Mémoire présenté par

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On singularity formation, long time behaviour and soliton dynamics for some evolution partial differential equations

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Abstract

This habilitation thesis presents a selection of my works which study phenomena of concentration, propagation, stability, and emergence of coherent structures, for solutions to some evolution partial differential equations. The results prove asymptotic expansions for large times, or close to the blow-up time for singular solutions. For most of these works these expansions decompose the solution as the sum of particular solutions, stationary states, backward self-similar solutions, and traveling waves, which are sometimes conveniently termed under the generic name of "solitons", despite the fact that this word has a precise (and sometimes different) meaning in physics. For a few works on weak wave turbulence these results prove asymptotic expansions that show the emergence of kinetic wave equations as effective equations to describe the statistics of solutions.

There are three main parts in this thesis. The first part is concerned with singularity formation in some equations with advection, diffusion, or reaction effects. Various blow-up patterns will be presented via a unified approach based on the type of self-similarity and solitons that are involved, for the semilinear heat equation, the parabolic-elliptic Keller-Segel system, and the two-dimensional Prandtl system and related systems. The second part studies the dynamics around solitons in some dispersive equations. After discussing the asymptotic stability of traveling waves and stationary states for the nonlinear Schrödinger equation and the Hartree equation, the soliton resolution for the energy critical wave equation will be described. The third part deals with weak wave turbulence. Results on the derivation of kinetic wave equations in certain weakly turbulent regimes of nonlinear Schrödinger equations will be presented, and we will conclude by the stability of a Kolmogorov-Zakharov spectrum for the steady four-waves kinetic equation.

Résumé

Formation de singularités, comportement en temps long et dynamiques près de solitons pour certaines équations aux dérivées partielles d'évolution

Cette thèse d'habilitation présente une sélection de mes travaux portant sur l'étude des phénomènes de concentration, de propagation, de stabilité et d'émergence de structures cohérentes, pour des solutions de certaines équations aux dérivées partielles d'évolution. Les résultats démontrent des développements asymptotiques en temps long, ou près du temps d'explosion pour des solutions singulières. Dans la plupart de ces travaux, ces développements décomposent la solution en une somme de solutions particulières, états stationnaires, solutions auto-similaires rétrogrades et ondes progressives, qui sont parfois appelées par le terme de solitons. Pour quelques travaux sur la turbulence faible d'ondes, ces résultats démontrent des développements asymptotiques qui mettent en évidence l'émergence d'équations d'ondes cinétiques pour décrire la statistique des solutions.

La première partie traite de la formation de singularités dans des équations avec effets d'advection, de diffusion ou de réaction. Divers mécanismes d'explosion seront présentés, selon le type d'autosimilarité et de solitons sous-jacents, pour l'équation de la chaleur semi-linéaire, le système de Keller-Segel, le système de Prandtl en deux dimensions et des systèmes apparentés. La deuxième partie étudie la dynamique autour de solitons pour des équations dispersives. Après avoir examiné la stabilité asymptotique des ondes progressives et des états stationnaires pour l'équation de Schrödinger non linéaire et l'équation de Hartree, la résolution en solitons de l'équation des ondes critique pour l'énergie sera décrite. La troisième partie portera sur la turbulence faible d'ondes, avec la dérivation d'équations d'ondes cinétiques dans certains régimes faiblement non linéaires d'équations de Schrödinger, et la stabilité d'un spectre pour l'équation cinétique à quatre ondes stationnaire.

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List of publications

Research articles and preprints presented in this habilitation thesis

The following papers are the main papers that are discussed in this habilitation thesis. The papers are ranked in the reverse chronological order taking into account the first release on the arXiv website. All the papers except one are available on arXiv. The last three are part of my PhD thesis, but are discussed in the present manuscript for coherency purpose.

— [96], with C. Prange and J. Tan, Stable self-similar singularity formation for infinite energy solutions of the incompressible porous medium equations, *submitted* (2025).

— [83], with P. Germain and E. Pacherie, Absence of embedded spectrum for nonlinear Schrödinger equations linearized around one dimensional ground states, *to appear in Proceedings of the American Mathematical Society* (2025).

— [84], with P. Germain and E. Pacherie, Estimates for the Gross-Pitaevskii equation linearized around a vortex, *submitted* (2025).

— [91], with T.-E. Ghoul, N. Masmoudi and V. T. Nguyen, Singularity formed by the collision of two collapsing solitons in interaction for the 2D Keller-Segel system, *submitted* (2024).

— [98], with K. Zhang, On the stability of Type I self-similar blowups for the Keller-Segel system in three dimensions and higher, *submitted* (2024).

— [80], with P. Germain, Asymptotic Stability of Solitary Waves for One Dimensional Nonlinear Schrödinger Equations, *Journal of the European Mathematical Society* (2025).

— [76], with C. Kenig, T. Duyckaerts and F. Merle, On classification of non-radiative solutions for various energy-critical wave equations, *Advances in Mathematics* (2023).

— [75], with C. Kenig, T. Duyckaerts and F. Merle, On channels of energy for the radial linearised energy critical wave equation in the degenerate case, *International Mathematics Research Notices* (2022).

— [74], with H. Dietert and P. Germain, Stability and cascades for the Kolmogorov-Zakharov spectrum of wave turbulence, *Archive for Rational Mechanics and Analysis* (2024).

— [77], with C. Kenig, T. Duyckaerts and F. Merle, Soliton resolution for the radial quadratic wave equation in six space dimensions, *Vietnam Journal of Mathematics* (2024).

— [90], with T.-E. Ghoul, N. Masmoudi and V. T. Nguyen, Collapsing-ring blowup solutions for the Keller-Segel system in three dimensions and higher, *Journal of Functional Analysis* (2023).

— [92], with S. Ibrahim and Q. Lin, Stable Singularity Formation for the Inviscid Primitive Equations, *Annales de l'Institut Henri Poincaré. Analyse Non Linéaire* (2023).

— [4], with I. Ampatzoglou and P. Germain, Derivation of the kinetic wave equation for quadratic dispersive problems in the inhomogeneous setting, *American Journal of Mathematics* (2025).

— [81], with P. Germain, Derivation of the homogeneous kinetic wave equation: longer time scales, *Journal of Functional Analysis* (2025).

— [73], with A.-S. de Suzonni, Stability of Steady States for Hartree and Schrodinger Equations for Infinitely Many Particles, *Annales Henri Lebesgue* (2022).

— [82], with P. Germain, On the derivation of the homogeneous kinetic wave equation, *Communications on Pure and Applied Mathematics* (2025).

— [89], with T.-E. Ghoul, N. Masmoudi and V.-T. Nguyen, Spectral analysis for singularity formation of the two dimensional Keller-Segel system, *Annals of PDEs* (2022).

— [88], with T.-E. Ghoul, N. Masmoudi and V.-T. Nguyen, Refined description and stability for singular solutions of the 2D Keller-Segel system, *Communications on pure and applied mathematics* (2022).

— [86], with T.-E. Ghoul and N. Masmoudi, Singularities and unsteady separation for the inviscid two-dimensional Prandtl system, *Archive for Rational Mechanics and Analysis* (2021).

— [85], with T.-E. Ghoul, S. Ibrahim and N. Masmoudi, On singularity formation for the two dimensional unsteady Prandtl system around the axis, *Journal of the European Mathematical Society* (2022).

— [87], with T.-E. Ghoul and N. Masmoudi, Singularity formation for Burgers equation with transversal viscosity, *Annales Scientifiques de l'école Normale Supérieure* (2022).

— [95], with F. Merle and P. Raphaël, Strongly anisotropic type II blow up at an isolated point, *Journal of the American Mathematical Society* (2020).

— [97], with P. Raphaël and J. Szeftel, On the stability of type I blow up for the energy super critical heat equation, *Memoirs of the American Mathematical Society* (2019).

— [94], with F. Merle and P. Raphaël, Stability of ODE blow-up for the energy critical semilinear heat equation, *Comptes Rendus Mathematique* (2017).

— [69], Nonradial type II blow up for the energy supercritical semilinear heat equation, *Analysis & PDE* (2017).

Other research articles and preprints

The following papers are not presented in this habilitation thesis. They are ranked in the reverse chronological order taking into account the first release on arXiv. All the papers except one are available on arXiv.

— [78], with T. Duyckaerts, C. Kenig and F. Merle, Remark on the energy channel property for the radial linear wave equation, *Comptes Rendus Mathematique* (2025).

— [71], with E. Danesi, A.-S. de Suzzoni and C. Maleze, Stability of homogeneous equilibria of the Hartree-Fock equation, for its equivalent formulation for random fields, *Probability and Mathematical Physics* (2025).

— [72], with A.-S. de Suzzoni, Stability of equilibria for a Hartree equation for random fields, *Journal de Mathématiques Pures et Appliquées* (2020).

— [93], with F. Merle and P. Raphaël, Dynamics near the ground state for the energy critical non-linear heat equation in large dimensions, *Communications in Mathematical Physics* (2017).

— [70], Type II blow up manifolds for a supercritical semi-linear wave equation, *Memoirs of the American Mathematical Society* (2018).

Contents

Notation

Norms and inequalities

The usual norm on $\mathbb{R}^d$ is the Euclidean one,

$$|x| = \sqrt{\sum_1^d x_i^2},$$

and the japanese bracket is

$$\langle x \rangle = \sqrt{1+|x|^2}.$$

The unit sphere is

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d,\ |x| = 1\}$$

and its volume is denoted by

$$|\mathbb{S}^{d-1}|.$$

For inequalities we will write

$$A \lesssim B$$

if there exists a constant C that is independent of the parameters at stake such that $A \leq CB$. The notation

$$A \approx B$$

means that simultaneously $A \lesssim B$ and $B \lesssim A$. We will write

$$A \ll B$$

in a particular limit if $A = o(B)$ in this limit.

Differential calculus

Partial derivatives of a function u with respect to a variable x will be denoted by

$$u_x \qquad \text{or} \qquad \partial_x u.$$

The nabla operator for a function over $\mathbb{R}^d$ is

$$\nabla u = (\partial_{x_1} u, ..., \partial_{x_d} u)$$

and for differentiation restricted only to a subvariable $y \in \mathbb{R}^k \subseteq \mathbb{R}^d$ we will write

$$\nabla_y u = (\partial_{y_1} u, ..., \partial_{y_k} u).$$

The notation $\nabla^m u$ will then stand for the vector consisting of all m-th order partial derivatives of the function u.

The standard Laplace operator in dimension d is

$$\Delta = \sum_1^d \partial_{x_i}^2$$

Function spaces

The weight for weighted Lebesgue spaces is indicated between parentheses, such as

$$L^2(w(y)dy) = \{u, \int u^2 w(y)dy < \infty\}$$

and similarly for weighted Sobolev spaces, such as

$$H^2(e^{-|y|^2/4}dy) = \{u, \sum_0^1 \int |\nabla^i u|^2 w(y)dy < \infty\}$$

To shorten notation, we introduce the following weighted space

$$L^{2,s} = \{u, \int_{\mathbb{R}^d} u^2 \langle y\rangle^{2s} dy < \infty\}$$

with its corresponding norm $\|\cdot\|_{L^{2,s}}$.

Operators

The usual Fourier and inverse Fourier transforms are denoted by

$$\mathscr{F} \qquad \text{and} \qquad \mathscr{F}^{-1}$$

with the shorthand notation

$$\hat{u}(\xi) = \mathscr{F}(u)(\xi)$$

for a given function u.

The commutator between two operators is

$$[\cdot,\cdot]$$

The convolution product on $\mathbb{R}^d$ is

$$*.$$

The expectation on a probability space is

$$\mathbb{E}.$$

Chapter 1

Introduction: the dynamics of some evolution partial differential equations

This introduction is aimed at presenting the works discussed in the present document without technical considerations and references in order to keep the discussion short. Technicalities and bibliographical details will be given in the forthcoming specific chapters.

1.1 Universal asymptotic phenomena in nonlinear evolution partial differential equations

Evolution partial differential equations (PDEs) are used as idealized models in various areas of natural sciences, and also arise in connexion with other branches of mathematical sciences. For all the PDEs that are addressed in the present document, the local well-posedness of the Cauchy problem, that is, the understanding of the conditions under which these equations do have solutions, is by now well-understood, as a result of intense works throughout the second part of the last century and the first decade of the current one.

These evolution PDEs arise as models to describe various phenomena. The common question that underlines all the works I did with my collaborators and which are presented here is the following: can one give a rigorous, proof-based, description of the properties of the solutions that gives a mathematical justification of these phenomena? On the one hand PDEs are by definition local, as they determine the way their solutions vary locally. On the other hand phenomena are described by the global or asymptotic properties of the solutions. Hence by finding a mathematical justification of the phenomena below, one aims at understanding in the most universal way how infinitesimal information given by PDEs structure their global properties.

The phenomena that are studied here are the following:

— *Small scale creation*. The first phenomenon that is addressed in this document is that of small scale formation. How can a PDE make a solution concentrate to smaller and smaller scales? When this phenomenon takes place for global-in-time solutions one speaks of infinite time blow-up. More drastic is the case in which the solution concentrates to an infinitesimally small scale in finite time, in which case the solution becomes singular and ceases to exist (at least in a smooth sense). This will be the content of Chapter 3.

— *Stationarity and nonlinear propagation*. The most common linear effects (diffusion, dispersion etc.) usually tend to make the solution spread spatially. With further nonlinear effects, the solution may remain spatially concentrated, while moving. The appearance of nonlinear motion is a key phenomenon in wave equations. It is intimately related to the existence of traveling wave solutions, or solitons, for the PDE at stake. It can also be the case that nonlinear effects counterbalance linear ones to produce stationary solutions, such as equilibria and vortices. These two instances will be discussed in the first two sections of Chapter 4.

