}{1+|x|^{2(\sigma-|\mu|)}}dx\leq C\|u\|^2_{\dot H^{\sigma}}<+\infty$$

and we are done. If $\sigma<\mu\leq 2i$ then by interpolation $u\in \dot H^{|\mu|}(\mathbb{R}^d)$ and then:

$$\int_{\mathbb{R}^d}\frac{|\partial^{\mu}u|^2}{1+|x|^{4i-2|\mu|+2\delta'}}dx\leq \int|\partial^{\mu}u|^2dx<+\infty.$$

Thus (*4.C.20*) holds, which together with (*4.C.19*) implies for all $\delta'\geq 0$:

$$\sum_{j=0}^{i}\int_{\mathbb{R}^d}\frac{|H^ju|^2}{1+|x|^{4i-4j+2\delta'}}dx+\frac{|\nabla H^{j-1}u|^2}{1+|x|^{4i-4j+2+2\delta'}}dx<+\infty \tag{4.C.21}$$

step 3 Intermediate coercivity. Let $\delta=\min(\delta_0,...,\delta_{n_0+1},\frac12)$ if $\delta_{n_0+1}\neq 0$ and $\delta=\min(\delta_0,...,\delta_{n_0},\frac12)$ if $\delta_{n_0+1}=0$. The conditions on the δ_n of the lemma implies $\delta>0$. We now claim that for all integer $1\leq l\leq i$ there holds:

$$C(\delta)\int_{\mathbb{R}^d}\frac{|H^{l-1}u|^2}{1+|x|^{4i-4(l-1)+2\delta'}}+C(\delta)\int_{\mathbb{R}^d}\frac{|\nabla H^{l-1}u|^2}{1+|x|^{4i-4l+2+2\delta'}}\leq\int_{\mathbb{R}^d}\frac{|H^lu|^2}{1+|x|^{4i-4l+2\delta'}}. \tag{4.C.22}$$

We now prove this estimate. We want to apply Lemma 4.C.2 to the function $H^{l-1}u$ with weight $q=\delta'+2(i-l)$. To use it, we have to check the orthogonality and integrability conditions that are required, and the conditions on the weight.

Integrability condition. It is true because of (*4.C.21*).

Condition on the weight. For the case $n\geq n_0+1$ one computes from (*4.1.6*):

$$\begin{array}{rl} & |\delta'+2(i-l)-(\frac{d}{2}-\gamma_n-2)| \\ = & |\delta'-2\delta_{n_0+1}-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1})|. \end{array} \tag{4.C.23}$$

One has $2(l-1)\geq 0$ as $l\geq 1$ and $2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1})\geq 0$ because $(m_n+\delta_n)_n$ is an increasing sequence from (*4.1.5*) and (*4.1.1*). For the subcase $\delta_{n_0+1}=0$, then as $m_{n_0+1}>i$ and m_{n_0+1} is an integer, $2(m_{n_0+1}-i)>2$. Therefore $-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1})=-a$ for $a\geq 2$, and injecting it in the above identity as $0<\delta'<1$ gives:

$$|\delta'+2(i-l)-(\frac{d}{2}-\gamma_n-2)|=|\delta'-a|\geq\delta'\geq\delta.$$

For the subcase $\delta_{n_0+1}\neq 0$, then $\delta'-2\delta_{n_0+1}\leq \delta-2\delta_{n_0+1}\leq -\delta_{n_0+1}\leq -\delta$. Moreover, $m_{n_0+1}\geq i$ and $-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1})\leq 0$, implying:

$$\delta'-2\delta_{n_0+1}-2(m_{n_0+1}-i)-2(l-1)-2(m_n+\delta_n-m_{n_0+1}-\delta_{n_0+1})\leq\delta'-2\delta_{n_0+1}\leq-\delta$$

and therefore from (*4.C.23*) this yields in that case:

$$|\delta' + 2(i-l) - (\frac{d}{2} - \gamma_n - 2)| \geq \delta.$$

In both subcases one has: $|\delta' + 2(i-l) - (\frac{d}{2} - \gamma_n - 2)| \geq \delta$. For the case $n \leq n_0$:

$$|\delta' + 2(i-l) - (\frac{d}{2} - \gamma_n - 2)| = |\delta' - 2\delta_n + 2(i-l+1-m_n)|.$$

In the above identity, $2(i-l+1-m_n)$ is an even integer, and $\delta'-2\delta_n$ is a number satisfying $\delta'-2\delta_n \leq \delta - 2\delta_n \leq -\delta$ and we recall that $\delta < 1$, and $\delta'-2\delta_n \geq -2\delta_n \geq -1$. Therefore $|\delta'-2\delta_n+2(i-l+1-m_n)| \geq \delta$, yielding:

$$|\delta' + 2(i-l) - (\frac{d}{2} - \gamma_n - 2)| \geq \delta.$$

Therefore, for each $n \in \mathbb{N}$, $|\delta' + 2(i-l) - (\frac{d}{2} - \gamma_n - 2)| \geq \delta$.
Orthogonality conditions. Let $n_0' = n_0'(l) \in \mathbb{N} \cup \{-1\}$ be the lowest number such that $2(i-l+1) + \delta' - 2(m_{n_0'+1} + \delta_{n_0'+1}) < 0$. By construction one has $n_0' \leq n_0$. If $n_0' = -1$ then we are done because no orthogonality condition is required. If $n_0' \neq -1$, let n be an integer, $0 \leq n \leq n_0'$. By definition of n_0' it means:

$$2(i-l+1) + \delta' - 2(m_n + \delta_n) > 0$$

which implies $0 \leq l-1 \leq i - m_n - 1$ as $\delta' - 2\delta_n \leq \delta - 2\delta_n \leq -\delta_n \leq 0$. The orthogonality conditions (*4.C.15*) then gives for any $1 \leq k \leq k(n)$:

$$\langle u, H^{l-1}\Phi_M^{(n,k)} \rangle = 0.$$

We have then proved that for all $0 \leq n \leq n_0'$, $1 \leq k \leq k(n)$ there holds:

$$\langle H^{l-1}u, \Phi_M^{(n,)k} \rangle = 0$$

which are the required orthogonality conditions.
Conclusion. One can apply Lemma 4.C.2 to $H^{l-1}u$ with weight $q = 2i - 2l + \delta'$, giving the desired coercivity estimate (*4.C.22*).
step 4 Iterations of coercivity estimates. We show the following bound by induction on $l = 0, ..., i$:

$$\int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{2\delta'}} dx \geq c(\delta, i) \sum_{0 \leq |\mu| \leq 2l} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{4i-2\mu+2\delta'}} dx. \qquad (4.C.24)$$

This property is naturally true for $l = 0$. We now suppose it is true for $l-1$ with $0 \leq l-1 \leq i-1$. From the formula (*4.C.18*) relating Δ^l to H^l we see that (using Cauchy-Schwarz and Young's inequalities):

$$\begin{array}{rcl} \int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{4(i-l)+2\delta'}} & \geq & C(i) \int_{\mathbb{R}^d} \frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} - C'(i) \sum_{0 \leq |\mu| \leq 2l-2} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{4i-2|\mu|+2\delta'}} \\ & \geq & C(i) \int_{\mathbb{R}^d} \frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} - C'(i) \int_{\mathbb{R}^d} \frac{|H^i u|^2}{1+|x|^{2\delta'}} \end{array}$$

where we used the induction hypothesis (*4.C.24*) for $l-1$ for the second line. We now use (*4.C.24*) and (*4.B.4*) to recover a control over all derivatives:

$$\begin{aligned}
&\int_{\mathbb{R}^d} \frac{|\Delta^l u|^2}{1+|x|^{4(i-l)+2\delta'}} \\
\geq\ & C(i) \sum_{1\leq|\mu|\leq 2} \int_{\mathbb{R}^d} \frac{|\partial^\mu \Delta^{l-1} u|^2}{1+|x|^{4(i-l)+4-2|\mu|}} - C'(i)\int_{\mathbb{R}^d} \frac{|\Delta^{l-1} u|^2}{1+|x|^{4(i-l)+4}} \\
\geq\ & C(i) \sum_{0\leq|\mu|\leq 2} \int_{\mathbb{R}^d} \frac{|\Delta^{l-1}\partial^\mu u|^2}{1+|x|^{4(i-(l-1))-2|\mu|}} - C'(\delta,i)\int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}} \\
\geq\ & C(i) \sum_{0\leq|\mu|\leq 2}\sum_{1\leq|\mu'|\leq 2} \int_{\mathbb{R}^d} \frac{|\partial^{\mu'}\Delta^{l-2}\partial^\mu u|^2}{1+|x|^{4(i-(l-1))+4-2|\mu|-2|\mu'|}} - C'(i)\int_{\mathbb{R}^d} \frac{|\Delta^{l-2} u|^2}{1+|x|^{4(i-l)+8}} \\
&-C'(\delta,i)\int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}} \\
\geq\ & C(i) \sum_{0\leq|\mu|\leq 4} \int_{\mathbb{R}^d} \frac{|\Delta^{l-2}\partial^\mu u|^2}{1+|x|^{2p+4(i-(l-2))-2\mu}} - C'(i,\delta)\int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}} \\
\geq\ & ... \\
\geq\ & C(i) \sum_{0\leq|\mu|\leq 2l} \int_{\mathbb{R}^d} \frac{|\partial^\mu u|^2}{1+|x|^{2p+4-2\mu+2\delta'}} - C'(\delta,i)\int_{\mathbb{R}^d} \frac{|H^{l-1} u|^2}{1+|x|^{2\delta'}}.
\end{aligned}$$

Injecting this last equation in the previous one we obtain:

$$\int_{\mathbb{R}^d} \frac{|H^l u|^2}{1+|x|^{4(i-l)+2\delta'}} \geq C(\delta,i)\sum_{0\leq|\mu|\leq 2l}\int_{\mathbb{R}^d} \frac{|\Delta^{l-2}\partial^\mu u|^2}{1+|x|^{2p+4-2\mu}} - C'(\delta,i)\int_{\mathbb{R}} \frac{|H^{l-1}u|^2}{1+|x|^{2\delta'}}.$$

This, together with (*4.C.22*), gives that (*4.C.24*) is true for l. Hence by induction it is true for i, which is precisely the estimate (*4.C.16*) we had to show and end the proof of the lemma.

□

4.D Specific bounds for the analysis

This section is dedicated to the statement and the proof of several estimates used in the analysis.

Lemma 4.D.1 (Specific bounds for the error in the trapped regime). *Let ε satisfy (4.4.25) and (4.4.11). We recall that $\mathcal{E}_\sigma$ and $\mathcal{E}_{2s_L}$ are defined by (4.4.9) and (4.4.7). Then the following bounds hold:*

(i) Interpolated Hardy type inequality: *For $\mu\in\mathbb{N}^d$ and $q>0$ satisfying $\sigma\leq|\mu|+q\leq 2s_L$ there holds:*

$$\int \frac{|\partial^\mu \varepsilon|^2}{1+|y|^{2q}}dy \leq C(M)\mathcal{E}_\sigma^{\frac{2s_L-(|\mu|+q)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{|\mu|+q-\sigma}{2s_L-\sigma}}, \qquad (4.D.1)$$

(ii) Weighted L^∞ bound for low order derivative: *for $0\leq a\leq 2$ and $\mu\in\mathbb{N}^d$ with $|\mu|\leq 1$ there holds*

$$\left\|\frac{\partial^\mu \epsilon}{1+|y|^a}\right\|_{L^\infty} \leq C(K_1,K_2,M)\sqrt{\mathcal{E}_\sigma}^{1+O\left(\frac{1}{L^2}\right)}\frac{1}{s^{a+|\mu|_1+\left(\frac{d}{2}-\sigma\right)+\frac{\left(\frac{2}{p-1}+a+|\mu|_1\right)\alpha}{L}+O\left(\frac{\sigma-s_c}{L}\right)}}. \qquad (4.D.2)$$

(iii) L^∞ bound for high order derivative: *for* $\mu\in\mathbb{N}^d$ *with* $|\mu|\leq s_L$ *there holds:*

$$\|\partial^\mu \varepsilon\|_{L^\infty}^2\leq C(M)\mathcal{E}_\sigma^{\frac{2s_L-|\mu|_1-\frac d2}{2s_L-\sigma}+O\left(\frac{1}{L^2}\right)}\mathcal{E}_{2s_L}^{\frac{|\mu|_1+\frac d2-\sigma}{2s_L-\sigma}+O\left(\frac{1}{L^2}\right)}. \tag{4.D.3}$$

Proof of Lemma 4.D.1

Proof of (i) We first recall that from the coercivity estimate (*4.C.16*) one has:

$$\|\nabla^\sigma\varepsilon\|_{L^2}^2=\mathcal{E}_\sigma,\quad \|\nabla^{2s_L}\varepsilon\|_{L^2}^2\leq C(M)\|H^{s_L}\varepsilon\|_{L^2}^2=C(M)\mathcal{E}_{2s_L}.$$

If the weight satisfies $q<\frac d2$, then the inequality (*4.D.1*) claimed in the lemma is a consequence of the standard Hardy inequality, followed by an interpolation:

$$\begin{array}{rcl}\|\frac{\partial^\mu\varepsilon}{1+|x|^q}\|_{L^2}^2 & \leq & C\|\nabla^{|\mu|_1+q}\varepsilon\|_{L^2}^2\leq C\|\nabla^\sigma\varepsilon\|_{L^2}^{2\frac{2s_L-(|\mu|_1+q)}{2s_L-\sigma}}\|\nabla^{2s_L}\varepsilon\|_{L^2}^{2\frac{|\mu|_1+q-\sigma}{2s_L-\sigma}}\\ & \leq & C(M)\mathcal{E}_\sigma^{\frac{2s_L-(|\mu|_1+q)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{|\mu|_1+q-\sigma}{2s_L-\sigma}}.\end{array}$$

If the potential satisfies $q=2s_L-|\mu|$, then the inequality (*4.D.1*) claimed in the lemma is a consequence of the coercivity estimate (*4.C.16*):

$$\left\|\frac{\partial^\mu\varepsilon}{1+|x|^q}\right\|_{L^2}^2\leq C(M)\mathcal{E}_{2s_L}.$$

For a weight that is in between, ie $\frac d2\leq q<2s_L-|\mu|_1$, the inequality (*4.D.1*) is then obtained by interpolating the two previous ones, as:

$$\frac{|\varepsilon|^2}{1+|x|^{2b}}\sim\left(\frac{|\varepsilon|^2}{1+|x|^{2a}}\right)^{\frac{c-b}{c-a}}\left(\frac{|\varepsilon|^2}{1+|x|^{2c}}\right)^{\frac{b-a}{c-a}}.$$

Proof of (ii). As the dimension is $d\geq 11$ and $L\gg 1$ is big, one has $\frac{\partial^\mu\varepsilon}{1+|x|^a}\in L^\infty$ with the following bound (using the bound (i) we just derived):

$$\begin{array}{rcl}\|\frac{\partial^\mu\varepsilon}{1+|x|^a}\|_{L^\infty} & \leq & C(z)(\|\nabla^{\frac d2-z}(\frac{\partial^\mu\varepsilon}{1+|x|^a})\|_{L^2}+\|\nabla^{\frac d2+z}(\frac{\partial^\mu\varepsilon}{1+|x|^a})\|_{L^2})\\ & \leq & C(z)(\|\nabla^{\frac d2-z+a+|\mu|_1}\varepsilon\|_{L^2}+\|\nabla^{\frac d2+a+|\mu|_1+z}\varepsilon\|_{L^2})\\ & \leq & C(M,z)\Big(\mathcal{E}_\sigma^{\frac{2s_L-(a+|\mu|_1+\frac d2-z)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{a+|\mu|_1+\frac d2-z-\sigma}{2s_L-\sigma}}\\ & & +\mathcal{E}_\sigma^{\frac{2s_L-(a+|\mu|_1+\frac d2+z)}{2s_L-\sigma}}\mathcal{E}_{2s_L}^{\frac{a+|\mu|_1+\frac d2+z-\sigma}{2s_L-\sigma}}\Big).\end{array}$$

for $z>0$ small enough. We then let z_1 be so close to 0 (of order L^{-1}) that its impact when using the bootstrap bounds (*4.4.25*) is of order $s^{-\frac{1}{L^2}}$ (the constant $C(M,z_1)$ exploding as z_1 approches 0 we cannot take $z_1=0$ but z_1 very close to $\frac d2$ is enough for our purpose). Injecting the bootstrap bounds (*4.4.25*) then yields the desired result (*4.D.2*).

Proof of (iii). It can be proved verbatim the same way we did for (ii).

□

Lemma 4.D.2 (A nonlinear estimate). *Let $d \in \mathbb{N}$, $a \geq 0$ and $b > \frac{d}{2}$. Let $\Omega \subset \mathbb{R}^d$ be a smooth bounded domain. There exists a constant $C > 0$ such that for any $u, v \in H^{max(a,b)}(\Omega)$ there holds*[22]*:*

$$\|uv\|_{H^a(\Omega)} \leq C \left(\|u\|_{H^a(\Omega)} \|v\|_{H^b(\Omega)} + \|u\|_{H^b(\Omega)} \|v\|_{H^a(\Omega)} \right). \tag{4.D.4}$$

Proof of Lemma 4.D.2

Without loss of generality one assumes $\frac{d}{2} < b \leq \frac{d}{2} + \frac{1}{4}$:

$$b := \frac{d}{2} + \delta_b, \quad \text{with } 0 < \delta_b \leq \frac{1}{4}. \tag{4.D.5}$$

Indeed, if (*4.D.4*) holds for all $b \in (\frac{d}{2}, \frac{d}{2} + \frac{1}{4}]$ then for any $b' > \frac{d}{2} + \frac{1}{4}$, applying (*4.D.4*) for the couple of parameters $(a, \frac{d}{2} + \frac{1}{4})$ and using the fact that $\|f\|_{H^{\frac{d}{2}+\frac{1}{4}}(\Omega)} \leq \|f\|_{H^b(\Omega)}$ for any $f \in H^b(\Omega)$ gives that (*4.D.4*) holds for the couple of parameters (a, b').

step 1 A scalar inequality. We claim that for all $(\nu_1, \nu_2) \in [0,1]^2$ with $\nu_1 + \nu_2 \geq 1$ and for all $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $\lambda_1 \leq \lambda_2$ and $\lambda_3 \leq \lambda_4$ there holds:

$$\lambda_1^{\nu_1} \lambda_2^{1-\nu_1} \lambda_3^{\nu_2} \lambda_4^{1-\nu_2} \leq \lambda_1 \lambda_4 + \lambda_2 \lambda_3. \tag{4.D.6}$$

We now prove this estimate. Since $1 - \nu_1 - \nu_2 \leq 0$ and $0 \leq 1 - \nu_2 \leq 1$ one has:

$$\forall (x, z) \in [1, +\infty) \times [0, +\infty), \quad x^{1-\nu_1-\nu_2} z^{1-\nu_2} \leq z^{1-\nu_2} \leq 1 + z.$$

Let $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $0 < \lambda_1 \leq \lambda_2$ and $0 < \lambda_3 \leq \lambda_4$. We apply the above estimate to $x = \frac{\lambda_2}{\lambda_1} \geq 1$ and $z = \frac{\lambda_1 \lambda_4}{\lambda_2 \lambda_3}$, and multiply both sides by $\lambda_2 \lambda_3$, yielding the desired estimate (*4.D.6*) after simplifications. If $\lambda_1 = 0$ or $\lambda_3 = 0$, (*4.D.6*) always hold. Consequently, (*4.D.6*) holds for all $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \in [0, +\infty)$ satisfying $0 < \lambda_1 \leq \lambda_2$ and $0 < \lambda_3 \leq \lambda_4$.

step 2 Proof in the case $\Omega = \mathbb{R}^d$ and $a \geq b$. We claim that for $u, v \in H^a(\mathbb{R}^d)$:

$$\|uv\|_{H^a(\mathbb{R}^d)} \leq C \left(\|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)} + \|u\|_{H^b(\mathbb{R}^d)} \|v\|_{H^a(\mathbb{R}^d)} \right). \tag{4.D.7}$$

We now show the above estimate. Let $u, v \in H^{s_2}(\mathbb{R}^d)$. First, one obtain a L^2 bound using Hölder and Sobolev embedding (as $b > \frac{d}{2}$):

$$\|uv\|_{L^2(\mathbb{R}^d)} \leq \|u\|_{L^2(\mathbb{R}^d)} \|v\|_{L^\infty(\mathbb{R}^d)} \leq C \|u\|_{H^a(\mathbb{R}^d)} \|v\|_{H^b(\mathbb{R}^d)}. \tag{4.D.8}$$

Secondly, one decomposes $a = A + \delta_a$ where $A := E[a] \in \mathbb{N}$ is the entire part of a and $0 \leq \delta_a < 1$. Using Leibniz rule one has the identity:

$$\|\nabla^a (uv)\|^2_{L^2(\mathbb{R}^d)} \leq C \sum_{(\mu_1, \mu_2) \in \mathbb{N}^{2d}, \ |\mu_1| + |\mu_2| = A} \|\nabla^{\delta_a} (\partial^{\mu_1} u \partial^{\mu_2} v)\|^2_{L^2(\mathbb{R}^d)}. \tag{4.D.9}$$

We fix $(\mu_1, \mu_2) \in \mathbb{N}^{2d}$ with $|\mu_1| + |\mu_2| = A$ in the sum and aim at estimating the corresponding term. We recall the commutator estimate:

$$\|\nabla^{\delta_a} (\partial^{\mu_1} u \partial^{\mu_2} v)\|_{L^2} \lesssim \|\nabla^{|\mu_1| + \delta_a} u\|_{L^{p_1}} \|\partial^{\mu_2} v\|_{L^{q_1}} + \|\nabla^{|\mu_2| + \delta_a} v\|_{L^{p_2}} \|\partial^{\mu_1} u\|_{L^{q_2}}, \tag{4.D.10}$$

[22] The product uv indeed belongs to $H^a(\Omega)$ as $H^{\max(a,b)}(\Omega)$ is an algebra since $b > \frac{d}{2}$.

for $\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_1'}+\frac{1}{p_2'}=\frac12$, provided $2\leq p_1,p_2<+\infty$ and $2\leq q_1,q_2\leq +\infty$. We now chose appropriate exponents p_1 and p_2 in several cases.

- *Case 1* If $|\mu_2|=0$. Then $|\mu_1|+\delta_a=a$ and using Sobolev embedding (as $b>\frac d2$):

$$\|\nabla^{|\mu_1|+\delta_a}u\|_{L^2(\mathbb{R}^d)}\|\partial^{\mu_2}v\|_{L^\infty(\mathbb{R}^d)}\leq C\|u\|_{H^a(\mathbb{R}^d)}\|v\|_{H^b(\mathbb{R}^d)}. \tag{4.D.11}$$

- *Case 2* If $1\leq |\mu_2|<a-\frac d2$ and $|\mu_1|+\delta_a<b$. Then $b<|\mu_2|+\frac d2<a$ from (*4.D.5*) and one computes using Sobolev embedding:

$$\|\nabla^{|\mu_1|+\delta_a}u\|_{L^2(\mathbb{R}^d)}\|\partial^{\mu_2}v\|_{L^\infty(\mathbb{R}^d)}\leq C\|u\|_{H^b(\mathbb{R}^d)}\|v\|_{H^a(\mathbb{R}^d)}. \tag{4.D.12}$$

- *Case 3* If $1\leq |\mu_2|<a-\frac d2$ and $b\leq |\mu_1|+\delta_a$. Then $b<|\mu_2|+\frac d2<a$ from (*4.D.5*) and $b\leq |\mu_1|+\delta_a\leq a$. We let $x:=\min(\frac{\delta_b}{2},a-|\mu_2|-\frac d2)>0$. One computes using Sobolev embedding, interpolation and (*4.D.6*) (since $b>\frac d2+x$ and $|\mu_1|+|\mu_2|+\delta_a=a$):

$$\begin{aligned}\|\nabla^{|\mu_1|+\delta_a}u\|_{L^2(\mathbb{R}^d)}\|\partial^{\mu_2}v\|_{L^\infty(\mathbb{R}^d)}&\leq C\|u\|_{H^{|\mu_1|+\delta_a}(\mathbb{R}^d)}\|v\|_{H^{|\mu_2|+\frac d2+x}(\mathbb{R}^d)}\\ &\leq C\|u\|_{H^b(\mathbb{R}^d)}^{\frac{a-|\mu_1|-\delta_a}{a-b}}\|u\|_{H^a(\mathbb{R}^d)}^{\frac{|\mu_1|+\delta_a-b}{a-b}}\|v\|_{H^b(\mathbb{R}^d)}^{\frac{a-|\mu_2|-\frac d2-x}{a-b}}\|v\|_{H^a(\mathbb{R}^d)}^{\frac{|\mu_2|+\frac d2+x-b}{a-b}}\\ &\leq C\left(\|u\|_{H^a(\mathbb{R}^d)}\|v\|_{H^b(\mathbb{R}^d)}+\|u\|_{H^b(\mathbb{R}^d)}\|v\|_{H^a(\mathbb{R}^d)}\right).\end{aligned} \tag{4.D.13}$$

- *Case 4* If $a-\frac d2\leq |\mu_2|<a$. Let $x:=\frac12\min(a-|\mu|_2,\delta_b)>0$. We define p_1, q_1 and s by $\frac{1}{q_1}:=\frac12-\frac{a-x-|\mu_2|}{d}$, $\frac{1}{p_1}=\frac12-\frac{1}{q_1}$ and $s=\frac{d}{q_1}$. One has $|\mu_1|+\delta_a+s=\frac d2+x<b$, and, using Sobolev embedding:

$$\|\nabla^{|\mu_1|+\delta_a}u\|_{L^{p_1}}\|\partial^{\mu_2}v\|_{L^{q_1}}\leq C\|u\|_{H^{|\mu_1|+\delta_a+s}}\|v\|_{H^{a-x}}\leq C\|u\|_{H^b}\|v\|_{H^a} \tag{4.D.14}$$

and $\frac{1}{p_1}+\frac{1}{q_1}=\frac12,\ \ p_1\neq +\infty$.

- *Case 5* If $|\mu_2|=a$. Then $|\mu_1|+\delta_a=0$ and using Sobolev embedding (as $b>\frac d2$):

$$\|\nabla^{|\mu_1|+\delta_a}u\|_{L^\infty(\mathbb{R}^d)}\|\partial^{\mu_2}v\|_{L^2(\mathbb{R}^d)}\leq C\|u\|_{H^b(\mathbb{R}^d)}\|v\|_{H^a(\mathbb{R}^d)}. \tag{4.D.15}$$

- ***Conclusion*** In all possible cases, from (*4.D.11*), (*4.D.12*), (*4.D.13*), (*4.D.14*) and (*4.D.15*) there always exist $p_1,q_1,p_2,q_2\in[2,+\infty)$ with $p_1,p_2\neq+\infty$, $\frac{1}{p_1}+\frac{1}{q_1}=\frac12$ and:

$$\begin{aligned}&\|\nabla^{|\mu_1|+\delta_a}u\|_{L^{p_1}(\mathbb{R}^d)}\|\partial^{\mu_2}v\|_{L^{q_1}(\mathbb{R}^d)}+\|\nabla^{|\mu_1|}u\|_{L^{q_2}v}\|\nabla^{|\mu_2|+\delta_a}v\|_{L^{p_2}(\mathbb{R}^d)}\\ \leq\ & C\|u\|_{H^b(\mathbb{R}^d)}\|v\|_{H^a(\mathbb{R}^d)}+C\|u\|_{H^a(\mathbb{R}^d)}\|v\|_{H^b(\mathbb{R}^d)}.\end{aligned}$$

where the estimate for the second term in the left hand side of the above equation comes from a symmetric reasoning. We now come back to (*4.D.9*), apply (*4.D.10*) and the above identity to obtain:

$$\|\nabla^a(uv)\|_{L^2(\mathbb{R}^d)}\leq C\|u\|_{H^b(\mathbb{R}^d)}\|v\|_{H^a(\mathbb{R}^d)}+C\|u\|_{H^a(\mathbb{R}^d)}\|v\|_{H^b(\mathbb{R}^d)}.$$

The above estimate and (*4.D.8*) imply the desired estimate (*4.D.7*) by interpolation.

step 3 Proof in the case $\Omega=\mathbb{R}^d$ and $a\leq b$. The proof is similar and simpler and we do not write it here. Therefore, (*4.D.7*) holds for all $a\geq 0$ and $b>\frac d2$.

step 4 Proof in the case of a smooth bounded domain Ω. There exists $\tilde{C} > 0$ such that for any $f \in H^{\max(a,b)}(\Omega)$ there exists an extension $\tilde{f} \in H^{\max(a,b)}(\mathbb{R}^d)$ with compact support, satisfying $\tilde{f} = f$ on Ω and:

$$\frac{1}{\tilde{C}} \|\tilde{f}\|_{H^c(\mathbb{R}^d)} \leq \|f\|_{H^c(\Omega)} \leq \tilde{C} \|\tilde{f}\|_{H^c(\mathbb{R}^d)}, \quad c = a, b,$$

see [2]. Let $u, v \in H^{\max(a,b)}(\Omega)$ and denote by $\tilde{u}$ and $\tilde{v}$ their respective extensions. Using (*4.D.7*) and the above estimate then yields:

$$\begin{array}{rcl} \|uv\|_{H^a(\Omega)} & \leq & \|\tilde{u}\tilde{v}\|_{H^a(\mathbb{R}^d)} \\ & \leq & C \left(\|\tilde{u}\|_{H^a(\mathbb{R}^d)} \|\tilde{v}\|_{H^b(\mathbb{R}^d)} + \|\tilde{u}\|_{H^b(\mathbb{R}^d)} \|\tilde{v}\|_{H^a(\mathbb{R}^d)} \right) \\ & \leq & C\tilde{C}^2 \left(\|u\|_{H^a(\Omega)} \|v\|_{H^b(\Omega)} + \|u\|_{H^b(\Omega)} \|v\|_{H^a(\Omega)} \right) \end{array}$$

and (*4.D.4*) is obtained.

□

4.E Geometrical decomposition

This section is devoted to the proof of Lemma 4.4.3 .

Lemma 4.E.1. *Let X denote the functional space*

$$X := \left\{ u \in L^\infty(\mathcal{B}^d(0, 4M)), \ \langle u - Q, H\Phi_M^{(0,1)} \rangle > \|u - Q\|_{L^\infty(\mathcal{B}^d(0,3M))} \right\}. \qquad (4.E.1)$$

There exists $\kappa, K > 0$ such that for all $u \in X \cap \{\|u - Q\|_{L^\infty(\mathcal{B}^d(0,4M)))} < \kappa\}$, there exists a unique choice of parameters $b \in \mathbb{R}^{\mathcal{I}}$ with $b_1^{(0,1)} > 0$, $\lambda > 0$ and $z \in \mathbb{R}^d$ such that the function $v := (\tau_{-z} u)_\lambda - \tilde{Q}_b$ satisfies:

$$\langle v, H^i \Phi_M^{(n,k)} \rangle = 0, \ \ \textit{for}\ 0 \leq n \leq n_0, \ 1 \leq k \leq k(n), \ 0 \leq i \leq L_n \qquad (4.E.2)$$

and such that:

$$|\lambda - 1| + |z| + \sum_{(n,k,i) \in \mathcal{I}} |b_i^{(n,k)}| \leq K. \qquad (4.E.3)$$

Moreover, b, λ and z are Fréchet differentiable[23] *and satisfy:*

$$|\lambda - 1| + |z| + \sum_{(n,k,i) \in \mathcal{I}} |b_i^{(n,k)}| \leq K \|u - Q\|_{L^\infty(\mathcal{B}^d(0,3M)))}. \qquad (4.E.4)$$

Proof of Lemma 4.E.1

We define first the application ξ as:

$$\begin{array}{rl} \xi : & L^\infty(\mathcal{B}^d(0, 3M)) \times (0, +\infty) \times \mathbb{R}^{d+\#\mathcal{I}} \to \mathbb{R}^{1+d+\#\mathcal{I}} \\ & (u, \tilde{\lambda}, \tilde{z}, \tilde{b}) \mapsto (\langle (\tau_{\tilde{z}} u)_{\frac{1}{\tilde{\lambda}}} - Q - \alpha_{\tilde{b}}, H^i \Phi_M^{(n,k)} \rangle)_{\substack{0 \leq n \leq n_0, 0 \leq i \leq L_n \\ 1 \leq k \leq k(n)}} \end{array} \qquad (4.E.5)$$

ξ is $\mathcal{C}^\infty$. From the definition (*4.3.7*) of α_b, and the orthogonality conditions (*4.4.3*), the differential of ξ with respect to the second variable at the point $(Q, 1, 0, ..., 0)$ is the diagonal matrix:

$$D^{(2)}\xi(Q, 1, 0, ..., 0) = - \begin{pmatrix} \langle T_0^{(0)}, \chi_M T_0^{(0)} \rangle \mathrm{Id}_{L+1} & & \\ & . & \\ & & \langle T_0^{(n_0)}, \chi_M T_0^{(n_0)} \rangle \mathrm{Id}_{L_{n_0}} \end{pmatrix} \qquad (4.E.6)$$

[23]For the ambient Banach space $L^\infty(\mathcal{B}^d(0, 3M))$.

where Id_{L_n} is the $L_n \times L_n$ identity matrix. $D^{(2)}\xi(Q,1,0,...,0)$ is invertible for M large from (*4.4.3*). Consequently, from the implicit functions theorem, there exist $\kappa, K>0$, such that for all $u \in X \cap \{\|u-Q\|_{L^\infty(\mathcal{B}^d(0,3M)))} < \kappa\}$, there exists a choice of the parameters $\tilde\lambda = \tilde\lambda(u)$, $\tilde z = \tilde z(u)$ and $\tilde b = \tilde b(u)$ such that:

$$\xi(u,\tilde\lambda,\tilde z,\tilde b)=0, \quad |\tilde\lambda-1|+|\tilde z|+\sum_{(n,k,i)\in\mathcal{I}} |\tilde b_i^{(n,k)}| \leq K\|u-Q\|_{L^\infty(\mathcal{B}^d(3M)))} \qquad (4.E.7)$$

and it is the unique solution of $\xi(u,\tilde\lambda,\tilde z,\tilde b)=0$ in the range

$$|\tilde\lambda-1|+|\tilde z|+\sum_{(n,k,i)\in\mathcal{I}} |\tilde b_i^{(n,k)}| \leq K.$$

Moreover, they are Fréchet differentiable, again from the implicit function theorem. Now, defining $\lambda = \frac{1}{\tilde\lambda}$, $b=\tilde b$ and $z=-\tilde z$, this means from (*4.E.5*) that the function $w := (\tau_{-z}u)_\lambda - Q - \alpha_b$ satisfies:

$$\langle w, H^i\Phi_M^{(n,k)}\rangle = 0, \quad \text{for } 0\leq n\leq n_0, \ 1\leq k\leq k(n), \ 0\leq i\leq L_n,$$

Finally, still from the implicit function theorem, from the identity for the differential (*4.E.6*), the definition (*4.E.1*) of X and (*4.4.3*):

$$\begin{array}{rcl} b_1^{(0,1)} & = & -[D^{(2)}\xi(Q,1,0,...,0)]^{-1}(\xi(u,1,0,...,0)) + o(\|u-Q\|_{L^\infty(\mathcal{B}^d(3M))}) \\ & = & \frac{\langle u-Q, H^1\Phi_M^{(0,1)}\rangle}{\langle T_0^{(0)}, \chi_M T_0^{(0)}\rangle} + o\left(\langle u-Q, H^1\Phi_M^{(0,1)}\rangle\right) > 0 \end{array}$$

where the $o()$ is as $\kappa\to 0$, and the strict positivity is then for κ small enough. Consequently, in that case $\tilde Q_b = Q + \chi_{(b_1^{(0,1)})^{-\frac{1+\eta}{2}}}\alpha_b$ is well defined, and one has $(b_1^{(0,1)})^{-\frac{1+\eta}{2}} \gg 2M$ for κ small enough. Thus, for $v := (\tau_{-z}u)_\lambda - \tilde Q_b$ there holds:

$$\langle v, H^i\Phi_M^{(n,k)}\rangle = \langle \tilde v, H^i\Phi_M^{(n,k)}\rangle = 0, \quad \text{for } 0\leq n\leq n_0, \ 1\leq k\leq k(n), \ 0\leq i\leq L_n$$

because the support of $v-\tilde v$ is outside $\mathcal{B}^d(0,2M)$. One has found a choice of the parameters λ, b and z such that $b_1^{(0,1)}>0$ and (*4.E.2*) and (*4.E.3*) hold. This choice is unique in the range (*4.E.3*) and the parameters are Fréchet differentiable since under (*4.E.3*), they are equal to the parameters given by the above inversion of ξ.

□

Lemma 4.E.2. *There exists $\kappa^*, \tilde K>0$ such that the following holds for all $0<\kappa<\kappa^*$. Let $\mathcal{O}$ be the open set of $L^\infty(\mathcal{B}^d(0,1))$ of functions u satisfying (4.4.4). For each $u\in\mathcal{O}$ there exists a unique choice of the parameters $\lambda\in\left(0,\frac{1}{4M}\right)$, $z\in\mathcal{B}^d\left(0,\frac14\right)$ and $b\in\mathbb{R}^{\mathcal{I}}$ such that $b_1^{(0,1)}>0$, $v=(\tau_{-z}u)_\lambda - \tilde Q_b \in L^\infty\left(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\})\right)$ satisfies*[24]*:*

$$\langle v, H^i\Phi_M^{(n,k)}\rangle = 0, \quad \textit{for } 0\leq n\leq n_0, \ 1\leq k\leq k(n), \ 0\leq i\leq L_n \qquad (4.E.8)$$

and

$$\sum_{(n,k,i)\in\mathcal{I}} |b_i^{(n,k)}| + \|v\|_{L^\infty\left(\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\})\right)} \leq \tilde K\kappa. \qquad (4.E.9)$$

Moreover, the functions λ, z and b defined this way are Fréchet differentiable on $\mathcal{O}$.

[24]The following assertions make sense as v is defined on $\frac{1}{\lambda}(\mathcal{B}^d(0,1)-\{z\})$ which indeed contains $\mathcal{B}^d(0,2M)$ since $0<\lambda<\frac{1}{4M}$ and $|z|\leq\frac14$, and as $\Phi_M^{(n,k)}$ is compactly supported in $\mathcal{B}^d(0,2M)$ from (*4.4.1*).

Proof of Lemma 4.E.2

Let K and κ_0 be the numbers associated to Lemma 4.E.1.

step 1 Existence. Let

$$(\tilde{\lambda}, \tilde{z}) \in \left(0, \frac{1}{8M}\right) \times \mathcal{B}^d\left(0, \frac{1}{8}\right) \tag{4.E.10}$$

be such that

$$\|u - Q_{\tilde{z}, \frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} < \frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}},$$
$$\|(\tau_{-\tilde{z}}u)_{\tilde{\lambda}} - Q\|_{L^\infty(\mathcal{B}^d(4M))} < \langle (\tau_{-\tilde{z}}u)_{\tilde{\lambda}} - Q, H\Phi_M^{(0,1)} \rangle,$$

which exists from (*4.4.4*). We define $w := (\tau_{-\tilde{z}}u)_{\tilde{\lambda}}$. It is defined on the set $\frac{1}{\tilde{\lambda}}(\mathcal{B}(1) - \tilde{z})$ which contains $\mathcal{B}^d(7M)$ as $0 < \tilde{\lambda} < \frac{1}{8M}$ and $|z| \leq \frac{1}{8}$. From this fact and the above estimates w satisfies:

$$\|w - Q\|_{L^\infty(\mathcal{B}(7M))} < \kappa, \quad \|w - Q\|_{L^\infty(\mathcal{B}^d(3M))} < \langle w - Q, H\Phi_M^{(0,1)} \rangle. \tag{4.E.11}$$

Thus for κ small enough one can apply Lemma 4.E.1: there exist a choice of the parameters z', b' and λ' such that $v' = (\tau_{-z'}w)_{\lambda'} - \tilde{Q}_{b'}$ satisfies (*4.E.8*) and $b_1^{'(0,1)} > 0$. This choice is unique in the range

$$|\lambda' - 1| + |z'| + \sum_{(n,k,i) \in \mathcal{I}} |b_i^{'(n,k)}| \leq K. \tag{4.E.12}$$

Moreover, there holds the estimate

$$|\lambda' - 1| + |z'| + \sum_{(n,k,i) \in \mathcal{I}} |b_i^{'(n,k)}| \leq K \|w - Q\|_{L^\infty(\mathcal{B}^d(0,3M)))} \leq K\kappa.$$

Now we define

$$b = b', \ z = \tilde{z} + \tilde{\lambda} z', \ \lambda = \tilde{\lambda}\lambda' \tag{4.E.13}$$

and $v = v'$. One has then $b_1^{(0,1)} > 0$, and from (*4.E.10*) and the above estimate:

$$\sum_{(n,k,i) \in \mathcal{I}} |b_i^{(n,k)}| \leq K\kappa, \quad |z| \leq \frac{1}{4}, \quad 0 < \lambda < \frac{1}{4M}$$

for κ small enough. From the definition of w, v' and v one has the identity:

$$u = (v + \tilde{Q}_b)_{z, \frac{1}{\lambda}}, \text{ with } v \text{ satisfying } (4.E.8).$$

From (*4.3.7*), (*4.3.29*) and the above estimate:

$$\begin{aligned} \|v\|_{L^\infty\left(\frac{1}{\lambda}(\mathcal{B}^d(1) - z)\right)} &= \lambda^{\frac{2}{p-1}} \|u - \tau_z(\tilde{Q}_{b, \frac{1}{\lambda}})\|_{L^\infty(\mathcal{B}^d(1))} \\ \leq \ & \lambda^{\frac{2}{p-1}} \|u - \tau_{\tilde{z}}(Q_{\frac{1}{\tilde{\lambda}}})\|_{L^\infty(\mathcal{B}^d(1))} + \lambda^{\frac{2}{p-1}} \|\tau_{\tilde{z}}(Q_{\frac{1}{\tilde{\lambda}}}) - \tau_z(\tilde{Q}_{b, \frac{1}{\lambda}})\|_{L^\infty(\mathcal{B}^d(1))} \leq CK\kappa \end{aligned}$$

for some constant $C > 1$ independent of the others. Therefore, one takes $\tilde{K} = CK$, and the choice of parameters λ, z and b that we just found provide the decomposition claimed by the Lemma and the existence is proven.

step 2 Differentiability. We claim that the parameters λ, b and z found in step 1 are unique, this will be proven in the next step. Therefore, from their construction using the auxiliary variables $\tilde{\lambda}$ and $\tilde{z}$ in step 1, and since the parameters λ', z' and b' provided by Lemma 4.E.1 are Fréchet differentiable, λ, b and z are Fréchet differentiable.

step 3 Unicity. Let $\hat{b}$, $\hat{\lambda}$, $\hat{z}$ be another choice of parameters with $\hat{b}_1^{(0,1)} > 0$, $0 < \lambda < \frac{1}{4M}$ and $|z| \leq \frac{1}{4}$ such that (*4.E.8*) and (*4.E.9*) hold for $\hat{v} = (\tau_{-\hat{z}}u)_{\hat{\lambda}} - \tilde{Q}_b$. The function $(\tau_{-\tilde{z}}u)_{\tilde{\lambda}}$, where $\tilde{\lambda}$ and $\tilde{z}$ were defined in (*4.E.10*) in the first step, then satisfy the bound:

$$\|(\tau_{-\tilde{z}}u)_{\tilde{\lambda}} - Q\|_{L^\infty(\mathcal{B}(3M))} < \kappa_0$$

for κ small enough from (*4.E.11*), and admits two decompositions:

$$(\tau_{-\tilde{z}}u)_{\tilde{\lambda}} = (\tilde{Q}_{b'} + v')_{z', \frac{1}{\lambda'}} = (\tilde{Q}_{\hat{b}} + \hat{v})_{\frac{\hat{z}-\tilde{z}}{\tilde{\lambda}}, \frac{\tilde{\lambda}}{\hat{\lambda}}},$$

such that v and v' satisfy (*4.E.8*). The first parameters satisfy from (*4.E.12*):

$$|\lambda' - 1| + |z'| + \sum_{(n,k,i)\in\mathcal{I}} |b_i'^{(n,k)}| \leq K\kappa_0.$$

We claim that the second parameters satisfy:

$$|\frac{\tilde{\lambda}}{\hat{\lambda}} - 1| + |\frac{\hat{z} - \tilde{z}}{\tilde{\lambda}}| + \sum_{(n,k,i)\in\mathcal{I}} |\hat{b}_i^{(n,k)}| \leq K\kappa_0, \tag{4.E.14}$$

which will be proven hereafter. Then, as such parameters are unique under the above bound from Lemma 4.E.1, one obtains:

$$\frac{\tilde{\lambda}}{\hat{\lambda}} = \frac{1}{\lambda'}, \ \frac{\hat{z} - \tilde{z}}{\tilde{\lambda}} = z', \ \hat{b} = b',$$

implying that $\hat{\lambda} = \lambda$, $\hat{z} = z$ and $\hat{b} = b$ where λ, z and b are the choice of the parameters given by the first step defined by (*4.E.13*). The unicity is obtained.

- *Proof of* (*4.E.14*). From the assumptions on $\hat{b}$, $\hat{\lambda}$ and $\hat{z}$, the definition of $\tilde{Q}_b$ (*4.3.29*) and (*4.E.9*) there holds for κ small enough:

$$\|u - Q_{\hat{z}, \frac{1}{\hat{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} \leq \frac{C\tilde{K}\kappa}{\hat{\lambda}^{\frac{2}{p-1}}}.$$

From (*4.E.10*) one has also:

$$\|u - Q_{\tilde{z}, \frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} \leq \frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}}.$$

From the two above estimates one deduces that:

$$\|Q_{\hat{z}, \frac{1}{\hat{\lambda}}} - Q_{\tilde{z}, \frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} \leq \frac{\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}} + \frac{C\tilde{K}\kappa}{\hat{\lambda}^{\frac{2}{p-1}}}. \tag{4.E.15}$$

Assume that $\hat{\lambda} \leq \tilde{\lambda}$. Then, since Q is radially symmetric and attains its maximum at the origin, and $\hat{z} \in \mathcal{B}^d(0,1)$ because $|\hat{z}| \leq \frac{1}{4}$, the above inequality at $x = \hat{z}$ implies:

$$\begin{aligned} Q(0)\left(\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}} - \frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\right) &= Q_{\hat{z}, \frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z}, \frac{1}{\tilde{\lambda}}}(\tilde{z}) \\ &\leq Q_{\hat{z}, \frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z}, \frac{1}{\tilde{\lambda}}}(\hat{z}) \\ &= |Q_{\hat{z}, \frac{1}{\hat{\lambda}}}(\hat{z}) - Q_{\tilde{z}, \frac{1}{\tilde{\lambda}}}(\hat{z})| \\ &\leq C\tilde{K}\kappa\left(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}} + \frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\right) \end{aligned}$$

which gives $\left|\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}-\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\right|\leq C\tilde{K}\kappa\left(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}+\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\right)$. The symmetric reasoning works in the case $\hat{\lambda}\geq\tilde{\lambda}$ and one obtains that in both cases:

$$\left|\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}-\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}\right|\leq C\tilde{K}\kappa\left(\frac{1}{\tilde{\lambda}^{\frac{2}{p-1}}}+\frac{1}{\hat{\lambda}^{\frac{2}{p-1}}}\right).$$

Basic computations show that for κ small enough the above identity implies:

$$\left|1-\frac{\hat{\lambda}}{\tilde{\lambda}}\right|\leq C\tilde{K}\kappa \ \text{ or } \ \hat{\lambda}=\tilde{\lambda}(1+O(\kappa)).$$

obtaining the first bound in (*4.E.14*) for κ small enough. We inject the above estimate in (*4.E.15*), yielding:

$$\begin{array}{rl} & \|Q_{\hat{z},\frac{1}{\hat{\lambda}}}-Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))} \\ \leq & \|Q_{\hat{z},\frac{1}{\hat{\lambda}}}-Q_{\hat{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\|+\|Q_{\hat{z},\frac{1}{\tilde{\lambda}}}-Q_{\tilde{z},\frac{1}{\tilde{\lambda}}}\|_{L^\infty(\mathcal{B}^d(1))}\|\leq\frac{C\tilde{K}\kappa}{\tilde{\lambda}^{\frac{2}{p-1}}}\end{array}$$

which implies in renormalized variables (as $|\hat{z}|\leq\frac{1}{8}$ and $\tilde{\lambda}\leq\frac{1}{8M}$):

$$\|Q-\tau_{\frac{\hat{z}-\tilde{z}}{\tilde{\lambda}}}Q\|_{L^\infty(\mathcal{B}^d(0,2M))}\leq C\tilde{K}\kappa.$$

As Q is smooth, radially symmetric and radially decreasing this implies:

$$\left|\frac{\hat{z}-\tilde{z}}{\tilde{\lambda}}\right|\leq C\tilde{K}\kappa \ \text{ or } \ \hat{z}=\tilde{z}+\tilde{\lambda}O(\kappa)$$

and the second bound in (*4.E.14*) is obtained.

$\square$

5

Dynamics near the ground state for the energy critical heat equation in large dimensions

5.1 Introduction

In this Chapter, we prove Theorems 2.3.4 and 2.3.5. This work has been done in collaboration with F. Merle and P. Raphaël, and is to appear for one part in Communications in Mathematical Physics [24], and for the second part in Compte Rendus Mathématiques de l'Académie des Sciences de Paris [25]. As in the other chapters focusing on the rigorous proofs of the result we obtained, we give first the notations that are specific to the problem and invite the reader to come back there whenever he or she has a doubt.

The chapter contains three main parts and an appendix. The first part is devoted to the analysis of solutions close to the manifold of ground states obtained by the action of the symmetry groups on Q. In Section 5.2 we recall the standard results on the semilinear heat equation and on Q, leading to a preliminary study of solutions near Q. The local well-posedness is addressed in Proposition 5.2.1. In Proposition 5.2.2 we describe the spectral structure of the linearized operator. In particular, for functions orthogonal to its instable eigenfunction and to its kernel, it displays some coercivity properties stated in Lemma 5.2.3. The nonlinear decomposition Lemma 5.2.5 then allows to decompose in a suitable way solutions around Q. Under this decomposition, the adapted variables are defined in Definition 5.2.4, the variation of the energy is studied in Lemma 5.2.7, the modulation equations are established in Lemma 5.2.6 and the energy bounds for the remainder on the infinite dimensional subspace are stated in Lemma 5.2.9. A direct consequence is the non-degeneracy of the scale and the central point, subject of the Lemma 5.2.11 ending the section.
Once this analysis is established, in Section 5.3 we first construct the unstable manifold near Q. These minimal solutions are constructed in Proposition 5.3.1 and then we give their uniqueness in a broader class of solutionsin the Liouville-type Theorem 2.3.5.
Finally, in Section 5.4 we give the proof of the classification Theorem 2.3.4. In Lemma 5.4.1 we characterize the instability time. If this time never happens, the solution is proved to dissipate to a soliton in Lemma 5.4.2. If it happens, then it blows up with type I blow up or dissipate according to Lemma 5.4.4. Eventually, Section 5.5 contains the proof of the stability of type I blow up.

The Appendix is organized as follows. First in Section 5.A we study the kernel of the linearized operator in Lemma 5.A.1. Then in Section 5.B we give the proof of the coercivity Lemma 5.2.3. In Section 5.C we prove the decomposition Lemma 5.2.5. Next, in Section 5.D, we state some useful estimate on the purely nonlinear estimate as $1 < p < 2$ in Lemma 5.D.1. In Section 5.E we give certain parabolic type results, especially Lemma 5.E.2 that allows to propagate exponential bounds.

Notations

We collect the main notations used throughout the chapter.

General notations. We will use a generic notation for the constants, C, whose value can change from line to line but just depends on d and not on the other variables used in the chapter. The notations $a \lesssim b$ (respectively $a \gtrsim b$) means $a \leq Cb$ (respectively $b \geq Ca$) for such a constant C. The notation $a = O(b)$ then means $|a| \lesssim b$. We employ the Kronecker δ notation:

$$\delta_{a,b} = 1 \text{ if } a = b, \ 0 \text{ otherwise}.$$

PDE notations. We let the heat kernel be:

$$K_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} \tag{5.1.1}$$

We recall that the elliptic equation

$$\Delta u + |u|^{p-1} = 0, \tag{5.1.2}$$

admits a unique [60] up to symmetries strictly positive $\dot{H}^1$ solution, equivalently

$$u = Q_{z,\lambda}, \ \lambda > 0, \ z \in \mathbb{R}^d. \tag{5.1.3}$$

where the radial soliton Q is explicitely given by (*2.2.2*). The linearized operator for (NLH) close to Q is the Schrödinger operator:

$$H := -\Delta - pQ^{p-1} = -\Delta + V \tag{5.1.4}$$

for the potential:

$$V := -pQ^{p-1} = -\frac{d+2}{d-2} \frac{1}{\left(1 + \frac{|x|^2}{d(d-2)}\right)^2}.$$

The operator H has only one negative eigenvalue $-e_0$, $e_0 > 0$ with multiplicity one associated to a non negative function $\mathcal{Y}$ that decays exponentially fast (and also its derivatives):

$$H\mathcal{Y} = -e_0 \mathcal{Y}, \ \ \mathcal{Y} > 0, \ \ \int_{\mathbb{R}^d} \mathcal{Y} = 1,$$

see Proposition 5.2.2 for more details. We denote the nonlinearity by:

$$f(u) := |u|^{p-1}u.$$

The constant in the constant in space ODE blow up solution κ_H in (*1.4.2*) will be simply denoted by κ:

$$\kappa := \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}} \tag{5.1.5}$$

Invariances. For $\lambda > 0$ and $u : \mathbb{R}^d \to \mathbb{R}$, we define the rescaling

$$u_\lambda \ : \ x \ \mapsto \ \frac{1}{\lambda^{\frac{d-2}{2}}} u\left(\frac{x}{\lambda}\right).$$

The infinitesimal generator of this tranformation is

$$\Lambda u := -\frac{\partial}{\partial \lambda}(u_\lambda)_{|\lambda=1} = \frac{d-2}{2} u + x.\nabla u.$$

Given a point $z \in \mathbb{R}^d$ and a function $u : \mathbb{R}^d \to \mathbb{R}$, we define the translation of vector z of u as:

$$\tau_z u \ : \ x \ \mapsto \ u(x-z).$$

The infinitesimal generator of this semi group is:

$$\left[\frac{\partial}{\partial z}(\tau_z u)\right]_{|z=0} = -\nabla u.$$

One has for any $\lambda > 0$ and $z \in \mathbb{R}^d$ that $\tau_z(u_\lambda) = \left(\tau_{\frac{z}{\lambda}} u\right)_\lambda$. For a function $u : \mathbb{R}^d \to \mathbb{R}$ we use the condensed notation:

$$u_{z,\lambda} := \tau_z(u_\lambda) = x \mapsto \frac{1}{\lambda^{\frac{d-2}{2}}} u\left(\frac{x-z}{\lambda}\right).$$

For $\lambda > 0$, the original space variable will be referred to as $x \in \mathbb{R}^d$, the renormalized space variable will be referred to as $y \in \mathbb{R}^d$, and they are related by

$$y = \frac{x-z}{\lambda}.$$

The manifold of ground states, being the orbit of Q under the action of the two previous groups of symmetries, is:

$$\mathcal{M} := \{Q_{z,\lambda}, \ z \in \mathbb{R}^d, \ \lambda > 0\}.$$

One has the following integration by parts and commutator formulas for smooth well localized functions :

$$\int (\Lambda u)v + \int u\Lambda v + 2\int uv = 0, \qquad (5.1.6)$$

$$H\Lambda = \Lambda H + 2H - (2V + x.\nabla V), \qquad (5.1.7)$$

$$H\nabla = \nabla H - \nabla V. \qquad (5.1.8)$$

Functional spaces. We will use the standard notation H^s, $\dot{H}^s$ and H^s_{loc} for inhomogeneous and homogeneous Sobolev spaces, and for the space of distributions that are locally in H^s. The distance between an element $u \in \dot{H}^1$ and a subset $X \subset \dot{H}^1$ is denoted by:

$$d(u,X) := \inf_{v \in X} \|u - v\|_{\dot{H}^1}.$$

For $u, v \in L^2(\mathbb{R}^d)$ real valued, the standard scalar product is

$$\langle u, v\rangle = \int_{\mathbb{R}^d} uv dx$$

and the orthogonality $u \perp v$ means $\langle u, v\rangle = 0$. We extend these notations whenever $uv \in L^1(\mathbb{R}^d)$. For a subspace $X \subset \mathbb{R}^d$ and $0 < \alpha < 1$, the space $C^{0,\alpha}(X)$ is the space of Hölder α-continuous functions on X. For $\alpha \in \mathbb{N}^d$, we will use the notations for the derivatives:

$$\partial^\alpha f = \partial^{\alpha_i}_{x_1} ... \partial^{\alpha_d}_{x_d} f, \quad \nabla f = (\partial_{x_i} f)_{1 \leq i \leq d}, \quad \nabla^2 f = (\partial_{x_i x_j} f)_{1 \leq i,j \leq d}.$$

5.2 Estimates for solutions trapped near $\mathcal{M}$

This section is devoted of the study of solutions which are globally trapped near the solitary wave. The heart of the proof is the coupling of the dissipative properties of the flow with coercivity estimates for the linearized operator H and energy bounds which imply the control of the scaling parameter, Lemma 5.2.11, and hence the impossibility of type II blow up near Q.

5.2.1 Cauchy theory

We first recall the standard Cauchy theory which takes care of the lack of differentiability of the nonlinearity in high dimensions. The Cauchy problem for (NLH) is well posed in the critical Lebesgue space $L^{\frac{2d}{d-2}}$, see [16, 155, 90]. A direct adaptation of the arguments used there implies that it is also well posed in $\dot{H}^1$. (NLH) being a parabolic evolution equation, it also possesses a regularizing effect. Namely, any solution starting from a singular initial datum will have an instantaneous gain of regularity, directly linked to the regularity of the nonlinearity which is only C^1 here as $1<p<2$. The proof of the following proposition is classical using for example the space time bounds of Lemma 5.E.5, and the details are left to the reader.

Proposition 5.2.1 (LWP of the energy critical heat equation in $\dot{H}^1$ and regularizing effects). *Let $d\geq 7$. For any $u_0\in \dot{H}^1(\mathbb{R}^d)$ there exists $T(u_0)>0$ and a weak solution $u\in \mathcal{C}([0,T(u_0)),\dot{H}^1(\mathbb{R}^d))$ of (NLH). In addition the following regularizing effects hold:*

(i) $u\in C^{(\frac32,3)}((0,T_{u_0})\times\mathbb{R}^d)$, *$u$ is a classical solution of (NLH) on $(0,T_{u_0})\times\mathbb{R}^d$.*

(ii) $u\in C((0,T(u_0)),W^{3,\infty}(\mathbb{R}^d))$.

(iii) $u\in C((0,T(u_0)),\dot{H}^3(\mathbb{R}^d))$, $u\in C^1((0,T(u_0)),\dot{H}^1)$.

For any $0<t_1<t_2<T_{u_0}$ the solution mapping is continuous from $\dot{H}^1$ into $C^{(\frac32,3)}([t_1,t_2]\times\mathbb{R}^d)$, $C([t_1,t_2],W^{3,\infty})$, $C((t_1,t_2),\dot{H}^1\cap\dot{H}^3)$ and $C^1((t_1,t_2),\dot{H}^1)$ at u_0.

Here $u\in C^{(\frac32,3)}((0,T_{u_0})\times\mathbb{R}^d)$ means that u is in the Hölder space $C^{\frac32}((0,T_{u_0})\times\mathbb{R}^d)$ and that u is three time differentiable with respect to the space variable x. The maximal time of existence of the solution u will then be denoted by T_{u_0}. From Proposition 5.2.1 we will always assume without loss of generality that the initial datum u_0 of a solution u of (NLH) belongs to $\dot{H}^1\cap\dot{H}^3\cap W^{3,+\infty}$. One has in particular the following blow-up criterion if u blows up at time T:

$$\|u(t)\|_{\dot{H}^2}\underset{t\to T}{}=+\infty. \tag{5.2.1}$$

5.2.2 The linearized operator H

We recall that all the properties of the radial stationary state Q are described in Lemma 2.2.1. The spectral properties of $H=-\Delta-pQ^{p-1}$, the linearized operator of (NLH) close to Q, are well known, see for example [147] and references therein.

Proposition 5.2.2 (Spectral theorem for the linearized operator). *Let $d \geq 3$. Then $H : H^2 \to L^2$ is self adjoint in L^2. It admits only one negative eigenvalue denoted by $-e_0$ where $e_0 > 0$, of multiplicity 1, associated to a profile $\mathcal{Y} > 0$ which decays together with its derivatives exponentially fast. For $d \geq 5$, H admits $d+1$ zeros in H^2 given by the invariances of (5.1.2):*

$$\mathit{Ker} H = \mathit{Span}(\Lambda Q, \partial_{x_1} Q, ..., \partial_{x_d} Q). \tag{5.2.2}$$

The rest of the spectrum is contained in $[0, +\infty)$.

Our aim being to study perturbations of Q, we will decompose such perturbations in three pieces: a main term in $\mathcal{M}$, a part on the instable direction $\mathcal{Y}$, and a remainder being "orthogonal" to these latter. Unfortunately, in dimensions $7 \leq d \leq 10$ the functions in the kernel do not decay quickly enough at infinity, what forces us to localize the natural orthogonality condition $u \in Ker(H)^{\perp}$. To do so we define:

$$\Psi_0 := \chi_M \Lambda Q - \langle \chi_M \Lambda Q, \mathcal{Y} \rangle \mathcal{Y}, \tag{5.2.3}$$

$$\Psi_i := \chi_M \partial_{x_i} Q, \quad \text{for } 1 \leq i \leq d. \tag{5.2.4}$$

for a constant $M \gg 1$ that can be chosen independently of the sequel. $\Psi_0, \Psi_1, ..., \Psi_d$ look like $\Lambda Q, \partial_{x_1} Q, ..., \partial_{x_d} Q$ for they satisfy the following orthogonality relations:

$$\begin{array}{l} \langle \Psi_i, \mathcal{Y} \rangle = 0, \quad \text{for } 0 \leq i \leq d, \\ \langle \Psi_0, \Lambda Q \rangle = \int \chi_M (\Lambda Q)^2, \quad \text{and for } 1 \leq i \leq d, \ \langle \Psi_i, \Lambda Q \rangle = 0, \\ \langle \Psi_i, \partial_{x_j} Q \rangle = \frac{1}{d} \int \chi_M |\nabla Q|^2 \delta_{i,j} \quad \text{for } 0 \leq i \leq d, \ 1 \leq j \leq d. \end{array} \tag{5.2.5}$$

The potential $V = -pQ^{p-1}$ is in the Kato class, implying that the essential spectrum of H is $[0, +\infty)$, the same as the one of the laplacian $-\Delta$. Consequently, H is not L^2-coercive for functions orthogonal to its zeros and $\mathcal{Y}$. However, a weighted coercivity holds, similar to the one of the laplacian given by the Hardy inequality and its higher Sobolev versions.

Lemma 5.2.3 (Weighted coercivity for H on the stable subspace). *Let $d \geq 7$. There exists a constant $C > 0$ such that for all $v \in \dot{H}^1(\mathbb{R}^d)$ satisfying*

$$v \in \mathit{Span}(\Psi_0, \Psi_1, ..., \Psi_d, \mathcal{Y})^{\perp}. \tag{5.2.6}$$

the following holds:
(i) Energy bound*:*

$$\frac{1}{C} \|v\|^2_{\dot{H}^1} \leq \int_{\mathbb{R}^d} |\nabla v|^2 - pQ^{p-1} v^2 \leq C \|v\|^2_{\dot{H}^1} \tag{5.2.7}$$

(ii) $\dot{H}^2$ bound*: If $v \in \dot{H}^2(\mathbb{R}^d)$, then*

$$\frac{1}{C} \|v\|^2_{\dot{H}^2} \leq \int_{\mathbb{R}^d} |Hv|^2 \leq C \|v\|^2_{\dot{H}^2} \tag{5.2.8}$$

(iii) $\dot{H}^3$ bound*: If $v \in \dot{H}^3(\mathbb{R}^d)$, then*

$$\frac{1}{C} \|v\|^2_{\dot{H}^3} \leq \int_{\mathbb{R}^d} |\nabla H v|^2 - pQ^{p-1} |Hv|^2 \leq C \|v\|^2_{\dot{H}^3} \tag{5.2.9}$$

This result follows the scheme of proof in [138] and details are given in Appendix 5.B.

5.2.3 Geometrical decomposition of trapped solutions

From now on and for the rest of this section, we study solutions that are trapped near the ground state manifold $\mathcal{M}$.

Definition 5.2.4 (Trapped solutions). Let $I \subset \mathbb{R}$ be a time interval containing 0 and $0 < \eta \ll 1$. A solution u of (NLH) is said to be trapped at distance η on I if:

$$\sup_{t\in I} d\left(u(t), \mathcal{M}\right) \leq \eta. \tag{5.2.10}$$

A classical consequence of the orbital stability assumption (*5.2.10*) is the existence of a geometrical decomposition of the flow adapted to the spectral structure of the linearized operator H stated in Proposition 5.2.2 and Lemma 5.2.3.

Lemma 5.2.5 (Geometrical decomposition). *Let $d \geq 7$. There exists $\delta > 0$ such that for any $u_0 \in \dot{H}^1(\mathbb{R}^d)$ with $\|u_0 - Q\|_{\dot{H}^1} < \delta$, the following holds. If the solution of (NLH) given by Proposition 5.2.1 is defined on some time interval $[0, T)$ and satisfies:*

$$\mathit{sup}_{0\leq t<T} d(u(t), \mathcal{M}) < \delta,$$

then there exists three functions $\lambda : [0, T) \to (0, +\infty)$, $z : [0, T) \to \mathbb{R}^d$, and $a : [0, T) \to \mathbb{R}$ that are $\mathcal{C}^1$ on $(0, T)$, such that

$$u = (Q + a\mathcal{Y} + v)_{z,\lambda}, \tag{5.2.11}$$

where the function v satisfies the orthogonality conditions (5.2.6). Moreover, one has the following estimate for each time $t \in [0, T)$:

$$|a| + \|v\|_{\dot{H}^1} \lesssim \mathit{inf}_{\lambda>0, z\in\mathbb{R}^d} \|u - Q_{z,\lambda}\|_{\dot{H}^1}. \tag{5.2.12}$$

The proof of this result is standard and sketched in section 5.C for the sake of completeness. We therefore introduce the C^1 in time decomposition:

$$u = (Q + a\mathcal{Y} + \varepsilon)_{z,\lambda}$$

with ε satisfying the orthogonality conditions (*5.2.6*). We define the renormalized time $s = s(t)$ is by

$$s(0) = 0, \quad \frac{ds}{dt} = \frac{1}{\lambda^2}. \tag{5.2.13}$$

and obtain the evolution equation in renormalized time

$$a_s\mathcal{Y} + \varepsilon_s - \frac{\lambda_s}{\lambda}\left(\Lambda Q + a\Lambda\mathcal{Y} + \Lambda\varepsilon\right) - \frac{z_s}{\lambda}.\nabla\left(Q + a\mathcal{Y} + \varepsilon\right) = e_0 a\mathcal{Y} - H\varepsilon + NL \tag{5.2.14}$$

where the nonlinear term is defined by:

$$NL := f(Q + a\mathcal{Y} + \varepsilon) - f(Q) - f'(Q)(a\mathcal{Y} + \varepsilon).$$

5.2.4 Modulation equations

We start the analysis of the flow near $\mathcal{M}$ by the computation of the modulation equations.

Lemma 5.2.6 (Modulation equations). *Let $d \geq 7$ and I be a time interval containing 0. There exists $0<\eta^* \ll 1$ such that if u is trapped at distance η for $0<\eta<\eta^*$ on I, then the following holds on $s(I)$:*
1. Modulation equations[1]*:*

$$a_s - e_0 a = O(a^2+\|\varepsilon\|_{\dot H^2}^2) = O(\eta^2), \tag{5.2.15}$$
$$\frac{\lambda_s}{\lambda} = O(a^2+\|\varepsilon\|_{\dot H^2}) = O(\eta), \tag{5.2.16}$$
$$\frac{z_s}{\lambda} = O(a^2+\|\varepsilon\|_{\dot H^2}) = O(\eta). \tag{5.2.17}$$

2. Refined identities[2]*:*

$$\left|\frac{\lambda_s}{\lambda}+\frac{d}{ds}O(\|\varepsilon\|_{\dot H^s})\right| \lesssim a^2+\|\varepsilon\|_{\dot H^2}^2 \ \textit{ for any } 1\leq s\leq \frac{4}{3}, \tag{5.2.18}$$

$$\left|\frac{z_s}{\lambda}+\frac{d}{ds}O(\|\varepsilon\|_{\dot H^s})\right| \lesssim a^2+\|\varepsilon\|_{\dot H^2}^2 \ \textit{ for any } 1\leq s\leq 2. \tag{5.2.19}$$

Proof of Lemma 5.2.6 We compute the modulation equations as a consequence of the orthogonality conditions (*5.2.6*) for ε. However, since in dimension $7\leq d\leq 10$ the Ψ_j are not exact elements of the kernel of H, we will need the refined space time bounds (*5.2.18*), (*5.2.19*), see [109, 141] for related issued.

step 1 First algebraic identities. We first prove (*5.2.15*), (*5.2.16*) and (*5.2.17*).

First identity for a. First, taking the scalar product between (*5.2.14*) and $\mathcal{Y}$ yields, using the orthogonality (*5.2.6*), (*5.2.2*), the pointwise estimate for the nonlinearity (*5.D.2*), the generalized Hardy inequality (*5.B.5*), the integration by parts formula (*5.1.6*) and the fact that $\mathcal{Y}$ decays exponentially fast:

$$\begin{aligned} a_s-e_0a &= \frac{1}{\|\mathcal{Y}\|_{L^2}^2}\left(\frac{\lambda_s}{\lambda}\int (a\Lambda\mathcal{Y}+\Lambda\varepsilon)\mathcal{Y}+\int NL\mathcal{Y}+\int\frac{z_s}{\lambda}.\nabla\varepsilon\mathcal{Y}\right)\\ &= \frac{\lambda_s}{\lambda}O\left(|a|\int e^{-C|x|}+\int|\varepsilon|e^{-C|x}\right)+O\left(\left|\frac{z_s}{\lambda}\right|\int e^{-C|x|}|\varepsilon|\right)\\ &\quad +O\left(a^2\int e^{-C|x|}Q^{p-2}+\int e^{-C|x|}Q^{p-2}|\varepsilon|^2\right)\\ &= O\left(\left|\frac{\lambda_s}{\lambda}\right|(a+\|\varepsilon\|_{\dot H^s})\right)+O(a^2+\|\varepsilon\|_{\dot H^s}^2)+O\left(\left|\frac{z_s}{\lambda}\right|\|\varepsilon\|_{\dot H^s}\right). \end{aligned} \tag{5.2.20}$$

for $s\in[1,2]$. Taking $s=1$ and injecting (*5.2.12*) gives:

$$a_s-e_0a=\frac{\lambda_s}{\lambda}O(\eta)+O(\eta^2)+O\left(\left|\frac{z_s}{\lambda}\right|\eta\right). \tag{5.2.21}$$

*First identity for λ.*Taking the scalar product between (*5.2.14*) and Ψ_0, using (*5.2.6*), (*5.2.5*) gives:

$$-\frac{\lambda_s}{\lambda}\int[\Lambda Q+a\Lambda\mathcal{Y}+\Lambda\varepsilon]\Psi_0-\int\frac{z_s}{\lambda}.\nabla\varepsilon\Psi_0=\int NL\Psi_0-\int H\varepsilon\Psi_0 \tag{5.2.22}$$

[1]The two "=O()" on each line represent two estimates, the first one being an explicit control in function of a and ε, and the second one a uniform bound related to the distance to the manifold of ground states.

[2]The notation $\frac{d}{ds}O(\cdot)$ means the derivative with respect to the renormalized time s of a quantity satisfying the estimate $O(\cdot)$.

where we used the fact that $\int \partial_{x_i} \mathcal{Y} \Psi_0 = 0$ as $\mathcal{Y}$ and Ψ_0 are radial. (*5.1.6*) and the Hardy inequality (*5.B.5*) give, as Ψ_0 is exponentially decaying:

$$\int \Lambda \varepsilon \Psi_0 = O\left(\int |\varepsilon| e^{-C|x|}\right) = O\left(\|\varepsilon\|_{\dot{H}^1}\right) = O(\eta).$$

Hence from (*5.2.12*) and (*5.2.5*) the first term in the left hand side of (*5.2.22*) is:

$$-\frac{\lambda_s}{\lambda} \int \left[\Lambda Q + a\Lambda \mathcal{Y} + \Lambda \varepsilon\right] \Psi_0 = \frac{\lambda_s}{\lambda}\left(-\int \chi_M (\Lambda Q)^2 + O(\eta)\right).$$

For the second term in the left hand side of (*5.2.22*) one uses Hardy inequality (*5.B.5*) for $1 \leq i \leq d$ and the fact that Ψ_0 is exponentially decreasing:

$$\int \partial_{y_i} \varepsilon \Psi_0 = O(\|\varepsilon\|_{\dot{H}^1}) = O(\eta).$$

For the right hand side we use the pointwise estimate (*5.D.2*) on the nonlinearity and the Hardy inequality (*5.B.5*) to obtain, as Ψ_0 is exponentially decreasing:

$$\int NL \Psi_0 = O\left(\int |\Psi_0| Q^{p-2} |a\mathcal{Y} + \varepsilon|^2\right) = O\left(a^2 + \|\varepsilon\|_{\dot{H}^s}^2\right)$$

for $s \in [1,2]$. The linear term is estimated using (*5.2.7*), (*5.2.8*) and the fact that Ψ_0 is exponentially decreasing:

$$\left|\int H\varepsilon \Psi_0\right| = \left|\int \varepsilon H \Psi_0\right| \lesssim \|\varepsilon\|_{\dot{H}^s}$$

for $s \in [1,2]$. The five previous equations give then:

$$\frac{\lambda_s}{\lambda} = O\left(a^2 + \|\varepsilon\|_{\dot{H}^s}\right) + O\left(\eta \left|\frac{z_s}{\lambda}\right|\right). \tag{5.2.23}$$

for $s \in [1,2]$. Taking $s = 1$ and injecting (*5.2.12*) one obtains:

$$\frac{\lambda_s}{\lambda} = O(\eta) + O\left(\eta \left|\frac{z_s}{\lambda}\right|\right). \tag{5.2.24}$$

First identity for z. We take the scalar product between (*5.2.14*) and Ψ_i for $1 \leq i \leq d$, using (*5.2.6*) and (*5.2.5*):

$$-\frac{\lambda_s}{\lambda}\int \Lambda \varepsilon \Psi_i - \frac{z_{i,s}}{\lambda}\int \partial_{y_i}(Q + a\mathcal{Y} + \varepsilon)\Psi_i - \sum_{i \neq j} \frac{z_{j,s}}{\lambda}\int \partial_{x_j} \varepsilon \Psi_i = \int NL \Psi_i - \int H\varepsilon \Psi_i.$$

For the first term, using (*5.1.6*) and Hardy inequality (*5.B.5*) as Ψ_i is compactly supported one finds:

$$\int \Lambda \varepsilon \Psi_i = O(\eta).$$

For the second term, using Hardy inequality and (*5.2.5*), as Ψ_i is compactly supported one gets:

$$\begin{array}{rll} & \int \partial_{y_i}(Q + a\mathcal{Y} + \varepsilon)\Psi_i & = \int \chi_M (\partial_{y_i} Q)^2 + O(|a|) - \int \varepsilon \partial_{x_i} \Psi_i \\ = & \int \chi_M (\partial_{y_i} Q)^2 + O(\eta) + O\left(\|\varepsilon\|_{\dot{H}^1}\right) & = \int \chi_M (\partial_{y_i} Q)^2 + O(\eta) \end{array}$$

and similarly for $1 \leq j \leq d$ with $j \neq i$:

$$\int \partial_{y_j} \varepsilon \Psi_i = O(\eta).$$

For the nonlinear term we use the pointwise estimate (*5.D.2*) on the nonlinearity, the Hardy inequality (*5.B.5*) and the fact that Ψ_i is compactly supported:

$$\begin{array}{rcl} \int NL\Psi_i & = & O\left(\int |\Psi_i| Q^{p-2}|a\mathcal{Y}+\varepsilon|^2\right) = O\left(a^2+\int \varepsilon^2|\Psi_i|Q^{p-2}\right) \\ & = & O\left(a^2+\|\varepsilon\|^2_{\dot{H}^s}\right) \end{array}$$

for $s\in[1,2]$. For the linear term, as Ψ_i is compactly supported, using Hardy inequality (*5.B.5*):

$$\int H\varepsilon\Psi_i = O(\|\varepsilon\|_{\dot{H}^s})$$

for $s\in[1,2]$. The four previous equations give then:

$$\frac{z_{i,s}}{\lambda} = O\left(a^2+\|\varepsilon\|_{\dot{H}^s}\right)+\sum_{j\neq i} O\left(\eta\left|\frac{z_{j,s}}{\lambda}\right|\right)+O\left(\eta\left|\frac{\lambda_s}{\lambda}\right|\right). \tag{5.2.25}$$

for $s\in[1,2]$.Taking $s=1$ and injecting (*5.2.12*) one gets:

$$\frac{z_{i,s}}{\lambda} = O(\eta)+\sum_{j\neq i} O\left(\eta\left|\frac{z_{j,s}}{\lambda}\right|\right)+O\left(\eta\left|\frac{\lambda_s}{\lambda}\right|\right). \tag{5.2.26}$$

Conclusion. We gather the primary identities (*5.2.20*), (*5.2.21*), (*5.2.23*), (*5.2.24*), (*5.2.25*), (*5.2.26*) and bootstrap the information they contained for the mixed terms, yielding the identities (*5.2.15*), (*5.2.16*) and (*5.2.17*).

step 2 Refined identities. We now show (*5.2.18*) and (*5.2.19*). We perform an integration by part in time, improving the modulation equations by removing the derivative with respect to time of the projection of ε onto ΛQ and ∇Q. Note that this procedure *requires* sufficient decay of ΛQ and hence the assumption $d\geq 7$.

Improved modulation equation for z. For $1\leq i\leq d$ the quantity $\int \varepsilon\partial_{y_i}Q$ is well defined for $7\leq d$ via Sobolev embedding. We compute the following identity:

$$\frac{d}{ds}\left[\frac{\langle\varepsilon,\partial_{y_i}Q\rangle}{\langle\partial_{y_i}(Q+a\mathcal{Y}+\varepsilon),\partial_{y_i}Q\rangle}\right] = \frac{\langle\varepsilon_s,\partial_{y_i}Q\rangle}{\langle\partial_{y_i}(Q+a\mathcal{Y}+\varepsilon),\partial_{y_i}Q\rangle}-\frac{\langle\varepsilon,\partial_{y_i}Q\rangle\langle\partial_{y_i}(a_s\mathcal{Y}+\varepsilon_s),\partial_{y_i}Q\rangle}{\langle\partial_{y_i}(Q+a\mathcal{Y}+\varepsilon),\partial_{y_i}Q\rangle^2} \tag{5.2.27}$$

Using Hardy inequality (*5.B.5*) and (*5.2.12*) one has as $7\leq d$:

$$\begin{array}{rcl} |\langle\varepsilon,\partial_{y_i}Q\rangle|\leq\int\frac{|\varepsilon|}{1+|y|^{d-1}} & \leq & \left(\int\frac{|\varepsilon|^2}{1+|y|^4}\right)^{\frac12}\left(\int\frac{1}{1+|y|^{2d-6}}\right)^{\frac12} \\ & = & O(\|\varepsilon\|_{\dot{H}^s}) \text{ for any } 1\leq s\leq 2, \end{array} \tag{5.2.28}$$

$$\langle\partial_{y_i}(Q+a\mathcal{Y}+\varepsilon),\partial_{y_i}Q\rangle = \int(\partial_{y_i}Q)^2+O(\eta), \tag{5.2.29}$$

and therefore the quantity on the left hand side of (*5.2.27*) is for η^* small enough:

$$\frac{\langle\varepsilon,\partial_{y_i}Q\rangle}{\langle\partial_{y_i}(Q+a\mathcal{Y}+\varepsilon),\partial_{y_i}Q\rangle} = O(\|\varepsilon\|_{\dot{H}^s}) \text{ for any } 1\leq s\leq 2 \tag{5.2.30}$$

From (*5.2.14*) the numerator of the first quantity in the right hand side of (*5.2.27*) is:

$$\begin{array}{rcl} \langle\varepsilon_s,\partial_{y_i}Q\rangle & = & \frac{\lambda_s}{\lambda}\langle\Lambda\varepsilon,\partial_{y_i}Q\rangle+\frac{z_{i,s}}{\lambda}\langle\partial_{y_i}(Q+a\mathcal{Y}+\varepsilon),\partial_{y_i}Q\rangle \\ & & +\sum_{j=1,\ j\neq i}^d\frac{z_{j,s}}{\lambda}\langle\partial_{y_j}(a\mathcal{Y}+\varepsilon),\partial_{y_i}Q\rangle-\langle H\varepsilon,\partial_{y_i}Q\rangle+\langle NL,\partial_{y_i}Q\rangle. \end{array}$$

Using (*5.2.16*), (*5.1.6*), (*5.2.12*) and Hardy inequality (*5.B.5*) one gets for the first term:

$$\begin{array}{lll} \left|\frac{\lambda_s}{\lambda}\langle \Lambda\varepsilon, \partial_{y_i}Q\rangle\right| & \leq & \left|\frac{\lambda_s}{\lambda}\right|\int \frac{|\varepsilon|}{1+|y|^{d-1}} \lesssim (a^2+\|\varepsilon\|_{\dot H^2})\left(\int \frac{|\varepsilon|^2}{1+|y|^4}\right)^{\frac 12}\left(\int \frac{1}{1+|y|^{2d-6}}\right)^{\frac 12} \\ & \lesssim & a^2+\|\varepsilon\|_{\dot H^2}^2. \end{array}$$

Similarly for the third term using (*5.2.17*), (*5.2.12*) and Hardy inequality (*5.B.5*):

$$\begin{array}{lll} \left|\sum_{j=1,\ j\neq i}^d \frac{z_{j,s}}{\lambda}\langle \partial_{y_j}(a\mathcal{Y}+\varepsilon), \partial_{y_i}Q\rangle\right| & \leq & \left|\frac{z_s}{\lambda}\right|\left(|a|+\int \frac{|\varepsilon|}{1+|y|^d}\right) \\ \lesssim \ (a^2+\|\varepsilon\|_{\dot H^2})\left(|a|+\int \frac{|\varepsilon|^2}{1+|y|^4}\right)^{\frac 12}\left(\int \frac{1}{1+|y|^{2d-4}}\right)^{\frac 12} & \lesssim & a^2+\|\varepsilon\|_{\dot H^2}^2. \end{array}$$

For the fourth term, performing an integration by parts:

$$\langle H\varepsilon, \partial_{y_i}Q\rangle = \int \varepsilon H\partial_{y_i}Q = 0.$$

For the last term, the nonlinear one, one computes using the hardy inequality (*5.B.5*):

$$\begin{array}{lll} \int NL\partial_{y_i}Q & = & O\left(\int |\partial_{y_i}Q|Q^{p-2}|a\mathcal{Y}+\varepsilon|^2\right) = O\left(a^2+\int \varepsilon^2|\partial_{y_i}Q|Q^{p-2}\right) \\ & = & O\left(a^2+\int \frac{\varepsilon^2}{1+|y|^5}\right) = O\left(a^2+\|\varepsilon\|_{\dot H^s}^2\right) \end{array}$$

The five identities above plus (*5.2.29*) imply that the second term in (*5.2.27*) is:

$$\frac{\langle \varepsilon_s, \partial_{y_i}Q\rangle}{\langle \partial_{y_i}(Q+a\mathcal{Y}+\varepsilon), \partial_{y_i}Q\rangle} = \frac{z_{i,s}}{\lambda}+O(a^2+\|\varepsilon\|_{\dot H^2}^2). \quad (5.2.31)$$

We now turn to the third term in (*5.2.27*). One computes from (*5.2.14*), using (*5.2.16*) and (*5.2.17*) that[3]:

$$\int \partial_{y_i}(a_s\mathcal{Y}+\varepsilon_s)\partial_{y_i}Q = O\left(\left|\frac{\lambda_s}{\lambda}\right|+\left|\frac{z_s}{\lambda}\right|+|a|+\|\varepsilon\|_{\dot H^2}\right) = O(|a|+\|\varepsilon\|_{\dot H^2}).$$

With (*5.2.29*) and (*5.2.28*) the above estimate yields for the third term in (*5.2.27*):

$$\left|\frac{\langle \varepsilon, \partial_{y_i}Q\rangle\langle \partial_{y_i}(a_s\mathcal{Y}+\varepsilon_s), \partial_{y_i}Q\rangle}{\langle \partial_{y_i}(Q+a\mathcal{Y}+\varepsilon), \partial_{y_i}Q\rangle^2}\right| \lesssim a^2+\|\varepsilon\|_{\dot H^2}^2. \quad (5.2.32)$$

We now come back to the identity (*5.2.27*), and inject the estimates (*5.2.30*), (*5.2.31*) and (*5.2.32*) we have found for each term, yielding the improved modulation equation (*5.2.19*) claimed in the lemma.