— *Pattern formation*. The formation of coherent structures remains a phenomenon that is very poorly understood at a mathematical level. The precise mathematical understanding of the three phenomena above (concentration, stationarity, propagation) will involve here specific functions whose simplest form is $f[\mu]((x-x^*)/\lambda)$ where x^* is the spatial position, λ is the spatial scale, and μ denotes possible additional parameters. The function $f[\mu]$ is a particular function that produces the phenomena at stake: a self-similar solution, a stationary state, a traveling wave, etc. which we will call by the generic name of soliton ; they have a particle-like behaviour for the equation. Such a function $f[\mu]$ is called a profile and represents a coherent structure for the equation; understanding their appearance is a challenge. In addition, a solution u can be a superposition of several of these solitons $u \approx \sum_1^N f_i[\mu_i]((x-x_i^*)/\lambda_i)$, and understanding patterns involve understanding the form of these multisoliton solutions. The streamline of Chapter 3 will be to present singularity formation for various PDEs with the universality of pattern formation as a common feature. The last section of Chapter 4 will discuss the soliton resolution, which is the proof of the appearance of solitons from general solutions.

— *Stability*. Once the existence of possible phenomena is proved (e.g. motion through space), by proving the existence of suitable solutions with specific properties (e.g. to leading order the solution is a traveling wave with fixed speed) that give precise mathematical meaning and description of the phenomena, it is of crucial importance to determine which phenomena are stable. A stable pattern will provide a physically relevant exemple since it should be easily observable. Even more important is to determine which pattern is generic, i.e. that appears for most solutions. Instable phenomena are nonetheless relevant; first they can describe transitions between stable patterns (in particular, co-dimensional one stability describes threshold solutions), and second an appealing roadmap for proving generic behaviour is to combine a result showing the appearance of a collection of patterns with for each one a result on its stability. Stability analysis will be involved in all the results discussed in this document.

— *Weak turbulence*. Weak wave turbulence aims at describing the statistics of solutions to dispersive equations in weakly nonlinear regimes. Universal phenomena appear, in the way linear waves interact, and in the patterns these interactions form. These phenomenas cannot be directly described by solitons as described above, which are strongly nonlinear solutions. However, in weak wave turbulence, kinetic equations arise to describe the leading way small nonlinear effects add up over time. An appealing roadmap, which has not been put fully on rigorous mathematical grounds so far, is to combine results showing the emergence of kinetic equations as effective equations to describe the statistics of weakly turbulent nonlinear waves with results showing the large time emergence of solitons for the associated kinetic equation. This will be the content of Chapter 5.

1.2 Classes of PDEs studied in this document

Numerous equations will be studied here. This is at the image of the field of PDEs itself, for which in order to describe mathematically a given phenomenon, certain classes of equations are investigated before others due to differences in the available techniques. The parallel investigation of various classes of equations leads to the emergence of methods that can be spread from one class to another.

Parabolic equations

These equations involve a diffusion term, and were investigated first due to parabolic regularizing effects and availability of powerful techniques such as comparison principle and Sturm-Liouville-type analysis. As one of the simplest parabolic equation modeling a highly idealized one-dimensional fluid, the dynamics of the viscous Burgers equation

$$u_t + uu_x = u_{xx} \tag{1.1}$$

is by now very well understood. It will be presented in Section 2.1, where the description of its asymptotic dynamics, in particular the general appearance of a single forward self-similar solution or of a single traveling wave, will serve as an exemple to motivate the next chapters.

The most famous semilinear parabolic equation is the focusing nonlinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1}u$$

where $p > 1$, as the simplest equation where diffusion competes with local nonlinear effects (serving as an idealized model for combustion, instabilities in viscous fluids etc.), and it will be introduced with more details in Section 3.1.2. While the understanding of its dynamics for energy subcritical nonlinearities (in low dimensions $d = 1$ and 2 and higher dimensions provided $p < 1 + 4/(d-2)$) and for general nonlinearities in the radial case is very advanced, there are still numerous open problems regarding the energy critical and supercritical dynamics, especially in the non-radial case. Blow-up by the concentration of exact backward self-similar solutions will be discussed in Section 3.1.2, the so called ODE blow-up (named after its similarities with the Ordinary Differential Equation $\partial_t u = |u|^{p-1}u$) will be discussed in Section 3.2.1, slower so-called type II blow-ups in Section 3.3.2 and anisotropic blow-up in Section 3.4.2.

A more delicate parabolic model is the parabolic-elliptic Keller-Segel system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \end{cases}$$

which arise in describing motion by chemotaxis in biology, and in stellar dynamics. An introduction will be given in Section 3.1.1. Its study is harder since nonlinear effects are nonlocal, which makes it similar to certain models in fluid mechanics, and since as a system it shares similarities with other wave equations since they too can be formulated as systems. Singularities formed by backward self-similar solutions will be discussed in Section 3.1.1, slower type II blow-up in Section 3.3.1 and the non-radial collision of collapsing solitons will be the content of Section 3.4.3.

Fluid mechanics

Small scales formation in fluids, at the mathematical level, remains only understood in particular instances, and no general enough theory is available. The problem of the formation of singularities for smooth solutions to the three dimensional Euler and Navier-Stokes equations remains a well-identified open problem. Understanding singularities of other fluid models is important both as a relevant description of small scale creation for these fluids, and in connexion with this problem. The inviscid Burgers equation

$$u_t + u u_x = 0,$$

presented in Section 2.1, will serve as the easiest toy model to present singularity formation and long time dynamics, with the emergence of backward or forward self-similar solutions respectively. Adding streamwise viscosity leads to the viscous Burgers equation (1.1). More subtle to understand is the effect of a transverse viscosity, and the corresponding equation

$$\partial_t u + u \partial_x u - \partial_{yy} u = 0$$

set on the plane $(x, y) \in \mathbb{R}^2$ and its singular solutions will be discussed in Section 3.4.1. Adding further a horizontal boundary with Dirichlet boundary conditions, and a normal transport making the flow incompressible leads to the two-dimensional Prandtl system

$$\begin{cases} u_t - u_{yy} + u u_x + v u_y = -p^E_x, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \qquad u|_{y \to \infty} = u^E, \end{cases}$$

that serves as an effective equation for describing boundary layers arising in the inviscid limit (large Reynolds number) of the Navier-Stokes system towards the Euler equations, the functions p^E and u^E above being the traces of the outer Eulerian flow. The system will be presented in Section 3.2.2, which will present the key phenomenon of boundary layer separation and how it is connected to singularity formation, after what rigorous results will be discussed. The inviscid Prandtl system, where the u_{yy} term above is removed, yields to a model for which much more can be described, and this will be done in Section 3.1.3.

This latter model shares similarities with the two dimensional inviscid primitive equations

$$\begin{cases} u_t + u\,u_X + wu_Z + p_X = 0, \\ p_Z = 0, \\ u_X + w_Z = 0, \end{cases}$$

describing ocean motion, and with the incompressible porous medium equation

$$\begin{cases} \partial_\tau \rho + u \cdot \nabla \rho = 0, \\ u + \nabla P = (0, \rho), \\ \operatorname{div} u = 0, \end{cases}$$

describing a flow in a porous medium with a gravity effect. More background on these equations will be given in Sections 3.2.3 and 3.1.4 respectively, and singularity formation for a reduced model describing solutions to these equations with symmetries will be described.

Dispersive equations

Some PDEs have in common that certain terms of the equation have the effect of "dispersing" the solution, typically by having the solution resembling locally the sum of wave packets moving with different group velocities. Such effect is however very different and far less stabilizing than the diffusion in parabolic equations, since the amplitude of the wave packets is not damped, and it enables richer dynamics. Perhaps the most famous nonlinear dispersive equation is the semilinear wave equation

$$\partial_t^2 u = \Delta u + |u|^{p-1} u,$$

for $p > 1$. While there is a large literature on this equation, which will be mentioned in the associated sections, we will focus here on the soliton resolution in Section 4.2.2 and the description of energy radiation in Section 4.2.3.

The nonlinear Schrödinger equation, that appears in many different physical contexts such as water surface motion, nonlinear optics, quantum dynamics etc.,

$$\partial_t u - i \Delta u - F(u) = 0,$$

with various nonlinearities, and variants corresponding to other quadratic dispersion relation, is to our taste one of the most complicated model for asymptotic dynamics. One-dimensional traveling waves will be discussed in Section 4.1.1, vortices of the Gross-Pitaevskii Equation in Section 4.1.2, homogeneous equilibria of the Hartree equation (where the above is understood in the sense of a system) in Section 4.1.3, and the kinetic limit of weak turbulence is addressed in Sections 5.1 and 5.2.

Kinetic equations

All the aforementioned equations are macroscopic equations in that the quantities only depend on the spatial variable x. Kinetic equations are mesoscopic equations with a further velocity variable

v, which appears in integral terms representing collisions. A unique kinetic equation will be studied in this document. As mentioned above, the four-wave kinetic equation

$$\partial_t f(t,\omega) = \mathscr{C}(f)(t,\omega),$$

for $\omega > 0$, where the collision operator is

$$\mathscr{C}(f)(\omega_1) = \iint_{\omega_2,\omega_3,\omega_4 \geq 0} W[(f(\omega_1)+f(\omega_2))f(\omega_3)f(\omega_4) - (f(\omega_3)+f(\omega_4))f(\omega_1)f(\omega_2)]\, d\omega_3\, d\omega_4,$$

where above $\omega_2 = \omega_3 + \omega_4 - \omega_1$ and $W = min(\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4})/\sqrt{\omega_1}$, arise as an effective equation for weakly turbulent solutions to the cubic nonlinear Schrödinger equation, as presented in Section 5.1. Kolmogorov-Zakharov spectra, as well as mass and energy cascades, will conclude the document in Section 5.3.

1.3 Main open problems

The following open problems are the ones that drove all the works of the present document:

— *Construct and describe the largest possible classes of blow-up patterns*. There is by now a vast literature dealing with the existence of particular singular solutions. Not so many patterns seem however to be possible, Chapter 3 describing most of the known ones, even for other equations than those discussed in this document. Discovering new blow-up patterns is interesting in itself; it may also give insights for studying singularity formation for other equations, and one needs to discover all possible patterns before trying to classify blow-up solutions for a given equation.

— *Determine stable and generic blow-up patterns*. For applications in natural sciences, these are the most important patterns. Known stable blow-up patterns are presented for all equations (when available) in Chapter 3.

— *Determine for an instable blow-up pattern the asymptotics produced by instabilities*. The final goal is to show that, once the instabilities have grown enough, they drive the solution towards another pattern. This identifies these patterns as attractors and this is similar to identifying heteroclinic orbits. The instability of certain patterns presented in Chapter 3 is presented. The fate of solutions once instabilities become large remains however mostly open for PDEs, as this cannot be reduced to a perturbative problem anymore.

— *Classify blow-up dynamics*. For a few equations with subcritical nonlinearities or radial symmetry, blow-up patterns are classified, see the discussion in Section 4.2.1. Proving that general singular solutions have only a few specific behaviours remains open for most equations. This is only obtained in the present document for singular solutions of the Burgers equation in Section 2.1 and for generic singular solutions of the inviscid two-dimensional Prandtl system in Section 3.1.3.

— *Prove soliton resolution*. A precise statement of this open problem is given in Section 4.2.1. This amounts to showing that general solutions, both in finite and infinite time, decompose asymptotically as the sum of particular solutions (without describing exactly where, at which scale, and other very fine properties of these solutions, which should be thought of a second more refined problem to investigate). This is the topic of Section 4.2.

— *Determine stable traveling waves and stationary states for dispersive equations*. Some problems in low dimensions, with spectral obstructions, or with nonzero asymptotics at spatial infinity remain open. This is discussed in particular for the one-dimensional Schrödinger equation in Section 4.1.1, the two-dimensional Gross-Pitaevskii equation in Section 4.1.2, and the Hartree equation in Section 4.1.3. Such solutions are of particular importance in physics since they should be the building blocks of the soliton resolution for generic solutions.

— *Justify kinetic limits in weak wave turbulence*. This amounts to show that the statistics of turbulent solutions to some wave equations can be described in suitable high frequency and weakly nonlinear limits by certain kinetic wave equations. This is here the content of Sections 5.1 and 5.2.

— *Show the appearance of spectra of turbulence*. The open problem is to determine the asymptotic behaviour of the aforementioned kinetic wave equations in out-of-equilibrium, and to show that self-similar solutions emerge. This corresponds to showing soliton resolution for forced kinetic equations.

1.4 Outline

This thesis is organized as follows. Chapter 2 is an introductory chapter presenting the asymptotic behaviour for simple evolution PDEs. Burgers' equations are discussed in Section 2.1, firstly the inviscid Burgers equation in Section 2.1.1 and secondly the viscous Burgers equation in Subsection 2.1.2. Three important linear equations are then presented in Section 2.2: linear transport in Subsection 2.2.1, the linear heat equation in Subsection 2.2.2 and the linear wave equation in Subsection 2.2.3.

The main part starts with Chapter 3 that describes various patterns of singularity formation for equations with advection, diffusion or reaction. Dynamics involving exact backward self-similar solutions are presented first in Section 3.1; their appearance for singular solutions of the Keller-Segel system will be discussed in Subsection 3.1.1, for the semilinear heat equation in Subsection 3.1.2, for the inviscid Prandtl system in Subsection 3.1.3 and for the incompressible porous medium equation in Subsection 3.1.4. Exact backward self-similar solutions can also emerge in a more perturbative way in which certain effects of the equation are asymptotically negligible, which will be the content of Section 3.2, this occurs for the so-called ODE blow-up for the semilinear heat equation in Subsection 3.2.1, the two dimensional Prandtl system in Subsection 3.2.2 and the primitive equations in Subsection 3.2.3. Type II blow-up, where a self-similar behaviour occurs that is not described by exact backward self-similar solutions, is then presented in Section 3.3, with several possible dynamics for the Keller-Segel system in Subsection 3.3.1 and the semilinear heat equation in Subsection 3.3.2. The chapter ends by describing non-radial dynamics in Section 3.4, anisotropic concentration of backward self-similar solutions for the Burgers equation with transverse viscosity in Subsection 3.4.1 or of stationary states for the semilinear heat equation in Subsection 3.4.2, and the non-radial collision of solitons at a single point for the critical Keller-Segel system in Subsection 3.4.3.

Then, Chapter 4 gathers result dealing with the asymptotic occurence and stability of solitons for certain dispersive equations. Some asymptotic stability problems are discussed first in Section 4.1, that of one-dimensional traveling waves for the nonlinear Schrödinger equation in Subsection 4.1.1, that of vortices for the Gross-Pitaevskii equation in Subsection 4.1.2, and that of homogeneous equilibria of the Hartree equation in Subsection 4.1.3. Second, Section 4.2 focuses on the soliton resolution problem. A precise formulation of the soliton resolution conjecture is intended in Subsection 4.2.1, its proof for the energy critical wave equation is the topic of Subsection 4.2.2, and some key results on the energy radiation at the heart of its proof are discussed in Section 4.2.3.

The last part, Chapter 5, adresses weak wave turbulence. The problem of the derivation of kinetic wave equations will be presented first in Section 5.1 for the cubic Schrödinger equation on the usual tori, and anisotropic or spatially inhomogeneous set-ups will then be presented in Section 5.2. Mass and energy cascades, as well as Kolmogorov-Zakharov spectra, for the isotropic four-wave kinetic equation form the content of Section 5.3

Chapter 2

Asymptotics for simple models

This chapter discusses simple toy models for which almost everything is by now understood. It is in particular aimed at a non specialized reader or one at an early stage of research. The behaviour for the two nonlinear equations will shed light on what to expect for other evolution nonlinear PDEs; for the linear equations we will see some common properties in the asymptotic behaviours, starting to shed light on universal asymptotic phenomena, and we will present some asymptotics and decay estimates which have key stabilizing properties that one should expect to be found in the linearization of other PDEs around particular solutions.

The first section 2.1 is devoted to the presentation of two model nonlinear PDEs for which everything is understood: the inviscid and viscous Burgers equation. We will see the for both equations, for finite time blow-up and global-in-time dynamics solutions are always attracted by backward self-similar solutions, forward self-similar solutions or traveling waves respectively. This proves that the asymptotic behaviour is always described by self-similar motion of some kind, corresponding to spatial concentration, spatial spreading, or spatial translation respectively. This also proves that nonlinear coherent structures always appear, which are these particular solutions that we will call by the generic name of solitons in this document. This is an instance of soliton resolution for evolution PDEs, and there are only integrable PDEs and a few other non-integrable ones for which this has been proved, see the discussion later in Section 4.2. This is however a "simple" occurence of soliton resolution, as multi-solitons never appear, and because no anisotropy effects can appear since these are one-dimensional equations; the dynamics of other PDEs can be much richer.

The second section 2.2 will discuss linear transport which will be similar to linearized evolutions for the fluid mechanics models presented in Section 1.2, the heat equation which will be similar to that of parabolic molels presented there, and the wave equation which will serve as a first attempt to understand nonlinear wave equations. All solutions will be proved to become self-similar for large times, what will show the ubiquity of self-similar motion. A few key estimates will be presented that served as a basis for our methods for stability analysis for the corresponding nonlinear evolution PDEs.

2.1 Burgers' equations

This section is devoted to the study of the simplest nonlinear transport equations. These are the inviscid Burgers equation

$$\begin{cases} u_t + uu_x = 0, & t > 0 \text{ and } x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases}$$

and the viscous Burgers equation

$$\begin{cases} u_t + uu_x = u_{xx}, & t > 0 \text{ and } x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

They serve as the simplest models for the nonlinear propagation of waves or of fluids, and we refer to [394], [64], for example for detailed discussions and studies.

Their solutions can be computed explicitly thanks to representation formulas: the method of characteristics for smooth solutions to the inviscid Burgers equation, the Lax-Oleinik formula for entropy solutions of the inviscid Burgers equation, and the Cole-Hopf formula for the viscous Burgers equation.

2.1.1 The inviscid Burgers equation

We focus in this subsection on the inviscid Burgers equation

$$\begin{cases} u_t + uu_x = 0, & t > 0 \text{ and } x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Solutions may be found thanks to the method of characteristics. Denoting by X the Lagrangian variables (the initial position of a particle), and by x the Eulerian variables, the characteristic lines are

$$x(t,X) = X + tu_0(X). \tag{2.2}$$

Given $u_0 \in C^1(\mathbb{R})$ with bounded derivative, a smooth solution to (2.1) exists up to the maximal time of existence

$$T = \begin{cases} \infty & \text{if } u_x \geq 0, \\ -\frac{1}{\inf_{\mathbb{R}} u_x} & \text{otherwise.} \end{cases}$$

and is given by the formula

$$u(t,x) = u_0(X(t,x)) \tag{2.3}$$

where $x \mapsto X(t,\cdot) = (X \mapsto x(t,X))^{-1}$ denotes the inverse map of the Lagrangian to Eulerian map (2.2). A straightforward computation then shows that when $T < \infty$, such solutions become singular in the sense that

$$\lim_{t \uparrow T} \|u_x(t,\cdot)\|_{L^\infty} = \infty,$$

modelling wave steepening.

The set of solutions enjoys numerous symmetries. Namely, if u is a solution, then for $(t_0, x_0, c, \lambda, \mu) \in \mathbb{R}^3 \times (0,\infty)^2$ by space and time scaling invariances, time and space translation and Galilean transformation, the function

$$(t,x) \mapsto \frac{\mu}{\lambda} u\left(\frac{t-t_0}{\lambda}, \frac{x-x_0-ct}{\mu}\right) + c$$

is also a solution.

Backward self-similar solutions are those defined on $(-\infty, 0) \times \mathbb{R}$ of the form

$$u(t,x) = (-t)^{\alpha-1}\Psi\left(\frac{x}{(-t)^\alpha}\right) \tag{2.4}$$

for some $\alpha \in \mathbb{R}$. Although they have appeared previously in the literature [152], they were not studied in details, perhaps because the Burgers equation is considered as too simple. We gave a detailed study of their role in singularity formation in [85].

Proposition 2.1 (Classification of smooth self-similar profiles [85])**.** *For all positive integers $i \in \mathbb{N}$, let $\Psi_i = (\mathscr{X} \mapsto -\mathscr{X} - \mathscr{X}^{2i+1})^{-1}$ be the inverse of the function $\mathscr{X} \mapsto -\mathscr{X} - \mathscr{X}^{2i+1}$. Then*

$$u(t,x) = (-t)^{\frac{1}{2i}}\Psi_i\left(\frac{x}{(-t)^{1+\frac{1}{2i}}}\right) \tag{2.5}$$

is a self-similar solutions to the Burgers equation (2.1).

Moreover, they are the only smooth self-similar solutions in the sense that for all nontrivial smooth solutions of the form (2.4) with $\Psi'(\mathscr{X}) \to 0$ as $|X| \to \infty$ we have $\alpha = 1 + 1/2i$ for some integer i and $u(t,x) = (-t)^{\frac{1}{2i}}\mu\Psi_i(x/\mu(-t)^{1+\frac{1}{2i}})$ for some $\mu > 0$.

They then describe the behaviour of classical solutions to the Burgers equation near singularities.

Theorem 2.2 (Self-similar resolution of singularities [85])**.** *Let u_0 be an analytic function with $\partial_x u_0(x) \to 0$ as $|x| \to \infty$ with finite time of existence $T > 0$. Then there exist $J \in \mathbb{N}^*$, J points $x_j^* \in \mathbb{R}$, integers $i_j \in \mathbb{N}^*$, scales $\mu_j > 0$ and velocities $c_j \in \mathbb{R}$ such that:*

(i) For all $j = 1, ..., J$, one has near (T, x_j^*) that

$$u(t,x) = \mu_j(T-t)^{\frac{1}{2i_j}}\Psi_{i_j}\left(\frac{x - x_j(t)}{\mu_j(T-t)^{1+\frac{1}{2i_j}}}\right) + c_j + \tilde{u}_j(t,x),$$

where $x_j(t) = x_j^* - (T-t)c_j$ and where

$$\frac{\tilde{u}_j(t,x)}{(T-t)^{\frac{1}{2i_j}}\Psi_{i_j}\left(\frac{x-x_j(t)}{\mu_j(T-t)^{1+\frac{1}{2i_j}}}\right)} \to 0 \qquad \text{as } (t,x) \to (T, x_j^*).$$

(ii) u *and all its derivatives remain locally uniformly bounded up to time* T *outside any neighbourhood of* $\{x_1, ..., x_J\}$.

Remark 2.3. — *Explanation of the appearance of self-similarity*. Theorem 2.2 asserts that all singularities are formed by concentrating self-similar solutions, and one may wonder what key features of the equation are responsible for this. For such a transport equation, knowing the solution is equivalent to knowing the characteristics, thanks to (2.3). When singularities form, the characteristics themselves actually become self-similar, implying that the solution also becomes self-similar.

— *On the role of regularity*. Solutions with limited regularity may form other patterns during singularity formation. For example, there exist discretely self-similar solutions with limited but arbitrarily high regularity, which are those such that $u(t,x) = \lambda^{\alpha-1}\Psi\left(t/\lambda \frac{x}{\lambda^\alpha}\right)$ for some $\lambda > 1$ and $\alpha \in \mathbb{R}$, but which are not of the form (2.4), see [85].

— *Generic singularity*. One can prove that for a dense and open set of C^3 initial data leading to finite time blow-up, the solution becomes singular at a single point and concentrates the profile Ψ_1.

From the above discussion, given a smooth and localized enough initial data u_0, it must become singular in finite time $T > 0$, near which self-similar solutions provide an asymptotic expansion. After T, it is possible to continue the solution in a weak sense as an entropy solution (for Burgers this is a classical result, see e.g. the textbook [167]). It is given by the Lax-Oleinik formula

$$u(t,x) = \frac{x - Y(t,x)}{t}. \tag{2.6}$$

where $Y(t,x)$ is the point at which the function $y \mapsto u_0(y) + \frac{1}{2t}(x-y)^2$ attains its minimum (which is unique almost everywhere).

Forward self-similar solutions are those defined on $(0,\infty)\times\mathbb{R}$ and of the form $u(t,x) = t^{\alpha-1}\Psi(x/t^\alpha)$ for $\alpha \in \mathbb{R}$. We restrict our study to L^1 initial data for simplicity, and recall that the mass $\int u(t,x)dx = \int u_0 dx$ is a conserved quantity. We are therefore interested in the scaling transformation that preserves the mass, corresponding to $\alpha = 1/2$.

There indeed exists a forward self-similar entropy solution, given for $p, q \geq 0$ by

$$u(t,x) = \frac{1}{\sqrt{t}}\Psi_{p,q}\left(\frac{x}{\sqrt{t}}\right)$$

where

$$\Psi_{p,q}(y) = \begin{cases} y & \text{for } -p \leq y \leq q, \\ 0 & \text{for } y \in (-\infty,-p)\cup(q,\infty). \end{cases} \tag{2.7}$$

This solution is called the N-wave. It is the universal attractor of entropy solutions as $t \to \infty$.

Theorem 2.4 (Convergence to self-similar solution for large times [275], see also [132])**.** *Let* $u_0 \in$

$L^\infty(\mathbb{R})$ with compact support and define

$$p^2 = -2 \min_{y\in\mathbb{R}} \int_{-\infty}^{y} u_0(x)dx \geq 0,$$
$$q^2 = 2 \max_{y\in\mathbb{R}} \int_{y}^{\infty} u_0(x)dx \geq 0.$$

Then

$$u(t,x) = \frac{1}{\sqrt{t}}\Psi_{p,q}\left(\frac{x}{\sqrt{t}}\right) + \tilde{u}(t,x)$$

where, for some $C > 0$,

$$\|\tilde{u}(t,\cdot)\|_{L^1(\mathbb{R})} \leq \frac{C}{\sqrt{t}} \to 0 \qquad \textit{as } t \to \infty.$$

Remark 2.5.

On the role of spatial decay. If one considers initial data with less decay near $\pm\infty$, the attractor may no longer be the above self-similar expander.

2.1.2 The viscous Burgers equation

We now consider the viscous Burgers equation

$$\begin{cases} u_t + uu_x = u_{xx}, & t > 0 \text{ and } x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.8}$$

Contrary to its inviscid counterpart (2.1), all solutions are global in time, and only the large time behaviour matters. Solutions may be found thanks to the Cole-Hopf formula,

$$u = -\frac{2\phi_x}{\phi} \tag{2.9}$$

where ϕ is the solution of the linear heat equation $\phi_t = \phi_{xx}$ with initial datum

$$\phi_0(x) = \exp\left(-\frac{1}{2}\int_0^x u_0(y)dy\right). \tag{2.10}$$

The group of symmetries is smaller than that for the inviscid equation. Namely, if u is a solution, then for $(t_0, x_0, c, \lambda) \in \mathbb{R}^3 \times (0,\infty)$, the function

$$(t,x) \mapsto \frac{1}{\lambda}u\left(\frac{t-t_0}{\lambda^2}, \frac{x-x_0-ct}{\lambda}\right) + c$$

is also a solution. The following forward self-similar solution is well known

Proposition 2.6 (Classification of self-similar solutions, see for example [416]). *For all $M \in \mathbb{R}$, the function*

$$\Psi_M(y) = \frac{c_M e^{-\frac{y^2}{4}}}{1 - \frac{c_M}{2}\int_0^y e^{-\frac{z^2}{4}}\,dz},$$

where $c_M = \frac{2}{\sqrt{\pi}}\tanh(\frac{M}{4})$, satisfies that $\int_{\mathbb{R}} \Psi = M$ and that

$$u(t,x) = \frac{1}{\sqrt{t}}\Psi_M\left(\frac{x}{\sqrt{t}}\right)$$

is a forward self-similar solution to the viscous Burgers equation (2.8). Moreover, for any other solution of this form $u(t,x) = \frac{1}{\sqrt{t}}\Psi\left(\frac{x}{\sqrt{t}}\right)$ for $\Psi \in C^2(\mathbb{R}) \cap L^1(\mathbb{R})$ we have $\Psi = \Psi_M$ for some $M \in \mathbb{R}$.

The above self-similar solution is the attractor of all solutions emanating from localized enough initial data.

Theorem 2.7 (Convergence to self-similar solution for large times, see e.g. [63]). *Let $u_0 \in L^1(\mathbb{R})$ with mass $M = \int_{\mathbb{R}} u_0(y)dy$, and u be the corresponding solution to the viscous Burgers equation (2.8). Then u can be decomposed for large times as*

$$u(t,x) = \frac{1}{\sqrt{t}}\Psi_M(\frac{x}{\sqrt{t}}) + \tilde{u}(t,x)$$

where for all $p \in [1,\infty]$

$$t^{\frac{1}{2}(1-\frac{1}{p})}\|\tilde{u}(t,\cdot)\|_{L^p(\mathbb{R})} \to 0$$

as $t \to \infty$.