Improved modulation equation for λ. The quantity $\int \varepsilon\Lambda Q$ is well defined for $d\geq 7$ from the Sobolev embedding of $\dot H^1$ into $L^{\frac{2d}{d-2}}$. From (*5.2.14*) one computes first the identity:

$$\frac{d}{ds}\left[\frac{\langle \varepsilon,\Lambda Q\rangle}{\langle \Lambda(Q+\varepsilon),\Lambda Q\rangle}\right] = \frac{\langle \varepsilon_s,\Lambda Q\rangle}{\langle \Lambda(Q+\varepsilon),\Lambda Q\rangle}-\frac{\langle \varepsilon,\Lambda Q\rangle\langle \Lambda(\varepsilon_s),\Lambda Q\rangle}{\langle \Lambda(Q+\varepsilon),\Lambda Q\rangle^2} \quad (5.2.33)$$

We now compute all the terms in the previous identity.
∘ *Left hand side of* (*5.2.33*). One computes:

$$\begin{array}{lll} |\langle \varepsilon, \Lambda Q\rangle| & \lesssim & \int \frac{|\varepsilon|}{1+|y|^{d-2}} \lesssim \left(\int \frac{|\varepsilon|^2}{1+|y|^{2+\frac 23}}\right)^{\frac 12}\left(\int \frac{1}{1+|y|^{2d-\frac{20}{3}}}\right)^{\frac 12} \\ & \lesssim & \|\varepsilon\|_{\dot H^s} \text{ for any } 1\leq s\leq 1+\frac 13 \end{array} \quad (5.2.34)$$

[3]We do not redo here the application of Hardy inequality and of (*5.D.3*) to control the linear and nonlinear term as we have used them numerous times before in the proof.

as since $d\geq 7$ one has indeed $\frac{1}{1+|x|^{2d-\frac{20}{3}}}\in L^1$. For the denominator one computes using (*5.1.6*), Hardy inequality (*5.B.5*) and (*5.2.12*):

$$
\begin{aligned}
\langle \Lambda(Q+\varepsilon),\Lambda Q\rangle &= \int (\Lambda Q)^2-\int \varepsilon \Lambda^2 Q-2\int \varepsilon Q=\int (\Lambda Q)^2+O(\|\varepsilon\|_{\dot H^1})\\
&= \int (\Lambda Q)^2+O(\eta).
\end{aligned} \tag{5.2.35}
$$

We then conclude that the quantity in the left hand side of (*5.2.33*) is:

$$
\frac{\langle \varepsilon,\Lambda Q\rangle}{\langle \Lambda(Q+\varepsilon),\Lambda Q\rangle}=O(\|\varepsilon v_{\dot H^s})\ \text{ for any } 1\leq s\leq 1+\frac{1}{3}. \tag{5.2.36}
$$

∘ *First term in the right hand side of* (*5.2.33*). Using (*5.2.14*) one has:

$$
\begin{aligned}
\langle \varepsilon_s,\Lambda Q\rangle &= \tfrac{\lambda_s}{\lambda}\langle \Lambda(Q+a\mathcal{Y}+\varepsilon),\Lambda Q)-\langle H\varepsilon,\Lambda Q\rangle\\
&\quad +\langle \tfrac{z_s}{\lambda}.\nabla(a\mathcal{Y}+\varepsilon),\Lambda Q\rangle+\langle NL,\Lambda Q\rangle.
\end{aligned}
$$

For the second term of the right hand side we perform an integration by parts:

$$
\langle H\varepsilon,\Lambda Q\rangle=\langle \varepsilon,H\Lambda Q\rangle=0.
$$

For the third term, performing an integration by parts, using (*5.2.17*) and Hardy inequality (*5.B.5*) one gets:

$$
\begin{aligned}
& |\textstyle\int \frac{z_s}{\lambda}.\nabla(a\mathcal{Y}+\varepsilon)\Lambda Q|\\
\leq\ & |\tfrac{z_s}{\lambda}|\textstyle\int(|a|\mathcal{Y}+|\varepsilon|)|\nabla \Lambda Q|\lesssim (a^2+\|\varepsilon\|_{\dot H^2})(|a|+\int \frac{|\varepsilon|}{1+|y|^{d-1}})\\
\lesssim\ & (a^2+\|\varepsilon\|_{\dot H^2})\left(|a|+\left(\textstyle\int \frac{|\varepsilon|^2}{1+|y|^2}\right)^{\frac12}\left(\int\frac{1}{1+|y|^{2d-6}}\right)\right)^{\frac12}\lesssim a^2+\|\varepsilon\|_{\dot H^2}^2.
\end{aligned}
$$

For the nonlinear term using the Hardy inequality (*5.B.5*)

$$
\begin{aligned}
\textstyle\int NL\Lambda Q &= O\left(\textstyle\int |\Lambda Q|Q^{p-2}|a\mathcal{Y}+\varepsilon|^2\right)=O\left(a^2+\textstyle\int \varepsilon^2|\Lambda Q|Q^{p-2}\right)\\
&= O\left(a^2+\textstyle\int\frac{\varepsilon^2}{1+|y|^4}\right)=O\left(a^2+\|\varepsilon\|_{\dot H^2}^2\right)
\end{aligned}
$$

The four identities above, plus (*5.2.16*), and (*5.2.35*) give that the first term in the right hand side of (*5.2.33*) is:

$$
\frac{\langle \varepsilon_s,\Lambda Q\rangle}{\langle \Lambda(Q+\varepsilon),\Lambda Q\rangle}=\frac{\lambda_s}{\lambda}+O(a^2+\|\varepsilon\|_{\dot H^2}^2). \tag{5.2.37}
$$

∘ *Second term in the right hand side of* (*5.2.33*). From the asymptotic of the solitary wave (*2.2.2*) there exists a constant $\tilde c\in\mathbb{R}$ such that:

$$
\Lambda^2 Q+2\Lambda Q=\tilde c(\Lambda Q+R)
$$

where R is a function satisfying an improved decay at infinity:

$$
|\partial^\alpha R|\lesssim \frac{1}{1+|x|^{d+|\alpha|}}.
$$

Using (*5.1.6*) one then computes that:

$$
\begin{aligned}
& -\frac{\langle \varepsilon, \Lambda Q\rangle\langle \Lambda(\varepsilon_s), \Lambda Q\rangle}{\langle \Lambda(Q+\varepsilon), \Lambda Q\rangle^2} = \tilde c \frac{\langle \varepsilon, \Lambda Q+R\rangle\langle \varepsilon_s, \Lambda Q+R\rangle}{\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle} - \tilde c \frac{\langle \varepsilon, R\rangle\langle \varepsilon_s, \Lambda Q+R\rangle}{\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle} \\
= & \frac{\tilde c}{2}\frac{d}{ds}\left[\frac{\langle \varepsilon, \Lambda Q+R\rangle^2}{\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle}\right] + \frac{\tilde c}{2}\frac{\langle \varepsilon, \Lambda Q+R\rangle^2\langle \varepsilon_s, \Lambda Q+R\rangle}{\left(\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle\right)^2} \\
& -\tilde c \frac{\langle \varepsilon, R\rangle\langle \varepsilon_s, \Lambda Q+R\rangle}{\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle} \qquad (5.2.38) \\
= & \frac{\tilde c}{2}\frac{d}{ds}\left[\frac{\langle \varepsilon, \Lambda Q+R\rangle^2}{\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle} + \frac{1}{3}\frac{\langle \varepsilon, \Lambda Q+R\rangle^3}{\left(\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle\right)^2}\right] \\
+ & \frac{\tilde c}{3}\frac{\langle \varepsilon, \Lambda Q+R\rangle^3\langle \varepsilon_s, \Lambda Q+R\rangle}{\left(\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle\right)^3} - \tilde c \frac{\langle \varepsilon, R\rangle\langle \varepsilon_s, \Lambda Q+R\rangle}{\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle}. \qquad (5.2.39)
\end{aligned}
$$

Using the Hardy inequality and (*5.2.36*) the first term in the right hand side is:

$$
\left|\frac{\langle \varepsilon, \Lambda Q+R\rangle^2}{\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle} + \frac{1}{3}\frac{\langle \varepsilon, \Lambda Q+R\rangle^3}{\left(\int(\Lambda Q)^2+\langle \varepsilon, \Lambda Q+R\rangle\right)^2}\right| \lesssim \eta\|\varepsilon\|_{\dot H^s}
$$

for any $1\leq s\leq 1+\frac13$. Next one computes from (*5.2.14*) using (*5.2.15*), (*5.2.16*) and (*5.2.17*) that:

$$
|\langle \varepsilon_s, \Lambda Q+R\rangle| \lesssim a^2+\|\varepsilon\|_{\dot H^2}.
$$

Using Hardy inequality (*5.B.5*) one has:

$$
\left|\int \varepsilon R\right| \lesssim \int \frac{|\varepsilon|}{1+|y|^d} \lesssim \left(\int \frac{|\varepsilon|}{1+|y|^4}\right)^{\frac12}\left(\int \frac{1}{1+|y|^{2d-4}}\right)^{\frac12} \lesssim \|\varepsilon\|_{\dot H^2}.
$$

Using the generalized Hardy inequality, interpolation and (*5.2.12*) one finds:

$$
\begin{aligned}
|\langle \varepsilon, \Lambda Q+R\rangle| & \lesssim \int \tfrac{|\varepsilon|}{1+|y|^{d-2}} \lesssim \left(\int \tfrac{|\varepsilon|^2}{1+|y|^{2+\frac23}}\right)^{\frac12}\left(\int \tfrac{1}{1+|y|^{2d-\frac{20}{3}}}\right)^{\frac12} \\
& \lesssim \|\varepsilon\|_{\dot H^{1+\frac13}} \lesssim \|\varepsilon\|_{\dot H^1}^{\frac23}\|\varepsilon\|_{\dot H^2}^{\frac13} \lesssim \eta^{\frac23}\|\varepsilon\|_{\dot H^2}^{\frac13}.
\end{aligned}
$$

Injecting the four equations above in (*5.2.38*) one finds that the second term in the right hand side of (*5.2.33*) can be rewritten as:

$$
-\frac{\langle \varepsilon, \Lambda Q\rangle\langle \Lambda(\varepsilon_s), \Lambda Q\rangle}{\langle \Lambda(Q+\varepsilon), \Lambda Q\rangle^2} = \frac{d}{ds}O\left(\eta\|\varepsilon\|_{\dot H^s}\right) + O(a^2+\|\varepsilon\|_{\dot H^2}^2). \qquad (5.2.40)
$$

∘ *End of the proof of the improved modulation equation for* λ. We now come back to the identity (*5.2.33*). We computed each term appearing in (*5.2.36*), (*5.2.37*) and (*5.2.40*), implying the improved modulation equation (*5.2.18*) claimed in the lemma and ending its proof.

□

5.2.5 Energy bounds for trapped solutions

We now provide the necessary pointwise and space time parabolic bounds on the flow which will allow us to close the control of the modulation equations of Lemma 5.2.6 in the trapped regime near $\mathcal{M}$. We first claim a global space time energy bound.

Lemma 5.2.7 (Global energy bound). *Let $d \geq 7$ and I be a time interval containing 0. There exists $0<\eta^* \ll 1$ such that if u is trapped at distance η for $0<\eta<\eta^*$ on I then:*

$$\int_{s(I)} \left(\|\varepsilon\|_{\dot H^2}^2 + a^2 \right) ds \lesssim \eta^2 \tag{5.2.41}$$

Remark 5.2.8. In dimension $d \geq 11$, one does not need to localize the orthogonality conditions in (*5.2.6*) and Q decays faster. One can then obtain simpler and stronger estimates, i.e. for (*5.2.16*) and (*5.2.17*) one would have $\|\varepsilon\|_{\dot H^2}^2$ instead of $\|\varepsilon\|_{\dot H^2}$, and for (*5.2.18*) and (*5.2.19*) one would remove the boundary term $\frac{d}{ds}O(\|\varepsilon\|_{\dot H^s})$, easing the sequel.

Proof of Lemma 5.2.7

As Q solves $\Delta Q + Q^p = 0$, from (*5.D.10*), Sobolev embedding and Hardy inequality, for any $v \in \dot H^1$:

$$\begin{aligned}
& |E(Q+v) - E(Q)| \\
= & \left| \frac12 \int |\nabla (Q+v)|^2 - \frac{1}{p+1}\int |Q+v|^{p+1} - \frac12 \int |\nabla Q|^2 + \frac{1}{p+1}\int Q^{p+1} \right| \\
\leq & \left| - \int (\Delta Q + Q^p) v + \frac12 \int |\nabla v|^2 - \frac{1}{p+1} \int (|Q+v|^{p+1} - Q^{p+1} - (p+1)Q^p v) \right| \\
\leq & \frac12 \int |\nabla v|^2 + \frac{1}{p+1} \int ||Q+v|^{p+1} - Q^{p+1} - (p+1)Q^p v| \\
\leq & C\|v\|_{\dot H^1}^2 + C\int |Q^{p-1}v^2| + |v|^{p+1} \leq C\|v\|_{\dot H^1}^2 + C\int \frac{v^2}{1+|x|^4} + \|v\|_{\dot H^1}^{p+1} \\
\leq & C\|v\|_{\dot H^1}^2 + \|v\|_{\dot H^1}^{p+1}.
\end{aligned}$$

From the above identity, the closeness assumption (*5.2.10*) and the invariance of the energy via scale and translation, we then estimate:

$$\forall t \in I, \ |E(u) - E(Q)| \lesssim \eta^2.$$

Using the regularizing effects from Proposition 5.2.1 and (*2.1.5*) one computes:

$$\int_I \|u_t\|_{L^2}^2 dt = \lim_{t \to \inf(I)} E(u(t)) - \lim_{t\to \sup(I)} E(u(t)) \lesssim \eta^2.$$

From (*5.2.13*) and (*5.2.14*), in renormalized variables, the left hand side is

$$\begin{aligned}
& \int_I \|u_t\|_{L^2}^2 dt = \int_I \int_{\mathbb{R}^d} \left(\Delta u + |u|^{p-1} u \right)^2 dt \\
= & \int_I \int_{\mathbb{R}^d} \left(\Delta (Q + a\mathcal{Y} + \varepsilon)_{z,\lambda} + |(Q + a\mathcal{Y} + \varepsilon)_{z,\lambda}|^{p-1} (Q + a\mathcal{Y} + \varepsilon)_{z,\lambda} \right)^2 dt \\
= & \int_I \int_{\mathbb{R}^d} \frac{1}{\lambda^2} \left(e_0 a \mathcal{Y} - H\varepsilon + NL \right)^2 dt = \int_{s(I)} \|e_0 a\mathcal{Y} - H\varepsilon + NL\|_{L^2}^2 ds.
\end{aligned}$$

Therefore the two above equations imply:

$$\int_{s(I)} \|e_0 a \mathcal{Y} - H\varepsilon + NL\|_{L^2}^2 ds \lesssim \eta^2. \tag{5.2.42}$$

We now show that the contribution of the two linear terms are decorelated, and that they control the nonlinear one. Young's inequality $ab \leq \frac{a}{4} + 2b$ yields:

$$\begin{aligned} \|e_0 a\mathcal{Y} - H\varepsilon + NL\|_{L^2}^2 &= \int (e_0 a\mathcal{Y} - H\varepsilon + NL)^2 \\ &= \int (e_0 a\mathcal{Y} - H\varepsilon)^2 + 2\int (e_0 a\mathcal{Y} - H\varepsilon)\, NL + \int NL^2 \\ &\geq \frac{1}{2}\int (e_0 a\mathcal{Y} - H\varepsilon)^2 - \int (NL)^2. \end{aligned}$$

From the orthogonality (*5.2.6*) and the coercivity estimate (*5.2.8*) the linear term controls the following quantities:

$$\int (e_0 a\mathcal{Y} - H\varepsilon)^2 \gtrsim a^2 + \|\varepsilon\|_{\dot{H}^2}^2.$$

From the estimate (*5.D.1*) on the nonlinearity, Sobolev embedding, interpolation and (*5.2.12*):

$$\begin{aligned} \int NL^2 &\lesssim \int |a\mathcal{Y} + \varepsilon|^{2p} \lesssim \int a^{2p}\mathcal{Y}^{2p} + |\varepsilon|^{2p} \lesssim a^{2p} + \|\varepsilon\|_{\dot{H}^{\frac{2d}{d+2}}}^{2p} \\ &\lesssim a^{2p} + \|\varepsilon\|_{\dot{H}^1}^{2p\times\frac{4}{d+2}} \|\varepsilon\|_{\dot{H}^2}^{2p\times\frac{d-2}{d+2}} \lesssim a^2\delta^{2(p-1)} + \delta^{\frac{8p}{d+2}}\|\varepsilon\|_{\dot{H}^2}^2. \end{aligned}$$

For δ small enough, the three previous equations give:

$$\|e_0 a\mathcal{Y} - H\varepsilon + (a\mathcal{Y} + \varepsilon)^2\|_{L^2}^2 \gtrsim a^2 + \|\varepsilon\|_{\dot{H}^2}^2.$$

In turn, injecting this estimate in the variation of energy formula (*5.2.42*) yields the identity (*5.2.41*) we claimed.

□

The $\dot{H}^1$ bound of Lemma 5.2.7 is not enough to control the modulation equations of Lemma 5.2.6, and we claim as a consequence of the coercivity bounds of Lemma 5.2.3 higher order $\dot{H}^2, \dot{H}^3$ bounds which lock the dynamics.

Lemma 5.2.9 (Higher order energy bounds). *Let* $d \geq 7$ *and* I *be a time interval containing* 0. *There exists* $0 < \eta^* \ll 1$ *such that if* u *is trapped at distance* η *for* $0 < \eta < \eta^*$ *on* I *then the following holds on* $s(I)$.

(i) $\dot{H}^1$ monotonicity:

$$\frac{d}{ds}\left[\frac{1}{2}\int \varepsilon H\varepsilon\right] \leq -\frac{1}{C}\int (H\varepsilon)^2 + Ca^4. \tag{5.2.43}$$

(ii) $\dot{H}^2$ monotonicity:

$$\frac{d}{ds}\left[\frac{1}{2}\int (H\varepsilon)^2\right] - \frac{\lambda_s}{\lambda}\int |H\varepsilon|^2 \leq -\frac{1}{C}\int H\varepsilon H^2\varepsilon + Ca^4 + Ca^2\|\varepsilon\|_{\dot{H}^2}^2 \tag{5.2.44}$$

Remark 5.2.10. The notation $\int H^2\varepsilon H\varepsilon$ means $\int |\nabla H\varepsilon|^2 - \int V|H\varepsilon|^2$, this later formula given by an integration by parts makes sense from Proposition 5.2.1 whereas the first one does not, but we keep it to ease notations. One also has from Proposition 5.2.1 that $u \in C^1((0,T), \dot{H}^1)$, hence the identity (*5.2.43*) makes sense. For (*5.2.44*), the quantity $\int |H\varepsilon|^2$ is well defined from Proposition 5.2.1 but this does not give its time differentiability. (*5.2.44*) should then be understood as an abuse of notation for its integral version using a standard procedure of regularization of the nonlinearity.

Proof of Lemma 5.2.9

step 1 Proof of the $\dot H^1$ bound. One computes from (*5.2.14*) using the orthogonality conditions (*5.2.6*), (*5.1.7*), (*5.1.6*) and (*5.1.8*):

$$\begin{array}{rcl} \frac{d}{ds}\left[\frac12\int \varepsilon H\varepsilon\right] & = & -\int (H\varepsilon)^2+\int NLH\varepsilon+\frac12\frac{\lambda_s}{\lambda}\int\varepsilon^2(V+y.\nabla V)\\ & & -\int\varepsilon^2\frac{z_s}{\lambda}.\nabla V+a\frac{\lambda_s}{\lambda}\int H\varepsilon\Lambda\mathcal{Y}+a\int H\varepsilon\frac{z_s}{\lambda}.\nabla\mathcal{Y}\end{array} \tag{5.2.45}$$

and we now estimate each term in the right hand side. Using Young inequality, the estimate (*5.D.3*) on the nonlinearity, Sobolev embedding, interpolation, (*5.2.12*) and the fact that $\mathcal{Y}$ is exponentially decreasing one has for any $0<\kappa\ll 1$:

$$\begin{array}{rl} & \left|\int NLH\varepsilon\right|\leq\int\frac{\kappa(H\varepsilon)^2}{2}+\frac{NL^2}{2\kappa}\lesssim\frac{\kappa}{2}\int(H\varepsilon)^2+\int\frac{a^4}{2\kappa}\mathcal{Y}^4Q^{2(p-2)}+\frac{1}{2\kappa}\int\varepsilon^{2p}\\ \lesssim & \frac{\kappa}{2}\int(H\varepsilon)^2+\frac{a^4}{2\kappa}+\frac{1}{2\kappa}\|\varepsilon\|^{2p}_{\dot H^{\frac{2d}{d+2}}}\lesssim\frac{\kappa}{2}\int(H\varepsilon)^2+\frac{a^4}{2\kappa}+\frac{1}{2\kappa}\|\varepsilon\|^{\frac{8p}{d+2}}_{\dot H^1}\|\varepsilon\|^2_{\dot H^2}\\ \lesssim & \frac{\kappa}{2}\int(H\varepsilon)^2+\frac{a^4}{2\kappa}+\frac{\eta^{\frac{8p}{d+2}}}{2\kappa}\int(H\varepsilon)^2.\end{array}$$

Now using (*5.2.16*), (*5.2.17*), and the coercivity estimate (*5.2.7*):

$$\left|\frac12\frac{\lambda_s}{\lambda}\int\varepsilon(V+y.\nabla V)\varepsilon-\int\varepsilon^2\frac{z_s}{\lambda}.\nabla V\right|\lesssim\eta\int\frac{|\varepsilon|^2}{1+|y|^4}\lesssim\eta\int(H\varepsilon)^2.$$

Using Young inequality, the estimates (*5.2.16*), (*5.2.17*), and (*5.2.12*) one has for any $0<\kappa\ll 1$:

$$\begin{array}{rcl}\left|a\frac{\lambda_s}{\lambda}\int H\varepsilon\Lambda\mathcal{Y}+a\int H\varepsilon\frac{z_s}{\lambda}.\nabla\mathcal{Y}\right| & \lesssim & \frac{a^2}{\kappa}\left(\left|\frac{\lambda_s}{\lambda}\right|^2+\left|\frac{z_s}{\lambda}\right|^2\right)+\kappa\int(H\varepsilon)^2\\ & \lesssim & \frac{a^6}{\kappa}+\left(\frac{\eta^2}{\kappa}+\kappa\right)\int(H\varepsilon)^2\end{array}$$

We now inject the three estimates above in the identity (*5.2.45*). One can choose κ small enough independently of η^* and then η^* small enough such that the desired identity (*5.2.43*) holds.

step 2 Proof of the $\dot H^2$ bound. One computes first the following identity using the evolution equation (*5.2.14*), the orthogonality (*5.2.6*), the identities (*5.1.6*), (*5.1.7*) and $H\nabla=\nabla H-\nabla V$:

$$\begin{array}{rl} & \frac{d}{ds}\left(\frac12\|H\varepsilon\|^2_{L^2}\right)\\ = & \frac{\lambda_s}{\lambda}\int H\varepsilon H(a\Lambda\mathcal{Y}+\Lambda\varepsilon)+\int H\varepsilon H(\frac{z_s}{\lambda}.\nabla(a\mathcal{Y}+\varepsilon))-\int\varepsilon H^3\varepsilon+\int H\varepsilon H(NL)\\ = & \frac{\lambda_s}{\lambda}\int|H\varepsilon|^2+\frac{\lambda_s}{\lambda}a\int H\varepsilon H\Lambda\mathcal{Y}-\int\varepsilon H^3\varepsilon+\int H\varepsilon H(NL)\\ + & \frac{\lambda_s}{\lambda}\int H\varepsilon(2V+y.\nabla V)\varepsilon+a\int H\varepsilon H(\frac{z_s}{\lambda}.\nabla\mathcal{Y})-\int H\varepsilon\frac{z_s}{\lambda}.\nabla V\varepsilon.\end{array}\tag{5.2.46}$$

Estimate for the nonlinear term. One first computes the influence of the nonlinear term. Performing an integration by parts we write:

$$\int H\varepsilon H(NL)=\int H\varepsilon VNL+\int\nabla H\varepsilon.\nabla(NL).\tag{5.2.47}$$

The potential term is estimated directly as it has a very strong decay, using the estimate (*5.D.3*) for the nonlinear term, the coercivity (*5.2.8*), (*5.2.12*), Sobolev embedding, interpolation and the fact that $\mathcal{Y}$ is exponentially decaying:

$$
\begin{aligned}
&\left|\int H(\varepsilon)VNL\right| \lesssim \int \frac{|H\varepsilon||NL|}{1+|y|^4} \lesssim \int \frac{|H\varepsilon|(|a|^2\mathcal{Y}^2Q^{p-2}+|\varepsilon|^p)}{1+|y|^4}\\
\lesssim\ & \|\varepsilon\|_{\dot{H}^3}a^2+\left(\int\frac{|H\varepsilon|^2}{1+|y|^2}\right)^{\frac12}\left(\int\frac{|\varepsilon|^{2p}}{1+|y|^6}\right)^{\frac12}\\
\lesssim\ & \|\varepsilon\|_{\dot{H}^3}a^2+\left(\int H\varepsilon H^2\varepsilon\right)^{\frac12}\left(\||\varepsilon|^{2p}\|_{L^{\frac{d}{d-2}}}\|\frac{1}{1+|y|^6}\|_{L^{\frac d2}}\right)^{\frac12}\\
\lesssim\ & \|\varepsilon\|_{\dot{H}^3}a^2+\left(\int H\varepsilon H^2\varepsilon\right)^{\frac12}\|\varepsilon\|^p_{L^{\frac{2d(d+2)}{(d-2)^2}}}\\
\lesssim\ & \|\varepsilon\|_{\dot{H}^3}a^2+\left(\int H\varepsilon H^2\varepsilon\right)^{\frac12}\|\varepsilon\|^p_{\dot{H}^{3-\frac{8}{d+2}}}\\
\lesssim\ & \|\varepsilon\|_{\dot{H}^3}a^2+\left(\int H\varepsilon H^2\varepsilon\right)^{\frac12}\|\varepsilon\|^{\frac{4}{d+2}p}_{\dot{H}^1}\|\varepsilon\|^{\frac{d-2}{d+2}p}_{\dot{H}^3}\\
\lesssim\ & \|\varepsilon\|_{\dot{H}^3}a^2+\left(\int H\varepsilon H^2\varepsilon\right)^{\frac12}\eta^{\frac{4}{d+2}p}\|\varepsilon\|_{\dot{H}^3}\\
\lesssim\ & \|\varepsilon\|_{\dot{H}^3}a^2+\eta^{\frac{4}{d+2}p}\int H\varepsilon H^2\varepsilon. \qquad (5.2.48)
\end{aligned}
$$

Now for the other term one first computes the first derivatives of the nonlinear term:

$$
\begin{aligned}
\nabla(NL) &= p\left(|Q+a\mathcal{Y}+\varepsilon|^{p-1}-Q^{p-1}\right)\nabla(a\mathcal{Y}+\varepsilon)\\
&\quad +p\left(|Q+a\mathcal{Y}+\varepsilon|^{p-1}-Q^{p-1}-(p-1)Q^{p-2}(a\mathcal{Y}+\varepsilon)\right)\nabla Q \qquad (5.2.49)\\
&=: A_1+A_2.
\end{aligned}
$$

We estimate the first term pointwise using the estimates on the nonlinearity (*5.D.4*) and (*5.D.5*):

$$
\begin{aligned}
|A_1| &= p\left|\left(|Q+a\mathcal{Y}+\varepsilon|^{p-1}-Q^{p-1}\right)\nabla(a\mathcal{Y}+\varepsilon)\right|\\
&\leq \left(Q^{p-2}|a|\mathcal{Y}+|\varepsilon|^{p-1}|\right)|a||\nabla\mathcal{Y}|+\left(Q^{p-2}|a|\mathcal{Y}+|\varepsilon|^{p-1}|\right)|\nabla\varepsilon|\\
&\lesssim \left(Q^{p-2}|a|\mathcal{Y}+|\varepsilon||\right)|a||\nabla\mathcal{Y}|+\left(Q^{p-2}|a|\mathcal{Y}+|\varepsilon|^{p-1}|\right)|\nabla\varepsilon|\\
&\lesssim Q^{p-2}a^2\mathcal{Y}|\nabla\mathcal{Y}|+Q^{p-2}|\varepsilon||a||\nabla\mathcal{Y}|+Q^{p-2}|a|\mathcal{Y}|\nabla\varepsilon|+|\varepsilon|^{p-1}|\nabla\varepsilon|
\end{aligned}
$$

Hence, using the coercivity (*5.2.9*) for the second and third terms, and Sobolev embedding plus (*5.2.12*) for the last one:

$$
\begin{aligned}
\int|A_1|^2 &\lesssim \int Q^{2(p-2)}a^4\mathcal{Y}^2|\nabla\mathcal{Y}|^2+Q^{2(p-2)}|\varepsilon|^2a^2|\nabla\mathcal{Y}|^2\\
&\quad+\int Q^{2(p-2)}a^2\mathcal{Y}^2|\nabla\varepsilon|^2+|\varepsilon|^{2(p-1)}|\nabla\varepsilon|^2\\
&\lesssim a^4+a^2\|\varepsilon\|^2_{\dot{H}^3}+\||\nabla\varepsilon|^2\|_{L^{\frac{d}{d-4}}}\||\varepsilon|^{2(p-1)}\|_{L^{\frac d4}}\\
&\lesssim a^4+a^2\|\varepsilon\|^2_{\dot{H}^3}+\|\nabla\varepsilon\|^2_{L^{\frac{2d}{d-4}}}\|\varepsilon\|^{2(p-1)}_{L^{\frac{2d}{d-2}}}\\
&\lesssim a^4+a^2\|\varepsilon\|^2_{\dot{H}^3}+\|\varepsilon\|^2_{\dot{H}^3}\|\varepsilon\|^{2(p-1)}_{\dot{H}^1}\\
&\lesssim a^4+(a^2+\eta^{2(p-1)})\|\varepsilon\|^2_{\dot{H}^3} \qquad (5.2.50)
\end{aligned}
$$

We now turn to the second term A_2. We use the pointwise estimate for the nonlinearity (*5.D.6*), yielding:

$$|A_2| \lesssim Q^{p-2-\frac{2}{d-2}}|\varepsilon|^{1+\frac{2}{d-2}}|\nabla Q| + a^2\mathcal{Y}^2 Q^{p-3}|\nabla Q|.$$

We then estimate A_2 using Sobolev embedding, the coercivity estimate (*5.2.9*) and (*5.2.12*):

$$\begin{array}{lll} \int |A_2|^2 & \lesssim & a^4 + \int \frac{|\varepsilon|^{\frac{2d}{d-2}}}{1+|y|^6} \lesssim a^4 + \| |\varepsilon|^{\frac{2d}{d-2}} \|_{L^{\frac{d}{d-4}}} \| \frac{1}{1+|y|^6} \|_{L^{\frac{d}{4}}} \\ & \lesssim & a^4 + \|\varepsilon\|^{\frac{2d}{d-2}}_{L^{\frac{2d^2}{(d-2)(d-4)}}} \lesssim a^4 + \|\varepsilon\|^{\frac{2d}{d-2}}_{\dot{H}^{3-\frac{4}{d}}} \\ & \lesssim & a^4 + \|\varepsilon\|^{\frac{2d}{d-2}\frac{2}{d}}_{\dot{H}^1} \|\varepsilon\|^{\frac{2d}{d-2}\frac{d-2}{d}}_{\dot{H}^3} \lesssim a^4 + \eta^{\frac{4}{d-2}}\|\varepsilon\|^2_{\dot{H}^3}. \end{array} \quad (5.2.51)$$

Putting together the two estimates (*5.2.50*) and (*5.2.51*) we have found for the two terms in (*5.2.49*) yields:

$$\int |\nabla NL|^2 \lesssim a^4 + (a^2 + \eta^{\frac{4}{d-2}})\|\varepsilon\|^2_{\dot{H}^3}.$$

We now come back to the original nonlinear term (*5.2.47*) we had to treat, and inject the above estimate and the estimate (*5.2.48*) we just found for each term, yielding:

$$\begin{array}{ll} & |\int H\varepsilon H(NL)| \lesssim a^2\|\varepsilon\|_{\dot{H}^3} + (a + \eta^{\frac{2}{d-2}})\|\varepsilon\|^2_{\dot{H}^3} \\ \lesssim & \frac{a^4}{\kappa} + (a + \eta^{\frac{2}{d-2}} + \kappa)\|\varepsilon\|^2_{\dot{H}^3} \end{array} \quad (5.2.52)$$

where we used Young inequality with a small parameter $0 < \kappa \ll 1$ to be chosen later on.

Remainders from scale and space change. We put some upper bounds on the following term appearing in (*5.2.46*). From (*5.2.16*) and the coercivity estimate (*5.2.9*):

$$\begin{array}{lll} \left| \frac{\lambda_s}{\lambda} \int H\varepsilon(2V + y.\nabla V)\varepsilon \right| & \lesssim & \eta \int \frac{|\varepsilon||H\varepsilon|}{1+|y|^4} \leq \eta \left(\int \frac{|\varepsilon|^2}{1+|y|^6} \right)^{\frac{1}{2}} \left(\int \frac{|H\varepsilon|^2}{1+|y|^2} \right)^{\frac{1}{2}} \\ & \lesssim & \eta \int H\varepsilon H^2\varepsilon. \end{array} \quad (5.2.53)$$

Similarly, from (*5.2.17*):

$$\left| \int H\varepsilon \frac{z_s}{\lambda}.\nabla V\varepsilon \right| \lesssim \eta \int \frac{|H\varepsilon|\varepsilon}{1+|x|^5} \lesssim \eta^2 \int H\varepsilon H^2\varepsilon. \quad (5.2.54)$$

Now, from the fact that $\mathcal{Y}$ decays exponentially fast and the coercivity estimate (*5.2.9*), one has for any $0 < \kappa \ll 1$ to be chosen later:

$$\begin{array}{ll} & \left| \frac{\lambda_s}{\lambda} a \int H\varepsilon H\Lambda\mathcal{Y} + a \int H\varepsilon H(\frac{z_s}{\lambda}.\nabla\mathcal{Y}) \right| \\ \lesssim & \frac{a^2}{\kappa} \left(\left|\frac{\lambda_s}{\lambda}\right|^2 + \left|\frac{z_s}{\lambda}\right|^2 \right) + \kappa \int H\varepsilon H^2\varepsilon. \end{array} \quad (5.2.55)$$

End of the proof of the energy identity. We come back to the identity (*5.2.46*) and inject all the bounds we found so far, (*5.2.52*), (*5.2.53*), (*5.2.54*) and (*5.2.55*). Using the two different types of estimates for the modulation of λ and z, (*5.2.16*) and (*5.2.17*), one sees that there exists κ and η^* such that the desired bound (*5.2.44*) holds. $\square$

5.2.6 No type II blow up near the soliton

A spectacular consequence of Lemma 5.2.6, Lemma 5.2.7 and Lemma 5.2.9 is the uniform control of the scale for trapped solutions.

Lemma 5.2.11 (Non degeneracy of the scale for trapped solutions). *Let $d \geq 7$ and I be a time interval containing 0. There exists $0 < \eta^* \ll 1$ such that if u is trapped at distance η for $0 < \eta < \eta^*$ on I, then:*

$$\lambda(t) = \lambda(0)(1 + O(\eta)) \tag{5.2.56}$$

In particular, this rules out type II blow up near the solitary wave Q as in [68] for the large homotopy number corotational harmonic heat flow.

Proof of Lemma 5.2.11

We reason in renormalized time. The modulation equation (*5.2.16*) for λ can be written as:

$$\left| \frac{d}{ds} \left[\log(\lambda) + O(\|\varepsilon(s)\|_{\dot{H}^1}) \right] \right| \lesssim a^2 + \|\varepsilon\|_{\dot{H}^2}^2$$

on $s(I)$. We now reintegrate it in time:

$$\left| \log\left(\frac{\lambda(s)}{\lambda(0)} \right) + \|\varepsilon(s)\|_{\dot{H}^1} + O(\|\varepsilon(0)\|_{\dot{H}^1}) \right| \lesssim \int_0^s a^2 + \|\varepsilon\|_{\dot{H}^2}^2$$

and inject the estimate (*5.2.41*) for the right hand side coming from the variation of energy and (*5.2.12*), yielding:

$$\lambda(s) = \lambda(0) e^{O(\eta)} = \lambda(0)(1 + O(\eta))$$

and (*5.2.56*) is proved. □

Remark 5.2.12. The estimate (*5.2.56*) implies that for solutions trapped at distance η on I, the original time variable t is equivalent to the renormalized time variable s, namely there exists $c > 0$ such that for all $t \in I$:

$$\lambda(0)(1 - c\eta)t \leq s \leq \lambda(0)(1 + c\eta)t. \tag{5.2.57}$$

5.3 Existence and uniqueness of minimal solutions

This section is devoted to the proof of existence and uniqueness of the minimal elements which are trapped backwards in time near the soliton manifold, together with the complete description of their forward behaviour.

5.3.1 Existence

The existence of $Q^{\pm}$ follows from a simple brute force fixed point argument near $-\infty$, while the derivation of their forward (exit) behavior follows from the maximum principle. We may here relax the dimensional assumption and assume $d \geq 3$ only.

Proposition 5.3.1 (Existence of $Q^{\pm}$). *Let $d \geq 3$. There exists two strictly positive, $\mathcal{C}^{\infty}$ and radial solutions of (NLH), Q^+ and Q^-, defined on $(-\infty, 0] \times \mathbb{R}^d$, such that*

$$\lim_{t \to -\infty} \|Q^{\pm} - Q\|_{\dot{H}^1} = 0.$$

Moreover:
1. Trapping near Q for $t \leq 0$*: there holds the expansion for $t \in (-\infty, 0]$:*

$$Q^{\pm}(t) = Q \pm \epsilon e^{e_0 t} \mathcal{Y} + v, \ \ \|v\|_{\dot{H}^1} + \|v\|_{L^{\infty}} \lesssim \epsilon^2 e^{2e_0 t} \tag{5.3.1}$$

for some $0 < \epsilon \ll 1$.
2. Forward (exit)*: Q^+ explodes forward in finite time in the ODE type I blow up regime, while Q^- is global and dissipates* $\lim_{t \to +\infty} \|Q^-\|_{\dot{H}^1} = 0$.

Proof of Proposition 5.3.1

We prove the existence of Q^+ and Q^- via a compactness argument. Let $0 < \epsilon \ll 1$ be a small enough strictly positive constant. We look at two solutions $u_n^{\pm}$ of (NLH) with initial data at the initial time $t_0(n) = -n$:

$$u_n^{\pm}(-n) = Q \pm \epsilon e^{-ne_0} \mathcal{Y}. \tag{5.3.2}$$

The sign $+$ (resp. $-$) will give an approximation of Q^+ (resp. Q^-). The results and techniques employed in this step being exactly the same for the two cases, we focus on the $+$ sign case. As $u_n^{\pm}(-n)$ is radial, $u_n^{\pm}$ is radial for all times.

step 1 Forward propagation of smallness. We claim that there exists constants $C_a, \ C_2, \ C_{\infty} > 0$ such that for ϵ small enough for any $n \in \mathbb{N}$ the solution is at least defined on $[-n, 0]$ and can be written on this interval under the form:

$$u_n^+(t) = Q + \epsilon(1+a) e^{te_0} \mathcal{Y} + \epsilon v, \ \ a : [-n, 0] \to \mathbb{R}, \ \ v \in L^2 \cap L^{\infty}, \ \ v \perp \mathcal{Y} \tag{5.3.3}$$

where the corrections a and v satisfy the following bounds on $[-n; 0]$:

$$\|a\|_{L^{\infty}} \leq \epsilon C_a e^{e_0 t}, \ \ \|v\|_{L^2} \leq \epsilon C_2 e^{2te_0}, \ \ \|v\|_{L^{\infty}} \leq \epsilon C_{\infty} e^{2te_0}. \tag{5.3.4}$$

Indeed, the decomposition and the regularity of the solution is ensured by the fact that one can solve the Cauchy problem for $H^1 \cap L^{\infty}$ perturbations of Q arguing like for the proof of Proposition 5.2.1. We now prove (*5.3.4*) using a bootstrap argument. Let $\mathcal{T} \subset [-n, 0]$ be the set of times $-n \leq t \leq 0$ such that (*5.3.4*) holds on $[-n, t]$. $\mathcal{T}$ is not empty as it contains $-n$, and it is closed from a continuity argument. We now show that it is open, implying $\mathcal{T} = [-n, 0]$ using connectedness. To do so, we are going to show that under some compatibility conditions on the constants $C_a, \ C_2, \ C_{\infty}$ and for ϵ sufficiently small, the inequalities in (*5.3.4*) are strict in $\mathcal{T}$, implying that this latter is open by continuity. From (NLH) the time evolution of v is given by:

$$v_t + Hv = -a_t e^{te_0} \mathcal{Y} + NL, \ \ NL := \frac{1}{\epsilon} \left[f(u_n^+) - p\epsilon Q^{p-1}(e^{te_0}(1+a)\mathcal{Y} + v) \right]. \tag{5.3.5}$$

Using the bound (*5.D.3*) on the nonlinearity, the bootstrap bounds (*5.3.4*), the fact that $\mathcal{Y}$ decays exponentially and interpolation, one gets for $q \in [2, +\infty]$ and $t \in \mathcal{T}$:

$$\begin{array}{rcl} \|NL\|_{L^q} & \leq & C\|Q^{p-2}\epsilon e^{2e_0 t}(1+a)^2\mathcal{Y}^2 + \epsilon^{p-1}|v|^p\|_{L^q} \\ & \leq & C\epsilon e^{2e_0 t} + CC_a^2\epsilon^2 e^{4e_0 t} + C\epsilon^{2p-1}C_2^{\frac{2}{q}}C_\infty^{\frac{q-2}{q}} \\ & \leq & C\epsilon e^{2te_0} \end{array} \qquad (5.3.6)$$

for a constant C independent of the others, for any choice of C_a, C_2 and C_∞ if ϵ is chosen small enough, as $t \leq 0$ and $p > 1$.

Bound for a. Projecting (*5.3.5*) onto $\mathcal{Y}$, using the orthogonality $v \perp \mathcal{Y}$ and the above estimate then yields that for $t \in \mathcal{T}$:

$$|a_t| \leq C\epsilon e^{e_0 t} \qquad (5.3.7)$$

for C independent of $C_a,\ C_2,\ C_\infty,\ \epsilon$. Reintegrated in time this gives:

$$|a(t)| \leq |a(-n)| + C\epsilon e^{e_0 t} \leq C\epsilon e^{e_0 t} < C_a e^{e_0 t} \ \text{ if } \ C < C_a \qquad (5.3.8)$$

for C independent of $C_a,\ C_2,\ C_\infty,\ \epsilon$ for any $t \in \mathcal{T}$ as $a(-n) = 0$ from (*5.3.2*) and (*5.3.3*).

L^2 ***bound for*** v. We multiply (*5.3.5*), by v and integrate using (*5.3.6*), (*5.3.7*), Cauchy-Schwarz inequality and the fact $\int vHv \geq 0$ from Proposition 5.2.2 to obtain the following energy estimate on $\mathcal{T}$:

$$\begin{array}{rcl} \frac{d}{dt}\|v\|_{L^2}^2 & = & -2\int vHv + 2\int v(-a_t e^{te_0}\mathcal{Y} + NL) \leq C\|v\|_{L^2}(\|NL\|_{L^2} + |a_t|) \\ & \leq & CC_2\epsilon e^{e_0 t}. \end{array}$$

Reintegrated in time, as $v(-n) = 0$ from (*5.3.2*) and (*5.3.3*) one obtains that for $t \in \mathcal{T}$:

$$\|v\|_{L^2} \leq C\sqrt{C_2}\epsilon e^{e_0 t} < C_2\epsilon e^{e_0 t} \ \text{ if } \ C < C_2 \qquad (5.3.9)$$

for C independent of $C_a,\ C_2,\ C_\infty,\ \epsilon$.

L^∞ ***bound for*** v. Using Duhamel formula one has:

$$v(t) = \int_{-n}^{t} K_{t-t'} * (-a_t e^{t' e_0}\mathcal{Y} + NL + Vv)dt', \qquad (5.3.10)$$

where K_t is defined by (*5.1.1*). For the first two terms, thanks to the bounds (*5.3.6*) and (*5.3.7*) on a_t and NL and the fact that $K_t* : L^\infty \to L^\infty$ is unitary, one gets:

$$\|\int_{-n}^{t} K_{t-t'} * (-a_t e^{t' e_0}\mathcal{Y} + NL)\|_{L^\infty} \leq \int_{-n}^{t} C\epsilon e^{2t' e_0} \leq C\epsilon e^{2te_0}$$

for C independent of $C_a,\ C_2,\ C_\infty,\ \epsilon$ and $t \in \mathcal{T}$. Now for the last term, for $\delta = \frac{1}{2\|V\|_{L^\infty}}$, making the abuse of notation $t - \delta = \min(t-\delta, -n)$, using Hölder inequalities, (*5.E.1*), the fact that $\|K_t\|_{L^1} = 1$ for all $t\geq 0$ and interpolation one computes for $t\in\mathcal{T}$:

$$\begin{aligned}
& \| \int_{-n}^t K_{t-t'}*(Vv)dt'\|_{L^\infty} \\
\leq & \int_{-n}^{t-\delta}\|K_{t-t'}*(Vv)\|_{L^\infty}dt'+\int_{t-\delta}^t\|K_{t-t'}*(Vv)\|_{L^\infty}dt' \\
\leq & \int_{-n}^{t-\delta}\|K_{t-t'}\|_{L^{\frac{d}{d-1}}}\|Vv\|_{L^d}dt'+\int_{t-\delta}^t\|K_{t-t'}\|_{L^1}\|Vv\|_{L^\infty}dt' \\
\leq & \int_{-n}^{t-\delta}\frac{C}{|t-t'|^{\frac12}}\|v\|_{L^{2d}}\|V\|_{L^{2d}}dt'+\int_{t-\delta}^t\|V\|_{L^\infty}\|v\|_{L^\infty}dt' \\
\leq & \int_{-n}^{t-\delta}\frac{C}{|t-t'|^{\frac12}}\epsilon e^{2e_0t'}C_2^{\frac1d}C_\infty^{1-\frac1d}dt'+\int_{t-\delta}^t\|V\|_{L^\infty}\epsilon C_\infty e^{2e_0t'}dt' \\
\leq & C\epsilon C_2^{\frac1d}C_\infty^{1-\frac1d}e^{2e_0t}+\delta\|V\|_{L^\infty}\epsilon C_\infty e^{2e_0t}\leq \epsilon e^{2e_0t}(CC_2^{\frac1d}C_\infty^{1-\frac1d}+\frac12 C_\infty).
\end{aligned}$$

The three previous equations then imply that for $t\in\mathcal{T}$:

$$\|v\|_{L^\infty}\leq \epsilon e^{2e_0t}(C+CC_2^{\frac1d}C_\infty^{1-\frac1d}+\frac12 C_\infty)<\epsilon C_\infty e^{2e_0t}C \tag{5.3.11}$$

provided $C+CC_2^{\frac1d}C_\infty^{1-\frac1d}<C_\infty$.

Conclusion. The estimates (*5.3.8*), (*5.3.9*) and (*5.3.11*) ensure the existence of $\epsilon_0>0$, $C_a,C_2,C_\infty>0$ such that for $0<\epsilon\leq\epsilon_0$, (*5.3.4*) is strict on $\mathcal{T}$ hence this latter is open, and (*5.3.4*) is proved.

step 2 Propagation of smoothness. We claim that one has the following additional bounds on $[-n+1,0]$:

$$\|v\|_{W^{2,\infty}}+\|v\|_{\dot H^2}\leq \epsilon e^{te_0}, \tag{5.3.12}$$

$$\|\nabla^2 v\|_{C^{0,\frac14}(\mathbb{R}^d\times[-n+1,0])}+\|\partial_t v\|_{L^\infty(\mathbb{R}^d\times[-n+1,0])}+\|\partial_t v\|_{C^{0,\frac14}(\mathbb{R}^d\times[-n+1,0])}\leq C \tag{5.3.13}$$

and that for all $t\in[-n,0]$:

$$\|v\|_{\dot H^1}\leq C\epsilon e^{2e_0t}. \tag{5.3.14}$$

where $C^{0,\frac14}$ denotes the Hölder $\frac14$-norm, for C independent of n and ϵ. The first two bounds (*5.3.12*) and (*5.3.13*) are direct consequences of (*5.3.3*), (*5.3.4*) and the parabolic estimates (*5.E.6*) and (*5.E.7*) from Lemma 5.E.3. To prove the last bound (*5.3.14*) one computes using (*5.3.10*), (*5.3.7*), (*5.3.4*), (*5.D.1*), Young and Hölder inequalities that for $t\in[-n,0]$:

$$\begin{aligned}
\|v(t)\|_{\dot H^1} \leq & \int_{-n}^t\|K_{t-t'}\|_{L^1}\|a_te^{t'e_0}\mathcal{Y}\|_{\dot H^1}dt' \\
& +\int_{-n}^t\|\nabla K_{t-t'}\|_{L^1}(\|NL\|_{L^2}+\|Vv\|_{L^2})dt' \\
\leq & C\epsilon\int_{-n}^t\left(e^{2e_0t'}+\frac{e^{2e_0t'}}{|t-t'|^{\frac12}}+\frac{e^{2e_0t'}}{|t-t'|^{\frac12}}\right)dt'\leq C\epsilon e^{2e_0t}
\end{aligned}$$

for $C>0$ independent of the other constants, ending the proof of the claim.

step 3 Maximum principle for $u_n^\pm$. We claim that

$$\left| \begin{array}{l} u_n^+ \geq Q, \quad \partial_t(u_n^+) \geq 0 \\ 0 \leq u_n^- \leq Q \ \text{ and } \ \partial_t(u_n^-) \leq 0 \end{array} \right. \tag{5.3.15}$$

We prove it for u_n^+, the proof being similar for u_n^-. This is true at initial time $-n$, because $\mathcal{Y} > 0$ and $\epsilon > 0$, implying $u_n^+(-n) - Q = \epsilon e^{-ne_0}\mathcal{Y} > 0$ and

$$\begin{array}{rcl} \partial_t(u_n^+)(-n) & = & -H(\epsilon e^{-ne_0}\mathcal{Y}) + [f(Q+\epsilon e^{-ne_0}\mathcal{Y}) - f(Q) - f'(Q)\epsilon e^{-ne_0}\mathcal{Y}] \\ & = & e_0\epsilon e^{-ne_0}\mathcal{Y} + [f(Q+\epsilon e^{-ne_0}\mathcal{Y}) - f(Q) - f'(Q)\epsilon e^{-ne_0}\mathcal{Y}] \\ & > & 0 \end{array}$$

as the nonlinearity f is strictly convex on $[0,+\infty)$ from $p>1$. Therefore, from the maximum principle, see Lemma 5.E.2, one has that $u_n^+ \geq 0$ and $\partial_t u_n^+ \geq 0$ on $[-n, 0]$.

step 4 Compactness. From (*5.3.4*), (*5.3.7*), (*5.3.12*), (*5.3.13*), Arzela-Ascoli theorem and a diagonal argument, there exist subsequences of $(u_n^+)_{n\in\mathbb{N}}$ and $(u_n^-)_{n\in\mathbb{N}}$ that converge in $C^{1,2}_{\text{loc}}((-\infty,0]\times\mathbb{R}^d)$ toward some functions that we call Q^+ and Q^- respectively. The equation (NLH) then passes to the limit using (*5.3.12*), (*5.3.13*), implying that Q^+ and Q^- are also solutions of (NLH) on $(-\infty,0]$. For each time $t\leq 0$, the L^∞ bound in (*5.3.4*) and (*5.3.12*) passes to the limit via pointwise convergence, and the $\dot{H}^1$ bound (*5.3.14*) passes to the limit via lower semi- continuity of the $\dot{H}^1$ norm. This implies that Q^+ and Q^- can be decomposed according to:

$$Q^\pm = Q \pm \epsilon e^{te_0}\mathcal{Y} + a\epsilon e^{te_0}\mathcal{Y} + \epsilon v \tag{5.3.16}$$

with $v \perp \mathcal{Y}$ satisfying:

$$|a| \lesssim \epsilon e^{2te_0}, \quad \|v\|_{\dot{H}^1} + \|v\|_{L^\infty} \lesssim \epsilon e^{2te_0}. \tag{5.3.17}$$

Moreover, as $u_n^\pm$ is radial, so is $Q^\pm$.

step 5 Q^+ blows up forward Type I. The estimate (*5.3.15*) ensures $Q^+(t) \geq Q(t)$ on the whole space-time domain $(-\infty,0]\times\mathbb{R}^d$. This then propagates forward in time according to the maximum principle, see Lemma 5.E.2, giving that $Q^+ \geq Q$ on $[0,T)$ where T denotes the maximal time of existence of Q^+. We recall that $\mathcal{Y} > 0$, $\int \mathcal{Y} = 1$, and hence since the function $g:[0,+\infty)\to\mathbb{R}$ defined by $g(x) = (Q+x)^p - Q^p - pQ^{p-1}x$ is convex from $p>1$, Jensen inequality implies:

$$g\left(\int (Q^+ - Q)\mathcal{Y}\right) \leq \int g(Q^+ - Q)\mathcal{Y}.$$

This gives the following polynomial lower bound for the derivative of the component along $\mathcal{Y}$:

$$\begin{array}{rcl} \partial_t\left(\int (Q^+ - Q)\mathcal{Y}\right) & = & e_0\int (Q^+ - Q)\mathcal{Y} + \int g(Q^+ - Q)\mathcal{Y} \\ & \geq & e_0\int (Q^+ - Q)\mathcal{Y} + g\left(\int (Q^+ - Q)\mathcal{Y}\right) \end{array}$$

As this quantity is strictly positive at time 0 from (*5.3.16*) and (*5.3.17*), this implies that Q^+ blows up in finite time because $g(x)\sim x^p$ as $x\to+\infty$. Moreover, from Theorem 1.7 in [97], $u>0$ implies that the blow up is of type 1. Hence Q^+ blows up with a type I blow up forward in time.

step 6 Q^- dissipates forward. As $\|u_n^-(0)-Q\|_{L^\infty}\geq \frac{\epsilon}{2}$ from (*5.3.3*) and (*5.3.4*), $0\leq u_n^-(0)\leq Q$ and $\partial_t u_n^-(0)\leq 0$, in the limit one obtains that $Q^-\neq Q$, $Q^-\neq 0$, $\partial_t Q^-(0)\leq 0$ and $0\leq Q^-(0)\leq Q$. Using the maximum principle for Q^- and $\partial_t Q^-$, see Lemma 5.E.2, one has that $0\leq Q^-\leq Q$ and $\partial_t Q^-\leq 0$ for all times $t\in[0,T)$ where T is the maximal time of existence of Q^-. The L^∞ Cauchy theory then ensures that Q^- is a global solution. As $0\leq Q^-\leq Q$ on $(0,+\infty)$, from the parabolic estimates (*5.E.6*) and (*5.E.7*) one deduces that for any $t>1$,

$$\|Q^-\|_{W^{2,\infty}}+\|\partial_t Q^-\|_{L^\infty}\lesssim 1, \tag{5.3.18}$$

$$\|\nabla^2 Q^-\|_{C^{0,\frac14}([t,t+1]\times\mathbb{R}^d)}+\|\partial_t Q^-\|_{C^{0,\frac14}([t,t+1]\times\mathbb{R}^d)}\lesssim 1 \tag{5.3.19}$$

where $C^{0,\frac14}$ denotes the Hölder norm. We define $u_\infty(x):=\lim\limits_{t\to+\infty}Q^-(t,x)$ which exists as $\partial_t Q^-\leq 0$ and $Q^-\geq 0$, satisfies $0\leq u_\infty\leq Q$, $u_\infty\neq Q$ and is radial. For $n\in\mathbb{N}$ let the sequence of functions

$$v_n(t,x)=Q^-(n+t,x),\quad (t,x)\in(0,1)\times\mathbb{R}^d. \tag{5.3.20}$$

As Q^- is decreasing, for any $0\leq t_1\leq t_2\leq 1$ and $x\in\mathbb{R}^d$ there holds

$$Q(n+t_1,x)=v_n(t_1,x)\geq v_n(t_2,x)=Q(n+t_2,x)$$

which implies:

$$\lim_{n\to+\infty}v_n(t_1,x)=\lim_{n\to+\infty}v_n(t_2,x)=\lim_{t\to+\infty}Q^-(t,x)=u_\infty(x),$$

meaning that v_n converges to the constant in time function u_∞. From its definition (*5.3.20*) and the bounds (*5.3.18*) and (*5.3.19*), using Arzela-Ascoli theorem, $u_\infty\in W^{2,\infty}$ and $v_n\to u_\infty$ in $C^{1,2}_{\text{loc}}((0,1)\times\mathbb{R}^d)$. From (*5.3.20*), v_n solves $\partial_t v_n=\Delta v_n+|v_n|^{p-1}v_n$, and therefore at the limit one obtains:

$$0=\partial_t u_\infty=\Delta u_\infty+|u_\infty|^{p-1}u_\infty,$$

u_∞ is consequently a stationary solution of (NLH). From the classification of all the radial solutions (*5.1.3*), one has that either $u_\infty=Q_\lambda$ for some $\lambda>0$, or $u_\infty=0$. For $\lambda_1>\lambda_2>0$ from the formula (*2.2.2*) one sees that the radial Q_{λ_1} and Q_{λ_2} must intersect at some radius r^* and that one is strictly above the other for $0<r<r^*$ and conversely for $r>r^*$. Since $u_\infty<Q$ necessarily $u_\infty=0$. One has proven that for any $x\in\mathbb{R}^d$, $Q^-(t,x)\to 0$.

One notices that as $Q^-\to 0$ and $0<Q^-<Q$, the nonlinear term $(Q^-)^p$ is in every Lebesgue space L^p for $p\geq 1$ in which it then converges to zero as $t\to+\infty$ from Lebesgue's dominated convergence theorem. For any $T>0$ and $t>T$, we write using Duhamel formula and Young inequality:

$$\begin{array}{rcl}
\nabla(Q^-) & = & \nabla(K_t*Q^-(0))+\int_0^t(\nabla K_{t-t'})*(Q^-)^p dt' \\
& = & o_{L^2,\ t\to+\infty}(1)+\int_0^{t-T}(\nabla K_{t-t'})*(Q^-)^p dt'+\int_{t-T}^t(\nabla K_{t-t'})*(Q^-)^p dt' \\
& = & o_{L^2,\ t\to+\infty}(1)+\int_0^{t-T}O_{L^2}\left(\|\nabla K_{t-t'}\|_{L^2}\|(Q^-)^p\|_{L^1}\right)dt' \\
& & +\int_{t-T}^t O_{L^2}\left(\|\nabla K_{t-t'}\|_{L^1}\|(Q^-)^p\|_{L^2}\right)dt' \\
& = & o_{L^2,\ t\to+\infty}(1)+\int_0^{t-T}O_{L^2}\left(\frac{1}{(t-t')^{\frac{d+2}{4}}}\times C\right)dt' \\
& & +\int_{t-T}^t O_{L^2}\left(\frac{1}{\sqrt{t-t'}}\sup\limits_{t'\in[t-T,t]}\|(Q^-)^p\|_{L^2}\right)dt' \\
& = & o_{L^2,\ t\to+\infty}(1)+O_{L^2}(T^{-\frac{d-2}{4}})+O_{L^2}\left(\sup\limits_{t'\in[t-T,t]}\|Q^-\|^p_{L^{2p}}\right) \\
& = & O_{L^2}(T^{-\frac{d-2}{4}})\ \text{ as }\ t\to+\infty.
\end{array}$$

As this is valid for any $T>0$, this implies that ∇Q^- tends to 0 in L^2. Hence $\lim_{t\to+\infty}\|Q^-(t)\|_{\dot H^1}=0$. □

5.3.2 Uniqueness

We now conlude the proof of Theorem 2.3.5 by proving that $Q^\pm$ are the only solutions uniformly trapped near $\mathcal{M}$ on $(-\infty,0]$ in the sense of Definition 5.2.4, up to the symmetries of the equation.

Proof of Theorem 2.3.5 let u be trapped at distance $0<\delta\ll 1$ of $\mathcal{M}$ on $(-\infty,0]$ in the sense of Definition 5.2.4. We follow the strategy designed in [142, 103], First we use a dissipation argument to show that the instability must be the main term at $-\infty$, the stable part of the perturbation being of quadratic size. This then implies an exponential decay of the whole perturbation, which hence enters the regime where we constructed the solution, and a simple contraction like argument will close the proof.

The primary information to notice is that the scale cannot diverge as $t\to-\infty$ from (*5.2.56*), i.e. there exists $0<\lambda_1<\lambda_2$ such that for $t\leq 0$:

$$\lambda_1\leq\lambda\leq\lambda_2.$$

We then rewrite the $\dot H^2$ energy bound (*5.2.44*) as:

$$\frac{d}{ds}\left[\frac{1}{\lambda^2}\int (H\varepsilon)^2\right]\leq -\frac{1}{C\lambda^2}\int H\varepsilon H^2\varepsilon+\frac{Ca^4}{\lambda^2}+\frac{Ca^2}{\lambda^2}\|\varepsilon\|_{\dot H^2}^2.$$

From the fact (*5.2.56*) that λ does not diverge we rewrite it as:

$$\frac{d}{ds}\left[\frac{1}{\lambda^2}\|H\varepsilon\|_{L^2}^2\right]\leq C(a^4+a^2\|\varepsilon\|_{\dot H^2}^2)-\frac{1}{C}\int\varepsilon H^3\varepsilon \tag{5.3.21}$$

Finally the global energy bound (*5.2.41*) ensures

$$\int_{-\infty}^0(\|\varepsilon(s)\|_{\dot H^2}^2+a^2(s))ds\lesssim\delta^2. \tag{5.3.22}$$

step 1 Dominance of the instability. We claim that there for any constant $K\gg 1$ for any $0<\delta\ll 1$ small enough there holds the following bound:

$$\forall t<0,\quad \|\varepsilon\|_{\dot H^2}^2(t)\leq\frac{|a(t)|}{K}. \tag{5.3.23}$$

Sequential control. We first claim that (*5.3.23*) holds on a subsequence in time $t_n\to-\infty$. We argue by contradiction and assume that there exists $s_0\leq 0$ such that:

$$\forall s\leq s_0,\quad \|\varepsilon(s)\|_{\dot H^2}^2>\frac{|a(s)|}{4K}. \tag{5.3.24}$$

On the one hand, there exists $s_n\to-\infty$ such that $\lim_{s_n\to-\infty}\|\varepsilon(s_n)\|_{\dot H^2}=0$ from (*5.3.22*). On the other hand, injecting (*5.3.24*) in the energy identity (*5.3.21*), using (*5.2.56*) and (*5.2.12*), yields:

$$\frac{d}{ds}\left[\frac{1}{\lambda^2}\|H\varepsilon\|_{L^2}^2\right]\leq C(K|a|^3+a^2)\|H\varepsilon\|_{L^2}^2\leq C(K\delta+1)a^2\|H\varepsilon\|_{L^2}^2\leq Ca^2\frac{\|H\varepsilon\|_{L^2}^2}{\lambda^2}$$

which, integrated in time on $[s,s_0]$ for any $s\leq s_0$, using the integrability of a^2 coming from the variation of the energy (*5.2.41*), gives:

$$\log\left(\frac{1}{\lambda^2(s_0)}\|H\varepsilon(s_0)\|_{L^2}^2\right)-\log\left(\frac{1}{\lambda^2(s)}\|H\varepsilon(s)\|_{L^2}^2\right)\leq C\int_s^{s_0}a^2ds\leq C\delta^2.$$

This can be rewritten, for any $s \leq s_0$ as:

$$\|H\varepsilon(s)\|_{L^2}^2 \geq e^{-C\delta^2} \frac{\lambda^2(s)}{\lambda^2(s_0)} \|H\varepsilon(s_0)\|_{L^2}^2 \geq c > 0$$

for some constant $c > 0$, from (*5.3.24*) and as the scale does not diverge from (*5.2.56*). The above identity then contradicts the convergence to 0 of $H\varepsilon$ along a subsequence $s_n \to -\infty$ and gives the desired contradiction.

No return. We claim that there exists $K, \delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, if at some time s_0 there holds:

$$\|\varepsilon\|_{\dot{H}^2}^2(s_0) \leq \frac{|a(s_0)|}{4K}, \tag{5.3.25}$$

then

$$\forall s \in (s_0, 0), \quad \|\varepsilon\|_{\dot{H}^2}^2(s) \leq \frac{|a(s)|}{K} \tag{5.3.26}$$

which concludes the proof of (*5.3.23*) given that we just showed (*5.3.25*) for a sequence of times $s_n \to -\infty$. Indeed, if $a(s_0) = 0$, then the solution is Q up to symmetries and the claim follows. We now assume $a(s_0) \neq 0$. We look at $\mathcal{S}$, the set of times $s_0 \leq s \leq 0$ such that:

$$|a(s)| \geq \frac{2K}{\lambda^2(s)} \|\varepsilon\|_{\dot{H}^2}^2$$

$\mathcal{S}$ is closed and non empty as it contains s_0 from (*5.3.25*) and as $|\lambda(s_0) - 1| \lesssim \delta$ from (*5.2.56*). Now inside $\mathcal{S}$ we compute from the modulation equation (*5.2.15*) for a, the modified energy estimate (*5.3.21*) for the error, the size estimate $|a| \lesssim \delta$ and the non divergence of the scale (*5.2.56*):

$$\begin{array}{rl} & \frac{d}{ds}\left(|a(s)| - K\frac{2}{\lambda^2(s)}\|\varepsilon\|_{\dot{H}^2}^2(s)\right) \\ \geq & |a(s)|\left(e_0 - C\delta - \frac{C}{K} - KC\delta^3 - C\delta^2\right) > 0 \end{array}$$

for K large enough and δ small enough. Consequently $\mathcal{S}$ is open, and via connectedness $\mathcal{S} = [s_0, 0]$. One then has on $[s_0, s]$ the bound $\|\varepsilon\|_{\dot{H}^2}^2 \leq \frac{\lambda^2(s)}{2K}|a(s)|$ which gives (*5.3.26*) from (*5.2.56*).

step 2 Primary refined asymptotics at $-\infty$. We claim that the solution enters the regime of exponential smallness: there exists $\lambda_\infty > 0$ and $z_\infty \in \mathbb{R}^d$ such that:

$$a = a_0 e^{\frac{e_0}{\lambda_\infty^2}t} + O(\delta^2 e^{\frac{2e_0}{\lambda_\infty^2}t}), \quad \|\varepsilon\|_{\dot{H}^2} \lesssim \delta^2 e^{\frac{2e_0}{\lambda_\infty^2}t}, \quad 0 \neq |a_0| \lesssim \delta, \tag{5.3.27}$$

$$\lambda = \lambda_\infty + O(\delta^2 e^{\frac{2e_0}{\lambda_\infty^2}t}), \quad z = z_\infty + O(\delta^2 e^{\frac{2e_0}{\lambda_\infty^2}t}). \tag{5.3.28}$$

Intermediate estimate. We first claim the intermediate estimate:

$$a(s) = a_0 e^{e_0 s} + O(\delta^2 e^{2e_0 s}), \quad \|\varepsilon\|_{\dot{H}^2} \lesssim \delta^2 e^{2e_0 s}, \quad 0 \neq |a_0| \lesssim \delta, \tag{5.3.29}$$

which we will now prove by bootstraping the informations one can obtain from (*5.2.15*) and (*5.3.21*) starting from the first bound (*5.3.23*). Taking K large enough and δ in a small enough range, (*5.2.15*) and (*5.3.23*) yields:

$$\frac{d}{ds}|a(s)| > \frac{2e_0}{3}|a(s)|.$$

Integrating this on $[s,0]$ using the global bound $|a(s)|\lesssim \delta$ from (*5.2.12*) ensures:

$$|a(s)|e^{-\frac{2e_0}{3}s}\lesssim \delta$$

and hence

$$|a|\lesssim \delta e^{\frac{2e_0}{3}s}. \tag{5.3.30}$$

From (*5.3.23*) this implies

$$\|\varepsilon\|^2_{\dot H^2}\leq C\delta e^{\frac{2e_0}{3}s}. \tag{5.3.31}$$

We inject these two estimates in (*5.3.21*) to find:

$$\frac{d}{ds}\left[O(1)\|H\varepsilon\|^2_{L^2}\right]\lesssim \delta^3 e^{2e_0 s}$$

which implies the refined estimate using the coercivity (*5.2.8*)

$$\|\varepsilon\|_{\dot H^2}\lesssim \delta^{\frac32}e^{e_0 s}. \tag{5.3.32}$$

In a similar way, we inject this and (*5.3.30*) in (*5.2.15*) to find after reintegration on $[s,0]$:

$$a=a_0e^{e_0 s}+O(\delta^2 e^{\frac43 e_0 s}+\delta^{\frac32}e^{2e_0 s}),\ \ a_0=O(\delta). \tag{5.3.33}$$

Again, injecting the above estimate and (*5.3.32*) in (*5.3.21*) gives:

$$\|\varepsilon\|_{\dot H^2}\lesssim \delta^2 e^{2e_0 s}. \tag{5.3.34}$$

Injecting this estimate and (*5.3.33*) in (*5.2.15*) gives:

$$a=a_0e^{e_0 s}+O(\delta^2 e^{2e_0 s}),\ \ a_0=O(\delta). \tag{5.3.35}$$

The above two estimates are the bound (*5.3.29*) we had to show.

Conclusion. We rewrite the modulation equations (*5.2.16*), and (*5.2.17*) and for λ and z using the exponential decay in renormalized time (*5.3.29*) as:

$$\frac{\lambda_s}{\lambda}=O(\delta^2e^{2e_0s}),\ \ \frac{z_s}{\lambda}=O(\delta^2e^{2e_0s}).$$

This implies that there exists λ_∞ and z_∞ such that:

$$\lambda=\lambda_\infty+O(\delta^2e^{2e_0s}),\ \ z=z_\infty+O(\delta^2e^{2e_0s}).$$

From the fact that s solves $\frac{ds}{dt}=\lambda^{-2}$, one gets $t=\lambda_\infty^2 s+O(\delta^2e^{2e_0s})$. The primary exponential bound (*5.3.29*) in renormalized time s and the above identity then become the desired bounds (*5.3.27*) and (*5.3.28*) in original time t.

step 3 Additional exponential decay. Up to scale change and translation, one can assume that $\lambda_\infty=1$ and $z_\infty=0$. Up to time translation[4] one can assume that u is not a ground state and that $a_0=\pm1$ in (*5.3.27*). We write the solution u as:

$$u(t)=Q\pm e^{e_0t}\mathcal{Y}+\tilde a e^{e_0t}\mathcal{Y}+v,\ \ v\perp\mathcal{Y}. \tag{5.3.36}$$

[4]As a consequence the solution is maybe no more defined on $(-\infty,0[$ but just on $(-\infty,t_0)$ for some $t_0\in\mathbb{R}$, which is not a problem as we are working at the asymptotic $t\to-\infty$.

Then from (*5.3.27*), (*5.3.28*) and (*5.2.16*) one has for small enough times:

$$|\tilde a|\lesssim e^{e_0t},\quad |\tilde a_t|\lesssim e^{e_0t},\quad \|v\|_{\dot H^1}\lesssim \delta,\quad \|v\|_{\dot H^2}\lesssim e^{2e_0t}. \tag{5.3.37}$$

We claim that in addition[5] that for small enough times:

$$\|v\|_{L^\infty}\lesssim e^{e_0t},\quad \|v\|_{\dot H^1}\lesssim e^{2e_0t}. \tag{5.3.38}$$

From (*5.3.37*) and Sobolev embedding v satisfies $\|v\|_{L^{\frac{2d}{d-4}}}\lesssim e^{2e_0t}$. This, with (*5.3.36*) and (*5.3.37*) implies that the whole perturbation satisfies $\|u-Q\|_{L^{\frac{2d}{d-4}}}\lesssim e^{e_0t}$. Therefore, one can apply the parabolic estimate of Lemma 5.E.3, (*5.E.6*), to obtain the L^∞ estimate in (*5.3.38*) on some time interval $(-\infty,T]$ for T small enough. We now prove the $\dot H^1$ bound in (*5.3.38*). The evolution equation for v is:

$$v_t+Hv=-\tilde a_te^{te_0}\mathcal Y+NL,\quad NL:=f(u)-f(Q)-f'(Q)u. \tag{5.3.39}$$

The Duhamel formula for the solution of (*5.3.39*) yields for t small enough and $t_0>0$:

$$\nabla v(t)=(\nabla K_{t_0})*v(t-t_0)+\int_{t-t_0}^t(\nabla K_{t-\tau}*(-Vv-\tilde a_te^{\tau e_0}\mathcal Y+NL)d\tau. \tag{5.3.40}$$

We estimate the nonlinear term using Young inequality, the estimate (*5.D.3*) on the nonlinearity, Sobolev embedding, interpolation, (*5.3.37*),(*5.2.12*) and the fact that $\mathcal Y$ is exponentially decreasing:

$$\begin{aligned}\int NL^2 &\lesssim \int(e^{e_0t}+\tilde a)^4\mathcal Y^4Q^{2(p-2)}+\int\varepsilon^{2p}\lesssim e^{4e_0t}+\|\varepsilon\|^{2p}_{\dot H^{\frac{2d}{d+2}}}\\ &\lesssim e^{4e_0t}+\|\varepsilon\|^{\frac{8p}{d+2}}_{\dot H^1}\|\varepsilon\|^2_{\dot H^2}\lesssim e^{4e_0t}+\delta^{\frac{8p}{d+2}}e^{4e_0t}.\end{aligned}$$

We come back to the above Duhamel formula. For the first term we use Hölder inequality and (*5.3.37*), for the second the fact that $\mathcal Y$ is exponentially decaying and (*5.3.37*) and for the third the estimate we just proved, yielding:

$$\begin{aligned}&\left\|\int_{t-t_0}^t(\nabla K_{t-\tau})*(-Vv-\tilde a_te^{\tau e_0}\mathcal Y+NL)d\tau\right\|_{L^2}\\ \le&\int_{t-t_0}^t\|\nabla K_{t-\tau}\|_{L^1}\|-Vv-\tilde a_te^{\tau e_0}\mathcal Y+NL\|_{L^2}d\tau\\ \lesssim&\int_{t-t_0}^t\frac{1}{\sqrt{t-\tau}}\left(\|V\|_{L^{\frac d2}}\|v\|_{L^{\frac{2d}{d-4}}}+|\tilde a_t|e^{e_0\tau}+\|NL\|^p_{L^{2p}}\right)d\tau\\ \lesssim&\int_{t-t_0}^t\frac{e^{2e_0\tau}}{\sqrt{t-\tau}}d\tau\lesssim e^{2e_0t}.\end{aligned}$$

The very same computations shows that the second term in (*5.3.40*) converges strongly at speed e^{2e_0t} in L^2 as $t_0\to+\infty$. This implies that $(\nabla K_{t_0})*v(t-t_0)$ converges strongly in L^2 as $t_0\to+\infty$. As v is uniformly bounded in L^4, one has that $(\nabla K_{t_0})*v(t-t_0)$ converges weakly to 0 as $t_0\to+\infty$. Therefore, $(\nabla K_{t_0})*v(t-t_0)$ converges strongly in L^2 to 0 as $t_0\to+\infty$ and one has the following formula:

$$\nabla v(t)=\int_{-\infty}^t(\nabla K_{t-\tau}*(-Vv-\tilde a_te^{\tau e_0}\mathcal Y+NL)d\tau=O(e^{2e_0t})$$

where the upper bound is implied by the above estimate.