We now consider solutions which are less localized, focusing for simplicity on those which converge fast enough to different limits as $x \to \infty$ and $x \to -\infty$. Novel coherent structures appear, which are the traveling waves.

Proposition 2.8. *For any $c \in \mathbb{R}$, the only smooth solution to the viscous Burgers equation (2.8) that is of the form $u(t,x) = W(x - ct)$ is, up to spatial translation, given by*

$$W_{c_-,c_+}(x) = c - \frac{c_- - c_+}{2}\tanh\left(\frac{c_- - c_+}{4}\right)$$

where $c_- = \lim_{x\to-\infty} W(x)$ and $c_+ = \lim_{x\to\infty} W(x)$.

The above traveling wave is the attractor of the dynamics as $t \to \infty$.

Theorem 2.9 (Convergence to traveling wave for large times [351]). *Let $u_0 \in L^\infty(\mathbb{R})$ be such that $\int_{-\infty}^0 |u(x) - c_-|dx < \infty$ and $\int_0^\infty |u(x) - c_+|dx < \infty$ for some $c_-, c_+ \in \mathbb{R}$. Then, there exists $x_0 \in \mathbb{R}$ such that the corresponding solution u of the viscous Burgers equation (2.8) can be decomposed for large times as*

$$u(t,x) = W_{c_-,c_+}(x - x_0 - ct) + \tilde{u}(t,x)$$

with $c = (c_- + c_+)/2$, and where $\|\tilde{u}(t,\cdot)\|_{L^\infty} \to 0$ as $t \to \infty$.

Remark 2.10.

On the role of spatial decay. If one considers initial data with less decay near $\pm\infty$, the attractors may no longer be the above self-similar expander or traveling waves. We refer to [196] for references.

2.2 Linear equations

2.2.1 Linear transport

Linear transport equations are the simplest linear partial differential equations. Understanding their properties is the first step towards understanding nonlinear transport equations that arise in fluid mechanics for example, and certain wave equations in regimes where transport effects occur.

Let us consider the following one dimensional transport equation on $[0,\infty)$,

$$\begin{cases} u_t + T(x)u_x = 0, & t>0 \text{ and } x\in[0,\infty),\\ u(0,x)=u_0(x), & x\in[0,\infty), \end{cases} \tag{2.11}$$

with a smooth transport field $T\in C^\infty([0,\infty))$ that presents a source at 0 and a sink at ∞,

$$T(0)=0, \quad T'(0)=1, \quad \text{and } T>0 \text{ on } (0,\infty).$$

Solutions are found by the method of characteristics, the Lagrangian variables X to Eulerian variables x being defined as the flow of the ODE $\partial_t x(t,X)=T(x(t,X))$ with $x(0,X)=X$, namely

$$u(t,x)=u_0(X(t,x))$$

where $X(t,\cdot)=(X\mapsto x(t,X))^{-1}$ is the Eulerian to Lagrangian map. This formula shows finite speed of propagation, in that if u_0 is supported in $[0,X_0]$ for some $X_0>0$ then $u(t)$ is supported in $[0,x(t,X_0)]$.

In order to obtain an asymptotic expansion for large time, one may look for eigenvalues λ and eigenfunctions φ_λ of the transport operator, solving

$$T(x)\partial_x\varphi_\lambda=\lambda\varphi_\lambda.$$

In $C^\infty([0,\infty))$, the eigenvalues are $\{0,1,2,...\}$ with eigenfunctions $\varphi_0=1$ and $\varphi_\lambda(x)=c_\lambda e^{\lambda\int_1^x\frac{dx'}{T(x')}}$ for $\lambda=1,2,...$ where $c_\lambda\in\mathbb{R}$ is picked to ensure the normalization $\varphi_\lambda(x)\sim x^\lambda$ as $x\to 0$. Given a smooth initial datum $u_0\in C^\infty([0,\infty))$, one then verifies that the solution to the linear transport equation (2.11) admits the following expansion for large times for any $n\in\mathbb{N}$,

$$u(t,x)=\sum_0^n c_\lambda(u_0)e^{-\lambda s}\varphi_\lambda(x)+\tilde u_n(t,x), \tag{2.12}$$

where $c_\lambda(u_0)=\partial_x^\lambda u_0(0)/\lambda!$ and

$$\|\tilde u_n(t,\cdot)\|_{L^\infty([0,R])}=O(e^{-(n+1)s}) \text{ for any } R>0.$$

The expansion (2.12) should be understood as a refined self-similar behaviour for large times. Two reasons explain this asymptotic self-similarity: the first one is the fact that on compact sets of the form $[0, R]$ for any $R > 0$ for large times only the behaviour of the initial datum u_0 near the origin (its Taylor expansion) matters since all particles are sent towards ∞, and the second is the smoothness of the initial datum u_0 (the smoothness of a function being by definition a self-similarity property at small scales) which ensures the quantization in (2.12). It is worth noting that without regularity assumptions on u_0, the expansion (2.12) fails and more complex behaviours as $t \to \infty$ are possible.

Let us now consider one dimensional transport on $[0, 1]$,

$$\begin{cases} u_t + T(x)u_x = 0, & t > 0 \text{ and } x \in [0,1], \\ u(0,x) = u_0(x), & x \in [0,1] \end{cases} \tag{2.13}$$

with smooth transport field $T \in C^\infty([0, \infty))$ that presents a source at 0 and a sink at 1,

$$T(0) = T(1) = 0, \quad T'(0) = 1, \quad T > 0 \text{ on } (0,1), \quad \text{and} \quad T'(1) = -\alpha < 0.$$

The previous discussion still applies to obtain an asymptotic expansion like (2.12) on intervals of the form $[0, R(t)]$ for $R(t) - 1 \gg e^{-\alpha t}$. The difference is now the appearance of a boundary layer of width $e^{-\alpha t}$ near the sink at 1, where the behaviour is described to leading order by the following asymptotic expansion,

$$u(t,x) = f\left(\frac{x-1}{e^{-\alpha t}}\right) + \tilde{u}(t,x) \tag{2.14}$$

where the function f can be determined from the initial datum u_0 and where

$$\|\tilde{u}\|_{L^\infty([1-Re^{-\alpha t},1])} \to 0 \text{ as } t \to \infty \text{ for any } R > 0.$$

The expansion (2.14) should also be understood as a self-similar behaviour for large times, due to the fact that the characteristic lines have a self-similar behaviour near the sink 1 as $t \to \infty$. A transport equation with sources and sinks like (2.13) thus shows two different self-similar asymptotic behaviours: the behaviour (2.12) describes a stabilizing mechanism away from the sink 1, while the behaviour (2.14) describes a small scale creation near the sink 1 which is an infinite time concentration that induces growth for the derivatives $\|\partial_x^k u(t)\|_{L^\infty} \approx e^{k\alpha t}$ (for nontrivial initial data u_0).

Finally, let us mention as a reference the article of Jia-Stewart-Sverak [248] where one dimensional transport is studied in depth, with similar features as in this Subsection.

2.2.2 The linear heat equation

The simplest diffusion equation is the linear heat equation

$$\begin{cases} u_t = \Delta u, & t > 0 \text{ and } x \in \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.15}$$

which serves as a toy model for understanding more complex advection-reaction-diffusion equations. Solutions are given by the convolution of the initial data with the heat kernel

$$u(t) = K(t) * u_0, \qquad K(t,x) = \frac{1}{t^{N/2}} G(\frac{x}{\sqrt{t}}) \text{ with } G(y) = \frac{1}{(4\pi)^{N/2}} e^{-|y|^2/4}.$$

This induces parabolic regularization: all solutions are smooth no matter the initial datum. This also induces the following self-similar asymptotic behaviour:

$$u(t,x) = \frac{M(u_0)}{t^{N/2}} G(\frac{x}{\sqrt{t}}) + \tilde{u}(t,x) \tag{2.16}$$

where $M(u_0) = \int u_0 dx$ is the mass and $\|\tilde{u}\|_{L^\infty} \lesssim t^{-(N+1)/2}$ if u_0 is localized enough. This self-similar behaviour can be refined by introducing the forward self-similar parabolic variables

$$u(t,x) = (1+t)^{-N/2} v(s,y) \quad s = \ln(t+1), \quad y = \frac{x}{\sqrt{t+1}}$$

where v solves the renormalized heat equation

$$\partial_s v = \Delta v + \frac{N}{2} v + \frac{y}{2} \cdot \nabla v.$$

The Fokker-Planck operator $\Delta + \frac{N}{2} + \frac{y}{2} \cdot \nabla$, via the conjugation $v = e^{-|y|^2/4} w$, can be conjugated to the operator

$$\Delta w - \frac{y}{2} \cdot w.$$

The above operator is self-adjoint from $H^2(e^{-|y|^2/4}dy)$ into $L^2(e^{-|y|^2/4}dy)$ with compact resolvent. Its spectrum is $\{0, 1/2, 1, 3/2, ...\}$ and the associated eigenfunctions are Hermite polynomials h_λ. This implies the refined asymptotic expansion

$$u(t,x) = \frac{M(u_0)}{t^{N/2}} G(\frac{x}{\sqrt{t}}) + \sum_1^\Lambda \frac{1}{t^{(N+2\lambda)/2}} h_\lambda(\frac{x}{\sqrt{t}}) G(\frac{x}{\sqrt{t}}) + \tilde{u}_n(t,x) \tag{2.17}$$

where h_λ is a Hermite polynomial solving $\Delta h_\lambda - \frac{y}{2} \cdot h_\lambda = \lambda h_\lambda$ whose coefficients depend on u_0 and where $\|\tilde{u}_n(t)\|_{L^\infty} \lesssim \langle t \rangle^{-(N+2\Lambda+1)/2}$. One notices that the asymptotic expansion (2.17) only holds for localized enough initial data, for example $u_0 \in L^2(e^{|y|^2/4})$, otherwise other asymptotic behaviours are possible. Thus, the possibility of a quantization (2.17) for large times is dictated by the decay of the initial datum near ∞.

In finite time $T > 0$ and near a point x_0, we know the solution is smooth at (T, x_0) by parabolic regularization, and it is possible to obtain a refined asymptotic expansion. We renormalize using the backward self-similar parabolic variables

$$u(t,x) = v'(s,y) \quad s = \ln(T-t) - \ln T, \quad y = \frac{x - x_0}{\sqrt{T-t}},$$

which leads to renormalized heat equation

$$\partial_s v' = \Delta v' - \frac{y}{2} \cdot \nabla v'.$$

This is the same operator as before. However, note that no conjugation was performed, so that $v'(0) = u_0 \in L^2(e^{-|y|^2/4})$ holds for very general initial data, without requiring decay assumption near ∞. This shows the asymptotic expansion

$$u(t,x) = \sum_{1}^{\Lambda} (T-t)^{\lambda} h_{\lambda}\left(\frac{x-x_0}{\sqrt{T-t}}\right) + \tilde{u}'_n(t,x) \tag{2.18}$$

where $\|\tilde{u}'_n(t)\|_{L^\infty(\{|x-x_0|<R\sqrt{T-t}\})} \lesssim (T-t)^{\Lambda+1/2}$ for any $R > 0$. This is again a self-similar behaviour, and we note that the quantization (2.18) in backward parabolic variables holds for all initial datum.

2.2.3 The linear wave equation

We now consider the simplest dispersive equation which is the linear wave equation

$$\begin{cases} u_{tt} = \Delta u, & t > 0 \text{ and } x \in \mathbb{R}^N, \\ (u(0,x), \partial_t u(0,x)) = (u_0(x), u_1(x)), & x \in \mathbb{R}^N. \end{cases} \tag{2.19}$$

The first key property of this equation is finite speed of propagation. The second is that solutions become asymptotically self-similar, in a form which is known as radiation field and was originally due to Friedlander [179]. More precisely, it is proved for example in Theorem 6.2.1 of [236] and Lemma A.3 in [147] that if (u_0, u_1) is smooth and compactly supported, then for $x \neq 0$,

$$\begin{aligned} u(t,x) &= \frac{1}{|x|^{\frac{N-1}{2}}} F\left(\frac{x}{|x|}, |x|-t, \frac{1}{|x|}\right) \\ &= \frac{1}{|x|^{\frac{N-1}{2}}} F_0\left(\frac{x}{|x|}, |x|-t\right) + \tilde{u}(t,x) \end{aligned}$$

where in the first identity $F \in C^\infty(\mathbb{S}^{N-1}, \mathbb{R}, [0,\infty))$ and in the second identity $F_0(\omega,\rho) = F(\omega,\rho,0)$ and $\tilde{u}$ is small in an appropriate sense as $t \to \infty$ depending on the zone under consideration (for example $|\tilde{u}(t,x)| \lesssim t^{\frac{N+1}{2}} \langle |x|-t\rangle^{-\frac{N-3}{2}}$ for $|x| > t/2 > 1$). Analogue identities hold as $t \to -\infty$ by time reversal symmetry.

The above formula shows that the solution concentrates near the light cone $\{|x| = |t| + O(1)\}$ as $t \to \infty$. Since the energy of the solution is a conservation law,

$$\int_{\mathbb{R}^2} |\nabla_{t,x} u(t,x)|^2 dx = \int_{\mathbb{R}^2} |\nabla_{t,x} u(0,x)|^2 dx,$$

this means that the energy concentrates in this region. The function $\nabla_{t,x} u$ enjoys a related self-similar asymptotic behaviour in the energy space that shows how its energy channels to null infinity close to the boundary of the light cone, where it is moreover locally equipartitioned, as is formulated in the following result.

Theorem 2.11 (see Theorem 2.1. in [147])**.** *Assume $N \geq 3$ and let u be a solution to the linear wave equation* (2.19) *with $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^N)$. Then there exists a unique function $G \in L^2(\mathbb{R} \times \mathbb{S}^{N-1})$ such that*

$$\lim_{t\to\infty} \int_0^\infty \int_{\mathbb{S}^{N-1}} \left| |x|^{\frac{N-1}{2}} \partial_t u(t, r\omega) - G(r-t,\omega) \right|^2 d\omega dr = 0$$

and

$$\lim_{t\to\infty} \int_0^\infty \int_{\mathbb{S}^{N-1}} \left| |x|^{\frac{N-1}{2}} \partial_r u(t, r\omega) + G(r-t,\omega) \right|^2 d\omega dr = 0.$$

Moreover,

$$\lim_{t\to\infty} \int_{\mathbb{R}^N} |\nabla_\omega u(t,x)|^2 dx = 0$$

where $\nabla_\omega u = \nabla u - \cdot \frac{\nabla u \cdot x}{|x|^2} x$ denotes the angular part of the gradient.

In addition, the map $(u_0, u_1) \to G$ is a bijective isometry between $\dot{H}^1 \times L^2(\mathbb{R}^N)$ and $L^2(\mathbb{R} \times \mathbb{S}^{N-1})$.

The function G is called the energy radiation field at $t \to +\infty$. A similar energy radiation field at $t \to -\infty$ exists by time reversal symmetry. There exist explicit formulas relating the radiation field at $t \to +\infty$ and the one at $t \to -\infty$, and relating them with the solution, we refer to [282] where such identities and a detailed bibliography can be found.

Measuring via precise estimates how energy can be radiated towards null infinity for nonlinear wave equations and linearized wave equations is crucial in the proof of the results of Sections 4.2.2 and 4.2.3.

Chapter 3

Singularity formation in equations with advection, diffusion, or reaction

In this chapter we will present certain results we obtained regarding singularity formation for equations with terms modelling advection, diffusion, and, or reaction effects. In all equations of this section, terms involving u model reaction effects, terms involving ∇u can be regarded as transport terms, and terms involving $\nabla^2 u$ are diffusion terms. The aim will be to describe how certain solutions become singular in finite time $T < \infty$. While reaction terms amplify the solution, transport terms displace, concentrate or spread it, and diffusion terms damp and spread it, so it is then interesting to investigate equations where all or only some of these terms are present, and to understand their individual and combined roles in singularity formation phenomena. We chose not to discuss singularity formation for other type of equations, such as wave equations for which only adequate references will be mentioned, in order to keep the presentation focused.

3.1 Exact backward self-similar solutions

We start by describing certain evolution PDEs that have singular solutions of the simplest possible form: backward self-similar solutions. These are exact singular solutions that are invariant by a one-dimensional subgroup of the scaling group of the equation. When the scaling group of the equation is one-dimensional, all relevant parameters (the spatial scale, the size of the solution etc.) can be easily determined from dimensional analysis, which is called self-similarity of the first kind. This is the content of Sections 3.1.1, 3.1.2 and 3.1.4 on the Keller-Segel system, the semilinear heat equation and the incompressible porous medium equation. When the scaling group is of greater dimension, one needs to determine what is the relevant one-dimensional subgroup for which there is a backward self-similar solution; the scaling parameters can no longer be determined from dimensional analysis and this is referred to as a type of self-similarity of the second kind. This was already the case for the inviscid Burgers equation in Section 2.1, and is the content of Section 3.1.3 for the inviscid Prandtl system.

3.1.1 Type I blow-up for the mass supercritical Keller-Segel system

The parabolic-elliptic Keller-Segel system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, & \text{in } \mathbb{R}^d, \\ u(t=0) = u_0 \geq 0, \end{cases} \tag{3.1}$$

is a model parabolic equation that arises in describing chemotactic aggregation in biology [252]. Here, $u(x,t)$ represents the cell density, while Φ_u stands for the concentration of chemoattractant.

The reason why we chose to present first this model is because it is (from our point of view) the simplest parabolic equation to possess an explicit and stable self-similar blow-up. The "simplest" is because, under radial symmetry $u(x) = u(|x|)$, the system (3.1) can be simplified, in that the partial mass of the solution

$$m_u(r) = \frac{1}{|\mathbb{S}^{d-1}|} \int_{|x|<r} u(x) dx$$

solves the following semilinear advection-diffusion equation

$$\partial_t m = \partial_r^2 m - \frac{1}{r} \partial_r m + \frac{1}{r^{d-1}} m \partial_r m. \tag{3.2}$$

It is to be noted that outside radial symmetry, no such transformation to a local parabolic equation is available, so that the study of the system (3.1) is much harder.

Self-similar blow up of the first kind refers to the following kind of solutions. The system (3.1) admits a scaling invariance, i.e., if u is a solution of (3.1), then so is

$$u_\lambda(t,x) = \frac{1}{\lambda^2} u\left(\frac{t}{\lambda^2}, \frac{x}{\lambda}\right)$$

for all $\lambda > 0$. Self-similarity of the first kind refers to exact backward self-similar solutions, which are solutions that are invariant by the one-dimensional scaling transformation, of the following form:

$$u(t,x) = \frac{1}{T-t} U\left(\frac{x}{\sqrt{T-t}}\right).$$

One verifies that in dimensions $d \geq 3$ there exists an explicit such backward self-similar solution:

$$U_0(x) = \frac{4(d-2)(2d+|x|^2)}{(2(d-2)+|x|^2)^2}, \tag{3.3}$$

see [229, 33]. Moreover, in dimensions $3 \leq d \leq 9$, there exists in fact a countable family of such self-similar solutions

$$u(t,x) = \frac{1}{T-t} U_n\left(\frac{x}{\sqrt{T-t}}\right) \tag{3.4}$$

for $n \geq 0$ an integer [229, 349] but which are no longer known in explicit form. There also exist additional self-similar solutions in higher dimensions, see [33].

For parabolic models, a notion of blow-up of type I has been introduced, which refers to solution whose size in L^∞ is comparable to that of self-similar solutions

$$\limsup_{t\to T^-} (T-t)\|u(t)\|_{L^\infty(\mathbb{R}^d)} < \infty$$

Note that by an application of the comparison principle, solutions blow up in finite time $T > 0$ if $\|u(t)\|_{L^\infty(\mathbb{R}^N)} \to \infty$ as $t \uparrow T$ and there holds $\|u(t)\|_{L^\infty} \geq (T-t)^{-1}$. Hence, above we always have $\limsup_{t\to T^-} (T-t)\|u(t)\|_{L^\infty(\mathbb{R}^d)} > 0$. Blow-up solutions which are not of type I are said to be of type II. Self-similar solutions are obviously type I blow-ups.

The above type I self-similar solutions are found in dimensions $d \geq 3$, which is intimately related to the fact that these are dimensions for which the system is mass-supercritical. Indeed, the system (3.1) preserves mass, that is, if $u_0 \in L^1$, then for all $t \in [0, T)$,

$$M := \int_{\mathbb{R}^d} u_0(x)dx = \int_{\mathbb{R}^d} u(t,x)dx.$$

The following computation describes how mass is affected by concentration according to the scaling of the equation:

$$\int_{\mathbb{R}^d} u_\lambda(0,x)dx = \lambda^{d-2}\int_{\mathbb{R}^d} u(0,x)dx \underset{\lambda\to 0}{\longrightarrow} \begin{cases} \infty & \text{if } d=1, \\ \int_{\mathbb{R}^d} u(0,x)dx & \text{if } d=2, \\ 0 & \text{if } d\geq 3. \end{cases}$$

This computation explains formally that in one dimension concentration is prohibited by the conservation of mass, that in high dimensions $d \geq 3$ it is compatible with mass conservation, and that the two dimensional case is the borderline case for which the mass is preserved by the scaling transformation. The system is thus termed mass subcritical in one dimension, mass critical in two dimensions, and mass supercritical in higher dimensions.

In general, for subcritical equations with a coercive conserved quantity (such as the Keller-Segel system in one dimension), solutions are expected to be global in time. This is indeed the case for the one dimensional Keller-Segel system [338]. For certain focusing supercritical equations, it is conjectured that self-similar blow-up of the first kind is always possible. Again, this is indeed the case for the Keller-Segel system in dimensions $d \geq 3$. Global existence is a stable phenomenon, and self-similar blow-up of the first kind is a stable phenomenon too (in a sense that we will describe below). Formally, this explains why self-similar blow-up of the first kind cannot happen for most equations in critical dimensions, since the critical dimension defines a threshold between two stable behaviours. This is indeed the case for the two dimensional Keller-Segel system, for which all blow-up solutions are type II [342, 393]. The more complicated phenomenon of type II blow-up will be described later in Section 3.3.1.

Once the existence of exact self-similar blow-up solutions such as (3.3)-(3.4) is established, one may ask wether they are stable or not. It is also interesting to know wether they can emerge from localize initial data, because due to their heavy tails at spatial infinity, functions like (3.3)-(3.4) do not belong to the critical Lebesgue space $L^{d/2}(\mathbb{R}^d)$ which is the Lebesgue space of smallest exponent in which the Cauchy problem is well-posed.

The self-similar blow-up solution U_0 has been shown to be stable by radial Schwartz perturbations with small H^3 norm by Glogić and Schörkhuber [204]. using a semigroup approach. Let N_0 denote the number of nonpositive eigenvalues of the linearized operator around U_0. It was showed numerically that $N_0 = 1$ for $d \geq 3$ in [33], and this was proved rigorously for $d = 3$ in [204].

We obtained in [98] the first stability result for U_0 by general perturbations.

Theorem 3.1 (Radial stability of U_0 [98])**.** *Assume $d = 3$, or $d \geq 4$ and $N_0 = 1$. Then there exist $\epsilon > 0$ such that for any radial $u_0 = U_0 + \tilde{u}_0$ with $\|\tilde{u}_0\|_{L^\infty(\mathbb{R}^d)} \leq \epsilon$, the solution to (3.1) blows up in finite time $0 < T < +\infty$ (depending on u_0) and can be decomposed as*

$$u(t, x) = \frac{1}{T-t} U_0 \left(\frac{x}{\sqrt{T-t}} \right) + \tilde{u}(t, x)$$

where:
(i) Behaviour of the maximum norm*: there holds the asymptotic stability of U_0,*

$$\lim_{t \to T} (T-t) ||\tilde{u}(t)||_{L^\infty} = 0. \tag{3.5}$$

In particular, the blow-up is type I.
(ii) Convergence of the solution: *there exists a function u^* such that $u(t, x) \to u^*(x)$ for any $x \neq 0$ as $t \to T$. For any $p \in [1, \frac{3}{2})$, assuming $u_0 \in L^p(\mathbb{R}^3)$, we have $u^* \in L^p(\mathbb{R}^3)$ and*

$$\lim_{t \to T} ||u(t) - u^*||_{L^p} = 0. \tag{3.6}$$

(iii) Regularity of the blow-up time*: the blow-up time T is a Lipschitz function of u_0 with respect to the L^∞ norm, i.e. there exists $C > 0$ such that*

$$|T(u_0) - T(u_0')| \leq C \|u_0 - u_0'\|_{L^\infty(\mathbb{R}^d)} \tag{3.7}$$

for any two such functions u_0 and u_0'.

Shortly after our work, Li and Zhou were able to prove the stability of U_0 without symmetry assumptions, that is in the full non-radial setting [284].

The other self-similar solutions (3.4) are unstable, due to the existence of instable eigenvalues for the associated linearized operator. Nonetheless, one can still construct a set of initial data for which they are stable, and this set is a Lipschitz deformation of the linear subspace that is orthogonal to these instable directions.

Namely, our second result shows the Lipschitz codimensional stability of the countable family of self-similar profiles (3.4), proving that U_n is the blow-up profile of a class of initial data residing on a manifold of radial initial data of codimension $N_n - 1$. The integer N_n denotes the number of radial instable modes ϕ_j of the linearized operator around U_n with the convention that $\phi_{N_n} = \Lambda U_n$, see [98]. We introduce their localization $\varphi_j = \phi_j \chi(x/R)$ for χ a cut-off function and a large enough R, and introduce

$$V_n = \left\{ v_0 \in L^\infty \text{ radial }, \quad \int_{\mathbb{R}^d} v_0 \varphi_j dx = 0 \quad \text{for } 1 \leq j \leq N_n - 1 \right\}.$$

Theorem 3.2. *(Finite Lipschitz codimensional radial stability of U_n [98]). For all $3 \le d \le 9$ and $n \ge 1$, there exist functions $c_j : V_n \to \mathbb{R}$ for $1 \le j \le N_n - 1$ that are Lipschitz with respect to the L^∞ topology and with $c_j(0) = 0$, such that the following hold true. For any $v_0 \in V_n$ with $\|v_0\|_{L^\infty}$ small enough, the initial data*

$$u_0 = U_n + v_0 + \sum_{j=1}^{N_n-1} c_j(v_0)\varphi_j$$

produces a solution to (3.1) that blows up in finite time $0 < T < +\infty$ (depending on v_0) and can be decomposed as

$$u(t,x) = \frac{1}{T-t} U_n\left(\frac{x}{\sqrt{T-t}}\right) + \tilde{u}(t,x)$$

where:
(i) Behaviour of the L^∞ norm*: there holds the asymptotic stability of the self similar profile*

$$\lim_{t\to T} (T-t)||\tilde{u}(t)||_{L^\infty} = 0. \tag{3.8}$$

In particular, the blow-up is type I.
(ii) Convergence of the solution and regularity of the blow-up time*: the analogues of (3.6)-(3.7) hold true.*

3.1.2 Type I blow-up for the semilinear heat equation

We now consider the most famous semilinear parabolic equation, which is the focusing nonlinear heat equation

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u, & (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ u_{|t=0} = u_0, \end{cases} \tag{3.9}$$

where $p > 1$. It is the simplest reaction-diffusion equation, and serves for example as a simplified model of combustion [367]. It has attracted a large attention since the pioneering work of Fujita [182].

Equation (3.9) admits a scaling invariance: if $u(t,x)$ is a solution, then so is

$$u_\lambda(t,x) = \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.$$

Analogously to the discussion for the Keller-Segel system in Section 3.1.1, self-similar blow-up of the first kind for Equation 3.9 refers to the existence of backward self-similar solutions that are invariant by the above scaling transformation, of the form

$$u(t,x) = \frac{1}{(T-t)^{\frac{1}{p-1}}} U\left(\frac{x}{\sqrt{T-t}}\right).$$

There exists an explicit blow-up solution that is space independent,

$$u(t,x) = \frac{\kappa}{(T-t)^{\frac{1}{p-1}}}, \quad \kappa = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}},$$

which is a backward self-similar solution with constant profile $U(y) = \kappa$. It solves the ordinary differential equation $\partial_t u = |u|^{p-1}u$.