[5]The L^∞ norm being finite from Proposition 5.2.1

step 5 Uniqueness via contraction argument. Let u_1 and u_2 be two solutions of (NLH) that are trapped on $(-\infty,0]$ at distance δ and that are not ground states. From all the previous results of the previous steps, we assume that they have been renormalized and from (*5.3.37*) and (*5.3.38*) they can be decomposed as:

$$u_i=Q\pm e^{e_0t}\mathcal{Y}+\tilde a_ie^{e_0t}\mathcal{Y}+v_i+b_i\Lambda Q+z_i.\nabla Q,\quad v_i\in \text{Span}(\mathcal{Y},\Psi_0,\Psi_1,...,\Psi_d)^{\perp}, \qquad (5.3.41)$$

the profiles $\Psi_0,...,\Psi_i$ being defined by (*5.2.3*) and (*5.2.4*) with

$$|b_i|+|z_i|+|\tilde a_i|e^{e_0t}+\|v_i\|_{\dot H^1}+\|v_i\|_{\dot H^2}\lesssim e^{2e_0t},\quad \|v_i\|_{L^\infty}\lesssim e^{e_0t}. \qquad (5.3.42)$$

Then if they have the same sign at first order on their projection onto the unstable mode (the $\pm$ in (*5.3.41*)) we claim $u_1=u_2$. This will end the proof of the proposition as Q^- and Q^+ are indeed trapped at any distance of $\mathcal{M}$ as $t\to-\infty$ from (*5.3.17*). Without loss of generality we chose a $+$ sign. For $T\ll 0$ we define the following norm for the difference:

$$\begin{array}{rcl}\|u_1-u_2\|_T & := & \sup\limits_{t\leq T} e^{-2e_0t}\|v_1-v_2\|_{\dot H^1}+\sup\limits_{t\leq T}e^{-e_0t}|\tilde a_1-\tilde a_2| \\ & & +\sup\limits_{t\leq T}e^{-2e_0t}|b_1-b_2|+\sup\limits_{t\leq T}e^{-2e_0t}|z_1-z_2|\end{array} \qquad (5.3.43)$$

which is finite from (*5.3.42*). The evolution of the difference u_1-u_2 is given by:

$$(v_1-v_2)_t+H(v_1-v_2)=-(\tilde a_1-\tilde a_2)_te^{e_0t}\mathcal{Y}-(b_1-b_2)_t\Lambda Q-(z_1-z_2)_t.\nabla Q+NL_1-NL_2, \qquad (5.3.44)$$

where:

$$NL_i:=f(u_i)-f(Q)-f'(Q)u_i.$$

From (*5.3.42*), (*5.D.7*), Hölder inequality and Sobolev embedding one gets the following bounds for the nonlinear term for $t\leq T$:

$$\begin{array}{rl} & \|NL_1-NL_2\|_{L^2} \\ \lesssim & \Big\||(\tilde a_1-\tilde a_2)e^{e_0t}\mathcal{Y}+v_1-v_2+(b_1-b_2)\Lambda Q+(z_1-z_2).\nabla Q| \\ & \times(|\tilde a_1e^{e_0t}\mathcal{Y}+v_1+b_1\Lambda Q+z_1.\nabla Q|^{p-1}+|\tilde a_2e^{e_0t}\mathcal{Y}+v_2+b_2\Lambda Q+z_2.\nabla Q|^{p-1}\Big\|_{L^2} \\ \lesssim & \|(\tilde a_1-\tilde a_2)e^{e_0t}\mathcal{Y}+v_1-v_2+(b_1-b_2)\Lambda Q+(z_1-z_2).\nabla Q\|_{L^{\frac{2d}{d-2}}} \\ & \times\Big(\|(1+\tilde a_1)e^{e_0t}\mathcal{Y}+v_1+b_1\Lambda Q+z_1.\nabla Q\|^{p-1}_{L^{(p-1)d}} \\ & +\|\tilde{(}1+\tilde a_2)e^{e_0t}\mathcal{Y}+v_2+b_2\Lambda Q+z_2.\nabla Q\|^{p-1}_{L^{(p-1)d}}\Big) \\ \lesssim & \|(\tilde a_1-\tilde a_2)e^{e_0t}\mathcal{Y}+v_1-v_2+(b_1-b_2)\Lambda Q+(z_1-z_2).\nabla Q\|_{\dot H^1}\times e^{(p-1)t} \\ \lesssim & \|u_1-u_2\|_Te^{(p+1)t}. \end{array}$$

Energy estimate for the difference of errors. From (*5.3.44*), the orthogonality conditions (*5.3.41*) and the bound (*5.3.45*) on the nonlinear term one gets the following energy estimate:

$$\begin{array}{rcl}\frac{d}{dt}\left[\int(v_1-v_2)H(v_1-v_2)\right] & = & 2\int H(v_1-v_2)(NL_1-NL_2)-2\int H(v_1-v_2)^2 \\ & \leq & \|NL_1-NL_2\|^2_{L^2}\lesssim e^{2(p+1)e_0t}\|u_1-u_2\|_T.\end{array}$$

From the coercivity property of the linearized operator (*5.2.8*) one has:

$$\int (v_1-v_2)H(v_1-v_2) \lesssim \|v_1-v_2\|_{\dot H^1}^2 \lesssim \int (v_1-v_2)H(v_1-v_2).$$

Therefore, $\int (v_1-v_2)H(v_1-v_2)$ goes to zero as $t\to-\infty$, and reintegrating in time the energy estimate gives from the coercivity:

$$\sup_{t\leq T}\|v_1-v_2\|_{\dot H^1}e^{-2e_0t} \lesssim e^{(p-1)e_0T}\|u_1-u_2\|_T. \tag{5.3.45}$$

Modulation equations for the differences of parameters. We take the scalar products of (*5.3.44*) with ΛQ, $\mathcal{Y}$ and ∇Q, using the bound (*5.3.45*) for the nonlinear term and obtain for any $t\leq T$

$$\begin{aligned}\left|\tfrac{d}{dt}\left[b_1-b_2+\textstyle\int (v_1-v_2)\Lambda Q\right]\right| + \left|\tfrac{d}{dt}\left[z_1-z_2+\textstyle\int(v_1-v_2)\nabla Q\right]\right| \\ +\left|\tfrac{d}{dt}(\tilde a_1-\tilde a_2)\right|e^{e_0t} \lesssim e^{(p+1)e_0t}\|u_1-u_2\|_T.\end{aligned}$$

The boundary term involving v_1-v_2 satisfies from Sobolev embedding as $d\geq 7$:

$$\begin{aligned}&\left|\textstyle\int (v_1-v_2)\Lambda Q\right| + \left|\textstyle\int (v_1-v_2)\nabla Q\right| \lesssim \textstyle\int \frac{|v_1-v_2|}{1+|x|^{d-2}} \\ \lesssim\ & \|v_1-v_2\|_{L^{\frac{2d}{d-2}}}\|(1+|x|)^{-d+2}\|_{L^{\frac{2d}{d+2}}} \lesssim \|v_1-v_2\|_{\dot H^1}.\end{aligned}$$

The two above equations, after reintegration in time, as the left hand side goes to 0 as $t\to-\infty$, imply:

$$\begin{aligned}&\sup_{t\leq T}(|b_1-b_2|+|z_1-z_2|+|\tilde a_1-\tilde a_2|e^{e_0t})e^{-2e_0t} \\ \lesssim\ & e^{(p-1)e_0T}\|u_1-u_2\|_T+\sup_{t\leq T}\|v\|_{\dot H^1}e^{-2e_0t} \lesssim e^{(p-1)e_0T}\|u_1-u_2\|_T\end{aligned} \tag{5.3.46}$$

where we used the estimate (*5.3.45*).

Conclusion. From the definition (*5.3.43*) of the norm of the difference that is adapted to the exponential decay, and the estimates (*5.3.46*) and (*5.3.45*) one obtains:

$$\|u_1-u_2\|_T \lesssim e^{(p-1)e_0T}\|u_1-u_2\|_T.$$

For $T\ll 0$ small enough this implies: $\|u_1-u_2\|_T=0$. This means that the solutions u_1 and u_2 are equal.

□

5.4 Classification of the flow near the ground state

We are now in position to conclude the proof of Theorem 2.3.4.

5.4.1 Set up

Let

$$0<\delta\ll\alpha\ll\alpha^*\ll 1.$$

be three small strictly positive constants to be fixed later on. Let $u_0\in\dot H^1$ with

$$\|u_0-Q\|_{\dot H^1}\leq \delta^4 \tag{5.4.1}$$

and let u be the solution of (NLH) starting from u_0 (see Proposition 5.2.1 for the local wellposedness result) with maximal time of existence T_{u_0}. To prove Theorem 2.3.4 we are going to study u for times where it is close to the manifold of ground states $\mathcal{M}$, that is to say in the set

$$\left\{ v \in \dot{H}^1, \ d(v, \mathcal{M}) = \inf_{\lambda>0, \ z\in\mathbb{R}^d} \|v - Q_{z,\lambda}\|_{\dot{H}^1} \leq \alpha^* \right\}$$

using the variables λ, s, z, a and ε introduced in Definition 5.2.4 to decompose it in a suitable way. We introduce three particular times related to the trajectory of the solution u starting from u_0. For a constant $K \gg 1$ big enough to be fixed later we define

$$T_{\text{ins}} := \sup \Big\{ \quad 0 \leq t < T_{u_0}, \quad \sup_{0\leq t'\leq t} d(u(t'), \mathcal{M}) \leq \delta^2, \\ \forall t' \in [0, t], \quad \|\varepsilon(t')\|^2_{\dot{H}^2} > \tfrac{|a(t')|}{K} \Big\}. \tag{5.4.2}$$

Let $\tilde{K} > 0$ be a constant, independent of the other constants, such that for any $\nu > 0$ with $0 < \tilde{K}\nu \leq \alpha^*$ and $v \in \dot{H}^1$,

$$\tilde{K}\nu \leq d(v, \mathcal{M}) \leq \alpha^* \quad \Rightarrow \quad \text{either} \ \|\varepsilon\|_{\dot{H}^1} \geq \nu \ \text{ or } \ |a| \geq \nu. \tag{5.4.3}$$

Such a constant $\tilde{K}$ exists since for $v \in \dot{H}^1$ with $d(v, \mathcal{M}) \leq \alpha$ one has from (5.2.11)

$$d(v, \mathcal{M}) \leq \|\varepsilon\|_{\dot{H}^1} + C|a|.$$

We then define:

$$T_{\text{trans}} := \sup \Big\{ \quad T_{\text{ins}} \leq t < T_{u_0}, \quad \sup_{T_{\text{ins}}\leq t'\leq t} d(u(t'), \mathcal{M}) \leq \tilde{K}\delta, \\ \forall t' \in [0, t], \quad |a(t')| \leq \delta \Big\}, \tag{5.4.4}$$

$$T_{\text{exit}} := \sup \Big\{ \quad T_{\text{trans}} \leq t < T_{u_0}, \quad \sup_{T_{\text{trans}}\leq t'\leq t} d(u(t'), \mathcal{M}) \leq \tilde{K}\alpha, \\ \forall t' \in [0, t], \quad |a(t')| \leq \alpha \Big\}. \tag{5.4.5}$$

with the convention that $\sup(\emptyset) = T_{u_0}$. Our strategy of the proof is the following. In Lemma 5.4.1 we characterize T_{ins} as the time, if not $+\infty$, for which the instability has started to take control over the solution. In Lemma 5.4.2 we show that if it never happens, i.e. $T_{\text{ins}} = +\infty$, then the solution converges to some soliton. Indeed, in that case the main part of the renormalized perturbation is located on the stable infinite dimensional direction of perturbation $\approx (\Lambda Q, \partial_{x_1} Q, ..., \partial_{x_d} Q, \mathcal{Y})^\perp$ and undergoes dissipation. In Lemma 5.4.4 we show that if it happens, i.e. $T_{\text{ins}} < +\infty$, then the instability will drive the solution toward type I blow up or dissipation. The analysis is done in three times. First we characterize the time interval $[T_{\text{ins}}, T_{\text{trans}}]$ as the transition period in which the solution stays trapped at distance δ^2 and is such that at T_{trans} the stable perturbation is quadratic compared to the instability. For later times, on $[T_{\text{trans}}, T_{\text{exit}}]$ this implies an exponential growth of the instability, with a stable perturbation being still quadratic. In this exponential instable regime we can compare the solution with the minimal solutions $Q^\pm$ introduced in Proposition 5.3.1 and compute that they are close at the exit time T_{exit}. As Type I blow up and dissipation are stable behaviors, u will undergo one or the other.

We now proceed to the detailed proof of Theorem 2.3.4.

5.4.2 Caracterization of T_{ins}

We first characterize the time T_{ins}.

Lemma 5.4.1 (T_{ins} as the instability time). *There exists $K^* \gg 1$, such that for any $K \geq K^*$, there exists $0 < \delta^*(K) \ll 1$, such that for any $0 < \delta < \delta^*(K)$, on $[0, T_{ins})$ there holds:*

$$|a(t)| \lesssim K\delta^4, \quad \|\varepsilon(t)\|_{\dot{H}^1} \lesssim \delta^4. \tag{5.4.6}$$

Moreover, if $T_{ins} < T_{u_0}$ then:

$$\|\varepsilon(T_{ins})\|_{\dot{H}^2}^2 = \frac{|a(T_{ins})|}{K}. \tag{5.4.7}$$

Proof of Lemma 5.4.1

To prove the lemma, from the definition (*5.4.2*) of T_{ins} , it suffices to show that (*5.4.6*) holds, which will automatically imply that the other identity (*5.4.7*) holds for δ small enough. We will prove it by computing the time evolution of a and performing an energy estimate adapted to the linear level for ε in the regime $0 \leq t \leq T_{\text{ins}}$. u being trapped at distance δ^2 on $[0, T_{\text{ins}})$ in the sense of Definition 5.2.4, we reason in renormalized time (*5.2.13*) and define: $S_{\text{ins}} := s(T_{\text{ins}})$.

step 1 Bound for a. From the definition (*5.4.2*) of T_{ins}, the modulation estimate (*5.2.15*) for a on $0 \leq s \leq S_{\text{ins}}$ and (*5.2.12*) one has:

$$|a_s| \leq e_0|a| + Ca^2 + C\|\varepsilon\|_{\dot{H}^2}^2 \leq \|\varepsilon\|_{\dot{H}^2}^2(K|e_0| + CK|\delta^2| + C) \lesssim K\|\varepsilon\|_{\dot{H}^2}^2$$

for K large enough. We reintegrate in time this identity using the variation of energy formula (*5.2.41*):

$$|a(s)| \leq |a(0)| + CK\int_0^{S_{\text{ins}}} \|\varepsilon(s)\|_{\dot{H}^2}^2 ds \leq C|\delta|^4 + CK\delta^4 \lesssim K\delta^4$$

for K large enough as initially $|a(0)| \lesssim \delta^4$ from (*5.4.1*) and (*5.2.12*).

step 2 Bound for ε. From the definition of T_{ins} (*5.4.2*), the $\dot{H}^1$ bound (*5.2.43*), (*5.2.12*) and the coercivity (*5.2.8*), one has for $0 \leq s \leq S_{\text{ins}}$:

$$\begin{aligned} \frac{d}{ds}\left[\frac{1}{2}\int \varepsilon H\varepsilon\right] &\leq -\frac{1}{C}\int (H\varepsilon)^2 + Ca^4 \leq -\frac{1}{C}\int (H\varepsilon)^2 + CK|a|^3\|\varepsilon\|_{\dot{H}^2}^2 \\ &\leq -\frac{1}{C}\int (H\varepsilon)^2 + CK\delta^6\int (H\varepsilon)^2 \leq 0 \end{aligned} \tag{5.4.8}$$

for δ small enough (depending on K) meaning that on $[0, S_{\text{ins}}]$, the quantity $\int \varepsilon H\varepsilon$ is a Lyapunov functional. We integrate in time using the coercivity (*5.2.7*) to find for $0 \leq s \leq S_{\text{ins}}$:

$$\|\varepsilon(s)\|_{\dot{H}^1} \lesssim \left(\int \varepsilon(s)H\varepsilon(s)\right)^{\frac{1}{2}} \leq \left(\int \varepsilon(0)H\varepsilon(0)\right)^{\frac{1}{2}} \lesssim \|\varepsilon(0)\|_{\dot{H}^1} \lesssim \delta^4$$

from (*5.4.1*) and (*5.2.12*), ending the proof of the lemma. □

5.4.3 Soliton regime

We now claim that $T_{\text{ins}} = T_{u_0}$ is the (Soliton) regime.

Lemma 5.4.2 (Soliton regime for $T_{\text{ins}} = T_{u_0}$). *There exists $K^* \gg 1$, such that for any $K \geq K^*$, there exists $0 < \delta^*(K) \ll 1$, such that for any $0 < \delta < \delta^*(K)$, if $T_{ins} = T_{u_0}$ then $T_{u_0} = +\infty$ and there exists $z_\infty \in \mathbb{R}^d$ and $\lambda_\infty > 0$ such that:*

$$u \to Q_{z_\infty,\lambda_\infty} \ \textit{ as } t \to +\infty \textit{ in } \dot{H}^1.$$

Proof of Lemma 5.4.2 Let u satisfy (*5.4.1*) such that $T_{\text{ins}} = T_{u_0}$. From (*5.4.6*), there exists $\tilde{C} > 0$ (depending on K) such that u is trapped at distance $\tilde{C}\delta^4$ in the sense of Definition 5.2.4 on $[0, T_{u_0})$. We reason in renormalized time (*5.2.13*) and define $S(u_0) = \lim_{t\to T_{u_0}} s(t)$.

step 1 Global existence. We claim that u is a global solution, i.e. that $T_{u_0} = +\infty$. Indeed, recall from (*5.2.57*) that the times t or s are equivalents, i.e. $\frac{s}{C} \leq t \leq Cs$ for $C > 0$. Injecting the bound (*5.4.6*) on a into (*5.2.44*) and using Gronwall's lemma ensures that $\|\varepsilon(t)\|_{\dot{H}^2} < c(t) < +\infty$ for all bounded time t, and hence $\|u(t)\|_{\dot{H}^2} < c(t) < +\infty$ an $T(u_0) = +\infty$ from the blow up criterion (*5.2.1*).

step 2 Convergence of the perturbation to 0 in $\dot{H}^2$. We claim that:

$$\|\varepsilon\|_{\dot{H}^2} \to 0 \ \text{ as } \ s \to +\infty. \tag{5.4.9}$$

Indeed, the $\dot{H}^2$ bound (*5.2.44*) and the smallness (*5.4.6*) ensure:

$$\frac{d}{ds}\left[\frac{1}{\lambda^2}\|H\varepsilon\|_{L^2}^2\right] \leq \frac{C}{\lambda^2}(a^4 + a^2\|\varepsilon\|_{\dot{H}^2}^2) - \frac{1}{C\lambda^2}\int \varepsilon H^3 \varepsilon \lesssim a^2 + \|\varepsilon\|_{\dot{H}^2}^2. \tag{5.4.10}$$

Now since $0 < \lambda_0 < \lambda(s) < \lambda_2$, the right hand side is in $L^1([0, +\infty))$ from (*5.2.41*), and hence there exists a sequence $s_n \to +\infty$ with $\|\varepsilon(s_n)\|_{\dot{H}^2} \to 0$, and integrating (*5.4.10*) on $[s_n, s]$ yields

$$\forall s \geq s_n, \ |\varepsilon(s)\|_{\dot{H}^2}^2 \lesssim \|\varepsilon(s_n)\|_{\dot{H}^2}^2 + \int_{s_n}^{+\infty} (a^2(\tau) + \|\varepsilon(\tau)\|_{\dot{H}^2}^2) d\tau$$

and (*5.4.9*) follows.

step 3 Convergence of the central point and the scale. We claim that there exist $\lambda_\infty > 0$ and $z_\infty \in \mathbb{R}^d$ such that:

$$\lambda \to \lambda_\infty, \quad z \to z_\infty \ \text{ as } s \to +\infty. \tag{5.4.11}$$

The evolution for these parameters is given by the modulation equations (*5.2.18*), (*5.2.19*). After integration in time this gives:

$$|\lambda(s) - \lambda(0)| = O(\|\varepsilon(s)\|_{\dot{H}^{1+\frac{1}{3}}}) + \int_0^s O(a^2(\tau) + \|\varepsilon(\tau)\|_{\dot{H}^2}^2) d\tau,$$

$$|z(s) - z(0)| = O(\|\varepsilon\|_{\dot{H}^{1+\frac{1}{3}}}) + \int_0^s O(a^2(\tau) + \|\varepsilon(\tau)\|_{\dot{H}^2}^2) d\tau.$$

From the uniform bound (*5.4.6*) at the $\dot{H}^1$ level and the convergence to 0 at the $\dot{H}^2$ level (*5.4.9*), using interpolation one has that the first term in the above identities converges to 0:

$$\|\varepsilon\|_{\dot{H}^{1+\frac{1}{3}}} \lesssim \|\varepsilon\|_{\dot{H}^1}^{\frac{2}{3}} \|\varepsilon\|_{\dot{H}^2}^{\frac{1}{3}} \leq (\tilde{C}\delta^4)^{\frac{2}{3}} \|\varepsilon\|_{\dot{H}^2}^{\frac{1}{3}} \rightarrow 0 \ \text{ as } s \rightarrow +\infty.$$

From the variation of energy identity (*5.2.41*) one has that $(a^2 + \|\varepsilon\|_{\dot{H}^2}^2) \in L^1([0,+\infty))$, implying that the second term is convergent. These two facts imply (*5.4.11*).

step 4 Convergence of the perturbation to 0 in $\dot{H}^1$. The convergence (*5.4.9*) and the bound $a \lesssim \|\varepsilon\|_{\dot{H}^2}^2$ from (*5.4.2*) ensure

$$a(s) \rightarrow 0 \ \text{ as } \ s \rightarrow +\infty.$$

It remains to show the convergence to 0 for ε in the $\dot{H}^1$ energy norm. We come back to the original time variable t. As a converges to 0 as $t \rightarrow +\infty$ with $\int_0^{+\infty} a^2(t)dt < +\infty$ from (*5.2.41*), and as ε converges to 0 as $t \rightarrow +\infty$ in $\dot{H}^2$ with $\int_0^{+\infty} \|\varepsilon(t)\|_{\dot{H}^2}^2 dt < +\infty$, and is bounded in $\dot{H}^1$ from (*5.2.12*) and the fact that the solution is trapped at distance $\tilde{C}\delta^4$ on $[0,+\infty)$ from (*5.4.6*), we can gather the instable and stable parts and write our solution as:

$$u = Q_{z,\lambda} + \tilde{\varepsilon}, \ \ \tilde{\varepsilon} = (a\mathcal{Y} + \varepsilon)_{z,\lambda} \tag{5.4.12}$$

with a perturbation $\tilde{\varepsilon}$ satisfying, as the scale does not diverge from (*5.2.56*):

$$\|\tilde{\varepsilon}\|_{\dot{H}^1} \ll 1, \ \ \lim_{t\rightarrow+\infty} \|\tilde{\varepsilon}(t)\|_{\dot{H}^2} = 0, \ \ \int_0^{+\infty} \|\tilde{\varepsilon}(t)\|_{\dot{H}^2}^2 dt < +\infty. \tag{5.4.13}$$

The last space time integrability property, via Sobolev embedding yields:

$$\int_0^{+\infty} \|\nabla\tilde{\varepsilon}\|_{L^{\frac{2d}{d-2}}}^2 < +\infty.$$

Hence one has the boundedness of Strichartz type norms for $\nabla\tilde{\varepsilon}$:

$$\|\nabla\tilde{\varepsilon}\|_{L^\infty([0,+\infty),L^2(\mathbb{R}^d))} + \|\nabla\tilde{\varepsilon}\|_{L^2([0,+\infty),L^{\frac{2d}{d-2}}(\mathbb{R}^d))} < +\infty. \tag{5.4.14}$$

The evolution of $\tilde{\varepsilon}$ is given by:

$$\tilde{\varepsilon}_t - \Delta\tilde{\varepsilon} = pQ_{z,\lambda}^{p-1}\tilde{\varepsilon} + \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} + NL, \ \ NL := f(u) - f(Q) - f'(Q)\tilde{\varepsilon}.$$

The Duhamel formula then gives, with K_t being defined by (*5.1.1*):

$$\begin{array}{rcl} \nabla\tilde{\varepsilon} & = & K_t * (\nabla\tilde{\varepsilon}(0)) + \int_0^t (\nabla K_{t-t'}) * (pQ_{z,\lambda}^{p-1}\tilde{\varepsilon})dt' + \int K_{t-t'} * (\nabla NL)dt' \\ & & + \int_0^t (\nabla K_{t-t'}) * (\frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda})dt'. \end{array} \tag{5.4.15}$$

We now estimate each term in the right hand side of the previous identity and prove that it goes to 0 in $\dot{H}^1$ as $t \rightarrow +\infty$.

Free evolution term. The first one undergoes dissipation:

$$K_t * (\nabla\tilde{\varepsilon}(0)) \rightarrow 0 \text{ in } L^2(\mathbb{R}^d) \text{ as } t \rightarrow +\infty. \tag{5.4.16}$$

Potential term. Let $0<\epsilon\ll 1$ be small enough and $T>0$. Using Young and Hölder inequalities, (*5.4.13*) and the fact that λ and z converge as $t\to+\infty$, for $t>T$ one computes:

$$
\begin{aligned}
& \| \int_0^t (\nabla K_{t-t'})*(pQ_{z,\lambda}^{p-1}\tilde\varepsilon)dt'\|_{L^2} \\
\leq & \int_0^{t-T}\|(\nabla K_{t-t'})*(pQ_{z,\lambda}^{p-1}\tilde\varepsilon)\|_{L^2}dt'+\int_{t-T}^t\|(\nabla K_{t-t'})*(pQ_{z,\lambda}^{p-1}\tilde\varepsilon)\|_{L^2}dt' \\
\lesssim & \int_0^{t-T}\|\nabla K_{t-t'}\|_{L^{1+\epsilon}}\|pQ_{z,\lambda}^{p-1}\tilde\varepsilon\|_{L^{\frac{2+2\epsilon}{1+3\epsilon}}}dt'+\int_{t-T}^t\|\nabla K_{t-t'}\|_{L^1}\|pQ_{z,\lambda}^{p-1}\tilde\varepsilon\|_{L^2} \\
\lesssim & \int_0^{t-T}\frac{1}{(t-t')^{\frac12+\epsilon\frac{d}{2+2\epsilon}}}\|pQ_{z,\lambda}^{p-1}\|_{L^{\frac{d+\epsilon d}{2+\epsilon(2+d)}}}\|\tilde\varepsilon\|_{L^{\frac{2d}{d-4}}}dt' \\
& +\int_{t-T}^t\frac{1}{\sqrt{t-t'}}\|pQ_{z,\lambda}^{p-1}\|_{L^{\frac d2}}\|\tilde\varepsilon\|_{L^{\frac{2d}{d-4}}}dt' \\
\lesssim & \int_0^{t-T}\frac{1}{(t-t')^{\frac12+\frac{\epsilon d}{2+2\epsilon}}}\|\tilde\varepsilon\|_{\dot H^2}dt'+\int_{t-T}^t\frac{1}{\sqrt{t-t'}}\|\tilde\varepsilon\|_{\dot H^2}dt' \\
\lesssim & \left(\int_0^{t-T}\frac{dt'}{(t-t')^{1+\frac{\epsilon d}{1+\epsilon}}}\right)^{\frac12}\left(\int_0^{t-T}\|\tilde\varepsilon\|_{\dot H^2}^2dt'\right)^{\frac12}+\int_{t-T}^t\frac{\sup_{t-T\leq t'\leq t}\|\tilde\varepsilon(t')\|_{\dot H^2}dt'}{\sqrt{t-t'}} \\
\lesssim & \frac{1}{T^{\frac{\epsilon d}{2+2\epsilon}}}\left(\int_0^{+\infty}\|\tilde\varepsilon(t')\|_{\dot H^2}^2dt'\right)^{\frac12}+C(T)\sup_{t-T\leq t'\leq t}\|\tilde\varepsilon(t')\|_{\dot H^2} \\
\lesssim & \left(\frac{1}{T^{\frac{\epsilon d}{2+2\epsilon}}}+C(T)\sup_{t-T\leq t'\leq t}\|\tilde\varepsilon(t')\|_{\dot H^2}\right)\to\frac{1}{T^{\varepsilon(d+1)}} \text{ as } t\to+\infty.
\end{aligned}
$$

Now, this computation being valid for any fixed $T>0$, one obtains the convergence to 0 for this term:

$$
\lim_{t\to+\infty}\left\|\int_0^t(\nabla K_{t-t'})*(pQ_{z,\lambda}^{p-1}\tilde\varepsilon)dt'\right\|_{L^2}=0. \qquad (5.4.17)
$$

Nonlinear term. We now turn to the third term in (*5.4.15*). Using the estimates (*5.D.8*) and (*5.D.9*) for the nonlinearity one obtains first the pointwise bound:

$$
\begin{aligned}
|\nabla NL| &= \Big|p(|Q_{z,\lambda}+\tilde\varepsilon|^{p-1}-Q_{z,\lambda}^{p-1})\nabla\tilde\varepsilon \\
&\quad +p(|Q_{z,\lambda}+\tilde\varepsilon|^{p-1}-Q_{z,\lambda}^{p-1}-(p-1)Q_{z,\lambda}^{p-2}\tilde\varepsilon)\nabla(Q_{z,\lambda})\Big| \\
&\lesssim |\tilde\varepsilon|^{p-1}|\nabla\tilde\varepsilon|+\tfrac{|\tilde\varepsilon|^p}{1+|x|}.
\end{aligned}
$$

Now, using Hölder inequality, the generalized Hardy inequality in Lebesgue spaces and Sobolev embedding we estimate this term via:

$$
\begin{aligned}
& \|\nabla NL\|_{L^{\frac{2d}{(d-2)+(d-4)(p-1)}}} \\
\lesssim & \||\tilde\varepsilon|^{p-1}|\nabla\tilde\varepsilon|\|_{L^{\frac{2d}{(d-2)+(d-4)(p-1)}}}+\left\|\frac{|\tilde\varepsilon|^p}{1+|x|}\right\|_{L^{\frac{2d}{(d-2)+(d-4)(p-1)}}} \\
\lesssim & \|\tilde\varepsilon\|_{L^{\frac{2d}{d-4}}}^{p-1}\|\nabla\tilde\varepsilon\|_{L^{\frac{2d}{d-2}}}+\left\|\frac{\tilde\varepsilon}{1+|x|}\right\|_{L^{\frac{2d}{d-2}}}\|\tilde\varepsilon\|_{L^{\frac{2d}{d-4}}}^{p-1} \\
\lesssim & \|\tilde\varepsilon\|_{\dot H^2}^{p-1}\|\tilde\varepsilon\|_{\dot H^2}+\|\nabla\tilde\varepsilon\|_{L^{\frac{2d}{d-2}}}\|\tilde\varepsilon\|_{\dot H^2}^{p-1}\lesssim\|\tilde\varepsilon\|_{\dot H^2}^p.
\end{aligned}
$$

As $\|\tilde\varepsilon\|_{\dot H^2}\in L^2([0,+\infty))$ this means that:

$$
\|\nabla NL\|_{L^{\frac2p}\left([0,+\infty),L^{\frac{2d}{(d-2)+(d-4)(p-1)}}(\mathbb R^d)\right)}<+\infty.
$$

We let (q,r) be the conjugated exponents of $\frac{2}{p}$ and $\frac{2d}{(d-2)+(d-4)(p-1)}$ respectively:

$$q=\frac{2}{2-p}>2, \quad r=\frac{2d}{d+2-(d-4)(p-1)}>2.$$

They satisfy the Strichartz relation $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$. Therefore, using (*5.E.22*) one gets for any $0\leq T\leq t$:

$$\left\| \int_0^T K_{T-t'} * (\nabla NL)dt' \right\|_{L^2} \leq \| \nabla NL \|_{L^{\frac{2}{p}}\left([0,T],L^{\frac{2d}{(d-2)+(d-4)(p-1)}}\right)}<+\infty,$$

$$\begin{aligned} \| \textstyle\int_T^t K_{t-t'} * (\nabla NL)dt' \|_{L^2} &\leq \| \nabla NL \|_{L^{\frac{2}{p}}\left([T,t],L^{\frac{2d}{(d-2)+(d-4)(p-1)}}\right)} \\ &\leq \| \nabla NL \|_{L^{\frac{2}{p}}\left([T,+\infty],L^{\frac{2d}{(d-2)+(d-4)(p-1)}}\right)} \\ &\to 0 \text{ as } T\to+\infty. \end{aligned}$$

We now write:

$$\int_0^t K_{t-t'} * (\nabla NL)dt' = K_{t-T} * \left(\int_0^T K_{T-t'} * (\nabla NL)dt' \right) + \int_T^t K_{t-t'} * (\nabla NL)dt'$$

and the two previous inequalities imply that, for T fixed the first term goes to 0 in $\dot{H}^1$ as $t\to+\infty$, and the second goes to 0 in $\dot{H}^1$ as $T\to+\infty$ uniformly in $t\geq T$. Therefore one gets the convergence to 0 of the nonlinear term in $\dot{H}^1$ as $t\to+\infty$:

$$\left\| \int K_{t-t'} * (\nabla NL)dt' \right\|_{L^2} \to 0 \text{ as } t\to+\infty. \tag{5.4.18}$$

Remainders from scale and space translations. We now turn to the last two terms in (*5.4.15*). From the modulation equations (*5.2.16*), (*5.2.17*), the variation of energy formula (*5.2.41*), the fact that λ converges and the convergence to 0 of a and ε in $\dot{H}^2$ (*5.4.11*) and (*5.4.9*) one has:

$$\lambda_t, z_t \in L^2([0,+\infty))\cap L^\infty([0,+\infty)), \quad \text{with } |\lambda_t|+|z_t|\to 0 \text{ as } t\to+\infty.$$

Moreover one has $\Lambda Q, \nabla Q\in L^2\cap L^{\frac{3}{2}}$ as $d\geq 7$. One then deduces that:

$$\left\| \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda}+\frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \right\|_{L^2} \to 0 \text{ as } t\to+\infty,$$

$$\left\| \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda}+\frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \right\|_{L^2} \in L^2([0,+\infty),L^{\frac{3}{2}}(\mathbb{R}^d)).$$

Therefore one has for any $t\geq T>0$, using Young inequality for convolution and Hölder inequality:

$$
\begin{aligned}
& \left\| \int_0^t (\nabla K_{t-t'}) * \left[\frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \right] dt' \right\|_{L^2} \\
\leq & \int_0^{t-T} \|\nabla K_{t-t'}\|_{L^{\frac{6}{5}}} \left\| \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \right\|_{L^{\frac{3}{2}}} dt' \\
& + \int_{t-T}^t \|\nabla K_{t-t'}\|_{L^1} \left\| \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \right\|_{L^2} dt' \\
\lesssim & \int_0^{t-T} \frac{dt'}{(t-t')^{\frac{1}{2}+\frac{d}{12}}} \left\| \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \right\|_{L^{\frac{3}{2}}} \\
& + \int_{t-T}^t \frac{dt'}{\sqrt{t-t'}} \| \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \|_{L^2} \\
\lesssim & T^{-\frac{d}{12}} \left(\int_0^{t-T} \| \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \|^2_{L^{\frac{3}{2}}} dt' \right)^{\frac{1}{2}} \\
& + \sqrt{T} \sup_{t\in[t-T,t]} \| \frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda} \|_{L^2} \to T^{-\frac{d}{12}} \text{ as } t\to +\infty.
\end{aligned}
$$

from what we deduce as this is valid for any $T>0$ that

$$
\| \textstyle\int_0^t (\nabla K_{t-t'}) * (\frac{\lambda_t}{\lambda}(\Lambda Q)_{z,\lambda} + \frac{z_t}{\lambda}.(\nabla Q)_{z,\lambda}) dt' \|_{L^2} \to 0 \text{ as } t\to+\infty. \tag{5.4.19}
$$

Conclusion We now come back to the Duhamel formula (*5.4.15*). We showed in (*5.4.16*), (*5.4.17*), (*5.4.18*) and (*5.4.19*) that each terms in the right hand side converges to zero strongly in L^2 as $t\to+\infty$. Hence $\tilde{\varepsilon}$ converges strongly to 0 in $\dot{H}^1$ as $t\to+\infty$. Going back to (*5.4.12*), this, with the convergence of λ and z as $t\to+\infty$ showed in Step 3, implies that $u\to Q_{z_\infty,\lambda_\infty}$ strongly in $\dot{H}^1$ as $t\to+\infty$, ending the proof of the Lemma.

□

5.4.4 Transition regime and no return

We now study the (exit) regime $T_{ins}<T_{u_0}$ and start with the fundamental no return lemma:

Lemma 5.4.3 (No return lemma). *There holds*

$$
T_{trans}<+\infty, \quad T_{trans}<T_{u_0}
$$

and the bounds for all $T_{ins}\leq t\leq T_{trans}$*:*

$$
\|\varepsilon(t)\|_{\dot{H}^1} + \|\varepsilon(t)\|_{\dot{H}^2} \lesssim \delta^2 \tag{5.4.20}
$$

$$
\|\varepsilon\|^2_{\dot{H}^2}(t) \lesssim \frac{|a(t)|}{K}. \tag{5.4.21}
$$

Moreover,

$$
|a(T_{trans})| = \delta. \tag{5.4.22}
$$

Proof of Lemma 5.4.4 First notice is from their definition (*5.4.4*) and (*5.4.5*) one has $T_{\text{trans}}\leq T_{\text{exit}}\leq T_{u_0}$ and that the solution is trapped at distance $\tilde K\alpha$ on $[0,T_{\text{exit}})$. From (*5.2.56*) the scale does not degenerate:

$$\lambda=1+O(\alpha). \tag{5.4.23}$$

To reason in renormalized time we define

$$S_{\text{trans}}:=\lim_{t\to T_{\text{trans}}}s(t),\quad S_{\text{exit}}:=\lim_{t\to T_{\text{exit}}}s(t).$$

We recall that from its definition, on $[0,S_{\text{trans}})$ the solution is trapped at distance $\tilde K\delta$.

step 1 Proof of the $\dot H^1$ bound. On $[0,S_{\text{trans}})$ the $\dot H^1$ energy bound (*5.2.43*) gives:

$$\frac{d}{ds}\left[\frac12\int\varepsilon H\varepsilon\right]\lesssim a^4.$$

We integrate it in time on using the fact that $\int_0^{S_{\text{trans}}}a^2\lesssim\delta^2$ from (*5.2.41*), the fact that $|a(s)|\leq\delta$ for all $0\leq s\leq S_{\text{trans}}$ from (*5.4.2*) and (*5.4.4*) and (*5.2.12*), the coercivity (*5.2.7*) and (*5.4.1*):

$$\|\varepsilon(s)\|_{\dot H^1}^2\lesssim\int\varepsilon(s)H\varepsilon(s)\lesssim\int\varepsilon(0)H\varepsilon(0)+\int_0^s a^4(s)ds\lesssim\|\varepsilon(0)\|_{\dot H^1}^2+\delta^4\lesssim\delta^4.$$

This proves the first bound in (*5.4.20*).

step 2 Proof of the $\dot H^2$ bound. On $[0,S_{\text{trans}})$ the $\dot H^2$ energy bound (*5.2.44*), gives:

$$\frac{d}{ds}\left[\frac{1}{\lambda^2}\int(H\varepsilon)^2\right]\lesssim\frac{1}{\lambda^2}\left(a^4+a^2\int(H\varepsilon)^2\right).$$

We integrate it in time on $[S_{\text{ins}},S_{\text{trans}})$ using the facts that

$$\int_{S_{\text{ins}}}^{S_{\text{trans}}}\left(\|H\varepsilon(s)\|^2+a^2(s)\right)ds\lesssim\delta^2$$

from (*5.2.41*) and because on $[0,S_{\text{trans}})$ the solution is trapped at distance $\tilde K\delta$ from (*5.4.4*), that $|a(s)|\leq\delta$ for all $S_{\text{ins}}\leq s\leq S_{\text{trans}}$ from (*5.4.4*) and (*5.2.12*), the non degeneracy of the scale (*5.4.23*) and (*5.2.8*):

$$\begin{array}{rcl}\|\varepsilon(s)\|_{\dot H^2}^2&\lesssim&\int(H\varepsilon(s))^2\lesssim\int(H\varepsilon(S_{\text{ins}}))^2+\int_{S_{\text{ins}}}^s\left(a^4+a^2\int(H\varepsilon)^2\right)ds\\&\lesssim&\|\varepsilon(S_{\text{ins}})\|_{\dot H^2}^2+\delta^2\int_{S_{\text{ins}}}^s(a^2+\|\varepsilon\|_{\dot H^2}^2)ds\lesssim\delta^4\end{array}$$

as $\|\varepsilon(T_{\text{ins}})\|_{\dot H^2}^2\lesssim\delta^4$ from (*5.4.7*) and (*5.4.6*). This proves the second bound in (*5.4.20*).

step 3 No return. We now turn to the proof of (*5.4.21*) using a bootstrap argument. Let $\tilde C>0$ be a constant such that:

$$\frac{1}{\tilde C}\|\varepsilon\|_{\dot H^2}^2\leq\int(H\varepsilon)^2\leq\tilde C\|\varepsilon\|_{\dot H^2}^2$$

which exists from (*5.2.8*) and is independent of the other constants. Let $\mathcal S$ be the set of times:

$$\mathcal S:=\left\{s\in[S_{\text{ins}},S_{\text{trans}}],\ \forall S_{\text{ins}}\leq s'\leq s,\ \int(H\varepsilon(s'))^2\leq\frac{\tilde C|a(s')|\lambda^2(s')}{K\lambda^2(S_{\text{ins}})}\right\}.$$

$\mathcal{S}$ is non empty as it contains S_{ins} from the two previous inequalities. It is closed by a continuity argument. We now show that it is open in $[S_{\text{ins}}, S_{\text{trans}})$. For each $s\in\mathcal{S}$ using (*5.2.44*), (*5.2.15*), (*5.4.23*), the fact that $|a(s)|\leq \delta$ for all $S_{\text{ins}}\leq s\leq S_{\text{trans}}$ from (*5.4.4*) and (*5.2.12*), and the coercivity (*5.2.8*):

$$
\begin{aligned}
&\frac{d}{ds}\left(|a(s)|-\frac{K\lambda^2(S_{\text{ins}})}{\tilde C\lambda^2}\int (H\varepsilon)^2\right)\\
\geq\ & e_0|a(s)|-C|a(s)|^2-C\|\varepsilon(s)\|^2_{\dot H^2}-\frac{CK}{\tilde C}\frac{\lambda^2(S_{\text{ins}})}{\lambda^2(s)}(|a(s)|^4+|a(s)|^2\|\varepsilon(s)\|^2_{\dot H^2})\\
\geq\ & |a(s)|\left(e_0-C\delta-\frac{C}{K}-CK\delta^3-C\delta^2\right)>0
\end{aligned}
$$

for K large enough and δ small enough, where the constant C is independent of the other constants. Consequently $\mathcal{S}$ is open, which implies that $\mathcal{S}=[S_{\text{ins}},S_{\text{trans}}]$. From the definition of $\mathcal{S}$, (*5.4.23*) and (*5.2.8*) one has proven (*5.4.21*).

step 4 Proof of $T_{\text{trans}}<T_{u_0}$. We claim that $T_{\text{trans}}<+\infty$. Indeed, from (*5.4.21*) and the modulation equation (*5.2.15*) for a one gets:

$$|a|_s\geq |a(s)|\left(e_0-C\delta-\frac{C}{K}\right)$$

for a constant C independent of the other constants. Hence for K large enough, the function $|a|$ satisfies $|a|_s>c>0$ on $[S_{\text{ins}},S_{\text{trans}}]$. Therefore $T_{\text{trans}}<+\infty$ because if not a would be unbounded, which is a contradiction to the very definition of T_{trans} (*5.4.4*). This implies $T_{\text{trans}}<T_{u_0}$. This is obvious if $T_{u_0}=+\infty$ and is otherwise a consequence of (*5.4.20*) and the control of the scale which implies a uniform $\dot H^2$ bound on u in $[0,T_{trans}]$ and hence $T_{trans}<T_{u_0}$ from (*5.2.1*). The estimate (*5.4.22*) now follows by continuity and (*5.4.21*). $\square$

5.4.5 (Exit) dynamics

We now classify the (Exit) dynamics and show that $Q^\pm$ are the attractors.

Lemma 5.4.4 (Classification of the (Exit) dynamics). *There exists $K^*\gg 1$, such that for any $K\geq K^*$, there exists $0<\delta^*(K)\ll 1$, such that for any $0<\delta<\delta^*(K)$, if $T_{ins}<T_{u_0}$ then either u will blow up with type I blow up forward in time, or u will converge to 0 strongly in $\dot H^1$.*

Proof of lemma 5.4.4 At time T_{trans}, there holds from (*5.4.22*), (*5.4.20*):

$$|a(T_{\text{trans}})|=\delta,\quad \|\varepsilon(t)\|_{\dot H^1}+\|\varepsilon(t)\|_{\dot H^2}\lesssim \delta^2.$$

This is the cruising instable regime. We recall that the exit time $T_{\text{trans}}<T_{\text{exit}}$ is defined by (*5.4.5*).

step 1 First exponential bounds in renormalized time. We claim that $T_{\text{exit}}<+\infty$, $T_{\text{exit}}\neq T_{u_0}$ and that the following holds on $[S_{\text{trans}},S_{\text{exit}}]$:

$$
\begin{aligned}
&a(s)=\pm(\delta+\tilde a)e^{e_0(s-S_{\text{trans}})},\ \text{with } |\tilde a|\leq \tilde C'\delta^2 e^{e_0(s-S_{\text{trans}})},\\
&\|\varepsilon\|_{\dot H^1}+\|\varepsilon\|_{\dot H^2}\leq \tilde C'\delta^2 e^{2e_0(s-S_{\text{trans}})}
\end{aligned}
\qquad (5.4.24)
$$

for some constant $\tilde{C}'>0$ independent of the other constants. We use a bootstrap method to prove the above bound. Fix $\tilde{C}>0$ and define $\mathcal{S}\subset[S_{\text{trans}},S_{\text{exit}}]$ as the set of times s such that (*5.4.24*) holds on $[S_{\text{trans}},s]$. For $\tilde{C}'$ large enough independently on the other constants, $\mathcal{S}$ is non empty as it contains S_{trans} from (*5.4.20*). It is closed by a continuity argument. We claim that for $\tilde{C}'$ big enough independently of the other constants, it is open. The first thing to notice is that from (*5.4.24*) and (*5.4.5*):

$$\delta e^{e_0(s-S_{\text{trans}})}\leq 2\alpha. \tag{5.4.25}$$

Estimate for $\tilde{a}$. For $s\in\mathcal{S}$ we compute from (*5.2.15*), using (*5.4.24*) and (*5.4.25*):

$$\begin{aligned} |\tilde{a}_s| &= e^{-e_0(s-S_{\text{trans}})}[O(a^2)+O(\|\varepsilon\|^2_{\dot{H}^2})] \\ &\leq e^{-e_0(s-S_{\text{trans}})}[C\delta^2e^{2(e_0s-S_{\text{trans}})}+C(\tilde{C}')^2\delta^2e^{2e_0(s-S_{\text{trans}})}\alpha^2] \\ &\leq \delta^2e^{e_0(s-S_{\text{trans}})}(C+C\alpha^2(\tilde{C}')^2). \end{aligned}$$

Reintegrating it in time between S_{trans} and $s\in\mathcal{S}$ we find, as $\tilde{a}(S_{\text{trans}})=0$:

$$|\tilde{a}(s)|\lesssim\delta^2e^{e_0(s-S_{\text{trans}})}(C+C\alpha^2(\tilde{C}')^2)<\delta^2\tilde{C}'e^{e_0(s-S_{\text{trans}})} \tag{5.4.26}$$

for $\tilde{C}'$ large enough and α small enough.
Estimate for $\|\varepsilon\|_{\dot{H}^1}$. For $s\in\mathcal{S}$ we compute from (*5.2.43*), using (*5.4.24*) and (*5.4.25*):

$$\frac{d}{ds}\left[\int\varepsilon H\varepsilon\right]\leq C|a|^4\leq C\delta^4e^{4e_0(s-S_{\text{trans}})}[1+(\tilde{C}')^4\alpha^4].$$

Reintegrating it in time between S_{trans} and $s\in\mathcal{S}$ we find, using (*5.4.20*) and the coercivity (*5.2.7*):

$$\begin{aligned} \|\varepsilon(s)\|^2_{\dot{H}^1} &\leq C\int\varepsilon(s)H\varepsilon(s) \\ &\leq C\int\varepsilon(S_{\text{trans}})H\varepsilon(S_{\text{trans}})+\int_{S_{\text{Trans}}}^sC\delta^4e^{4e_0(s-S_{\text{trans}})}[1+(\tilde{C}')^4\alpha^4]ds \\ &\leq C\|\varepsilon(S_{\text{trans}})\|^2_{\dot{H}^1}+\delta^4e^{4e_0(s-S_{\text{trans}})}(C+C\alpha^4(\tilde{C}')^2) \\ &\leq C\delta^4+\delta^4e^{4e_0(s-S_{\text{trans}})}(C+C\alpha^4(\tilde{C}')^2)<\delta^4\tilde{C}'e^{4e_0(s-S_{\text{trans}})} \end{aligned} \tag{5.4.27}$$

for $\tilde{C}'$ large enough and α small enough.

Estimate for $\|\varepsilon\|_{\dot{H}^2}$. For $s\in\mathcal{S}$ we compute from (*5.2.44*), using (*5.4.24*), (*5.4.25*) and (*5.4.23*):

$$\frac{d}{ds}[O(1)\int(H\varepsilon)^2]\leq Ca^4+Ca^2\|\varepsilon\|^2_{\dot{H}^2}\leq C\delta^4e^{4e_0(s-S_{\text{trans}})}[1+(\tilde{C}')^4\alpha^4].$$

where $O(1)=\frac{1}{\lambda^2}=1+O(\alpha)$ from (*5.4.23*). Reintegrating it in time between S_{trans} and $s\in\mathcal{S}$ we find, using (*5.4.20*) and the coercivity (*5.2.8*):

$$\begin{aligned} \|\varepsilon(s)\|^2_{\dot{H}^2} &\leq C\|\varepsilon(S_{\text{trans}})\|^2_{\dot{H}^2}+\delta^4e^{4e_0(s-S_{\text{trans}})}(C+C\alpha^2(\tilde{C}')^2) \\ &\leq C\delta^4+\delta^4e^{4e_0(s-S_{\text{trans}})}(C+C\alpha^4(\tilde{C}')^2)<\delta^4\tilde{C}'e^{4e_0(s-S_{\text{trans}})} \end{aligned} \tag{5.4.28}$$

for $\tilde{C}'$ large enough and α small enough.

Conclusion. From (*5.4.26*), (*5.4.27*) and (*5.4.28*) one obtains that $\mathcal{S}$ is open, hence $\mathcal{S}=[S_{\text{trans}},S_{\text{exit}})$ which ends the proof of (*5.4.24*). The law for a (*5.4.24*) implies $T_{\text{exit}}<+\infty$, and the $\dot{H}^2$ bound (*5.4.24*) and the control of the scale now imply $T_{\text{exit}} < T_{u_0}$.

step 2 Exponential bounds in original time variable. Let now the constant $\tilde{C}'$ used in the first substep in (*5.4.24*) be fixed. We claim that one has the following estimates[6] in original time variables on $[T_{\text{trans}}, T_{\text{exit}}]$:

$$|z(t) - z(T_{\text{trans}})| + \left|\frac{\lambda(t)}{\lambda(T_{\text{trans}})} - 1\right| \lesssim \delta^2 e^{\frac{2e_0}{\lambda^2(T_{\text{trans}})}(t-T_{\text{trans}})}, \tag{5.4.29}$$

$$\begin{aligned} &a(t) = \pm(\delta+\tilde{a})e^{e_0\frac{(t-T_{\text{trans}})}{\lambda^2(T_{\text{trans}})}}, \quad |\tilde{a}| \lesssim \delta^2 e^{\frac{e_0}{\lambda^2(T_{\text{trans}})}(t-T_{\text{trans}})}, \\ &\|\varepsilon\|_{\dot{H}^1} + \|\varepsilon\|_{\dot{H}^2} \lesssim \delta^2 e^{\frac{2e_0}{\lambda^2(T_{\text{trans}})}(t-T_{\text{trans}})}. \end{aligned} \tag{5.4.30}$$

Bound for λ and estimate on s. The modulation equation (*5.2.18*), using (*5.4.24*), can be rewritten as:

$$\left|\frac{d}{ds}\left[\log(\lambda) + O(\delta^2 e^{2e_0(s-S_{\text{trans}})})\right]\right| \lesssim \delta^2 e^{2e_0(s-S_{\text{trans}})}$$

After reintegration in time this becomes, using (*5.4.23*) and (*5.4.25*):

$$\lambda(s) = \lambda(S_{\text{trans}}) + O(\delta^2 e^{2e_0(s-S_{\text{trans}})}). \tag{5.4.31}$$

The definition of the renormalized time s (*5.2.13*) then implies:

$$\frac{dt}{ds} = \lambda^2(T_{\text{trans}}) + O(\delta^2 e^{2e_0(s-S_{\text{trans}})}).$$

Reintegrated in time this gives:

$$t - T_{\text{trans}} = \lambda^2(T_{\text{trans}})(s - S_{\text{trans}}) + O(\delta^2 e^{2e_0(s-S_{\text{trans}})}).$$

From (*5.4.25*) this implies:

$$e^{s-S_{\text{trans}}} = e^{\frac{t-T_{\text{trans}}}{\lambda^2(T_{\text{trans}})} + O(\delta^2 e^{2e_0(s-S_{\text{trans}})})} = e^{\frac{t-T_{\text{trans}}}{\lambda^2(T_{\text{trans}})}}(1+O(\alpha))$$

and therefore

$$t - T_{\text{trans}} = \lambda^2(T_{\text{trans}})(s - S_{\text{trans}}) + O\left(\delta^2 e^{\frac{2e_0}{\lambda^2(T_{\text{trans}})}(t-T_{\text{trans}})}\right). \tag{5.4.32}$$

We inject the above identity in (*5.4.31*), yielding the estimate for λ in (*5.4.29*).

Bound for z. The modulation equation (*5.2.19*) for z, using (*5.4.24*), can be rewritten as:

$$\left|\frac{d}{ds}\left[z + O(\delta^2 e^{2e_0(s-S_{\text{trans}})})\right]\right| \lesssim \delta^2 e^{2e_0(s-S_{\text{trans}})}$$

After reintegration in time this becomes:

$$|z(s) - z(S_{\text{trans}})| \lesssim \delta^2 e^{2e_0(s-S_{\text{trans}})}. \tag{5.4.33}$$

We inject (*5.4.32*) in the above equation, giving the estimate for z in (*5.4.29*).
Bounds for a and ε. We inject (*5.4.32*) in (*5.4.24*), giving (*5.4.30*).

[6]The constants involved in the $\lesssim$ may depend on $\tilde{C}'$ defined earlier on the first substep of Step 2 but this is not a problem as it is fixed from now on independently of the other constants.

step 3 Setting up the comparison with $Q^{\pm}$. From now on, without loss of generality, we treat the case of a "plus" sign in (*5.4.30*), i.e. $a=\delta e^{e_0\frac{t-T_{\text{trans}}}{\lambda^2(T_{\text{trans}})}}$ at the leading order on $[T_{\text{trans}},T_{\text{exit}}]$, which will correspond to an exit close to a renormalized[7] version of Q^+. The case of a "minus" sign corresponds to an exit close to a renormalized[8] version of Q^- and can be treated with exactly the same techniques. From the definition (*5.4.5*) of the exit time T_{exit} and the law for a (*5.4.30*) on $[T_{\text{trans}},T_{\text{exit}}]$ at this time there holds:

$$\begin{aligned}
&\delta e^{\frac{e_0}{\lambda^2(T_{\text{trans}})}(T_{\text{exit}}-T_{\text{trans}})}+O\left(\delta^2 e^{\frac{2e_0}{\lambda^2(T_{\text{trans}})}(T_{\text{exit}}-T_{\text{trans}})}\right)=a(T_{\text{exit}})=\alpha,\\
&\delta e^{\frac{e_0}{\lambda^2(T_{\text{trans}})}(T_{\text{exit}}-T_{\text{trans}})}=O(\alpha),
\end{aligned}$$

from what one obtains the following formula for T_{exit}:

$$T_{\text{exit}}=T_{\text{trans}}+\frac{\lambda^2(T_{\text{trans}})}{e_0}\log\left(\frac{\alpha(1+O(\alpha))}{\delta}\right). \tag{5.4.34}$$

To ease the writing of the estimates, we first renormalize the function u at time T_{trans}. For $t\in[0,\frac{T_{\text{exit}}-T_{\text{trans}}}{\lambda^2(T_{\text{trans}})}]$ we define the renormalized time:

$$t'(t):=\lambda^2(T_{\text{trans}})t+T_{\text{trans}}\in[T_{\text{trans}},T_{\text{exit}}]. \tag{5.4.35}$$

We define the renormalized versions of u and of the adapted variables as:

$$\hat{u}(t,\cdot):=\left(\tau_{-z(T_{\text{trans}})}u\left(t',\cdot\right)\right)_{\frac{1}{\lambda(T_{\text{trans}})}}, \tag{5.4.36}$$

$$\bar{\varepsilon}(t):=\varepsilon(t'),\quad \bar{a}(t):=a(t'),\quad \bar{z}(t):=\tfrac{z(t')-z(T_{\text{trans}})}{\lambda(T_{\text{trans}}}, \quad \bar{\lambda}(t):=\tfrac{\lambda(t')}{\lambda(T_{\text{trans}})}. \tag{5.4.37}$$

We define the renormalized exit time $\hat{T}_{\text{exit}}$ by

$$\hat{T}_{\text{exit}}:=\frac{T_{\text{exit}}-T_{\text{trans}}}{\lambda^2(T_{\text{trans}})}=\frac{1}{e_0}\log\left(\frac{\alpha(1+O(\alpha))}{\delta}\right)\gg 1 \tag{5.4.38}$$

from (*5.4.34*). From the invariances of the equation, $\hat{u}$ is also a solution of (NLH), at least defined on $[0,\hat{T}_{\text{exit}}]$, and u blows up with type I if and only if $\hat{u}$ blows up with type I. We will then show the result for $\hat{u}$. As a,z,λ and ε are the adapted variables for u given by Definition 5.2.4 and as $(\tau_z f)_\lambda=\tau_{\lambda z}(f_\lambda)$:

$$\begin{aligned}
\hat{u}(t)&=\left(\tau_{-z(T_{\text{trans}})}u\left(t'\right)\right)_{\frac{1}{\lambda(T_{\text{trans}})}}=\left(\tau_{-z(T_{\text{trans}})}\tau_{z(t')}(Q+a(t')\mathcal{Y}+\varepsilon(t'))\right)_{\frac{1}{\lambda(T_{\text{trans}})}}\\
&=(Q+a(t')\mathcal{Y}+\varepsilon(t'))_{\frac{z(t')-z(T_{\text{trans}})}{\lambda(T_{\text{trans}})},\frac{\lambda(t')}{\lambda(T_{\text{trans}})}}=(Q+\bar{a}(t)\mathcal{Y}+\bar{\varepsilon}(t))_{\bar{z}(t),\bar{\lambda}(t)}
\end{aligned} \tag{5.4.39}$$

with $\bar{\varepsilon}$ satisfying the orthogonality conditions (*5.2.6*), from (*5.4.36*), (*5.4.37*), (*5.4.35*), (*5.2.11*) and (*5.2.6*). Therefore, $(\bar{\lambda},\bar{z},\bar{a},\bar{\varepsilon})$ are the variables associated to the decomposition of $\hat{u}$ given by Definition 5.2.4. (*5.4.35*), (*5.4.37*) and the bounds (*5.4.29*) and (*5.4.30*) imply:

$$|1-\bar{\lambda}|+|\bar{z}|+|\bar{a}-\delta e^{e_0 t}|+\|\bar{\varepsilon}\|_{\dot{H}^1}+\|\bar{\varepsilon}\|_{\dot{H}^2}\lesssim\delta^2 e^{2e_0 t} \tag{5.4.40}$$

The change of variable we did thus simplified the estimates. As we aim at comparing $\hat{u}$ to $Q^{\pm}$, the scale and central points $\bar{\lambda}$ and $\bar{z}$ might be adapted for $\hat{u}$ but they are not for $Q^{\pm}$. We perform a second change

[7] i.e. an element of the orbit of Q^+ under the symmetries of the flow: scaling and space and scale translations.

[8] Idem.

of variables to treat these two profiles as perturbations of Q under an affine adapted decomposition. To do so we define:

$$\begin{array}{ll} \hat{a}=\frac{e^{-e_0 t}}{\|\mathcal{Y}\|_{L^2}^2}\langle \hat{u}-Q,\mathcal{Y}\rangle-\delta, & \hat{z}_i=\frac{d}{\int \chi_M |\nabla Q|^2}\langle \hat{u}-Q,\Psi_i\rangle \ \text{ for } \ 1\leq i\leq d, \\ \hat{b}=\frac{1}{\int \chi_M \Lambda Q^2}\langle \hat{u}-Q,\Psi_0\rangle, & \hat{v}=\hat{u}-Q-(\hat{a}+\delta e^{e_0 t})\mathcal{Y}-\hat{b}\Lambda Q-\hat{z}.\nabla Q. \end{array} \quad (5.4.41)$$

From (*5.2.5*) these new variables produce the following decomposition for $\hat{u}$:

$$\hat{u}=Q+(\delta+\hat{a})e^{e_0 t}\mathcal{Y}+\hat{v}+\hat{b}\Lambda Q+\hat{z}.\nabla Q, \quad \hat{v}\in \text{Span}(\mathcal{Y},\Psi_0,\Psi_1,...,\Psi_d)^{\perp} \quad (5.4.42)$$

and from (*5.4.39*), (*5.4.40*), (*5.4.41*) and (*5.2.5*) they enjoy the following bounds:

$$|\hat{b}|+|\hat{z}|+|\hat{a}|e^{e_0 t}+\|\hat{v}\|_{\dot{H}^1}+\|\hat{v}\|_{\dot{H}^2}\lesssim \delta^2 e^{2e_0 t}. \quad (5.4.43)$$

This, from Sobolev embedding, implies that:

$$\|\hat{u}-Q\|_{L^{\frac{2d}{d-4}}}\lesssim \delta e^{e_0 t}.$$

By the parabolic regularization estimate (*5.E.6*) this implies that on $[1,\hat{T}_{\text{exit}}]$:

$$\|\hat{u}-Q\|_{L^\infty}\lesssim \delta e^{e_0 t}.$$

In turn, using again (*5.4.42*) and (*5.4.43*) this implies that in addition to (*5.4.43*) for all $t\in[1,\hat{T}_{\text{exit}}]$ one has an exponential L^∞ bound:

$$\|\hat{v}\|_{L^\infty}\lesssim \delta e^{e_0 t}. \quad (5.4.44)$$

We now aim at comparing $\hat{u}$, under the decomposition (*5.4.42*), the a priori bounds (*5.4.43*) and (*5.4.44*), to Q^+ in the time interval $[0,\hat{T}_{\text{exit}}]$. We need to provide a similar decomposition for Q^+. We recall that from (*5.3.17*), Q^+ satisfies on $(-\infty,0]$:

$$Q^+=Q+(\epsilon+O(\epsilon^2 e^{e_0 t}))e^{te_0}\mathcal{Y}+w, \quad \|w\|_{L^\infty}+\|w\|_{\dot{H}^1}\lesssim \epsilon^2 e^{2e_0 t} \quad (5.4.45)$$

for some $0<\epsilon\ll 1$ fixed independent of α and δ. One can therefore assume:

$$\alpha\ll\epsilon \quad (5.4.46)$$

First, we perform a time translation so that at time 0, the projection of Q^+ onto the unstable mode $\mathcal{Y}$ matches the one of $\hat{u}$, i.e. is δ. To do so we define the time:

$$t_0:=\frac{1}{e_0}\log\left(\frac{\epsilon}{\delta}\right)\gg \hat{T}_{\text{exit}} \quad (5.4.47)$$

from (*5.4.38*) and (*5.4.46*). We let $\hat{Q}^+$ be the time translated version of Q^+ defined by:

$$\hat{Q}^+(t,x)=Q^+(t-t_0,x), \quad (t,x)\in(-\infty,t_0]\times\mathbb{R}^d. \quad (5.4.48)$$

We then decompose $\hat{Q}^+$ in a similar way we decomposed $\hat{u}$, introducing the three associated parameters $\hat{a}'$, $\hat{b}'$ and $\hat{z}'$, and the profile $\hat{v}'$:

$$\hat{Q}^+=Q+(\delta+\hat{a}')e^{e_0 t}\mathcal{Y}+\hat{v}'+\hat{b}'\Lambda Q+\hat{z}'.\nabla Q, \quad \hat{v}'\in\text{Span}(\mathcal{Y},\Psi_0,\Psi_1,...,\Psi_d)^{\perp}$$

with the following bounds on $[0,t_0]$ from (*5.4.45*) and (*5.4.47*):

$$|\hat{b}'|+|\hat{z}'|+|\hat{a}'|e^{e_0t}+\|\hat{v}'\|_{\dot{H}^1}+\|\hat{v}'\|_{L^\infty}\lesssim \delta^2e^{2e_0t}. \tag{5.4.49}$$

Our aim is to compare $\hat{u}$ and $\hat{Q}^+$ and we claim that at the exit time there holds:

$$\|\hat{u}(\hat{T}_{\text{exit}})-\hat{Q}^+(\hat{T}_{\text{exit}})\|_{\dot{H}^1}\lesssim \delta. \tag{5.4.50}$$

Similarly, in the case $a(T_{\text{trans}})=-\delta$, the above bound holds replacing Q^+ with Q^-.

step 5 Proof of (*5.4.50*). The evolution of the difference $\hat{u}-\hat{Q}^+$ on $[1,\hat{T}_{\text{exit}}]$ is:

$$(\hat{v}-\hat{v}')_t+H(\hat{v}-\hat{v}')=-(\hat{a}-\hat{a}')_te^{e_0t}\mathcal{Y}-(\hat{b}-\hat{b}')_t\Lambda Q-(\hat{z}-\hat{z}')_t.\nabla Q+NL-NL', \tag{5.4.51}$$

where:

$$NL:=f(\hat{u})-f(Q)-f'(Q)\hat{u} \ \text{ and } \ NL':=f(\hat{Q}^+)-f(Q)-f'(Q)\hat{Q}^+$$

We define the weighted distance between $\hat{u}$ and $\hat{Q}^+$ on $[1,\hat{T}_{\text{exit}}]$ as:

$$D:=\sup_{1\leq t\leq \hat{T}_{\text{exit}}}\frac{e^{-e_0t}}{\delta}\left(\|\hat{v}-\hat{v}'\|_{\dot{H}^1}+\frac{|\hat{a}-\hat{a}'|}{\delta}e^{e_0t}+|\hat{b}-\hat{b}'|+|\hat{z}-\hat{z}'|\right) \tag{5.4.52}$$

From (*5.4.49*) and (*5.4.43*), (*5.4.44*), (*5.D.7*), Hölder inequality and interpolation one gets the following bounds for the difference of nonlinear terms on $[1,\hat{T}_{\text{exit}}]$:

$$\begin{array}{rl}
& \|NL-NL'\|_{L^2}\lesssim \||\hat{u}-\hat{Q}^+|(|\hat{u}-Q|^{p-1}+|\hat{Q}^+-Q|^{p-1})\|_{L^2}\\
\leq & \|\hat{u}-\hat{Q}^+\|_{L^{\frac{2d}{d-2}}}(\||\hat{u}-Q|^{p-1}\|_{L^d}+\||\hat{Q}^+-Q|^{p-1}\|_{L^d})\\
\leq & \|\hat{u}-\hat{Q}^+\|_{\dot{H}^1}(\|\hat{u}-Q\|^{p-1}_{L^{d(p-1)}}+\|\hat{Q}^+-Q\|^{p-1}_{L^{d(p-1)}})\\
\leq & \delta e^{e_0t}D\Big(\|\hat{u}-Q\|^{\frac{p-1}{2}}_{L^{\frac{2d}{d-2}}}\|\hat{u}-Q\|^{\frac{p-1}{2}}_{L^\infty}+\|\hat{Q}^+-Q\|^{\frac{p-1}{2}}_{L^{\frac{2d}{d-2}}}\|\hat{Q}^+-Q\|^{\frac{p-1}{2}}_{L^\infty}\Big)\\
\leq & \delta e^{e_0t}D\Big(\|\hat{u}-Q\|^{\frac{p-1}{2}}_{\dot{H}^1}\|\hat{u}-Q\|^{\frac{p-1}{2}}_{L^\infty}+\|\hat{Q}^+-Q\|^{\frac{p-1}{2}}_{\dot{H}^1}\|\hat{Q}^+-Q\|^{\frac{p-1}{2}}_{L^\infty}\Big)\\
\lesssim & \delta^pe^{pe_0t}D.
\end{array} \tag{5.4.53}$$

Energy estimate for the difference of errors. From (*5.4.51*), the orthogonality conditions (*5.3.41*) and the bound (*5.3.45*) on the nonlinear term one gets the following energy estimate:

$$\begin{array}{rcl}
\frac{d}{dt}\left[\int(\hat{v}-\hat{v}')H(\hat{v}-\hat{v}')\right] & = & 2\int H(\hat{v}-\hat{v}')(NL-NL')-2\int H(\hat{v}-\hat{v}')^2\\
& \lesssim & \|NL-NL'\|^2_{L^2}\lesssim D^2\delta^{2p}e^{2e_0pt}.
\end{array}$$

From the coercivity of the linearized operator (*5.2.8*), we reintegrate in time the above inequality to obtain that on $[1,\hat{T}_{\text{exit}}]$:

$$\begin{array}{rcl}
\|\hat{v}-\hat{v}'\|^2_{\dot{H}^1} & \lesssim & \int(\hat{v}-\hat{v}')H(\hat{v}-\hat{v}')\\
& \leq & \int(\hat{v}(1)-\hat{v}'(1))H(\hat{v}(1)-\hat{v}'(1))+\int_1^tD^2\delta^{2p}e^{2e_0pt}dt\\
& \lesssim & \|\hat{v}(1)\|^2_{\dot{H}^1}+\|\hat{v}'(1)\|^2_{\dot{H}^1}+D^2\delta^{2p}e^{2e_0pt}\lesssim \delta^4+D^2\delta^{2p}e^{2e_0pt}
\end{array} \tag{5.4.54}$$

where we used (*5.4.43*) and (*5.4.49*). Using (*5.4.38*) this means:

$$\begin{array}{rcl}
\sup\limits_{1\leq t\leq \hat{T}_{\text{exit}}}\|\hat{v}-\hat{v}'\|_{\dot{H}^1}\frac{e^{-e_0t}}{\delta} & \lesssim & \sup\limits_{1\leq t\leq \hat{T}_{\text{exit}}}\delta^{p-1}e^{(p-1)e_0t}D+\sup\limits_{1\leq t\leq \hat{T}_{\text{exit}}}\frac{e^{-e_0t}\delta^2}{\delta}\\
& \lesssim & \alpha^{p-1}D+\delta
\end{array} \tag{5.4.55}$$

Modulation equations for the differences of parameters. We take the scalar products of (*5.4.51*) with ΛQ, $\mathcal{Y}$ and ∇Q, using the bound (*5.4.53*) for the nonlinear term and obtain for any $1 \leq t \leq \hat{T}_{\text{exit}}$:

$$\begin{aligned} & \left| \frac{d}{dt} \left[\hat{b} - \hat{b}' + \int (\hat{v} - \hat{v}') \Lambda Q \right] \right| + \left| \frac{d}{dt} \left[\hat{z} - \hat{z}' + \int (\hat{v} - \hat{v}') \nabla Q \right] \right| + |(\hat{a} - \hat{a}')_t| e^{e_0 t} \\ \lesssim \ & \delta^p e^{p e_0 t} D. \end{aligned}$$

We integrate in time the two above equations on $[1, \hat{T}_{\text{exit}}]$, using (*5.4.54*), (*5.4.43*) and (*5.4.49*), giving for $t \in [1, \hat{T}_{\text{exit}}]$:

$$\begin{aligned} & |\hat{b}(t) - \hat{b}'(t)| + |\hat{z}(t) - \hat{z}'(t)| + |\hat{a}(t) - \hat{a}'(t)| e^{e_0 t} \\ \lesssim \ & |\hat{b}(1) - \hat{b}'(1)| + |\hat{z}(1) - \hat{z}'(1)| + |\hat{a}(1) - \hat{a}'(1)| e^{e_0 t} + e^{e_0 t} \textstyle\int_1^t \delta^p e^{(p-1) e_0 t} D dt \\ & + |\textstyle\int (\hat{v}(t) - \hat{v}'(t)) \Lambda Q| + |\textstyle\int (\hat{v}(t) - \hat{v}'(t)) \Lambda Q| + \textstyle\int_1^t \delta^p e^{p e_0 t} D dt \\ \lesssim \ & \delta^2 + \delta^2 e^{e_0 t} + \|\hat{v}(t) - \hat{v}'(t)\|_{\dot{H}^1} + \delta^p e^{p e_0 t} D \lesssim \delta^2 e^{e_0 t} + \delta^p e^{p e_0 t} D. \end{aligned}$$

From this, we deduce using (*5.4.38*) that:

$$\sup_{1 \leq t \leq \hat{T}_{\text{exit}}} (|\hat{b} - \hat{b}'| + |\hat{z} - \hat{z}'| + |\hat{a} - \hat{a}'| e^{e_0 t}) \frac{e^{-e_0 t}}{\delta} \lesssim \delta + \alpha^{p-1} D. \tag{5.4.56}$$

End of the proof of (*5.4.50*). From the definition (*5.4.52*) of the weighted norm of the difference and the estimates (*5.4.56*) and (*5.3.45*) one obtains:

$$D \lesssim \delta + \alpha^{p-1} D.$$

We then conclude that $D \lesssim \delta$. From this fact and the definition (*5.4.52*) of D, the definition (*5.4.38*) of $\hat{T}_{\text{exit}}$, at time $\hat{T}_{\text{exit}}$ there holds:

$$\left(|\hat{b} - \hat{b}'| + |\hat{z} - \hat{z}'| + |\hat{a} - \hat{a}'| e^{e_0 \hat{T}_{\text{exit}}} + \|\hat{v} - \hat{v}'\|_{\dot{H}^1} \right) \lesssim D \delta e^{e_0 \hat{T}_{\text{exit}}} \lesssim \delta \alpha$$

which implies the estimate (*5.4.50*) we had to prove.
End of the proof. From (*5.4.48*) and (*5.4.38*) one has:

$$\hat{Q}^+(\hat{T}_{\text{exit}}) = Q^+ \left(\log \left(\left(\frac{\alpha + O(\alpha^2)}{\epsilon} \right)^{\frac{1}{e_0}} \right) \right)$$

which implies that there exists $C > 0$ such that:

$$\hat{Q}^+(\hat{T}_{\text{exit}}) \in \cup_{-C \leq \mu \leq C} Q^+ \left(\log \left(\left(\frac{\alpha + \mu \alpha}{\epsilon} \right)^{\frac{1}{e_0}} \right) \right) =: \mathcal{K}.$$

$\mathcal{K}$ is a compact set of $\dot{H}^1$ functions that will explode according to type I blow up forward in time, and $\mathcal{K}$ does not depend on δ. We notice that as $\delta \to 0$, from (*5.4.50*):

$$\inf_{f \in \mathcal{K}} \|\hat{u}(\hat{T}_{\text{exit}}) - f\|_{\dot{H}^1} \lesssim \delta \to 0 \text{ uniformly as } \delta \to 0.$$

As the set of functions exploding with type I blow up forward in time is an open set of $\dot{H}^1$ from Proposition 5.5.1 and Proposition 5.2.1, for δ small enough, $\hat{u}$ is blowing up with type I blow up forward in time.

By the symmetries of the equation this means that u is also blowing up with type I blow up forward in time.

The very same reasoning applies if $a(T_{\text{trans}}) = -\delta$, and in that case at the exit time the solution is arbitrarily close in $\dot{H}^1$ to a compact set of renormalized versions of Q^- going to 0 in the energy topology because of dissipation. As the set of initial data such that the solution goes to 0 in $\dot{H}^1$ as $t \to +\infty$ is also an open set of $\dot{H}^1$, one obtains that u is global with $\|u\|_{\dot{H}^1} \to 0$ as $t \to +\infty$. This ends the proof of Theorem 2.3.4.

□

5.5 Stability of type I blow up

In this section we give some properties of solutions blowing up with type I blow up, and we prove the stability of this behavior. We follow the lines of [54] where the authors prove it in the energy subcritical case $1 < p < \frac{d+2}{d-2}$. Their proof adapts almost automatically but we write a proof here for the sake of completeness and for some estimates are more subtle in the energy critical case, mainly Proposition 5.5.7. We recall that type I blow up is defined in Definition 2.1.7.

Proposition 5.5.1 (Stability of type I blow-up). *Let $u_0 \in W^{3,\infty}$ be an initial datum such that the solution u of (NLH) starting from u_0 blows up with type I. Then there exists $\delta = \delta(u_0) > 0$ such that for any $v_0 \in W^{3,\infty}$ with*

$$\|u_0 - v_0\|_{W^{3,\infty}} \leq \delta \tag{5.5.1}$$

the solution v of (NLH) starting from v_0 blows up with type I.

Remark 5.5.2. The topology $W^{3,\infty}$ is convenient for our purpose but is not essential because of the parabolic regularizing effects, see Proposition 5.2.1.

The section is organized as follows. In Subsection 5.5.1 we recall without proof some already known facts on type I blow-up, then we introduce the self-similar renormalization at a blow up point in Subsection 5.5.2, before giving the proof of Proposition 5.5.1 in Subsection 5.5.3.

5.5.1 Properties of type I blowing-up solution

A point $x \in \mathbb{R}^d$ is said to be a blow up point for u blowing up at time T if there exists $(t_n, x) \to (T, x)$ such that:

$$|u(t_n, x_n)| \to +\infty \text{ as } n \to +\infty.$$

A fundamental fact is the rigidity for solutions satisfying the type I blow up estimate of Definition 2.1.7 that are global backward in time, this is the result of Proposition 5.5.3. Then in Lemma 5.5.4 we give a precise description of type I blow up, with an asymptotic at a blow up point and an ODE type caracterization.

Proposition 5.5.3 (Liouville type theorem fot type I blow up [122, 117]). *If u be a solution of (NLH) on $(-\infty, 0] \times \mathbb{R}^d$ such that $\|u\|_{L^\infty} \leq C(-t)^{\frac{1}{p-1}}$ for some constant $C > 0$, then there exists $T \geq 0$ such that $u = \frac{\kappa}{(T-t)^{\frac{1}{p-1}}}$ where κ is defined in (5.1.5).*

Lemma 5.5.4 (Description of type I blow up [63, 122, 117]). *Let u solve (NLH) with $u_0 \in W^{2,\infty}$ blowing up at $T>0$. The three following properties are equivalent:*

$$(i) \quad \text{The blow-up is of type } I.$$

$$(ii) \quad \exists K>0, \quad |\Delta u| \leq \frac{1}{2}|u|^p + K \text{ on } \mathbb{R}^d \times [0,T). \tag{5.5.2}$$

$$(iii) \quad \|u\|_{L^\infty}(T-t)^{\frac{1}{p-1}} \to \kappa \text{ as } t \to T. \tag{5.5.3}$$

Moreover, if u blows up with type I at x then:

$$|u(t,x)| \sim \frac{\kappa}{(T-t)^{\frac{1}{p-1}}} \quad \text{as } t \to T, \tag{5.5.4}$$

and if $u_n(0) \to u(0)$ in $W^{2,\infty}$, for large n, u_n blows up at time T_n with $T_n \to T$.

The above results are stated in [54, 63, 122, 117] in the case $1<p<\frac{d+2}{d-2}$. They are however still valid in the energy critical case because the main argument is that the only bounded stationary solutions of (5.5.7) are κ, $-\kappa$ and 0, which is still true for $p=\frac{d+2}{d-2}$. Indeed for a bounded stationary solution of (5.5.7), the Pohozaev identity

$$(d+2-p(d-2))\int_{\mathbb{R}^d} |\nabla w|^2 e^{-\frac{|y|^2}{4}} dy + \frac{p-1}{2}\int_{\mathbb{R}^d} |y|^2 |\nabla w|^2 e^{-\frac{|y|^2}{4}} dy = 0$$

gives that $\nabla w = 0$ if $1<p\leq \frac{d+2}{d-2}$. Hence w is constant in space, meaning that it is one of the aforementioned solutions.

5.5.2 Self-similar variables

We follow the method introduced in [61, 62, 63] to study type I blow-up locally. The results the ideas of their proof are either contained in [62] or similar to the results there. A sharp blow-up criterion and other preliminary bounds are given by Lemma 5.5.5 and a condition for local boundedness is given in Proposition 5.5.7. For u defined on $[0,T_{u_0})\times\mathbb{R}^d$, $a\in\mathbb{R}^d$ and $T>0$ we define the self-similar renormalization of u at (T,a):

$$w_{a,T}(y,t) := (T-t)^{\frac{1}{p-1}} u(t, a+\sqrt{T-t}y) \tag{5.5.5}$$

for $(t,y)\in[0,\min(T_{u_0},T))\times\mathbb{R}^d$. Introducing the self-similar renormalized time:

$$s := -\log(T-t) \tag{5.5.6}$$

one sees that if u solves (NLH) then $w_{a,T}$ solves:

$$\partial_s w_{a,T} - \Delta w_{a,T} - |w_{a,T}|^{p-1} w_{a,T} + \frac{1}{2}\Lambda w_{a,T} = 0. \tag{5.5.7}$$

Equation (5.5.7) admits a natural Lyapunov functional,

$$E(w) = \int_{\mathbb{R}^d} \left(\frac{1}{2}|\nabla w(y)|^2 + \frac{1}{2(p-1)}|w(y)|^2 - \frac{1}{p+1}|w(y)|^{p+1} \right) \rho(y) dy, \tag{5.5.8}$$

where $\rho(y) := \frac{1}{(4\pi)^{\frac{d}{2}}}e^{-\frac{|y|^2}{4}}$ from the fact that for its solutions there holds:

$$\frac{d}{ds}E(w) = -\int_{\mathbb{R}^d} w_s^2 \rho dy \leq 0. \tag{5.5.9}$$

Another quantity that will prove to be helpful is the following:

$$I(w) := -2E(w) + \frac{p-1}{p+1}\left(\int_{\mathbb{R}^d} w^2\rho dy\right)^{\frac{p+1}{2}}. \tag{5.5.10}$$

Lemma 5.5.5 ([61, 122]). *Let w be a global solution of (5.5.7) with $E(w(0)) = E_0$, then*[9] *for $s \geq 0$:*

$$I(w(s)) \leq 0, \tag{5.5.11}$$

$$\int_0^{+\infty}\int_{\mathbb{R}^d} w_s^2\rho dy ds \leq E_0. \tag{5.5.12}$$

If moreover $E_0 := E(w(0)) \leq 1$, then[10] *for any $s \geq 0$:*

$$\int_{\mathbb{R}^d} w^2\rho dy \leq CE_0^{\frac{2}{p+1}}, \tag{5.5.13}$$

$$\int_s^{s+1}\left(\int_{\mathbb{R}^d}(|\nabla w|^2 + w^2 + |w|^{p+1})\rho dy\right)^2 ds \leq CE_0^{\frac{p+3}{p+1}}. \tag{5.5.14}$$

Remark 5.5.6. If $I(w(s)) > 0$ holds for $w_{a,T}$ associated by (5.5.5) to a solution u of (NLH), then u blows up before T from (5.5.6) since w is not global from (5.5.11).

Proof of Lemma 5.5.5

step 1 Proof of (5.5.11). We argue by contradiction and assume that $I(w(s_0)) > 0$ for some $s_0 \geq 0$. The set $\mathcal{S} := \{s \geq s_0,\ I(s) \geq I(s_0)\}$ is closed by continuity. For any solution of (5.5.7) one has:

$$\frac{d}{ds}\left(\int_{\mathbb{R}^d} w^2\rho dy\right) = 2\int_{\mathbb{R}^d} ww_s\rho dy = -4E(w) + \frac{2(p-1)}{p+1}\int_{\mathbb{R}^d}|w|^{p+1}\rho dy. \tag{5.5.15}$$

Therefore, for any $s \in \mathcal{S}$, from (5.5.10) and Jensen inequality this gives:

$$\frac{d}{ds}\left(\int_{\mathbb{R}^d} w^2\rho dy\right) \geq -4E(w(s)) + \frac{2(p-1)}{p+1}\left(\int_{\mathbb{R}^d} w^2\rho dy\right)^{\frac{p+1}{2}} = I(w(s)) > 0 \tag{5.5.16}$$

as $I(w(s)) \geq I(w(s_0))$ which with (5.5.9) and (5.5.10) imply $\frac{d}{ds}I(w(s)) > 0$. Hence $\mathcal{S}$ is open and therefore $\mathcal{S} = [s_0, +\infty)$. From (5.5.16) and (5.5.9) there exists s_1 such that $E(w(s)) \leq \frac{p-1}{2(p+1)}\left(\int_{\mathbb{R}^d} w^2\rho dy\right)^{\frac{p+1}{2}}$ for all $s \geq s_1$, implying from (5.5.16):

$$\frac{d}{ds}\left(\int_{\mathbb{R}^d} w^2\rho dy\right) \geq 2\frac{p-1}{p+1}\left(\int_{\mathbb{R}^d} w^2\rho dy\right)^{\frac{p+1}{2}}.$$

This quantity must then tend to $+\infty$ in finite time, which is a contradiction.

[9] From the definition (5.5.10) of I and (5.5.11) one has that for all $s \geq 0$, $E(w(s)) \geq 0$. Hence the right hand side in (5.5.12) is nonnegative.