For general solutions, by comparison principle there holds the following blow-up criterion: a solution blows up at time $T > 0$ if $\lim_{t\uparrow T} \|u(t)\|_{L^\infty(\mathbb{R}^d)} = \infty$ and in this case $\|u(t)\|_{L^\infty} \geq \kappa(T-t)^{-1/(p-1)}$. Like for the Keller-Segel system, type I blow-up solutions refer to those whose blow-up rate is comparable to that of self-similar solutions

$$\lim_{t\uparrow T}(T-t)^{\frac{1}{p-1}}\|u(t)\|_{L^\infty(\mathbb{R}^d)} \in (0,\infty),$$

and blow-up solutions which are not of type I are said to be of type II. For a singular solution u, a blow-up point x^* is a point for which there exists a sequence $(t_n, x_n) \to (T, x^*)$ such that $|u(t_n, x_n)| \to \infty$ as $n \to \infty$. Although ODE blow-up is a terminology that is used in the literature, there is no precise definition of what it means. Hence we shall say here that a solution is said to undergo ODE blow-up at x^* if

$$(T-t)^{\frac{1}{p-1}} u(t, x^* + \sqrt{T-t}y) \to \pm\kappa$$

in L^∞_{loc} with respect to the y variable.

Equation (3.9) dissipates the total energy

$$E(u) = \frac{1}{2}\int |\nabla u|^2 - \frac{1}{p+1}\int |u|^{p+1}, \quad \frac{dE}{dt} = -2\int (\partial_t u)^2.$$

Considering the first term in the energy, it is affected the following way by the scaling transformation:

$$\begin{aligned} E(u_\lambda(0,x))dx &= \lambda^{d-2-\frac{4}{p-1}}\left(\frac{1}{2}\int_{\mathbb{R}^d}|\nabla u(0,x)|^2dx - \frac{1}{p+1}\int_{\mathbb{R}^d}|u(0,x)|^{p+1}dx\right) \\ &\xrightarrow[\lambda\to 0]{} \begin{cases} \pm\infty \text{ unless } E(u(0)) = 0, & \text{if } d = 1, 2, \text{ or } p < 1 + \frac{4}{d-2} =: p_c, \\ E(u(0,x)) & \text{if } d \geq 3 \text{ and } p = p_c, \\ 0 & \text{if } d \geq 3 \text{ and } p > p_c. \end{cases} \end{aligned}$$

When $d = 1, 2$ or $d \geq 3$ and $p < p_c$, the equation is termed as energy subcritical, when $d \geq 3$ and $p = p_c$ it is energy critical and supercritical when $p > p_c$, the exponent p_c being called the energy critical exponent.

In the subcritical case $p < p_c$, unlike the discussion in Section 3.1.1 for the Keller-Segel system that has a coercive conserved quantity, here the decreasing energy E does not control the size of the solution. Therefore, the formal argument that solutions of the subcritical equation should be global does not work, as the decrease of the energy is not a priori incompatible with concentration to small scales. However, blow-up in the subcritical case is rigid and the following was obtained in [199, 198, 200, 251, 201]: type II blow-up is impossible and only type I is possible (which is somewhat linked to the above energy and scaling transformation). Moreover $u(t,x) = \kappa(T-t)^{-1/(p-1)}$ is the only one backward self-similar solution, it is additionnally stable and in fact all blow-up solutions converge to this solution $\kappa(T-t)^{-1/(p-1)}$ as $t\uparrow T$ after zooming near the singularity, in the sense that any blow-up solution undergoes ODE blow-up near its blow-up points.

In the energy critical case, $u(t,x)=\kappa(T-t)^{-1/(p-1)}$ is still the only backward self-similar solution [199]. However, there exists type II blow-up solutions, which do not undergo ODE blow-up near their singular point. We will comment more on type II blow-up in Section 3.3.2. If u is a type I blow-up solution, then it undergoes ODE blow-up near its blow-up points. Despite the fact that type II blow-up exists, a natural question was whether type I self-similar remained stable, as in the subcritical case. We could adapt the subcritical proof of [251] and indeed show the stability of type I blow-up.

Theorem 3.3 (Stability of type I blow-up in the energy critical case [94])**.** *The set of initial data giving rise to solutions of the energy critical semilinear heat equation* (3.9) *blowing-up in finite time with type I blow-up is open in $L^\infty(\mathbb{R}^d)$.*

Note that the stability result in [94] is stated for $W^{3,\infty}$ perturbations, but that it also holds true for L^∞ perturbations, since when small enough they become instantaneously $W^{3,\infty}$ perturbations thanks to parabolic regularizing effects.

In the energy subcritical and critical cases, type I blow-up is thus stable and always corresponds to ODE blow-up. However, this does not provide a complete asymptotic description of type I blow-up solutions, as knowing ODE blow-up occurs at x^* only means that $u\sim\kappa(T-t)^{-1/(p-1)}$ in the backward parabola from (T,x^*), but this does not describe how u departs from this behaviour away from x^*. This will involve perturbed self-similarity of the first kind and will be described in Section 3.2.1. This is why we believe the Keller-Segel system (3.1) is a simpler model than the semilinear heat equation as far as the stable self-similar blow-up of the first kind is concerned, see Theorem 3.1.

We know consider energy supercritical exponents $p>p_c$. New backward self-similar solutions appear, whose profile solve the stationary problem in backward self-similar variables:

$$\Delta U-\frac{1}{2}\Lambda U+|U|^{p-1}U=0,\ \ \Lambda U=\frac{2}{p-1}U+y\cdot\nabla U. \tag{3.10}$$

The constant κ solves the above equation. There is also a singular solution

$$U_*=\frac{c_\infty}{|x|^{\frac{2}{p-1}}},\ \ c_\infty=\left(\frac{2}{p-1}\left(d-2-\frac{2}{p-1}\right)\right)^{\frac{1}{p-1}};$$

Finally, for

$$1+\frac{4}{d-2}<p<p_{JL}=\begin{cases}+\infty & \text{for } d\le 10,\\ 1+\frac{4}{d-4-2\sqrt{d-1}} & \text{for } d\ge 11,\end{cases} \tag{3.11}$$

where p_{JL} is the Joseph-Lundgren exponent [249], there exists a quantized sequence of smooth radially symmetric solutions U_n to (3.10) which behave like

$$U_n(r)\sim\frac{c_n}{r^{\frac{2}{p-1}}}\ \text{ as }\ r\to+\infty.$$

These solutions have been constructed using shooting methods and global Lyapounov functionals methods for ordinary differential equations, [404, 37, 38, 276], and no self-similar solution apart from κ exists above the Lepin exponent [329].

Note that all these profiles have infinite energy due to their heavy tails as $r \to \infty$, and one may wonder whether they can model singularity formation for smooth and localized initial data. In addition, one may ask what is their stability property. For the sake of simplicity, we restrict ourselves to

$$d = 3, \quad p > 5, \quad p_{JL} = +\infty.$$

despite the method of the proof being valid for any $p_c < p < p_{JL}$ in any dimension. We obtained first a novel construction of the self-similar profiles, allowing for a precise description, and a precise estimate on the number of instable eigenmodes.

Proposition 3.4 (Existence and asymptotic of excited self similar solutions [97]). *Assume $d \geq 3$ and $p > 5$. For all $n > N$ large enough, there exist a smooth radially symmetric solution U_n to the self similar equation* (3.10) *such that*

$$\Lambda U_n \ \text{ vanishes exactly } n \text{ times on } \ (0, +\infty).$$

Moreover, there exists a small enough constant $r_0 > 0$ independent of n such that:
1. Behavior at infinity*:*

$$\lim_{n\to+\infty} \sup_{r\geq r_0} \left(1 + r^{\frac{2}{p-1}}\right) |U_n(r) - U_*(r)| = 0.$$

2. Behaviour at the origin*: there exists a sequence $\mu_n > 0$ with $\mu_n \to 0$ as $n \to +\infty$ such that*

$$\lim_{n\to+\infty} \sup_{r\leq r_0} \left| U_n(r) - \frac{1}{\mu_n^{\frac{2}{p-1}}} Q\left(\frac{r}{\mu_n}\right) \right| = 0.$$

where Q is the radial stationary state solving $\Delta Q + Q^p = 0$ (unique up to scaling).
3. Number of instable eigenmodes*: the linearized operator around U_n in self-similar variables*

$$-\Delta + \frac{1}{2}\Lambda + pU_n^{p-1}$$

from $H^2(e^{-|y|^2/4})$ to $L^2(e^{-|y|^2/4})$ is essentially self-adjoint with compact resolvent. Moreover, it admits exactly n negative eigenvalues.

This precise construction enabled us to prove the precise codimensional stability of these self-similar profiles.

Theorem 3.5 (Lipschitz codimensional stability of backward self-similar solutions [97]). *Keeping the assumptions and notation from the previous Proposition, there exists a Lipschitz codimension n manifold* [1] *of non radial initial data with finite energy*

$$u_0 = \chi_{A_0} \Phi_n + w_0$$

1. In a sense similar to the statement of Theorem 3.2.

where $\chi_{A_0}(x) = \chi(x/A_0)$ *for some large enough* A_0 *and* w_0 *is small enough*

$$\|w_0\|_{H^2_\rho} + \|\Delta w_0\|_{L^2} + \|w_0\|_{\dot{H}^{s_c}} \ll 1, \tag{3.12}$$

such that the corresponding solution to (3.9) *blows up in finite time* $0 < T < +\infty$ *with a decomposition*

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} (\Phi_n + v) \left(t, \frac{x - x(t)}{\lambda(t)}\right)$$

where:

1. Control of the geometrical parameters*: the blow up speed is self similar*

$$\lambda(t) = \sqrt{(T-t)}(1 + o(1)) \ \ as \ \ t \to T$$

and the blow up point converges

$$x(t) \to x(T) \ \ as \ \ t \to T.$$

2. Stability*: there holds* [2]

$$(T-t)^{1/(p-1)} \|v(t)\|_{L^\infty(\mathbb{R}^3)} \to 0$$

as $t \to T$ *and the asymptotic stability of the self similar profile above scaling*

$$\lim_{t \to T} \|v(t)\|_{\dot{H}^s} = 0 \ \ for \ \ s_c < s \leq 2,$$

the boundedness of norms below scaling

$$\limsup_{t \to T} \|u(t)\|_{\dot{H}^s} < +\infty \ \ for \ \ 1 \leq s < s_c,$$

and the logarithmic growth of the critical norm

$$\|u(t)\|_{\dot{H}^{s_c}} = c_n(1 + o_{t\to T}(1))\sqrt{|\log(T-t)|}, \ \ c_n \neq 0.$$

3.1.3 Generic separation induced by blow-up for the inviscid Prandtl system

After having considered parabolic models, let us turn to nonlinear transport equation emanating from fluid mechanics. We first consider the inviscid Prandtl equations on the upper half-plane $\mathscr{H} := \mathbb{R} \times [0, \infty)$:

$$\begin{cases} u_t + uu_x + vu_y = -p^E_x \\ u_x + v_y = 0, \\ v|_{y=0} = 0, \quad \lim_{y\to\infty} u(t,x,y) = u^E(t,x), \end{cases} \qquad (t,x,y) \in [0,T) \times \mathscr{H}, \tag{3.13}$$

2. In [97], this stability in L^∞ was not stated in the main Theorem, but it follows from the Sobolev estimates and parabolic regularization.

where p^E and u^E are the traces of the Eulerian pressure and tangential flow at the boundary $\mathbb{R}\times\{0\}$ induced by the Eulerian flow at infinity. The functions p^E and u^E are prescribed, and then act as forcing terms for u. They are linked through Bernoulli's equation,

$$u_t^E + u^E u_x^E = -p_x^E,$$

whose solutions have to be global in two dimensions. The system (3.13) is obtained from the usual two-dimensional Prandtl system (3.28) which is a limit equation obtained from the Navier-Stokes equations at high Reynolds number near a boundary (see Section 3.2.2 for more details on this system), by discarding the viscosity term u_{yy}.

In case of zero pressure $p^E = 0$, and forgetting about the boundary condition at $y \to \infty$, the system (3.13) becomes

$$\begin{cases} u_t + uu_x + vu_y = 0 \\ u_x + v_y = 0, \qquad v|_{y=0} = 0, \end{cases} \qquad (t,x,y) \in [0,T) \times \mathscr{H}. \tag{3.14}$$

It thus appears as the simplest incompressible two-dimensional fluid flow, in which tangential transport is given by the Burgers equation (2.1) and normal transport is found by requiring no penetration condition at the boundary and incompressibility, namely

$$v(t,x,y) = -\int_0^y \partial_x u(t,x,\tilde{y})d\tilde{y}.$$

Despite the simplicity of this model, the Cauchy problem and singularity formation was not studied in depth prior to our work.

The only local existence with general smooth enough initial data for the inviscid Prandlt system (3.13) was due to Hong and Hunter, in [235] (see also [31, 209]), but their maximal time of existence was not sharp. We obtained a sharp estimate for the maximal time of existence thanks to a careful use of the method of characteristics.

We denote by (X,Y) the Lagrangian variables and (x,y) the Eulerian ones for equation (3.13). The position at time t of a particle with initial position (X,Y) and that is transported by the flow, is denoted by $(x(t,X,Y), y(t,X,Y))$. Using dots for differentiation with time t, they are related by the characteristics ODE:

$$\begin{cases} \dot{x} = u, \\ \dot{y} = v(t,x,y), \\ \dot{u} = -p_x^E(t,x), \end{cases} \qquad (x,y,u)(0) = (X,Y,u_0(X,Y)). \tag{3.15}$$

We notice that the sets $\{u_y = 0\}$ and $\{y = 0\}$ are transported by the characteristics, and that on both of them along the characteristics the tangential derivative solves

$$\dot{u}_x = -(u_x)^2 - p_{xx}^E(t,x).$$

The solution of this Riccati type equation may not exist for all times, and we denote the corresponding maximal time of existence by $T(X,Y)$.

The remarkable feature of (3.15) is that one can solve the first and third equations, without solving the second equation for y. We can compute first the tangential component of the Eulerian variables $(X,Y)\mapsto x(t,X,Y)$. Then, the normal component is retrieved by requiring that the mapping $(X,Y)\mapsto (x(t,X,Y),y(t,X,Y))$ is volume-preserving, which in two dimensions is given by the following explicit formula

$$y(t,X,Y)=\int_0^{s[t,x](X,Y)}\frac{d\tilde s}{|\nabla x(t,\gamma[t,x](\tilde s))|}.$$

Above, provided that the set $\{(\tilde X,\tilde Y),\ x(t,\tilde X,\tilde Y)=x(t,X,Y)\}$ is a curve that intersects the boundary $\{Y=0\}$, we parametrize by arclength $\gamma[t,x]:s\mapsto\gamma[t,x](s)$ the portion of this curves that connects the boundary to the point (X,Y).

Theorem 3.6 (Sharp local well-posedness in Hölder spaces [86])**.** *For any $(u^E,p^E_x)\in\mathscr{C}^2$ and $u_0\in\mathscr{C}^2$ with certain (very mild) suitable bounds, there exists a unique solution $u\in C^1([0,T)\times\mathscr{H})$ of (3.13) where $T>0$ is defined as*

$$T=\min\left\{T(X,Y),\ \ (X,Y)\in\{\partial_Y u_0(X,Y)=0\}\cup\{Y=0\}\right\}.$$

It satisfies $\sup_{t\in[0,\tilde T]}\|\nabla u(t)\|_{L^\infty(\mathscr{H})}<\infty$ for any $\tilde T<T$. The above formula defines the maximal time of existence since if T is finite then:

$$\lim_{t\uparrow T}\|u_x(t)\|_{L^\infty(\mathscr{H})}=\infty.$$

Moreover, there is $C^k\to C^{k-1}$ propagation of regularity for any $k\geq 3$.

We now aim at understanding singularity formation. Singularity formation for the original Prandtl system is linked to boundary layer separation. Prandtl describes in [364] the phenomenon of boundary layer separation, when "a fluid layer, set rotating as a result of friction at the wall, moves out into the free stream". For the steady Prandtl system, Goldstein [206] found (with a physicist's level of rigour) that separation was equivalent to having a singular solution. Dalibard and Masmoudi [110] gave recently a mathematically rigorous description of the Goldstein singularity.

For the unsteady Prandtl system, still with a physicist's level of rigour, Moore, Rott and Sears [335, 378, 390] came up with what is now known as the MRS conditions for separation, and Van Dommelen and Shen [407, 405] showed the equivalence between separation and singularity formation. We refer to the introduction of [49] for an historic perspective for criteria for steady and unsteady separation. It is noted in [407, 406] that the layer becomes inviscid to leading order during separation. This justifies our study of the inviscid Prandtl system (3.13).

The homogeneous inviscid Prandtl system (3.14) has the following invariances. If u is a solution then so is

$$\frac{\mu}{\lambda}\iota u\left(\frac{t}{\lambda},\iota\frac{x-ct}{\mu},\frac{y}{\nu}\right)+c \tag{3.16}$$

for $(\iota,\lambda,\mu,\nu,c)\in\{-1,1\}\times(0,\infty)^3\times\mathbb{R}$. Backward self-similar solutions are special solutions living in the orbit of the initial datum under the action of a one dimensional scaling subgroup, of the form

$$u(t,x,y) = (T-t)^{\alpha-1}\Theta(x/(T-t)^{\alpha}, y/(T-t)^{\beta}).$$

To have a solution to (3.14) of this form is equivalent to have a solution of the stationary equation:

$$\begin{cases} (1-\alpha)\Theta + (\alpha\mathscr{X} + \Theta)\partial_{\mathscr{X}}\Theta + (\beta\mathscr{Y} + \upsilon)\partial_{\mathscr{Y}}\Theta = 0, \\ \partial_{\mathscr{X}}\Theta + \partial_{\mathscr{Y}}\Upsilon = 0, \end{cases} \qquad (\mathscr{X},\mathscr{Y}) \in \Omega, \tag{3.17}$$

subject to the condition $\lim_{\mathscr{Y}\to 0}\Upsilon = 0$, on an open set $\Omega \subseteq \mathscr{H}$ (which might be different than $\mathscr{H}$, as we shall see below). This equation seems at first hard to solve, because the tangential component of the velocity is nonlocal $\Upsilon(y) = -\int \partial_X \Theta$. In addition, since the scaling group is three-dimensional, it is not clear a priori what the values of α and β should be. While solutions to (3.17) were studied by physicists in [50], we found a transformation of (3.17) that allows to get (almost) explicit formulas for the solutions. We developed a method to find the admissible exponents α and β (which are found by requiring smoothness on the corresponding self-similar solution), and an explicit formula for Θ. We found that equation (3.17) can be transformed into the following linear equation.

Lemma 3.7 (Transformation to solve the self-similar equation [86])**.** *A couple (Θ,Υ) solves (3.17) near a point $(\mathscr{X}_0,\mathscr{Y}_0)$ at which $\Theta(\mathscr{X}_0,\mathscr{Y}_0) \neq 0$ if and only if there exists a local volume preserving change of coordinates $(\mathscr{X},\mathscr{Y}) \mapsto (\mathscr{a},\mathscr{b})$ with $\mathscr{a} = -\Theta$ such that the function $\mathscr{X} : (\mathscr{a},\mathscr{b}) \mapsto \mathscr{X}(\mathscr{a},\mathscr{b})$ solves the linear equation*

$$(\alpha-1)\mathscr{a}\partial_{\mathscr{a}}\mathscr{X} + (1+\beta)\mathscr{b}\partial_{\mathscr{b}}\mathscr{X} = \alpha\mathscr{X} - \mathscr{a}.$$

The variables $(\mathscr{a},\mathscr{b})$ can be interpreted as self-similar Lagrangian variables, see [86]. We obtained a particular solution, the generic self-similar profile, corresponding to $\alpha = 3/2$, $\beta = -1/4$ and it is the one related to the so-called Van-Dommelen and Shen singularity [407]. We introduce (Γ being the Gamma function)

$$p^* := \frac{4}{9\pi^3}\Gamma\left(\frac{1}{4}\right)^4. \tag{3.18}$$

Proposition 3.8 (The generic backward self-similar profile [86])**.** *The mapping $\Phi : (\mathscr{a},\mathscr{b}) \mapsto (\mathscr{X},\mathscr{Y})$ given by:*

$$\Phi(\mathscr{a},\mathscr{b}) = \left(\mathscr{a} + \mathscr{b}^2 + p^{*2}\mathscr{a}^3\,,\ \int_{-\infty}^{\mathscr{b}} \frac{d\tilde{\mathscr{b}}}{1 + 3\Psi_1^2\left(p^*\left(\mathscr{a} + p^{*2}\mathscr{a}^3 + \mathscr{b}^2 - \tilde{\mathscr{b}}^2\right)\right)}\right),$$

where Ψ_1 is the self-similar profile of the inviscid Burgers equation as defined in Proposition 2.1 and p^* is given by (3.18), satisfies the following properties:

(i) It is an analytic volume preserving diffeomorphism between $\mathbb{R}^2$ and the subset of the upper half plane $\{(\mathscr{X},\mathscr{Y}) \in \mathscr{H},\ 0 < \mathscr{Y} < 2\mathscr{Y}^*(\mathscr{X})\}$ where $\mathscr{Y}^* > 0$ is the following analytic function:

$$\mathscr{Y}^*(\mathscr{X}) = \int_0^{\infty} \frac{d\tilde{\mathscr{b}}}{1 + 3\Psi_1^2(p^*(\mathscr{X} - \tilde{\mathscr{b}}^2))}.$$

(ii) The opposite of the first component of its inverse $\Phi^{-1} = (\Phi_1^{-1}, \Phi_2^{-1})$*:*

$$\Theta := -\Phi_1^{-1} : (\mathscr{X}, \mathscr{Y}) \mapsto -a,$$

is a self-similar profile, that is, the following is a solution of (3.14) *on its support:*

$$u(t,x,y) = (T-t)^{\frac{1}{2}} \Theta \left(\frac{x}{(T-t)^{\frac{3}{2}}}, \frac{y}{(T-t)^{-\frac{1}{4}}} \right).$$

(iii) U *has the following singular behaviour near the boundary of its domain:*

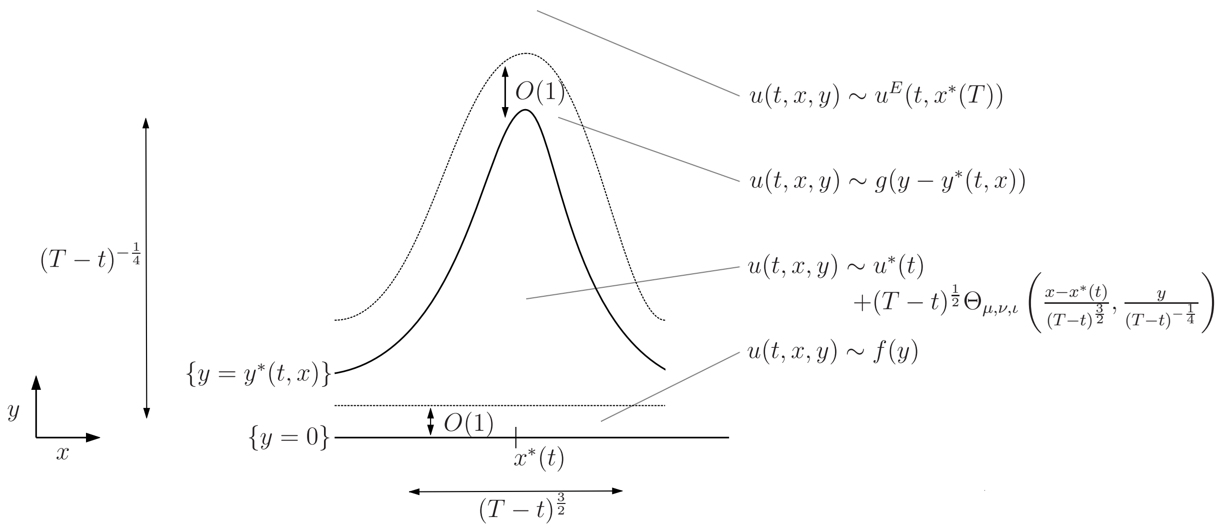
$$\Theta(\mathscr{X}, \mathscr{Y}) = p^{*-2}\mathscr{Y}^{-2} + O(|\mathscr{X}|\mathscr{Y}^4 + \mathscr{Y}^2) \qquad \text{for } 0 < \mathscr{Y} \le \mathscr{Y}^*(\mathscr{X})$$
$$\Theta(\mathscr{X}, 2\mathscr{Y}^*(\mathscr{X}) - \mathscr{Y}) = p^{*-2}\mathscr{Y}^{-2} + O(|\mathscr{X}|\mathscr{Y}^4 + \mathscr{Y}^2) \qquad \text{for } 0 < \mathscr{Y} \le \mathscr{Y}^*(\mathscr{X}),$$

Further properties are given in [86], but let us mention that the striking ones are the fact that the domain of the self-similar profile U is not the whole upper half plane and that it is singular near the boundary of its domain. By sign reversal and scaling invariance,

$$\Theta_{\iota,\mu,\nu}(\mathscr{X}, \mathscr{Y}) = \iota\mu\Theta(\iota\frac{\mathscr{X}}{\mu}, \frac{\mathscr{Y}}{\nu})$$

is also a self-similar solution for any $(\iota, \mu, \nu) \in \{\pm 1\} \times (0, \infty)^2$.

The self-similar profile Θ given by Proposition 3.8 is at the heart of the singularity formation for the inviscid Prandtl equations, triggering the separation of the boundary layer. We showed the appearance of this self-similar profile from a generic set (dense and open, among data producing singularities) of initial data u_0. This shows that there is a Burgers type compression along the tangential variable, simultaneously with an expansion induced by incompressibility that ejects the boundary layer to $y = \infty$. We justify the following picture, which was first derived with a physicist's level of rigour and numerical simulations in [407, 405]:



We denote by $d_{C^k_{\mathrm{loc}}}$ the standard C^k_{loc} distance for functions on $\mathscr{H}$ that are k times continuously differentiable, and equip $\mathscr{E}^k$ with the topology associated to the distance $d_{\mathscr{E}^k}(u_0,u_0') = d_{C^k_{\mathrm{loc}}}(u_0,u_0') + \|\nabla(u_0-u_0')\|_{L^\infty(\mathscr{H})}$. We define $\mathscr{Y}^*_{\mu,\nu,\iota}(\mathscr{X}) = \nu\mathscr{Y}^*(\iota\mathscr{X}/\mu)$.