[10] Idem for the right hand side of (5.5.13) and (5.5.14).

step 2 End of the proof. (*5.5.12*) and (*5.5.13*) are consequences of (*5.5.9*), (*5.5.10*) and (*5.5.11*). To prove (*5.5.14*), from (*5.5.15*), (*5.5.9*), (*5.5.13*) and Hölder one obtains:

$$\int_s^{s+1}\left(\int_{\mathbb{R}^d}|w|^{p+1}\rho dy\right)^2 ds \leq \int_s^{s+1}\left(CE_0^2+C\int_{\mathbb{R}^d}w_s^2\rho dy\int_{\mathbb{R}^d}w^2\rho dy\right)ds \leq CE_0^{\frac{p+3}{p+1}}$$

as $E_0 \leq 1$. This identity, using (*5.5.8*), (*5.5.9*) and as $E_0 \leq 1$ implies (*5.5.14*).

□

Proposition 5.5.7 (Condition for local boundedness). *Let $R > 0$, $0 < T_- < T_+$ and $\delta > 0$. There exists $\eta > 0$ and $0 < r \leq R$ such that for any $T \in [T_-, T_+]$ and u solution of (NLH) on $[0,T) \times \mathbb{R}^d$ with $u_0 \in W^{2,\infty}$ satisfying:*

$$\forall a \in B(0,R), \quad E(w_{a,T}(0,\cdot)) \leq \eta, \tag{5.5.17}$$

$$\forall (t,x) \in [0,T) \times \mathbb{R}^d, \quad |\Delta u(t,x)| \leq \frac{1}{2}|u(t,x)|^p + \eta, \tag{5.5.18}$$

there holds

$$\forall t \in \left[\frac{T_-}{2}, T\right), \quad \|u(t)\|_{W^{2,\infty}(B(0,r))} \leq \delta. \tag{5.5.19}$$

The proof Proposition 5.5.7 is done at the end of this subsection. We need intermediate results: Proposition 5.5.8 gives local smallness in self-similar variables, Lemma 5.5.12 and its Corollary 5.5.13 give local boundedness in L^∞ in original variables.

Proposition 5.5.8. *For any $R, s_0, \delta > 0$ there exists $\eta > 0$ such that for any w global solution of (5.5.7), with $w(0) \in W^{2,\infty}(\mathbb{R}^d)$ satisfying*

$$E(w(0)) \leq \eta \ \text{ and } \ \forall (s,y) \in [0,+\infty) \times \mathbb{R}^d, \ |\Delta w(s,y)| \leq \frac{1}{2}|w(s,y)|^p + \eta, \tag{5.5.20}$$

then there holds:

$$\forall (s,y) \in [s_0,+\infty) \times B(0,R), \quad |w(s,y)| \leq \delta. \tag{5.5.21}$$

Proof of Proposition 5.5.7

It is a direct consequence of Lemma 5.5.9 and Lemma 5.5.10.

□

Lemma 5.5.9. *For any $R, s_0, \eta' > 0$ there exists $\eta > 0$ such that for w a global solution of (5.5.7), with $w(0) \in W^{2,\infty}(\mathbb{R}^d)$, satisfying (5.5.20), there holds*

$$\forall s \in [s_0,+\infty), \quad \int_{B(0,R)} (|w|^2 + |\nabla w|^2)dy \leq \eta'. \tag{5.5.22}$$

Lemma 5.5.10. *For any $R, \delta > 0$, $0 < s_0 < s_1$ there exists $\eta, \eta' > 0$ and $0 < r \leq R$ such that for w a global solution of (5.5.7) with $w(0) \in W^{2,\infty}$, satisfying (5.5.20) and (5.5.22) there holds:*

$$\forall (s,y) \in [s_1,+\infty) \times B(0,r), \quad |w(s,y)| \leq \delta. \tag{5.5.23}$$

We now prove the two above lemmas. In what follows we will often have to localize the function w. Let χ be a smooth cut-off function, $\chi=1$ on $B(0,1)$ and $\chi=0$ outside $B(0,2)$. For $R>0$ we define $\chi_R(x)=\chi\left(\frac{x}{R}\right)$ and:

$$v:=\chi_R w \tag{5.5.24}$$

(we will forget the dependence in R in the notations to ease writing, and will write χ instead of χ_R). From (*5.5.7*) the evolution of v is then given by:

$$v_s-\Delta v=\chi|w|^{p-1}w+\left(\left[\frac{1}{p-1}-\frac{d}{2}\right]\chi-\frac{1}{2}\nabla\chi.y+\Delta\chi\right)w+\nabla.\left(\left[\frac{1}{2}\chi y-2\nabla\chi\right]w\right) \tag{5.5.25}$$

Proof of Lemma 5.5.9

We will prove that (*5.5.22*) holds at time s_0, which will imply (*5.5.22*) at any time $s\in[s_0,+\infty)$ because of time invariance.

step 1 An estimate for Δw. First one notices that the results of Lemma 5.5.5 apply. From (*5.5.20*) and (*5.5.7*) there exists a constant $C>0$ such that:

$$|w|^{2p}\leq C(|w|^{p-1}w+\Delta w)^2+C\eta^2\leq C|w_s|^2+C|y|^2|\nabla w|^2+Cw^2+C\eta^2.$$

We integrate this in time, using (*5.5.12*), (*5.5.13*), (*5.5.14*) and (*5.5.20*), yielding for $s\geq 0$:

$$\int_s^{s+1}\int_{B(0,2R)}|w|^{2p}dyds\leq C\eta+C\eta^{\frac{p+3}{p+1}}+C\eta^{\frac{2}{p+1}}+C\eta^2\leq C\eta^{\frac{2}{p+1}}. \tag{5.5.26}$$

Injecting the above estimate in (*5.5.20*), using (*5.5.13*) and (*5.5.14*) we obtain for $s\geq 0$:

$$\begin{array}{rl} & \int_s^{s+1}\|w\|^2_{H^2(B(0,2R))}ds\leq \int_s^{s+1}\int_{B(0,2R)}(|\Delta w|^2+|\nabla w|^2+w^2)dyds \\ \leq & \int_s^{s+1}\int_{B(0,2R)}C(|w|^{2p}+|\nabla w|^2+w^2)dyds+C\eta^2\leq C\eta^{\frac{2}{p+1}}. \end{array} \tag{5.5.27}$$

step 2 Localization. We localize at scale R and define v by (*5.5.24*). From (*5.5.24*), (*5.5.14*) and (*5.5.13*) one obtains that there exists $\tilde s_0\in[\max(0,s_0-1),s_0]$ such that:

$$\|v(\tilde s_0)\|^2_{H^1(\mathbb{R}^d)}\lesssim\int_{B(0,2R)}(w(\tilde s_0)^2+|\nabla w(\tilde s_0)|^2)dy\leq C\eta^{\frac{2}{p+1}}+C\eta^{\frac{p+3}{p+1}}\leq C\eta^{\frac{2}{p+1}} \tag{5.5.28}$$

We apply Duhamel formula to (*5.5.25*) to find that $v(s_0)$ is given by:

$$\begin{array}{rcl} v(s_0) & = & \int_{\tilde s_0}^{s_0}K_{s_0-s}*\left\{\chi|w|^{p-1}w+\left(\left[\frac{1}{p-1}-\frac{d}{2}\right]\chi-\frac12\nabla\chi.y+\Delta\chi\right)w\right\}ds \\ & & +\int_{\tilde s_0}^{s_0}\nabla.K_{s_0-s}*\left(\left[\frac12\chi y-2\nabla\chi\right]w\right)ds+K_{s_0-\tilde s_0}*v(\tilde s_0). \end{array} \tag{5.5.29}$$

We now estimate the $\dot H^1$ norm of each term in the previous identity, using (*5.5.28*), (*5.5.14*), (*5.E.1*), Young and Hölder inequalities:

$$\|K_{s_0-\tilde s_0}*v(\tilde s_0)\|_{\dot H^1(\mathbb{R}^d)}\leq\|v(\tilde s_0)\|_{\dot H^1(\mathbb{R}^d)}\leq C\eta^{\frac{1}{p+1}}, \tag{5.5.30}$$

$$\begin{array}{rl} & \left\|\int_{\tilde s_0}^{s_0}K_{s_0-s}*\{([\frac{1}{p-1}-\frac d2]\chi-\frac{\nabla\chi.y}{2}+\Delta\chi)w\}+\nabla.K_{s_0-s}*([\frac{\chi y}{2}-2\nabla\chi]w)\right\|_{\dot H^1} \\ \leq & C\int_{\tilde s_0}^{s_0}\|w\|_{H^1(B(0,2R))}ds+C\int_{\tilde s_0}^{s_0}\frac{1}{|s_0-s|^{\frac12}}\|w\|_{H^1(B(0,2R))}ds \\ \leq & C\eta^{\frac{p+3}{4(p+1)}}+C\left(\int_{\tilde s_0}^{s_0}\frac{ds}{|\tilde s_1-s|^{\frac12\times\frac43}}\right)^{\frac34}\left(\int_{\tilde s_0}^{s_0}\|w\|^4_{H^1(B(0,2R))}ds\right)^{\frac14}\leq C\eta^{\frac{p+3}{4(p+1)}} \end{array} \tag{5.5.31}$$

For the non linear term in (*5.5.29*), one first compute from (*5.5.24*) that:

$$\nabla(\chi|w|^{p-1}w)=p\chi|w|^{p-1}\nabla w+\nabla\chi|w|^{p-1}w. \tag{5.5.32}$$

For the first term in the previous identity, using Sobolev embedding one obtains:

$$\begin{aligned}\||w|^{p-1}\nabla w\|_{L^{\frac{2d}{d-2+(d-4)(p-1)}}(B(0,2R))} &\leq C\|w\|^{p-1}_{L^{\frac{2d}{d-4}}(B(0,2R))}\|\nabla w\|_{L^{\frac{2d}{d-2}}(B(0,2R))}\\ &\leq C\|w\|^{p}_{H^2(B(0,2R))}\end{aligned}$$

Therefore, from (*5.5.27*) this force term satisfies:

$$\int_{\tilde s_0}^{s_0}\||w|^{p-1}\nabla w\|^{\frac{2}{p}}_{L^{\frac{2d}{d-2+(d-4)(p-1)}}(B(0,2R))}ds\leq\int_{\tilde s_0}^{s_0}\|w\|^2_{H^2(B(0,2R))}ds\leq C\eta^{\frac{2}{p+1}}.$$

We let (q,r) be the Lebesgue conjugated exponents of $\frac{2}{p}$ and $\frac{2d}{(d-2)+(d-4)(p-1)}$:

$$q=\frac{2}{2-p}>2,\quad r=\frac{2d}{d+2-(d-4)(p-1)}>2.$$

They satisfy the Strichartz relation $\frac{2}{q}+\frac{d}{r}=\frac{d}{2}$. Therefore, using (*5.E.22*) one obtains:

$$\begin{aligned}&\left\|\int_{\tilde s_0}^{s_0}K_{s_0-s}*(p\chi|w(s)|^{p-1}\nabla w(s))ds\right\|_{L^2}\\ \leq\ & C\left(\int_{\tilde s_0}^{s_0}\||w|^{p-1}\nabla w\|^{\frac{2}{p}}_{L^{\frac{2d}{d-2+(d-4)(p-1)}}(B(0,2R))}ds\right)^{\frac{p}{2}}\leq C\eta^{\frac{p}{(p+1)}}.\end{aligned}$$

For the second term in (*5.5.32*) using (*5.5.26*), (*5.E.1*) and Hölder one has:

$$\left\|\int_{\tilde s_0}^{s_0}K_{s_0-s}*(\nabla\chi|w|^{p-1}w)ds\right\|_{L^2}\leq C\int_{\tilde s_0}^{s_0}\|w\|^p_{L^{2p}(B(0,2R))}\leq C\eta^{\frac{1}{p+1}}.$$

The two above estimates and the identity (*5.5.32*) imply the following bound:

$$\left\|\int_{\tilde s_0}^{s_0}K_{s_0-s}*(\chi|w|^{p-1}w)ds\right\|_{\dot H^1}\leq C\eta^{\frac{1}{p+1}}$$

We come back to (*5.5.29*) where we found estimates for each term in the right hand side in (*5.5.30*), (*5.5.31*) and the above identity, yielding $\|v(s_0)\|_{\dot H^1}\leq C\eta^{\frac{1}{p+1}}$. From (*5.5.24*), as v is compactly supported in $B(0,2R)$, the above estimate implies the desired estimate (*5.5.22*) at time s_0.

□

To prove Lemma 5.5.10 we need the following parabolic regularization result.

Lemma 5.5.11 (Parabolic regularization). *Let* $R,M>0$, $0<s_0\leq 1$ *and* w *be a global solution of* (*5.5.7*) *satisfying:*

$$\forall(s,y)\in[0,+\infty)\times\mathbb{R}^d,\quad\|w(s,y)\|_{H^2(B(0,R))}\leq M. \tag{5.5.33}$$

Then there exists $0<r\leq R$*, a constant* $C=C(R,s_0)$ *and* $\alpha>1$ *such that:*

$$\forall(s,y)\in[s_0,+\infty)\times B(0,r),\quad|w(s,y)|\leq C(M+M^\alpha). \tag{5.5.34}$$

Proof of Lemma 5.5.11

The proof is very similar to the proof of Lemma 5.E.3, therefore we do not give it here.

□

Proof of Lemma 5.5.10

Without loss of generality we take $\eta'=\eta$, $s_0=0$, localize at scale $\frac{R}{2}$ by defining v by (*5.5.24*). The assumption (*5.5.22*) implies that for $s\geq 0$:

$$\int_{\mathbb{R}^d}(|v(s)|^2+|\nabla v(s)|^2)dy\leq C\eta. \tag{5.5.35}$$

We claim that for all $s\geq \frac{s_1}{2}$,

$$\|v\|_{H^2}\leq C\eta.$$

This will give the desired result (*5.5.23*) by applying Lemma 5.5.11 from (*5.5.24*). We now prove the above bound. By time invariance, we just have to prove it at time $\frac{s_1}{2}$.

step 1 First estimate on v_s. Since w is a global solution starting in $W^{2,\infty}(\mathbb{R}^d)$ with $E(w(0))\leq\eta$, from (*5.5.12*) one obtains:

$$\int_0^{+\infty}\int_{\mathbb{R}^d}|v_s|^2dyds\leq C\eta. \tag{5.5.36}$$

step 2 Second estimate on v_s. Let $u=v_s$. From (*5.5.7*) and (*5.5.24*) the evolution of u is given by:

$$u_s-\Delta u=p|w|^{p-1}u+\left(\left[\frac{1}{p-1}-\frac{d}{2}\right]\chi-\frac{1}{2}\nabla\chi.y+\Delta\chi\right)w_s+\nabla.\left(\left[\frac{1}{2}\chi y-2\nabla\chi\right]w_s\right). \tag{5.5.37}$$

We first state a non linear estimate. Using Sobolev embedding, Hölder inequality and (*5.5.22*), one obtains:

$$\int_{\mathbb{R}^d}|u|^2|w|^{p-1}dy\leq \|u\|^2_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)}\|w\|^{p-1}_{L^{\frac{2d}{d-2}}(B(0,R))}\leq C\eta^{\frac{p-1}{2}}\int_{\mathbb{R}^d}|\nabla u|^2dy.$$

We now perform an energy estimate. We multiply (*5.5.37*) by u and integrate in space using Young inequality for any $\kappa>0$ and the above inequality:

$$\begin{array}{rcl}
\frac{1}{2}\frac{d}{ds}\left[\int_{\mathbb{R}^d}|u|^2dy\right] & = & -\int_{\mathbb{R}^d}|\nabla u|^2dy+\int_{\mathbb{R}^d}\left(\left[\frac{1}{p-1}-\frac{d}{2}\right]\chi-\frac{1}{2}\nabla\chi.y+\Delta\chi\right)w_sudy\\
 & & +\int\left(\left[\frac{1}{2}\chi y-2\nabla\chi\right]w_s\right).\nabla udy+\int_{\mathbb{R}^d}u^2|w|^{2(p-1)}dy\\
 & \leq & -\int_{\mathbb{R}^d}|\nabla u|^2dy+C\int_{B(0,R)}(w_s^2+u^2)dy+\frac{C}{\kappa}\int_{B(0,R)}w_s^2dy\\
 & & +C\kappa\int_{\mathbb{R}^d}|\nabla u|^2dy+C\eta^{\frac{p-1}{2}}\int_{\mathbb{R}^d}|\nabla u|^2dy\\
 & \leq & -\int_{\mathbb{R}^d}|\nabla u|^2dy+C(\kappa)\int_{B(0,R)}w_s^2dy
\end{array}$$

if κ and η have been chosen small enough. Now because of the integrability (*5.5.36*) there exists at least one $\tilde{s}\in[\max(0,\frac{s_1}{2}-1),\frac{s_1}{2}]$ such that:

$$\int_{\mathbb{R}^d}|v_s(\tilde{s})|^2dy\leq C(s_1)\eta.$$

One then obtains from the two previous inequalities and (*5.5.12*):

$$\int_{\mathbb{R}^d}|v_s(s)|^2dy\leq\int_{\mathbb{R}^d}|v_s(\tilde{s})|^2dy+C\int_{\tilde{s}}^{\frac{s_1}{2}}\int_{B(0,R)}w_s^2dyds\leq C\eta. \tag{5.5.38}$$

step 3 Estimate on Δv. Applying Sobolev embedding and Hölder inequality, using the fact that $\left(\frac{\frac{2d}{d-4}}{2}\right)'=\frac{d}{4}=\frac{\frac{2d}{d-2}}{2(p-1)}$ one gets that for any $s\geq 0$:

$$\begin{array}{rcl} & & \int_{\mathbb{R}^d} v^2|w|^{2(p-1)}dy \leq \|v^2\|_{L^{\frac{\frac{2d}{d-4}}{2}}(\mathbb{R}^d)} \||w|^{2(p-1)}\|_{L^{\frac{\frac{2d}{d-2}}{2(p-1)}}(B(0,R))} \\ & = & \|v\|^2_{L^{\frac{2d}{d-4}}(\mathbb{R}^d)} \|w\|^{2(p-1)}_{L^{\frac{2d}{d-2}}(B(0,R))} \leq C\|v\|^2_{\dot{H}^2(\mathbb{R}^d)}\|w\|^{2(p-1)}_{H^1(B(0,R))} \\ & \leq & C\eta^{p-1}\int_{\mathbb{R}^d}|\Delta v|^2dy \end{array} \quad (5.5.39)$$

where injected the estimate (*5.5.22*). We inject the above estimate in (*5.5.25*), using (*5.5.24*), yielding for all $s\geq 0$:

$$\begin{array}{rcl} \int_{\mathbb{R}^d}|\Delta v|^2dy & \leq & C\left(\int_{\mathbb{R}^d}(|v_s|^2+|w|^2+|\nabla w|^2+v^2|w|^{2(p-1)})dy\right) \\ & \leq & C\int_{\mathbb{R}^d}|v_s|^2dy+C\eta+C\eta^{p-1}\int_{\mathbb{R}^d}|\Delta v|^2dy \end{array}$$

where we used (*5.5.33*). Injecting (*5.5.38*), for η small enough:

$$\int_{\mathbb{R}^d}\left|\Delta v\left(\frac{s_1}{2}\right)\right|^2dy \leq C\int_{\mathbb{R}^d}\left|v_s\left(\frac{s_1}{2}\right)\right|^2dy+C\eta\leq C\eta. \quad (5.5.40)$$

step 4 Conclusion. From (*5.5.35*) and (*5.5.40*) we infer $\|v(\frac{s_1}{2})\|_{\dot{H}^2}\leq C\eta$ which is exactly the bound we had to prove.

□

We now go from boundedness in L^∞ in self-similar provided by Proposition 5.5.8 to boundedness in L^∞ in original variables.

Lemma 5.5.12 ([63]). *Let $0\leq a\leq\frac{1}{p-1}$ and $R,\epsilon_0>0$. Let $0<\epsilon\leq\epsilon_0$ and u be a solution of (NLH) on $[-1,0)\times\mathbb{R}^d$ satisfying*

$$\forall (t,x)\in[-1,0)\times B(0,R),\ \ |u(t,x)|\leq\frac{\epsilon}{|t|^{\frac{1}{p-1}-a}}. \quad (5.5.41)$$

For ϵ_0 small enough the following holds for all $(t,x)\in[-1,0)\times B\left(0,\frac{R}{2}\right)$.

$$\text{If } \tfrac{1}{p-1}-a<\tfrac12,\qquad |u(t,x)|\leq C(a)\epsilon. \quad (5.5.42)$$

$$\text{If } \tfrac{1}{p-1}-a=\tfrac12,\qquad |u(t,x)|\leq C\epsilon(1+|\mathit{ln}(t)|) \quad (5.5.43)$$

$$\text{If } \tfrac{1}{p-1}-a>\tfrac12,\qquad |u(t,x)|\leq \frac{C(a)\epsilon}{|t|^{\frac{1}{p-1}-a-\frac12}} \quad (5.5.44)$$

Applying several times Lemma 5.5.12, via scale change and time invariance, one obtains the following corollary.

Corollary 5.5.13. *Let $R>0$ and $0<T_-<T_+$. There exists $\epsilon_0>0$, $0<r\leq R$ and $C>0$ such that the following holds. For any $0<\epsilon<\epsilon_0$, $T\in[T_-,T_+]$ and u solution of (NLH) on $[0,T)\times\mathbb{R}^d$ satisfying*

$$\forall (t,x)\in[0,T)\times B(0,R),\ \ |u(t,x)|\leq\frac{\epsilon}{(T-t)^{\frac{1}{p-1}}} \quad (5.5.45)$$

one has:

$$\forall (t,x)\in[0,T)\times B(0,r),\ \ |u(t,x)|\leq C\epsilon \quad (5.5.46)$$

To prove Lemma 5.5.12 we need two technical Lemmas taken from [63] whose proof can be found there.

Lemma 5.5.14 ([63]). *Define for* $0<\alpha<1$ *and* $0<\theta<h<1$ *the integral* $I(h)=\int_h^1 (s-h)^{-\alpha}s^{\theta}ds$*. It satisfies*

$$\text{If } \alpha+\theta>1, \qquad I(h)\leq \left(\frac{1}{1-\alpha}+\frac{1}{\alpha+\theta-1}\right)h^{1-\alpha-\theta}. \tag{5.5.47}$$

$$\text{If } \alpha+\theta=1, \qquad I(h)\leq \frac{1}{1-\alpha}+|log(h)|. \tag{5.5.48}$$

$$\text{If } \alpha+\theta<1, \qquad I(h)\leq \frac{1}{1-\alpha-\theta}. \tag{5.5.49}$$

Lemma 5.5.15 ([63]). *If* y, r *and* q *are continuous functions defined on* $[t_0,t_1]$ *with*

$$y(t)\leq y_0+\int_{t_0}^t y(s)r(s)ds+\int_{t_0}^t q(s)ds$$

for $t_0\leq t\leq t_1$, *then for all* $t_0\leq t\leq t_1$*:*

$$y(t)\leq e^{\int_{t_0}^t r(\tau)d\tau}\left[y_0+\int_{t_0}^t q(\tau)e^{-\int_{t_0}^{\tau} r(\sigma)d\sigma}d\tau\right]. \tag{5.5.50}$$

Proof of Lemma 5.5.12

We first localize the problem, with χ a smooth cut-off function, with $\chi=1$ on $B\left(0,\frac{R}{2}\right)$, $\chi=0$ outside $B(0,R)$ and $|\chi|\leq 1$. We define

$$v:=\chi u \tag{5.5.51}$$

whose evolution, from (NLH), is given by:

$$v_t=\Delta v+|u|^{p-1}v+\Delta\chi u-2\nabla.(\nabla\chi u). \tag{5.5.52}$$

We apply Duhamel formula to (*5.5.52*) to find that for $t\in[-1,0)$:

$$v(t)=K_{t+1}*v(-1)+\int_{-1}^t K_{t-s}*(|u|^{p-1}v+\Delta\chi u-2\nabla.(\nabla\chi u))ds. \tag{5.5.53}$$

From (*5.5.41*) and (*5.5.51*) one has for free evolution term:

$$\|K_{t+1}*v(-1)\|_{L^\infty}\leq \epsilon. \tag{5.5.54}$$

We now find an upper bound for the other terms in the previous equation.

step 1 Case (i). For the linear terms, as $\frac{1}{p-1}-a+\frac{1}{2}<1$, from (*5.5.49*) one has:

$$\begin{array}{rcl}\|\int_{-1}^t K_{t-s}*(\Delta\chi u-2\nabla.(\nabla\chi u))ds\|_{L^\infty} & \leq & C\int_{-1}^t \frac{1}{(t-s)^{\frac12}}\|u\|_{L^\infty(B(0,R))} \\ & \leq & C\epsilon\int_{-1}^t \frac{1}{(t-s)^{\frac12}}\frac{1}{|s|^{\frac{1}{p-1}-a}}\leq C(a)\epsilon.\end{array} \tag{5.5.55}$$

For the nonlinear term, as $\frac{1}{p-1}-a<\frac12<\frac{1}{2(p-1)}=\frac{d-2}{8}$ because $d\geq 7$ we compute using (*5.5.41*):

$$\begin{array}{rcl}\|\int_{-1}^t K_{t-s}*(\chi|u|^{p-1}v)ds\|_{L^\infty} & \leq & \int_{-1}^t \|u\|^{p-1}_{L^\infty(B(0,R))}\|v\|_{L^\infty}ds \\ & \leq & \epsilon^{p-1}\int_{-1}^t \frac{1}{|s|^{\frac12}}\|v\|_{L^\infty}ds.\end{array} \tag{5.5.56}$$

Gathering (*5.5.54*), (*5.5.55*) and (*5.5.56*), from (*5.5.53*) one has:

$$\|v(t)\|_{L^\infty}\leq C(a)\epsilon+\epsilon^{p-1}\int_{-1}^t \frac{1}{|s|^{\frac12}}\|v\|_{L^\infty}.$$

Applying (*5.5.50*) one obtains:

$$\|v(t)\|_{L^\infty}\leq C(a)\epsilon e^{\int_{-1}^t|s|^{-\frac12}ds}\leq C(a)\epsilon$$

which from (*5.5.51*) implies the bound (*5.5.42*) we had to prove.

step 2 Case (ii). For the linear terms, as $\frac{1}{p-1}-a=\frac12$, from (*5.5.48*) one has:

$$\begin{array}{lll}\| \int_{-1}^t K_{t-s}*(\Delta\chi u-2\nabla.(\nabla\chi u))ds\|_{L^\infty} & \leq & C\int_{-1}^t \frac{1}{(t-s)^{\frac12}}\|u\|_{L^\infty(B(0,R))}\\ & \leq & C\epsilon\int_{-1}^t \frac{1}{(t-s)^{\frac12}}\frac{1}{|s|^{\frac12}}\leq C\epsilon(1+|\log(t)|).\end{array} \quad (5.5.57)$$

For the nonlinear term, as $\frac{1}{p-1}-a<\frac12<\frac{1}{2(p-1)}=\frac{d-2}{8}$ as $d\geq 7$, using (*5.5.41*):

$$\begin{array}{lll}\| \int_{-1}^t K_{t-s}*(\chi|u|^{p-1}v)ds\|_{L^\infty} & \leq & \int_{-1}^t \|u\|^{p-1}_{L^\infty(B(0,R))}\|v\|_{L^\infty}\\ & \leq & \epsilon^{p-1}\int_{-1}^t\frac{1}{|s|^{\frac12}}\|v\|_{L^\infty}.\end{array} \quad (5.5.58)$$

Gathering (*5.5.54*), (*5.5.57*) and (*5.5.58*), from (*5.5.53*) one has:

$$\|v(t)\|_{L^\infty}\leq C\epsilon+C\epsilon|\log(t)|+\epsilon^{p-1}\int_{-1}^t \frac{1}{|s|^{\frac12}}\|v\|_{L^\infty}.$$

Applying (*5.5.50*) one obtains:

$$\|v(t)\|_{L^\infty}\leq C\epsilon e^{\int_{-1}^t|s|^{-\frac12}ds}\left[1+\int_{-1}^t\frac{ds}{|s|}e^{-\int_{-1}^s|\tau|^{-\frac12}d\tau}\right]\leq C\epsilon(1+|\log(t)|)$$

which from (*5.5.51*) implies (*5.5.43*).

step 3 Case (iii). For the linear terms, as $\frac{1}{p-1}-a>\frac12$, from (*5.5.48*) one has:

$$\begin{array}{lll}\| \int_{-1}^t K_{t-s}*(\Delta\chi u-2\nabla.(\nabla\chi u))ds\|_{L^\infty} & \leq & C\int_{-1}^t \frac{1}{(t-s)^{\frac12}}\|u\|_{L^\infty(B(0,R))}\\ & \leq & C\epsilon\int_{-1}^t \frac{1}{(t-s)^{\frac12}}\frac{1}{|s|^{\frac{1}{p-1}-a}}\leq \frac{C(a)\epsilon}{|t|^{\frac{1}{p-1}-a-\frac12}}.\end{array} \quad (5.5.59)$$

For the nonlinear term, as $\frac{1}{p-1}-a\leq\frac{1}{p-1}$ we compute using (*5.5.41*):

$$\begin{array}{lll}\| \int_{-1}^t K_{t-s}*(\chi|u|^{p-1}v)ds\|_{L^\infty} & \leq & \int_{-1}^t \|u\|^{p-1}_{L^\infty(B(0,R))}\|v\|_{L^\infty}ds\\ & \leq & \epsilon^{p-1}\int_{-1}^t\frac{1}{|s|}\|v\|_{L^\infty}.\end{array} \quad (5.5.60)$$

Gathering (*5.5.54*), (*5.5.59*) and (*5.5.60*), from (*5.5.53*) one has:

$$\|v(t)\|_{L^\infty}\leq \frac{C(a)\epsilon}{|t|^{\frac{1}{p-1}-a-\frac12}}+\epsilon^{p-1}\int_{-1}^t\frac{1}{|s|}\|v\|_{L^\infty}ds.$$

Applying (*5.5.50*) one obtains if $\epsilon^{p-1} < \frac{1}{p-1} - a - \frac{1}{2}$:

$$\|v(t)\|_{L^\infty} \leq C(a)\epsilon e^{\epsilon^{p-1}\int_{-1}^t \frac{ds}{|s|}} \left[1 + \int_{-1}^t \frac{1}{|s|^{\frac{1}{p-1}-a+\frac{1}{2}}} e^{-\epsilon^{p-1}\int_{-1}^s \frac{d\tau}{\tau}} ds\right] \leq \frac{C(a)\epsilon}{|t|^{\frac{1}{p-1}-a-\frac{1}{2}}}$$

implying (*5.5.44*) from (*5.5.51*) .

□

We can now end the proof of Proposition 5.5.7.

Proof of Proposition 5.5.7

For any $a \in B(0,R)$, from (*5.5.5*), (*5.5.17*) and (*5.5.18*) $w_{a,T}$ satisfies $E(w_{a,T}(0,\cdot)) \leq \eta$ and:

$$|\Delta w_{a,T}| \leq \frac{1}{2}|w_{a,T}|^p + \eta T_+^{\frac{p}{p-1}}.$$

Applying Proposition 5.5.8 to $w_{a,T}$ one obtains that for any $\eta' > 0$ if η is small enough:

$$\forall s \geq s\left(\frac{T_-}{4}\right), \quad |w_{a,T}(s,0)| \leq \eta'.$$

In original variables this means:

$$\forall (t,x) \in B(0,R) \times [\frac{T_-}{4}, T), \quad |u(t,x)| \leq \frac{\eta'}{(T-t)^{\frac{1}{p-1}}}.$$

Applying Corollary 5.5.13 for η' small enough there exists $r > 0$ such that

$$\forall (t,x) \in B(0,R) \times [\frac{T_-}{4}, T), \quad |u(t,x)| \leq C\eta'.$$

Then, a standard parabolic estimate, similar to these in Lemma 5.E.3 propagates this bound for higher derivatives, yielding the result (*5.5.19*).

□

5.5.3 Proof of Proposition 5.5.1

We now prove Proposition 5.5.1 by contradiction, following [54]. Assume the result is false. From Lemma 5.5.4 and from the Cauchy theory in $W^{2,\infty}$ the negation of the result of Proposition 5.5.1 means the following. There exists $u_0 \in W^{3,\infty}$ such that the solution of (NLH) starting from u_0 blows up at time 1 with:

$$\|u(t)\| \sim \kappa(1-t)^{-\frac{1}{p-1}} \text{ as } t \to 1, \tag{5.5.61}$$

and satisfies:

$$|\Delta u| \leq \frac{1}{2}|u|^p + K \text{ on } \mathbb{R}^d \times [0,1). \tag{5.5.62}$$

There exists a sequence u_n of solutions of (NLH) blowing up at time T_n with:

$$T_n \to 1 \text{ and } u_n \to u \text{ in } \mathcal{C}_{\text{loc}}([0,1), W^{3,\infty}(\mathbb{R}^d)) \tag{5.5.63}$$

for any $0 \leq T < 1$ and there exists two sequences $0 \leq t_n < T_n$ and x_n such that:

$$|\Delta u_n| \leq \frac{1}{2}|u_n|^p + 2K \text{ on } \mathbb{R}^d \times [0, t_n), \tag{5.5.64}$$

$$|\Delta u_n(t_n,x_n)| = \frac{1}{2}|u_n(t_n,x_n)|^p + 2K. \tag{5.5.65}$$

The strategy is the following. First we centralize the problem, showing in Lemma 5.5.16 that one can assume without loss of generality $x_n = 0$. One also obtains that u and u_n become singular near 0 as $t \to 1$ and $n \to +\infty$. In view of Lemma 5.5.4, the ODE type bound (*5.5.64*) means that u_n behaves approximately as a type I blowing up solution until t_n. This intuition is made rigorous by the second Lemma 5.5.17, stating that if we renormalize u_n before t_n, one converges to the constant in space blow up profile associated to type I blow up and that the L^∞ norm grows as during type I blow up. We end the proof by showing that the inequality (*5.5.65*) then passes to the limit, contradicting (*5.5.62*).

Lemma 5.5.16. *Let u, u_n be solutions of (NLH), t_n and x_n satisfy (5.5.61), (5.5.62), (5.5.67), (5.5.64) and (5.5.70). Then:*

$$t_n \to 1 \tag{5.5.66}$$

and there exist $\hat{u}$ and $\hat{u}_n$ solutions of (NLH) satisfying (5.5.61), (5.5.62), (5.5.64) and (5.5.70) with $\hat{x}_n = 0$, $\hat{u}(t_n,0) \to +\infty$. In addition, $\hat{u}$ blows up with type I at $(1,0)$ and $\hat{u}_n$ blows up at time T_n. One has the following asymptotics:

$$\|\hat{u}_n(0)\|_{W^{2,\infty}} \lesssim 1, \quad T_n \to 1 \;\; \textit{and} \;\; u_n \to u \textit{ in } \mathcal{C}^{1,2}_{loc}([0,1)\times\mathbb{R}^d) \tag{5.5.67}$$

Proof of Lemma 5.5.16

step 1 Proof of (*5.5.66*). At time t_n, u satisfies the inequality (*5.5.62*) whereas u_n doesn't from (*5.5.65*). As u_n converges to u in $C^{1,2}_{\text{loc}}([0,1)\times\mathbb{R}^d)$ from (*5.5.67*) this forces t_n to tend to 1.

step 2 Centering and limit objects. Define $\hat{u}_n(t,x) = u_n(t,x+x_n)$. Then $\hat{u}_n$ is a solution satisfying (*5.5.64*), (*5.5.65*) with $\hat{x}_n = 0$, and blowing up at time $T_n \to 1$ from (*5.5.67*). From the Cauchy theory, see Proposition 5.2.1, $(t,x) \mapsto u(t,x_n+x)$ is uniformly bounded in $C^{\frac{3}{2},3}_{\text{loc}}([0,1),\mathbb{R}^d)$, hence as $n \to +\infty$ using Arzela Ascoli theorem it converges to a function $\hat{u}$ that also solves (NLH), satisfies (*5.5.62*) and

$$\|\hat{u}(t)\| \lesssim \kappa(1-t)^{-\frac{1}{p-1}}. \tag{5.5.68}$$

As u_n converges to u in $C_{\text{loc}}([0,1),W^{3,\infty}(\mathbb{R}^d))$ from (*5.5.67*), $\hat{u}_n$ converges to $\hat{u}$ in $C^{1,2}_{\text{loc}}([0,1)\times\mathbb{R}^d))$, establishing (*5.5.67*).

step 3 Conditions for boundedness. We claim two facts. 1) If $\hat{u}$ does not blow up at $(1,0)$ then there exists $r, C > 0$ such that for all $(t,y) \in [0,t_n]\times B(0,r)$, $|\hat{u}_n(t,y)| \leq C$. 2) If there exists $C > 0$ such that for all $0 \leq t \leq t_n$, $|\hat{u}_n(t,0)| \leq C$ then $\hat{u}$ does not blow up at $(0,1)$.

Proof of the first fact. We reason by contradiction. If $\hat{u}$ does not blow up at $(1,0)$ there exists $r, C > 0$ such that for all $(t,y) \in [0,1)\times B(0,r)$, $|\hat{u}(t,y)| \leq C$. Assume that there exists $(\tilde{x}_n,\tilde{t}_n)$ such that $\tilde{x}_n \in B(0,r)$ and $|\hat{u}_n(\tilde{x}_n,\tilde{t}_n)| \to +\infty$. As $\hat{u}_n$ solves (NLH), from (*5.5.65*) one then has that:

$$\forall t \in [0,\tilde{t}_n], \quad \partial_t|\hat{u}_n(t,\tilde{x}_n)| \leq \frac{3}{2}|\hat{u}_n(t,\tilde{x}_n)|^p + 2K, \quad |\hat{u}_n(\tilde{x}_n,\tilde{t}_n)| \to +\infty$$

This then implies that for any $M > 0$ there exists $s > 0$ such that for n large enough, $|\hat{u}_n(\tilde{x}_n,t)| \geq M$ on $[\max(0,\tilde{t}_n - s),\tilde{t}_n]$. But this contradicts the convergence in $C_{\text{loc}}([0,1)\times B(0,r))$ established in Step 2 to the bounded function $\hat{u}$.

Proof of the second fact. We also prove it by contradiction. Assume that $\hat{u}$ blows up at $(0,1)$ and $|\hat{u}_n(t_n,0)|\leq C$. Then we claim that

$$\forall t\in[0,t_n),\quad |\hat{u}_n(t,0)|\leq \max((4K)^{\frac{1}{p}},C)$$

Indeed, as $\hat{u}_n$ is a solution of (NLH) satisfying (*5.5.64*) one has that:

$$\forall t\in[0,t_n],\quad \partial_t|\hat{u}_n(t,0)|\geq \frac{1}{2}|\tilde{u}_n(t,0)|^p-2K.$$

So if the bound we claim is violated at some time $0\leq t_0\leq \tau_n'$, then $|\hat{u}_n(t,0)|$ is non decreasing on $[t_0,\tau_n']$, strictly greater than C, which at time t_n is a contradiction. But now as this bound is independent of n, valid on $[0,t_n)$ with $t_n\to 1$, and as $\hat{u}_n(t,0)\to\hat{u}(t,0)$ on $[0,1)$ one obtains at the limit that $\hat{u}(t,0)$ is bounded on $[0,1)$. From (*5.5.4*) this contradicts the blow up of $\hat{u}$ at $(1,0)$.

step 4 End of the proof. It remains to prove the singular behavior near 0: that $\hat{u}$ blows up at $(1,0)$ and that $|\hat{u}_n(t_n,0|\to+\infty$. We reason by contradiction. From Step 3 we assume that there exists $C,r>0$ such that $|\hat{u}|+|\hat{u}_n|\leq C$ on $[0,1)\times B(0,r)$. A standard parabolic estimate, similar to these in Lemma 5.E.3 then implies that

$$\|\hat{u}(t)\|_{W^{3,\infty}(B(0,r'))}+\|\hat{u}_n(t)\|_{W^{3,\infty}(B(0,r'))}+\leq C' \qquad (5.5.69)$$

for all $t\in[\frac{1}{2},1)$ for some $0<r'\leq r$ and. Let χ be a cut-off function, $\chi=1$ on $B(0,\frac{r'}{2})$, $\chi=0$ outside $B(0,r')$. The evolution of $\tilde{u}_n=\chi\hat{u}_n$ is given by:

$$\tilde{u}_{n,\tau}-\Delta\tilde{u}_n=\chi|\hat{u}_n|^{p-1}\hat{u}_n+\Delta\chi\hat{u}_n-2\nabla.(\nabla\chi\hat{u}_n)=F_n$$

with $\|F_n\|_{W^{1,\infty}}\leq C$ from (*5.5.86*). Fix $0<s\ll 1$. One has:

$$\begin{array}{rcl}\Delta\hat{u}_n(t_n,0) & = & K_s*(\Delta\tilde{u}_n(t_n-s))(0)+\sum_1^d\int_0^s\left[\partial_{x_i}K_{s-s'}*\partial_{x_i}F(t_n-s+s')\right](0)\\ & = & \Delta\hat{u}(t_n-s,0)+o_{n\to+\infty}(1)+o_{s\to 0}(1)\end{array}$$

from (*5.5.67*), the estimate on F_n and (*5.5.69*). Similarly,

$$\hat{u}_n(t_n,0)=\hat{u}(t_n,0)+o_{n\to+\infty}(1)+o_{s\to 0}(1).$$

The equality (*5.5.65*) and the two above identities imply the following asymptotics: lim inf$|\Delta\hat{u}(t_n)|-\frac{|\hat{u}(t_n,0)|^p}{2}\geq 2K$, which is in contradiction with (*5.5.62*). Hence $\hat{u}$ blows up at $(1,0)$ with type I blow up from (*5.5.68*) and $|\hat{u}(t_n,0)|\to+\infty$.

□

We return to the study of u and u_n introduced at the begining of this subsection to prove Proposition 5.5.1 by contradiction. From Lemma 5.5.16, keeping the the notation u and u_n for $\hat{u}$ and $\hat{u}_n$ introduced there, one can assume without loss of generality that in addition to (*5.5.61*), (*5.5.62*) and (*5.5.64*), u and u_n satisfy (*5.5.66*), (*5.5.67*) and:

$$|\Delta u_n(t_n,0)|=\frac{1}{2}|u_n(t_n,0)|^p+2K, \qquad (5.5.70)$$

$$u_n(t_n,0)\to+\infty, \qquad (5.5.71)$$

$$|u(t,0)| \sim \frac{\kappa}{(1-t)^{\frac{1}{p-1}}}. \tag{5.5.72}$$

To renormalize appropriately u_n near $(1,0)$ we do the following. Define:

$$M_n(t) := \left(\frac{\kappa}{\|u_n(t)\|_{L^\infty}}\right)^{p-1}. \tag{5.5.73}$$

For $(\tilde{t}_n)_{n\in\mathbb{N}}$ a sequence of times, $0 \leq \tilde{t}_n < T_n$, the renormalization near $(\tilde{t}_n, 0)$ is:

$$v_n(\tau, y) := M_n^{\frac{1}{p-1}}(\tilde{t}_n) u_n \left(M_n^{\frac{1}{2}}(\tilde{t}_n) y, \tilde{t}_n + \tau M_n(\tilde{t}_n)\right) \tag{5.5.74}$$

for $(\tau, y) \in [-\frac{\tilde{t}_n}{M_n(\tilde{t}_n)}, \frac{T_n - \tilde{t}_n}{M_n(\tilde{t}_n)}] \times \mathbb{R}^d$. One has the following asymptotics.

Lemma 5.5.17. *Assume* $0 \leq \tilde{t}_n \leq t_n$ *and* $\tilde{t}_n \to 1$. *Then*

$$\|u_n(\tilde{t}_n)\|_{L^\infty} \sim \frac{\kappa}{(T_n - \tilde{t}_n)^{\frac{1}{p-1}}}, \quad \textit{i.e.}\ M_n(\tilde{t}_n) \sim (T_n - \tilde{t}_n). \tag{5.5.75}$$

Moreover, up to a subsequence[11]*:*

$$v_n \to \frac{\kappa}{\left[\left(\textit{lim}\ \frac{u_n(\tilde{t}_n, 0)}{\|u_n(\tilde{t}_n)\|_{L^\infty}}\right)^{1-p} - t\right]^{\frac{1}{p-1}}} \quad \textit{in}\ C^{1,2}_{loc}(]-\infty, 1) \times \mathbb{R}^d). \tag{5.5.76}$$

Proof of Lemma 5.5.17

step 1 Upper bound for $M_n(\tilde{t}_n)$. We claim that one always has $\|u_n(\tilde{t}_n)\|_{L^\infty} \geq \frac{\kappa}{(T_n - \tilde{t}_n)^{\frac{1}{p-1}}}$, i.e.

$$M_n(\tilde{t}_n) \leq (T_n - \tilde{t}_n). \tag{5.5.77}$$

Indeed if it is false then there exists $\delta > 0$ such that $\|u_n(\tilde{t}_n)\|_{L^\infty} < \frac{\kappa}{(T_n + \delta - \tilde{t}_n)^{\frac{1}{p-1}}}$. Therefore, from a parabolic comparison argument this inequality propagates for the solutions, yielding that $-\frac{\kappa}{(T_n+\delta-t)^{\frac{1}{p-1}}} \leq u_n \leq \frac{\kappa}{(T_n+\delta-t)^{\frac{1}{p-1}}}$ for all times $t \geq \tilde{t}_n$. This implies that u_n stays bounded up to T_n, which is a contradiction.

step 2 Proof of (*5.5.76*). Let $(x_n)_{n\in\mathbb{N}} \in (\mathbb{R}^d)^{\mathbb{N}}$ and define:

$$\tilde{v}_n(\tau, y) := M_n^{\frac{1}{p-1}}(\tilde{t}_n) u_n \left(x_n + M_n^{\frac{1}{2}}(\tilde{t}_n) y, \tilde{t}_n + \tau M_n(\tilde{t}_n)\right) \tag{5.5.78}$$

From (*5.5.74*), $\tilde{v}_n$ is defined on $[-\frac{\tilde{t}_n}{M_n(\tilde{t}_n)}, \frac{T_n - \tilde{t}_n}{M_n(\tilde{t}_n)}] \times \mathbb{R}^d$. The lower bound, $-\frac{\tilde{t}_n}{M_n(\tilde{t}_n)}$, then goes to $-\infty$ from (*5.5.77*). $\tilde{v}_n$ is a solution of (NLH) satisfying:

$$\|\tilde{v}_n(0)\|_{L^\infty} \leq \kappa, \tag{5.5.79}$$

$$\forall (\tau, y) \in [-\frac{\tilde{t}_n}{M_n(\tilde{t}_n)}, 0] \times \mathbb{R}^d, \text{ă} \quad |\Delta \tilde{v}_n| \leq \frac{1}{2}|\tilde{v}_n|^p + 2K M_n^{\frac{p}{p-1}}(\tilde{t}_n), \tag{5.5.80}$$

from (*5.5.64*) and (*5.5.74*).

[11]With the convention that if the limit in the denominator is 0 the limit function is 0.

Precompactness of the renormalized functions. We claim that $\tilde v_n$ is uniformly bounded in $C^{\frac32,3}_{\text{loc}}(]-\infty,1)\times\mathbb{R}^d)$. We now prove this result. First, we claim that

$$|\tilde v_n|\leq \max((4K)^{\frac1p}M_n^{\frac{1}{p-1}}(\tilde t_n),\kappa) \tag{5.5.81}$$

Indeed, as $\tilde v_n$ is a solution of (NLH) satisfying (*5.5.80*) one has that:

$$\partial_t|\tilde v_n|\geq \frac12|\tilde v_n|^p-2KM_n^{\frac{p}{p-1}}(\tilde t_n).$$

So if the bound we claim is violated, then $\|\tilde v_n\|_{L^\infty}$ is strictly increasing, greater than κ, which at time 0 is a contradiction to (*5.5.79*). Moreover, as $\|\tilde v_n(0)\|_{L^\infty}\leq\kappa$, from a comparison argument, for $0\leq t<1$, on has that $\|\tilde v_n(0)\|_{L^\infty}\leq\kappa(1-t)^{-\frac{1}{p-1}}$. This and the above bound implies that for any $T<1$, $\tilde v_n$ is uniformly bounded, independently of n, in $L^\infty((-\frac{\tilde t_n}{M_n(\tilde t_n)},T]\times\mathbb{R}^d)$. Applying Lemma 5.E.3, it is uniformly bounded in $C^{\frac32,3}((-\frac{\tilde t_n}{M_n}+1,T)\times\mathbb{R}^d)$, yielding the desired result.

Rigidity at the limit. From Step 2 and Arzela Ascoli theorem, up to a subsequence, v_n converges in $C^{1,2}_{\text{loc}}((-\infty,0]\times\mathbb{R}^d)$ to a function v. The equation (NLH) passes to the limit and v also solves (NLH). (*5.5.81*) and (*5.5.77*) imply that $|v|\leq\kappa$. (NLH), (*5.5.77*) and (*5.5.80*) imply that:

$$\partial_t|v|\geq\frac12|v|^p.$$

Reintegrating this differential inequality one obtains that $|v|\leq\frac{C}{|c-\tau|^{\frac{1}{p-1}}}$ for some $C,c>0$. Applying the Liouville Lemma 5.5.3, one has that v is constant in space. Up to a subsequence, $v(0,x_n)=\kappa\lim\frac{u_n(\tilde t_n,x_n)}{\|u_n(\tilde t_n)\|_{L^\infty}}$. Taking $x_n=0$, $\tilde v_n=v_n$ defined by (*5.5.74*) and v is then given by (*5.5.76*), ending the proof of this identity.

step 3 Lower bound on M_n. We claim that lim inf $\frac{M_n}{T_n-\tilde t_n}\geq1$. We prove it by contradiction. From (*5.5.73*), and up to a subsequence, assume that there exists $0<\delta\ll1$ and $x_n\in\mathbb{R}^d$ such that $u_n(\tilde t_n,x_n)>\frac{(1+\delta)\kappa}{(T_n-\tilde t_n)^{\frac{1}{p-1}}}$ and $\frac{u_n(\tilde t_n,x_n)}{\|u_n(\tilde t_n)\|_{L^\infty}}\to1$. Therefore the renormalized function $\tilde v_n$ defined by (*5.5.78*) blows up at $\frac{T_n-\tilde t_n}{M_n(\tilde t_n)}\geq(1+\delta)^{p-1}$. From Step 2 $v(0,\cdot)$ is uniformly bounded and converges to κ. Hence, defining the self similar renormalization near $((1+\delta)^{p-1},0)$,

$$w^{(n)}_{0,(1+\delta)^{p-1}}(t,y)=((1+\delta)^{p-1}-t)^{\frac{1}{p-1}}\tilde v_n(t,\sqrt{(1+\delta)^{p-1}-t}y),$$

one has that $I(w_{0,(1+\delta)^{p-1}}(0,\cdot))\to I((1+\delta)^{p-1}\kappa)>0$ where I is defined by (*5.5.10*). From (*5.5.11*) for n large enough $\tilde v_n$ should have blown up before $(1+\delta)^{p-1}$ which yields the desired contradiction.

□

To end the proof of Proposition 5.5.1, we now distinguish two cases for which one has to find a contradiction (which cover all possible cases up to subsequence):

$$\text{Case 1:}\quad \lim\frac{|u_n(x_n,t_n)|}{\|u_n(t_n)\|_{L^\infty}}>0, \tag{5.5.82}$$

$$\text{Case 2:}\quad \lim\frac{|u_n(x_n,t_n)|}{\|u_n(t_n)\|_{L^\infty}}=0 \tag{5.5.83}$$

Proof of Proposition 5.5.1 in Case 1

In this case we can renormalize at time t_n. Let $\tilde{t}_n = t_n$ and define v_n and $M_n(\tilde{t}_n)$ by (*5.5.74*) and (*5.5.73*). (*5.5.76*) and (*5.5.82*) imply that $\Delta v_n(0,0) \to 0$ and $v_n(0,0) \to v(0,0) > 0$. From (*5.5.70*) v_n satisfies at the origin:

$$|\Delta v_n(0,0)| = \frac{1}{2}|v_n(0,0)|^p + 2KM_n^{\frac{p}{p-1}}(t_n).$$

As $M_n(t_n) \to 0$ from (*5.5.75*), at the limit we get $0 = \frac{1}{2}v(0,0) > 0$ which is a contradiction. This ends the proof of Proposition 5.5.1 in Case 1.

□

Proof of Proposition 5.5.1 in Case 2

step 1 Suitable renormalization before t_n. We claim that for any $0 < \kappa_0 \ll 1$ one can find a sequence of times $\tilde{t}_n$ such that $0 \leq \tilde{t}_n \leq t_n$, $\tilde{t}_n \to 1$ and such that v_n defined by (*5.5.74*) satisfy up to a subsequence:

$$v_n \to \frac{\kappa}{\left[\left(\frac{\kappa}{\kappa_0}\right)^{p-1} - 1 - t\right]^{\frac{1}{p-1}}} \text{ in } C^{1,2}_{\text{loc}}(]-\infty,1) \times \mathbb{R}^d). \tag{5.5.84}$$

We now prove this fact. On one hand, $\frac{|u(t,0)|}{\|u(t)\|_{L^\infty}} \to 1$ as $t \to 1$ (from (*5.5.72*) and (*5.5.3*) as u blow up with type I at 0) and for any $0 \leq T < 1$ u_n converges to u in $\mathcal{C}([0,T], L^\infty(\mathbb{R}^d))$ from (*5.5.67*). As $t_n \to 1$, using a diagonal argument, up to a subsequence there exists a sequence of times $0 \leq t_n' \leq t_n$ such that $\frac{|u_n(t_n',0)|}{\|u(t_n')\|_{L^\infty}} \to 1$. On the other hand, from the assumption (*5.5.83*) and (*5.5.66*), $\lim \frac{|u_n(t_n,0)|}{\|u_n(t_n)\|_{L^\infty}} = 0$ and $t_n \to 1$. From a continuity argument, for κ_0 small enough, there exists a sequence $t_n' \leq \tilde{t}_n \leq t_n$ such that $\lim \frac{|u_n(\tilde{t}_n,0)|}{\|u_n(\tilde{t}_n)\|_{L^\infty}} = \frac{1}{\left[\left(\frac{\kappa}{\kappa_0}\right)^{p-1} - 1\right]^{\frac{1}{p-1}}}$. From Lemma (5.5.17) one obtains the desired result (*5.5.84*).

step 2 Boundedness via smallness of the energy. Take $\tilde{t}_n$ and v_n as in Step 1. From (*5.5.74*) and (*5.5.75*) v_n blows up at time $\tau_n = \frac{T_n - \tilde{t}_n}{M_n(\tilde{t}_n)} \to 1$. Up to time $\tau_n' = \frac{T_n - t_n}{M_n(\tilde{t}_n)}$, $0 \leq \tau_n' \leq 0$, v_n satisfies:

$$|\Delta v_n| \leq \frac{1}{2}|v_n|^p + 2KM_n^{\frac{p}{p-1}}(\tilde{t}_n) \tag{5.5.85}$$

and we recall that $M_n(\tilde{t}_n) \to 0$ from (*5.5.75*). Let $R > 0$ and $a \in B(0,R)$. Define

$$w^{(n)}_{a,\tau_n}(y,t) := (\tau_n - t)^{\frac{1}{p-1}} v_n(t, a + \sqrt{\tau_n - t}y).$$

Then as $v_n(-1) \to \kappa_0$ from (*5.5.84*), one has that for n large enough

$$E[w^{(n)}_{a,\tau_n}(-1,\cdot)] = O(\kappa_0^2)$$

where the energy is defined by (*5.5.8*). One can then apply the result (*5.5.19*) of Proposition 5.5.7: there exists $r > 0$ such that for κ_0 small enough and n large enough one has:

$$\forall t \in [0,\tau_n'], \quad \|v_n(t)\|_{W^{2,\infty}(B(0,r))} \leq C. \tag{5.5.86}$$

step 3 End of the proof. Let χ be a cut-off function, $\chi = 1$ on $B(0,\frac{R}{16})$ and $\chi = 0$ outside $B(0,\frac{R}{8})$. The evolution of $\tilde{v}_n = \chi v_n$ is given by:

$$\tilde{v}_{n,\tau} - \Delta\tilde{v}_n = \chi|v_n|^{p-1}v_n + \Delta\chi v_n - 2\nabla.(\nabla\chi v_n) = F_n$$

with $\|F_n\|_{W^{1,\infty}} \leq C$ from (*5.5.86*). Fix $0 < s \ll 1$. One has:

$$\begin{array}{rcl} \Delta v_n(\tau_n', 0) & = & K_s * (\Delta \tilde{v}_n(\tau_n' - s))(0) + \sum_1^d \int_0^s \left[\partial_{x_i} K_{s-s'} * \partial_{x_i} F(\tau_n' - s + s')\right](0) \\ & = & o_{n\to+\infty}(1) + o_{s\to 0}(1) \end{array}$$

from (*5.5.84*) and the estimate on F_n. Hence $\Delta v_n(\tau_n', 0) \to 0$ as $n \to +\infty$. On the other hand, $\lim v_n(\tau_n', 0) = v(\tau_n', 0) > 0$ from (*5.5.84*) and the fact that $0 \leq \tau_n' \leq 1$. We recall that at time τ_n' v_n satisfies:

$$|\Delta v_n(\tau_n', 0)| = \frac{1}{2}|v_n(\tau_n', 0)|^p + 2K M_n^{\frac{p}{p-1}}(\tilde{t}_n).$$

As $M_n^{\frac{p}{p-1}}(\tilde{t}_n) \to 0$ from (*5.5.75*) at the limit one has $0 = \frac{1}{2}|v(\tau_n', 0)|^p > 0$ which is a contradiction. This ends the proof of Proposition 5.5.1 in Case 2.

□

5.A Kernel of the linearized operator $-\Delta - pQ^{p-1}$

In this section we prove study the linearized operator $-\Delta - pQ^{p-1}$. We characterize its kernel on each spherical harmonics in the following lemma. This will be useful in the next section to derive suitable coercivity properties for this operator.

Lemma 5.A.1 (Zeros of $-\Delta - pQ^{p-1}$ on spherical harmonics). *Let $n \in \mathbb{N}$ and $f \in \mathcal{C}^2((0,+\infty),\mathbb{R})$ satisfy $H^{(n)}f = 0$. Then $f \in Span(T^{(n)}, \Gamma^{(n)})$ where $T^{(n)}$ and $\Gamma^{(n)}$ are smooth and satisfy:*

(i) Radial case: $T^{(0)} = \Lambda Q$ *and* $\Gamma^{(0)}(r) \sim r^{-d+2}$ *as* $r \to 0$.

(ii) First spherical harmonics: $T^{(1)} = -\partial_r Q$ *and* $\Gamma^{(1)}(r) \sim r^{-d+1}$ *as* $r \to 0$.

(iii) Higher spherical harmonics: *for* $n \geq 2$, $T^{(n)} > 0$, $T^{(n)} \sim r^n$ *as* $r \to +\infty$ *and* $\Gamma^{(n)}(r) \sim r^{-d+2-n}$ *as* $r \to 0$.

Proof of Lemma 5.A.1

Let $n \in \mathbb{N}$ and f satisfy $H^{(n)}f = 0$. First we rewrite the equation as an almost constant coefficient ODE. Setting $w(t) = f(e^t)$, f solves $H^{(n)}f = 0$ if and only if w solves:

$$w'' + (d-2)w' - \left[e^{2t}V(e^t) + n(d+n-2)\right] w = 0. \quad (5.A.1)$$

First, as $|V(r)| \lesssim (1+|r|^4)^{-1}$, one gets that $|e^{2t}V(e^t| \lesssim e^{-2|t|}$. This implies that asymptotically, as $t \to \pm\infty$, (*5.A.1*) is almost the constant coefficients ODE

$$w'' + (d-2)w' - n(d+n-2)w. \quad (5.A.2)$$

One also has the bound

$$\forall t \in \mathbb{R}, |e^{2t}V(e^t)| = r^2|V(r)| \leq (\sqrt{d(d-2)})^2 V(\sqrt{d(d-2)}) = \frac{d(d+2)}{4} \quad (5.A.3)$$

which is obtained by maximizing this function.

step 1 Existence of a solution behaving like e^{nt} as $t\to-\infty$. We claim that for any $n\in\mathbb{N}$, there exists a solution $a^{(n)}$ of (*5.A.1*) such that $a^{(n)}(t)=e^{nt}+v(t)$ with $|v(t)|+|v'(t)|\lesssim e^{(n+1)t}$ as $t\to-\infty$. We now prove this fact. For $n=0$, the function $(\Lambda Q(0))^{-1}\Lambda Q(e^t)$ satisfies the desired property. We now assume $n\geq 1$. We use a standard fixed point argument to construct this solution as a perturbation of $t\mapsto e^{nt}$ which solves the asymptotic ODE (*5.A.2*). Using Duhamel formula, $a^{(n)}$ is a solution of (*5.A.1*) if and only if v is a solution of:

$$v(t)=\frac{1}{2n+d-2}\int_{-\infty}^{t}\left(e^{n(t-t')}-e^{-(d+n-2)(t-t')}\right)e^{2t'}V(t')\left[e^{nt'}+v(t')\right]dt'.$$

For $t_0\in\mathbb{R}$ we define the following functional space:

$$X_{t_0}:=\left\{v\in\mathcal{C}((-\infty,t_0],\mathbb{R}),\ \ \sup_{t\leq t_0}|v(t)|e^{-(n+1)t}<+\infty\right\}$$

on which we define the following canonical weighted L^∞ norm:

$$\|v\|_{X_{t_0}}:=\sup_{t\leq t_0}|v(t)|e^{-(n+1)t}.$$

$(X_{t_0},\|\cdot\|_{X_{t_0}})$ is a Banach space. We define the following function Φ on X_{t_0}:

$$(\Phi(v))(t):=\frac{1}{2n+d-2}\int_{-\infty}^{t}\left(e^{n(t-t')}-e^{-(d+n-2)(t-t')}\right)e^{2t'}V(t')(e^{nt'}+v(t'))dt'.$$

As $|e^{2t'}V(t')|\lesssim e^{2t'}$, for $t_0\ll 0$ small enough, estimating by brute force, for $v\in X_{t_0}$, $t_0\ll 0$ and $t\leq t_0$:

$$\begin{array}{rcl}|(\Phi(v))(t)| & \lesssim & e^{nt}\int_{-\infty}^{t}e^{(2-n)t'}|V(t')|(e^{nt'}+|v(t')|)dt' \\ & & +e^{-(d+n-2)t}\int_{-\infty}^{t}e^{(d+n)t'}|V(t')|(e^{nt'}+|v(t')|)dt' \\ & \lesssim & e^{nt}\int_{-\infty}^{t}e^{(2-n)t'}(e^{nt'}+e^{(n+1)t'}\|v\|_{X_{t_0}})dt' \\ & & +e^{-(d+n-2)t}\int_{-\infty}^{t}e^{(d+n)t'}(e^{nt'}+e^{(n+1)t'}\|v\|_{X_{t_0}})dt' \\ & \lesssim & e^{(n+2)t}+e^{(n+3)t}\|v\|_{X_{t_0}}\end{array}$$

so that $\|\Phi(v)\|_{X_{t_0}}\lesssim e^{t_0}+e^{2t_0}\|v\|_{X_{t_0}}$. This implies that for t_0 small enough, Φ maps $B_{X_{t_0}}(0,1)$, the unit ball of X_{t_0}, into itself. Now, as Φ is an affine function one computes similarly for $t_0\ll 1$, $v_1,v_2\in X_{t_0}$ and $t\leq t_0$:

$$\begin{array}{cc} & |(\Phi(v_1)-\Phi(v_2))(t)| \\ \lesssim & e^{nt}\int_{-\infty}^{t}e^{(2-n)t'}|V(t')||v_1-v_2|dt'+e^{-(d+n-2)t}\int_{-\infty}^{t}e^{(d+n)t'}|V(t')||v_1-v_2|)dt' \\ \lesssim & e^{((2+n)t}\|v_1-v_2\|_{X_{t_0}},\end{array}$$

implying that $\|\Phi(v_1)-\Phi(v_2)\|_{X_{t_0}}\lesssim e^{2t_0}\|v_1-v_2\|_{X_{t_0}}$, meaning that Φ is a contraction on $B_{X_{t_0}}(0,1)$. By Banach fixed point theorem, one gets that there exists a unique fixed point v_0 of Φ. The function $a^{(n)}(t)=e^{nt}+v_0(t)$ is then a solution of (*5.A.1*) on $(-\infty,t_0)$. There exists a unique global solution of (*5.A.1*) that coincides with it on $(-\infty,t_0]$ that we still denote by $a^{(n)}$. As v is a fixed point of Φ using verbatim the same type of computations we just did one sees that $|v'(t)|\lesssim e^{(n+1)t}$ as $t\to-\infty$. Hence $a^{(n)}$ has the properties we claimed in this step.

step 2 Existence of a solution behaving like $e^{-(d+n-2)t}$ as $t\to+\infty$. We claim that for any $n\in\mathbb{N}$, there exists a solution $b^{(n)}$ of (*5.A.1*) such that $b^{(n)}(t)=e^{-(d+n-2)t}+v(t)$ with $|v(t)|+|v'(t)|\lesssim e^{-(d+n-1)t}$ as

$t \to +\infty$. We now prove this fact. We reverse time and let $\tilde{t} = -t$, $\tilde{w}(\tilde{t}) = w(t)$. Then, w solves (*5.A.1*) if and only if $\tilde{w}$ solves

$$\tilde{w}'' - (d-2)\tilde{w}' - \left[e^{-2\tilde{t}}V(e^{\tilde{t}}) + n(d+n-2)\right]\tilde{w} = 0$$

and $w = e^{-(d+n-2)\cdot} + v$ with $|v(t)| + |v'(t)| \lesssim e^{-(d+n-1)t}$ as $t \to +\infty$ if and only if $\tilde{w} = e^{(d+n-2)} + \tilde{v}$ with $|\tilde{v}(\tilde{t})| + |\tilde{v}'(\tilde{t})| \lesssim e^{(d+n-1)\tilde{t}}$ as $\tilde{t} \to -\infty$. One notices that the eigenvalues of the constant coefficients part of the ODE, (*5.A.2*), are $-n$ and $d+n-2$, and that $|e^{-2\tilde{t}}V(e^{\tilde{t}})| \lesssim e^{2\tilde{t}}$ as $\tilde{t} \to -\infty$. Therefore, we are again in the context of a constant coefficient second order ODE with a strictly positive and a strictly negative eigenvalue, plus a exponentially small perturbative linear term. One obtains the result claimed in this step by verbatim the same techniques we just employed in Step 1.

step 3 Existence of a solution behaving like e^{nt} as $t \to +\infty$. We claim that for any $n \geq 1$, there exists a solution $c^{(n)}$ of (*5.A.1*) such that $c^{(n)}(t) = e^{nt} + v(t)$ with $|v(t)| \leq \frac{e^{nt}}{2}$ as $t \to +\infty$. We now prove this fact. Let $t_0 \in \mathbb{R}$ and w be the solution of (*5.A.1*) with initial condition $w(t_0) = e^{nt_0}$ and $w'(t_0) = ne^{nt_0}$. Then we claim that there exists $0 \ll t_0$ large enough, such that $|w - e^{nt}| \leq \frac{e^{nt}}{2}$ for all $t_0 \leq t$. To prove it, we use a connectedness argument. We define $\mathcal{T} \subset [t_0, +\infty)$ as the set of times $t \geq t_0$ such that this inequality holds on $[t_0, t]$. $\mathcal{T}$ is non empty as it contains t_0. It is closed by continuity. Then using Duhamel formula, one has for any $t \in \mathcal{T}$:

$$\begin{array}{rcl} |w(t) - e^{nt}| & \lesssim & \int_{t_0}^{t} \left| \left(e^{n(t-t')} - e^{-(d+n-2)(t-t')}\right) e^{2t'}V(t')(w(t'))\right| dt' \\ & \lesssim & e^{nt}e^{-2t_0}. \end{array}$$

This implies that for t_0 large enough, $\mathcal{T}$ is open. By connectedness, one has $\mathcal{T} = [t_0, +\infty)$, which means that $|w(t) - e^{nt}| \leq \frac{e^{nt}}{2}$ for all times $t \geq t_0$. We extend w backward in times to obtain a global solution of (*5.A.1*), it then has the properties we claimed in this third step.

step 4 Existence of a solution behaving like $e^{-(d+n-2)t}$ as $t \to -\infty$. We claim that for any $n \in \mathbb{N}$, there exists a solution $d^{(n)}$ of (*5.A.1*) such that $d^{(n)}(t) = e^{-(d+n-2)t} + v(t)$ with $|v(t)| \leq \frac{e^{-(d+n-2)t}}{2}$ as $t \to +\infty$. This can be proved using verbatim the same techniques we already employed: first by reversing time as in Step 2, then by performing a bootstrap argument as in Step 3.

step 5 $a^{(n)} \neq b^{(n)}$ for $n \geq 2$. Let $n \geq 2$, $a^{(n)}$ and $b^{(n)}$ be the two solutions we defined in Step 1 and Step 2 respectively. Then we claim that $a^{(n)} \neq b^{(n)}$. To show this, we see (*5.A.1*) as a planar dynamical system. We associate to $w \in \mathcal{C}^2(]-\infty, +\infty), \mathbb{R})$ the vector $W := \begin{pmatrix} w \\ w' \end{pmatrix}$. Then w solves (*5.A.1*) if an only if W solves:

$$W' = \begin{pmatrix} 0 & 1 \\ e^{2t}V(e^t) + n(d+n-2) & -(d-2) \end{pmatrix} W.$$

We denote by $A^{(n)}$ and $B^{(n)}$ the vectors associated to $a^{(n)}$ and $b^{(n)}$. From the results of Step 1 and Step 2, one has that:

$$A^{(n)} = e^{nt}\begin{pmatrix} 1 \\ n \end{pmatrix} + O(e^{(n+1)t}) \text{ as } t \to -\infty,$$

$$B^{(n)} = e^{-(d+n-2)t}\begin{pmatrix} 1 \\ -(d+n-2) \end{pmatrix} + O(e^{-(d+n-1)t}) \text{ as } t \to +\infty.$$

We claim that solutions starting in $Z := \{(x,y) \in \mathbb{R}^2,\ x \geq 0 \text{ and } y \geq -\frac{d-2}{2}x\}$ cannot escape this zone. Once we have proven this, the result we claim follows as from their asymptotic behaviors, $A^{(n)}$ is in Z for small enough times and $B^{(n)}$ is not in Z for large times. To prove this, one computes the direction of the flow at the boundary $\partial Z := Z_1 \cup Z_2 \cup \{(0,0)\}$, where $Z_1 := \{(0,y),\ y > 0\}$ and $Z_2 := \{(x, -\frac{d-2}{2}x),\ x > 0\}$. On Z_1, one takes $(1,0)$ as the unit vector orthogonal to Z_1 pointing inside Z. At a time $t \in \mathbb{R}$, one computes the entering flux at the point $(0,1)$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & 1 \\ e^{2t}V(e^t) + n(d+n-2) & -(d-2) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = 1.$$

By linearity, the flux entering Z through Z_1 is also always strictly positive at each point of Z_1 for each time $t \in \mathbb{R}$. On Z_2 one takes $(\frac{d-2}{2}, 1)$ as an orthogonal vector pointing inside Z, and one computes the entering flux at the point $(\frac{d-2}{2})$ using (*5.A.3*):

$$\begin{array}{rc} & \begin{pmatrix} \frac{d-2}{2} \\ 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & 1 \\ e^{2t}V(e^t) + n(d+n-2) & -(d-2) \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{d-2}{2} \end{pmatrix} \right) \\ = & \frac{(d-2)^2}{4} + e^{2t}V(e^t) + n(d+n-2) \\ \geq & \frac{(d-2)^2}{4} - \frac{d(d+2)}{4} + 2d \\ = & \frac{d}{2} + 1. \end{array}$$

By linearity, the flux entering Z through Z_2 is also always strictly positive at each point of Z_2 for each time $t \in \mathbb{R}$. Consequently we have proven that Z is forward in time stable by the flow. This ends the proof of this step.

step 6 Conclusion. We collect the results proved for the equivalent ODE (*5.A.1*) in the five previous steps and go back to original variables. We set for $n \in \mathbb{N}$, $T^{(n)}(r) = a^{(n)}(\log(r))$, $\Gamma^{(n)}(r) = b^{(n)}(\log(r))$ and $\tilde{\Gamma}^{(n)}(r) = d^{(n)}(\log(r))$, and for $n \geq 1$, $\tilde{T}^{(n)}(r) = c^{(n)}(\log(r))$. From the previous steps there hold

$$\forall n \geq 1,\ T^{(n)} \underset{r \to 0}{\sim} r^n,\ \Gamma^{(n)} \underset{r \to 0}{\sim} r^{-(d+n-2)},\ \tilde{T}^{(n)} \underset{r \to +\infty}{\sim} r^n,\ \tilde{\Gamma}^{(n)} \underset{r \to +\infty}{\sim} r^{-(d+n-2)}.$$

Case $n = 0$. From a direct computation, one has $H^{(0)}\Lambda Q = 0$. This comes from the invariance by scale change of the equation. Together with the asymptotic of the other solution $\Gamma^{(0)}$ at the origin, this proves the lemma for the remaining case $n = 0$.

Case $n = 1$. From a direct computation, one has $H^{(1)}\partial_r Q = 0$. This comes from the invariance by translation of the equation. We claim that there exists $a_1, a_2 \in \mathbb{R}$ such that $T^{(1)} = a_1 \partial_r Q = a_2 \tilde{\Gamma}^{(1)}$. Indeed, as the equation is a second order linear ODE, there exists $b_1, b_2 \in \mathbb{R}$ such that $\partial_r Q = b_1 T^{(1)} + b_2 \Gamma^{(1)}$, and as $\Gamma^{(1)}$ is singular at the origin, one has that $b_2 = 0$. We apply the same reasoning at $+\infty$ to prove that $\partial_r Q$ is collinear with $\tilde{\Gamma}^{(1)}$. This, together with the asymptotic of the other solution $\Gamma^{(1)}$ at the origin proves the lemma for $n = 1$.

Case $n \geq 2$. We proved in Step 5 that $T^{(n)}$ and $\tilde{\Gamma}^{(n)}$ are not collinear. Hence there exists $c^{(n)}, c^{(n)'}$ with $c^{(n)} \neq 0$ such that $T^{(n)} = c^{(n)}\tilde{T}^n + c^{(n)'}\tilde{\Gamma}^{(n)} \sim c^{(n)} r^n$ as $r \to +\infty$. This, together with the asymptotic of the other solution $\Gamma^{(n)}$ at the origin proves the lemma for $n \geq 2$.

$\square$

5.B Proof of the coercivity lemma 5.2.3

This Appendix is devoted to the proof of Lemma 5.2.3 which adapts to the non radial setting the related proof in [138].
f Thanks to Lemma 5.A.1 and Proposition 5.2.2, we can state and prove the following coercivity property for the linearized operator $-\Delta - pQ^{p-1}$ under suitable orthogonality conditions. We keep the notations for the spherical harmonics introduced in Appendix 5.A.

Proof of Proposition 5.2.2 For each $n \geq 1$ we define the following first order operator on radial functions:

$$A^{(n)} := -\partial_r + W^{(n)} \tag{5.B.1}$$

where the potential is $W^{(n)} := \partial_y (\log T^{(n)})$, and where $T^{(n)}$ is defined in Lemma 5.A.1, with the convention $T^{(1)} = -\partial_r Q$. As $H^{(n)}$ for $n \geq 1$ has a positive zero eigenfunction, this implies the following factorization property for smooth enough functions:

$$\int u^{(n,k)} H^{(n)} u^{(n,k)} r^{d-1} dr = \int |A^{(n)} u^{(n,k)}|^2 r^{d-1} dr.$$

In turn, this gives the following formula for all functions $u \in \dot{H}^1 \cap \dot{H}^2$:

$$\begin{aligned} \int uHudx &= \sum_{n\in\mathbb{N},\ 1\leq k\leq k(n)} \int_0^{+\infty} u^{(n,k)} H^{(n)} u^{(n,k)} r^{d-1} dr \\ &= \int_0^{+\infty} u^{(0,1)} H^{(0)} u^{(0,1)} r^{d-1} dr + \sum_{1\leq n,\ 1\leq k\leq k(n)} \int_0^{+\infty} |A^{(n)} u^{(n,k)}|^2 r^{d-1} dr. \end{aligned} \tag{5.B.2}$$

The second term in the right hand side is always non negative, and the first term is non negative if $u \perp \mathcal{Y}$ from the first part of the proof. This ends the proof of Proposition 5.2.2. $\square$

Proof of Lemma 5.2.3

We recall that the first order operarors factorizing H on each spherical harmonics are defined by (5.B.1). If $u \in \dot{H}^1(\mathbb{R}^d)$, or $u \in \dot{H}^1 \cap \dot{H}^2(\mathbb{R}^d)$ or $u \in \dot{H}^1 \cap \dot{H}^3(\mathbb{R}^d)$, with decomposition into spherical harmonics

$$u(x) = \sum_{n\in\mathbb{N},\ 1\leq k\leq k(n)} u^{(n,k)}(|x|) Y^{(n,k)} \left(\frac{x}{|x|}\right)$$

one deduces respectively:

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla u|^2 - pQ^{p-1}u^2 &= \sum_{n\geq 1,\ 1\leq k\leq k(n)} \int_0^{+\infty} |A^{(n)} u^{(n,k)}|^2 r^{d-1} dr \\ &\quad + \int_0^{+\infty} (|\partial_r u^{(0,1)}|^2 - pQ^{p-1}|u^{(0,1)}|^2) r^{d-1} dr \end{aligned}$$

$$\int_{\mathbb{R}^d} |Hu|^2 = \sum_{n\in\mathbb{N},\ 1\leq k\leq k(n)} \int_0^{+\infty} |H^{(n)} u^{(n,k)}|^2 r^{d-1} dr$$

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla Hu|^2 - pQ^{p-1}|Hu|^2 &= \sum_{n\geq 1,\ 1\leq k\leq k(n)} \int_0^{+\infty} |A^{(n)} H^{(n)} u^{(n,k)}|^2 r^{d-1} dr \\ &\quad \int_0^{+\infty} (|\partial_r H^{(0)} u^{(0,1)}|^2 - pQ^{p-1}|H^{(0)} u^{(0,1)}|^2) r^{d-1} dr. \end{aligned} \tag{5.B.3}$$

We first show the estimate (*5.2.9*) for which the proof is a bit more delicate than the proof of (*5.2.7*) and (*5.2.8*).

step 1 Subcoercivity. We claim that for any $d\geq 7$ there exists a constant $C=C(d)>0$ such that for any $u\in \dot H^1\cap \dot H^3(\mathbb{R}^d)$ there holds:

$$\begin{array}{rcl} \int_{\mathbb{R}^d}|\nabla Hu|^2-pQ^{p-1}|Hu|^2 & \geq & \frac{1}{C}\left(\int_{\mathbb{R}^d}|\nabla^3 u|^2+\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{|x|^2}+\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^4}+\int_{\mathbb{R}^d}\frac{u^2}{|x|^6}\right) \\ & & -C\left(\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^4}+\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^6}+\int_{\mathbb{R}^d}\frac{u^2}{1+|x|^8}\right). \end{array} \tag{5.B.4}$$

Indeed, first recall the standard Hardy inequality for $f\in \dot H^s$ for $0\leq s<\frac d2$:

$$\int_{\mathbb{R}^d}\frac{|f|^2}{|x|^{2s}}\lesssim \|f\|_{\dot H^s}^2. \tag{5.B.5}$$

In particular, as we are in dimension $d\geq 7$, we can apply this inequality for $s=1,2,3$. As $H=-\Delta-pQ^{p-1}$, with a potential decays faster than the Hardy potential, i.e. for all $j\in\mathbb{N}$, $|\partial_r^j Q^{p-1}|\lesssim (1+|x|)^{-4-j}$, using the above Hardy inequality plus Young and Cauchy-Schwarz inequalities:

$$\begin{array}{rl} & \int_{\mathbb{R}^d}|\nabla Hu|^2-pQ^{p-1}|Hu|^2 \\ = & \int_{\mathbb{R}^d}|\nabla\Delta u|^2+2\nabla\Delta u.\nabla(pQ^{p-1}u)+p^2|\nabla(Q^{p-1}u)|^2 \\ & -\int_{\mathbb{R}^d}pQ^{p-1}|\Delta u|^2-2p^2Q^{2(p-1)}u\Delta u-p^3Q^{3(p-1)}u^2 \\ \geq & \frac12\int_{\mathbb{R}^d}|\nabla\Delta u|^2-\int_{\mathbb{R}^d}p^2|\nabla(Q^{p-1}u)|^2+pQ^{p-1}|\Delta u|^2 \\ & -\int_{\mathbb{R}^d}2p^2Q^{2(p-1)}u\Delta u+p^3Q^{3(p-1)}u^2 \\ \geq & \frac12\int_{\mathbb{R}^d}|\nabla\Delta u|^2-C\left(\int_{\mathbb{R}^d}\frac{|u|^2}{1+|x|^{10}}+\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{1+|x|^8}+\int_{\mathbb{R}^d}\frac{|\nabla^2 u|^2}{1+|x|^4}\right) \\ \geq & c\left(\int_{\mathbb{R}^d}|\nabla\Delta u|^2+\frac{|u|^2}{|x|^6}+\frac{|\nabla u|^2}{|x|^4}+\frac{|\nabla^2 u|^2}{|x|^2}\right) \\ & -C\left(\int_{\mathbb{R}^d}\frac{|u|^2}{1+|x|^{10}}+\frac{|\nabla u|^2}{1+|x|^8}+\frac{|\nabla^2 u|^2}{1+|x|^4}\right) \end{array}$$

for some constants $C,c>0$. This gives the estimate (*5.B.4*) we claimed in this step.

step 2 Orthogonality for Hu. We claim that for any $d\geq 7$, if $u\in\dot H^3(\mathbb{R}^d)$ is such that $u\perp \mathcal{Y}$ then

$$Hu\in \text{Span}(\partial_{x_1}Q,...,\partial_{x_n}Q,\Lambda Q,\mathcal{Y})^{\perp}. \tag{5.B.6}$$

We now prove this. The linear form

$$\Phi: u\mapsto (\langle Hu,\partial_{x_1}Q\rangle,...,\langle Hu,\partial_{x_d}Q\rangle,\langle Hu,\Lambda Q\rangle,\langle Hu,\mathcal{Y}\rangle)$$

is well defined on $\dot H^3(\mathbb{R}^d)$. To see this we estimate each term via Cauchy-Schwarz inequality and Hardy inequalities (*5.B.5*), using the asymptotic of the solitary wave (*2.2.2*) and the fact that $\mathcal{Y}$ decays exponentially fast:

$$\begin{array}{rl} & \sum_1^d |\langle Hu, \partial_{x_i} Q\rangle| + |\langle Hu, \Lambda Q\rangle| + |\langle Hu, \mathcal{Y}\rangle| \lesssim \int_{\mathbb{R}^d} \frac{|\Delta u|}{1+|x|^{d-2}} + \int_{\mathbb{R}^d} \frac{|u|}{1+|x|^{d+2}} \\ \lesssim & \left(\int_{\mathbb{R}^d} \frac{|\Delta u|^2}{1+|x|^2}\right)^{\frac12} \left(\int_{\mathbb{R}^d} \frac{1}{1+|x|^{2d-6}}\right)^{\frac12} + \left(\int_{\mathbb{R}^d} \frac{u^2}{1+|x|^6}\right)^{\frac12} \left(\int_{\mathbb{R}^d} \frac{1}{1+|x|^{2d-2}}\right)^{\frac12} \\ \lesssim & \|u\|_{\dot H^3}. \end{array}$$

This also gives the continuity of Φ for the natural topology on $\dot H^3(\mathbb{R}^d)$. For u smooth and compactly supported satisfying the orthogonality condition $u \perp \mathcal{Y}$ one can perform the following integrations by parts:

$$\begin{array}{rcl} \Phi(u) & = & (\langle u, H\partial_{x_1} Q\rangle, ..., \langle u, H\partial_{x_d} Q\rangle, \langle u, H\Lambda Q\rangle, \langle u, H\mathcal{Y}\rangle) \\ & = & (\langle u, 0\rangle, ..., \langle u, 0\rangle, \langle u, 0\rangle, \langle u, -e_0\mathcal{Y}\rangle) \\ & = & (0, ..., 0). \end{array}$$

By density of such functions, one then gets the desired orthogonality (*5.B.6*) for all functions $u \in \dot H^3(\mathbb{R}^d)$ satisfying $u \perp \mathcal{Y}$.

step 3 Proof of the coercivity estimate. First, from the orthogonality condition (*5.B.6*), the fact that on radial functions H is self adjoint and admits only $\mathcal{Y}$ as an eigenfunction associated to a negative eigenvalue, and the formula (*5.B.3*), one obtains the nonegativity of the quantity:

$$\int_{\mathbb{R}^d} |\nabla Hu|^2 - pQ^{p-1}|Hu|^2 \geq 0.$$

We now argue by contradiction and assume that the estimate (*5.2.9*) does not hold for functions on $u \in \dot H^3(\mathbb{R}^d)$ satisfying the orthogonality conditions (*5.2.6*). Up to renormalization, this amounts to say that there exists a sequence $(u_n)_{n\in\mathbb{N}} \in [\dot H^3(\mathbb{R}^n)]^{\mathbb{N}}$ such that for each n, u_n satisfies the orthogonality conditions (*5.2.6*), $H(u_n)$ satisfies (*5.B.6*),

$$\int_{\mathbb{R}^d} |\nabla^3 u_n|^2 + \int_{\mathbb{R}^d} \frac{|\nabla^2 u_n|^2}{|x|^2} + \int_{\mathbb{R}^d} \frac{|\nabla u_n|^2}{|x|^4} + \int_{\mathbb{R}^d} \frac{u_n^2}{|x|^6} = 1 \tag{5.B.7}$$

and:

$$\int_{\mathbb{R}^d} |\nabla Hu_n|^2 - pQ^{p-1}|Hu_n|^2 \to 0 \text{ as } n \to +\infty. \tag{5.B.8}$$

From the subcoercivity formula (*5.B.4*) from Step 1, the convergence to zero of its left hand side (*5.B.8*) and the order 1 size for the first terms of the right hand side (*5.B.7*) one deduces that there exists $c > 0$ such that for all $n \in \mathbb{N}$:

$$\int_{\mathbb{R}^d} \frac{|\nabla^2 u_n|^2}{1+|x|^4} + \int_{\mathbb{R}^d} \frac{|\nabla u_n|^2}{1+|x|^6} + \int_{\mathbb{R}^d} \frac{u_n^2}{1+|x|^8} > c. \tag{5.B.9}$$

From weak compactness of $\dot H^3(\mathbb{R}^d)$ and the compactness of the embedding for localized Sobolev spaces, there exists $u_\infty \in \dot H^3$ such that u_n converges toward u_∞ weakly in $\dot H^3(\mathbb{R}^d)$ and strongly in $H^2_{\text{loc}}(\mathbb{R}^d)$. The above lower bound, together with (*5.B.7*) means that the mass of $(u_n)_{n\in\mathbb{N}}$ cannot go to infinity. Combined with the strong local convergence, this implies:

$$\int_{\mathbb{R}^d} \frac{|\nabla^2 u_n|^2}{1+|x|^4} + \frac{|\nabla u_n|^2}{1+|x|^6} + \frac{u_n^2}{1+|x|^8} \to \int_{\mathbb{R}^d} \frac{|\nabla^2 u_\infty|^2}{1+|x|^4} + \frac{|\nabla u_\infty|^2}{1+|x|^6} + \frac{u_\infty^2}{1+|x|^8}.$$

Combined with (*5.B.9*) one gets the non nullity of the limit: $u_\infty \neq 0$. From the weak convergence, u_∞ satisfies also the orthogonality conditions (*5.2.6*) as Ψ_0 is exponentially decaying and $\Psi_1, ..., \Psi_d$ are compactly supported. Hu_∞ then satisfies the orthogonality condition (*5.B.6*), from the result of Step 2. From (*5.B.8*), Fatou lemma and the formula (*5.B.3*) , one gets:

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\nabla H u_\infty|^2 - pQ^{p-1}|Hu_\infty|^2 \\
= &\sum_{n\geq 1,\ 1\leq k\leq k(n)} \int_0^{+\infty} |A^{(n)}H^{(n)}u_\infty^{(n,k)}|^2 r^{d-1}dr \\
&+ \int_0^{+\infty} (|\partial_r H^{(0)}u_\infty^{(0,1)}|^2 - pQ^{p-1}|H^{(0)}u_\infty^{(0,1)}|^2) r^{d-1}dr = 0.
\end{aligned} \qquad (5.B.10)$$

We now decompose u_∞ on spherical harmonics:

$$u_\infty(x) = \sum_{n\in\mathbb{N},\ 1\leq k\leq k(n)} u_\infty^{(n,k)}(|x|)Y^{(n,k)}\left(\frac{x}{|x|}\right)$$

and prove the nullity of each component $u_\infty^{(n,k)}$, which will be a contradiction to the fact that we just obtained the non nullity of u_∞.

Case $n \geq 2$. Let $n \geq 2$ and $1 \leq k \leq k(n)$, then we claim that $u_\infty^{(n,k)} = 0$. From (*5.B.10*) and the fact that all the terms in the right hand side are nonnegative thanks to the spectral Proposition 5.2.2 for H one gets:

$$\int_0^{+\infty} |A^{(n)}H^{(n)}u_\infty^{(n,k)}|^2 r^{d-1}dr = 0$$

which implies $A^{(n)}H^{(n)}u_\infty^{(n,k)} = 0$. From the definition (*5.B.1*) of $A^{(n)}$ this implies that there exists $c^{(n,k)} \in \mathbb{R}$ such that:

$$H^{(n)}u_\infty^{(n,k)} = c^{(n,k)}T^{(n)}$$

where $T^{(n)}$ is defined in Lemma (5.A.1). However still from this Lemma one has that $T^{(n)}(r) \sim r^n$ as $r \to +\infty$. Hence $\int_0^{+\infty} \frac{|T^{(n)}|^2}{1+r^4} r^{d-1}dr = +\infty$. From $u_\infty \in \dot{H}^3(\mathbb{R}^d)$ and Sobolev embedding, one gets $\int_0^{+\infty} \frac{|H^{(n)}u_\infty^{(n,k)}|^2}{1+r^4} r^{d-1}dr < +\infty$. Hence $c^{(n,k)} = 0$, which means that $H^{(n)}u_\infty = 0$. In turn, again from Lemma 5.A.1, this means that there exists two constants $c_1^{(n,k)}$ and $c_2^{(n,k)}$ such that:

$$u_\infty^{(n,k)} = c_1^{(n,k)}T^{(n)} + c_2^{(n,k)}\Gamma^{(n)}.$$

As $u_\infty, T^{(n)} \in L^2_{\text{loc}}(\mathbb{R}^d)$ and as $\Gamma^{(n)}$ is singular at the origin, with in particular $\Gamma^{(n)} \notin L^2_{\text{loc}}(\mathbb{R}^d)$ one gets that $c_2^{(n,k)} = 0$. As $\int_{\mathbb{R}^d} \frac{|u_\infty^{(n,k)}|^2}{|x|^6} < +\infty$ from Fatou Lemma, and as $\int_{\mathbb{R}^d} \frac{|T^{(n)}|^2}{|x|^6} = +\infty$ because $T^{(n)} \sim |x|^n$ at infinity from Lemma 5.A.1, one obtains that $c_1^{(n,k)} = 0$ too. Therefore $u_\infty^{(n,k)} = 0$ which is the fact we claimed.

Case $n = 1$. Let $1 \leq k \leq d$, we claim that $u_\infty^{(1,k)} = 0$. Similarly, from (*5.B.10*) and the fact that all the terms in the right hand side are nonnegative thanks to the spectral Proposition 5.2.2 for H one gets:

$$\int_0^{+\infty} |A^{(1)}H^{(1)}u_\infty^{(1,k)}|^2 r^{d-1}dr = 0$$

which implies $A^{(1)}H^{(n)}u_\infty^{(1,k)}=0$. From its definition (*5.B.1*) and Lemma (5.A.1) this implies that there exists $c^{(1,k)}\in\mathbb{R}$ such that:

$$H^{(1)}u_\infty^{(1,k)}=c^{(1,k)}\partial_rQ.$$

From the orthogonality conditions (*5.B.6*) for Hu one gets $c^{(1,k)}=0$, meaning that $H^{(1)}u_\infty^{(1,k)}=0$. From Lemma 5.A.1, there exists two constants $c_1^{(1,k)}$ and $c_2^{(1,k)}$ such that:

$$u_\infty^{(n,k)}=c_1^{(n,k)}\partial_rQ+c_2^{(n,k)}\Gamma^{(1)}.$$

As $u_\infty,\partial_rQ\in L^2_{\text{loc}}(\mathbb{R}^d)$ and $\Gamma^{(n)}\notin L^2_{\text{loc}}(\mathbb{R}^d)$ for it is singular at the origin from Lemma 5.A.1, the second integration constant is nul: $c_2^{(1,k)}=0$. As u_∞ satisfies from (*5.2.6*) $\int u_\infty\Psi_k=0$, which reads $\int_0^{+\infty}u_\infty^{(1,k)}\chi_M\partial_rQr^{d-1}dr=0$ in spherical harmonics, one gets that the first integration constant is nul: $c_1^{(1,k)}=0$. Hence $u_\infty^{(1,k)}=0$ which is the fact we claimed.

Case $n=0$. From the two previous points one has that u_∞ is a radial function and so Hu_∞ is also radial. As Hu_∞ enjoys the orthogonality conditions $\langle Hu_\infty,\Lambda Q\rangle=0=\langle Hu_\infty,\mathcal{Y}\rangle$ from (*5.B.6*), Proposition 5.2.2 for H implies that $Hu_\infty=0$. From Lemma 5.A.1, this means $u_\infty\in\text{Span}(\Lambda Q,\Gamma^{(0)})$. As u_∞ and ΛQ are square integrable at the origin, whereas $\Gamma^{(0)}$ is not (it is singular from Lemma 5.A.1), one gets that $u_\infty\in\text{Span}(\Lambda Q)$. The orthogonality conditions $\langle u_\infty,\Psi_0\rangle=0$ then imply $u_\infty=0$.

Consequently, we have proven that u_∞ is 0, which is the desired contradiction. Hence the coercivity property (*5.2.9*) is true. The proofs of the first two coercivity properties (*5.2.7*) and (*5.2.8*) follow the same line and is left to the reader. □

5.C Adapted decomposition close to the manifold of ground states

In this section we give the proof of the decomposition Lemma 5.2.5. Such result is standard in modulation theory, and we give the proof here for the sake of completeness.

Proof of Lemma 5.2.5

We give a classical proof relying on the implicit function theorem.

step 1 Stationary decomposition. We define the following function:

$$\begin{array}{llll}\Phi: & \dot H^1(\mathbb{R}^d)\times\mathbb{R}^{d+1}\times(0,+\infty) & \to & \mathbb{R}^{d+2}\\ & (u,z,a,\lambda) & \mapsto & (\langle v,\Psi_1\rangle,...,\langle v,\Psi_dQ\rangle,\langle v,\mathcal{Y}\rangle,\langle v,\Psi_0\rangle)\end{array}$$

where:

$$v=(\tau_{-z}u)_{\frac1\lambda}+(\tau_{-z}Q)_{\frac1\lambda}-Q-a\mathcal{Y}.$$

The function Φ is well defined, as the orthogonality conditions are taken against functions that are either exponentially decaying or compactly supported, and as $\dot H^1$ is continuously embedded in $L^{\frac{2d}{d-2}}$ from Sobolev inequalities. It is $\mathcal{C}^\infty$. One computes the Jacobian matrix with respect to the last arguments at the point $(0,0,0,1)$:

$$
\begin{aligned}
J\Phi(0,0,0,1) &= \left(\frac{\partial \Phi}{\partial z_1}, ..., \frac{\partial \Phi}{\partial z_d}, \frac{\partial \Phi}{\partial a}, \frac{\partial \Phi}{\partial \lambda}\right)(0,0,0,1) \\
&= \begin{pmatrix} \int \frac{1}{d}\chi_M |\nabla Q|^2 & & & & (0) \\ & \cdot & & & \\ & & \int \frac{1}{d}\chi_M |\nabla Q|^2 & & \\ & & & \int \mathcal{Y}^2 & \\ (0) & & & & \int \chi_M \Lambda Q^2 \end{pmatrix}
\end{aligned}
$$

The implicit function theorem then gives that there exists $\delta > 0$ and unique smooth functions z, a and λ on $B_{\dot{H}^1}(0,\delta)$ such that for any $u \in B_{\dot{H}^1}(0,\delta)$ one has $\Phi(u,z,a,\lambda) = 0$, meaning that $v \in \text{Span}(\Psi_1, ..., \Psi_d, \mathcal{Y}, \Psi_0)^{\perp}$.

step 2 Dynamical decomposition. Now, by invariance of the $\dot{H}^1$ norm by scaling and translation, for any $z \in \mathbb{R}^d$ and $\lambda > 0$ there exists such a decomposition for all balls $B_{\dot{H}^1}(Q_{z,\lambda}, \delta)$, and they coincide when the balls overlap. Therefore, there exist smooth functions that we still denote by λ, a, and z, defined on $(Q_{z,\lambda})_{\lambda>0, z\in\mathbb{R}^d} + B_{\dot{H}^1}(0,\delta)$, such that $v \in \text{Span}(\Psi_1, ..., \Psi_d, \mathcal{Y}, \Psi_0)^{\perp}$.

To finish, assume $u_0 \in \dot{H}^1(\mathbb{R}^d)$ with $\|u_0 - Q\|_{\dot{H}^1} < \delta$ and that the solution of (NLH) given by Proposition 5.2.1 is defined on some time interval $[0,T)$ and satisfies:

$$\sup_{0\leq t<T} \inf_{\lambda>0, z\in\mathbb{R}^d} \|u - Q_{z,\lambda}\|_{\dot{H}^1} < \delta.$$

Then, from Proposition 5.2.1, $u \in \mathcal{C}^1((0,T), \dot{H}^1)$, hence, the functions $\lambda(u(t))$, $z(u(t))$ and $a(u(t))$ give the desired decomposition and are C^1 on $(0,T)$.

□

5.D Nonlinear inequalities

In this section, we state certain estimates on the nonlinear term $f(Q+u) - f'(Q)u - f(Q)$ and on its derivatives.

Lemma 5.D.1 (Pointwise estimates on the purely nonlinear term). *For any $x, y, y_1, y_2 \in \mathbb{R}$ there holds the following estimates:*

$$\left| |1+x|^{p-1}(1+x) - px - 1 \right| \lesssim |x|^p. \tag{5.D.1}$$

$$\left| |1+x|^{p-1}(1+x) - px - 1 \right| \lesssim |x|^2. \tag{5.D.2}$$

$$\left| |1+x+y|^{p-1}(1+x+y) - p(x+y) - 1 \right| \lesssim |x|^2 + |y|^p. \tag{5.D.3}$$

$$\left| |1+x+y|^{p-1} - 1 \right| \lesssim |x| + |y|^{p-1}. \tag{5.D.4}$$

$$\left| |1+x+y|^{p-1} - 1 \right| \lesssim |x| + |y|. \tag{5.D.5}$$

$$\left| |1+x+y|^{p-1} - 1 - (p-1)(x+y) \right| \lesssim |x|^{1+\frac{2}{d-2}} + |y|^2. \tag{5.D.6}$$

If $x>0$:

$$\Big||x+y_1|^{p-1}(x+y_1)-|x+y_2|^{p-1}(x+y_2)-px^{p-1}(y_1-y_2)\Big|\lesssim|y_1-y_2|(|y_1|^{p-1}+|y_2|^{p-1}) \quad (5.D.7)$$

$$\Big||x+y|^{p-1}-|x|^{p-1}-(p-1)|x|^{p-2}y\Big|\,|x|\lesssim|y|^p \quad (5.D.8)$$

$$\Big||x+y|^{p-1}-|x|^{p-1}\Big|\lesssim|y|^{p-1} \quad (5.D.9)$$

$$\Big||1+x|^{p+1}-1-(p+1)x\Big|\lesssim|x|^{p+1}+|x|^2 \quad (5.D.10)$$

Proof of Lemma 5.D.1

- *Proof of (5.D.1).* As $g:x\mapsto|1+x|^{p-1}(1+x)$ is $\mathcal{C}^2$ on $[-\frac12,\frac12]$ at $x=0$, with $g(0)=1$ and $g'(0)=p$ there exists $C>0$ such that:

$$\forall x\in[-\frac12,\frac12],\quad \Big||1+x|^{p-1}(1+x)-px-1\Big|\leq C|x|^2\leq C|x|^p. \quad (5.D.11)$$

as $1<p<2$. Now as

$$\Big||1+x|^{p-1}(1+x)-px-1\Big|\sim|x|^p \text{ as } |x|\to+\infty, \quad (5.D.12)$$

there exists $C'>0$ such that:

$$\forall|x|\geq\frac12,\quad \Big||1+x|^{p-1}(1+x)-px-1\Big|\leq C'|x|^p. \quad (5.D.13)$$

The two estimates (*5.D.11*) and (*5.D.13*) then imply (*5.D.1*).

- *Proof of (5.D.2).* From (*5.D.12*), as $1<p<2$, there exists $C'>0$ such that:

$$\forall|x|\geq\frac12,\quad \Big||1+x|^{p-1}(1+x)-px-1\Big|\leq C'|x|^2.$$

This, combined with (*5.D.11*), implies (*5.D.2*).

- *Proof of (5.D.3).* As $g:x\mapsto|1+x|^{p-1}(1+x)$ is $\mathcal{C}^2$ on $[-\frac12,\frac12]$ at $x=0$, with $g(0)=1$ and $g'(0)=p$ there exists $C>0$ such that:

$$\forall x,y\in[-\frac12,\frac12],\quad \Big||1+x|^{p-1}(1+x)-px-1\Big|\leq C(|x|^2+|y|^2)\leq C(|x|^2+|x|^p). \quad (5.D.14)$$

as $1<p<2$. Now, for $|x|,|y|\geq\frac14$ one has:

$$\Big||1+x+y|^{p-1}(1+x+y)-p(x+y)-1\Big|\leq|1+x+y|^p+1+|x|+|y|\lesssim|x|^p+|y|^p\leq|x|^2+|y|^p \quad (5.D.15)$$

as $1<p<2$. (*5.D.14*) and (*5.D.15*) then imply (*5.D.3*).

- *Proof of (5.D.4) and (5.D.5).* They can be proved using the same arguments used in the proof of (*5.D.3*).

- *Proof of (5.D.6).* It can be proved using the same arguments of (*5.D.3*), using the fact that:

$$1<1+\frac{2}{d-2}<p<2.$$

- *Proof of (5.D.7).* First, by dividing everything by x^p (*5.D.7*) is equivalent to:

$$\Big||1+y_1|^{p-1}(1+y_1)-|1+y_2|^{p-1}(1+y_2)-p(y_1-y_2)\Big|\lesssim|y_1-y_2|(|y_1|^{p-1}+|y_2|^{p-1}). \quad (5.D.16)$$

If $|y_i| \geq \frac{1}{2}$ for $i = 1, 2$ and $|y_1| \geq 2|y_2|$ or $|y_1| \geq 2|y_2|$ (as the estimate is symmetric we assume $|y_1| \geq 2|y_2|$) then one has:

$$\begin{array}{rl} & \Big||1+y_1|^{p-1}(1+y_1)-|1+y_2|^{p-1}(1+y_2)-p(y_1-y_2)\Big| \\ = & \Big||1+y_1|^{p-1}(1+y_1)-|1+y_2|^{p-1}(1+y_2)-p((y_1+1)-(y_2+1))\Big| \\ \leq & |1+y_1|^p+|1+y_2|^p+p|1+y_1|+p|1+y_2| \leq 2|y_1|^p+2|y_2|^p+p|y_1|+p|y_2| \\ \leq & (4+2p)|y_1|^p \leq (8+4p)|y_1-y_2|(|y_1|^{p-1}+|y_2|^{p-1}). \end{array}$$

If $|y_i| \geq \frac{1}{2}$ for $i = 1, 2$ and $\frac{1}{2}|y_1| \leq |y_2| \leq 2|y_1|$, then using (*5.D.1*):

$$\begin{array}{rl} & \Big||1+y_1|^{p-1}(1+y_1)-|1+y_2|^{p-1}(1+y_2)-p(y_1-y_2)\Big| \\ = & \Big||1+y_2+(y_1-y_2)|^{p-1}(1+y_2+(y_1-y_2))-|1+y_2|^{p-1}(1+y_2) \\ & -p|1+y_2|^{p-1}(y_1-y_2)+p|1+y_2|^{p-1}(y_1-y_2)-p(y_1-y_2)\Big| \\ \leq & \Big||1+y_2+(y_1-y_2)|^{p-1}(1+y_2+(y_1-y_2))-|1+y_2|^{p-1}(1+y_2) \\ & -p|1+y_2|^{p-1}(y_1-y_2)\Big|+p|1+y_2|^{p-1}|y_1-y_2|+p|y_1-y_2| \\ \lesssim & |y_1-y_2|^p+|1+y_2|^{p-1}|y_1-y_2|+|y_1-y_2| \\ \lesssim & |y_1-y_2|(1+|y_1-y_2|^{p-1}+|1+y_2|^{p-1}) \\ \lesssim & |y_1-y_2|(1+|y_1|^{p-1}+|y_2|^{p-1}) \lesssim |y_1-y_2|(|y_1|^{p-1}+|y_2|^{p-1}). \end{array}$$

If $|y_i| \leq \frac{1}{2}$ for $i = 1, 2$, then as $g : x \mapsto |1+x|^{p-1}(1+x)$ is $\mathcal{C}^2$ on $[-\frac{1}{2}, \frac{1}{2}]$ there exists a constant $C > 0$ such that:

$$\begin{array}{rl} & ||1+y_1|^{p-1}(1+y_1)-|1+y_2|^{p-1}(1+y_2)-p(y_1-y_2)| \\ \leq & C(y_1-y_2)^2 \leq C|y_1-y_2|(|y_1|+|y_2|) \leq C|y_1-y_2|(|y_1|^{p-1}+|y_2|^{p-1}) \end{array}$$

as $0 < p-1 < 1$. The three above estimates in established in a partition of $\mathbb{R}^2$ in three zones, imply (*5.D.16*).

- ***Proof of*** (*5.D.8*). This can be proved using the same reasoning we did to prove (*5.D.1*).
- ***Proof of*** (*5.D.9*). It is a direct consequence of (*5.D.4*).
- ***Proof of*** (*5.D.10*). The function $g(x) = ||1+x|^{p+1}-1-(p+1)x|$ is smooth near 0 and one has $g(0) = g'(0) = 0$. Hence there exists $C > 0$ such that $|g(x)| \leq C|x|^2$ for $|x| \leq \frac{1}{2}$. As $|g(x)| \sim |x|^{p+1}$ as $x \to \pm\infty$, there exists a constant C' such that $|g(x)| \leq C'|x|^{p+1}$ for $|x| \geq \frac{1}{2}$. Therefore, for all $x \in \mathbb{R}$, $|g(x)| \leq C|x|^2 + C'|x|^{p+1}$ and (*5.D.10*) is proven.

□

5.E Parabolic estimates

This last section is devoted to the results coming from the parabolic properties of the equation. First we state some estimates on the heat kernel K_t defined in (*5.1.1*) in Lemma 5.E.1. We then recall the maximum principle in Lemma 5.E.1. Some parabolic estimates that are used several times in the chapter are stated in Lemma 5.E.3 and eventually we state some Strichartz-type inequalities in Lemma 5.E.5. We prove here most of the results for the sake of completeness, and refer to [65] and [90] for more informations about elliptic and parabolic equations.

Lemma 5.E.1 (Estimates for the heat kernel). *For any $d\in\mathbb{N}$ and $t>0$ one has:*

$$\forall j\in\mathbb{N},\ \forall q\in[1,+\infty],\quad \|\nabla^j K_t\|_{L^q}\leq \frac{C(d,j)}{t^{\frac{d}{2q'}+\frac{j}{2}}} \tag{5.E.1}$$

where q' is the Lebesgue conjugated exponent of q,

$$\forall y\in\mathbb{R}^d,\quad \frac{1}{|y|^{\frac14}}\int|\nabla K_t(x)-\nabla K_t(x+y)|\leq \frac{C(d)}{t^{\frac58}}, \tag{5.E.2}$$

and for $t'>t$:

$$\frac{1}{|t'-t|^{\frac14}}\int|\nabla K_t(x)-\nabla K_{t'}(x)|\leq \frac{C(d)}{t^{\frac34}}. \tag{5.E.3}$$

Proof of Lemma 5.E.1

(*5.E.1*) is a standard computation that we do not write here. To prove (*5.E.2*) we change variables and let $\tilde y=\frac{y}{2\sqrt t}$ and $\tilde x=\frac{x}{2\sqrt t}$:

$$\begin{array}{cl} & \frac{1}{|y|^{\frac14}}\int|\nabla K_t(x)-\nabla K_t(x+y)| \\ = & \frac{1}{|y|^{\frac14}}\frac{1}{Ct^{\frac d2+1}}\int|xe^{-\frac{|x|^2}{4t}}-(x+y)e^{\frac{-|x+y|^2}{4t}}|dx \\ = & \frac{1}{|\tilde y|^{\frac12}}\frac{1}{Ct^{\frac58}}\int|\tilde xe^{-|\tilde x|^2}-(\tilde x+\tilde y)e^{-|\tilde x+\tilde y|^2}|d\tilde x. \end{array}$$

Now, if $\tilde y\geq 1$, then:

$$\frac{1}{|\tilde y|^{\frac12}}\int|\tilde xe^{-|\tilde x|^2}-(\tilde x+\tilde y)e^{-|\tilde x+\tilde y|^2}|d\tilde x\leq 2\int|xe^{-|x|^2}|\leq C$$

and if $y\leq 1$:

$$\begin{array}{cl} & \frac{1}{|\tilde y|^{\frac12}}\int|\tilde xe^{-|\tilde x|^2}-(\tilde x+\tilde y)e^{-|\tilde x+\tilde y|^2}|d\tilde x \\ \leq & |\tilde y|^{-\frac12}\int|\tilde y|\sup\limits_{|x'-\tilde x|\leq 1}|\nabla(x'e^{-|x'|^2})|d\tilde x \\ \leq & |\tilde y|^{\frac12}\int 3(|\tilde x|+1)^2e^{-(|\tilde x|-1)^2}d\tilde x\leq C \end{array}$$

The three previous equations then imply (*5.E.2*). To prove (*5.E.3*) we change variables and let $\tilde x=\frac{x}{2\sqrt t}$ and $\tilde t=\frac{t'}{t}$:

$$\begin{array}{cl} & \frac{1}{|t'-t|^{\frac14}}\int|\nabla K_t(x)-\nabla K_{t'}(x)|dx \\ \leq & \frac{1}{t^{\frac34}}\frac{1}{(\tilde t-1)^{\frac14}}\int_{\mathbb{R}^d}|\tilde xe^{-|\tilde x|^2}-\frac{\tilde xe^{-\frac{|\tilde x|^2}{\tilde t}}}{\tilde t^{\frac d2+1}}|d\tilde x \end{array}$$

Now, $\frac{1}{(\tilde t-1)^{\frac14}}\int_{\mathbb{R}^d}|\tilde xe^{-|\tilde x|^2}|-\frac{\tilde xe^{-\frac{|\tilde x|^2}{\tilde t}}}{\tilde t^{\frac d2+1}}|d\tilde x\to 0$ as $\tilde t\to+\infty$ and $\tilde t\to 1$ (using Lebesgue differentiation theorem), hence this quantity is bounded, and the above estimate implies (*5.E.3*).

□

We now state a comparison principle adapted to our problem.

Lemma 5.E.2 (Comparison principle). *Let u be a solution of (NLH) given by Proposition 5.2.1 on $[0,T)$ with $u_0 \in W^{2,\infty} \cap \dot{H}^1$ and $u_0 \geq 0$. Then:*

(i) $u(t) \geq 0$ for all $t \in [0,T)$.

(ii) If $\partial_t u(0) \geq 0$ (resp. $\partial_t u(0) \leq 0$) then $\partial_t u(t) \geq 0$ (resp. $\partial_t u(t) \leq 0$) for all $t \in [0,T)$.

(iii) If $u(0) \leq Q$ (resp. $u(0) \geq Q$) then $u(t) \leq Q$ (resp. $u(t) \geq Q$) for all $t \in [0,T)$.

Proof.

We do not do a detailed proof here, just sketch the main arguments.
To prove (i), one notices that the semi-group $(K_t * \cdot)_{t\geq 0}$ and the nonlinearity $f(\cdot)$ preserve the cones of positive and negative functions and hence so does the solution mapping of (NLH). Now, to prove (ii), one notices that $\partial_t u$ solves $\partial_t(\partial_t u) = \Delta \partial_t u + pu^{p-1}\partial_t u$ which is again a parabolic equation with a force term $pu^{p-1}\cdot$ that preserves the cones of positive and negative functions as u is positive from (i). The same argument as in (i) then yields that $\partial_t u$ stays positive (resp. negative) if it is so initially.

To prove (iii), notice that the difference $u(t)-Q$ solves $\partial_t(u-Q) = \Delta(u-Q)+f(Q+(u-Q))-f(Q)$ as Q is a solution of (NLH). The force term, again, preserves the cones of positive and negative functions: if $u-Q \geq 0$ (resp. $u-Q \leq 0$ then $f(Q+(u-Q))-f(Q) \geq 0$ (resp. $f(Q+(u-Q))-f(Q) \leq 0$) as f is increasing on $\mathbb{R}$. For the same arguments $u(t)-Q$ then has to stay positive (resp. negative) if it is so initially.

$\square$

We now state some estimates that we use several time in the chapter. The main purpose is to propagate some pointwise in time space-averaged exponential bounds at a regularity level to higher regularity levels.

Lemma 5.E.3 (Parabolic estimates). *Let $\mu \geq 0$ and $I = (t_0, t_1)$, with $-\infty \leq t_0 \leq t_0 + 1 < t_1 < +\infty$. There exists $C > 0$ such that for $0 < \delta \lesssim \min(1, e^{-\mu t})$, for any u solution of (NLH) on (t_0, t_1) of the form:*

$$u(t) = Q + v \tag{5.E.4}$$

satisfying for some $q \geq \frac{2d}{d-4}$ for any $t \in I$

$$\|v\|_{L^q} \leq \delta e^{\mu t} \tag{5.E.5}$$

there holds for any[12] $t \in (t_0 + \tilde{t}, t_1)$:

$$\|v\|_{W^{2,\infty}(\mathbb{R}^d)} + \|\partial_t v\|_{L^\infty(\mathbb{R}^d)} \leq C\delta e^{\mu t}, \tag{5.E.6}$$

and if $\mu > 0$ or $t_0 \neq -\infty$ for $C'(t_1) > 0$:

$$\|\nabla^2 v\|_{C^{0,\frac14}((t_0+\tilde{t},t_1)\times\mathbb{R}^d)} + \|\partial_t v\|_{C^{0,\frac14}((t_0+\tilde{t},t_1)\times\mathbb{R}^d)} \leq C'. \tag{5.E.7}$$

where $C^{0,\frac14}$ denotes the Hölder $\frac14$-norm.

The first step to prove Lemma 5.E.3 is to obtain the L^∞ bound, which is the purpose of the following lemma.

[12] With the convention that $-\infty + \tilde{t} = -\infty$.

Lemma 5.E.4 (Parabolic bootstrap). *There exists $\nu>0$ such that the following holds. Let $I=(t_0,t_1)$, with $-\infty\leq t_0\leq t_0+1<t_1<+\infty$ and $\mu\geq 0$. For any $0<\tilde t<1$ there exists $\delta^*=\delta(t_1,\tilde t)>0$ and $C=C(\tilde t)>0$ such that for any $0<\delta<\delta^*$ and u solution of (NLH) of the form:*

$$u(t)=Q+v \tag{5.E.8}$$

satisfying for some $q\geq \frac{2d}{d-4}$ for any $t\in I$:

$$\|v\|_{L^q}\leq \delta e^{\mu t} \tag{5.E.9}$$

there holds for any[13] $t\in(t_0+\tilde t,t_1)$:

(i) If $q\leq pd$,

$$\|v\|_{L^{(1+\nu)q}}\leq C\delta e^{\mu t}. \tag{5.E.10}$$

(i) If $q>pd$,

$$\|v\|_{L^\infty}\leq \delta e^{\mu t}. \tag{5.E.11}$$

Proof of Lemma 5.E.4

We take $\nu=\frac{1}{d^2}$ and $\delta^*=\min(1,e^{\mu t_1})$. From (*5.E.8*) and (NLH) the evolution of v is given by:

$$v_t=\Delta v-Vv+NL,\quad NL:=f(Q+v)-f(Q)-vf'(Q) \tag{5.E.12}$$

which implies using Duhamel formula that for any $t\in(t_0+\tilde t,t_1)$

$$v(t)=K_{\tilde t}*v(t-\tilde t)+\int_0^{\tilde t}K_{\tilde t-s}*[-Vv(t-\tilde t+s)+NL(t-\tilde t+s)]ds. \tag{5.E.13}$$

We now estimate each term in the above identity.

step 1 Case $\frac{2d}{d-4}\leq q\leq pd$. Using Young inequality for convolution, from (*5.E.9*) and (*5.E.1*) for the free evolution term:

$$\|K_{\tilde t}*v(t-\tilde t)\|_{L^{(1+\nu)q}}\leq C(\tilde t)\delta e^{\mu(t-\tilde t)}\leq C(\tilde t)\delta e^{\mu t}.$$

For the linear force term associated to the potential, using (*5.E.9*), Young inequality and (*5.E.1*):

$$\begin{array}{rl} & \|\int_0^{\tilde t}K_{\tilde t-s}*(-Vv(t-\tilde t+s))ds\|_{L^{(1+\nu)q}} \\ \leq & \int_0^{\tilde t}\|K_{\tilde t-s}\|_{L^{\frac{1}{1+\frac{1}{(1+\nu)q}-\frac1q}}}\|v(t-\tilde t+s))ds\|_{L^q}\|V\|_{L^\infty}ds \\ \leq & C\int_0^{\tilde t}\frac{1}{|\tilde t-s|^{\frac{d}{2q}\frac{\nu}{1+\nu}}}\delta e^{\mu(t-\tilde t+s)}ds\leq C\delta e^{\mu t}\end{array}$$

because $\frac{d}{2q}\frac{\nu}{1+\nu}<\frac14$ as $\nu=\frac{1}{d^2}$ and $q\geq\frac{2d}{d-4}$, and $0<\tilde t\leq 1$. For the nonlinear term, using (*5.E.9*), Young inequality, (*5.E.1*) and (*5.D.1*):

$$\begin{array}{rl} & \|\int_0^{\tilde t}K_{\tilde t-s}*(NL(t-\tilde t+s))ds\|_{L^{(1+\nu)q}} \\ \leq & C\int_0^{\tilde t}\|K_{\tilde t-s}\|_{L^{\frac{1}{1+\frac{1}{(1+\nu)q}-\frac pq}}}\|NL(t-\tilde t+s))ds\|_{L^{\frac qp}}ds \\ \leq & C\int_0^{\tilde t}\|K_{\tilde t-s}\|_{L^{\frac{1}{1+\frac{1}{(1+\nu)q}-\frac pq}}}\|v(t-\tilde t+s))ds\|_{L^q}^p ds \\ \leq & C\int_0^{\tilde t}\frac{1}{|\tilde t-s|^{\frac{d}{2q}\frac{p-1+p\nu}{1+\nu}}}\delta^p e^{p\mu(t-\tilde t+s)}ds \\ \leq & C\int_0^{\tilde t}\frac{1}{|\tilde t-s|^{\frac{d}{2q}\frac{p-1+p\nu}{1+\nu}}}\delta^p e^{p\mu(t-\tilde t+s)}ds\leq C\delta e^{\mu t}\end{array}$$

[13] With the convention that $-\infty+\tilde t=-\infty$.

from the fact that $\frac{d}{2q}\frac{p-1+p\nu}{1+\nu}\leq \kappa(d)<1$ as $\nu=\frac{1}{d^2}$ and $q\geq \frac{2d}{d-4}$, and $0<\delta\leq 1$. The three previous estimates then imply (*5.E.10*).

step 2 Case $q>pd$. Using Hölder inequality, (*5.E.9*) and (*5.E.1*) for the first term:

$$\|K_{\tilde{t}}*v(t-\tilde{t})\|_{L^\infty}\leq C(\tilde{t})\delta e^{\mu(t-\tilde{t})}\leq C(\tilde{t})\delta e^{\mu t}.$$

For the second, using (*5.E.9*), Hölder inequality and (*5.E.1*):

$$\begin{array}{rcl} \| \int_0^{\tilde{t}} K_{\tilde{t}-s}*(-Vv(t-\tilde{t}+s))ds\|_{L^\infty} & \leq & C\int_0^{\tilde{t}}\|K_{\tilde{t}-s}\|_{L^{q'}}\|v(t-\tilde{t}+s))ds\|_{L^q}ds \\ & \leq & C\int_0^{\tilde{t}}\frac{1}{|\tilde{t}-s|^{\frac{d}{2q}}}\delta e^{\mu(t-\tilde{t}+s)}ds\leq C\delta e^{\mu t} \end{array}$$

from the fact that $\frac{d}{2q}<\frac12$. For the nonlinear term, using again (*5.E.9*), Hölder inequality, (*5.E.1*) and (*5.D.1*):

$$\begin{array}{rcl} \| \int_0^{\tilde{t}} K_{\tilde{t}-s}*(NL(t-\tilde{t}+s))ds\|_{L^\infty} & \leq & C\int_0^{\tilde{t}}\|K_{\tilde{t}-s}\|_{L^{(\frac{q}{p})'}}\|v(t-\tilde{t}+s))ds\|_{L^q}^p ds \\ & \leq & C\int_0^{\tilde{t}}\frac{1}{|\tilde{t}-s|^{\frac{dp}{2q}}}\delta^p e^{p\mu(t-\tilde{t}+s)}ds \\ & \leq & C\int_0^{\tilde{t}}\frac{1}{|\tilde{t}-s|^{\frac{dp}{2q}}}\delta^p e^{p\mu(t-\tilde{t}+s)}ds\leq C(\tilde{t})\delta e^{\mu t} \end{array}$$

from the fact that $\frac{dp}{2q}<\frac12$. The three above estimates give (*5.E.11*).

□

We can now end the proof of Lemma 5.E.3.

Proof of Lemma 5.E.3

First, iterating several times Lemma 5.E.4 one obtains that for any $t\in(t_0+\frac{\tilde{t}}{4},t_1)$:

$$\|v\|_{L^\infty}\leq C\delta e^{\mu t}. \tag{5.E.14}$$

We define for $t\in(t_0,t_1)$:

$$F(t)=-Vv(t)+NL(t),\quad NL:=f(Q+v)-f(Q)-vf'(Q) \tag{5.E.15}$$

so that v solves $v_t=\Delta v+F(t)$ and the Duhamel formula writes:

$$v(t+t')=K_{t'}*v(t)+\int_0^{t'}K_{t'-s}*F(t+s)ds. \tag{5.E.16}$$

step 1 Proof of the $W^{1,\infty}$ bound. From (*5.E.14*), (*5.E.15*), (*5.D.1*) one has that for $t\in]t_0+\frac{\tilde{t}}{4},t_1)$:

$$\|F\|_{L^\infty}\leq C(\delta e^{\mu t}+\delta^p e^{\mu t})\leq C\delta e^{\mu t}.$$

Using this, (*5.E.14*), (*5.E.16*), (*5.D.1*), (*5.E.1*) and Hölder inequality one has that for $t\in(t_0+\frac{\tilde{t}}{2},t_1)$:

$$\begin{array}{rcl} \|\nabla v(t)\|_{L^\infty} & \leq & \|\nabla K_{\frac{\tilde{t}}{4}}*v(t-\frac{\tilde{t}}{4})\|_{L^\infty}+\int_0^{\frac{\tilde{t}}{4}}\|\nabla K_{\frac{\tilde{t}}{4}-s}\|_{L^1}\|F(t-\frac{\tilde{t}}{4}+s)\|_{L^\infty}ds \\ & \leq & C(\tilde{t})e^{\mu t}+\int_0^{\frac{\tilde{t}}{4}}C\frac{C(\delta e^{\mu t}+\delta^p e^{p\mu t})}{|\frac{\tilde{t}}{4}-s|^{\frac12}}\leq C(\tilde{t})\delta e^{\mu t}. \end{array}$$

which with (*5.E.14*) means that on $t\in(t_0+\frac{\tilde t}{2},t_1)$

$$\|v\|_{W^{1,\infty}}\leq C\delta e^{\mu t}. \qquad (5.E.17)$$

step 2 Proof of the $W^{2,\infty}$ bound. From (*5.E.15*) one has:

$$\nabla(NL)=p\left(|Q+v|^{p-1}-Q^{p-1}\right)\nabla v+p\left(|Q+v|^{p-1}-Q^{p-1}-(p-1)Q^{p-2}v\right)\nabla Q$$

This, (*5.E.17*) and (*5.E.15*) then imply that for $t\in(t_0+\frac{\tilde t}{2},t_1)$:

$$\|\nabla F\|_{L^\infty}\leq C(\delta e^{\mu t}+\delta^p e^{\mu t})\leq C\delta e^{\mu t}. \qquad (5.E.18)$$

One then computes from (*5.E.16*), (*5.E.17*), Hölder inequality, (*5.E.1*) and the above estimate that for $t\in(t_0+\frac{3\tilde t}{4},t_1)$ and $1\leq i,j\leq d$:

$$\begin{array}{rcl}\|\partial_{x_ix_j}v(t)\|_{L^\infty} & \leq & \|\partial_{x_i}K_{\frac{\tilde t}{4}}*\partial_{x_j}v(t-\frac{\tilde t}{4})\|_{L^\infty}\\ & & +\int_0^{\frac{\tilde t}{4}}\|\partial_{x_i}K_{\frac{\tilde t}{4}-s}\|_{L^1}\|\partial_{x_j}F(t-\frac{\tilde t}{4}+s)\|_{L^\infty}ds\\ & \leq & C(\tilde t)\delta e^{\mu t}+\int_0^{\frac{\tilde t}{4}}\frac{C(\delta e^{\mu(t-\frac{\tilde t}{4}+s)}+\delta^p e^{p\mu(t-\frac{\tilde t}{4}+s)})}{|\frac{\tilde t}{4}-s|^{\frac12}}\leq C(\tilde t)e^{\mu t}\end{array}$$

which with (*5.E.17*) means that on $t\in(t_0+\frac{3\tilde t}{4},t_1)$

$$\|v\|_{W^{2,\infty}}\leq C\delta e^{\mu t}. \qquad (5.E.19)$$

step 3 Proof of the Hölder bound. For t fixed in $(t_0+\tilde t,t_1)$ and $1\leq i,j\leq d$, using (*5.E.18*) and one computes that:

$$\begin{array}{rl} & \|\partial_{x_ix_j}v\|_{C^{0,\frac14}(\{t\}\times\mathbb{R})}\\ \leq & \sup_{y,x\in\mathbb{R}^d}|y|^{-\frac14}\left|\int_0^{\frac{\tilde t}{4}}\int_{\mathbb{R}^d}(\nabla K_{\frac{\tilde t}{4}-s}(x+y+z)-\nabla K_{\frac{\tilde t}{4}-s}(x+z))\nabla F(t-\frac{\tilde t}{4}+s)dz\right|\\ & +\|\partial_{x_i}K_{\frac{\tilde t}{4}}*\partial_{x_j}v(t-\frac{\tilde t}{4})\|_{C^{0,\frac14}(\mathbb{R}^d)}\\ \leq & \int_0^{\frac{\tilde t}{4}}\|\nabla F(t-\frac{\tilde t}{4}+s)\|_{L^\infty}\sup_{y,x\in\mathbb{R}^d}|y|^{-\frac14}\int_{\mathbb{R}^d}|\nabla K_{\frac{\tilde t}{4}-s}(y+z)-\nabla K_{\frac{\tilde t}{4}-s}(z)|dzds\\ & +C(\tilde t)\delta e^{\mu t}\\ \leq & \int_0^{\frac{\tilde t}{4}}\frac{C\delta e^{\mu t'}}{|t-t'|^{\frac34}}dt'+C(\tilde t)\delta e^{\mu t}\leq C(\tilde t)\delta e^{\mu t}.\end{array}$$

Fix $x\in\mathbb{R}^d$ and $t_0+\tilde t<t<t'<t_1$. We treat the case $\mu>0$, the cases $\mu=0$ and $t_0\neq+\infty$ being similar. Using (*5.E.18*), Hölder inequality, (*5.E.1*), (*5.E.3*) and (*5.E.16*):

$$\begin{array}{rl} & \frac{|\partial_{x_ix_j}v(t',x)-\partial_{x_ix_j}v(t,x)|}{|t'-t|^{\frac14}}\\ \leq & \frac{1}{|t'-t|^{\frac14}}\left|(\partial_{x_i}K_{t'-t+\frac{\tilde t}{4}}-\partial_{x_i}K_{\frac{\tilde t}{4}})*(\partial_{x_j}v(t-\frac{\tilde t}{4}))\right|\\ & +\frac{1}{|t'-t|^{\frac14}}\left|\int_t^{t'}\partial_{x_i}K_{t'-s}*\partial_{x_j}F(s)ds\right|\\ & +\frac{1}{|t'-t|^{\frac14}}\left|\int_{t-\frac{\tilde t}{4}}^t(\partial_{x_i}K_{t'-s}-\partial_{x_i}K_{t-s})*\partial_{x_j}F(s)ds\right|\\ \leq & \frac{1}{|t'-t|^{\frac14}}\|\partial_{x_i}K_{t'-t+\frac{\tilde t}{4}}-\partial_{x_i}K_{\frac{\tilde t}{4}}\|_{L^1}\|\partial_{x_j}v(t-\frac{\tilde t}{4})\|_{L^\infty}\\ & +\frac{1}{|t'-t|^{\frac14}}\int_t^{t'}\|\partial_{x_i}K_{t'-s}\|_{L^1}\|\partial_{x_j}F(s)\|_{L^\infty}ds\\ & +\int_{t-\frac{\tilde t}{4}}^t\frac{1}{|t'-t|^{\frac14}}\|\partial_{x_i}K_{t'-s}-\partial_{x_i}K_{t-s}\|_{L^1}\|\partial_{x_j}F(s)\|_{L^\infty}ds\\ \leq & \frac{C}{\tilde t^{\frac34}}\delta e^{\mu t}+\frac{1}{|t'-t|^{\frac14}}\int_t^{t'}\frac{C}{|t'-s|^{\frac12}}\delta e^{\mu s}ds+\int_{t-\frac{\tilde t}{4}}^t\frac{1}{|t-s|^{\frac34}}C\delta e^{\mu s}ds\leq\delta C(\tilde t,t_1).\end{array}$$

The two above bounds imply that:

$$\|\nabla^2 v\|_{C^{0,\frac{1}{4}}((t_0+\tilde{t},t_1)\times\mathbb{R}^d)} \leq \delta C(\tilde{t}).$$

The above bound and (*5.E.19*), as v_t is related to $\nabla^2 v$ via (*5.E.12*), implies the desired bound (*5.E.7*).

□

We end this section with some Strichartz type estimates for the heat equation. Though such estimates are more often used in the context of dispersive equations, the present chapter deals with an energy critical equation in the energy space, and Strichartz type norms appear naturally as in (*5.4.14*). We do not write here a proof of the following Lemma 5.E.5 and refer to the proof of Theorem 8.18 in [6]. Let $d \geq 2$. We say that a couple of real numbers (q,r) is admissible if they satisfy:

$$q,r \geq 2, \ (q,r,d) \neq (2,+\infty,2) \text{ and } \frac{2}{q}+\frac{d}{r}=\frac{d}{2}. \tag{5.E.20}$$

For any exponent $p \geq 1$, we denote by $p' = \frac{p-1}{p}$ its Lebesgue conjugated exponent.

Lemma 5.E.5 (Strichartz type estimates for solutions to the heat equation). *Let $d \geq 2$ be an integer. The following inequalities hold.*

(i) The homogeneous case. *For any couple (q,r) satisfying (5.E.20) there exists constant $C = C(d,q) > 0$ such that for any initial datum $u \in L^2(\mathbb{R}^d)$:*

$$\|K_t * u\|_{L^q([0,+\infty),L^r(\mathbb{R}^d))} \leq C\|u\|_{L^2(\mathbb{R}^d)}. \tag{5.E.21}$$

(ii) The inhomogeneous case. *For any couples (q_1,r_1), (q_2,r_2) satisfying (5.E.20) there exists a constant $C = C(d,q_1,q_2)$ such that for any source term $f \in L^{q_2'}([0,+\infty),L^{r_2'}(\mathbb{R}^d))$:*

$$\left\| t \mapsto \int_0^t K_{t-t'} * f(t')dt' \right\|_{L^{q_1}([0,+\infty),L^{r_1}(\mathbb{R}^d))} \leq C\|f\|_{L^{q_2'}([0,+\infty),L^{r_2'}(\mathbb{R}^d))}. \tag{5.E.22}$$

6

On the stability of non-constant self-similar solutions for the supercritical heat equation

6.1 Introduction

In this chapter, we give a complete proof of Theorem 2.4.4. We recall that this result was introduced with related results in Section 2.4 of Chapter 2 and a sketch of this proof was given in Subsection 2.4.2. It has been done in collaboration with P. Raphaël and J. Szeftel and has been accepted for publication in Memoirs of the American Mathematical Society. The chapter is divided in two parts.

We first revisit the construction of self similar blow up solutions of [153, 19] and implement an abstract bifurcation argument which relies on the sole existence of the stationary profile Q given by (*2.2.1*). Note that this kind of argument is classical in the ODE literature, see for example [18, 33, 28], and relies on the oscillatory nature of the eigenfunctions of the linearized operator close to Φ^* for $p < p_{JL}$. The first result is then a rigorous proof of Proposition 2.4.5 stated in Subsection 2.4.2. For this purpose, in Section 6.2 we construct the family of self similar solutions Φ_n using a nonlinear matching argument. The argument is classical, but requires a careful track of various estimates to obtain the sharp bounds (*2.4.9*), (*2.4.10*). This gives the proof of the existence part of the result in Theorem 2.4.4. Then in section 6.3, we show how these bounds coupled with Sturm-Liouville like arguments allow for a sharp counting of the number of instabilities of the linearized operator close to Φ_n which is self adjoint against the confining measure $\rho(y)dy$, Proposition 6.3.1.

We then study the stability of the self-similar solution Φ_n in a second part. In section 6.4, we turn to the heart of the dynamical argument and show how the spectral estimates in the weighted space coupled with the control of the super critical $\dot{H}^2$ norm design a stability zone for well localized initial data. This gives the proof of the stability part of the result in Theorem 2.4.4.

Notations

From now on and for the rest of this chapter we fix

$$d = 3, \quad p > 5.$$

The homogeneous and singular radial self-similar solution is

$$\Phi^*(x) := \frac{c_\infty}{|x|^{\frac{2}{p-1}}} \tag{6.1.1}$$

where c_∞ is defined in (*2.2.4*). *The ground state expansion.* We let $Q(r)$ denote the unique radially symmetric solution to

$$\begin{cases} Q'' + \frac{2}{r}Q' + Q^p = 0, \\ Q(0) = 1, \quad Q'(0) = 0, \end{cases}$$

which asymptotic behavior at infinity is from standard ODE argument[1] given by

$$Q(r) = (1 + o_{r\to+\infty}(1))\Phi_*(r).$$

We already gave some properties of Q in Lemma 2.2.2 but we need some more complete details that we give here. The next term is this expansion relates to the p_{JL} exponent (*1.4.1*) which is infinite in dimension $d = 3$. Hence the quadratic polynomial

$$\gamma^2 - \gamma + pc_\infty^{p-1} = 0$$

has complex roots

$$\gamma = \frac{1}{2} \pm i\omega, \quad \Delta := 1 - 4pc_\infty^{p-1} < 0, \quad \omega := \frac{\sqrt{-\Delta}}{2} \tag{6.1.2}$$

and the asymptotic behavior of Q may be precised[2]:

$$Q(r) = \Phi_*(r) + \frac{c_1 \sin(\omega \log(r) + c_2)}{r^{\frac{1}{2}}} + o\left(\frac{1}{r^{\frac{1}{2}}}\right) \text{ as } r \to +\infty \tag{6.1.3}$$

where $c_1 \neq 0$ and $c_2 \in \mathbb{R}$. Note that

$$\frac{1}{2} - \frac{2}{p-1} = s_c - 1 > 0$$

so that the second term in the expansion of Q is indeed a correction term.

Weighted spaces. We define the derivation operator

$$D^k := \begin{cases} \Delta^m & \text{for } m = 2k, \\ \nabla\Delta^k & \text{for } m = 2k+1. \end{cases}$$

We define the scalar product

$$(f,g)_\rho = \int_{\mathbb{R}^3} f(x)g(x)\rho dx, \quad \rho = e^{-\frac{|x|^2}{2}} \tag{6.1.4}$$

and let L^2_ρ be the corresponded weighted L^2 space. We let H^k_ρ be the completion of $\mathcal{C}^\infty_c(\mathbb{R}^d)$ for the norm

$$\|u\|_{H^k_\rho} = \sqrt{\sum_{j=0}^{k} \|D^j u\|^2_{L^2_\rho}}.$$

Linearized operators. The scaling semi-group on functions $u : \mathbb{R}^d \to \mathbb{R}$:

$$u_\lambda(x) := \lambda^{\frac{2}{p-1}} u(\lambda x) \tag{6.1.5}$$

has for infinitesimal generator the linear operator

$$\Lambda u := \frac{2}{p-1}u + x.\nabla u = \frac{\partial}{\partial\lambda}(u_\lambda)_{|\lambda=1}.$$

[1] see [35, 77, 93].
[2] see [35, 77, 93].

We define the linearized operator corresponding to (*2.4.1*) around respectively Φ_* and Φ_n by

$$\mathcal{L}_\infty := -\Delta + \Lambda - \frac{pc_\infty^{p-1}}{r^2}, \quad \mathcal{L}_n := -\Delta + \Lambda - p\Phi_n^{p-1}$$

and their projection onto spherical harmonics:

$$\begin{aligned}
\mathcal{L}_{\infty,m} &:= -\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{p-1} + r\partial_r + \frac{m(m+1)}{r^2} - p\Phi_*^{p-1}, \quad m \in \mathbb{N},\\
\mathcal{L}_{n,m} &:= -\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{p-1} + r\partial_r + \frac{m(m+1)}{r^2} - p\Phi_n^{p-1}, \quad m \in \mathbb{N}.
\end{aligned}$$

Note that $\mathcal{L}_\infty$ is formally self adjoint for the L^2_ρ scalar product but (*6.1.2*) implies that the associated quadratic form is not bounded from below[3] on H^1_ρ. We similarly define the linearized operator corresponding to (*2.2.1*) around Q:

$$\begin{aligned}
H &:= -\Delta - pQ^{p-1}\\
H_m &:= -\partial_{rr} - \frac{2}{r}\partial_r + \frac{m(m+1)}{r^2} - pQ^{p-1}, \quad m \in \mathbb{N}.
\end{aligned}$$

and again H is not bounded from below on $\dot{H}^1$.

General notation. We let $\chi(x)$ denote a smooth radially symmetric function with

$$\chi(x) := \begin{cases} 1 & \text{for } |x| \le \frac{1}{4}, \\ 0 & \text{for } |x| \ge \frac{1}{2}, \end{cases}$$

and for $A > 0$ (note the difference with (*6.1.5*)),

$$\chi_A(x) = \chi\left(\frac{x}{A}\right).$$

6.2 Construction of self-similar profiles

Our aim in this section is to construct radially symmetric solutions to the self similar equation

$$\Delta v - \Lambda v + v^p = 0, \tag{6.2.1}$$

by using the classical strategy of gluing solutions which behave like Φ_* at infinity, and like Q at the origin. As in [8, 33, 18], the matching is made possible by the oscillatory behaviour (*6.1.3*) for $p < p_{JL}$. The strength of this approach it that it relies on the implicit function theorem and not on fine monotonicity properties, and in this sense it goes far beyond the scalar parabolic setting, see for example [81] for a deeply related approach. The sharp control of the obtained solution (*2.4.9*), (*2.4.10*) will allow us to control the eigenvalues of the associated linearized operator in suitable exponentially weighted spaces, see Proposition 6.3.1.

[3] this is a limit point circle case as $r \to 0$, [143].

6.2.1 Exterior solutions

Recall that Φ_* given by (*6.1.1*) is a solution to (*6.2.1*) on $(0,+\infty)$. Our aim in this section is to construct the full family of solutions to (*6.2.1*) on $[r_0,+\infty)$ for some small $r_0>0$ with the suitable behaviour at infinity. The argument is a simple application of the implicit function theorem and continuity properties of the resolvent of $\mathcal{L}_\infty$ in suitable weighted spaces.

Given $0<r_0<1$, we define X_{r_0} as the space of functions on $(r_0,+\infty)$ such that the following norm is finite

$$\|w\|_{X_{r_0}}=\sup_{r_0\le r\le 1} r^{\frac12}|w|+\sup_{r\ge 1} r^{\frac{2}{p-1}+2}|w|.$$

Lemma 6.2.1 (Outer resolvent of $\mathcal{L}_\infty$). 1. Basis of fundamental solutions*: there exists two solutions ψ_1 and ψ_2 of*

$$\mathcal{L}_\infty(\psi_j)=0 \textit{ for } j=1,2 \quad \textit{on} \quad (0,+\infty) \tag{6.2.2}$$

with the following asymptotic behavior:

$$\psi_1=\frac{1}{r^{\frac{2}{p-1}}}\left(1+O\left(\frac{1}{r^2}\right)\right),\ \psi_2=r^{\frac{2}{p-1}-3}e^{\frac{r^2}{2}}\left(1+O\left(\frac{1}{r^2}\right)\right),\ \textit{as } r\to+\infty \tag{6.2.3}$$

and

$$\psi_1=\frac{c_3\sin\left(\omega\log(r)+c_4\right)}{r^{\frac12}}+O\left(r^{\frac32}\right),\ \psi_2=\frac{c_5\sin\left(\omega\log(r)+c_6\right)}{r^{\frac12}}+O\left(r^{\frac32}\right)\ \textit{as } r\to 0 \tag{6.2.4}$$

where $c_3,c_5\neq 0$ and $c_4,c_6\in\mathbb{R}$. Moreover, there exists $c\neq 0$ such that

$$\Lambda\psi_1=\frac{c}{r^{\frac{2}{p-1}+2}}\left(1+O\left(\frac{1}{r^2}\right)\right)\ \textit{as } r\to+\infty. \tag{6.2.5}$$

2. Continuity of the resolvent*: let the inverse*

$$\mathcal{T}(f)=\left(\int_r^{+\infty} f\psi_2 r'^2 e^{-\frac{r'^2}{2}}dr'\right)\psi_1-\left(\int_r^{+\infty} f\psi_1 r'^2 e^{-\frac{r'^2}{2}}dr'\right)\psi_2,$$

then

$$\mathcal{L}_\infty(\mathcal{T}(f))=f$$

and

$$\|\mathcal{T}(f)\|_{X_{r_0}}\lesssim\int_{r_0}^1|f|r'^{\frac32}dr'+\sup_{r\ge1} r^{\frac{2}{p-1}+2}|f|. \tag{6.2.6}$$

Proof. The proof is classical and we sketch the details for the reader's convenience.

step 1 Basis of homogeneous solutions. Recall (*6.1.2*). Let the change of variable and unknown

$$\psi(r)=\frac{1}{y^{\frac{\gamma}{2}}}\phi(y),\quad y=r^2,$$

then

$$\partial_r=2r\partial_y,\quad \partial_r^2=4r\partial_y(r\partial_y)=4r^2\partial_y^2+4r\partial_y(r)\partial_y=4y\partial_y^2+2\partial_y,\quad r\partial_r=2y\partial_y.$$

This yields

$$\mathcal{L}_\infty(\psi) = \left(-4y\partial_y^2 - 2\partial_y - 4\partial_y + \frac{2}{p-1} + 2y\partial_y - \frac{pc_\infty^{p-1}}{y}\right)\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi(y)\right).$$

Since

$$\begin{aligned}\partial_y\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi(y)\right) &= \frac{1}{y^{\frac{\gamma}{2}}}\phi'(y) - \frac{\gamma}{2y^{\frac{\gamma}{2}+1}}\phi(y),\\ \partial_y^2\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi(y)\right) &= \frac{1}{y^{\frac{\gamma}{2}}}\phi''(y) - \frac{\gamma}{y^{\frac{\gamma}{2}+1}}\phi'(y) + \frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{1}{y^{\frac{\gamma}{2}+2}}\phi(y),\end{aligned}$$

we infer

$$\begin{aligned}\mathcal{L}_\infty(\psi) &= \Bigg\{-4y\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi''(y) - \frac{\gamma}{y^{\frac{\gamma}{2}+1}}\phi'(y) + \frac{\gamma}{2}\left(\frac{\gamma}{2}+1\right)\frac{1}{y^{\frac{\gamma}{2}+2}}\phi(y)\right)\\ &+ (-6+2y)\left(\frac{1}{y^{\frac{\gamma}{2}}}\phi'(y) - \frac{\gamma}{2y^{\frac{\gamma}{2}+1}}\phi(y)\right) + \left(\frac{2}{p-1} - \frac{pc_\infty^{p-1}}{y}\right)\frac{1}{y^{\frac{\gamma}{2}}}\phi(y)\Bigg\}\\ &= \frac{1}{y^{\frac{\gamma}{2}}}\Bigg\{-4y\phi''(y) + \Big(4\gamma - 6 + 2y\Big)\phi'(y)\\ &+ \left(\frac{2}{p-1} - \gamma + \Big(3\gamma - \gamma(\gamma+2) - pc_\infty^{p-1}\Big)\frac{1}{y}\right)\phi(y)\Bigg\}.\end{aligned}$$

Since γ satisfies

$$\gamma^2 - \gamma + pc_\infty^{p-1} = 0,$$

we infer

$$\mathcal{L}_\infty(\psi) = -\frac{4}{y^{\frac{\gamma}{2}}}\left\{y\phi''(y) + \left(-\gamma + \frac{3}{2} - \frac{y}{2}\right)\phi'(y) + \frac{1}{4}\left(-\frac{2}{p-1} + \gamma\right)\phi(y)\right\}.$$

We change again variable by setting

$$\phi(y) = w(z), \quad z = \frac{y}{2}.$$

We have

$$\phi'(y) = \frac{1}{2}w'(z), \quad \phi''(y) = \frac{1}{4}w''(z)$$

and obtain

$$\mathcal{L}_\infty(\psi) = -\frac{2}{y^{\frac{\gamma}{2}}}\left(zw''(z) + \left(-\gamma + \frac{3}{2} - z\right)w'(z) - \left(\frac{1}{p-1} - \frac{\gamma}{2}\right)w(z)\right).$$

Thus, $\mathcal{L}_\infty(\psi) = 0$ if and only if

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0 \tag{6.2.7}$$

where we have used the notations

$$a = \frac{1}{p-1} - \frac{\gamma}{2}, \quad b = -\gamma + \frac{3}{2}. \tag{6.2.8}$$

(*6.2.7*) is known as Kummer's equation. As long as a is not a negative integer - which holds in particular for our choice of a in (*6.2.8*) -, a basis of solutions to Kummer's equation consists of the Kummer's function $M(a,b,z)$ and the Tricomi function $U(a,b,z)$. These special functions have the following asymptotic behavior for $z \geq 0$ (see for example [135])

$$M(a,b,z) = \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z (1 + O(z^{-1})), \; U(a,b,z) = z^{-a}(1 + O(z^{-1})) \text{ as } z \to +\infty, \tag{6.2.9}$$

$$M(a,b,z) = 1 + O(z) \text{ as } z \to 0, \tag{6.2.10}$$

and[4] for $1 \leq \Re(b) < 2$ with $b \neq 1$,

$$U(a,b,z) = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + O(z^{2-\Re(b)}) \text{ as } z \to 0. \tag{6.2.11}$$

Since w is a linear combination of $M(a,b,z)$ and $U(a,b,z)$, we immediately infer from (*6.2.9*), (*6.2.10*) and (*6.2.11*) the asymptotic of w both as $z \to +\infty$ and $z \to 0_+$. Finally, since

$$\psi(r) = \frac{1}{r^\gamma} w\left(\frac{r^2}{2}\right),$$

we infer from the asymptotic of w the claimed asymptotic for ψ both as $r \to +\infty$ and $r \to 0_+$. This concludes the proof of (*6.2.3*), (*6.2.4*).

step 2 Estimate on the resolvent. The Wronskian

$$W \;\; := \;\; \psi_1'\psi_2 - \psi_2'\psi_1.$$

satisfies

$$W' = \left(-\frac{2}{r} + r\right) W, \;\; W = \frac{C}{r^2} e^{\frac{r^2}{2}}$$

where we may without loss of generality assume $C = 1$. We then solve

$$\mathcal{L}_\infty(w) = f$$

using the variation of constants which yields

$$w = \left(a_1 + \int_r^{+\infty} f\psi_2 r'^2 e^{-\frac{r'^2}{2}} dr'\right)\psi_1 + \left(a_2 - \int_r^{+\infty} f\psi_1 r'^2 e^{-\frac{r'^2}{2}} dr'\right)\psi_2. \tag{6.2.12}$$

In particular, $\mathcal{T}(f)$ corresponds to the choice $a_1 = a_2 = 0$ and thus satisfies

$$\mathcal{L}_\infty(\mathcal{T}(f)) = f.$$

[4]Note that our choice of b in (*6.2.8*) is such that $\Re(b) = 1$ and $b \neq 1$.

Next, we estimate $\mathcal{T}(f)$ using the asymptotic behavior (*6.2.3*) and (*6.2.4*) of ψ_1 and ψ_2 as $r \to 0_+$ and $r \to +\infty$. For $r \geq 1$, we have

$$\begin{aligned}
& r^{\frac{2}{p-1}+2}|\mathcal{T}(f)| \\
= \; & r^{\frac{2}{p-1}+2}\left|\left(\int_r^{+\infty} f\psi_2 r'^2 e^{-\frac{r'^2}{2}}dr'\right)\psi_1 - \left(\int_r^{+\infty} f\psi_1 r'^2 e^{-\frac{r'^2}{2}}dr'\right)\psi_2\right| \\
\lesssim \; & r^2\left(\int_r^{+\infty}|f|r'^{\frac{2}{p-1}-1}dr'\right) + r^{\frac{4}{p-1}-1}e^{\frac{r^2}{2}}\left(\int_r^{+\infty}|f|\frac{1}{r'^{\frac{2}{p-1}}}r'^2 e^{-\frac{r'^2}{2}}dr'\right) \\
\lesssim \; & \left\{\sup_{r>1}\left(r^2\left(\int_r^{+\infty}\frac{dr'}{r'^3}\right) + r^{\frac{4}{p-1}-1}e^{\frac{r^2}{2}}\left(\int_r^{+\infty} r'^{-\frac{4}{p-1}}e^{-\frac{r'^2}{2}}dr'\right)\right)\right\}\sup_{r\geq 1} r^{\frac{2}{p-1}+2}|f| \\
\lesssim \; & \sup_{r\geq 1} r^{\frac{2}{p-1}+2}|f|.
\end{aligned}$$

Also, for $r_0 \leq r \leq 1$, we have

$$\begin{aligned}
& r^{\frac{1}{2}}\left|\left(\int_r^{+\infty} f\psi_2 r'^2 e^{-\frac{r'^2}{2}}dr'\right)\psi_1 - \left(\int_r^{+\infty} f\psi_1 r'^2 e^{-\frac{r'^2}{2}}dr'\right)\psi_2\right| \\
\lesssim \; & \int_r^1 |f|r'^{\frac{3}{2}}dr' + \int_1^{+\infty} r'^{\frac{2}{p-1}-1}|f|dr' \lesssim \int_{r_0}^1 |f|r'^{\frac{3}{2}}dr' + \sup_{r\geq 1} r^{\frac{2}{p-1}+2}|f|
\end{aligned}$$

and (*6.2.6*) is proved.

step 3 Refined control of ψ_1. We now turn to the proof of (*6.2.5*). We decompose

$$\psi_1 = \frac{1}{r^{\frac{2}{p-1}}} + \widetilde{\psi}_1. \tag{6.2.13}$$

Since $\mathcal{L}_\infty(\psi_1) = 0$, we infer

$$\mathcal{L}_\infty(\widetilde{\psi}_1) = f$$

where f is given by

$$\begin{aligned}
f &= -\mathcal{L}_\infty\left(\frac{1}{r^{\frac{2}{p-1}}}\right) = \partial_r^2\left(\frac{1}{r^{\frac{2}{p-1}}}\right) + \frac{2}{r}\partial_r\left(\frac{1}{r^{\frac{2}{p-1}}}\right) + \frac{pc_\infty^{p-1}}{r^2}\frac{1}{r^{\frac{2}{p-1}}} \\
&= \frac{2(p-3)}{p-1)}\frac{1}{r^{\frac{2}{p-1}+2}}.
\end{aligned}$$

In view of (*6.2.12*), we infer

$$\widetilde{\psi}_1 = \left(a_1 + \frac{2(p-3)}{p-1)}\int_r^{+\infty}\psi_2\frac{e^{-\frac{r'^2}{2}}}{r'^{\frac{2}{p-1}}}dr'\right)\psi_1 + \left(a_2 - \frac{2(p-3)}{p-1}\int_r^{+\infty}\psi_1\frac{e^{-\frac{r'^2}{2}}}{r'^{\frac{2}{p-1}}}dr'\right)\psi_2.$$

On the other hand, we deduce from the asymptotic behavior of ψ_1

$$\widetilde{\psi}_1 = o\left(\frac{1}{r^{\frac{2}{p-1}}}\right) \text{ as } r \to +\infty.$$

In view of the asymptotic behavior of ψ_1 and ψ_2 as $r \to +\infty$, this forces $a_1 = a_2 = 0$ and hence

$$\widetilde{\psi}_1 = \frac{2(p-3)}{p-1}\left(\int_r^{+\infty}\psi_2\frac{e^{-\frac{r'^2}{2}}}{r'^{\frac{2}{p-1}}}dr'\right)\psi_1 - \frac{2(p-3)}{p-1}\left(\int_r^{+\infty}\psi_1\frac{e^{-\frac{r'^2}{2}}}{r'^{\frac{2}{p-1}}}dr'\right)\psi_2.$$

Then, applying Λ to both sides, and using the asymptotic behavior of ψ_1 and ψ_2 as $r \to +\infty$ yields

$$\Lambda\widetilde{\psi}_1 = \frac{c}{r^{\frac{2}{p-1}+2}}\left(1 + O\left(\frac{1}{r^2}\right)\right) \text{ as } r \to +\infty$$

for some constant[5] $c \neq 0$. Injecting this into (*6.2.13*) yields

$$\Lambda\psi_1 = \Lambda\widetilde{\psi}_1 = \frac{c}{r^{\frac{2}{p-1}+2}}\left(1 + O\left(\frac{1}{r^2}\right)\right) \text{ as } r \to +\infty$$

for some constant $c \neq 0$ and concludes the proof of Lemma 6.2.1. □

We are now in position to construct the family of outer self similar solutions as a classical consequence of the implicit function theorem.

Proposition 6.2.2 (Exterior solutions). *Let $0 < r_0 < 1$ a small enough universal constant. For all*

$$0 < \varepsilon \ll r_0^{s_c - 1}, \tag{6.2.14}$$

there exists a solution u to

$$\Delta u - \Lambda u + u^p = 0 \ \textit{ on } \ (r_0, +\infty) \tag{6.2.15}$$

of the form

$$u = \Phi_* + \varepsilon\psi_1 + \varepsilon w$$

with the bounds:

$$\|w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}, \quad \|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}. \tag{6.2.16}$$

Furthermore,

$$w_{|\varepsilon=0} = 0 \textit{ and } \|\partial_\varepsilon w_{|\varepsilon=0}\|_{X_{r_0}} \lesssim r_0^{1-s_c}.$$

Proof. This a classical consequence of Lemma 6.2.1.

step 1. Setting up the Banach fixed point. Let v such that

$$u = \Phi_* + \varepsilon v,$$

then u solves (*6.2.15*) iff:

$$\mathcal{L}_\infty(v) = \varepsilon\frac{p(p-1)}{2}\Phi_*^{p-2}v^2 + \varepsilon F(\Phi_*, v, \varepsilon) \text{ on } r > r_0,$$

where

$$F(\Phi_*, v, \varepsilon) = \frac{1}{\varepsilon^2}\left((\Phi_* + \varepsilon v)^p - \Phi_*^p - p\Phi_*^{p-1}\varepsilon v - \frac{p(p-1)}{2}\Phi_*^{p-2}\varepsilon^2 v^2\right).$$

Furthermore, we decompose

$$v = \psi_1 + w$$

[5]Actually, c is explicitly given by

$$c = -\frac{p-3}{p-1} \neq 0.$$

and hence, using in particular the fact that $\mathcal{L}_\infty(\psi_1) = 0$, w is a solution to

$$\mathcal{L}_\infty(w) = p(p-1)\varepsilon G[\Phi_*, \psi_1, \varepsilon]w \text{ on } r > r_0$$

where we defined the map:

$$G[\Phi_*, \psi_1, \varepsilon]w = \left(\int_0^1 (1-s)(\Phi_* + s\varepsilon(\psi_1 + w))^{p-2} ds \right) (\psi_1 + w)^2.$$

We claim the non linear bounds: assume that

$$\|w\|_{X_{r_0}} \le 1,$$

then

$$\int_{r_0}^1 |G[\Phi_*, \psi_1, \varepsilon]w| r'^{\frac{3}{2}} dr' + \sup_{r \ge 1} r^{\frac{2}{p-1}+2} |G[\Phi_*, \psi_1, \varepsilon]w| \lesssim r_0^{1-s_c} \tag{6.2.17}$$

and

$$\begin{aligned} & \int_{r_0}^1 |G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| r'^{\frac{3}{2}} dr' \\ & + \sup_{r \ge 1} r^{\frac{2}{p-1}+2} |G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| \\ \lesssim \quad & r_0^{1-s_c} \|w_1 - w_2\|_{X_{r_0}}. \end{aligned} \tag{6.2.18}$$

Assume (*6.2.17*), (*6.2.18*), then we look for w as the solution of the following fixed point

$$w = \varepsilon p(p-1) \mathcal{T}\Big(G[\Phi_*, \psi_1, \varepsilon]w \Big), \quad w \in X_{r_0}. \tag{6.2.19}$$

In view of the assumption $\varepsilon r_0^{1-s_c} \ll 1$, the continuity estimate on the resolvent (*6.2.6*) and the nonlinear estimates (*6.2.17*), (*6.2.18*), the Banach fixed point theorem applies and yields a unique solution w to (*6.2.19*) with

$$\|w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}.$$

Differentiating (*6.2.19*) in space, we immediately infer

$$\|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}.$$

Finally, we compute $w_{|\varepsilon=0}$ and $\partial_\varepsilon w_{|\varepsilon=0}$. In view of (*6.2.19*), we have

$$w_{|\varepsilon=0} = 0.$$

Also, we have

$$\partial_\varepsilon w = p(p-1)\mathcal{T}\Big(G[\Phi_*, \psi_1, \varepsilon]w \Big) + \varepsilon p(p-1) \mathcal{T}\Big(\partial_\varepsilon G[\Phi_*, \psi_1, \varepsilon]w \Big)$$

and hence

$$\partial_\varepsilon w_{|\varepsilon=0} = p(p-1)\mathcal{T}\Big(G[\Phi_*, \psi_1, \varepsilon]w \Big)_{|\varepsilon=0}.$$

We have

$$G[\Phi_*, \psi_1, \varepsilon]w_{|\varepsilon=0} = \left(\int_0^1 (1-s)\Phi_*^{p-2} ds\right)\psi_1^2 = \frac{1}{2}\Phi_*^{p-2}\psi_1^2$$

which yields

$$\partial_\varepsilon w_{|\varepsilon=0} = \frac{p(p-1)}{2}\mathcal{T}\Big(\Phi_*^{p-2}\psi_1^2\Big).$$

The continuity estimate (*6.2.6*) and the asymptotic behavior of ψ_1 (*6.2.3*) (*6.2.4*) yield

$$\|\partial_\varepsilon w_{|\varepsilon=0}\|_{X_{r_0}} \lesssim r_0^{1-s_c}.$$

step 2 Proof of the nonlinear estimates (*6.2.17*), (*6.2.18*). Note first that in view of Lemma 6.2.1 and the definition of $\|\cdot\|_{X_{r_0}}$, we have for $r_0 \le r \le 1$,

$$|w(r)| + |\psi_1(r)| \lesssim r^{-\frac{1}{2}} = r^{1-\frac{2}{p-1}-s_c} \lesssim r^{1-s_c}|\Phi_*(r)| \le r_0^{1-s_c}|\Phi_*(r)|$$

while for $r \ge 1$, we have

$$|w(r)| + |\psi_1(r)| \lesssim |\Phi_*(r)|,$$

and hence, our choice of ε yields for all $r \ge r_0$

$$\varepsilon|\psi_1(r)| + \varepsilon|w(r)| \lesssim |\Phi_*(r)|.$$

Next, we estimate $G[\Phi_*, \psi_1, \varepsilon]w$. For $r_0 \le r \le 1$, we have

$$\begin{aligned} &|G[\Phi_*, \psi_1, \varepsilon]w| \le (|\Phi_*(r)| + \varepsilon(|\psi_1(r)| + |w(r)|))^{p-2}(|\psi_1(r)| + |w(r)|)^2 \\ \lesssim\ & |\Phi_*(r)|^{p-2}(|\psi_1(r)| + |w(r)|)^2 \lesssim \left(\frac{1}{r^{\frac{2}{p-1}}}\right)^{p-2}\left(\frac{1}{r^{\frac{1}{2}}}\right)^2 (1 + \|w\|_{X_{r_0}})^2 \lesssim r^{\frac{2}{p-1}-3} \end{aligned}$$

and hence

$$\int_{r_0}^1 |G[\Phi_*, \psi_1, \varepsilon]w| {r'}^{\frac{3}{2}} dr' \lesssim \left(\int_{r_0}^1 {r'}^{-s_c} dr'\right) \lesssim r_0^{1-s_c}.$$

Also, for $r \ge 1$, we have

$$\begin{aligned} |G[\Phi_*, \psi_1, \varepsilon]w| &\le (|\Phi_*(r)| + \varepsilon(|\psi_1(r)| + |w(r)|))^{p-2}(|\psi_1(r)| + |w(r)|)^2 \\ &\lesssim \left(\frac{1}{r^{\frac{2}{p-1}}}\right)^p (1 + \|w\|_{X_{r_0}})^2 \lesssim \frac{1}{r^{2+\frac{2}{p-1}}} \end{aligned}$$

and hence

$$\sup_{r\ge 1} r^{\frac{2}{p-1}+2}|G[\Phi_*, \psi_1, \varepsilon]w| \lesssim 1$$

and (*6.2.17*) is proved. We now prove the contraction estimate:

$$
\begin{aligned}
& G[\Phi_*,\psi_1,\varepsilon]w_1 - G[\Phi_*,\psi_1,\varepsilon]w_2 \\
= \ & \left(\int_0^1 (1-s)(\Phi_* + s\varepsilon(\psi_1+w_1))^{p-2}ds\right)(\psi_1+w_1)^2 \\
- \ & \left(\int_0^1 (1-s)(\Phi_* + s\varepsilon(\psi_1+w_2))^{p-2}ds\right)(\psi_1+w_2)^2 \\
= \ & \left(\int_0^1 (1-s)(\Phi_* + s\varepsilon(\psi_1+w_1))^{p-2}ds\right)\Big((\psi_1+w_1)^2 - (\psi_1+w_2)^2\Big) \\
+ \ & \left(\int_0^1 (1-s)(\Phi_* + s\varepsilon(\psi_1+w_1))^{p-2}ds\right. \\
- \ & \left.\int_0^1 (1-s)(\Phi_* + s\varepsilon(\psi_1+w_2))^{p-2}ds\right)(\psi_1+w_2)^2 \\
= \ & \left(\int_0^1 (1-s)(\Phi_* + s\varepsilon(\psi_1+w_1))^{p-2}ds\right)\Big(2\psi_1+w_1+w_2\Big)(w_1-w_2) \\
+ \ & (p-2)\left(\int_0^1 s(1-s)\int_0^1 (\Phi_* + s\varepsilon(\psi_1+w_1)+\sigma s\varepsilon(w_2-w_1))^{p-3}d\sigma ds\right) \\
\times \ & (\psi_1+w_2)^2\varepsilon(w_1-w_2)
\end{aligned}
$$

and hence

$$
\begin{aligned}
& |G[\Phi_*,\psi_1,\varepsilon]w_1 - G[\Phi_*,\psi_1,\varepsilon]w_2| \\
\lesssim \ & (|\Phi_*(r)| + \varepsilon(|\psi_1(r)|+|w_1(r)|))^{p-2}\Big(|\psi_1(r)|+|w_1(r)|+|w_2(r)|\Big)|w_1(r)-w_2(r)| \\
& +(|\Phi_*(r)| + \varepsilon(|\psi_1(r)|+|w_1(r)|))^{p-3}(|\psi_1(r)|+|w_2(r)|)^2\varepsilon|w_1(r)-w_2(r)| \\
\lesssim \ & \Big\{|\Phi_*(r)|^{p-2}(|\psi_1(r)|+|w_1(r)|+|w_2(r)|) + \varepsilon|\Phi_*(r)|^{p-3}(|\psi_1(r)|+|w_2(r)|)^2\Big\}\,|w_1(r)-w_2(r)|.
\end{aligned}
$$

For $r_0 \le r \le 1$, we have

$$
\begin{aligned}
& \Big|G[\Phi_*,\psi_1,\varepsilon]w_1 - G[\Phi_*,\psi_1,\varepsilon]w_2\Big| \\
\lesssim \ & \left(\frac{1}{r^{\frac{2}{p-1}}}\right)^{p-2}\left(\frac{1}{r^{\frac{1}{2}}}\right)^2 (1+\|w_1\|_{X_{r_0}}+\|w_2\|_{X_{r_0}})\|w_1-w_2\|_{X_{r_0}} \\
+ \ & \varepsilon\left(\frac{1}{r^{\frac{2}{p-1}}}\right)^{p-3}\left(\frac{1}{r^{\frac{1}{2}}}\right)^3 (1+\|w_1\|_{X_{r_0}}+\|w_2\|_{X_{r_0}})^2\|w_1-w_2\|_{X_{r_0}} \\
\lesssim \ & \left(r^{\frac{2}{p-1}-3} + \varepsilon r^{\frac{4}{p-1}-\frac{7}{2}}\right)\|w_1-w_2\|_{X_{r_0}}
\end{aligned}
$$

and hence

$$
\begin{aligned}
& \int_{r_0}^1 |G[\Phi_*,\psi_1,\varepsilon]w_1 - G[\Phi_*,\psi_1,\varepsilon]w_2| {r'}^{\frac{3}{2}}dr' \\
\lesssim \ & \left(\int_{r_0}^1 {r'}^{-s_c}dr' + \varepsilon\int_{r_0}^1 {r'}^{1-2s_c}dr'\right)\|w_1-w_2\|_{X_{r_0}} \\
\lesssim \ & r_0^{1-s_c}(1+\varepsilon r_0^{1-s_c})\|w_1-w_2\|_{X_{r_0}} \lesssim r_0^{1-s_c}\|w_1-w_2\|_{X_{r_0}}.
\end{aligned}
$$

Similarly, for $r \geq 1$,

$$\begin{aligned}
& |G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| \\
\lesssim & \left(\frac{1}{r^{\frac{2}{p-1}}}\right)^p (1 + \|w_1\|_{X_{r_0}} + \|w_2\|_{X_{r_0}})^3 \|w_1 - w_2\|_{X_{r_0}} \\
\lesssim & \frac{1}{r^{2+\frac{2}{p-1}}} \|w_1 - w_2\|_{X_{r_0}}
\end{aligned}$$

and hence

$$\sup_{r\geq 1} r^{\frac{2}{p-1}+2} |G[\Phi_*, \psi_1, \varepsilon]w_1 - G[\Phi_*, \psi_1, \varepsilon]w_2| \lesssim \|w_1 - w_2\|_{X_{r_0}}.$$

This concludes the proof of (*6.2.17*), (*6.2.18*) and of Proposition 6.2.2. □

6.2.2 Constructing interior self-similar solutions

We now construct the family of inner solutions to (*6.2.1*) in $[0, r_0]$ which after renormalization bifurcate from the *stationary* equation and the ground state solution Q.

We start with the continuity of the resolvent of the linearized operator H close to Q in suitable weighted spaces. Given $r_1 \gg 1$, we define Y_{r_1} as the space of functions on $(0, r_1)$ such that the following norm is finite

$$\|w\|_{Y_{r_1}} = \sup_{0\leq r\leq r_1} (1+r)^{-\frac{3}{2}} (|w| + r|\partial_r w|).$$

Lemma 6.2.3 (Interior resolvent of H). 1. Basis of fundamental solutions*: we have*

$$H(\Lambda Q) = 0, \quad H\rho = 0$$

with the following asymptotic behavior as $r \to +\infty$

$$\Lambda Q(r) = \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right), \quad \rho(r) = \frac{c_9 \sin(\omega \log(r) + c_{10})}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right),$$

where $c_7, c_9 \neq 0$, $c_8, c_{10} \in \mathbb{R}$.
2. Continuity of the resolvent*: let the inverse*

$$\mathrm{SS}(f) = \left(\int_0^r f\rho r'^2 dr'\right) \Lambda Q - \left(\int_0^r f\Lambda Q r'^2 dr'\right) \rho$$

then

$$\|\mathrm{SS}(f)\|_{Y_{r_1}} \lesssim \sup_{0\leq r\leq r_1} (1+r)^{\frac{1}{2}} |f|. \tag{6.2.20}$$

Proof. step 1 Fundamental solutions. Define

$$Q_\lambda(r) = \lambda^{\frac{2}{p-1}} Q(\lambda r), \quad \lambda > 0,$$

then

$$\Delta Q_\lambda + Q_\lambda^p = 0 \text{ for all } \lambda > 0$$

and differentiating w.r.t. λ and evaluating at $\lambda = 1$ yields

$$H(\Lambda Q) = 0.$$

Let ρ be another solution to $H(\rho) = 0$ which does not depend linearly on ΛQ, we aim at deriving the asymptotic of both ΛQ and ρ as $r \to +\infty$.

Limiting problem We first solve

$$-\partial_r^2 \varphi - \frac{2}{r}\partial_r \varphi - \frac{pc_\infty^{p-1}}{r^2}\varphi = f. \tag{6.2.21}$$

The homogeneous problem admits the explicit basis of solutions

$$\varphi_1 = \frac{\sin(\omega \log(r))}{r^{\frac{1}{2}}}, \quad \varphi_2 = \frac{\cos(\omega \log(r))}{r^{\frac{1}{2}}}, \tag{6.2.22}$$

and the corresponding Wronskian is given by

$$W(r) = \varphi_1'(r)\varphi_2(r) - \varphi_2'(r)\varphi_1(r) = \frac{\omega}{r^2}.$$

Using the variation of constants, the solutions to (*6.2.21*) are given by

$$\varphi(r) = \left(a_{1,0} + \int_r^{+\infty} f\varphi_2 \frac{r'^2}{\omega}dr'\right)\varphi_1 + \left(a_{2,0} - \int_r^{+\infty} f\varphi_1\frac{r'^2}{\omega}dr'\right)\varphi_2.$$

Inverting H. We now claim that all solutions to $H(\phi) = 0$ admit an expansion

$$\phi(r) = a_{1,0}\varphi_1 + a_{2,0}\varphi_2 + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right) \text{ as } r \to +\infty. \tag{6.2.23}$$

Indeed, we rewrite the equation

$$-\partial_r^2 \phi - \frac{2}{r}\partial_r \phi - \frac{pc_\infty^{p-1}}{r^2}\phi = f, \quad f = p\left(Q^{p-1}(r) - \frac{c_\infty^{p-1}}{r^2}\right)\phi(r),$$

and hence

$$\phi = a_{1,0}\varphi_1 + a_{2,0}\varphi_2 + \widetilde{\phi}, \quad \widetilde{\phi} = \mathcal{F}\left(\widetilde{\phi}\right) \tag{6.2.24}$$

where

$$\begin{aligned}\mathcal{F}\left(\widetilde{\phi}\right)(r) \quad = \quad & -\left(\int_r^{+\infty} p\left(Q^{p-1}(r') - \frac{c_\infty^{p-1}}{r'^2}\right)\left(a_{1,0}\varphi_1 + a_{2,0}\varphi_2 + \widetilde{\phi}\right)(r')\varphi_2 \frac{r'^2}{\omega}dr'\right)\varphi_1 \\ & + \left(\int_r^{+\infty} p\left(Q^{p-1}(r') - \frac{c_\infty^{p-1}}{r'^2}\right)\left(a_{1,0}\varphi_1 + a_{2,0}\varphi_2 + \widetilde{\phi}\right)(r')\varphi_1 \frac{r'^2}{\omega}dr'\right)\varphi_2.\end{aligned}$$

Recall that

$$Q(r) = \frac{c_\infty}{r^{\frac{2}{p-1}}} + O\left(\frac{1}{r^{\frac{1}{2}}}\right) \text{ as } r \to +\infty$$

so that

$$\left|p\left(Q^{p-1}(r)-\frac{c_\infty^{p-1}}{r^2}\right)\right|\lesssim\frac{1}{r^{1+s_c}}\text{ for }r\geq 1.$$

We infer for $r\geq 1$

$$\begin{aligned}\left|\mathcal{F}\left(\widetilde{\phi}\right)(r)\right| &\lesssim \frac{1}{r^{\frac12}}\left(\int_r^{+\infty}\left(\frac{1}{r'^{s_c}}+\frac{1}{r'^{s_c-\frac12}}\left|\widetilde{\phi}\right|(r')\right)dr'\right)\\ &\lesssim \frac{1}{r^{s_c-\frac12}}+\frac{1}{r^{\frac12}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-\frac12}}\left|\widetilde{\phi}\right|(r')dr'\right)\end{aligned}$$

and

$$\left|\mathcal{F}\left(\widetilde{\phi}_1\right)(r)-\mathcal{F}\left(\widetilde{\phi}_2\right)(r)\right| \lesssim \frac{1}{r^{\frac12}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-\frac12}}\left|\widetilde{\phi}_1-\widetilde{\phi}_2\right|(r')dr'\right).$$

Thus, for $R\geq 1$ large enough, the Banach fixed point theorem applies in the space corresponding to the norm

$$\sup_{r\geq R} r^{s_c-\frac12}\left|\widetilde{\phi}\right|(r)$$

and yields a unique solution $\widetilde{\phi}$ to (*6.2.24*) with

$$\sup_{r\geq R} r^{s_c-\frac12}\left|\widetilde{\phi}\right|(r)\leq 1,$$

and (*6.2.23*) is proved.

In particular, in view of the explicit formula (*6.2.22*) for φ_1 and φ_2, and in view of the fact that $H(\Lambda Q)=0$ and $H(\rho)=0$, we infer as $r\to+\infty$

$$\Lambda Q(r)=\frac{c_7\sin\left(\omega\log(r)+c_8\right)}{r^{\frac12}}+O\left(\frac{1}{r^{s_c-\frac12}}\right),\ \ \rho=\frac{c_9\sin\left(\omega\log(r)+c_{10}\right)}{r^{\frac12}}+O\left(\frac{1}{r^{s_c-\frac12}}\right)\qquad (6.2.25)$$

where $c_7,c_9\neq 0$, $c_8,c_{10}\in\mathbb{R}$.

step 2 Continuity of the resolvent. We compute

$$W:=\Lambda Q'\rho-\rho'\Lambda Q,\ \ W'=-\frac{2}{r}W,\ \ W=\frac{-1}{r^2},$$

without loss of generality. Still without loss of generality for $R_0>0$ small enough such that $\Lambda Q>0$ on $[0,R_0]$ the integration of the Wronskian law yields

$$\rho=-\Lambda Q\int_r^{R_0}\frac{1}{(\Lambda Q)^2r'^2}dr'$$

on $(0,R_0]$ which ensures

$$|\rho(r)|\lesssim\frac{1}{r},\ \ |\partial_r\rho(r)|\lesssim\frac{1}{r^2}\text{ as }\ r\to 0.\qquad (6.2.26)$$

We now solve

$$H(w)=f,$$

using the variation of constants which yields

$$w\ \ =\ \ \left(a_1+\int_0^r f\rho r'^2dr'\right)\Lambda Q+\left(a_2-\int_0^r f\Lambda Q r'^2dr'\right)\rho.$$

In particular, $\mathrm{SS}(f)$ corresponds to the choice $a_1 = a_2 = 0$ and thus

$$H(\mathrm{SS}(f)) = f.$$

Finally, using the estimates (*6.2.25*), (*6.2.26*), we estimate for $0 \leq r \leq 1$:

$$\begin{aligned}|\mathrm{SS}(f)| &= \left|\left(\int_0^r f\rho r'^2 dr'\right)\Lambda Q - \left(\int_0^r f\Lambda Q r'^2 dr'\right)\rho\right|\\ &\lesssim \left(\int_0^r r' dr' + \frac{1}{r}\int_0^r r'^2 dr'\right)\sup_{0\leq r\leq 1}|f| \lesssim \sup_{0\leq r\leq r_1}(1+r)^{\frac12}|f|,\end{aligned}$$

$$\begin{aligned}|r\partial_r\mathrm{SS}(f)| &= \left|\left(\int_0^r f\rho r'^2 dr'\right)r\partial_r\Lambda Q - \left(\int_0^r f\Lambda Q r'^2 dr'\right)r\partial_r\rho\right|\\ &\lesssim \left(r^2\int_0^r r' dr' + \frac{1}{r}\int_0^r r'^2 dr'\right)\sup_{0\leq r\leq 1}|f| \lesssim \sup_{0\leq r\leq r_1}(1+r)^{\frac12}|f|,\end{aligned}$$

and for $1 \leq r \leq r_1$:

$$\begin{aligned}(1+r)^{-\frac32}|\mathrm{SS}(f)| &= (1+r)^{-\frac32}\left|\left(\int_0^r f\rho r'^2 dr'\right)\Lambda Q - \left(\int_0^r f\Lambda Q r'^2 dr'\right)\rho\right|\\ &\lesssim (1+r)^{-2}\left(\int_0^r f(1+r')^{\frac32}dr'\right) \lesssim (1+r)^{-2}\left(\int_0^r (1+r')dr'\right)\sup_{0\leq r\leq r_1}(1+r)^{\frac12}|f|\\ &\lesssim \sup_{0\leq r\leq r_1}(1+r)^{\frac12}|f|\end{aligned}$$

$$\begin{aligned}(1+r)^{-\frac32}|r\partial_r\mathrm{SS}(f)| &= (1+r)^{-\frac32}\left|\left(\int_0^r f\rho r'^2 dr'\right)r\partial_r\Lambda Q - \left(\int_0^r f\Lambda Q r'^2 dr'\right)r\partial_r\rho\right|\\ &\lesssim (1+r)^{-2}\left(\int_0^r f(1+r')^{\frac32}dr'\right) \lesssim (1+r)^{-2}\left(\int_0^r (1+r')dr'\right)\sup_{0\leq r\leq r_1}(1+r)^{\frac12}|f|\\ &\lesssim \sup_{0\leq r\leq r_1}(1+r)^{\frac12}|f|,\end{aligned}$$

which concludes the proof of (*6.2.20*) and Lemma 6.2.3. ☐

We are now in position to build the family of interior solutions:

Proposition 6.2.4 (Construction of the interior solution). *Let $r_0 > 0$ small enough and let $0 < \lambda \leq r_0$. Then, there exists a solution u to*

$$\Delta u - \Lambda u + u^p = 0 \textit{ on } 0 \leq r \leq r_0$$

of the form

$$u = \frac{1}{\lambda^{\frac{2}{p-1}}}(Q + \lambda^2 T_1)\left(\frac{r}{\lambda}\right)$$

with

$$\|T_1\|_{Y_{\frac{r_0}{\lambda}}} + \|\Lambda T_1\|_{Y_{\frac{r_0}{\lambda}}} + \|\Lambda^2 T_1\|_{Y_{\frac{r_0}{\lambda}}} \lesssim 1. \tag{6.2.27}$$

Proof. This is again a classical consequence of Lemma 6.2.3.

step 1 Setting up the Banach fixed point. We look for u of the form

$$u = \frac{1}{\lambda^{\frac{2}{p-1}}}(Q + \lambda^2 T_1)\left(\frac{r}{\lambda}\right)$$

so that u solves $\Delta u - \Lambda u + u^p = 0$ on $[0, r_0]$ if and only if

$$H(T_1) = J[Q, \lambda^2]T_1 \text{ on } 0 \leq r \leq r_1$$

where

$$r_1 = \frac{r_0}{\lambda} \geq 1$$

so that

$$\lambda^2 r_1^2 = r_0^2 \ll 1$$

and with

$$J[Q, \lambda^2]T_1 \quad = \quad -\Lambda Q - \lambda^2 \Lambda T_1 + p(p-1)\lambda^2 \left(\int_0^1 (1-s)(Q + s\lambda^2 T_1)^{p-2} ds\right) T_1^2.$$

We claim the nonlinear estimates: assume $\|w\|_{Y_{r_1}} \lesssim 1$, then

$$\sup_{0 \leq r \leq r_1} (1+r)^{\frac{1}{2}} |J[Q, \lambda^2]w| \lesssim 1, \tag{6.2.28}$$

$$\sup_{0 \leq r \leq r_1} (1+r)^{\frac{1}{2}} |J[Q, \lambda^2]w_1 - J[Q, \lambda^2]w_2| \lesssim r_1^2 \lambda^2 \|w_1 - w_2\|_{Y_{r_1}}. \tag{6.2.29}$$

Assume (*6.2.28*), (*6.2.29*), we then look for T_1 as the solution to the fixed point

$$T_1 = \mathrm{SS}(J[Q, \lambda^2]T_1). \tag{6.2.30}$$

In view of the bound $\lambda^2 r_1^2 \ll 1$, the resolvent estimate (*6.2.20*) and the nonlinear estimates (*6.2.28*), (*6.2.29*), the Banach fixed point theorem applies and yields a unique solution T_1 to (*6.2.30*) which furthermore satisfies:

$$\|T_1\|_{Y_{\frac{r_0}{\lambda}}} \lesssim 1.$$

step 2 Proof of (*6.2.28*), (*6.2.29*). Note first that for $0 \leq r \leq r_1$, we have

$$|w(r)| \lesssim (1+r)^{\frac{3}{2}} = r_1^2 (1+r)^{-\frac{1}{2}} \lesssim r_1^2 |Q(r)|.$$

Thus, we infer for all $0 \leq r \leq r_1$

$$\lambda^2 |w(r)| \lesssim \lambda^2 r_1^2 |Q(r)|$$

and hence, our choice of λ yields for all $0 \leq r \leq r_1$

$$\lambda^2 |w(r)| \lesssim |Q(r)|.$$

Next, we estimate $J[Q,\lambda^2]w$. For $0 \leq r \leq r_1$, we have

$$\begin{aligned}
&|J[Q,\lambda^2]w| \leq |\Lambda Q| + p(p-1)\lambda^2(|Q|+\lambda^2|w|)^{p-2}|w|^2 + \lambda^2\left|\frac{1}{2}w + r\partial_r w\right| \\
\lesssim\ & |\Lambda Q| + \lambda^2|Q|^{p-2}|w|^2 + \lambda^2|\frac{1}{2}w + r\partial_r w| \\
\lesssim\ & (1+r)^{-\frac{1}{2}} + \lambda^2(1+r)^{-\frac{2(p-2)}{p-1}}(1+r)^3\|w\|^2_{Y_{r_1}} + \lambda^2(1+r)^{\frac{3}{2}}\|w\|^2_{Y_{r_1}} \\
\lesssim\ & (1+r)^{-\frac{1}{2}}\left(1+\lambda^2(1+r)^{\frac{2}{p-1}+\frac{3}{2}} + \lambda^2(1+r)^2\right) \\
\lesssim\ & (1+r)^{-\frac{1}{2}}\left(1+\lambda^2(1+r)^{-s_c+3} + \lambda^2 r_1^2\right) \lesssim (1+r)^{-\frac{1}{2}}\left(1+\lambda^2 r_1^2\right) \lesssim (1+r)^{-\frac{1}{2}}
\end{aligned}$$

and hence

$$\sup_{0\leq r\leq r_1}(1+r)^{\frac{1}{2}}|J[Q,\lambda^2]w| \quad \lesssim 1.$$

Next, we estimate $|J[Q,\lambda^2]w_1 - J[Q,\lambda^2]w_2|$. We have

$$\begin{aligned}
&J[Q,\lambda^2]w_1 - J[Q,\lambda^2]w_2 \\
=\ & p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2w_1)^{p-2}ds\right)w_1^2 - p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2w_2)^{p-2}ds\right)w_2^2 \\
& +\lambda^2\left(\frac{1}{2}(w_1-w_2) + r(\partial_r w_1 - \partial_r w_2)\right)w \\
=\ & p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2w_1)^{p-2}ds\right)(w_1^2-w_2^2) \\
+\ & p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2w_1)^{p-2}ds - \int_0^1(1-s)(Q+s\lambda^2w_2)^{p-2}ds\right)w_2^2 \\
& +\lambda^2\left(\frac{1}{2}(w_1-w_2) + r(\partial_r w_1 - \partial_r w_2)\right)w \\
=\ & p(p-1)\lambda^2\left(\int_0^1(1-s)(Q+s\lambda^2w_1)^{p-2}ds\right)(w_1+w_2)(w_1-w_2) \\
+\ & p(p-1)(p-2)\lambda^4\left(\int_0^1 s(1-s)\int_0^1(Q+s\lambda^2w_1+\sigma s\lambda^2(w_2-w_1))^{p-3}d\sigma ds\right)w_2^2(w_1-w_2) \\
& +\lambda^2\left(\frac{1}{2}(w_1-w_2) + r(\partial_r w_1 - \partial_r w_2)\right)w
\end{aligned}$$

and hence

$$\begin{aligned}
&|J[Q,\lambda^2]w_1 - J[Q,\lambda^2]w_2| \lesssim \lambda^2(|Q(r)|+\lambda^2|w_1(r)|)^{p-2}(|w_1(r)|+|w_2(r)|)|w_1(r)-w_2(r)| \\
&+\lambda^4(|Q(r)|+\lambda^2|w_1(r)|+\lambda^2|w_2(r)|)^{p-3}|w_2(r)|^2|w_1(r)-w_2(r)| \\
&+\lambda^2\left(\frac{1}{2}(w_1-w_2) + r(\partial_r w_1 - \partial_r w_2)\right)w \\
\lesssim\ & \lambda^2|Q(r)|^{p-2}(|w_1(r)|+|w_2(r)|)|w_1(r)-w_2(r)| + \lambda^4|Q(r)|^{p-3}|w_2(r)|^2|w_1(r)-w_2(r)| \\
&+\lambda^2\left(\frac{1}{2}(w_1-w_2) + r(\partial_r w_1 - \partial_r w_2)\right)w.
\end{aligned}$$

This yields

$$
\begin{aligned}
& |J[Q,\lambda^2]w_1 - J[Q,\lambda^2]w_2| \lesssim \lambda^2(1+r)^{-\frac{2(p-2)}{p-1}}(1+r)^3(\|w_1\|_{Y_{r_1}} + \|w_2\|_{Y_{r_1}})\|w_1-w_2\|_{Y_{r_1}} \\
& \lambda^4(1+r)^{-\frac{2(p-3)}{p-1}}(1+r)^{\frac{9}{2}}\|w_2\|^2_{Y_{r_1}}\|w_1-w_2\|_{Y_{r_1}} + \lambda^2(1+r)^{\frac{3}{2}}\|w_1-w_2\|_{Y_{r_1}} \\
\lesssim\ & \lambda^2(1+r)^{-\frac{1}{2}}\Big((1+r)^{\frac{2}{p-1}+\frac{3}{2}} + \lambda^2(1+r)^{\frac{4}{p-1}+3} + (1+r)^2\Big)\|w_1-w_2\|_{Y_{r_1}} \\
\lesssim\ & \lambda^2(1+r)^{-\frac{1}{2}}\Big((1+r)^{-s_c+3} + \lambda^2(1+r)^{-2s_c+6} + (1+r)^2\Big)\|w_1-w_2\|_{Y_{r_1}} \\
\lesssim\ & r_1^2\lambda^2(1+r)^{-\frac{1}{2}}\Big(1+\lambda^2 r_1^2\Big)\|w_1-w_2\|_{Y_{r_1}} \lesssim r_1^2\lambda^2(1+r)^{-\frac{1}{2}}\|w_1-w_2\|_{Y_{r_1}}
\end{aligned}
$$

which concludes the proof of (*6.2.29*) and Proposition 6.2.4. □

6.2.3 The matching

We now construct a solution to (*6.2.1*) by matching the exterior solution to (*6.2.1*) constructed in section 6.2.1 on $[r_0,+\infty)$ to the interior solution to (*6.2.1*) constructed in section 6.2.2 on $[0,r_0]$. The oscillations (*6.1.3*) allow to perform the matching at r_0 for a quantized sequence of the small parameter ε introduced in Proposition 6.2.2.

Proposition 6.2.5 (Existence of a countable number of smooth selfsimilar profiles). *There exists $N \in \mathbb{N}$ large enough so that for all $n \geq N$, there exists a smooth solution Φ_n to (6.2.1) such that $\Lambda\Phi_n$ vanishes exactly n times.*

Proof. step 1 Initialization. Since

$$
\psi_1(r) = \frac{c_3\sin(\omega\log(r)+c_4)}{r^{\frac{1}{2}}} + O\left(r^{\frac{3}{2}}\right) \text{ as } r\to 0, \quad c_3\neq 0
$$

we compute

$$
\Lambda\psi_1(r) = c_3\frac{(1-s_c)\sin(\omega\log(r)+c_4)+\omega\cos(\omega\log(r)+c_4)}{r^{\frac{1}{2}}} + O\left(r^{\frac{3}{2}}\right) \text{ as } r\to 0.
$$

We may therefore choose $0<r_0\ll 1$ such that

$$
\psi_1(r_0) = \frac{c_3}{r_0^{\frac{1}{2}}} + O\left(r_0^{\frac{3}{2}}\right), \quad \Lambda\psi_1(r_0) = \frac{c_3(1-s_c)}{r_0^{\frac{1}{2}}} + O\left(r_0^{\frac{3}{2}}\right), \tag{6.2.31}
$$

and Proposition 6.2.2 and Proposition 6.2.4 apply. We therefore choose ε and λ such that

$$
0<\varepsilon\ll r_0^{s_c-1}, \quad 0<\lambda\leq r_0,
$$

and have from Proposition 6.2.2 an exterior solution u_{ext} to

$$
-\Delta u_{ext} + \Lambda u_{ext} - u_{ext}^p, \quad r\geq r_0
$$

such that

$$
u_{ext}[\varepsilon] = \Phi_* + \varepsilon\psi_1 + \varepsilon w
$$

and

$$
\|w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}, \quad \|\Lambda w\|_{X_{r_0}} \lesssim \varepsilon r_0^{1-s_c}. \tag{6.2.32}
$$

We also have from Proposition 6.2.4 an interior solution u_{int} to

$$-\Delta u_{int} + \Lambda u_{int} - u_{int}^p, \ \ 0 \le r \le r_0$$

such that

$$u_{int}[\lambda] \ \ = \ \ \frac{1}{\lambda^{\frac{2}{p-1}}}(Q + \lambda^2 T_1)\left(\frac{r}{\lambda}\right).$$

with

$$\|T_1\|_{Y_{\frac{r_0}{\lambda}}} \lesssim 1. \tag{6.2.33}$$

We now would like to match the two solutions at $r = r_0$ which is equivalent to requiring that

$$u_{ext}(r_0) - u_{int}(r_0) = 0 \ \text{ and } \ u'_{ext}(r_0) - u'_{int}(r_0).$$

step 2 Matching the functions. We introduce the map

$$\mathcal{F}[r_0](\varepsilon, \lambda) := u_{ext}[\varepsilon](r_0) - u_{int}[\lambda](r_0).$$

We compute

$$\partial_\varepsilon \mathcal{F}[r_0](\varepsilon, \lambda) \ \ = \ \ \partial_\varepsilon u_{ext}[\varepsilon](r_0) = \psi_1(r_0) + w(r_0) + \varepsilon \partial_\varepsilon w(r_0).$$

In particular, since $w_{|\varepsilon=0} = 0$ and $\|\partial_\varepsilon w_{|\varepsilon=0}\|_{X_{r_0}} \lesssim r_0^{1-s_c}$ in view of Proposition 6.2.2, we have

$$\partial_\varepsilon \mathcal{F}[r_0](0,0) = \psi_1(r_0) \neq 0$$

since we assumed that $\psi_1(r_0) \neq 0$. Also, in view of the asymptotic behavior of Q at infinity, we have as $\lambda \to 0_+$

$$\left|\frac{1}{\lambda^{\frac{2}{p-1}}}(Q - \Phi_* + \lambda^2 T_1)\left(\frac{r_0}{\lambda}\right)\right| \ \ \lesssim \ \ \frac{1}{\lambda^{\frac{2}{p-1}}}\left(\frac{1}{r^{\frac{1}{2}}} + \frac{\lambda^2 r^2}{r^{\frac{1}{2}}}\right)\left(\frac{r_0}{\lambda}\right) \lesssim \frac{\lambda^{\frac{1}{2}-\frac{2}{p-1}}}{r_0^{\frac{1}{2}}} \lesssim \frac{\lambda^{s_c-1}}{r_0^{\frac{1}{2}}}$$

and hence, since $s_c > 1$, we infer

$$\lim_{\lambda \to 0_+} \frac{1}{\lambda^{\frac{2}{p-1}}}(Q - \Phi_* + \lambda^2 T_1)\left(\frac{r_0}{\lambda}\right) = 0.$$

Since

$$\frac{1}{\lambda^{\frac{2}{p-1}}}\Phi_*\left(\frac{r_0}{\lambda}\right) = \Phi_*(r_0),$$

this yields

$$\mathcal{F}[r_0](0,0) = \Phi_*(r_0) - \Phi_*(r_0) = 0.$$

We may thus apply the implicit function theorem[6] which yields the existence of $\lambda_0 > 0$ and a $\mathcal{C}^{\min(1,(s_c-1)_-)}$ function $\varepsilon(\lambda)$ defined on $[0, \lambda_0)$ such that $\mathcal{F}(\varepsilon(\lambda), \lambda) = 0$ and hence

$$u_{ext}[\varepsilon(\lambda)](r_0) = u_{int}[\lambda](r_0) \ \text{ on } [0, \lambda_0).$$

[6]We actually apply the implicit function theorem to

$$\widetilde{\mathcal{F}}[r_0](\varepsilon, \mu) := \mathcal{F}(\varepsilon, \mu^{\frac{1}{s_c-1-\delta}})$$

for any $0 < \delta < s_c - 1$ so that $\widetilde{\mathcal{F}} \in \mathcal{C}^1$. This yields the existence of $\widetilde{\varepsilon} \in \mathcal{C}^1$ and we choose $\varepsilon(\lambda) = \widetilde{\varepsilon}(\lambda^{s_c-1-\delta})$ so that ε belongs indeed to $\mathcal{C}^{\min(1,(s_c-1)_-)}$.

step 3 Control of $\varepsilon(\lambda)$. We claim for $\lambda \in [0, \lambda_0)$

$$\varepsilon(\lambda) = \frac{1}{\psi_1(r_0)\lambda^{\frac{2}{p-1}}}(Q - \Phi_*)\left(\frac{r_0}{\lambda}\right) + O\left[\lambda^{s_c-1}(r_0^2 + \lambda^{s_c-1} r_0^{1-s_c})\right]. \tag{6.2.34}$$

Indeed, by construction

$$u_{ext}[\varepsilon(\lambda)](r_0) = u_{int}[\lambda](r_0)$$

which is equivalent to

$$\varepsilon(\lambda)\psi_1(r_0) + \varepsilon(\lambda)w(r_0) = \frac{1}{\lambda^{\frac{2}{p-1}}}(Q - \Phi_* + \lambda^2 T_1)\left(\frac{r_0}{\lambda}\right). \tag{6.2.35}$$

We infer from (6.2.31), (6.2.32), (6.2.33) and the asymptotic of Q:

$$\varepsilon(\lambda)\psi_1(r_0) + \varepsilon(\lambda)w(r_0) = \varepsilon(\lambda)\frac{c_3}{r_0^{\frac{1}{2}}}\left(1 + O(r_0^2) + O(\varepsilon(\lambda) r_0^{1-s_c})\right),$$

$$\frac{1}{\lambda^{\frac{2}{p-1}}}\left|Q - \Phi_* + \lambda^2 T_1)\left(\frac{r_0}{\lambda}\right)\right| \lesssim \frac{\lambda^{s_c-1}}{r_0^{\frac{1}{2}}}(1 + O(r_0^2)).$$

This first yields using (6.2.14)

$$|\varepsilon(\lambda)| \lesssim \lambda^{s_c-1}. \tag{6.2.36}$$

which reinjected into (6.2.35) yields (6.2.34).

step 4 Computation of the spatial derivatives. We consider the difference of spatial derivatives at r_0 for $\lambda \in [0, \lambda_0)$

$$\mathcal{G}[r_0](\lambda) := u_{ext}[\varepsilon(\lambda)]'(r_0) - u_{int}[\lambda]'(r_0)$$

and claim the leading order expansion:

$$\begin{aligned} \mathcal{G}[r_0](\lambda) \quad &= \quad \lambda^{s_c-1}\left[\frac{c_1 c_3 \omega}{\psi_1(r_0) r_0^2}\sin\left(-\omega\log(\lambda) + c_2 - c_4\right)\right. \\ &+ \quad \left. O\left(r_0^{-s_c-\frac{1}{2}}\lambda^{s_c-1} + r_0^{\frac{1}{2}}\right)\right]. \end{aligned} \tag{6.2.37}$$

Indeed,

$$\mathcal{G}[r_0](\lambda) \quad = \quad \varepsilon(\lambda)\psi_1'(r_0) + \varepsilon(\lambda)w'(r_0) - \frac{1}{\lambda^{\frac{2}{p-1}+1}}(Q' - \Phi_*' + \lambda^2 T_1')\left(\frac{r_0}{\lambda}\right).$$

From (6.2.36), (6.2.16):

$$|\varepsilon(\lambda)w'(r_0)| \quad \lesssim \quad \lambda^{s_c-1}|w'(r_0)| \lesssim \lambda^{2(s_c-1)} r_0^{-\frac{1}{2}-s_c}$$

and from (6.2.27)

$$\left|\frac{1}{\lambda^{\frac{2}{p-1}+1}}\lambda^2 T_1'\left(\frac{r_0}{\lambda}\right)\right| \quad \lesssim \quad r_0^{\frac{1}{2}}\lambda^{s_c-1}$$

and hence using (*6.2.34*), (*6.2.31*):

$$\begin{aligned}
\mathcal{G}[r_0](\lambda) &= \varepsilon(\lambda)\psi_1'(r_0) - \lambda^{s_c-1}\frac{1}{\lambda^{\frac32}}(Q'-\Phi_*')\left(\frac{r_0}{\lambda}\right) + O\left(\left(r_0^{-\frac32}\lambda^{s_c-1}+r_0^{\frac12}\right)\lambda^{s_c-1}\right)\\
&= \lambda^{s_c-1}\left(\frac{1}{\lambda^{\frac12}\psi_1(r_0)}(Q-\Phi_*)\left(\frac{r_0}{\lambda}\right)\psi_1'(r_0) - \frac{1}{\lambda^{\frac32}}(Q'-\Phi_*')\left(\frac{r_0}{\lambda}\right)\right)\\
&\quad + O\left(\left(r_0^{-s_c-\frac12}\lambda^{s_c-1}+r_0^{\frac12}\right)\lambda^{s_c-1}\right)\\
&= \frac{1}{r_0^{\frac12}\psi_1(r_0)}\lambda^{s_c-1}\left\{\left(\frac{r_0}{\lambda}\right)^{\frac12}(Q-\Phi_*)\left(\frac{r_0}{\lambda}\right)\psi_1'(r_0) - \left(\frac{r_0}{\lambda}\right)^{\frac32}(Q'-\Phi_*')\left(\frac{r_0}{\lambda}\right)\frac{\psi_1(r_0)}{r_0}\right\}\\
&\quad + O\left(\left(r_0^{-s_c-\frac12}\lambda^{s_c-1}+r_0^{\frac12}\right)\lambda^{s_c-1}\right).
\end{aligned}$$

Recall that

$$\begin{aligned}
\psi_1(r) &= \frac{c_3\sin(\omega\log(r)+c_4)}{r^{\frac12}} + O\left(r^{\frac32}\right) \text{ as } r\to 0,\\
\psi_1'(r) &= -\frac{c_3\sin(\omega\log(r)+c_4)}{2r^{\frac32}} + \frac{c_3\omega\cos(\omega\log(r)+c_4)}{r^{\frac32}} + O\left(r^{\frac12}\right) \text{ as } r\to 0,\\
Q(r)-\Phi_*(r) &= \frac{c_1\sin(\omega\log(r)+c_2)}{r^{\frac12}} + O\left(\frac{1}{r^{s_c-\frac12}}\right) \text{ as } r\to +\infty,\\
Q'(r)-\Phi_*'(r) &= -\frac{c_1\sin(\omega\log(r)+c_2)}{2r^{\frac32}} + \frac{c_1\omega\cos(\omega\log(r)+c_2)}{r^{\frac32}} + O\left(\frac{1}{r^{s_c+\frac12}}\right) \text{ as } r\to +\infty
\end{aligned}$$

and hence:

$$\begin{aligned}
&\left(\frac{r_0}{\lambda}\right)^{\frac12}(Q-\Phi_*)\left(\frac{r_0}{\lambda}\right)\psi_1'(r_0) - \left(\frac{r_0}{\lambda}\right)^{\frac32}(Q'-\Phi_*')\left(\frac{r_0}{\lambda}\right)\frac{\psi_1(r_0)}{r_0}\\
=\ & \frac{c_1c_3}{r_0^{\frac32}}\left(\sin(\omega\log(r_0)-\omega\log(\lambda)+c_2)\left(-\frac{\sin(\omega\log(r_0)+c_4)}{2}+\omega\cos(\omega\log(r_0)+c_4)\right)\right)\\
&-\left(-\frac{\sin(\omega\log(r_0)-\omega\log(\lambda)+c_2)}{2}+\omega\cos(\omega\log(r_0)-\omega\log(\lambda)+c_2)\right)\sin(\omega\log(r_0)+c_4)\\
&+O\left(r_0^{\frac12}+\lambda^{s_c-1}r_0^{-s_c-\frac12}\right)\\
=\ & \frac{c_1c_3\omega}{r_0^{\frac32}}\Bigg(\sin(\omega\log(r_0)-\omega\log(\lambda)+c_2)\cos(\omega\log(r_0)+c_4)\\
&-\cos(\omega\log(r_0)-\omega\log(\lambda)+c_2)\sin(\omega\log(r_0)+c_4)\Bigg)+O\left(r_0^{\frac12}+\lambda^{s_c-1}r_0^{-s_c-\frac12}\right)\\
=\ & \frac{c_1c_3\omega}{r_0^{\frac32}}\sin(-\omega\log(\lambda)+c_2-c_4)+O\left(r_0^{\frac12}+\lambda^{s_c-1}r_0^{-s_c-\frac12}\right).
\end{aligned}$$

The collection of above bounds and (*6.2.31*) yields (*6.2.37*).

step 5 Discrete matching. For $\delta_0>0$ a small enough universal constant such that $\delta_0\geq r_0$ to be chosen later, we consider

$$\lambda_{k,+}=\exp\left(\frac{-k\pi-c_4+c_2+\delta_0}{\omega}\right),\quad \lambda_{k,-}=\exp\left(\frac{-k\pi-c_4+c_2-\delta_0}{\omega}\right). \qquad (6.2.38)$$

From
$$\lim_{k\to+\infty}\lambda_{k,\pm}=0,$$
there holds for $k\geq k_0$ large enough:
$$0<\cdots<\lambda_{k,+}<\lambda_{k,-}<\cdots<\lambda_{k_0,+}<\lambda_{k_0,-}\leq\lambda_0$$
With the above definition of $\lambda_{k,\pm}$, we have for all $k\geq k_0$
$$\sin\left(-\omega\log(\lambda_{k,+})+c_2-c_4\right)=(-1)^k\sin(\delta_0),\quad \sin\left(-\omega\log(\lambda_{k,-})+c_2-c_4\right)=-(-1)^k\sin(\delta_0),$$
and hence
$$\mathcal{G}[r_0](\lambda_{k,\pm}) \quad = \quad \pm(-1)^k\lambda_{k,\pm}^{s_c-1}\left(\frac{c_1c_3\omega}{\psi_1(r_0)r_0^2}\sin(\delta_0)+O\left(r_0^{-s_c-\frac{1}{2}}\lambda_{k,\pm}^{s_c-1}+r_0^{\frac{1}{2}}\right)\right).$$
Since $\delta_0\geq r_0$, this yields for r_0 small enough and for any $k\geq k_0$ large enough:
$$\mathcal{G}[r_0](\lambda_{k,-})\mathcal{G}[r_0](\lambda_{k,+})<0.$$
Since the function $\lambda\to\mathcal{G}[r_0](\lambda)$ is continuous, we infer from the mean value theorem applied to the intervals $[\lambda_{k,+},\lambda_{k,-}]$ the existence of μ_k such that
$$\lambda_{k,+}<\mu_k<\lambda_{k,-}\text{ and }\mathcal{G}[r_0](\mu_k)=0\text{ for all }k\geq k_0.$$
Finally, for $k\geq k_0$, we have
$$\mathcal{F}[r_0](\varepsilon(\mu_k),\mu_k)=0\text{ and }\mathcal{G}[r_0](\mu_k)=0$$
which yields
$$u_{ext}[\varepsilon(\mu_k)](r_0)=u_{int}[\mu_k](r_0)\text{ and }u_{ext}[\varepsilon(\mu_k)]'(r_0)=u_{int}[\mu_k]'(r_0).$$
and hence the function
$$u_k(r):=\begin{cases}u_{int}[\mu_k](r) & \text{for }0\leq r\leq r_0,\\ u_{ext}[\varepsilon(\mu_k)](r) & \text{for }r>r_0\end{cases}$$
is smooth and satisfies (*6.2.1*).
The rest of the proof is devoted to counting the number of zeroes of Λu_k and showing that this number is an unambiguous way of counting the number of self similar solutions u_k as $k\to+\infty$.

step 6 Zeroes of $\Lambda u_{ext}[\varepsilon]$. We claim that
$$\Lambda u_{ext}[\varepsilon]\ \text{ has as many zeros as }\ \Lambda\psi_1\ \text{ on }\ r\geq r_0. \tag{6.2.39}$$
Indeed, $\Lambda\psi_1+\Lambda w$ does not vanish on $[R_0,+\infty)$ for R_0 large enough from (*6.2.5*) and the uniform bound (*6.2.16*). Moreover, $\Lambda\psi_1(r_0)\neq 0$ from the normalization (*6.2.31*), and the absolute derivative of $\Lambda\psi_1$ at any of its zeroes is uniformly lower bounded using (*6.2.2*), (*6.2.4*), and hence the uniform smallness (*6.2.16*)
$$\|\Lambda w\|_{X_{r_0}}\lesssim\varepsilon r_0^{1-s_c}\ll 1$$
yields the claim.

step 7 Zeroes of $\Lambda u_{int}[\mu_k]$. We now claim that

$$\Lambda u_{int}[\mu_k] \text{ has as many zeros as } \Lambda Q \text{ on } 0 \le r \le r_0/\mu_k. \tag{6.2.40}$$

Indeed, recall that

$$\Lambda u_{int}[\mu_k](r) = \frac{1}{\mu_k^{\frac{2}{p-1}}}(\Lambda Q + \mu_k^2 \Lambda T_1)\left(\frac{r}{\mu_k}\right).$$

We now claim

$$\left(\frac{r_0}{\mu_k}\right)^{\frac{1}{2}} \left|\Lambda Q\left(\frac{r_0}{\mu_k}\right)\right| \gtrsim 1. \tag{6.2.41}$$

Assume (*6.2.41*), then since the zeros of ΛQ are simple, since we have

$$\Lambda Q(r) = \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right) \text{ as } r \to +\infty,$$

since

$$\|\Lambda T_1\|_{Y_{\frac{r_0}{\mu_k}}} = \sup_{0 \le r \le \frac{r_0}{\mu_k}} (1+r)^{-\frac{3}{2}} |\Lambda T_1| \lesssim 1$$

so that

$$\sup_{0 \le r \le \frac{r_0}{\mu_k}} (1+r)^{\frac{1}{2}} |\mu_k^2 \Lambda T_1| \lesssim r_0^2,$$

and similarly for $\Lambda^2 T_1$, and since

$$\Lambda Q(0) = \frac{2}{p-1} \neq 0,$$

we conclude that $\Lambda Q + \mu_k^2 \Lambda T_1$ has as many zeros as ΛQ on $0 \le r \le r_0/\mu_k$. We deduce that on $0 \le r \le r_0$, $\Lambda u_{int}[\mu_k]$ has as many zeros as ΛQ on $0 \le r \le r_0/\mu_k$.

Proof of (*6.2.41*): Recall that

$$u_{ext}[\varepsilon(\mu_k)](r_0) = u_{int}[\mu_k](r_0) \text{ and } u_{ext}[\varepsilon(\mu_k)]'(r_0) = u_{int}[\mu_k]'(r_0),$$

which implies

$$\Lambda u_{ext}[\varepsilon(\mu_k)](r_0) = \Lambda u_{int}[\mu_k](r_0).$$

This yields using (*6.2.34*):

$$\frac{\varepsilon(\mu_k)}{\mu_k^{s_c-1}} = \frac{1}{\psi_1(r_0)\mu_k^{\frac{1}{2}}}(Q - \Phi_*)\left(\frac{r_0}{\mu_k}\right) + O\Big(\mu_k^{s_c-1} r_0^{s_c-1} + r_0^2\Big)$$

and differentiating (*6.2.35*):

$$\frac{\varepsilon(\mu_k)}{\mu_k^{s_c-1}} = \frac{1}{\Lambda\psi_1(r_0)\mu_k^{\frac{1}{2}}}\Lambda Q\left(\frac{r_0}{\mu_k}\right) + O\Big(\mu_k^{s_c-1} r_0^{s_c-1} + r_0^2\Big).$$

We infer

$$\frac{1}{\psi_1(r_0)\mu_k^{\frac{1}{2}}}(Q - \Phi_*)\left(\frac{r_0}{\mu_k}\right) = \frac{1}{\Lambda\psi_1(r_0)\mu_k^{\frac{1}{2}}}\Lambda Q\left(\frac{r_0}{\mu_k}\right) + O\Big(\mu_k^{s_c-1} r_0^{s_c-1} + r_0^2\Big).$$

In view of (*6.2.31*) which we recall below

$$\psi_1(r_0) = \frac{c_3}{r_0^{\frac{1}{2}}} + O\left(r_0^{\frac{3}{2}}\right), \quad \Lambda\psi_1(r_0) = \frac{c_3(1-s_c)}{r_0^{\frac{1}{2}}} + O\left(r_0^{\frac{3}{2}}\right),$$

this yields

$$\left|\left(\frac{r_0}{\mu_k}\right)^{\frac{1}{2}}(Q-\Phi_*)\left(\frac{r_0}{\mu_k}\right)\right| \le \frac{2}{s_c-1}\left|\left(\frac{r_0}{\mu_k}\right)^{\frac{1}{2}}\Lambda Q\left(\frac{r_0}{\mu_k}\right)\right| + O\Big(\mu_k^{s_c-1} + r_0^2\Big). \qquad (6.2.42)$$

On the other hand,

$$Q(r) - \Phi_*(r) = \frac{c_1\sin\left(\omega\log(r)+c_2\right)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c-\frac{1}{2}}}\right) \text{ as } r \to +\infty \qquad (6.2.43)$$

and hence as $r \to +\infty$

$$\begin{aligned} \Lambda Q(r) &= c_1\frac{(1-s_c)\sin(\omega\log(r)+c_2)+\omega\cos(\omega\log(r)+c_2)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c-\frac{1}{2}}}\right) \\ &= c_1\sqrt{(s_c-1)^2+\omega^2}\,\frac{\sin(\omega\log(r)+c_2+\alpha_0)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c-\frac{1}{2}}}\right) \end{aligned} \qquad (6.2.44)$$

where

$$\cos(\alpha_0) = \frac{1-s_c}{\sqrt{(s_c-1)^2+\omega^2}}, \quad \sin(\alpha_0) = \frac{\omega}{\sqrt{(s_c-1)^2+\omega^2}}, \quad \alpha_0 \in \left(\frac{\pi}{2}, \pi\right).$$

Thus there exists $r_2 > 0$ sufficiently small and a constant $\delta_1 > 0$ sufficiently small only depending on ω and $s_c - 1$ such that for $0 < r < r_2$, we have

$$\text{dist}\Big(\omega\log(r)+c_2+\alpha_0, \pi\mathbb{Z}\Big) < \delta_1 \;\Rightarrow\; r^{\frac{1}{2}}|Q(r)-\Phi_*(r)| \ge \frac{4}{s_c-1}r^{\frac{1}{2}}|\Lambda Q(r)| + \frac{c_1\sin(\alpha_0)}{2}.$$

In view of (*6.2.42*), we infer for $k \ge k_1$ large enough

$$\text{dist}\left(\omega\log\left(\frac{r_0}{\mu_k}\right) + c_2 + \alpha_0, \pi\mathbb{Z}\right) \ge \delta_1 \qquad (6.2.45)$$

and (*6.2.41*) is proved.

step 8 Counting. We have so far obtained

$$\begin{aligned} &\#\{r \ge 0 \text{ such that } \Lambda u_k(r) = 0\} \\ =\;& \#\left\{0 \le r \le \frac{r_0}{\mu_k} \text{ such that } \Lambda Q(r) = 0\right\} + \#\{r > r_0 \text{ such that } \Lambda\psi_1(r) = 0\} \end{aligned}$$

which implies

$$\#\{r \ge 0 \text{ such that } \Lambda u_{k+1}(r) = 0\} \;=\; \#\{r \ge 0 \text{ such that } \Lambda u_k(r) = 0\} + \#A_k,$$

with

$$A_k := \left\{\frac{r_0}{\mu_k} < r \le \frac{r_0}{\mu_{k+1}} \text{ such that } \Lambda Q(r) = 0\right\}.$$

We claim for $k \geq k_0$ large enough:

$$\#A_k = 1 \tag{6.2.46}$$

which by possibly shifting the numerotation by a fixed amount ensures that Λu_k vanishes exactly k times.

Upper bound. We first claim

$$\#A_k \leq 1 \tag{6.2.47}$$

Recall that

$$\Lambda Q(r) = \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} + O\left(\frac{1}{r^{s_c - \frac{1}{2}}}\right) \text{ as } r \to +\infty, \tag{6.2.48}$$

so that there exists $R \geq 1$ large enough such that

$$\{r \geq R \,/\, \Lambda Q(r) = 0\} = \{r_q,\;\; q \geq q_1\},\;\; \omega \log(r_q) + c_8 = q\pi + O\left(\frac{1}{r_q^{s_c - 1}}\right). \tag{6.2.49}$$

In view of (*6.2.44*) and (*6.2.48*), we have

$$c_2 + \alpha_0 = c_8$$

and hence, together with (*6.2.45*) and (*6.2.49*), we infer

$$\inf_{q \geq q_1, k \geq k_1} \left|\log\left(\frac{r_0}{\mu_k}\right) - \log(r_q)\right| \geq \frac{\delta_1}{2\omega}. \tag{6.2.50}$$

This implies for $k \geq k_1$

$$\begin{aligned} A_k &= \left\{q \geq q_1 \text{ such that } r_q \in \left(\frac{r_0}{\mu_k}, \frac{r_0}{\mu_{k+1}}\right)\right\} \\ &\subset \left\{q \geq q_1 \text{ such that } \log\left(\frac{r_0}{\mu_k}\right) + \frac{\delta_1}{2\omega} \leq \log(r_q) \leq \log\left(\frac{r_0}{\mu_{k+1}}\right) - \frac{\delta_1}{2\omega}\right\}. \end{aligned} \tag{6.2.51}$$

Since $\lambda_{k,+} < \mu_k < \lambda_{k,-}$ with $\lambda_{k,\pm}$ given by (*6.2.38*), we have for $k \geq k_1$

$$\begin{aligned} &\log\left(\frac{r_0}{\mu_{k+1}}\right) - \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_k}\right) + \frac{\delta_1}{2\omega}\right) = \log(\mu_k) - \log(\mu_{k+1}) - \frac{\delta_1}{\omega} \\ \leq\;& \log(\lambda_{k_+}) - \log(\lambda_{k+1,-}) - \frac{\delta_1}{\omega} \leq \frac{\pi + 2\delta_0 - \delta_1}{\omega}. \end{aligned}$$

Also, we have for $q \geq q_1$

$$\log(r_{q+1}) - \log(r_q) \;=\; \frac{\pi}{\omega} + O\left(\frac{1}{r_q^{s_c - 1}}\right).$$

We now choose δ_0 such that

$$0 < \delta_0 < \frac{\delta_1}{4}. \tag{6.2.52}$$

Then, we infer for $k \geq k_1$

$$\log\left(\frac{r_0}{\mu_{k+1}}\right) - \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_k}\right) + \frac{\delta_1}{2\omega}\right) \leq \frac{\pi}{\omega} - \frac{\delta_1}{2\omega}$$

and hence for $k \geq k_1$ and $q \geq q_1$, we have

$$\log(r_{q+1}) - \log(r_q) \;>\; \log\left(\frac{r_0}{\mu_{k+1}}\right) - \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_k}\right) + \frac{\delta_1}{2\omega}\right)$$

which in view of (*6.2.51*) implies (*6.2.47*).

Lower bound. We now prove (*6.2.46*) and assume by contradiction:

$$\# A_{k_2} = 0.$$

Then, let $q_2 \geq q_1$ such that

$$r_{q_2} < \frac{r_0}{\mu_{k_2}} < \frac{r_0}{\mu_{k_2+1}} < r_{q_2+1}.$$

We infer from (*6.2.50*):

$$\log(r_{q_2}) \leq \log\left(\frac{r_0}{\mu_{k_2}}\right) - \frac{\delta_1}{2\omega} < \log\left(\frac{r_0}{\mu_{k_2+1}}\right) + \frac{\delta_1}{2\omega} \leq \log(r_{q_2+1}). \tag{6.2.53}$$

However, we have for $k \geq k_1$

$$\begin{aligned}
& \log\left(\frac{r_0}{\mu_{k_2+1}}\right) + \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_{k_2}}\right) - \frac{\delta_1}{2\omega}\right) = \log(\mu_{k_2}) - \log(\mu_{k_2+1}) + \frac{\delta_1}{\omega} \\
\geq \quad & \log(\lambda_{k_2,-}) - \log(\lambda_{k_2+1,+}) + \frac{\delta_1}{\omega} \geq \frac{\pi - 2\delta_0 + \delta_1}{\omega} \geq \frac{\pi}{\omega} + \frac{\delta_1}{2\omega}
\end{aligned}$$

in view of our choice (*6.2.52*). Hence, we infer

$$\log\left(\frac{r_0}{\mu_{k_2+1}}\right) + \frac{\delta_1}{2\omega} - \left(\log\left(\frac{r_0}{\mu_{k_2}}\right) - \frac{\delta_1}{2\omega}\right) > \log(r_{q_2+1}) - \log(r_{q_2})$$

which contradicts (*6.2.53*).
This concludes the proof of Proposition 6.2.5. □

We now collect final estimates on the constructed solution Φ_n which conclude the proof of Proposition 2.4.5.

Corollary 6.2.6. *Let Φ_n the solution to (6.2.1) constructed in Proposition 6.2.5. Then there exists a small enough constant $r_0 > 0$ independent of n such that:*
1. Convergence to Φ_* as $n \to +\infty$*:*

$$\lim_{n\to+\infty} \sup_{r\geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_*(r)| = 0. \tag{6.2.54}$$

2. Convergence to Q at the origin*: there holds for some $\mu_n \to 0$ as $n \to +\infty$:*

$$\lim_{n\to+\infty} \sup_{r\leq r_0} \left|\Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n}\right)\right| = 0. \tag{6.2.55}$$

3. Last zeroes*: let $r_{0,n} < r_0$ denote the last zero of $\Lambda\Phi_n$ before r_0. Then, for $n \geq N$ large enough, we have*

$$e^{-\frac{2\pi}{\omega}} r_0 < r_{0,n} < r_0.$$

Let $r_{\Lambda Q,n} < r_0/\mu_n$ denote the last zero of ΛQ before r_0/μ_n, then

$$r_{0,n} = \mu_n r_{\Lambda Q,n}(1 + O(r_0^2)).$$

Proof. We choose $r_0 > 0$ small enough as in the proof of Proposition 6.2.5. We start with the proof of the first claim. Recall from the proof of Proposition 6.2.5 that we have for $r \geq r_0$

$$\Phi_n(r) \quad = \quad \Phi_*(r) + \varepsilon(\mu_n)\psi_1(r) + \varepsilon(\mu_n)w(r)$$

where we have in particular

$$\sup_{r_0 \leq r \leq 1} r^{\frac{1}{2}}(|\psi_1| + |w|) + \sup_{r \geq 1} r^{\frac{2}{p-1}}(|\psi_1| + |w|) \lesssim 1$$

and

$$\lim_{n \to +\infty} \varepsilon(\mu_n) = 0.$$

We infer

$$\begin{aligned}
&\sup_{r \geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_*(r)| \\
\lesssim \;\; & \varepsilon(\mu_n) \left(\sup_{r \geq r_0} (|\psi_1(r)| + |w(r)|) + \sup_{r \geq 1} r^{\frac{2}{p-1}} (|\psi_1(r)| + |w(r)|) \right) \\
\lesssim \;\; & \varepsilon(\mu_n) r_0^{-\frac{1}{2}}
\end{aligned}$$

and hence

$$\lim_{n \to +\infty} \sup_{r \geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |\Phi_n(r) - \Phi_*(r)| = 0.$$

Next, recall from the proof of Proposition 6.2.5 that we have for $r \leq r_0$

$$\Phi_n(r) \quad = \quad \frac{1}{\mu_n^{\frac{2}{p-1}}} (Q + \mu_n^2 T_1) \left(\frac{r}{\mu_n} \right)$$

with

$$\sup_{0 \leq r \leq \frac{r_0}{\mu_n}} (1 + r)^{-\frac{3}{2}} |T_1| \lesssim 1.$$

We infer for $r \leq r_0$

$$\left| \Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n} \right) \right| \quad \leq \quad \mu_n^{2 - \frac{2}{p-1}} |T_1| \left(\frac{r}{\mu_n} \right) \lesssim \mu_n^{\frac{1}{2} - \frac{2}{p-1}}$$

and hence

$$\sup_{r \leq r_0} \left| \Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n} \right) \right| \lesssim \mu_n^{s_c - 1}. \tag{6.2.56}$$

and since $\mu_n \to 0$ as $n \to +\infty$, (*6.2.55*) is proved.

We now estimate the localization of the last zeroes of Φ_n and ΛQ before r_0. Recall that

$$\Lambda Q(r) \sim \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} \text{ as } r \to +\infty.$$

Since $\sin(\omega \log(r) + c_8)$ changes sign on the interval

$$e^{-\frac{3\pi}{2\omega}} \frac{r_0}{\mu_n} \leq r \leq \frac{r_0}{\mu_n},$$

and since $r \gg 1$ on this interval, we infer by the mean value theorem that $\Lambda Q(r)$ has a zero on this interval. In particular, this yields

$$e^{-\frac{3\pi}{2\omega}}\frac{r_0}{\mu_n} \le r_{\Lambda Q,n} \le \frac{r_0}{\mu_n}.$$

Also, recall from the proof of Proposition 6.2.5 that we have for $r \le r_0$

$$\Lambda\Phi_n(r) = \frac{1}{\mu_n^{\frac{2}{p-1}}}(\Lambda Q + \mu_n^2 \Lambda T_1)\left(\frac{r}{\mu_n}\right),$$

Since

$$\Lambda Q(r) \sim \frac{c_7 \sin(\omega \log(r) + c_8)}{r^{\frac{1}{2}}} \text{ as } r \to +\infty,$$

and

$$\sup_{0\le r\le \frac{r_0}{\mu_n}} (1+r)^{\frac{3}{2}}|\Lambda T_1| \lesssim 1,$$

and since

$$e^{-\frac{2\pi}{\omega}} r_0 \le r \le r_0,$$

we have $r/\mu_n \sim r_0/\mu_n \gg 1$ for $n \ge N$ large enough, we infer

$$\Lambda\Phi_n(r) \sim \frac{c_7 \sin(\omega\log(r) - \omega\log(\mu_n) + c_8) + O(r_0^2)}{\mu_n^{\frac{2}{p-1}}\left(\frac{r}{\mu_n}\right)^{\frac{1}{2}}}.$$

This yields

$$\Big|\omega\log(r_{0,n}) - \omega\log(\mu_n) + c_8 - (\omega\log(r_{\Lambda Q,n}) + c_8)\Big| \lesssim r_0^2$$

and hence

$$\begin{aligned} r_{0,n} &= \mu_n r_{\Lambda Q,n} e^{O(r_0^2)} \\ &= \mu_n r_{\Lambda Q,n}(1+O(r_0^2)). \end{aligned}$$

Furthermore, since we have

$$e^{-\frac{3\pi}{2\omega}}\frac{r_0}{\mu_n} \le r_{\Lambda Q,n} \le \frac{r_0}{\mu_n},$$

we deduce

$$e^{-\frac{2\pi}{\omega}} r_0 \le r_{0,n} \le r_0.$$

This concludes the proof of the corollary. □

6.3 Spectral gap in weighted norms

Our aim in this section is to produce a spectral gap for the linearized operator corresponding to (*2.4.1*) around Φ_n:

$$\mathcal{L}_n := -\Delta + \Lambda - p\Phi_n^{p-1}. \qquad (6.3.1)$$

Recall (*6.1.4*), then $\mathcal{L}_n$ is self adjoint for the L^2_ρ scalar product. Moreover, from (*6.A.1*) and the local compactness of the Sobolev embeddings $H^1(|x| \le R) \hookrightarrow L^2(|x| \le R)$, and the fact that $\Phi_n \in L^\infty$, the selfadjoint operator $\mathcal{L}_n + M_n$ for the measure ρdx is for $M_n \ge 1$ large enough invertible with compact resolvent. Hence $\mathcal{L}_n$ is diagonalizable in a Hilbert basis of L^2_ρ, and we claim the following sharp spectral gap estimate:

Proposition 6.3.1 (Spectral gap for $\mathcal{L}_n$). *Let $n > N$ with $N \gg 1$ large enough, then the following holds:*
1. Eigenvalues. *The spectrum of $\mathcal{L}_n$ is given by*

$$-\mu_{n+1,n} < \cdots < -\mu_{2,n} < -\mu_{1,n} = -2 < -\mu_{-1,n} = -1 < 0 < \lambda_{0,n} < \lambda_{1,n} < \dots \tag{6.3.2}$$

with

$$\lambda_{j,n} > 0 \text{ for all } j \geq 0 \text{ and } \lim_{j \to +\infty} \lambda_{j,n} = +\infty. \tag{6.3.3}$$

The eigenvalues $(-\mu_{j,n})_{1 \leq j \leq n+1}$ are simple and associated to spherically symmetric eigenvectors

$$\psi_{j,n}, \quad \|\psi_{j,n}\|_{L^2_\rho} = 1, \quad \psi_{1,n} = \frac{\Lambda \Phi_n}{\|\Lambda \Phi_n\|_\rho},$$

and the eigenspace for $\mu_{-1,n}$ is spanned by

$$\psi^k_{-1,n} = \frac{\partial_k \Phi_n}{\|\partial_k \Phi_n\|_\rho}, \quad 1 \leq k \leq 3. \tag{6.3.4}$$

Moreover, there holds as $r \to +\infty$

$$|\partial_k \psi_{j,n}(r)| \lesssim (1+r)^{-\frac{2}{p-1} - \mu_{j,n} - k}, \quad 1 \leq j \leq n+1, \quad k \geq 0. \tag{6.3.5}$$

2. Spectral gap. *There holds for some constant $c_n > 0$:*

$$\forall \varepsilon \in H^1_\rho, \ (\mathcal{L}_n \varepsilon, \varepsilon)_\rho \geq c_n \|\varepsilon\|^2_{H^1_\rho} - \frac{1}{c_n} \left[\sum_{j=1}^{n+1} (\varepsilon, \psi_{j,n})^2_\rho + \sum_{k=1}^{3} (\varepsilon, \psi^k_{0,n})^2_\rho \right]. \tag{6.3.6}$$

In other words, $\mathcal{L}_n$ admits $n+1$ instability directions when $\Lambda \Phi_n$ vanishes n times, and 0 is never in the spectrum. Moreover, there are no additional non radial instabilities apart from the trivial translation invariance (*6.3.4*).

The rest of this section is devoted to preparing the proof of Proposition 6.3.1 which is completed in section 6.3.4.

6.3.1 Decomposition in spherical harmonics

We recall that the spherical harmonics are defined by (*4.1.17*). In particular, $u \in H^1_\rho$ is decomposed as

$$u = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} u_{m,k} Y^{(m,k)}$$

where $u_{m,k}$ are radial functions satisfying the Parseval formula

$$\|u\|^2_\rho = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} \|u_{m,k}\|^2_\rho.$$

This allows us to write

$$(\mathcal{L}_n(u), u)_\rho = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} (\mathcal{L}_{n,m}(u_{m,k}), u_{m,k})_\rho \tag{6.3.7}$$

where we recall

$$\mathcal{L}_{n,m} := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{p-1} + r\partial_r + \frac{m(m+1)}{r^2} - p\Phi_n^{p-1}.$$

We also recall for further use the definition of the operators:

$$\mathcal{L}_{\infty,m} := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{2}{p-1} + r\partial_r + \frac{m(m+1)}{r^2} - p\Phi_*^{p-1},$$
$$H_m := -\partial_{rr} - \frac{2}{r}\partial_r + \frac{m(m+1)}{r^2} - pQ^{p-1}.$$

6.3.2 Linear ODE analysis

We compute in this section the fundamental solutions of $\mathcal{L}_{n,m}$, H_m and we recall the behavior of the eigenvalues of $\mathcal{L}_\infty$. The claims are standard and follow from a classical ODE perturbation analysis using in an essential way the uniform bound (*2.4.10*).

Lemma 6.3.2 (Fundamental solution for $\mathcal{L}_{n,m}$, H_m). *Let $m \geq 1$. Let $\Delta_m > 0$ be given by (6.C.1).*
1. Basis for $\mathcal{L}_{n,m}$. *Let $\phi_{n,m}$ be the solution to $\mathcal{L}_{n,m}\phi_{n,m} = 0$ with the behaviour at the origin*

$$\varphi_{n,m} = r^m[1 + O(r^2)] \quad \textit{as} \quad r \to 0, \tag{6.3.8}$$

then

$$\varphi_{n,m} \sim \frac{c_1}{r^{\frac{2}{p-1}}} + c_2 r^{\frac{2}{p-1}-3} e^{\frac{r^2}{2}} \ \textit{as}\ r \to +\infty, \quad (c_1, c_2) \neq (0,0). \tag{6.3.9}$$

2. Basis for H_1*: let $m = 1$, then there exists a fundamental basis (ν_1, ϕ_1) with*

$$\nu_1(r) = \frac{Q'(r)}{Q''(0)} \left| \begin{array}{ll} = r[1 + O(r^2)] & \textit{as} \quad r \to 0 \\ \sim \frac{c_{1,+}}{r^{\frac{1+\sqrt{\Delta_1}}{2}}} & \textit{as} \quad r \to +\infty \end{array} \right. \tag{6.3.10}$$

and

$$\phi_1(r) = \left| \begin{array}{ll} \frac{1}{r^2}[1 + O(r^2)] & \textit{as} \quad r \to 0 \\ \sim \frac{c_{1,-}}{r^{\frac{1-\sqrt{\Delta_1}}{2}}} & \textit{as} \quad r \to +\infty, \quad c_{1,-} \neq 0. \end{array} \right. \tag{6.3.11}$$

2. Basis for H_m*: let $m \geq 2$, then there exists a fundamental basis (ν_m, ϕ_m) with*

$$\nu_m \left| \begin{array}{ll} = r^m[1 + O(r^2)] & \textit{as} \quad r \to 0 \\ \sim \frac{c_{m,-}}{r^{\frac{1-\sqrt{\Delta_m}}{2}}} & \textit{as}\ r \to +\infty, \quad c_{m,-} > 0 \end{array} \right. \tag{6.3.12}$$

and

$$\phi_m(r) = \left| \begin{array}{ll} \frac{1}{r^{1+m}}[1 + O(r^2)] & \textit{as} \quad r \to 0 \\ \sim \frac{c_{m,+}}{r^{\frac{1+\sqrt{\Delta_m}}{2}}} & \textit{as} \quad r \to +\infty, \quad c_{m,+} \neq 0. \end{array} \right. \tag{6.3.13}$$

4. Positivity*:*

$$\nu_m(r) > 0 \quad \textit{on} \quad (0, +\infty). \tag{6.3.14}$$

5. Uniform closeness*: Fix $m \geq 1$. There exists a sequence*[7] *$\mu_n \to 0$ as $n \to +\infty$ such that for $n \geq N$ large enough*

$$\sup_{0 \leq r \leq r_0} \frac{\left|\mu_n^{-m}\varphi_{n,m}(r) - \nu_m\left(\frac{r}{\mu_n}\right)\right|}{\left|\nu_m\left(\frac{r}{\mu_n}\right)\right|} + \sup_{0 \leq r \leq r_0} \frac{\left|\mu_n^{-m+1}\varphi'_{n,m}(r) - \nu'_m\left(\frac{r}{\mu_n}\right)\right|}{\left|\nu'_m\left(\frac{r}{\mu_n}\right)\right|} \lesssim r_0^2. \tag{6.3.15}$$

[7] $(\mu_n)_{n \geq N}$ is the same sequence of scales as in (*2.4.10*) in Proposition 2.4.5 and Corollary 6.2.6.

The uniform in n bound (*6.3.15*) follows from the uniform control (*2.4.10*) using a standard ODE analysis. We provide a detailed proof of Lemma 6.3.2 in Appendix 6.C for the sake of completeness.

We now detail the structure of the smooth zero of $\mathcal{L}_{n,0}$ which is the key to the counting of non positive eigenvalues. Let $\varphi_{n,0}$ be the solution to

$$\mathcal{L}_{n,0}(\varphi_{n,0}) = 0, \quad \varphi_{n,0}(0) = 1, \quad \varphi'_{n,0}(0) = 0. \tag{6.3.16}$$

We recall that $r_{0,n} < r_0$ denotes the last zero of $\Lambda\Phi_n$ before r_0, and we let $r_{1,n} < r_0$ denote the last zero of $\varphi_{n,0}$ before r_0. We claim:

Lemma 6.3.3 (Zeroes of $\Phi_{n,0}$). *There holds*

$$\sup_{0 \le r \le r_0} \left(1 + \frac{r}{\mu_n}\right)^{\frac{1}{2}} \left|\varphi_{n,0}(r) - \frac{p-1}{2}\Lambda Q\left(\frac{r}{\mu_n}\right)\right| \lesssim r_0^2 \tag{6.3.17}$$

and

$$r_{1,n} = r_{0,n} + O(r_0^3), \quad e^{-\frac{2\pi}{\omega}} r_0 \le r_{1,n} \le r_0. \tag{6.3.18}$$

This is again a simple perturbative analysis which proof is detailed in Appendix 6.D.

We now claim the following classical result which relies on the standard analysis of explicit special functions:

Lemma 6.3.4 (Special functions lemma). *Let $\lambda \in \mathbb{R}$. The solutions to*

$$\mathcal{L}_\infty(\psi) = \lambda\psi, \quad \psi \in H^1_\rho(1, +\infty)$$

behaves for $r \to +\infty$ as

$$\psi \sim r^{-\frac{2}{p-1}+\lambda}$$

and for $r \to 0_+$ as

$$\psi = \frac{1}{r^{\frac{1}{2}}} \cos(\omega \log(r) - \Phi(\lambda)) + O\left(r^{\frac{3}{2}}\right) \tag{6.3.19}$$

where

$$\Phi(\lambda) = \arg\left(\frac{2^{\frac{i\omega}{2}}\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2}\right)}\right).$$

Proof. We consider the solution ψ to

$$\mathcal{L}_\infty(\psi) = \lambda\psi.$$

The change of variable and unknown

$$\psi(r) = \frac{1}{(2z)^{\frac{\gamma}{2}}} w(z), \quad z = \frac{r^2}{2}$$

leads to

$$\mathcal{L}_\infty(\psi) - \lambda\psi = -\frac{2}{(2z)^{\frac{\gamma}{2}}}\left(zw''(z) + \left(-\gamma + \frac{3}{2} - z\right)w'(z) - \left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{\gamma}{2}\right)w(z)\right)$$

and thus $\mathcal{L}_\infty(\psi) = \lambda\psi$ if and only if

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0$$

with

$$a = \frac{1}{p-1} - \frac{\lambda}{2} - \frac{\gamma}{2}, \quad b = -\gamma + \frac{3}{2}. \tag{6.3.20}$$

Hence w is a linear combination of the special functions $M(a,b,z)$ and $U(a,b,z)$ whose asymptotic at infinity is given by (*6.2.9*):

$$M(a,b,z) \sim \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z, \quad U(a,b,z) \sim z^{-a} \text{ as } z \to +\infty,$$

In particular, a non zero contribution of $M(a,b,z)$ to w would yield for $\psi(r)$ the following asymptotic

$$\psi(r) \sim r^{\frac{2}{p-1}-3-\lambda} e^{\frac{r^2}{2}} \text{ as } r \to +\infty.$$

which contradicts $\psi \in H^1_\rho(1,+\infty)$. Hence

$$w(z) = U(a,b,z).$$

In view of the asymptotic of U recalled in (*6.2.9*), we have

$$w(z) \sim z^{-a} \text{ as } z \to +\infty.$$

Since

$$\psi(r) = \frac{1}{r^\gamma} w\left(\frac{r^2}{2}\right),$$

this yields

$$\psi \sim r^{-\frac{2}{p-1}+\lambda} \text{ as } r \to +\infty.$$

Also, in view of the asymptotic of U recalled in (*6.2.11*), we have

$$w(z) = \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + O(z^{2-\Re(b)}) \text{ as } z \to 0,$$

which in view of (*6.3.20*) and the fact that $\gamma = 1/2 + i\omega$ yields

$$w(z) = \frac{\Gamma(-i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} - \frac{i\omega}{2}\right)} z^{i\omega} + \frac{\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2}\right)} + O(z) \text{ as } z \to 0.$$

Since

$$\psi(r) = \frac{1}{r^\gamma} w\left(\frac{r^2}{2}\right),$$

this yields

$$\begin{aligned}\psi(r) \quad &= \quad \frac{2^{-\frac{i\omega}{2}}}{r^{\frac{1}{2}}} \left(\frac{2^{-\frac{i\omega}{2}}\Gamma(-i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} - \frac{i\omega}{2}\right)} r^{i\omega} + \frac{2^{\frac{i\omega}{2}}\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2}\right)} r^{-i\omega} \right) \\ &\quad + O\left(r^{\frac{3}{2}}\right) \text{ as } r \to 0,\end{aligned}$$

and since ψ is real valued, we infer[8]

$$\psi(r) = \frac{\cos(\omega\log(r) - \Phi(\lambda))}{r^{\frac{1}{2}}} + O\left(r^{\frac{3}{2}}\right) \text{ as } r \to 0,$$

where

$$\Phi(\lambda) = \arg\left(\frac{2^{\frac{i\omega}{2}}\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2}\right)}\right).$$

This concludes the proof of the lemma. □

6.3.3 Perturbative spectral analysis

We now prove elementary spectral analysis perturbation results based on the uniform bounds (*2.4.9*), (*2.4.10*) which allow us to precisely count the number of instabilities of $\mathcal{L}_{n,0}$.

Lemma 6.3.5 (Control of the outside spectrum). *Let $r_0 > 0$ and let $r_{n,2}$ such that $r_{n,2} > e^{-\frac{2\pi}{\omega}}r_0$. Let us define the operators*

$$\left|\begin{array}{l} A_n[r_{n,2}](f) = \mathcal{L}_{n,0}(f) \textit{ on } r > r_{n,2}, \quad f(r_{n,2}) = 0, \\ A_\infty[r_{n,2}](f) = \mathcal{L}_\infty(f) \textit{ on } r > r_{n,2}, \quad f(r_{n,2}) = 0, \end{array}\right. \tag{6.3.21}$$

then

$$\sup_{\lambda \in Spec(A_n[r_{n,2}])} \inf_{\mu \in Spec(A_\infty[r_{n,2}])} |\lambda - \mu| + \sup_{\mu \in Spec(A_\infty[r_{n,2}])} \inf_{\lambda \in Spec(A_n[r_{n,2}])} |\lambda - \mu| \to 0 \tag{6.3.22}$$

as $n \to +\infty$.

Proof. In view of (*6.A.1*), the local compactness of the Sobolev embeddings

$$H^1(|x| \le R) \hookrightarrow L^2(|x| \le R) \text{ for all } 1 \le R < +\infty,$$

and the fact that $\Phi_n \in L^\infty$ and , the selfadjoint operators $A_n[r_{n,2}] + M_n$ for the measure ρdx are for $M_n \ge 1$ large enough invertible with compact resolvent, and $A_n[r_{n,2}]$ is diagonalizable. Since $\Phi_* \in L^\infty(r > r_0)$, we deduce similarly that $A_\infty[r_{n,2}]$ is diagonalizable. Let then λ_n be an eigenvalue of $A_n[r_{n,2}]$ with normalized eigenvector w_n:

$$\mathcal{L}_n(w_n) = 0 \text{ on } r > r_{n,2}, \quad w_n(r_{n,2}) = 0, \quad \|w_n\|_{L^2_\rho(r > r_{n,2})} = 1.$$

Since $A_\infty[r_{n,2}]$ is diagonalizable in a Hilbert basis of L^2_ρ, we have

$$\begin{aligned} \|A_\infty[r_{n,2}](w_n) - \lambda_n w_n\|_{L^2_\rho(r>r_{n,2})} &\ge \text{dist}(\lambda_n, \text{spec}(A_\infty[r_{n,2}]))\|w_n\|_{L^2_\rho(r>r_{n,2})} \\ &= \text{dist}(\lambda_n, \text{spec}(A_\infty[r_{n,2}])). \end{aligned}$$

On the other hand,

$$\|A_\infty[r_{n,2}](w_n) - \lambda_n w_n\|_{L^2_\rho(r>r_{n,2})} = \|(A_\infty[r_{n,2}] - A_n[r_{n,2}])(w_n)\|_{L^2_\rho(r>r_{n,2})}$$

[8] Note in particular that Γ satisfies $\overline{\Gamma(z)} = \Gamma(\overline{z})$ for all $z \in \mathbb{C}$.

from which:

$$
\begin{aligned}
&\mathrm{dist}(\lambda_n, \mathrm{spec}(A_\infty[r_{n,2}])) \leq \|(A_\infty[r_{n,2}] - A_n[r_{n,2}])(w_n)\|_{L^2_\rho(r>r_{n,2})} \\
\leq\ & \left(\sup_{r\geq r_{n,2}} \left(p|\Phi_n(r) - \Phi_*(r)|^{p-1}\right)\right)^{\frac{1}{2}} \|w_n\|_{L^2_\rho(r>r_{n,2})} \leq \left(\sup_{r\geq r_{n,2}} \left(p|\Phi_n(r) - \Phi_*(r)|^{p-1}\right)\right)^{\frac{1}{2}} \\
\to\ & 0 \ \text{ as } \ n\to+\infty
\end{aligned}
$$

from (*2.4.9*). (*6.3.22*) follows by exchanging the role $A_n[r_{n,2}]$ and $A_\infty[r_{n,2}]$. □

Lemma 6.3.6 (Local continuity of the spectrum). *Let $r_0 > 0$ and let r_1 and r_2 such that*

$$e^{-\frac{2\pi}{\omega}} r_0 \leq r_1, r_2 \leq r_0$$

and

$$r_1 = r_2 + O(r_0^3).$$

Then, for any eigenvalue λ_1 of $A_\infty[r_1]$ such that $\lambda_1 \in [-3, 1]$, we have

$$\mathit{dist}(\lambda_1, \mathit{Spec}(A_\infty[r_2])) \lesssim r_0^{\frac{3}{2}}. \tag{6.3.23}$$

Proof. Recall from the proof of Lemma 6.3.5 that both $A_\infty[r_1]$ and $A_\infty[r_2]$ are diagonalizable. Furthermore, by Sturm-Liouville, their eigenvalues are simple. Let λ_1 be an eigenvalue of $A_\infty[r_1]$. We claim the existence of a nearby eigenvalue λ_2 of $A_\infty[r_2]$ using a classical Lyapunov Schmidt procedure.
Let φ_1 the normalized eigenfunction of $A_\infty[r_1]$ associated to λ_1 so that

$$A_\infty[r_1](\varphi_1) = \lambda_1\varphi_1, \quad \|\varphi_1\|_\rho = 1.$$

The eigenvalue equation

$$A_\infty[r_2](\varphi_2) = \lambda_2\varphi_2$$

is equivalent to

$$A_\infty[r_1](g) = \lambda_2 g + hg + (r_2 - r_1)\partial_r g \tag{6.3.24}$$

where

$$g(r) = \varphi_2(r + r_2 - r_1), \quad h(r) = \frac{pc_\infty^{p-1}}{(r + r_2 - r_1)^2} - \frac{pc_\infty^{p-1}}{r^2}.$$

We decompose

$$g = \varphi_1 + r_0\tilde{g}, \quad \lambda_2 = \lambda_1 + cr_0$$

where the constant c will be chosen later. Then, g satisfies (*6.3.24*) if and only if $\tilde{g}$ satisfies

$$(A_\infty[r_1] - \lambda_1)(\tilde{g}) = c\varphi_1 + cr_0\tilde{g} + \frac{h}{r_0}(\varphi_1 + r_0\tilde{g}) + \frac{r_2 - r_1}{r_0}\partial_r g. \tag{6.3.25}$$

We choose c such that

$$c(\varphi_1, r_0, \tilde{g}) := -\frac{1}{1 + r_0(\tilde{g}, \varphi_1)_\rho}\left(\frac{h}{r_0}(\varphi_1 + r_0\tilde{g}) + \frac{r_2 - r_1}{r_0}\partial_r(\varphi_1 + r_0\tilde{g}), \varphi_1\right)_\rho.$$

Then, the right-hand side of (*6.3.25*) is orthogonal to φ_1 and hence to the kernel of $A_\infty[r_1]-\lambda_1$ since λ_1 is a simple eigenvalue. Thus, we infer

$$\tilde{g}=\mathcal{F}(\tilde{g}) \tag{6.3.26}$$

where

$$\mathcal{F}(\tilde{g}):=B_\infty[r_1,\lambda_1]^{-1}\Big(c(\varphi_1,r_0,\tilde{g})(\varphi_1+r_0\tilde{g})+\frac{h}{r_0}(\varphi_1+r_0\tilde{g})+\frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde{g})\Big)$$

with the operator $B_\infty[r_1,\lambda_1]$ being the restriction of $A_\infty[r_1]-\lambda_1$ to the orthogonal complement of the kernel of $A_\infty[r_1]-\lambda_1$, i.e.

$$B_\infty[r_1,\lambda_1]=(A_\infty[r_1]-\lambda_1)_{|_{\varphi_1^\perp}}.$$

Since λ_1 is an eigenvalue of $A_\infty[r_1]$, from the explicit behavior (*6.3.19*) of the eigenfunctions of $\mathcal{L}_\infty$ and the boundary condition (*6.3.21*) at r_1 one deduces that there exists $k\in\mathbb{Z}$ such that

$$\omega\log(r_1)-\Phi(\lambda_1)=k\pi+\frac{\pi}{2}+O(r_0^2).$$

Let λ_1' be the smallest eigenvalue of $A_\infty[r_1]$ greater than λ_1. It then satisfies:

$$\omega\log(r_1)-\Phi(\lambda_1')=k\pi+\frac{\pi}{2}\pm\pi+O(r_0^2)$$

and so

$$|\Phi(\lambda_1)-\Phi(\lambda_1')|=\pi+O(r_0^2)\geq\frac{\pi}{2}.$$

As Φ is a continuous function we deduce that there exists $c>0$ independent of r_0 such that $\lambda_1'\geq\lambda_1+c$ and we infer

$$\inf\{|\lambda-\lambda_1|,\ \lambda\in\mathrm{Spec}(A_\infty[r_1]),\ \lambda>\lambda_1\}\geq c.$$

Similarly

$$\inf\{|\lambda-\lambda_1|,\ \lambda\in\mathrm{Spec}(A_\infty[r_1]),\ \lambda<\lambda_1\}\geq c',\quad c'>0$$

and we conclude that

$$\|B_\infty[r_1,\lambda_1]^{-1}\|_{\mathcal{L}(L^2_\rho,H^2_\rho)}\lesssim 1$$

with a bound that does not depend on r_0. Also, note that

$$\frac{h}{r_0}=\frac{pc_\infty^{p-1}}{r_0r^2}\left(\frac{1}{(1+\frac{r_2-r_1}{r})^2}-1\right)=\frac{h_1(r)}{r^2}$$

where

$$h_1(r)=-pc_\infty^{p-1}\frac{\left(\frac{2(r_1-r_2)}{r_0r}+\frac{(r_1-r_2)^2}{r_0r^2}\right)}{(1+\frac{r_2-r_1}{r})^2}.$$

Since

$$e^{-\frac{2\pi}{\omega}}r_0\leq r_1\leq r_0 \text{ and } r_1=r_2+O(r_0^3),$$

we infer

$$\|h_1\|_{L^\infty(r>r_1)} \lesssim \|h_1\|_{L^\infty(r>e^{-\frac{2\pi}{\omega}}r_0)} \lesssim r_0.$$

Moreover,

$$\|r^{-2}\|_{L^2(r_1<r<1)} \lesssim \left(\int_{r_1}^1 \frac{dr}{r^2}\right)^{\frac{1}{2}} \lesssim \frac{1}{r_1^{\frac{1}{2}}} \lesssim \frac{1}{r_0^{\frac{1}{2}}}.$$

Collecting the previous estimates, we infer

$$\begin{aligned}
& \|\mathcal{F}(\tilde{g})\|_{H^2_\rho(r>r_1)} \\
\lesssim\ & \|B_\infty[r_1,\lambda_1]^{-1}\|_{\mathcal{L}(L^2_\rho,H^2_\rho)}\left\|c(\varphi_1,r_0,\tilde{g})(\varphi_1+r_0\tilde{g})+\frac{h}{r_0}(\varphi_1+r_0\tilde{g})+\frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde{g})\right\|_{L^2_\rho} \\
\lesssim\ & |c(\varphi_1,r_0,\tilde{g})|(1+r_0\|\tilde{g}\|_{L^2_\rho})+r_0\|\tilde{g}\|_{H^1_\rho} \\
& +\|h_1\|_{L^\infty(r>r_1)}(1+r_0\|\tilde{g}\|_{L^2_\rho}+\|\varphi_1+r_0\tilde{g}\|_{L^\infty_{(r_1<r<1)}}\|r^{-2}\|_{L^2(r_1<r<1)}) \\
\lesssim\ & \frac{r_0^{\frac{1}{2}}}{1-r_0\|\tilde{g}\|_{L^2_\rho}}(1+r_0\|\tilde{g}\|_{L^2_\rho})+r_0\|\tilde{g}\|_{H^1_\rho}
\end{aligned}$$

and

$$\|\mathcal{F}(\tilde{g}_1)-\mathcal{F}(\tilde{g}_2)\|_{H^2_\rho(r>r_1)} \lesssim \frac{r_0^{\frac{3}{2}}}{1-r_0\|\tilde{g}\|_{L^2_\rho}}(1+r_0\|\tilde{g}\|_{L^2_\rho})\|\tilde{g}_1-\tilde{g}_2\|_{L^2_\rho}+r_0\|\tilde{g}_1-\tilde{g}_2\|_{H^1_\rho}.$$

Thus, for $r_0>0$ small enough, the Banach fixed point theorem applies in the space $H^2_\rho(r>r_1)$ and yields a unique solution $\tilde{g}$ to (*6.3.26*) with

$$\|\tilde{g}\|_{H^2_\rho(r>r_1)} \lesssim r_0^{\frac{1}{2}}.$$

Hence, φ_2 with

$$\varphi_2(r)=g(r+r_1-r_2), \quad g=\varphi_1+r_0\tilde{g}$$

satisfies

$$A_\infty[r_2](\varphi_2)=\lambda_2\varphi_2$$

where

$$\begin{aligned}
\lambda_2 &= \lambda_1+c(\varphi_1,r_0,\tilde{g})r_0 \\
&= \lambda_1-\frac{r_0}{1+r_0(\tilde{g},\varphi_1)_\rho}\left(\frac{h}{r_0}(\varphi_1+r_0\tilde{g})+\frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde{g}),\varphi_1\right)_\rho.
\end{aligned}$$

Thus, λ_2 belongs to the spectrum of $A_\infty[r_2]$ and hence

$$\begin{aligned}
\mathrm{dist}(\lambda_1,\mathrm{Spec}(A_\infty[r_2])) &\leq |\lambda_2-\lambda_1| \\
&\leq \left|\frac{r_0}{1+r_0(\tilde{g},\varphi_1)_\rho}\left(\frac{h}{r_0}(\varphi_1+r_0\tilde{g})+\frac{r_2-r_1}{r_0}\partial_r(\varphi_1+r_0\tilde{g}),\varphi_1\right)_\rho\right|.
\end{aligned}$$

In view of the previous estimates, we infer

$$\mathrm{dist}(\lambda_1,\mathrm{Spec}(A_\infty[r_2])) \lesssim \frac{r_0^{\frac{3}{2}}}{1-r_0\|\tilde{g}\|_{L^2_\rho}}(1+r_0\|\tilde{g}\|_{L^2_\rho}) \lesssim r_0^{\frac{3}{2}}.$$

and (*6.3.23*) is proved. □

6.3.4 Proof of Proposition 6.3.1

Recall that $\mathcal{L}_n$ is diagonalizable in a Hilbertian basis of L^2_ρ, and hence the spectral gap estimate (*6.3.6*) follows from the explicit distribution of eigenvalues (*6.3.2*) which we now prove. Observe that the symmetry group of dilations and translations generates the explicit eigenmodes

$$\mathcal{L}_n\Lambda\Phi_n = -2\Lambda\Phi_n, \quad \mathcal{L}_n\nabla\Phi_n = -\nabla\Phi_n. \tag{6.3.27}$$

Using the decomposition into spherical harmonics (*6.3.7*), the further study of the quadratic form $(\mathcal{L}_n(u), u)_\rho$ reduces to the study of the quadratic form $(\mathcal{L}_{n,m}(u), u)_\rho$ for $m \geq 0$ for which classical Strum Liouville arguments are now at hand.

step 1 The case $m = 1$. Let $\varphi_{n,1}$ be defined in Lemma 6.3.2. In particular, $\varphi_{n,1}$ satisfies

$$\mathcal{L}_{n,1}(\varphi_{n,1}) = 0, \quad \varphi_{n,1}(0) = 0, \quad \varphi'_{n,1}(0) = 1.$$

Then from standard Sturm Liouville oscillation argument for central potentials, [143], the number of zeros of $\varphi_{n,1}$ in $r > 0$ correspond to the number of strictly negative eigenvalues of $\mathcal{L}_{n,1}$.

Since we have

$$\nabla\Phi_n(x) = \Phi'_n(r)\frac{x}{r} = \Phi'_n(r)(Y^{(1,-1)}, Y^{(1,1)}, Y^{(1,0)})$$

and hence

$$\mathcal{L}_n(\nabla\Phi_n) = -\nabla\Phi_n \ \text{ implies } \ \mathcal{L}_{n,1}(\Phi'_n) = -\Phi'_n.$$

Thus, $\mathcal{L}_{n,1}$ has at least one strictly negative eigenvalue, and hence $\varphi_{n,1}$ has at least one zero which we denote by $r_{n,1} > 0$. On $[0, r_0]$, we have by (*6.3.15*):

$$\sup_{0\leq r\leq r_0} \frac{\left|\mu_n^{-1}\varphi_{n,1}(r) - \nu_1\left(\frac{r}{\mu_n}\right)\right|}{\left|\nu_1\left(\frac{r}{\mu_n}\right)\right|} \lesssim r_0^2$$

Since $\nu_1(r) > 0$ for all $r > 0$, we infer that $\varphi_{n,1}$ can not vanish on $[0, r_0]$. Hence, $r_{n,1} \geq r_0$.

No other zero. Assume by contradiction that there exists a second zero $r_{n,2} > r_{n,1}$. Let $f_{n,1}$ being given as

$$f_{n,1} := \begin{cases} \varphi_{n,1} & \text{on } r_{n,1} < r < r_{n,2}, \\ 0 & \text{on } r < r_{n,1}, \\ 0 & \text{on } r > r_{n,2}. \end{cases}$$

Then, we have $f_{n,1} \in H^1_\rho$ and

$$(\mathcal{L}_{n,1}(f_{n,1}), f_{n,1})_\rho = 0. \tag{6.3.28}$$

On the other hand, using (*6.1.1*):

$$\begin{aligned} (\mathcal{L}_{\infty,1}(u), u)_\rho &= \|u'\|^2_\rho + \int_0^{+\infty} \frac{2 - pc_\infty^{p-1}}{r^2} u^2 r^2 \rho dr \\ &= \|u'\|^2_\rho + \frac{2(p+1)}{(p-1)^2}\left(\int_0^{+\infty} \frac{u^2}{r^2} r^2 \rho dr\right) \gtrsim \|\frac{u}{r}\|^2_{L^2_\rho}. \end{aligned} \tag{6.3.29}$$

We now estimate from (*2.4.10*)

$$\sup_{r\geq r_0} r^2|\Phi_n^{p-1}-(\Phi_*)^{p-1}| = o_{n\to+\infty}(1) \tag{6.3.30}$$

and hence for u supported in $(r_0,+\infty)$:

$$\begin{aligned} |(\mathcal{L}_{\infty,1}(u),u)_\rho-(\mathcal{L}_{n,1}(u),u)_\rho| &\lesssim \int_{r_0}^{+\infty}\left|\Phi_n^{p-1}-\Phi_*^{p-1}\right|u^2r^2\rho(r)dr \\ &\leq o_{n\to+\infty}(1)\left\|\frac{u}{r}\right\|_{L^2_\rho}^2. \end{aligned} \tag{6.3.31}$$

Since $f_{n,1}$ is supported in $(r_{n,1},r_{n,2})\subset(r_0,+\infty)$, (*6.3.29*), (*6.3.31*) applied to $f_{n,1}$ and (*6.3.28*) yield a contradiction for $n\geq N$ large enough. Thus, $r_{n,2}$ can not exist, and hence $\varphi_{n,1}$ vanishes only once.

$\varphi_{n,1}$ ***is not an eigenstate.*** Since $\varphi_{n,1}$ vanishes only once, $\mathcal{L}_{n,1}$ has exactly one strictly negative eigenvalue. It remains to check the $\varphi_{n,1}\notin L^2_\rho$, i.e. $\varphi_{n,1}$ is not an eigenvector associated to the eigenvalue 0. To this end, note that $\varphi_{n,1}$ is strictly positive on $(0,r_{n,1})$ from (*6.3.8*) and strictly negative on $(r_{n,1},+\infty)$. In particular, we have

$$\varphi_{n,1}'(r_{n,1})<0.$$

Since $\mathcal{L}_{n,1}(\varphi_{n,1})=0$, we have

$$(r^2\rho\varphi_{n,1}')'=r^2\rho\left[\frac{2}{p-1}+\frac{(2-pr^2\Phi_n^{p-1})}{r^2}\right]\varphi_{n,1}$$

and from (*6.3.31*) for $r\geq r_{n,1}\geq r_0$:

$$2-r^2p\Phi_n^{p-1}=2-pc_\infty^{p-1}+pc_\infty^{p-1}-r^2p\Phi_n^{p-1}\geq\frac{2(p+1)}{(p-1)^2}+o(1)>0. \tag{6.3.32}$$

Since $\varphi_{n,1}$ is strictly negative on $(r_{n,1},+\infty)$, we deduce

$$r^2\rho\varphi_{n,1}'(r)\leq r_{n,1}^2\rho(r_{n,1})\varphi_{n,1}'(r_{n,1})=c_1<0 \text{ on } (r_{n,1},+\infty)$$

which implies

$$\int_{r_{n,1}}^{+\infty}|\varphi_{n,1}'(r)|^2\rho r^2dr\gtrsim\int_{r_{n,1}}^{+\infty}\frac{dr}{r^2\rho}=+\infty$$

and hence $\varphi_{n,1}\notin H^1_\rho$ and is therefore not an eigenvector.

Conclusion. We conclude that -1 is the only negative eigenvalue of $\mathcal{L}_{n,1}$, and is associated to the single eigenvector Φ_n'. Hence, there exists a constant $c_n>0$ such that for all $u\in H^1_\rho$:

$$(\mathcal{L}_{n,1}(u),u)_\rho\geq c_n\|u\|_{L^2_\rho}^2-\frac{1}{c_n}(u,\Phi_n')_\rho^2. \tag{6.3.33}$$

step 2 The case $m\geq 2$. Let $\varphi_{n,m}$ be defined in Lemma 6.3.2. In particular, $\varphi_{n,m}$ satisfies

$$\mathcal{L}_{n,m}(\varphi_{n,m})=0 \text{ and } \varphi_{n,m}=r^m(1+O(r^2)) \text{ as } r\to 0_+.$$

Then, the number of zeros of $\varphi_{n,m}$ in $r>0$ corresponds to the number of strictly negative eigenvalues of $\mathcal{L}_{n,m}$. On $[0,r_0]$, we have by Lemma 6.3.2.

$$\sup_{0\le r\le r_0}\frac{\left|\mu_n^{-m}\varphi_{n,m}(r)-\nu_m\left(\frac{r}{\mu_n}\right)\right|}{\left|\nu_m\left(\frac{r}{\mu_n}\right)\right|}\lesssim r_0^2$$

and $\nu_m(r)>0$ for all $r>0$, and hence $\varphi_{n,m}$ cannot vanish on $[0,r_0]$:

$$\varphi_{n,m}(r)>0 \text{ on } [0,r_0].$$

Next, we investigate the sign of $\varphi_{n,m}'(r_0)$. Recall (*6.3.12*):

$$\nu_m(r)\sim\frac{c_{m,-}}{r^{\frac{1-\sqrt{\Delta_m}}{2}}} \text{ as } r\to+\infty \quad c_{m.-}>0$$

and hence

$$\nu_m'(r)\sim\frac{c_{m,-}(\sqrt{\Delta_m}-1)}{r^{\frac{3-\sqrt{\Delta_m}}{2}}} \text{ as } r\to+\infty.$$

We infer for $n\ge N$ large enough

$$\varphi_{n,m}(r_0)=\frac{c_{m,-}(1+O(r_0^2))\mu_n^m}{\left(\frac{r_0}{\mu_n}\right)^{\frac{1-\sqrt{\Delta_m}}{2}}}$$

and

$$\varphi_{n,m}'(r_0)=\frac{c_{m,-}(\sqrt{\Delta_m}-1)(1+O(r_0^2))\mu_n^{m-1}}{\left(\frac{r_0}{\mu_n}\right)^{\frac{3-\sqrt{\Delta_m}}{2}}}.$$

Thus, taking also into account that $\varphi_{n,m}(r)>0$ on $[0,r_0]$, we infer from the identity for $\varphi_{n,m}(r_0)$ that

$$c_{m,-}>0.$$

Since $\sqrt{\Delta_m}\ge\sqrt{\Delta_1}=\frac{p+3}{p-1}>1$, we conclude:

$$\phi_{n,m}(r_0)>0,\quad \phi_{n,m}'(r_0)>0. \tag{6.3.34}$$

Since $\mathcal{L}_{n,m}(\varphi_{n,m})=0$, we have

$$(r^2\rho\varphi_{n,m}')'=r^2\rho\left[\frac{2}{p-1}+\frac{(m(m+1)-pr^2\Phi_n^{p-1})}{r^2}\right]\varphi_{n,m} \tag{6.3.35}$$

which together with (*6.3.34*), (*6.3.32*) and the fact that $m\ge2$, and an elementary continuity argument ensures

$$\phi_{m,n}'(r)>0,\quad \phi_{n,m}(r)\ge\phi_{n,m}(r_0)>0 \text{ for } r\ge r_0.$$

Hence $\phi_{n,m}$ does not vanish on $(0,+\infty)$ and using (*6.3.35*):

$$r^2\phi_{n,m}'\rho(r)\ge r_0^2\phi_{n,m}'\rho(r_0)=c_0>0$$

which implies

$$\int_{r_0}^{+\infty}(\phi_{n,m}')^2\rho r^2dr\gtrsim\int_{r_0}^{+\infty}\frac{dr}{r^2\rho}=+\infty$$

and hence $\phi_{n,m}$ is not eigenvector. We finally conclude that for $m = 2$ and all $n \geq N$ large enough, $\mathcal{L}_{n,2}$ has a spectral gap and there exists a constant $c_n > 0$ such that we have for all $u \in H^1_\rho$

$$(\mathcal{L}_{n,2}(u), u)_\rho \geq c_n \|u\|^2_{L^2_\rho}.$$

Since we have for all $m \geq 2$

$$(\mathcal{L}_{n,m}(u), u)_\rho \geq (\mathcal{L}_{n,2}(u), u)_\rho,$$

we infer for all $m \geq 2$ and for all $u \in H^1_\rho$

$$(\mathcal{L}_{n,m}(u), u)_\rho \geq c_n \|u\|^2_{L^2_\rho}. \tag{6.3.36}$$

step 3. The case $m = 0$. We now focus onto $\mathcal{L}_{n,0}$ which is the most delicate case, and we claim that $\mathcal{L}_{n,0}$ has exactly $n+1$ strictly negative eigenvalues, and that 0 is not in the spectrum. The key is to combine the uniform bounds (*2.4.9*) with the explicit knowledge of the limiting outer spectrum, Lemma 6.3.4, as nicely suggested at the formal level in [8].

Let $\varphi_{n,0}$ be the solution to (*6.3.16*) so that the number of strictly negative eigenvalues of $\mathcal{L}_{n,0}$ coincides with the numbers of zeroes of $\varphi_{n,0}$. We count the number of zeros of $\varphi_{n,0}$ by comparing them with the number of zeros of $\Lambda\Phi_n$.

Lower bound. First, since $\Lambda\Phi_n$ is an eigenvector of $\mathcal{L}_{n,0}$ corresponding to the eigenvalue -2 and since $\Lambda\Phi_n$ vanishes n times from Proposition 6.2.5, we infer from Sturm Liouville

$$\#\text{Spec}(\mathcal{L}_{n,0} + 2) \cap (-\infty, 0] = n + 1.$$

In particular, since the number of strictly negative eigenvalues of $\mathcal{L}_{n,0}$ coincides with the number of zeroes of $\varphi_{n,0}$, we infer

$$\#\{r \geq 0 \text{ such that } \varphi_{n,0}(r) = 0\} \geq n + 1.$$

Upper bound. Recall (*6.3.17*):

$$\sup_{0 \leq r \leq r_0} \left(1 + \frac{r}{\mu_n}\right)^{\frac{1}{2}} \left|\varphi_{n,0}(r) - \frac{p-1}{2}\Lambda Q\left(\frac{r}{\mu_n}\right)\right| \lesssim r_0^2.$$

Also, we have $\Lambda Q(0) \neq 0$ and from (*6.2.41*):

$$\left(\frac{r_0}{\mu_n}\right)^{\frac{1}{2}} \left|\Lambda Q\left(\frac{r_0}{\mu_n}\right)\right| \geq c > 0$$

for some constant $c > 0$ independent of n. Hence $\varphi_{n,0}$ and ΛQ vanish the same number of times on $[0, r_0]$. Since on the other hand ΛQ and $\Lambda\Phi_n$ vanish the same number of times on $[0, r_0]$ from (*6.2.40*), $\varphi_{n,0}$ and $\Lambda\Phi_n$ vanish the same number of times of $[0, r_0]$.

Let now $r_{n,0}$ to be the last zero of $\Lambda\Phi_n$ before r_0. In view of Corollary 6.2.6, we have

$$e^{-\frac{2\pi}{\omega}} r_0 \leq r_{n,0} \leq r_0.$$

Let us now consider the operators (*6.3.21*):

$$A_n[r_{n,0}](f) = \mathcal{L}_{n,0}(f) \text{ on } r > r_{0,n}, \quad f(r_{n,0}) = 0,$$
$$A_\infty[r_{n,0}](f) = \mathcal{L}_\infty(f) \text{ on } r > r_{n,0}, \quad f(r_{n,0}) = 0,$$

then

$$\mathcal{L}_{n,0}(\Lambda\Phi_n) = -2\Lambda\Phi_n \text{ and } \Lambda\Phi_n(r_{n,0}) = 0,$$

implies

$$A_n[r_{n,0}](\Lambda\Phi_n) = -2\Lambda\Phi_n.$$

In particular, -2 belongs to the spectrum of $A_n[r_{n,0}]$. In view of Lemma 6.3.5, we deduce for $n \geq N$ large enough that the exists an eigenvalue λ_0 of $A_\infty[r_{n,0}]$ such that $\lambda_0 = -2 + o(1)$. On the other hand, in view of Lemma 6.3.4, the solutions to

$$\mathcal{L}_\infty(f) = \lambda f$$

with $f \in H^1_\rho$ are completely explicit and behave for $r \to 0$ as

$$f \sim \frac{1}{r^{\frac{1}{2}}} \cos(\omega \log(r) - \Phi(\lambda))$$

with

$$\Phi(\lambda) = \arg\left(\frac{2^{\frac{i\omega}{2}}\Gamma(i\omega)}{\Gamma\left(\frac{1}{p-1} - \frac{\lambda}{2} - \frac{1}{4} + \frac{i\omega}{2}\right)}\right).$$

In order for f to be an eigenfunction of $A_\infty[r_{n,0}]$, we need $f(r_{n,0}) = 0$ and hence there should exists $k \in \mathbb{Z}$ such that

$$\omega \log(r_{n,0}) - \Phi(\lambda) \sim \frac{\pi}{2} + k\pi.$$

Recall that $\lambda_0 = -2 + o(1)$ is an eigenvalue of $A_\infty[r_{n,0}]$, and let $\lambda_1 > \lambda_0$ be the next eigenvalue of $A_\infty[r_{n,0}]$. Then, there exists $k_0 \in \mathbb{R}$ such that

$$\omega \log(r_{n,0}) - \Phi(\lambda_0) \sim \frac{\pi}{2} + k_0\pi, \quad \omega \log(r_{n,0}) - \Phi(\lambda_1) \sim \frac{\pi}{2} + (k_0 - 1)\pi$$

and hence

$$\Phi(\lambda_1) = \Phi(-2) + \pi + o(1). \tag{6.3.37}$$

Now, by numerical check, we have[9]

$$\sup_{5 \leq p < +\infty} \sup_{-2 \leq \lambda \leq 0.5} (\Phi(\lambda) - \Phi(-2) - \pi) \sim -0.5945 < 0,$$

and hence, the solution λ_1 to (*6.3.37*) satisfies

$$\inf_{5 \leq p < +\infty} \lambda_1 \geq 0.5 > 0.$$

We infer that $A_\infty[r_{n,0}]$ has no eigenvalue between $\lambda_0 = -2 + o(1)$ and $\lambda_1 \geq 0.5$. Hence, using again Lemma 6.3.5, $A_n[r_{n,0}]$ has no eigenvalue between -2 and $\lambda_1 + o(1) \geq 0.25$. Thus, we have

$$\#\mathrm{Spec}(A_n[r_{n,0}]) \cap (-\infty, 0] = \#\mathrm{Spec}(A_n[r_{n,0}] + 2) \cap (-\infty, 0].$$

[9]Notice that $\Phi(\lambda)$ has a well defined limit as $p \to +\infty$ given by

$$\Phi_\infty(\lambda) = \arg\left(\frac{2^{\frac{i}{4}}\Gamma(\frac{i}{2})}{\Gamma\left(-\frac{\lambda}{2} - \frac{1}{4} + \frac{i}{4}\right)}\right).$$

Our numerics are carried out using Matlab and indicate that $\Phi_p(\lambda)$ is increasing on $[-2, 0.5]$ for all $p \geq 5$ so that the maximum on $[-2, 0.5]$ is achieved at $\lambda = 0.5$. Also, this maximum appears to be a growing function of p so that the maximum in p is given by $\Phi_\infty(0.5) - \Phi_\infty(-2) - \pi \sim -0.5945$. See [8] for a similar numerical computation.

On the other hand, we have

$$\#\mathrm{Spec}(A_n[r_{n,0}]+2)\cap(-\infty,0] = \#\{r>r_{n,0} \text{ such that } \Lambda\Phi_n(r)=0\}+1$$

since $\Lambda\Phi_n$ is in the kernel of $A_n[r_{n,0}]+2$, and hence

$$\#\mathrm{Spec}(A_n)\cap(-\infty,0] = \#\{r>r_{n,0} \text{ such that } \Lambda\Phi_n(r)=0\}+1.$$

Also, since $\varphi_{n,0}$ can not be an eigenvector of A_n[10], we have

$$\#\mathrm{Spec}(A_n[r_{n,0}])\cap(-\infty,0] = \#\{r>r_{n,0} \text{ such that } \varphi_{n,0}(r)=0\}.$$

We infer

$$\#\{r>r_{n,0} \text{ such that } \varphi_{n,0}(r)=0\} = \#\{r>r_{n,0} \text{ such that } \Lambda\Phi_n(r)=0\}+1.$$

But since $r_{n,0}$ has been chosen to be the last zero of $\Lambda\Phi_n$ before r_0, we have

$$\#\{r>r_{n,0} \text{ such that } \Lambda\Phi_n(r)=0\} = \#\{r>r_0 \text{ such that } \Lambda\Phi_n(r)=0\}$$

and hence

$$\#\{r>r_{n,0} \text{ such that } \varphi_{n,0}(r)=0\} = \#\{r>r_0 \text{ such that } \Lambda\Phi_n(r)=0\}+1.$$

Next, together with the fact that $\varphi_{n,0}$ and $\Lambda\Phi_n$ vanish the same number of times of $[0,r_0]$, we infer

$$\begin{aligned}
&\#\{r>0 \text{ such that } \varphi_{n,0}(r)=0\}\\
\leq\ &\#\{0\leq r\leq r_0 \text{ such that } \varphi_{n,0}(r)=0\}+\#\{r>r_{n,0} \text{ such that } \varphi_{n,0}(r)=0\}\\
=\ &\#\{0\leq r\leq r_0 \text{ such that } \Lambda\Phi_n(r)=0\}+\#\{r>r_0 \text{ such that } \Lambda\Phi_n(r)=0\}+1\\
=\ &\#\{r>0 \text{ such that } \Lambda\Phi_n(r)=0\}+1\\
=\ &n+1
\end{aligned}$$

and since

$$\#\{r\geq 0 \text{ such that } \varphi_{n,0}(r)=0\}\geq n+1.$$

$\phi_{n,0}$ ***is not an eigenstate.*** We conclude that

$$\#\{r\geq 0 \text{ such that } \varphi_{n,0}(r)=0\}= n+1.$$

Assume now by contradiction that $\varphi_{n,0}$ is in the kernel of $\mathcal{L}_{n,0}$. Recall that $r_{0,n}<r_0$ is the last 0 of $\Lambda\Phi_n$ and let $r_{1,n}<r_0$ be the last 0 of $\varphi_{n,0}$. In particular, we have from Lemma 6.3.3:

$$e^{-\frac{2\pi}{\omega}}r_0\leq r_{0,n}, r_{1,n}\leq r_0 \text{ and } r_{1,n}=r_{0,n}+O(r_0^3).$$

Also, since $\varphi_{n,0}$ is in the kernel of $\mathcal{L}_{n,0}$ and $\varphi_{n,0}(r_{1,n})=0$, we infer that 0 is in the spectrum of $A_n[r_{1,n}]$, and hence applying Lemma 6.3.5 twice as well as Lemma 6.3.6, we obtain that

$$\mathrm{dist}(\mathrm{Spec}(A_n[r_{0,n}]),0)\lesssim r_0^{\frac{3}{2}}+o(1)$$

[10]Indeed, $\varphi_{n,0}$ would be an eigenvector for the eigenvalue 0, but 0 is not in the spectrum of A_n as seen above.

as $n \to +\infty$. In particular, we have for $r_0 > 0$ small enough and $n \geq N$ large enough

$$\mathrm{dist}(\mathrm{Spec}(A_n[r_{0,n}]), 0) \leq 0.2.$$

On the other hand, we have proved above that $A_n[r_{n,0}]$ has no eigenvalue between -2 and $\lambda_1 + o(1) \geq 0.25$ so that

$$\mathrm{dist}(\mathrm{Spec}(A_n[r_{0,n}]), 0) \geq 0.25$$

which is a contradiction. Hence $\varphi_{0,n}$ is not in the kernel of $\mathcal{L}_{n,0}$.

Conclusion. We conclude that $\mathcal{L}_{n,0}$ has exactly $n+1$ strictly negative eigenvalues. On the other hand, since $\Lambda\Phi_n$ is an eigenvector of $\mathcal{L}_{n,0}$ corresponding to the eigenvalue -2 and since $\Lambda\Phi_n$ vanishes n times, we infer

$$\#\mathrm{Spec}(\mathcal{L}_{n,0} + 2) \cap (-\infty, 0] = n+1,$$

and hence $\mathcal{L}_{n,0}$ has exactly $n+1$ negative eigenvalues and the largest negative eigenvalue is -2. We denote these eigenvalues by

$$-\mu_{n+1,n} < \cdots < -\mu_{2,n} < -\mu_{1,n} = -2.$$

By Sturm Liouville, these eigenvalues are simple and associated to eigenvectors

$$\psi_{j,n}, \quad \|\psi_{j,n}\|_{L^2_\rho} = 1, \quad \psi_{1,n} = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_\rho}.$$

Also, there holds for some constant $c_n > 0$ and for all $u \in H^1_\rho$

$$(\mathcal{L}_{n,0}(u), u)_\rho \geq c_n \|u\|^2_{L^2_\rho} - \frac{1}{c_n}\left[\sum_{j=1}^{n+1} (u, \psi_{j,n})^2_\rho\right]. \qquad (6.3.38)$$

The behavior as $r \to +\infty$ of the eigenstates (*6.3.5*) follows from the asymptotic in Lemma 6.3.4 and a standard ODE argument using the variation of constants formula, this is left to the reader.

step 4 Conclusion. We decompose $u \in H^1_\rho$ as

$$u = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} u_{m,k} Y^{(m,k)}$$

where $u_{m,k}$ are radial functions satisfying

$$\|u\|^2_\rho = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} \|u_{m,k}\|^2_\rho.$$

We have

$$(\mathcal{L}_n(u), u)_\rho = \sum_{m=0}^{+\infty} \sum_{k=-m}^{m} (\mathcal{L}_{n,m}(u_{m,k}), u_{m,k})_\rho.$$

Together with (*6.3.33*), (*6.3.36*) and (*6.3.38*), we infer for all $u \in H^1_\rho$

$$\begin{aligned}(\mathcal{L}_n(u), u)_\rho &= (\mathcal{L}_{n,0}(u_{0,0}), u_{0,0})_\rho + \sum_{k=-1}^{1} (\mathcal{L}_{n,1}(u_{1,k}), u_{1,k})_\rho + \sum_{m=2}^{+\infty}\sum_{k=-m}^{m} (\mathcal{L}_{n,m}(u_{m,k}), u_{m,k})_\rho \\ &\geq c_n \|u\|_\rho^2 - \frac{1}{c_n}\left[\sum_{j=1}^{n+1}(u_{0,0}, \psi_{j,n})_\rho^2 + \sum_{k=1}^{3}(u_{1,k}, \Phi_n')_\rho^2\right].\end{aligned}$$

Since $\psi_{j,n}$ are all radial, we have

$$(u_{0,0}, \psi_{j,n})_\rho = (u, \psi_{j,n})_\rho.$$

Also, since

$$\nabla \Phi_n(x) = \Phi_n'(r)\frac{x}{r} = \Phi_n'(r)(Y^{(1,-1)}, Y^{(1,1)}, Y^{(1,0)}),$$

we infer

$$\sum_{k=1}^{3}(u_{1,k}, \Phi_n')_\rho^2 = \sum_{k=1}^{3}(u, \partial_k \Phi_n)_\rho^2.$$

Finally, there holds for some constant $c_n > 0$ and for all $u \in H^1_\rho$

$$(\mathcal{L}_n u, u)_\rho \geq c_n \|u\|_{H^1_\rho}^2 - \frac{1}{c_n}\left[\sum_{j=0}^{n}(u, \psi_{j,n})_\rho^2 + \sum_{k=1}^{3}(u, \partial_k \Phi_n)_\rho^2\right].$$

This concludes the proof of Proposition 6.3.1.

6.4 Dynamical control of the flow

We now turn to the question of the stability of the self similar solution, and more precisely the construction of a manifold of *finite energy* initial data such that the corresponding solution to (NLH) blows up in finite time with Φ_n profile in the self similar regime described by Theorem 2.4.4. n is now fixed.

6.4.1 Setting of the bootstrap

We set up in this section the bootstrap analysis of the flow for a suitable set of finite energy initial data. The solution will be decomposed in a suitable way with standard technique, see [100, 111].

Geometrical decomposition. We start by showing the existence of the suitable decomposition. Recall the spectral Proposition 6.3.1. To ease notations we now omit the n subscript and write ψ_j, μ_j and λ_j instead.

Define the L^∞ tube around the renormalized versions of Φ_n:

$$X_\delta = \left\{ u = \frac{1}{\lambda^{\frac{2}{p-1}}}(\Phi_n + v)\left(\frac{x-y}{\lambda}\right),\ y \in \mathbb{R}^d,\ \lambda > 0,\ \|v\|_{L^\infty} < \delta \right\}$$

Lemma 6.4.1 (Geometrical decomposition). *There exists $\delta > 0$ and $C > 0$ such that any $u \in X_\delta$ has a unique decomposition*

$$u = \frac{1}{\lambda^{\frac{2}{p-1}}}(\Phi_n + \sum_{j=2}^{n+1} a_j \psi_j + \varepsilon)\left(\frac{x-\bar{x}}{\lambda}\right),$$

where ε satisfies the orthogonality conditions

$$(\varepsilon, \psi_j)_\rho = (\varepsilon, \partial_k \Phi_n)_\rho = 0, \quad 1 \le j \le n+1, \quad 1 \le k \le 3,$$

the parameters λ, $\bar{x}$ and a_j being Fréchet differentiable on X_δ, and with

$$\|\varepsilon\|_{L^\infty} + \sum |a_j| \le C. \tag{6.4.1}$$

Proof. It is a classical consequence of the implicit function theorem.

step 1 Decomposition near $\lambda = 1$, $\bar{x} = 0$. We introduce the smooth maps

$$F(v, \mu, x, b_1, \dots, b_n) = \mu^{\frac{2}{p-1}} (\Phi_n + v)(\mu y + x) - \Phi_n - \sum_{j=2}^{n+1} b_j \psi_j$$

and

$$G = ((F, \Lambda\Phi_n), (F, \partial_1\Phi_n), (F, \partial_2\Phi_n), (F, \partial_3\Phi_n), (F, \psi_2), \dots, (F, \psi_{n+1})).$$

We immediately check that $G(\Phi_n, 1, 0, \dots, 0) = 0$ and that

$$\frac{\partial G}{\partial(\mu, x, b_2, \dots, b_{n+1})}_{|(\Phi_n, 1, 0, \dots, 0)}$$

is invertible. In view of the implicit function theorem, for $\kappa > 0$ small enough, for any

$$\|v\|_{L^\infty} \le \kappa$$

there exists $(\mu, z, a_2, \dots, a_{n+1})$ and

$$\varepsilon = F(v, \mu, z, a_2, \dots, a_{n+1})$$

such that

$$u = \Phi_n + v = \frac{1}{\mu^{\frac{2}{p-1}}} \left(\Phi_n + \sum_{j=2}^{n+1} a_j \psi_j + \varepsilon \right) \left(\frac{x - z}{\mu} \right),$$

$$(\varepsilon, \psi_j) = (\varepsilon, \partial_k \Phi_n) = 0, \quad 1 \le j \le n, \quad 1 \le k \le 3,$$

and there exist two universal constants $K, \tilde{K} > 0$ such that

$$\|\varepsilon\|_{L^\infty} + \sum_{j=2}^{n+1} |a_j| + |\mu - 1| + |z| \le K \|v\|_{L^\infty}$$

and such that the decomposition is unique under the bound

$$\|\varepsilon\|_{L^\infty} + \sum_{j=2}^{n+1} |a_j| + |\mu - 1| + |z| \le \tilde{K}. \tag{6.4.2}$$

step 2 Decomposition near any $\lambda, \bar{x}$. For any $\delta > 0$, we take $C = C(\delta) := K\delta$. Let $u \in X_\delta$ then for some $\lambda' > 0$ and y one has

$$u(x) = \frac{1}{\lambda'^{\frac{2}{p-1}}} (\Phi_n + v) \left(\frac{x - y}{\lambda'} \right), \ \|v\|_{L^\infty} < \delta.$$

The first step then provides the decomposition claimed in the lemma for δ small enough via the formulas $\lambda = \lambda'\mu(v)$, $\bar{x} = y - \lambda' z(v)$, $a_j = a_j(v)$ and $\varepsilon = \varepsilon(v)$. We will show in the next step that the decomposition is unique, implying that the parameters are Fréchet differentiable on X_δ for those of step 1 are.

step 3 Uniqueness of the decomposition. First, from a continuity argument, for any $\epsilon > 0$, there exists $\delta > 0$ such that if

$$(\Phi_n + v)(x) = \frac{1}{\mu^{\frac{2}{p-1}}}(\Phi_n + v')\left(\frac{x-y}{\mu}\right), \ \|v\|_{L^\infty} + \|v'\|_{L^\infty} \leq \delta$$

then

$$|\mu - 1| + |y| \leq \epsilon.$$

Now recall that $C = K\delta$ and assume that we are given a second decomposition for $u \in X_\delta$. In view of step 2, performing a change of variable, this amount to say that $\Phi_n + v$ admits another decomposition:

$$(\Phi_n + v)(x) = \frac{1}{\bar{\mu}^{\frac{2}{p-1}}}(\Phi_n + \sum_{j=2}^{n+1} \bar{a}_j \psi_j + \bar{\varepsilon})\left(\frac{x - \bar{z}}{\bar{\mu}}\right)$$

and the bound (*6.4.1*) gives

$$\sum_{j=2}^{n+1} |\bar{a}_j| + \|\bar{\varepsilon}\|_{L^\infty} \leq K\delta.$$

Using the above continuity estimate, one obtains that for δ small enough

$$|\bar{z}| + |\bar{\mu} - 1| \ll \tilde{K}.$$

Therefore, for δ small enough the second decomposition associated with $\bar{\mu}$, $\bar{z}$, $\bar{a}_j$ and $\bar{\varepsilon}$ satisfies (*6.4.2*), and is therefore the one given by step 2 by uniqueness. □

Description of the initial datum. We will now focus on solutions of (NLH) that are a suitable perturbation of Φ_n at initial time:

$$u_0 = \frac{1}{\lambda_0^{\frac{2}{p-1}}}(\Phi_n + v_0)\left(\frac{x}{\lambda_0}\right) \tag{6.4.3}$$

with

$$v_0 = \sum_{j=2}^{n+1} a_j \psi_j + \varepsilon_0, \quad (\varepsilon_0, \psi_j)_\rho = (\varepsilon_0, \partial_k \Phi_n)_\rho = 0, \quad 1 \leq j \leq n+1, \quad 1 \leq k \leq 3. \tag{6.4.4}$$

For $s_0 \gg 1$ and $\mu, K_0 > 0$ three constants to be defined later on, the parameters λ_0, a_j and the profile ε_0 satisfy the bounds

- rescaled solution:

$$\lambda_0 = e^{-s_0}; \tag{6.4.5}$$

- initial control of the unstable modes:

$$\sum_{j=2}^{n+1} |a_j|^2 \leq e^{-2\mu s_0}; \tag{6.4.6}$$

- smallness of suitable initial norms:

$$\|\varepsilon_0\|_{H^2_\rho} + \|\Delta v_0\|_{L^2} + \|w_0\|_{\dot{H}^{s_c}} \leq K_0 e^{-\mu s_0}; \tag{6.4.7}$$

where w_0 is given by

$$w_0 = \left(1 - \chi_{\frac{1}{\lambda_0}}\right)\Phi_n + v_0.$$

Note that in view of the L^∞ bound (*6.4.23*), the decomposition (*6.4.3*) is precisely the one given by Lemma 6.4.1.

Renormalized flow. As long as the solution $u(t)$ starting from (*6.4.3*) belongs to X_δ, Lemma 6.4.1 applies and it can be written

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}}(\Phi_n + \psi + \varepsilon)(s,z), \quad y = \frac{x - x(t)}{\lambda(t)} \tag{6.4.8}$$

with

$$\psi = \sum_{j=2}^{n+1} a_j \psi_j, \quad (\varepsilon, \psi_j)_\rho = (\varepsilon, \partial_k \Phi_n)_\rho = 0, \quad 1 \leq j \leq n+1, \quad 1 \leq k \leq 3. \tag{6.4.9}$$

Moreover, as the parameters are Fréchet differentiable in L^∞, and as $u \in C^1((0,T), L^\infty)$ from parabolic regularizing effects, the above decomposition is differentiable with respect to time. We also introduce a further decomposition

$$v = \psi + \varepsilon, \quad \Phi_n + v = \chi_{\frac{1}{\lambda}}\Phi_n + w. \tag{6.4.10}$$

Consider the renormalized time

$$s(t) = \int_0^t \frac{d\tau}{\lambda^2(\tau)} + s_0.$$

Injecting (*6.4.8*) into (NLH) yields the renormalized equation

$$\partial_s \varepsilon + \mathcal{L}_n \varepsilon = F - \text{Mod} \tag{6.4.11}$$

with the modulation term

$$\text{Mod} = \sum_{j=2}^{n+1} \left[(a_j)_s - \mu_j a_j\right] \psi_j - \left(\frac{\lambda_s}{\lambda} + 1\right)(\Lambda \Phi_n + \Lambda \psi) - \frac{x_s}{\lambda} \cdot (\nabla \Phi_n + \nabla \psi) \tag{6.4.12}$$

and the force terms

$$F = L(\varepsilon) + \text{NL}, \quad L(\varepsilon) = \left(\frac{\lambda_s}{\lambda} + 1\right)\Lambda \varepsilon + \frac{x_s}{\lambda} \cdot \nabla \varepsilon \tag{6.4.13}$$

$$\text{NL} = g(\varepsilon + \psi), \quad g(v) = (\Phi_n + v)^p - \Phi_n^p - p\Phi_n^{p-1} v. \tag{6.4.14}$$

We claim the following bootstrap proposition.

Proposition 6.4.2 (Bootstrap). *There exist universal constants $0 < \mu, \eta \ll 1$, $K \gg 1$ such that for all $s_0 \geq s_0(K, \mu, \eta) \gg 1$ large enough the following holds. For any λ_0 and ε_0 satisfying (6.4.5), (6.4.4) and*

$$\|(1 - \chi_{\frac{1}{\lambda_0}})\Phi_n + \varepsilon_0\|_{\dot{H}^{s_c}} + \|\varepsilon_0\|_{H^2_\rho} + \|\Delta \varepsilon_0\|_{L^2} \leq e^{-2\mu s_0}, \tag{6.4.15}$$

there exist $(a_2(0), \dots, a_{n+1}(0))$ satisfying (6.4.6) such that the solution starting from u_0 given by (6.4.3), decomposed according to (6.4.8) satisfies for all $s \geq s_0$:

- *control of the scaling:*
$$0<\lambda(s)<e^{-\mu s}; \tag{6.4.16}$$

- *control of the unstable modes:*
$$\sum_{j=2}^{n+1}|a_j|^2\leq e^{-2\mu s}; \tag{6.4.17}$$

- *control of the exponentially weighted norm:*
$$\|\varepsilon\|_{H^2_\rho}<Ke^{-\mu s}; \tag{6.4.18}$$

- *control of a Sobolev norm above scaling:*
$$\|\Delta v\|_{L^2}<Ke^{-\mu s}; \tag{6.4.19}$$

- *control of the critical norm:*
$$\|w\|_{\dot H^{s_c}}<\eta. \tag{6.4.20}$$

Proposition 6.4.2 is the heart of the analysis, and the corresponding solutions are easily shown to satisfy the conclusions of Theorem 2.4.4. The strategy of the proof follows [31, 114]: we prove Proposition 6.4.2 by contradiction using a topological argument à la Brouwer: given $(\varepsilon_0,\lambda_0)$ satisfying (*6.4.5*), (*6.4.15*) and (*6.4.4*), we assume that for all $(a_2(0),\dots,a_{n+1}(0))$ satisfying (*6.4.6*), the exit time

$$s^*=\sup\{s\geq s_0\ \text{ such that }\ (6.4.16),(6.4.17),(6.4.18),(6.4.19),(6.4.20)\text{ holds on }[s_0,s)\} \tag{6.4.21}$$

is finite

$$s^*<+\infty \tag{6.4.22}$$

and look for a contradiction for $0<\mu,\eta,\frac{1}{K}$ small enough and $s_0\geq s_0(K,\mu)$ large enough. From now on, we therefore study the flow on $[s_0,s^*]$ where (*6.4.16*), (*6.4.17*), (*6.4.18*), (*6.4.19*) and (*6.4.20*) hold. Using a bootstrap method we show that the bounds (*6.4.16*), (*6.4.18*), (*6.4.19*) and (*6.4.20*) can be improved, implying that at time s^* necessarily the unstable modes have grown and (*6.4.17*) is violated. Since 0 is a linear repulsive equilibrium for these modes, this would contradict Brouwer fixed point theorem.

From the asymptotic (*6.3.5*) of ψ_j for $2\leq j\leq n+1$, (*6.4.6*) and (*6.4.15*), one can fix the constant K_0 independently of $(s_0,\mu,)$ such that (*6.4.7*) holds. Also, note that the bootstrap bounds (*6.4.17*), (*6.4.18*), (*6.4.19*) and (*6.4.20*) imply the L^∞ bound (*6.4.23*), and therefore the decomposition used in the Proposition is well defined since Lemma 6.4.1 applies.

6.4.2 L^∞ bound

We start with the derivations of *unweighted* L^∞ and Sobolev bounds on v,w which will be essential to control nonlinear terms in the sequel and follow from (*6.4.19*), (*6.4.20*).

Lemma 6.4.3 (L^∞ smallness). *There holds*

$$\|v\|_{L^\infty}+\|w\|_{L^\infty}\leq e^{-c\mu s}\leq\eta\ll 1 \tag{6.4.23}$$

for some universal constants $c>0$, $0<\eta\ll 1$.

Proof. We compute from (*6.4.10*):

$$w = (1-\chi_{\frac{1}{\lambda}})\Phi_n + v. \tag{6.4.24}$$

The self similar decay (*2.4.9*) and (*6.4.19*) yield:

$$\|w\|_{\dot{H}^2} \lesssim \|v\|_{\dot{H}^2} + \|(1-\chi_{\frac{1}{\lambda}})\Phi_n\|_{\dot{H}^2} \lesssim K\left[e^{-\mu s} + \lambda(s)^{2-s_c}\right] \leq e^{-c\mu s}.$$

Hence by interpolation using $s_c = \frac{3}{2} - \frac{2}{p-1} < \frac{3}{2} < 2$:

$$\|w\|_{L^\infty} \lesssim \|\hat{w}\|_{L^1} \lesssim \|w\|^{1-\alpha}_{\dot{H}^{s_c}} \|w\|^{\alpha}_{\dot{H}^2}, \quad \alpha = \frac{\frac{3}{2} - s_c}{2 - s_c}$$

which together with (*6.4.20*) ensures:

$$\|w\|_{L^\infty} \lesssim e^{-c\mu s}.$$

The decay (*2.4.9*) and (*6.4.16*), (*6.4.24*) yield the L^∞ smallness for v and conclude the proof. □

6.4.3 Modulation equations

We now compute the modulation equations which describe the time evolution of the parameters. They are computed in the self-similar zone, and involve the ρ weighted norm.

Lemma 6.4.4 (Modulation equations). *There holds the bounds*

$$\left|\frac{\lambda_s}{\lambda} + 1\right| + \left|\frac{x_s}{\lambda}\right| + \sum_{j=2}^{n+1} |(a_j)_s - \mu_j a_j| \lesssim \|\varepsilon\|^2_{H^1_\rho} + \|\Delta v\|^2_{L^2} + \sum_{j=2}^{n+1} |a_j|^2. \tag{6.4.25}$$

Proof. This lemma is a classical consequence of the choice of orthogonality conditions (*6.4.9*), but the control of the nonlinear term relies in an essential way on the L^∞ smallness (*6.4.23*).

step 1 Law for a_j. Take the L^2_ρ scalar product of (*6.4.11*) with ψ_j for $2 \leq j \leq n+1$, then using (*6.4.9*) and the orthogonality

$$(\psi_j, \psi_k)_\rho = \delta_{jk}, \quad \psi_1 = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_{L^2_\rho}}, \tag{6.4.26}$$

we obtain

$$(a_j)_s - \mu_j a_j = \left(\frac{\lambda_s}{\lambda} + 1\right)(\Lambda\psi, \psi_j)_\rho + (F, \psi_j)_\rho.$$

First, from (*6.4.17*) one has

$$|(\Lambda\psi, \psi_j)_\rho| \lesssim e^{-\mu s} \ll \eta.$$

We now estimate the F-term given by (*6.4.13*) . We use the bound from $p > 5$:

$$\left||1+z|^p - 1 - pz^{p-1}\right| \lesssim |z|^p + |z|^2$$

to estimate from the L^∞ bound (*6.4.23*):

$$|\mathrm{NL}| \lesssim |\varepsilon + \psi|^p + \Phi_n^{p-2}(\varepsilon + \psi)^2 \lesssim (\varepsilon + \psi)^2 = v^2. \tag{6.4.27}$$

We estimate from the Hardy inequality (*6.A.5*):

$$\int \frac{|\nabla v|^2}{1+|y|^2}+\frac{|v|^2}{1+|y|^4}\lesssim \int |\Delta v|^2+\|v\|^2_{H^1_\rho}\lesssim \int |\Delta v|^2+\|\varepsilon\|^2_{H^1_\rho}+\sum_{j=2}^{n+1}|a_j|^2 \qquad (6.4.28)$$

and hence using the polynomial bound (*6.3.5*):

$$\begin{aligned}|(\mathrm{NL},\psi_j)_\rho| &\lesssim \int v^2|\psi_j|\rho\lesssim\int\frac{|v|^2}{1+|y|^4}\lesssim\int|\Delta v|^2+\|v\|^2_{H^1_\rho}\\ &\lesssim \|\varepsilon\|^2_{H^1_\rho}+\|\Delta v\|^2_{L^2}+\sum_{j=2}^{n+1}|a_j|^2.\end{aligned}$$

Next, we integrate by parts and use Cauchy Schwarz and (*6.3.5*) to estimate:

$$\left|\left(\left(\frac{\lambda_s}{\lambda}+1\right)\Lambda\varepsilon+\frac{x_s}{\lambda}\cdot\nabla\varepsilon,\psi_j\right)_\rho\right|\lesssim\left[\left|\frac{\lambda_s}{\lambda}+1\right|+\left|\frac{x_s}{\lambda}\right|\right]\|\varepsilon\|_{L^2_\rho}$$

and hence the first bound

$$|(a_j)_s-\mu_ja_j|\lesssim\left(\left|\frac{\lambda_s}{\lambda}+1\right|+\left|\frac{x_s}{\lambda}\right|\right)\eta+\|\varepsilon\|^2_{H^1_\rho}+\sum_{j=2}^{n+1}|a_j|^2+\|\Delta v\|^2_{L^2}.$$

step 2 Law for scaling and translation. We scalarize (*6.4.11*) with $\psi_1=\frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_{L^2_\rho}}$ and $\frac{\partial_k\Phi_n}{\|\partial_k\Phi_n\|_{L^2}}$ and obtain in a completely similar way

$$\left|\frac{\lambda_s}{\lambda}+1\right|+\left|\frac{x_s}{\lambda}\right|\lesssim\left(\left|\frac{\lambda_s}{\lambda}+1\right|+\left|\frac{x_s}{\lambda}\right|\right)\eta+\|\varepsilon\|^2_{H^1_\rho}+\sum_{j=2}^{n+1}|a_j|^2+\|\Delta v\|^2_{L^2}.$$

Summing the above estimates and using the smallness of η yields (*6.4.25*). □

6.4.4 Energy estimates with exponential weights

We now turn to the proof of exponential decay which is an elementary consequence of the spectral gap estimate (*6.3.6*), the dissipative structure of the flow *and* the L^∞ bound (*6.4.23*) to control the non linear term.

Lemma 6.4.5 (Lyapounov control of exponentially weighed norms). *There holds the differential bound*

$$\frac{d}{ds}\|\varepsilon\|^2_{L^2_\rho}+c_n\|\varepsilon\|^2_{H^1_\rho}\lesssim\sum_{j=2}^{n+1}|a_j|^4+\|\Delta v\|^4_{L^2}+\|v\|^2_{L^\infty}\left[\|\Delta v\|^2_{L^2}+\sum_{j=2}^{n+1}|a_j|^2\right], \qquad (6.4.29)$$

$$\begin{aligned}\frac{d}{ds}\|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho}+c_n\|\mathcal{L}_n\varepsilon\|^2_{H^1_\rho} &\lesssim \|\varepsilon\|^2_{H^1_\rho}+\sum_{j=2}^{n+1}|a_j|^4+\|\Delta v\|^4_{L^2} \qquad (6.4.30)\\ &+ \|v\|^2_{L^\infty}\left[\|\Delta v\|^2_{L^2}+\|\varepsilon\|^2_{H^1_\rho}+\sum_{j=2}^{n+1}|a_j|^2\right],\end{aligned}$$

with $c_n>0$ given by (6.3.6).

Proof. **step 1** L^2 weighted bound. We compute from (*6.4.11*):

$$\frac{1}{2}\frac{d}{ds}\|\varepsilon\|_{L^2_\rho}^2 = (\varepsilon, \partial_s\varepsilon)_\rho = -(\mathcal{L}_n\varepsilon, \varepsilon)_\rho + (F - \text{Mod}, \varepsilon)_\rho. \tag{6.4.31}$$

From (*6.4.12*), (*6.4.25*):

$$\begin{aligned}|(\varepsilon, \text{Mod})_\rho| \lesssim \|\varepsilon\|_{L^2_\rho}\|\text{Mod}\|_{L^2_\rho} &\lesssim \|\varepsilon\|_{L^2_\rho}\left(\|\varepsilon\|_{H^1_\rho}^2 + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|_{L^2}^2\right)\\ &\lesssim \delta\|\varepsilon\|_{L^2_\rho}^2 + C_\delta\left(\|\varepsilon\|_{H^1_\rho}^4 + \sum_{j=2}^{n+1}|a_j|^4 + \|\Delta v\|_{L^2}^4\right)\end{aligned}$$

for any $\delta > 0$. Integrating by parts and using (*6.A.1*), we estimate

$$|(\varepsilon, \Lambda\varepsilon)_\rho| + |(\nabla\varepsilon, \varepsilon)_\rho| \lesssim \int(1+|y|^2)\varepsilon^2\rho dy \lesssim \|\varepsilon\|_{H^1_\rho}^2 \tag{6.4.32}$$

from which using (*6.4.25*):

$$\left|(L(\varepsilon), \varepsilon)_\rho\right| \lesssim \|\varepsilon\|_{H^1_\rho}^2\left(\|\varepsilon\|_{L^2_\rho}^2 + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|_{L^2}^2\right).$$

Finally using (*6.4.27*), (*6.4.28*):

$$\begin{aligned}|(\text{NL}, \varepsilon)_\rho| &\lesssim \int|\varepsilon|v^2\rho dy \le \delta\int|\varepsilon|^2\rho + C_\delta\int|v|^4\rho dy\\ &\le \delta\int|\varepsilon|^2\rho + C_\delta\|v\|_{L^\infty}^2\int\frac{|v|^2}{1+|y|^4}dy\\ &\le \delta\|\varepsilon\|_{L^2_\rho}^2 + C_\delta\|v\|_{L^\infty}^2\left[\int|\Delta v|^2 + \|\varepsilon\|_{H^1_\rho}^2 + \sum_{j=2}^{n+1}|a_j|^2\right].\end{aligned}$$

Injecting the collection of above bounds into (*6.4.31*) and using the spectral gap estimate (*6.3.6*) with the choice of orthogonality conditions (*6.4.9*) yields

$$\begin{aligned}\frac{d}{ds}\|\varepsilon\|^2 \le\ & -2c_n\|\varepsilon\|_{H^1_\rho}\left(1 - C(\|\varepsilon\|_{H^1_\rho}^2 - \sum_{j=2}^{n+1}|a_j|^2 - \|ă\Delta v\|_{L^2}^2) - C\delta - C_\delta\|\varepsilon ă\|_{H^1_\rho}\right)\\ &+C_\delta\|v\|_{L^\infty}\left[\int\Delta v^2 + \|\varepsilon\|_{H^1_\rho}^2 + \sum_{j=2}^{n+1}|a_j|^2\right]\end{aligned}$$

which using the bootstrap bounds (*6.4.17*), (*6.4.18*) and (*6.4.19*) gives (*6.4.29*) for s_0 large enough and δ small enough.

step 2 H^2 weighted bound. Let

$$\varepsilon_2 = \mathcal{L}_n\varepsilon,$$

then ε_2 satisfies the orthogonality conditions (*6.4.9*):

$$(\varepsilon_2, \psi_j) = (\varepsilon_2, \partial_k\Phi_n) = 0, \quad 1 \le j \le n+1, \quad 1 \le k \le 3, \tag{6.4.33}$$

and the equation from (6.4.11):

$$\partial_s \varepsilon_2 + \mathcal{L}_n \varepsilon_2 = \mathcal{L}_n(F - \text{Mod}).$$

Hence:

$$\frac{1}{2}\frac{d}{ds}\|\varepsilon_2\|^2_{L^2_\rho} = -(\mathcal{L}_n\varepsilon_2, \varepsilon_2)_\rho + (\mathcal{L}_n(F-\text{Mod}), \varepsilon_2)_\rho. \tag{6.4.34}$$

We estimate from (6.4.25):

$$\|\mathcal{L}_n \text{Mod}\|_{L^2_\rho} \lesssim \left|\frac{\lambda_s}{\lambda} - 1\right| + \left|\frac{x_s}{\lambda}\right| + \sum_{j=2}^{n+1} |(a_j)_s - a_j| \lesssim \|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1} |a_j|^2 + \|\Delta v\|^2_{L^2}.$$

We now use the commutator relation

$$[\Delta, \Lambda] = 2\Delta$$

to compute

$$[\mathcal{L}_n, \Lambda] = [-\Delta + \Lambda - p\Phi_n^{p-1}, \Lambda] = -2\Delta + p(p-1)\Phi_n^{p-2} r\partial_r \Phi_n = 2(\mathcal{L}_n - \Lambda + p\Phi_n^{p-1}) + p(p-1)\Phi_n^{p-2} r\partial_r\Phi_n$$

from which using (6.4.32), (6.A.1):

$$\begin{array}{rcl} |(\varepsilon_2, \mathcal{L}_n\Lambda\varepsilon)_\rho| & = & |(\varepsilon_2, [\mathcal{L}_n, \Lambda]\varepsilon)_\rho + (\varepsilon_2, \Lambda\varepsilon_2)_\rho| \\ & \lesssim & \|\varepsilon_2\|^2_{H^1_\rho} + |(\varepsilon_2, \Lambda\varepsilon)_\rho| + |(\varepsilon_2, \Phi_n^{p-1}\varepsilon)_\rho| + |(\varepsilon_2, \Phi_n^{p-2}\Lambda\Phi_n\varepsilon)_\rho| \\ & \lesssim & \|\varepsilon_2\|^2_{H^1_\rho} + \|\varepsilon\|^2_{H^1_\rho} \end{array}$$

and similarly

$$|(\varepsilon_2, \mathcal{L}_n\partial_k\varepsilon)_\rho| \lesssim \|\varepsilon_2\|^2_{H^1_\rho} + \|\varepsilon\|^2_{H^1_\rho}.$$

Hence from (6.4.25):

$$|(\varepsilon_2, \mathcal{L}_n L(\varepsilon))_\rho| \lesssim (\|\varepsilon_2\|^2_{H^1_\rho} + \|\varepsilon\|^2_{H^1_\rho}) \left(\|\varepsilon\|^2_\rho + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|^2_{L^2}\right).$$

It remains to estimate the nonlinear term. We first integrate by parts since $\mathcal{L}_n$ is self adjoint for $(\cdot,\cdot)_\rho$ to estimate using the notation (6.4.14):

$$\begin{array}{rcl} |(\mathcal{L}_n \text{NL}, \varepsilon_2)_\rho| & = & \left|(\nabla \text{NL}, \nabla\varepsilon_2)_\rho + \left(\frac{2}{p-1}\text{NL} - p\Phi_n^{p-1}\text{NL}, \varepsilon_2\right)_\rho\right| \\ & \lesssim & |(\nabla g(v), \nabla\varepsilon_2)_\rho| + \left|\left(\frac{2}{p-1}g(v) - p\Phi_n^{p-1}g(v), \varepsilon_2\right)_\rho\right|. \end{array}$$

We now compute explicitly

$$\begin{array}{rcl} \nabla g(v) & = & p\nabla v\left[(\Phi_n + v)^{p-1} - \Phi_n^{p-1}\right] \\ & + & p\nabla\Phi_n\left[(\Phi_n+v)^{p-1} - \Phi_n^{p-1} - (p-1)\Phi_n^{p-2}v\right]. \end{array} \tag{6.4.35}$$

We estimate by homogeneity with the L^∞ bound (6.4.23):

$$|g(v)| \lesssim |v|^2, \quad |\nabla g(v)| \lesssim |\nabla v||v| + |v|^2$$

and hence the bound using (*6.4.23*) again:

$$\begin{aligned}
& |(\nabla g(v), \nabla \varepsilon_2)_\rho| + \left| \left(\frac{2}{p-1} g(v) - p\Phi_n^{p-1} g(v), \varepsilon_2 \right)_\rho \right| \\
\lesssim & \int \left[|v||\nabla(v)| + |v|^2 \right] |\nabla \varepsilon_2| \rho dy + \int |\varepsilon_2||v|^2 \rho dy \\
\leq & \ \delta \|\varepsilon_2\|^2_{H^1_\rho} + C_\delta \left[\int |v|^2 |\nabla v|^2 \rho dy + \int |v|^4 \rho dy \right] \\
\leq & \ \delta \|\nabla \varepsilon_2\|^2_{L^2_\rho} + C_\delta \|v\|^2_{L^\infty} \left[\int \frac{|\nabla v|^2}{1+|y|^2} dy + \int \frac{|v|^2}{1+|y|^4} dy \right] \\
\leq & \ \delta \|\nabla \varepsilon_2\|^2_{L^2_\rho} + C_\delta \|v\|^2_{L^\infty} \left[\|\varepsilon\|^2_{H^1_\rho} + \sum_{j=2}^{n+1} |a_j|^2 + \|\Delta v\|^2_{L^2} \right].
\end{aligned}$$

The collection of above bounds together with the spectral gap estimate (*6.3.6*) and the orthogonality conditions (*6.4.33*) injected into (*6.4.34*) yields (*6.4.30*). □

Remark 6.4.6. The proof of (*6.4.29*) is elementary but requires in an essential way the L^∞ smallness bound[11] (*6.4.23*), and in particular the sole control of the H^1_ρ norm cannot suffice to control the nonlinear term $\int |\varepsilon|^{p+1}\rho$ due to both the energy super critical nature of the problem and the exponential weight.

6.4.5 Outer global $\dot{H}^2$ bound

We recall

$$v = \varepsilon + \psi$$

and now aim at propagating an ***unweighted global*** $\dot{H}^2$ decay estimate for v. We have

$$\partial_s v - \Delta v - \frac{\lambda_s}{\lambda}\Lambda v - \frac{x_s}{\lambda} \cdot \nabla v = G$$

with

$$G = \left[\left(\frac{\lambda_s}{\lambda} + 1 \right) \Lambda \Phi_n + \frac{x_s}{\lambda} \cdot \nabla \Phi_n \right] + \widehat{NL}, \quad \widehat{NL} = (\Phi_n + v)^p - \Phi_n^p.$$

Lemma 6.4.7 (Global $\dot{H}^2$ bound). *There holds the Lyapounov type monotonicity formula*

$$\frac{d}{ds}\left[\frac{1}{\lambda^{4-\delta-2s_c}} \int |\Delta v|^2 dy \right] + \frac{1}{\lambda^{4-\delta-2s_c}} \int |\nabla \Delta v|^2 dy \lesssim \frac{1}{\lambda^{4-2s_c-\delta}} \left[\|\varepsilon\|^2_{H^2_\rho} + \sum_{j=2}^{n+1} |a_j|^2 \right] \qquad (6.4.36)$$

for some universal constant $0 < \delta \ll 1$.

Proof. We compute the $\dot{H}^2$ energy identity:

$$\begin{aligned}
\frac{1}{2}\frac{d}{ds} \int |\Delta v|^2 dy &= \int \Delta v \Delta \left[\Delta v + \frac{\lambda_s}{\lambda}\Lambda v + \frac{x_s}{\lambda} \cdot \nabla v + G \right] dy \\
&= -\int |\nabla \Delta v|^2 dy + \int \Delta v \Delta \left[\frac{\lambda_s}{\lambda}\Lambda v + \frac{x_s}{\lambda} \cdot \nabla v + G \right] dy
\end{aligned}$$

[11] or anything above or equal scaling in terms of regularity.

and estimate all terms.

step 1 Parameters terms. For any $\mu > 0$, let $v_\mu = \frac{1}{\mu^{\frac{2}{p-1}}} v\left(\frac{y}{\mu}\right)$, then:

$$\int |\Delta v_\mu|^2 dy = \frac{1}{\mu^{4-2s_c}} \int |\Delta v|^2 dy$$

and hence differentiating and evaluating at $\mu = 1$:

$$-2 \int \Delta v \Delta(\Lambda v) dy = -(4-2s_c) \int |\Delta v|^2 dy.$$

Hence

$$\frac{\lambda_s}{\lambda} \int \Delta v \Delta(\Lambda v) = (2-s_c)\frac{\lambda_s}{\lambda} \int |\Delta v|^2 dy.$$

Also, integrating by parts:

$$\int \Delta v \Delta \left(\frac{x_s}{\lambda} \cdot \nabla v\right) dy = 0.$$

step 2 G terms. Thanks to the decay of the self similar solution from (*2.4.9*):

$$\int |\Delta \Lambda \Phi_n|^2 dy + \int |\Delta \nabla \Phi_n|^2 dy < +\infty,$$

we estimate in brute force using (*6.4.25*) the terms induced by the self similar solution:

$$\begin{aligned}
&\left| \int \Delta v \Delta \left\{ \left[\left(\frac{\lambda_s}{\lambda} + 1\right) \Lambda \Phi_n + \frac{x_s}{\lambda} \cdot \nabla \Phi_n \right] \right\} \right| \\
\lesssim \ & \left[\left|\frac{\lambda_s}{\lambda} + 1\right| + \left|\frac{x_s}{\lambda}\right| \right] \|\Delta v\|_{L^2} \leq \delta \|\Delta v\|_{L^2}^2 + C_\delta \left(\|\varepsilon\|_{H^1_\rho}^2 + \|\Delta v\|_{L^2}^2 + \sum_{j=2}^{n+1} |a_j|^2 \right)^2 \\
\leq \ & \delta \|\Delta v\|_{L^2}^2 + C_\delta \left(\|\varepsilon\|_{H^1_\rho}^2 + \sum_{j=2}^{n+1} |a_j|^2 \right).
\end{aligned}$$

It remains to estimate the nonlinear term. We estimate by homogeneity:

$$\begin{aligned}
|\Delta \widehat{NL}| \ = \ & \Big| p\Delta\Phi_n \left[(\Phi_n + v)^{p-1} - \Phi_n^{p-1} \right] + p(\Phi_n + v)^{p-1} \Delta v \\
+ \ & p(p-1)|\nabla \Phi_n|^2 \left[(\Phi_n + v)^{p-2} - \Phi_n^{p-2} \right] + p(p-1)|\nabla v|^2 (\Phi_n + v)^{p-1} \\
+ \ & 2p(p-1)(\Phi_n + v)^{p-2} \nabla \Phi_n \cdot \nabla v \Big| \\
\lesssim \ & |\Delta \Phi_n|(|v|^{p-1} + |\Phi_n|^{p-2}|v|) + |\Delta v|(|v|^{p-1} + |\Phi_n|^{p-1}) \\
+ \ & |\nabla \Phi_n|^2 (|v|^{p-2} + |\Phi_n|^{p-3}|v|) + |\nabla v|^2 (|v|^{p-1} + |\Phi_n|^{p-1}) + |\nabla v||\nabla \Phi_n|(|\Phi_n|^{p-2} + |v|^{p-2})
\end{aligned}$$

and hence using the self similar decay of Φ_n and the L^∞ smallness (*6.4.23*):

$$\begin{aligned}
|\Delta \widehat{NL}| \ \lesssim \ & \left[\frac{|\Delta v|}{1+|y|^2} + \frac{|\nabla v|}{1+|y|^3} + \frac{|v|}{1+|y|^4} \right] + \eta \left[|\Delta v| + \frac{|\nabla v|}{1+|y|} + \frac{|v|}{1+|y|^2} \right] \\
+ \ & |\nabla v|^2 (|v|^{p-1} + |\Phi_n|^{p-1}).
\end{aligned}$$

The linear term is estimated using (*6.A.5*):

$$\begin{aligned}\int \left|\frac{|\Delta v|}{1+|y|^2}+\frac{|\nabla v|}{1+|y|^3}+\frac{|v|}{1+|y|^4}\right|^2 &\lesssim \frac{1}{A^4}\int_{|y|\geq A}|\Delta v|^2+C_A\|v\|^2_{H^2_\rho}\\ &\leq \delta\int|\Delta v|^2+C_\delta\left(\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2\right)\end{aligned}$$

and using (*6.A.5*) again:

$$\int\left|\eta\left[|\Delta v|+\frac{|\nabla v|}{1+|y|}+\frac{|v|}{1+|y|^2}\right]\right|^2\lesssim \eta\|\Delta v\|^2_{L^2}+\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2.$$

To estimate the nonlinear term, we let

$$q_c=\frac{3(p-1)}{2}\quad\text{so that }\ \dot H^{s_c}\subset L^{q_c}.$$

We estimate using (*6.4.23*) with $6(p-2)>q_c$ and Sobolev:

$$\begin{aligned}&\int|\nabla v|^4(|v|^{2(p-2)}+|\Phi_n|^{2(p-2)})\lesssim\|\nabla v\|^4_{L^6}\left[\|v\|^{2(p-2)}_{L^{6(p-2)}}+\|\Phi_n\|^{2(p-2)}_{L^{6(p-2)}}\right]\\ \lesssim\ &\|\Delta v\|^4_{L^2}\left[\|\Phi_n\|^{2(p-2)}_{L^{6(p-2)}}+\|w\|^{2(p-2)}_{L^{6(p-2)}}\right]\lesssim\|\Delta v\|^4_{L^2}\left[1+\|w\|^{\frac{p-1}{2}}_{\dot H^{s_c}}\right]\leq\delta\|\Delta v\|^2_{L^2}.\end{aligned}$$

We have therefore obtained

$$\int|\Delta\widehat{NL}|^2\leq\delta\|\Delta v\|^2_{L^2}+C_\delta\left(\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2\right).$$

The collection of above bounds and (*6.4.25*) yields (*6.4.36*). □

6.4.6 Control of the critical norm

We now claim the control of the critical norm of w (defined by (*6.4.10*)).

Lemma 6.4.8 (Control of the critical norm). *There holds the Lyapounov type control*

$$\frac{d}{ds}\int|\nabla^{s_c}w|^2dy+\int|\nabla^{s_c+1}w|^2dy\lesssim\|\varepsilon\|^2_{H^2_\rho}+\sum_{j=2}^{n+1}|a_j|^2+\lambda^{\delta(2-s_c)}+\|\Delta v\|^\delta_{L^2}. \tag{6.4.37}$$

for some small enough universal constant $0<\delta=\delta(p)\ll 1$.

Proof. Let

$$\widetilde{\Phi_n}=\chi_{\frac{1}{\lambda}}\Phi_n, \tag{6.4.38}$$

we compute the evolution equation of w:

$$\partial_s w-\Delta w=\frac{\lambda_s}{\lambda}\Lambda w+\frac{x_s}{\lambda}\cdot\nabla w+\widetilde G \tag{6.4.39}$$

with

$$\begin{aligned}\widetilde G &= \left(\frac{\lambda_s}{\lambda}+1\right)\chi_{\frac{1}{\lambda}}\Lambda\Phi_n+\frac{x_s}{\lambda}\cdot\nabla\widetilde\Phi_n+2\nabla\chi_{\frac{1}{\lambda}}\cdot\nabla\Phi_n+\Delta\chi_{\frac{1}{\lambda}}\Phi_n-(\chi_{\frac{1}{\lambda}}-\chi^p_{\frac{1}{\lambda}})\Phi_n^p+\widetilde{\text{NL}},\\ \widetilde{\text{NL}} &= (\widetilde{\Phi_n}+w)^p-(\widetilde{\Phi_n})^p.\end{aligned}$$

Observe from the space localization of the cut, from the decay of the self similar solution, and from (*6.4.19*) and (*6.4.20*):

$$\forall s_c \leq s \leq 2, \quad \|w\|_{\dot{H}^s} \lesssim \eta. \tag{6.4.40}$$

We compute:

$$\begin{aligned} \frac{1}{2}\frac{d}{ds}\int |\nabla^{s_c} w|^2 dy &= \int \nabla^{s_c} w \cdot \nabla^{s_c}\left[\Delta w + \frac{\lambda_s}{\lambda}\Lambda w + \frac{x_s}{\lambda}\cdot\nabla w + \widetilde{G}\right] dy \\ &= -\int |\nabla^{s_c+1} w|^2 + \int \nabla^{s_c} w \cdot \nabla^{s_c}\left[\frac{\lambda_s}{\lambda}\Lambda w + \frac{x_s}{\lambda}\cdot\nabla w + \widetilde{G}\right] dy \end{aligned}$$

and estimate all terms.

step 1 Parameters terms. For any $\mu > 0$, let $w_\mu = \frac{1}{\mu^{\frac{2}{p-1}}} w\left(\frac{y}{\mu}\right)$, then :

$$\int |\nabla^{s_c} w_\mu|^2 dy = \int |\nabla^{s_c} w|^2 dy$$

and hence differentiating at $\mu = 1$:

$$-2\int \nabla^{s_c} w \cdot \nabla^{s_c}(\Lambda w) dy = 0.$$

Integrating by parts:

$$\int \nabla^{s_c} w \cdot \nabla^{s_c}\left(\frac{x_s}{\lambda}\cdot\nabla w\right) dy = 0.$$

step 2 $\widetilde{G}$ terms. The decay of the self similar solution and the space localization of the cut ensure using $1 < s_c < 2$:

$$\begin{aligned} & \left\|2\nabla\chi_{\frac{1}{\lambda}}\cdot\nabla\Phi_n + \Delta\chi_{\frac{1}{\lambda}}\Phi_n\right\|_{\dot{H}^{s_c}} \\ \lesssim & \left\|2\nabla\chi_{\frac{1}{\lambda}}\cdot\nabla\Phi_n + \Delta\chi_{\frac{1}{\lambda}}\Phi_n\right\|_{\dot{H}^1}^{2-s_c} \left\|2\nabla\chi_{\frac{1}{\lambda}}\cdot\nabla\Phi_n + \Delta\chi_{\frac{1}{\lambda}}\Phi_n\right\|_{\dot{H}^2}^{s_c-1} \\ \lesssim & \left(\frac{\lambda^2}{\lambda^{s_c-1}}\right)^{2-s_c}\left(\frac{\lambda^2}{\lambda^{s_c-2}}\right)^{s_c-1} \lesssim \lambda^2, \end{aligned}$$

and similarly

$$\begin{aligned} & \left\|\left(\chi_{\frac{1}{\lambda}} - \chi_{\frac{1}{\lambda}}^p\right)\Phi_n^p\right\|_{\dot{H}^{s_c}} \lesssim \left\|\left(\chi_{\frac{1}{\lambda}} - \chi_{\frac{1}{\lambda}}^p\right)\Phi_n^p\right\|_{\dot{H}^1}^{2-s_c}\left\|\left(\chi_{\frac{1}{\lambda}} - \chi_{\frac{1}{\lambda}}^p\right)\Phi_n^p\right\|_{\dot{H}^2}^{s_c-1} \\ \lesssim & (\lambda^{3-s_c})^{2-s_c}(\lambda^{4-s_c})^{s_c-1} \lesssim \lambda^2. \end{aligned}$$

Using (*6.4.25*):

$$\begin{aligned} & \left\|\left(\frac{\lambda_s}{\lambda}+1\right)\chi_{\frac{1}{\lambda}}\Lambda\Phi_n + \frac{x_s}{\lambda}\cdot\nabla(\chi_{\frac{1}{\lambda}}\Phi_n)\right\|_{\dot{H}^{s_c}} \\ \lesssim & \left|\frac{x_s}{\lambda}\right| + \left|\frac{\lambda_s}{\lambda}+1\right| \lesssim \|\varepsilon\|_{L^2_\rho}^2 + \sum_{j=2}^{n+1}|a_j|^2 + \|\Delta v\|_{L^2}^2. \end{aligned}$$

We now turn to the control of the nonlinear term and claim the bound:

$$\|\nabla^{s_c}\widetilde{NL}\|_{L^2} \lesssim \|\nabla^{s_c+\alpha} w\|_{L^2} \tag{6.4.41}$$

for some small enough universal constant $0 < \alpha = \alpha(p) \ll 1$. Assume (*6.4.41*), we then interpolate with $\delta = \frac{\alpha}{2-s_c}$ and use (*6.4.24*), (*6.4.20*) and the decay of the self similar solution to estimate:

$$\|\nabla^{s_c+\alpha} w\|_{L^2} \lesssim \|\nabla^{s_c} w\|_{L^2}^{1-\delta} \|\Delta w\|_{L^2}^{\delta} \lesssim \lambda^{\delta(2-s_c)} + \|\Delta v\|_{L^2}^{\delta},$$

and the collection of above bounds yields (*6.4.37*).

Proof of (*6.4.41*). We compute

$$\begin{aligned} \nabla \widetilde{NL} &= p\nabla(\widetilde{\Phi_n} + w)(\widetilde{\Phi_n} + w)^{p-1} - p\nabla\widetilde{\Phi_n}\widetilde{\Phi_n}^{p-1} \\ &= p\nabla\widetilde{\Phi_n}\left[(\widetilde{\Phi_n} + w)^{p-1} - \widetilde{\Phi_n}^{p-1}\right] + p\nabla w(\widetilde{\Phi_n} + w)^{p-1} \\ &= pg_1(w)\nabla(\widetilde{\Phi_n} + w) + p\widetilde{\Phi_n}^{p-1}\nabla w \end{aligned}$$

with

$$g_1(w) = (\widetilde{\Phi_n} + w)^{p-1} - \widetilde{\Phi_n}^{p-1}.$$

Hence letting

$$s_c = 1 + \nu, \quad 0 < \nu = \frac{1}{2} - \frac{2}{p-1} < \frac{1}{2},$$

we estimate:

$$\|\nabla^{s_c}\widetilde{\mathrm{NL}}\|_{L^2} \lesssim \left\|\nabla^{\nu}\left[g_1(w)\nabla(\widetilde{\Phi_n} + w)\right]\right\|_{L^2} + \left\|\nabla^{\nu}\left(\widetilde{\Phi_n}^{p-1}\nabla w\right)\right\|_{L^2}. \tag{6.4.42}$$

For the first term, we use the following commutator estimate proved in Appendix 6.B: let

$$0 < \nu < 1, \quad 1 < p_1, p_2, p_3, p_4 < +\infty, \quad \frac{1}{2} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}$$

then

$$\|\nabla^{\nu}(uv)\|_{L^2} \lesssim \|u\|_{\dot{B}^{\nu}_{p_1,2}}\|v\|_{L^{p_2}} + \|u\|_{L^{p_4}}\|v\|_{\dot{B}^{\nu}_{p_3,2}}, \tag{6.4.43}$$

where we use here the standard space formulation of Besov norms for $0 < s < 1$ and $1 \le p < +\infty$[12]:

$$\|u\|_{\dot{B}^s_{p,2}} \sim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|u(\cdot - y) - u(\cdot)\|_{L^p}}{t^s}\right)^2 \frac{dt}{t}\right)^{\frac{1}{2}}. \tag{6.4.44}$$

We pick a small enough $0 < \alpha \ll 1$ to be chosen later and

$$\begin{aligned} \frac{1}{p_1} &= \frac{1}{3} + \frac{\alpha}{3}, \quad \frac{1}{p_2} = \frac{1}{6} - \frac{\alpha}{3} \\ \frac{1}{p_3} &= \frac{1 + \alpha + \nu}{3}, \quad \frac{1}{p_4} = \frac{1 - 2(\alpha + \nu)}{6}. \end{aligned}$$

Observe that

$$-\nu + \frac{3}{p_2} = \frac{3}{p_4}$$

[12] see for example [21].

and hence from (*6.4.43*), the embedding of $\dot H^{s,p}$ in $\dot B^s_{p,2}$, and Sobolev[13]:

$$
\begin{aligned}
&\left\|\nabla^\nu\left[g_1(w)\nabla(\widetilde{\Phi_n}+w)\right]\right\|_{L^2}\\
\lesssim\;& \|\nabla(\widetilde{\Phi_n}+w)\|_{L^{p_1}}\|g_1(w)\|_{\dot B^\nu_{p_2,2}}+\|\nabla(\widetilde{\Phi_n}+w)\|_{\dot B^\nu_{p_3,2}}\|g_1(w)\|_{L^{p_4}}\\
\lesssim\;& \|\nabla^{1+\frac32-\frac{3}{p_1}}(\widetilde{\Phi_n}+w)\|_{L^2}\|g_1(w)\|_{\dot B^\nu_{p_2,2}}+\|\nabla^{1+\nu+\frac32-\frac{3}{p_3}}(\widetilde{\Phi_n}+w)\|_{L^2}\|\nabla^\nu g_1(w)\|_{L^{p_2}}\\
\lesssim\;& \|\nabla^{\frac32-\alpha}(\widetilde{\Phi_n}+w)\|_{L^2}\|g_1(w)\|_{\dot B^\nu_{p_2,2}}.
\end{aligned}
$$

Since $s_c=\frac32-\frac{2}{p-1}<\frac32$, we may pick $0<\alpha\ll 1$ with $\frac32-\alpha>s_c$ and hence using (*6.4.40*) and the decay of the self similar solution:

$$\|\nabla^{\frac32-\alpha}(\widetilde{\Phi_n}+w)\|_{L^2}\lesssim 1.$$

Let now

$$f(z)=(1+z)^{p-1}-1$$

then $f(0)=0$ and

$$|f(z_2)-f(z_1)|=\left|\int_{z_1}^{z_2}f'(\tau)d\tau\right|\lesssim\int_{z_1}^{z_2}(1+|\tau|^{p-2})d\tau\lesssim|z_2-z_1|(1+|z_1|^{p-2}+|z_2|^{p-2})$$

and hence by homogeneity:

$$|g_1(w_2)-g_1(w_1)|\lesssim|w_2-w_1|(|\widetilde{\Phi_n}|^{p-2}+|w_2|^{p-2}+|w_1|^{p-2}).$$

Using the L^∞ bound (*6.4.23*), (*6.4.44*), and Sobolev[14]

$$
\begin{aligned}
\|g_1(w)\|_{\dot B^\nu_{p_2,2}} &\lesssim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|g_1(w(\cdot-y))-g_1(w(\cdot))\|_{L^{p_2}}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12}\\
&\lesssim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\le t}\|w(\cdot-y)-w(\cdot)\|_{L^{p_2}}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12}\sim\|w\|_{\dot B^\nu_{p_2,2}}\\
&\lesssim \|\nabla^{\nu+\frac32-\frac{3}{p_2}}w\|_{L^2}=\|\nabla^{s_c+\alpha}w\|_{L^2}.
\end{aligned}
$$

The collection of above bounds yields the control of the first term of (*6.4.42*):

$$\|\nabla^\nu\left[g_1(w)\nabla(\widetilde{\Phi_n}+w)\right]\|_{L^2}\lesssim\|\nabla^{s_c+\alpha}w\|_{L^2}.$$

For the second term in (*6.4.42*), we recall the following estimate proved in [114]: let $0<\nu<1$ and $\mu>0$ with $\mu+\nu<\frac32$, let f smooth radially symmetric with

$$|\partial_r^k f|\lesssim\frac{1}{1+r^{\mu+k}},\quad k=0,1, \tag{6.4.45}$$

then there holds the generalized Hardy bound

$$\|\nabla^\nu(uf)\|_{L^2}\lesssim\|\nabla^{\nu+\mu}f\|_{L^2}. \tag{6.4.46}$$

[13]using $\frac32-\frac{3}{p_3}=\frac12-(\alpha+\nu)>0$.

[14]Here we use that $\dot B^s_{2,2}$ embeds in $\dot B^t_{p,2}$ with $s-3/2=t-3/p$ for $p\ge 2$, and $\dot B^s_{2,2}=\dot H^s$.

We then pick again a small enough $0 < \alpha \ll 1$ and let

$$\mu = \alpha, \quad \mu + \nu = \nu + \alpha = s_c - 1 + \alpha < \frac{3}{2}$$

for $0 < \alpha \ll 1$ small enough, and $f = (\chi_{\frac{1}{\lambda}} \Phi_n)^{p-1}$ satisfies

$$|\partial_r^k f| \lesssim \frac{1}{1 + r^{2+k}} \lesssim \frac{1}{1 + r^{\mu+k}}.$$

Hence

$$\|\nabla^\nu \left(\widetilde{\Phi_n}^{p-1} \nabla w\right) \|_{L^2} \lesssim \|\nabla^{\nu+\mu+1} w\|_{L^2} = \|\nabla^{s_c+\alpha} w\|_{L^2}.$$

This concludes the proof of (*6.4.41*). □

6.4.7 Conclusion

We are now in position to conclude the proof of Proposition 6.4.2 which then easily implies Theorem 2.4.4.

Proof of Proposition 6.4.2 We recall that we are arguing by contradiction assuming (*6.4.22*). We first show that the bounds (*6.4.16*), (*6.4.18*), (*6.4.19*) and (*6.4.20*) can be improved on $[s_0, s^*]$, and then, the existence of the data $(a_j(0))_{2 \le j \le n+1}$ follows from a classical topological argument à la Brouwer.

step 1 Improved scaling control. We estimate from (*6.4.17*), (*6.4.18*), (*6.4.19*), (*6.4.25*):

$$\left|\frac{\lambda_s}{\lambda} + 1\right| \lesssim K^2 e^{-2\mu s} \tag{6.4.47}$$

and hence after integration:

$$\left|\log\left(\frac{\lambda(s)}{\lambda_0}\right) + s - s_0\right| \lesssim \int_{s_0}^{+\infty} K^2 e^{-2\mu\tau} d\tau \lesssim 1 + o(1)$$

for s_0 large enough, which together with (*6.4.5*) implies:

$$\lambda(s) = (\lambda(s_0) e^{s_0})\, e^{-s}(1 + o(1)) \;\text{ and hence }\; \frac{e^{-s}}{2} \le \lambda(s) \le 2e^{-s}. \tag{6.4.48}$$

step 2 Improved Sobolev bounds.
L^2_ρ ***bound.*** From (*6.4.29*), (*6.4.17*), (*6.4.19*), (*6.4.23*):

$$\frac{d}{ds}\|\varepsilon\|^2_{L^2_\rho} + c_n\|\varepsilon\|^2_{H^1_\rho} \lesssim (1 + K^4)e^{-4\mu s} + K^2 e^{-2\mu s} e^{-2c\mu s} \le e^{-(2+c)\mu s}$$

for $s \ge s_0$ large enough. From now on, we may fix once and for all the value

$$\mu = \frac{c_n}{4} \tag{6.4.49}$$

and hence

$$\frac{d}{ds}\|\varepsilon\|^2_{L^2_\rho} + 4\mu\|\varepsilon\|^2_{H^1_\rho} \le e^{-(2+c)\mu s} \tag{6.4.50}$$

which time integration yields using (*6.4.7*):

$$\begin{aligned}\|\varepsilon(s)\|^2_{L^2_\rho}+2\mu e^{-2\mu s}\int_{s_0}^{s}e^{2\mu\sigma}\|\varepsilon\|^2_{H^1_\rho}d\sigma &\leq \left(e^{2\mu s_0}\|\varepsilon(s_0)\|^2_{L^2_\rho}\right)e^{-2\mu s}+e^{-2\mu s}\int_{s_0}^{s}e^{-\mu c\tau}d\tau\\ &\lesssim K_0^2e^{-2\mu s}.\end{aligned} \tag{6.4.51}$$

H^2_ρ ***bound.*** We estimate from (*6.4.30*) like for the proof of (*6.4.50*):

$$\frac{d}{ds}\|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho}+4\mu\|\mathcal{L}_n\varepsilon\|^2_{H^1_\rho}\lesssim\|\varepsilon\|^2_{H^1_\rho}+e^{-(2+c)\mu s}$$

whose time integration with the initial bound (*6.4.7*) and the bound (*6.4.51*) ensures:

$$\|\mathcal{L}_n\varepsilon(s)\|^2_{L^2_\rho}\lesssim K_0^2e^{-2\mu s}.$$

We recall

$$(\mathcal{L}_n\varepsilon,\varepsilon)_\rho=\|\nabla\varepsilon\|^2_{L^2_\rho}+\int\left(\frac{2}{p-1}-p\Phi_n^{p-1}\right)|\varepsilon|^2\rho dy$$

and hence we first estimate from the spectral bound (*6.3.6*), the orthogonality conditions (*6.4.9*), and Cauchy-Schwarz:

$$\|\nabla\varepsilon\|^2_{L^2_\rho}\leq(\mathcal{L}_n\varepsilon,\varepsilon)_\rho+C\|\varepsilon\|^2_{L^2_\rho}\lesssim\|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho}+\|\varepsilon\|^2_{L^2_\rho}\lesssim K_0^2e^{-2\mu s}. \tag{6.4.52}$$

This yields using (*6.A.2*):

$$\|\varepsilon\|^2_{H^2_\rho}\lesssim\|\mathcal{L}_n\varepsilon\|^2_{L^2_\rho}+\|\varepsilon\|^2_{H^1_\rho} \tag{6.4.53}$$

and hence the improved bound

$$\|\varepsilon\|^2_{H^2_\rho}\lesssim K_0^2e^{-2\mu s}. \tag{6.4.54}$$

$\dot{H}^2$ ***bound.*** We rewrite (*6.4.36*) using (*6.4.17*), (*6.4.25*), (*6.4.54*)

$$\frac{d}{ds}\|\Delta v\|^2_{L^2}+(4-\delta-2s_c)\|\Delta v\|^2_{L^2}\lesssim K_0^2e^{-2\mu s}.$$

By possibly diminishing the value of c_n, we may always assume

$$4-\delta-2s_c>c_n=4\mu$$

and hence from (*6.4.7*):

$$\|\Delta v\|^2_{L^2}\leq K_0^2e^{-4\mu s}e^{4\mu s_0}e^{-2\mu s_0}+e^{-4\mu s}\int_{s_0}^{s}K_0^2e^{4\mu\tau}e^{-2\mu\tau}d\tau\lesssim K_0^2e^{-2\mu s}. \tag{6.4.55}$$

$\dot{H}^{s_c}$ ***bound.*** We now rewrite (*6.4.37*) using (*6.4.16*)-(*6.4.20*):

$$\frac{d}{ds}\int\|\nabla^{s_c}w\|^2_{L^2}\leq e^{-c\mu s}$$

for some universal constant $c>0$ which time integration using (*6.4.7*) ensures:

$$\|\nabla^{s_c}w(s)\|^2_{L^2}\lesssim\|\nabla^{s_c}w(s_0)\|^2_{L^2}+e^{-cs_0}<\frac{\eta}{2} \tag{6.4.56}$$

for s_0 large enough.

step 3 The Brouwer fixed point argument. We conclude from (*6.4.48*), (*6.4.54*), (*6.4.55*), (*6.4.56*), the definition (*6.4.21*) of s^* and a simple continuity argument that the contradiction assumption (*6.4.22*) implies from (*6.4.17*):

$$\sum_{j=2}^{n+1}|a_j(s^*)|^2 = e^{-2\mu s^*}. \qquad (6.4.57)$$

Moreover, the vector field is strictly outgoing from (*6.4.25*), (*6.4.17*), (*6.4.18*), (*6.4.19*):

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\sum_{j=2}^{n+1}|a_je^{\mu s}|^2 &= \sum_{j=2}^{n+1}a_je^{2\mu s}((a_j)_s+\mu a_j)=\sum_{j=2}^{n+1}a_je^{2\mu s}\left[(\mu+\mu_j)a_j+O\left(K^2e^{-2\mu s}\right)\right]\\ &\geq \mu\sum_{j=2}^{n+1}|a_je^{\mu s}|^2+O\left(K^2e^{-\mu s}\right)\end{aligned}$$

from which

$$\left(\frac{d}{ds}\sum_{j=2}^{n+1}|a_je^{\mu s}|^2\right)(s^*)>\mu+O(K^2e^{-\mu s_0})>0$$

for s_0 large enough. We conclude from standard argument that the map

$$(a_j(0)e^{\mu s_0})_{2\leq j\leq n+1}\mapsto (a_j(s^*)e^{\mu s^*})_{2\leq j\leq n+1}$$

is continuous in the unit ball of $\mathbb{R}^n$, and the identity on its boundary, a contradiction to Brouwer's theorem. This concludes the proof of Proposition 6.4.2. $\square$

We are now in position to conclude the proof of Theorem 2.4.4.

Proof of Theorem 2.4.4 Let an initial data as in Proposition 6.4.2, then the corresponding solution $u(s,y)$ admits on $[s_0,+\infty)$ a decomposition (*6.4.8*) with the bounds (*6.4.17*), (*6.4.23*), (*6.4.19*), (*6.4.20*), (*6.4.48*).

step 1 Self similar time blow up. Using (*6.4.48*), the life space of the solution u is finite

$$T=\int_{s_0}^{+\infty}\lambda^2(s)ds\lesssim\int_{s_0}^{+\infty}e^{-2s}ds<+\infty,$$

and hence

$$T-t=\int_s^{+\infty}\lambda^2(s)ds\sim e^{-2s}.$$

We may therefore rewrite (*6.4.47*):

$$|\lambda\lambda_t+1|\lesssim(T-t)^\mu$$

and integrating in time using $\lambda(T)=0$ yields

$$\lambda(t)=\sqrt{(2+o(1))(T-t)}. \qquad (6.4.58)$$

Also from (*6.4.25*):

$$\int_0^T|x_t|=\int_{s_0}^{+\infty}|x_s|ds\lesssim\int_{s_0}^{+\infty}e^{-s-2\mu s}ds<+\infty$$

and (*2.4.5*) is proved.

step 2 Asymptotic stability above scaling. We now prove (*2.4.6*) and (*2.4.8*). We first estimate from (*6.4.24*) using the self similar decay of Φ_n:

$$\begin{aligned}\|w\|_{\dot{H}^2} &\lesssim \|v\|_{\dot{H}^2}+\|(1-\chi_{\frac{1}{\lambda}})\Phi_n\|_{\dot{H}^2}\lesssim e^{-2\mu s}+\lambda^{2-s_c}(s)\\ &\to 0 \;\text{ as }\; t\to T.\end{aligned}$$

Hence from (*6.4.20*):

$$\forall s_c<\sigma\le 2,\quad \lim_{s\to+\infty}\|w(s)\|_{\dot{H}^\sigma}=0$$

which using (*6.4.24*) and the self similar decay of Φ_n again implies

$$\forall s_c<\sigma\le 2,\quad \lim_{s\to+\infty}\|v(s)\|_{\dot{H}^\sigma}=0,$$

this is (*2.4.6*). At the critical level, we have from (*6.4.8*), (*6.4.10*) and the sharp self similar decay from Proposition 6.2.2:

$$\|u(t)\|_{\dot{H}^{s_c}}=\|\chi_{\frac{1}{\lambda}}\Phi_n+w\|_{\dot{H}^{s_c}}=c_n(1+o(1))\sqrt{|\log\lambda|},\quad c_n\neq 0,$$

and (*6.4.58*) now yields (*2.4.8*).

step 3 Boundedness below scaling. We now prove (*2.4.7*).

Control of the Dirichlet energy. Recall the notation (*6.4.38*) and compute by rescaling using the self similar decay of Φ_n:

$$\lambda^{2(s_c-1)}\left[\|\nabla\widetilde{\Phi_n}\|^2_{L^2}+\|\widetilde{\Phi_n}\|^{p+1}_{L^{p+1}}\right]\lesssim 1.$$

Hence the dissipation of energy which is translation invariant ensures

$$\begin{aligned}\lambda^{2(s_c-1)}\|\nabla w\|^2_{L^2} &\lesssim \lambda^{2(s_c-1)}\left[\|\nabla(\widetilde{\Phi_n}+w)\|^2_{L^2}+\|\nabla\widetilde{\Phi_n}\|^2_{L^2}\right]\lesssim 1+2E(u)+\frac{2}{p+1}\|u\|^{p+1}_{L^{p+1}}\\ &\lesssim 1+|E_0|+\lambda^{2(s_c-1)}\|w\|^{p+1}_{L^{p+1}}.\end{aligned}$$

We now interpolate using the smallness[15] (*6.4.20*)

$$\|w\|^{p+1}_{L^{p+1}}\lesssim\|w\|^{p-1}_{\dot{H}^{s_c}}\|\nabla w\|^2_{L^2}\lesssim\eta\|\nabla w\|^2_{L^2}$$

and hence

$$\lambda^{2(s_c-1)}\|\nabla w\|^2_{L^2}\lesssim C(u_0)\tag{6.4.59}$$

and

$$\|\nabla u\|^2_{L^2}\lesssim\lambda^{2(s_c-1)}\left[\|\nabla\widetilde{\Phi_n}\|^2_{L^2}+\|\nabla w\|^2_{L^2}\right]\lesssim 1.$$

Proof of (*2.4.7*). Let now $1\le\sigma<s_c$, then using (*6.4.20*), (*6.4.59*) and interpolation:

$$\begin{aligned}\|\nabla^\sigma u\|_{L^2} &\lesssim \lambda^{s_c-\sigma}\|\nabla^\sigma\widetilde{\Phi_n}\|_{L^2}+\lambda^{s_c-\sigma}\|\nabla^\sigma w\|_{L^2}\lesssim 1+\lambda^{s_c-\sigma}\|\nabla w\|^{\frac{s_c-\sigma}{s_c-1}}_{L^2}\|\nabla^{s_c}w\|^{\frac{\sigma-1}{s_c-1}}_{L^2}\\ &\lesssim 1+\left(\lambda^{s_c-1}\|\nabla w\|_{L^2}\right)^{\frac{s_c-\sigma}{s_c-1}}\lesssim C(u_0)\end{aligned}$$

and (*2.4.7*) is proved. This concludes the proof of Theorem 2.4.4. □

[15] this is the only place in the proof where we use that the critical norm is small, bounded suffices everywhere else.

6.4.8 The Lipschitz dependence

We now state the Lipschitz aspect of the set of solutions constructed in this chapter.

Proposition 6.4.9 (Lipschitz dependence). *Let $s_0 \gg 1$, $\varepsilon_0^{(1)}$ and $\varepsilon_0^{(2)}$ satisfy (6.4.4) and (6.4.15), and take $\lambda_0^{(1)} = \lambda_0^{(2)} = e^{-s_0}$. Then the parameters $(a_j^{(1)}(0))_{2\leq j\leq n+1}$ and $(a_j^{(2)}(0))_{2\leq j\leq n+1}$, associated by Proposition 6.4.2 to $(\varepsilon^{(1)}, \lambda_0^{(1)})$ and $(\varepsilon^{(2)}, \lambda_0^{(2)})$ respectively, satisfy:*

$$\sum_{j=2}^{n+1} \left| a_j^{(1)}(0) - a_j^{(2)}(0) \right|^2 \lesssim \left\| \varepsilon_0^{(1)} - \varepsilon_0^{(2)} \right\|^2_{L^2_\rho}. \tag{6.4.60}$$

Proof. The idea of the proof is classical, see for instance [42]. We study the difference of two solutions, and use the bounds we already derived in the existence result as a priori bounds now. This allows us to control the difference of solutions at a low regularity level which is sufficient to conclude.

We use the superscripts (i), $i = 1, 2$ for all variables associated to the two solutions respectively: $u^{(i)}$ for (6.4.8), $v^{(i)}$ for (6.4.10), $\psi^{(i)}$ for (6.4.9), $\lambda^{(i)}$ for the scales and $x^{(i)}$ for the central points. The differences are denoted by

$$\triangle\varepsilon := \varepsilon^{(1)} - \varepsilon^{(2)}, \ \triangle a_j := a_j^{(1)} - a_j^{(2)}, \ \triangle v := v^{(1)} - v^{(2)}.$$

We compare the two renormalized solutions at the same renormalized time s. The time evolution for the difference is given by

$$\begin{aligned} \triangle\varepsilon_s + \mathcal{L}_n \triangle\varepsilon \ &= \ \tfrac{d}{ds}\left[\log\left(\tfrac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right] \Lambda(\Phi_n + v^{(2)}) + \left(\tfrac{x_s^{(1)}}{\lambda^{(1)}} - \tfrac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla(\Phi_n + v^{(2)}) \\ &\quad - \sum_{j=2}^{n+1} (\triangle a_{j,s} - \mu_j \triangle a_j)\psi_j + \left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}} + 1\right)\Lambda\triangle v \\ &\quad + \tfrac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle v + \left[(\Phi_n + v^{(1)})^p - (\Phi_n + v^{(2)})^p - p\Phi_n^{p-1}\triangle v\right]. \end{aligned} \tag{6.4.61}$$

step 1 Modulation equations. We claim that

$$\begin{aligned} &\left|\frac{d}{ds}\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right| + \left|\frac{x_s^{(1)}}{\lambda^{(1)}} - \frac{x_s^{(2)}}{\lambda^{(2)}}\right| + \sum_{j=2}^{n+1} |\triangle a_{j,s} - \mu_j \triangle a_j| \\ \lesssim \ & e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho} + \sum_{j=2}^{n+1} |\triangle a_j|\right). \end{aligned} \tag{6.4.62}$$

We now show this estimate. Taking the scalar product of (6.4.61) with $\psi_1 = \frac{\Lambda\Phi_n}{\|\Lambda\Phi_n\|_{L^2_\rho}}$, using the orthogonality conditions (6.4.9) and (6.4.26) and the fact that ψ_j is radial for $1 \leq j \leq n+1$, yields the identity

$$\begin{aligned} &\tfrac{d}{ds}\left[\log\left(\tfrac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right] (\Lambda(\Phi_n + v^{(2)}), \psi_1)_\rho \\ = \ & -\left(\left(\tfrac{x_s^{(1)}}{\lambda^{(1)}} - \tfrac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla\varepsilon^{(2)}, \psi_1\right)_\rho - \left(\tfrac{\lambda_s^{(1)}}{\lambda^{(1)}} + 1\right)(\Lambda\triangle v, \psi_1)_\rho - \left(\tfrac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle\varepsilon, \psi_1\right)_\rho \\ & - \left((\Phi_n + v^{(1)})^p - (\Phi_n + v^{(2)})^p - p\Phi_n^{p-1}\triangle v, \psi_1\right)_\rho \end{aligned} \tag{6.4.63}$$

and we now estimate each term. The coercivity (*6.A.1*) and the bounds (*6.4.17*) and (*6.4.18*) yields

$$(\Lambda(\Phi_n+v^{(2)}),\psi_1)_\rho = 1+O(e^{-\mu s}),$$

$$\left|\left(\left(\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla\varepsilon^{(2)},\psi_1\right)_\rho\right| \lesssim e^{-\mu s}\left|\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right|.$$

The modulation estimate (*6.4.25*), with (*6.4.17*), (*6.4.18*) and (*6.4.19*) and an integration by parts yields

$$\left|\left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}}+1\right)(\Lambda\triangle v,\psi_1)_\rho-\left(\frac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle\varepsilon,\psi_1\right)_\rho\right| \lesssim e^{-\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right).$$

Eventually, for the difference of the nonlinear terms the nonlinear inequality

$$\left|(x+y)^p-(x+z)^p-px^{p-1}(y-z)\right| \lesssim (|x|^{p-2}+|y|^{p-2}+|z|^{p-2})(|y|+|z|)|y-z|$$

for any x,y,z and the bound (*6.4.23*) yields the pointwise estimate

$$\left|(\Phi_n+v^{(1)})^p-(\Phi_n+v^{(2)})^p-p\Phi_n^{p-1}\triangle v\right| \lesssim e^{-c\mu s}|\triangle v|, \tag{6.4.64}$$

which implies

$$\left|\left((\Phi_n+v^{(1)})^p-(\Phi_n+v^{(2)})^p-p\Phi_n^{p-1}\triangle v,\psi_1\right)_\rho\right| \lesssim e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right). \tag{6.4.65}$$

The collection of the above bounds, when plugged in (*6.4.63*), yields

$$\left|\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right]\right| \lesssim e^{-\mu s}\left|\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right|+e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right).$$

With the same techniques, taking the scalar product of (*6.4.61*) with $\partial^k\Phi_n$, $k=1,2,3$ implies

$$\left|\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right| \lesssim e^{-\mu s}\left|\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right]\right|+e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right).$$

The two above equations, when put together, imply the estimate

$$\left|\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right]\right|+\left|\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right| \lesssim e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right).$$

The corresponding estimate for $|\triangle a_{j,s}+\mu_j\triangle a_j|$ follows along the same lines, and therefore (*6.4.62*) is proven.

step 2 Localized energy estimate. We claim the differential bound

$$\frac{d}{ds}\|\triangle\varepsilon\|_{L^2_\rho}^2+c_n\|\triangle\varepsilon\|_{L^2_\rho}^2 \lesssim e^{-c\mu s}\sum_{j=2}^{n+1}|\triangle a_j|^2 \tag{6.4.66}$$

which we now prove. From the evolution equation (*6.4.61*) and the orthogonality conditions (*6.4.9*) one obtains first the identity

$$\begin{aligned}
\frac{d}{ds}\frac{1}{2}\|\triangle\varepsilon\|^2_{L^2_\rho} = & -(\mathcal{L}_n\triangle\varepsilon,\triangle\varepsilon)_\rho + \frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right](\Lambda v^{(2)},\triangle\varepsilon)_\rho \\
& + \left(\left(\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla v^{(2)},\triangle\varepsilon\right)_\rho + \left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}}+1\right)(\Lambda\triangle v,\triangle\varepsilon)_\rho \\
& + \left(\frac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle v,\triangle\varepsilon\right)_\rho + \left((\Phi_n+v^{(1)})^p-(\Phi_n+v^{(2)})^p-p\Phi_n^{p-1}\triangle v,\triangle\varepsilon\right)_\rho
\end{aligned} \tag{6.4.67}$$

and we now estimate each term. The spectral gap (*6.3.6*) and (*6.4.9*) imply

$$-(\mathcal{L}_n\triangle\varepsilon,\triangle\varepsilon)_\rho\leq -c_n\|\triangle\varepsilon\|^2_{L^2_\rho}.$$

The modulation estimates (*6.4.62*) of step 1 and Cauchy-Schwarz imply

$$\begin{aligned}
& \left|\frac{d}{ds}\left[\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right](\Lambda v^{(2)},\triangle\varepsilon)_\rho + \left(\left(\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right).\nabla v^{(2)},\triangle\varepsilon\right)_\rho\right| \\
\lesssim\ & \left(\left|\frac{d}{ds}\log\left(\frac{\lambda^{(1)}}{\lambda^{(2)}}\right)\right|\|\Lambda v^{(2)}\|_{L^2_\rho}+\left|\frac{x_s^{(1)}}{\lambda^{(1)}}-\frac{x_s^{(2)}}{\lambda^{(2)}}\right|\|\nabla v^{(2)}\|_{L^2_\rho}\right)\|\triangle\varepsilon\|_{L^2_\rho} \\
\lesssim\ & \|v^{(2)}\|_{H^2_\rho}e^{-c\mu s}\left(\|\triangle\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|\right)\|\triangle\varepsilon\|_{L^2_\rho} \\
\lesssim\ & e^{-(1+c)\mu s}\left(\|\triangle\varepsilon\|^2_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|^2\right)
\end{aligned}$$

where we used (*6.A.1*), (*6.4.17*) and (*6.4.18*) to control $v^{(2)}$. Using the modulation estimate (*6.4.25*), with (*6.4.17*), (*6.4.18*) and (*6.4.19*) for $u^{(1)}$, integrating by parts and applying Cauchy-Schwarz and (*6.A.1*) yields

$$\begin{aligned}
& \left|\left(\frac{\lambda_s^{(1)}}{\lambda^{(1)}}+1\right)(\Lambda\triangle v,\triangle\varepsilon)_\rho+\left(\frac{x_s^{(1)}}{\lambda^{(1)}}.\nabla\triangle v,\triangle\varepsilon\right)_\rho\right| \\
\lesssim\ & \left(\left|\frac{\lambda_s^{(1)}}{\lambda^{(1)}}+1\right|+\left|\frac{x_s^{(1)}}{\lambda^{(1)}}\right|\right)\left(|(\Lambda\triangle\psi,\triangle\varepsilon)_\rho|+|(\Lambda\triangle\varepsilon,\triangle\varepsilon)_\rho|+|(\nabla\triangle\psi,\triangle\varepsilon)_\rho|+|(\nabla\triangle\varepsilon,\triangle\varepsilon)_\rho|\right) \\
\lesssim\ & e^{-2\mu s}\left(\sum_{j=2}^{n+1}|\triangle a_j|^2+\|\triangle\varepsilon\|^2_{H^1_\rho}\right).
\end{aligned}$$

Finally, the pointwise estimate (*6.4.64*) and Cauchy-Schwarz imply for the nonlinear term

$$\left|\left((\Phi_n+v^{(1)})^p-(\Phi_n+v^{(2)})^p-p\Phi_n^{p-1}\triangle v,\triangle\varepsilon\right)_\rho\right|\lesssim e^{-c\mu s}\left(\|\triangle\varepsilon\|^2_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|^2\right).$$

We inject all the above bounds in the identity (*6.4.67*), which for s_0 large enough imply the desired estimate (*6.4.66*) since $0<c\leq 1$.

step 3 Lipschitz bound by reintegration. We define

$$A:=\sup_{s\geq s_0}\sum_{j=2}^{n+1}|\triangle a_j(s)|e^{\mu s}<+\infty,\quad \mathcal{E}:=\sup_{s\geq s_0}\|\triangle\varepsilon\|^2_{L^2_\rho}e^{2\mu s}<+\infty, \tag{6.4.68}$$

which are finite from (*6.4.17*) and (*6.4.18*).

Identity for $\triangle a_j$. Fix j with $2\leq j\leq n+1$. Reintegrating the modulation equation (*6.4.62*) yields

$$\begin{aligned}
\triangle a_j &= \triangle a_j(0)e^{\mu_j(s-s_0)}+e^{\mu_j s}\int_{s_0}^s e^{-\mu_j s'}O(e^{-c\mu s'}(\|\Delta\varepsilon\|_{L^2_\rho}+\sum_{j=2}^{n+1}|\triangle a_j|))ds' \\
&= \triangle a_j(0)e^{\mu_j(s-s_0)}+e^{\mu_j s}\int_{s_0}^s O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds' \\
&= \left(\triangle a_j(0)e^{-\mu_j s_0}+\int_{s_0}^{+\infty} O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds'\right)e^{\mu_j s} \\
&\quad -e^{\mu_j s}\int_s^{+\infty} O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds'. \qquad (6.4.69)
\end{aligned}$$

The integral appearing in this identity is indeed convergent and satisfies:

$$\left|\int_s^{+\infty} O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds'\right|\lesssim e^{-(\mu_j+(c+1)\mu)s}(A+\sqrt{\mathcal{E}}).$$

From (*6.4.68*) one gets $|\triangle a_j|\lesssim e^{-\mu s}$ and from the two above identities one necessarily must have that the parameter in front of the diverging term $e^{\mu_j s}$ is 0:

$$\triangle a_j(0)e^{-\mu_j s_0}+\int_{s_0}^{+\infty} O(e^{-(\mu_j+(c+1)\mu)s'}(A+\sqrt{\mathcal{E}}))ds'=0$$

which gives the first bound

$$|\triangle a_j(0)|\lesssim e^{-(c+1)\mu s_0}(A+\sqrt{\mathcal{E}}), \qquad (6.4.70)$$

and going back to the identity (*6.4.69*) one obtains:

$$|\triangle a_j|\lesssim e^{-((c+1)\mu)s}(A+\sqrt{\mathcal{E}})$$

which implies from the definition (*6.4.68*) of A the bound

$$A\lesssim e^{-c\mu s_0}\sqrt{\mathcal{E}}. \qquad (6.4.71)$$

Identity for $\triangle\varepsilon$. We reintegrate the energy bound (*6.4.66*) to find

$$\begin{aligned}
\|\triangle\varepsilon\|^2_{L^2_\rho} &\lesssim \|\triangle\varepsilon(0)\|^2_{L^2_\rho}e^{-c_n(s-s_0)}+e^{-c_n s}\textstyle\int_{s_0}^s e^{c_n s'}\sum_{j=2}^{n+1}|\triangle a_j|^2e^{-\mu c s'}ds' \\
&\lesssim \|\triangle\varepsilon(0)\|^2_{L^2_\rho}e^{-c_n(s-s_0)}+A^2e^{-(c+2)\mu s}
\end{aligned}$$

since $\mu=\frac{c_n}{4}$ from (*6.4.49*) and $0<c\ll 1$ can be chosen arbitrarily small. Injecting (*6.4.71*) in the above identity yields

$$\mathcal{E}\lesssim\|\triangle\varepsilon(0)\|^2_{L^2_\rho}e^{2\mu s_0}$$

so that (*6.4.71*) can be rewritten as $A\lesssim\|\triangle\varepsilon(0)\|_{L^2_\rho}e^{(1-c)\mu s_0}$. We inject these two last bounds in (*6.4.70*) which finally yields the desired estimate (*6.4.60*). □

6.A Coercivity estimates

Lemma 6.A.1 (Weighted L^2 estimate). *Let $u, \partial_r u \in L^2_\rho(\mathbb{R}^3)$, then*

$$\|ru\|_\rho \lesssim \|u\|_{H^1_\rho}. \tag{6.A.1}$$

Moreover,

$$\|\Delta u\|^2_{L^2_\rho} \lesssim \| -\Delta u + y\cdot\nabla u\|^2_{L^2_\rho} + \|u\|^2_{H^1_\rho}. \tag{6.A.2}$$

Proof. We may assume by density $u \in \mathcal{D}(\mathbb{R}^3)$.

step 1 Proof of (*6.A.1*). We use $\partial_r\rho = -r\rho$ and integrate by parts to compute:

$$\begin{aligned}
& \int_0^{+\infty}\left(\partial_r u - \frac{1}{2}ru\right)^2\rho r^2 dr \\
= & \int_0^{+\infty}(\partial_r u)^2\rho r^2 dr + \frac{1}{4}\int_0^{+\infty} r^2u^2\rho r^2 dr - \int_0^{+\infty} ru\partial_r u\rho r^2 dr \\
= & \int_0^{+\infty}(\partial_r u)^2\rho r^2 dr + \frac{1}{4}\int_0^{+\infty} r^2u^2\rho r^2 dr - \frac{1}{2}\left[r^3\rho u^2\right]_0^{+\infty} \\
& + \frac{1}{2}\int_0^{+\infty} u^2(3-r^2)\rho r^2 dr \\
= & \int_0^{+\infty}(\partial_r u)^2\rho r^2 dr - \frac{1}{4}\int_0^{+\infty} r^2u^2\rho r^2 dr + \frac{3}{2}\int_0^{+\infty} u^2\rho r^2 dr
\end{aligned}$$

and hence

$$\|ru\|^2_{L^2_\rho} = \int_0^{+\infty} r^2u^2\rho r^2 dr \le 4\int_0^{+\infty}(\partial_r u)^2\rho r^2 dr + 6\int_0^{+\infty} u^2\rho r^2 dr \lesssim \|u\|^2_{H^1_\rho}$$

which concludes the proof of (*6.A.1*).

step 2. Proof of (*6.A.2*). We compute:

$$\| -\Delta u + y\cdot\nabla u\|^2_{L^2_\rho} = \|\Delta u\|^2_{L^2_\rho} + \|y\cdot\nabla u\|^2_{L^2_\rho} - 2\int(\Delta u)y\cdot\nabla u\rho dy.$$

To compute the crossed term, let $u_\lambda(y) = u(\lambda y)$, then

$$\int|\nabla u_\lambda(y)|^2\rho dy = \frac{1}{\lambda}\int|\nabla u(y)|^2\rho\left(\frac{y}{\lambda}\right)dy$$

and hence differentiating in λ and evaluating at $\lambda = 1$:

$$2\int\nabla u\cdot\nabla(y\cdot\nabla u)\rho dy = \int|\nabla u|^2(-\rho - y\cdot\nabla\rho)dy$$

i.e.

$$2\int y\cdot\nabla u(\rho\Delta u + \nabla u\cdot\nabla\rho) = \int|\nabla u|^2(\rho + y\cdot\nabla\rho)dy$$

which using $\nabla\rho = -y\rho$ becomes:

$$-2\int(\Delta u)y\cdot\nabla u\rho dy = \int|\nabla u|^2\rho|y|^2 - 2\int|y\cdot\nabla u|^2\rho - \int\rho|\nabla u|^2.$$

Hence:

$$\begin{aligned} \| -\Delta u + y\cdot\nabla u\|^2_{L^2_\rho} &= \|\Delta u\|^2_{L^2_\rho} + \int \rho(|y|^2|\nabla u|^2 - |y\cdot\nabla u|^2) - \int \rho|\nabla u|^2 \\ &\geq \|\Delta u\|^2_{L^2_\rho} - \|\nabla u\|^2_{L^2_\rho} \end{aligned}$$

which concludes the proof of (*6.A.2*). □

We now turn to the proof of Hardy type inequalities. All proofs are more or less standard and we give the argument for the sake of completeness.

Lemma 6.A.2 (Radial Hardy with best constants). *Let* $u \in \mathcal{C}^\infty_c(r>1)$ *and*

$$\gamma \neq -1, \tag{6.A.3}$$

then

$$\int_1^{+\infty} \frac{(\partial_r u)^2}{r^\gamma}dr \geq \left(\frac{\gamma+1}{2}\right)^2 \int_1^{+\infty} \frac{u^2}{r^{\gamma+2}}dr. \tag{6.A.4}$$

Proof. We integrate by parts:

$$\int_1^{+\infty} \frac{u^2}{r^{\gamma+2}}dr = \frac{2}{\gamma+1}\int_1^{+\infty} \frac{u\partial_r u}{r^{\gamma+1}}dr \leq \frac{2}{|\gamma+1|}\left(\int_1^{+\infty}\frac{u^2}{r^{\gamma+2}}dr\right)^{\frac12}\left(\int_1^{+\infty}\frac{(\partial_r u)^2}{r^\gamma}dr\right)^{\frac12}$$

and (*6.A.4*) follows. □

Lemma 6.A.3 (Global Hardy for Δ). *Then there exists* $c>0$ *such that* $\forall u \in C^\infty_c(|x|>1)$,

$$\int |\Delta u|^2 dx \geq c\int \left(\frac{|\nabla u|^2}{|x|^2} + \frac{|u|^2}{|x|^4}\right)dx. \tag{6.A.5}$$

Proof. We decompose u in spherical harmonics and consider

$$\Delta_m u_m = \partial_r^2 u_m + \frac{2}{r}\partial_r u_m - \frac{m(m+1)}{r^2}, \quad m\in\mathbb{N}.$$

We claim that for all $v \in \mathcal{C}^\infty_c((1,+\infty))$,

$$\int_1^{+\infty} |\Delta_m v|^2 r^2 dr \geq c\int_1^{+\infty}\left(\frac{|\partial_r v|^2}{r^2} + \frac{(1+m^4)|v|^2}{r^4}\right)r^2 dr \tag{6.A.6}$$

with c independent of m. Assume (*6.A.6*), then

$$\int \frac{|\nabla u|^2}{r^2}dx \sim \sum_{m\geq 0}\sum_{k=-m}^{m}\int\left(\frac{|\partial_r u_{m,k}|^2}{r^2} + \frac{m^2|u_{m,k}|^2}{r^4}\right)r^2 dr$$

and hence summing (*6.A.6*) ensures (*6.A.5*).
To prove (*6.A.6*), we factorize the Laplace operator:

$$\Delta_m = -A_m^* A_m \quad \text{with} \quad \left| \begin{array}{l} A_m = -\partial_r - \frac{\gamma_m}{r} = -\frac{1}{r^{\gamma_m}}\partial_r(r^{\gamma_m}), \quad \gamma_m = -m, \\ A_m^* = \partial_r + \frac{2-\gamma_m}{r}\partial_r = \frac{1}{r^{2-\gamma_m}}\partial_r(r^{2-\gamma_m}). \end{array}\right.$$

Hence from (*6.A.4*):

$$\begin{aligned}
\int_1^{+\infty}(\Delta_m v)^2 r^2 dr &= \int_1^{+\infty}(A_m^* A_m v)^2 r^2 dr = \int_1^{+\infty}\frac{1}{r^{2-2\gamma_m}}(\partial_r(r^{2-\gamma_m}A_m v))^2 dr\\
\geq \ & \left(\frac{2-2\gamma_m+1}{2}\right)^2\int_1^{+\infty}(A_m v)^2 dr = \left(\frac{2-2\gamma_m+1}{2}\right)^2\int_1^{+\infty}\frac{1}{r^{2\gamma_m}}(\partial_r(r^{\gamma_m}v))^2 dr\\
\geq \ & \left(\frac{2-2\gamma_m+1}{2}\right)^2\left(\frac{2\gamma_m+1}{2}\right)^2\int_1^{+\infty}\frac{v^2}{r^2}dr
\end{aligned}$$

since $\gamma_m = -m$ with $m\in\mathbb{N}$ which ensures that the forbidden value (*6.A.3*) is never attained. We conclude that for some universal constant $\delta>0$ independent of m:

$$\int_1^{+\infty}(\Delta_m v)^2 r^2 dr \geq \delta(1+m^4)\int_1^{+\infty}\frac{v^2}{r^4}r^2 dr.$$

Also, since we have also proved that

$$\int_1^{+\infty}|A_m v|^2 dr \lesssim \int_1^{+\infty}(\Delta_m v)^2 r^2 dr,$$

we infer

$$\begin{aligned}
\int_1^{+\infty}\frac{(\partial_r v)^2}{r^2}r^2 dr &\lesssim \int_1^{+\infty}|A_m v|^2 dr + \gamma_m^2\int\frac{v^2}{r^4}r^2 dr\\
&\lesssim \int_1^{+\infty}(\Delta_m v)^2 r^2 dr
\end{aligned}$$

and (*6.A.6*) follows. □

6.B Proof of (*6.4.43*)

Let

$$0<\nu<1,\quad 1<p_1,p_2,p_3,p_4<+\infty,\quad \frac{1}{2}=\frac{1}{p_1}+\frac{1}{p_2}=\frac{1}{p_3}+\frac{1}{p_4}.$$

Using (*6.4.44*), we have

$$\begin{aligned}
\|\nabla^\nu(uv)\|_{L^2} &\sim \|uv\|_{\dot B^\nu_{2,2}}\\
&\sim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\leq t}\|uv(\cdot-y)-uv(\cdot)\|_{L^2}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12}\\
&\lesssim \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\leq t}\|u(\cdot-y)(v(\cdot-y)-v(\cdot))\|_{L^2}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12}\\
&\quad + \left(\int_0^{+\infty}\left(\frac{\sup_{|y|\leq t}\|v(\cdot)(u(\cdot-y)-u(\cdot))\|_{L^2}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12}\\
&\lesssim \|u\|_{L^{p_4}}\left(\int_0^{+\infty}\left(\frac{\sup_{|y|\leq t}\|v(\cdot-y)-v(\cdot)\|_{L^{p_3}}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12}\\
&\quad +\|v\|_{L^{p_2}}\left(\int_0^{+\infty}\left(\frac{\sup_{|y|\leq t}\|u(\cdot-y)-u(\cdot)\|_{L^{p_1}}}{t^\nu}\right)^2\frac{dt}{t}\right)^{\frac12}\\
&\lesssim \|u\|_{\dot B^\nu_{p_1,2}}\|v\|_{L^{p_2}}+\|u\|_{L^{p_4}}\|v\|_{\dot B^\nu_{p_3,2}}
\end{aligned}$$

which concludes the proof of (*6.4.43*).

6.C Proof of Lemma 6.3.2

The existence and uniqueness of $\phi_{n,m}, \nu_m$ satisfying (*6.3.8*) and (*6.3.12*) is well known. Thus, we focus on their behaviour as $r \to +\infty$.

step 1 Inverting $\mathcal{L}_{m,\infty}$. Let γ_m be the solution to

$$\gamma_m^2 - \gamma_m + pc_\infty^{p-1} - m(m+1) = 0,$$

the corresponding discriminant Δ_m is given by

$$\Delta_m := 1 - 4pc_\infty^{p-1} + 4m(m+1). \tag{6.C.1}$$

For $m = 1$,

$$\Delta_1 = \left(\frac{p+3}{p-1}\right)^2 > 0 \tag{6.C.2}$$

and hence for all $m \geq 1$

$$\Delta_m \geq \Delta_1 > 0.$$

Therefore, γ_m is real and we choose the smallest root[16] so that γ_m is given by

$$\gamma_m = \frac{1 - \sqrt{\Delta_m}}{2}.$$

We now solve

$$\mathcal{L}_{\infty,m}(\psi) = 0$$

through the change of variable and unknown

$$\psi(r) = \frac{1}{(2z)^{\frac{\gamma_m}{2}}} w(z), \quad z = \frac{r^2}{2}$$

which leads to

$$\mathcal{L}_{\infty,m}(\psi) \quad = \quad -\frac{2}{(2z)^{\frac{\gamma}{2}}} \left(zw''(z) + \left(-\gamma_m + \frac{3}{2} - z\right) w'(z) - \left(\frac{1}{p-1} - \frac{\gamma_m}{2}\right) w(z)\right).$$

Thus, $\mathcal{L}_{\infty,m}(\psi) = 0$ if and only if

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} - aw = 0$$

where we have used the notations

$$a = \frac{1}{p-1} - \frac{\gamma_m}{2}, \quad b = -\gamma_m + \frac{3}{2}.$$

[16] This is motivated by the fact that we obtain below the Kummer's equation with $b = -\gamma_m + 1/2$. This is equivalent to $-b = \pm\sqrt{\Delta_m}$. Since the Kummer function is not defined for $-b \in \mathbb{N}$, this justifies to consider the smallest root γ_m.

Hence w is a linear combination of two special functions, the Kummer's function $M(a,b,z)$ and the Tricomi function $U(a,b,z)$. These special functions have the following asymptotic behavior at infinity (see for example [135]):

$$M(a,b,z) \sim \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z, \quad U(a,b,z) \sim z^{-a} \text{ as } z \to +\infty.$$

This allows us to infer the asymptotic for w for $z \to 0_+$. Finally, since

$$\psi(r) = \frac{1}{r^{\gamma_m}} w\left(\frac{r^2}{2}\right),$$

we infer from the asymptotic of w the following asymptotic behavior for $\psi_{1,m}$ and $\psi_{2,m}$

$$\psi_{1,m} \sim \frac{1}{r^{\frac{2}{p-1}}} \text{ and } \psi_{2,m} \sim r^{\frac{2}{p-1}-3} e^{\frac{r^2}{2}} \text{ as } r \to +\infty.$$

Consider the Wronskian W which is defined as

$$W \quad := \quad \psi_{1,m}' \psi_{2,m} - \psi_{2,m}' \psi_{1,m},$$

then without loss of generality since $W' = \left(r - \frac{2}{r}\right) W$

$$W = \frac{1}{r^2} e^{\frac{r^2}{2}}.$$

We deduce using the variation of constants that the solution w to

$$\mathcal{L}_{\infty,m}(u) = f,$$

is given by

$$u \quad = \quad \left(a_1 + \int_r^{+\infty} f \psi_{2,m} r'^2 e^{-\frac{r'^2}{2}} dr'\right) \psi_{1,m} + \left(a_2 - \int_r^{+\infty} f \psi_{1,m} r'^2 e^{-\frac{r'^2}{2}} dr'\right) \psi_{2,m}.$$

step 2 Basis of $\mathcal{L}_{m,n}$ near $+\infty$. We now construct a solution to $\mathcal{L}_{n,m}(\varphi) = 0$ near $+\infty$ by solving:

$$\mathcal{L}_{\infty,m}(\varphi) \quad = \quad \mathcal{L}_{n,m}(\varphi) + p(\Phi_n^{p-1} - \Phi_*^{p-1}) = p(\Phi_n^{p-1} - \Phi_*^{p-1})\varphi$$

ie

$$\begin{aligned} \varphi \quad = \quad & \left(a_1 + \int_r^{+\infty} p(\Phi_n^{p-1} - \Phi_*^{p-1}) \varphi \psi_{2,m} r'^2 e^{-\frac{r'^2}{2}} dr'\right) \psi_{1,m} \\ & + \left(a_2 - \int_r^{+\infty} p(\Phi_n^{p-1} - \Phi_*^{p-1}) \varphi \psi_{1,m} r'^2 e^{-\frac{r'^2}{2}} dr'\right) \psi_{2,m}. \end{aligned}$$

To construct the solution φ_1 with the choice $a_1 = 1$ and $a_2 = 0$ we solve the fixed point equation

$$\varphi_1 = \psi_{1,m} + \widetilde{\varphi}_1, \quad \widetilde{\varphi}_1 = \mathcal{G}(\widetilde{\varphi}_1) \tag{6.C.3}$$

where

$$\begin{aligned} \mathcal{G}(\widetilde{\varphi})(r) \quad = \quad & \left(\int_r^{+\infty} p(\Phi_n^{p-1} - \Phi_*^{p-1})(\psi_{1,m} + \widetilde{\varphi})(r') \psi_{2,m} r'^2 e^{-\frac{r'^2}{2}} dr'\right) \psi_{1,m} \\ & - \left(\int_r^{+\infty} p(\Phi_n^{p-1} - \Phi_*^{p-1})(\psi_{1,m} + \widetilde{\varphi})(r') \psi_{1,m} r'^2 e^{-\frac{r'^2}{2}} dr'\right) \psi_{2,m}. \end{aligned}$$

Recall that we have in view of Corollary 6.2.6

$$\lim_{n\to+\infty}\sup_{r\geq 1} r^{\frac{2}{p-1}}|\Phi_n(r)-\Phi_*(r)|=0.$$

Thus, for $n\geq N$ large enough, we infer

$$|\Phi_n(r)-\Phi_*(r)|\leq \frac{1}{r^{\frac{2}{p-1}}} \text{ for } r\geq 1.$$

so that

$$|p(\Phi_n^{p-1}-\Phi_*^{p-1})|\lesssim \frac{1}{r^2}.$$

We infer for $r\geq 1$

$$\begin{aligned}|\mathcal{G}(\widetilde{\varphi})(r)| &\lesssim \frac{1}{r^{\frac{2}{p-1}}}\left(\int_r^{+\infty} r'^{\frac{2}{p-1}-3}\left(\frac{1}{r'^{\frac{2}{p-1}}}+|\widetilde{\varphi}(r')|\right)dr'\right)\\ &+ r^{\frac{2}{p-1}-3}e^{\frac{r^2}{2}}\left(\int_r^{+\infty}\frac{1}{r'^{\frac{2}{p-1}}}e^{-\frac{r'^2}{2}}\left(\frac{1}{r'^{\frac{2}{p-1}}}+|\widetilde{\varphi}(r')|\right)dr'\right)\\ &\lesssim \frac{1}{r^{2+\frac{2}{p-1}}}+\frac{1}{r^{\frac{2}{p-1}}}\left(\int_r^{+\infty} r'^{\frac{2}{p-1}-3}|\widetilde{\varphi}(r')|dr'\right)\\ &+ r^{\frac{2}{p-1}-3}e^{\frac{r^2}{2}}\left(\int_r^{+\infty}\frac{1}{r'^{\frac{2}{p-1}}}e^{-\frac{r'^2}{2}}|\widetilde{\varphi}(r')|dr'\right)\end{aligned}$$

and

$$\begin{aligned}\left|\mathcal{G}\left(\widetilde{\varphi}_{(1)}\right)(r)-\mathcal{G}\left(\widetilde{\varphi}_{(2)}\right)(r)\right| &\lesssim \frac{1}{r^{\frac{2}{p-1}}}\left(\int_r^{+\infty} r'^{\frac{2}{p-1}-3}|\widetilde{\varphi}_{(1)}(r')-\widetilde{\varphi}_{(2)}(r')|dr'\right)\\ &+ r^{\frac{2}{p-1}-3}e^{\frac{r^2}{2}}\left(\int_r^{+\infty}\frac{1}{r'^{\frac{2}{p-1}}}e^{-\frac{r'^2}{2}}|\widetilde{\varphi}_{(1)}(r')-\widetilde{\varphi}_{(2)}(r')|dr'\right)\end{aligned}$$

Thus, for $R\geq 1$ large enough, the Banach fixed point theorem applies in the space corresponding to the norm

$$\sup_{r\geq R} r^{1+\frac{2}{p-1}}\left|\widetilde{\varphi}\right|(r).$$

Hence, there exists a unique solution $\widetilde{\varphi}_1$ to (*6.C.3*) and

$$\sup_{r\geq R} r^{1+\frac{2}{p-1}}\left|\widetilde{\varphi}_1\right|(r)\lesssim 1.$$

Hence, φ_1 satisfies $\mathcal{L}_{n,m}(\varphi_1)=0$ and

$$\varphi_1\sim\frac{1}{r^{\frac{2}{p-1}}}, \text{ as } r\to+\infty.$$

The behaviour of the other solution at infinity is computed using the Wronskian relation

$$W=\varphi_1'\varphi_2-\varphi_2'\varphi_1=-\frac{1}{r^2}e^{\frac{r^2}{2}}$$

and hence

$$\left(\frac{\varphi_2}{\varphi_1}\right)'=-\frac{W}{\varphi_1^2}=\frac{1}{r^2\varphi_1^2}e^{\frac{r^2}{2}}$$

from which

$$\varphi_2(r) = \varphi_1(r)\int_1^r \frac{1}{r'^2\varphi_1^2(r')}e^{\frac{r'^2}{2}}dr' \sim r^{\frac{2}{p-1}-3}e^{\frac{r^2}{2}} \text{ as } r \to +\infty$$

and (*6.3.9*) is proved.

step 3 Behaviour of ν_m at $+\infty$. First, consider the solution φ to

$$-\partial_r^2\varphi - \frac{2}{r}\partial_r\varphi + \frac{m(m+1)}{r^2} - \frac{pc_\infty^{p-1}}{r^2}\varphi = f. \tag{6.C.4}$$

The homogeneous equation admits the basis of solutions

$$\varphi_+ = \frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}, \quad \varphi_- = \frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}$$

and the corresponding Wronskian is given by

$$W(r) = \varphi_+'(r)\varphi_-(r) - \varphi_-'(r)\varphi_+(r) = -\frac{1}{r^2}.$$

Using the variation of constants, the solutions to (*6.C.4*) are given by

$$\varphi(r) = \left(a_1 - \int_r^{+\infty} f\varphi_- r'^2 dr'\right)\varphi_+ + \left(a_2 + \int_r^{+\infty} f\varphi_+ r'^2 dr'\right)\varphi_-.$$

Now, the equation $H_m(\phi) = 0$ can be written as

$$-\partial_r^2\phi - \frac{2}{r}\partial_r\phi + \frac{m(m+1)}{r^2}\phi - \frac{pc_\infty^{p-1}}{r^2}\phi = p\left(Q^{p-1}(r) - \frac{c_\infty^{p-1}}{r^2}\right)\phi(r),$$

i.e. (*6.C.4*) with

$$f = p\left(Q^{p-1}(r) - \frac{c_\infty^{p-1}}{r^2}\right)\phi(r).$$

We construct the solution $\phi_{m,1}$ to $H_m(\phi_{m,1}) = 0$ with the choice $a_1 = 1$ and $a_2 = 0$ by solving the fixed point equation

$$\phi_{m,1} = \varphi_+ + \widetilde{\phi}, \quad \widetilde{\phi} = \mathcal{F}\left(\widetilde{\phi}\right) \tag{6.C.5}$$

where

$$\begin{aligned}\mathcal{F}\left(\widetilde{\phi}\right)(r) \quad = \quad & -\left(\int_r^{+\infty} p\left(Q^{p-1}(r') - \frac{c_\infty^{p-1}}{r'^2}\right)\left(\varphi_+ + \widetilde{\phi}\right)(r')\varphi_- r'^2 dr'\right)\varphi_+ \\ & + \left(\int_r^{+\infty} p\left(Q^{p-1}(r') - \frac{c_\infty^{p-1}}{r'^2}\right)\left(\varphi_+ + \widetilde{\phi}\right)(r')\varphi_+ r'^2 dr'\right)\varphi_-.\end{aligned}$$

Recall that

$$Q(r) = \frac{c_\infty}{r^{\frac{2}{p-1}}} + \frac{c_1\sin\left(\omega\log(r) + c_2\right)}{r^{\frac{1}{2}}} + o\left(\frac{1}{r^{\frac{1}{2}}}\right) \text{ as } r \to +\infty$$

so that

$$\left|p\left(Q^{p-1}(r) - \frac{c_\infty^{p-1}}{r^2}\right)\right| \lesssim \frac{1}{r^{1+s_c}} \text{ for } r \geq 1.$$

We infer for $r \geq 1$

$$\begin{aligned}
\left|\mathcal{F}(\widetilde{\phi})(r)\right| \lesssim & \frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\left(\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}+\left|\widetilde{\phi}\right|(r')\right)\frac{1}{r'^{\frac{1-\sqrt{\Delta_m}}{2}}}dr'\right)\\
&+\frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\left(\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}+\left|\widetilde{\phi}\right|(r')\right)\frac{1}{r'^{\frac{1+\sqrt{\Delta_m}}{2}}}dr'\right)\\
\lesssim & \frac{1}{r^{s_c-1}}\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}+\frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\frac{1}{r'^{\frac{1-\sqrt{\Delta_m}}{2}}}\left|\widetilde{\phi}\right|(r')dr'\right)\\
&+\frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\frac{1}{r'^{\frac{1+\sqrt{\Delta_m}}{2}}}\left|\widetilde{\phi}\right|(r')dr'\right)
\end{aligned}$$

and

$$\begin{aligned}
\left|\mathcal{F}(\widetilde{\phi}_1)(r)-\mathcal{F}(\widetilde{\phi}_2)(r)\right| \lesssim & \frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\frac{1}{r'^{\frac{1-\sqrt{\Delta_m}}{2}}}\left|\widetilde{\phi}_1-\widetilde{\phi}_2\right|(r')dr'\right)\\
&+\frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}}\left(\int_r^{+\infty}\frac{1}{r'^{s_c-1}}\frac{1}{r'^{\frac{1+\sqrt{\Delta_m}}{2}}}\left|\widetilde{\phi}_1-\widetilde{\phi}_2\right|(r')dr'\right).
\end{aligned}$$

Thus, for $R \geq 1$ large enough, the Banach fixed point theorem applies in the space corresponding to the norm

$$\sup_{r\geq R} r^{\frac{s_c-1}{2}}r^{\frac{1+\sqrt{\Delta_m}}{2}}|\widetilde{\phi}|(r)$$

and yields a unique solution $\widetilde{\phi}$ to (*6.C.5*) with

$$\sup_{r\geq R} r^{\frac{s_c-1}{2}}r^{\frac{1+\sqrt{\Delta_m}}{2}}|\widetilde{\phi}|(r) \leq 1.$$

Hence, $\phi_{m,1}$ satisfies $H_m(\phi_{m,1}) = 0$ and

$$\phi_{m,1} \sim \frac{1}{r^{\frac{1+\sqrt{\Delta_m}}{2}}}, \text{ as } r \to +\infty. \tag{6.C.6}$$

The other independent solution $\phi_{m,2}$ to $H_m(\phi_{m,2}) = 0$ is computed through the Wronskian relation

$$W \quad := \quad \phi'_{m,1}\phi_{m,2} - \phi'_{m,2}\phi_{m,1} = -\frac{1}{r^2}$$

ie

$$\phi_{m,2}(r) = \phi_{m,1}(r)\int_1^r \frac{1}{r'^2\phi_{m,1}^2(r')}dr' \sim \frac{1}{r^{\frac{1-\sqrt{\Delta_m}}{2}}} \text{ as } r \to +\infty.$$

Since ν_m is a linear combination of $\phi_{m,1}$ and $\phi_{m,2}$, we infer

$$\nu_m(r) \sim \frac{c_{m,+}}{r^{\frac{1+\sqrt{\Delta_m}}{2}}} + \frac{c_{m,-}}{r^{\frac{1-\sqrt{\Delta_m}}{2}}} \text{ as } r \to +\infty \tag{6.C.7}$$

for some constant $c_{m,+}$ and $c_{m,-}$.
case $m = 1$: By translation invariance

$$H_1(Q') = 0 \text{ and } Q'(r) = Q''(0)r(1 + O(r^2)) \tag{6.C.8}$$

Hence, by uniqueness of ν_1, we infer

$$\nu_1(r) = \frac{Q'(r)}{Q''(0)} < 0 \quad \text{on} \quad (0, +\infty)$$

where we used from standard ODE arguments $Q''(0) < 0$ and

$$Q' < 0 \quad \text{on} \quad (0, +\infty). \tag{6.C.9}$$

case $m = 2$: From (*6.C.8*), (*6.C.9*) and standard Sturm Liouville oscillation arguments for central potentials [143], the quadratic form $(H_1 u, u)$ is positive on $\dot{H}^1_{\text{rad}}(0, +\infty)$ and hence for $m \geq 2$, $H_m > H_1$ is definite positive, and hence $\nu_m > 0$ on $(0, +\infty)$. Moreover, If $c_{m,-} = 0$ in (*6.C.7*), then $\nu_m \in \dot{H}^1_{\text{rad}}$ satisfies $(H_m \nu_m, \nu_m) = 0$ which is a contradiction, hence the leading order behaviour (*6.3.12*).

step 4 Completing the basis.
case $m = 2$. Let ϕ_m be the solution to $H_m(\phi_m) = 0$ constructed above with the behaviour (*6.C.6*). At the origin, the equation $H_m \psi$ reads

$$A_m^* A_m \psi = V \psi,$$

with

$$A_m v = r^m \partial_r \left(\frac{v}{r^m} \right), \quad A_m^* = \frac{v}{r^{m+1}} \partial_r (r^{m+1} v)$$

and $V \in L^\infty$ and hence all solutions on $(0, \delta)$ with $0 < \delta \ll 1$ are of the form

$$\psi = c_0 r^m + \frac{c_1}{r^{m+1}} + r^m \int_r^\delta \frac{d\tau}{\tau^{2m+1}} \int_0^r \tau^{m+1} V \psi d\tau$$

through an elementary fixed point argument. Hence

$$\phi_m = \frac{c_1 + O(r^2)}{r^{m+1}}. \tag{6.C.10}$$

Assume by contradiction that $c_1 = 0$. Then, the fixed point above leads to $\phi_m = O(r^m)$. Hence ϕ_m is a zero of H_m in $\dot{H^1}_{\text{rad}}$ which is a contradiction. Thus, $c_1 \neq 0$ and together with (*6.C.10*), we have obtained (*6.3.13*).

case $m = 1$. We let ϕ_1 be given by the Wronskian relation

$$\phi_1 = \nu_1(r) \int_r^1 \frac{d\tau}{\tau^2 \nu_1^2(\tau)} d\tau \sim \left| \begin{array}{ll} \frac{c}{r^2} & \text{as } r \to 0, \quad c \neq 0, \\ \frac{1}{r^{\frac{1-\sqrt{\Delta_1}}{2}}} & \text{as } r \to +\infty, \end{array} \right.$$

which is (*6.3.11*).

step 5 Proof of (*6.3.15*). Let

$$\kappa_{n,m} := \mu_n^{-m} \varphi_{n,m}(\mu_n r).$$

Then, since $\varphi_{n,m}$ satisfies $\mathcal{L}_{n,m}(\varphi_{n,m}) = 0$, we infer

$$-\partial_r^2 \kappa_{n,m} - \frac{2}{r} \partial_r \kappa_{n,m} + \frac{m(m+1)}{r^2} \kappa_{n,m} - p \left(\mu_n^{\frac{2}{p-1}} \Phi_n(\mu_n r) \right)^{p-1} \kappa_{n,m} = -\mu_n^2 \Lambda \kappa_{n,m}.$$

This yields

$$H_m(\kappa_{n,m}) \quad = \quad f_{n,m} := p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\kappa_{n,m} - \mu_n^2\Lambda\kappa_{n,m}.$$

Since $H_m(\nu_m) = 0$, we infer

$$H_m\left(\kappa_{n,m} - \nu_m\right) \quad = \quad f_{n,m}.$$

We let (ν_m, ϕ_m) be the completed fundamental basis for H_m so that

$$\kappa_{n,m} - \nu_m = \left(a_1 - \int_0^r f_{n,m}\phi_m r'^2 dr'\right)\nu_m + \left(a_2 + \int_0^r f_{n,m}\nu_m r'^2 dr'\right)\phi_m.$$

Since

$$\nu_m(r) = r^m(1 + O(r^2)) \text{ and } \varphi_{n,m}(r) = r^m(1 + O(r^2)) \text{ as } r \to 0_+,$$

we infer

$$\kappa_{n,m}(r) - \nu(r) = O(r^{m+2})$$

and hence (*6.3.11*), (*6.3.13*) implies $a_1 = a_2 = 0$ and:

$$\kappa_{n,m} - \nu_m = -\left(\int_0^r f_{n,m}\phi_m r'^2 dr'\right)\nu_m + \left(\int_0^r f_{n,m}\nu_m r'^2 dr'\right)\phi_m.$$

In order to estimate $f_{n,m}$, recall from Corollary 6.2.6 that we have

$$\sup_{r \le r_0}\left|\Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}}Q\left(\frac{r}{\mu_n}\right)\right| \lesssim \mu_n^{s_c - 1}$$

This yields

$$\sup_{r \le \frac{r_0}{\mu_n}}\left|p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\right| \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}. \tag{6.C.11}$$

Also, we rewrite $f_{n,m}$ as

$$\begin{aligned} f_{n,m} \quad = \quad & p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\nu_m - \mu_n^2\Lambda\nu_m \\ & + p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)(\kappa_{n,m} - \nu_m) - \mu_n^2\Lambda(\kappa_{n,m} - \nu_m). \end{aligned} \tag{6.C.12}$$

$0 \le r \le 1$. In view of the asymptotic behavior as $r \to 0_+$ (*6.3.11*), (*6.3.13*) of the basis of solutions ν_m, ϕ_m, and after integrating by parts the term $\Lambda(\kappa_{n,m} - \nu_m)$, we have for $0 \le r \le 1$ using (*6.C.11*) and (*6.C.12*):

$$\begin{aligned} |\kappa_{n,m} - \nu_m|(r) \quad \lesssim \quad & \mu_n^2 r^2 |\kappa_{n,m} - \nu_m|(r) \\ + \quad & \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2\right)\left(r^{m+2} + r^m\left(\int_0^r |\kappa_{n,m} - \nu_m| r'^{1-m} dr'\right)\right. \\ + \quad & \left. r^{-m-1}\left(\int_0^r |\kappa_{n,m} - \nu_m| r'^{m+2} dr'\right)\right). \end{aligned}$$

Using again the asymptotic behavior of ν_m as $r \to 0_+$, we infer for all $m \geq 1$

$$\sup_{0\leq r\leq 1} \frac{|(\kappa_{n,m}-\nu_m)(r)|}{|\nu_m(r)|} \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2. \tag{6.C.13}$$

In particular, this yields

$$\int_0^1 |f_{n,m}| r'^{1-m} dr' + \int_0^1 |f_{n,m}| r'^{m+2} dr' \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2. \tag{6.C.14}$$

Next, we consider the region $r \geq 1$. In view of the asymptotic behavior at infinity (*6.3.11*), (*6.3.13*), (*6.3.10*), (*6.3.12*), after integrating by parts the term $\Lambda(\kappa_{n,m}-\nu_m)$ and using also (*6.C.14*), we have

$$\begin{aligned} |\kappa_{n,m}-\nu_m| \lesssim\ & \mu_n^2 r^2 |\kappa_{n,m}-\nu_m| \\ & + \frac{1}{(1+r)^{\frac{1+\sqrt{\Delta_m}}{2}}} \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 + \int_1^r |f_{n,m}| \frac{r'^2}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} dr' \right) \\ & + \frac{1}{(1+r)^{\frac{1-\sqrt{\Delta_m}}{2}}} \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 + \int_1^r |f_{n,m}| \frac{r'^2}{(1+r')^{\frac{1+\sqrt{\Delta_m}}{2}}} dr' \right). \end{aligned}$$

After integrating by parts the term $\Lambda(\kappa_{n,m}-\nu_m)$, and in view of the asymptotic behavior of ν_m as $r \to +\infty$ as well as (*6.C.11*), we deduce

$$\begin{aligned} & |(\kappa_{n,m}-\nu_m)(r)| \\ \lesssim\ & \frac{1}{(1+r)^{\frac{1+\sqrt{\Delta_m}}{2}}} \left(\int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} |\nu_m| + \mu_n^2 |\Lambda \nu_m| + \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) |\kappa_{n,m}-\nu_m| \right) \right. \\ & \left. \times \frac{r'^2}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} dr' + \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) \\ & + \frac{1}{(1+r)^{\frac{1-\sqrt{\Delta_m}}{2}}} \left(\int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} |\nu_m| + \mu_n^2 |\Lambda \nu_m| + \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) |\kappa_{n,m}-\nu_m| \right) \right. \\ & \left. \times \frac{r'^2}{(1+r')^{\frac{1+\sqrt{\Delta_m}}{2}}} dr' + \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right). \end{aligned}$$

case $m \geq 2$: We estimate from (*6.3.12*):

$$\begin{aligned} & \frac{|(\kappa_{n,m}-\nu_m)(r)|}{|\nu_m(r)|} \\ \lesssim\ & \frac{1}{(1+r)^{\sqrt{\Delta_m}}} \left\{ \int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) \left(\frac{1}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} + |\kappa_{n,m}-\nu_m| \right) \right. \\ & \left. \times \frac{r'^2}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} dr' + \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right\} \\ & + \int_1^r \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2 \right) \left(\frac{1}{(1+r')^{\frac{1-\sqrt{\Delta_m}}{2}}} + |\kappa_{n,m}-\nu_m| \right) \frac{r'^2}{(1+r')^{\frac{1+\sqrt{\Delta_m}}{2}}} dr' \\ & + \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}} + \mu_n^2. \end{aligned}$$

This yields

$$\sup_{1\le r\le \frac{r_0}{\mu_n}} \frac{|(\kappa_{n,m}-\nu_m)(r)|}{|\nu_m(r)|} \lesssim r_0^2\left(1+\frac{\mu_n^{s_c-1}}{r_0^{2-\frac{2}{p-1}}}\right)$$

which together with (*6.C.13*) concludes the proof of (*6.3.15*) for $n\ge N$ large enough and $m\ge 2$.
case $m=1$ We estimate using (*6.3.10*), (*6.3.11*):

$$\begin{aligned}
&\frac{|(\kappa_{n,1}-\nu_1)(r)|}{|\nu_1(r)|}\\
\lesssim\ & \int_1^r\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(\frac{1}{(1+r')^{\frac{1+\sqrt{\Delta_1}}{2}}}+|\kappa_{n,1}-\nu_1|\right)\frac{r'^2}{(1+r')^{\frac{1-\sqrt{\Delta_1}}{2}}}dr'\\
&+\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\\
&+(1+r)^{\sqrt{\Delta_1}}\left(\int_1^r\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(\frac{1}{(1+r')^{\frac{1+\sqrt{\Delta_1}}{2}}}+|\kappa_{n,1}-\nu_1|\right)\frac{r'^2}{(1+r')^{\frac{1+\sqrt{\Delta_1}}{2}}}dr'\right.\\
&\left.+\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right).
\end{aligned}$$

This yields[17]

$$\sup_{1\le r\le \frac{r_0}{\mu_n}} \frac{|(\kappa_{n,1}-\nu_1)(r)|}{|\nu_1(r)|} \lesssim r_0^2\left(1+\frac{\mu_n^{s_c-1}}{r_0^{2-\frac{2}{p-1}}}\right)+\left(\frac{r_0}{\mu_n}\right)^{\sqrt{\Delta_1}}\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)$$

and hence, together with (*6.C.13*) and the fact that[18] $\sqrt{\Delta_1}<2$, we have for $n\ge N$ large enough

$$\sup_{0\le r\le \frac{r_0}{\mu_n}} \frac{|(\kappa_{n,1}-\nu_1)(r)|}{|\nu_1(r)|} \lesssim r_0^2.$$

The corresponding estimates for first order derivatives are obtained in the same way, and (*6.3.15*) is proved.

6.D Proof of Lemma 6.3.3

step 1 Proof of (*6.3.17*). Let

$$\kappa_n := \varphi_{n,0}(\mu_n r).$$

[17] Here, we use the fact that

$$\sqrt{\Delta_1}-1=\frac{4}{p-1}<1$$

since $p>5$, so that

$$\int_0^r \frac{r'^2}{(1+r')^{1+\sqrt{\Delta_1}}}\lesssim (1+r)^{2-\sqrt{\Delta_1}}.$$

[18] Indeed, we have in view of (*6.C.2*)

$$\sqrt{\Delta_1}=\frac{p+3}{p-1}=2-\frac{p-5}{p-1}<2$$

since $p>5$.

Then, since $\varphi_{n,0}$ satisfies $\mathcal{L}_{n,0}(\varphi_{n,0}) = 0$, we infer

$$-\partial_r^2\kappa_n - \frac{2}{r}\partial_r\kappa_n - p\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1}\kappa_n = -\mu_n^2\Lambda\kappa_n.$$

This yields

$$H(\kappa_n) \quad = \quad f_n$$

where we have introduced the notation

$$f_n \quad := \quad p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1} - Q^{p-1}(r)\right)\kappa_n - \mu_n^2\Lambda\kappa_n.$$

Since $H(\Lambda Q) = 0$, we infer

$$H\left(\kappa_n - \frac{p-1}{2}\Lambda Q\right) \quad = \quad f_n.$$

Recall the solution ρ to $H(\rho) = 0$ constructed in Lemma 6.2.3 such that $(\Lambda Q, \rho)$ forms a basis of solutions of $H(w) = 0$, then the solution to

$$H(w) = f$$

is given by

$$w \quad = \quad \left(a_1 + \int_0^r f\rho r'^2 dr'\right)\Lambda Q + \left(a_2 - \int_0^r f\Lambda Q r'^2 dr'\right)\rho.$$

We infer

$$\kappa_n - \frac{p-1}{2}\Lambda Q \quad = \quad \left(a_1 + \int_0^r f_n\rho r'^2 dr'\right)\Lambda Q + \left(a_2 - \int_0^r f_n\Lambda Q r'^2 dr'\right)\rho.$$

Since ΛQ is a smooth function at $r = 0$ with

$$\Lambda Q(0) = \frac{2}{p-1} \neq 0,$$

we infer from the Wronskian relation that ρ has the following asymptotic behavior

$$\rho \sim \frac{c}{r} \text{ as } r \to 0_+$$

for some constant $c \neq 0$, and hence, we must have $a_2 = 0$. Furthermore, since we have

$$\left(\kappa_n - \frac{p-1}{2}\Lambda Q\right)(0) = 0, \quad \Lambda Q(0) = \frac{2}{p-1} \neq 0$$

we infer $a_1 = 0$. Hence, we have

$$\kappa_n - \frac{p-1}{2}\Lambda Q \quad = \quad \left(\int_0^r f_n\rho r'^2 dr'\right)\Lambda Q - \left(\int_0^r f_n\Lambda Q r'^2 dr'\right)\rho.$$

In order to estimate f_n, recall from Corollary 6.2.6 that we have

$$\sup_{r\leq r_0}\left|\Phi_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}}Q\left(\frac{r}{\mu_n}\right)\right| \lesssim \mu_n^{s_c-1}$$

This yields

$$\sup_{r\leq\frac{r_0}{\mu_n}}\left|p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1}-Q^{p-1}(r)\right)\right|\lesssim\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}. \tag{6.D.1}$$

Also, we rewrite f_n as

$$\begin{aligned} f_n &= p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1}-Q^{p-1}(r)\right)\frac{p-1}{2}\Lambda Q-\mu_n^2\frac{p-1}{2}\Lambda^2 Q \\ &\quad +p\left(\left(\mu_n^{\frac{2}{p-1}}\Phi_n(\mu_n r)\right)^{p-1}-Q^{p-1}(r)\right)\left(\kappa_n-\frac{p-1}{2}\Lambda Q\right)-\mu_n^2\Lambda\left(\kappa_n-\frac{p-1}{2}\Lambda Q\right). \end{aligned} \tag{6.D.2}$$

We start with the region $0\leq r\leq 1$. In view of the asymptotic behavior for ΛQ and ρ:

$$\Lambda Q\sim\frac{2}{p-1}\text{ and }\rho\sim\frac{c}{r}\text{ as }r\to 0+,$$

we infer

$$\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right| \lesssim \int_0^r|f_n|r'dr'+\frac{1}{r}\left(\int_0^r|f_n|r'^2dr'\right).$$

Together with (*6.D.1*) and (*6.D.2*) and integrating by parts the term $\Lambda(\kappa_n-(p-1)/2\Lambda Q)$, we deduce

$$\begin{aligned}\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right| &\lesssim \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(1+\int_0^r\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|r'dr'\right.\\ &\quad\left.+\frac{1}{r}\left(\int_0^r\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|r'^2dr'\right)\right).\end{aligned}$$

We infer

$$\sup_{0\leq r\leq 1}\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right| \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2. \tag{6.D.3}$$

In particular, this yields

$$\int_0^1|f_n|r'dr'+\int_0^1|f_n|r'^2dr' \lesssim \frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2. \tag{6.D.4}$$

Next, we consider the region $r\geq 1$. Recall the asymptotic behavior at infinity of ΛQ and ρ given by Lemma 6.2.3

$$\Lambda Q(r)\sim\frac{c_7\sin\left(\omega\log(r)+c_8\right)}{r^{\frac{1}{2}}},\quad \rho(r)\sim\frac{c_9\sin\left(\omega\log(r)+c_{10}\right)}{r^{\frac{1}{2}}}\text{ as }r\to+\infty,$$

where $c_7,c_9\neq 0$, $c_8,c_{10}\in\mathbb{R}$. We infer for $r\geq 1$

$$\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right| \lesssim \left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2+\int_1^r|f_n|\frac{r'^2}{(1+r')^{\frac{1}{2}}}dr'\right)\frac{1}{(1+r)^{\frac{1}{2}}}.$$

After integrating by parts the term $\Lambda(\kappa_n-(p-1)/2\Lambda Q)$, and together with (*6.D.1*) and (*6.D.2*), we deduce

$$\begin{aligned}
&(1+r)^{\frac{1}{2}}\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\\
\lesssim\ &\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(1+\int_1^r\left(\frac{1}{(1+r')^{\frac{1}{2}}}+\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\right)\frac{r'^2}{(1+r')^{\frac{1}{2}}}dr'\right)\\
\lesssim\ &\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)(1+r)^2+\left(\frac{\mu_n^{s_c+1}}{r_0^{2-\frac{2}{p-1}}}+\mu_n^2\right)\left(\int_1^r\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\frac{r'^2}{(1+r')^{\frac{1}{2}}}dr'\right).
\end{aligned}$$

This yields

$$\sup_{1\le r\le\frac{r_0}{\mu_n}}(1+r)^{\frac{1}{2}}\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\ \lesssim\ r_0^2\left(1+\frac{\mu_n^{s_c-1}}{r_0^{2-\frac{2}{p-1}}}\right)$$

which together with (*6.D.3*) implies

$$\sup_{0\le r\le\frac{r_0}{\mu_n}}(1+r)^{\frac{1}{2}}\left|\kappa_n-\frac{p-1}{2}\Lambda Q\right|\ \lesssim\ r_0^2\left(1+\frac{\mu_n^{s_c-1}}{r_0^{2-\frac{2}{p-1}}}\right).$$

Hence, we have for $n\ge N$ large enough

$$\sup_{0\le r\le r_0}\left(1+\frac{r}{\mu_n}\right)^{\frac{1}{2}}\left|\varphi_{n,0}(r)-\frac{p-1}{2}\Lambda Q\left(\frac{r}{\mu_n}\right)\right|\ \lesssim\ r_0^2.$$

step 2 Proof of (*6.3.18*). Recall from Lemma 6.3.3 that we have for $n\ge N$ large enough

$$\sup_{0\le r\le r_0}\left(1+\frac{r}{\mu_n}\right)^{\frac{1}{2}}\left|\varphi_{n,0}(r)-\frac{p-1}{2}\Lambda Q\left(\frac{r}{\mu_n}\right)\right|\ \lesssim\ r_0^2.$$

Also, recall that

$$\Lambda Q(r)\sim\frac{c_7\sin(\omega\log(r)+c_8)}{r^{\frac{1}{2}}}\text{ as }r\to+\infty$$

and that $r_{\Lambda Q,n}<r_0/\mu_n$ introduced in Corollary 6.2.6 denotes the last zero of ΛQ before r_0/μ_n. This yields

$$\Big|\omega\log(r_{1,n})-\omega\log(\mu_n)+c_8-(\omega\log(r_{\Lambda Q,n})+c_8)\Big|\lesssim r_0^2$$

and hence

$$r_{1,n}\ =\ \mu_n r_{\Lambda Q,n}e^{O(r_0^2)}=\mu_n r_{\Lambda Q,n}(1+O(r_0^2)).$$

Furthermore, since we have from the proof of Corollary 6.2.6 that

$$e^{-\frac{3\pi}{2\omega}}\frac{r_0}{\mu_n}\le r_{\Lambda Q,n}\le\frac{r_0}{\mu_n},$$

and

$$r_{0,n}\ =\ \mu_n r_{\Lambda Q,n}(1+O(r_0^2)),$$

we deduce

$$r_{1,n}=r_{0,n}+O(r_0^3)$$

and

$$e^{-\frac{2\pi}{\omega}}r_0\le r_{1,n}\le r_0.$$

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