Theorem 3.9 (Generic Separation [86]). *Let $(p_x^E,u^E)\in\mathscr{C}^4$ with suitable (very mild) bounds. In the subset of C^4 of initial data u_0 (again with additional suitable very mild bounds) such that $T<\infty$ and such that T, defined as an infimum by Theorem 3.6, is attained on the set $\{(X,Y),\partial_y u_0(X,Y)=0 \text{ and } Y>0\}$, there exists a dense open set for which the corresponding solution satisfies the following. There exist parameters $(\mu,\nu,\iota)\in(0,\infty)^2\times\{-1,1\}$ and two constants $\kappa,C>0$ such that:*

— Location of the singularity. *There exists $x^*\in C^4([0,T],\mathbb{R})$, regular up to time T such that $\nabla u(t)$ remains bounded in $\{(x,y,t),\ 0\le t\le T,\ |x-x^*(t)|\ge\epsilon\}$ for any $\epsilon>0$.*

— Displacement line. *There exists $y^*\in C^3([0,T)\times\mathbb{R})$ for which the properties below hold true, with:*

$$y^*(t,x) = \frac{2}{(T-t)^{\frac14}}\mathscr{Y}^*_{\mu,\nu,\iota}\left(\frac{x-x^*}{(T-t)^{\frac32}}\right)(1+O((1-t)^\kappa+|x-x^*(t)|^\kappa)).$$

— Self-similarity. *Let $\eta(t,x,y)=((T-t)^\kappa+|x-x^*(t)|^\kappa+y^{-\kappa}+|y^*(t,x)-y|^{-\kappa})$ and let $u^*(t)=\partial_t x^*(t)$. For any $\epsilon>0$, there exists $\tilde\epsilon>0$ such that for $|x-x^*|\le\tilde\epsilon$ and $y\le(1-\epsilon)y^*(t,x)$:*

$$u(t,x,y) = u^*(t)+(T-t)^{\frac12}\left(\Theta_{\mu,\nu,\iota}+\tilde u\right)(\mathscr{X},\mathscr{Y}),$$

where $\mathscr{X}=(T-t)^{-3/2}(x-x^*(t))$, $\mathscr{Y}=(T-t)^{1/4}y$, and where $\tilde u$ satisfies:

$$|\tilde u(t,\mathscr{X},\mathscr{Y})| \le C\left(|\mathscr{X}|^{\frac13}+|\mathscr{Y}|^{-2}+\left((T-t)^{\frac14}y^*(t,x)-\mathscr{Y}\right)^{-2}\right)\eta(t,x,y)$$
$$|\partial_{\mathscr{X}}\tilde u(t,\mathscr{X},\mathscr{Y})| \le C\left(|\mathscr{Y}|^4+(1+|\mathscr{X}|)^{-\frac76}\left((T-t)^{\frac14}y^*(t,x)-\mathscr{Y}\right)^{-3}\right)\eta(t,x,y)$$
$$|\partial_{\mathscr{Y}}\tilde u(t,\mathscr{X},\mathscr{Y})| \le C\left(|\mathscr{Y}|^{-3}+\left((T-t)^{\frac14}y^*(t,x)-\mathscr{Y}\right)^{-3}\right)\eta(t,x,y).$$

— Reconnections close to the displacement line, below and above. *One also has precise leading order asymptotic expansions for y close to or above $y^*(t,x)$, for $y=O(1)$, and for x close to x^* with $|x-x^*|/(T-t)^{3/2}$ large.*

3.1.4 Self-similar singularity for the incompressible porous medium equation

The key feature of the inviscid Prandtl system (3.13) we previously studied is that the pressure is prescribed, and is then a local quantity. Let us now consider another inviscid fluid for which the pressure depends on the solution in a nonlocal way, the two dimensional incompressible porous medium equation. The equation describes evolution of density carried by the flow of incompressible fluid that is determined via Darcy's law in the field of gravity:

$$\begin{cases} \partial_\tau \rho + u \cdot \nabla \rho = 0, \\ u + \nabla P = (0, \rho), \\ \operatorname{div} u = 0 \text{ and } u \cdot n = 0 \text{ for } x = \pm\pi \end{cases} \qquad (x,y) \in [-\pi,\pi] \times \mathbb{R} \tag{3.19}$$

Here $t \geq 0$ is the time variable, ρ is the transported density and u is the vector field describing the fluid motion, P is the pressure, and $n = (1,0)$. For more on flows in porous media, we refer to [12, 350, 168]. Throughout this paper, we consider the spatial domain to be the infinite strip $(x,y) \in [-\pi,\pi] \times \mathbb{R}$, or $\mathbb{T} \times \mathbb{R}$ that is periodic for $x \in [-\pi,\pi]$. In the first case, due to the presence of boundaries, u also satisfies the boundary condition $u \cdot n = 0$ for $x = \pm\pi$.

Whether smooth and localized solutions to (3.19) can blow up in finite time remains a challenging open problem, while numerical simulations indicate absence of singularities [100]. In Castro-Córdoba-Gancedo-Orive [51] a class of infinite energy solutions has been considered on $\mathbb{T} \times \mathbb{R}$, for which the stream function ψ of the velocity field has the form $\psi(t,x,y) = yf(t,x)$. After a few computations, they found that the quantity b defined by $f(t,x) = \int_{-\pi}^{x} b(t,\bar{x})\, d\bar{x}$, satisfies the following 1D nonlocal nonlinear transport equation:

$$\partial_t b + \left(\int_{-\pi}^{x} b(t,\bar{x})\, d\bar{x} \right) \partial_x b - b^2 + \frac{1}{\pi} \int_{-\pi}^{\pi} b^2\, dx = \frac{1}{\pi} \left(\int_0^t \int_{-\pi}^{\pi} b^2(\bar{t},\bar{x})\, d\bar{x}\, d\bar{t} \right) b, \tag{3.20}$$

supplemented with mean-free boundary condition

$$\int_{-\pi}^{\pi} b(\tau, x)\, dx = 0. \tag{3.21}$$

Note that the solution constructed in this way will also give a solution of (3.19) on the domain $[-\pi,\pi] \times \mathbb{R}$ with prescribed boundary condition.

It is found in [51] that Equation (3.20) has a class of explicit blow-up solutions of the form

$$b(t,x) = \frac{\mu}{\cos(\mu t)} \cos(x)$$

for $\mu \neq 0$. The above solution blows up at time $T = \frac{\pi}{2\mu}$ with the following self-similar behavior to leading order

$$b(t,x) = \frac{1}{T-t} \cos(x) + O(T-t)$$

as $t \to T$. We proved that this solution is in fact stable by smoother than C^2 perturbations (the statement below being formulated in C^3 for simplicity, but the method of the proof being valid for $C^{2+\epsilon}$ for any $\epsilon > 0$), and instable by rougher than C^2 perturbations.

Theorem 3.10 (Stable backward self-similar blow-up for infinite energy solutions [96])**.** Stability. *There exists $\delta > 0$ such that for any $\mu > 0$ the following hold. Let $b_0 \in C^3(\Omega)$ satisfy the mean free condition $\int_{-\pi}^{\pi} b_0 = 0$ and such that*

$$\|b_0(\cdot) - \mu \cos(\cdot)\|_{C^3([-\pi,\pi])} \leq \delta\mu,$$

then the unique C^3 solution $b(t,x)$ of the equation (3.20) with $b(0,x) = b_0(x)$ blows up in finite time $T > 0$ and can be decomposed as

$$b(t,x) = \frac{1}{T-t}\cos(x) + \tilde{b}(t,x), \tag{3.22}$$

where $\|\tilde{b}(t)\|_{C^1([-\pi,\pi])} \lesssim (T-t)^{-3/4}$.

Instability. *For any $\epsilon, \delta > 0$ and $\mu > 0$ there exists an initial data b_0 which satisfies the mean free condition $\int_{-\pi}^{\pi} b_0 = 0$ and $\|b_0 - \mu\cos(x)\|_{C^{2-\epsilon}([-\pi,\pi])} \leq \delta\mu$, such that the problem (3.20) cannot have a solution blowing up at some time $T > 0$ that can be decomposed as (3.22) with $(T-t)\|\tilde{b}(t)\|_{L^\infty([-\pi,\pi])} \to 0$ as $t \to T$.*

Since the solutions of Theorem 3.10 have infinite energy, it remains an open question whether their stability that is obtained for the reduced equation (3.20) can be extended to a full two-dimensional analysis that would enable to localize them to show blow-up for finite energy solutions.

Our interest in Equation (3.20) in fact lied in the following transformation. Given b a solution to (3.20) satisfying (3.21), the following change of variables:

$$a(\tau,x) = \frac{1}{\nu(t)} b(t,x), \qquad \tau = \int_0^t \nu(\bar{t})d\bar{t}, \qquad \frac{d}{dt}\nu = \frac{\nu}{\pi}\int_0^t \int_{-\pi}^{\pi} b^2(\bar{t},x)\,dx\,d\bar{t}$$

with $\nu(0) = \nu_0 > 0$ is such that the new function a solves

$$\begin{cases} \partial_\tau a + \left(\int_{-\pi}^{x} a(\tau,\bar{x})\,d\bar{x}\right)\partial_x a - a^2 + \frac{1}{\pi}\int_{-\pi}^{\pi} a^2\,dx = 0, \\ a(0,x) = a_0(x), \end{cases} \tag{3.23}$$

on $[0,\tau^*) \times [-\pi,\pi]$, supplemented by the mean-free condition

$$\int_{-\pi}^{\pi} a(t,x)dx = 0. \tag{3.24}$$

Equation (3.23) is the so-called Proudman-Johnson equation [365] that appears as a reduced model for special classes of solutions both for the $2D$ incompressible primitive equations (also known as the hydrostatic Euler equations), as we shall explain in Section 3.2.3, and the $2D$ Euler equations. Local-well posedness and finite time blow-up (which is possible depending on the initial data) for the Proudman-Johnson equation was studied in [65, 99, 353, 354] and references therein. In [383, 384, 382] it is found that smooth periodic initial data lead to global solutions, while the absence of periodicity or the lack of smoothness may lead to finite time blow-up; the authors develop an elegant Lagrangian approach based on the characteristics. We will describe more the behaviour of singular solutions to the Proudman-Johnson equation in Section 3.2.3.

Our proof of Theorem 3.10 in fact relied on the proof of the stability/instabililty of the family of steady states $\{\mu\cos\}_{\mu>0}$ for the Proudman Johnson equation (3.23), a result of independent interest that was stated as Theorem 1.3. in [96].

3.2 Perturbed backward self-similar blow-up

The previous section 3.1 was devoted to the study of solutions becoming singular by concentrating a backward self-similar solution of the equation. We will now describe what we will refer to as *perturbed* backward self-similarity. This will refer to singularities for which the effects some terms in the equation are subleading, and the solution concentrates a backward self-similar solution, but for the reduced equation that is obtained from the original equation by discarding the subleading terms. The backward self-similar blow-up solution of the reduced equation is then perturbed by the other subleading effects in the original equation. These effect may not be negligible however, as we shall see, resulting sometimes in anomalous self-similarity.

3.2.1 ODE blow-up for the semilinear heat equation

We explained in Section 3.1.2 that the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u_{|t=0} = u_0, \end{cases} \tag{3.25}$$

admits an explicit constant in space blow-up solution

$$u(t,x) = \frac{\kappa}{(T-t)^{\frac{1}{p-1}}}, \qquad \kappa = \left(\frac{1}{p-1}\right)^{\frac{1}{p-1}}$$

that solves the ODE $u_t = |u|^{p-1}u$, and that singular solutions undergoing ODE blow-up near a singular point x^* are those who are equal to $\pm\kappa(T-t)^{-\frac{1}{p-1}}$ to leading order in the backward parabola from x^*. We now address the following issue: how do such solutions u transition from the singular behaviour $\pm\kappa(T-t)^{-\frac{1}{p-1}}$ near x^* to a non-singular behaviour away from x^*?

Let us for simplicity take

$$d = 1 \text{ and } p = 2,$$

for the discussion. Since u is flat near x^*, the diffusion effect should be subleading with respect to the nonlinear term. Hence, we discard it and obtain

$$\begin{cases} \partial_t u = |u|u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u_{|t=0} = u_0. \end{cases} \tag{3.26}$$

The backward self-similar solutions of the above equation are easily found to be $u(t,x) = \frac{1}{T-t}U_{\alpha,\mu}(\frac{x}{(T-t)^{1-\frac{1}{2\alpha}}})$ for $U_{\alpha,\mu}(y) = \frac{1}{1+|y/\mu|^{2\alpha}}$. If $\alpha \notin \mathbb{N}$, these are singular at $y=0$, and cannot appear in a parabolic model like (3.25) due to parabolic regularization. Hence those who should appear are the analytic ones:

$$u(t,x) = \frac{1}{T-t}U_{k,\mu}(\frac{x}{(T-t)^{1-\frac{1}{2k}}}), \qquad U_{k,\mu}(y) = \frac{1}{1+|y/\mu|^{2k}}$$

for $k \in \mathbb{N}$. A straightforward computation shows that for an analytic initial data, the solution to (3.26) blows up at the points at which it attains its maxima and minima, where it concentrates the profile $U_{k,\mu}$ with the integer k being determined from u_0 having a Taylor expansion $C + C'|y|^{2k}$ near such points. The most stable configuration corresponds to $k = 1$, with backward self-similar solution $\frac{1}{T-t}U_{2,\mu}(\frac{x}{\sqrt{T-t}})$. Unfortunately, the diffusion effect does not seem to be negligible with respect to the nonlinear term, as

$$\left|\frac{1}{T-t}U_{2,\mu}(\frac{x}{\sqrt{T-t}})\right|^2 \approx \frac{1}{(T-t)^2} \quad \text{and} \quad \left|\Delta\left(\frac{1}{T-t}U_{2,\mu}(\frac{x}{\sqrt{T-t}})\right)\right| \approx \frac{1}{\mu^2(T-t)^2}.$$

The subleading term of the viscosity is in fact not negligible, and its effect is to alter the scaling parameter μ and make it diverge logarithmically close to the singularity. More precisely, it is known [20, 34, 231, 320] that there exists a stable solution to (3.26) of the form

$$u(t,x) = \frac{1}{T-t}\frac{1}{1+\frac{x^2}{8(T-t)|\log(T-t)|}} + \tilde{u}(t,x), \qquad \lim_{t\uparrow T}(T-t)\|\tilde{u}(t)\|_{L^\infty} = 0.$$

The construction and stability of [320] is in fact valid in any dimension d and for any nonlinearity exponent $p > 1$.

The existence of singular solutions blowing up with the flatter profile $U_{k,\mu}$ for $k \geq 2$ was done in [34, 227]. For the work [85] on the Burgers equation with transverse viscosity, see Section 3.4.1, we had to show additional weighted estimates than those showed in these articles. In particular, we revisited the proof in [34, 227, 320] and obtained a true improvement for the "flat" unstable blow-ups. Indeed, we avoided the use of maximum principle as in [227] or of Feynman-Kac formula as in [34, 227], and obtained sharp stability estimates. Namely, for flat blow-up the convergence of the solution to the blow-up profile holds in a spatial region that is of size one in original y variables, see estimate (3.27). For example, this estimate directly implies the existence of a profile at blow-up time $u(t,y) \to U^*(y)$ as $t \to T$ for $y \neq 0$, and that it satisfies $U^*(y) \sim (ay^{2k})^{-1}$ as $y \to 0$ (this fact would not be obtained directly in previous works).

Theorem 3.11 (Construction with precise stability estimates of ODE blow-up solutions [85]). *Let $J \in \mathbb{N}$. There exists an open set for a suitable topology of even solutions to (3.26) blowing up in finite time $T > 0$ with*

$$u(t,x) = \frac{1}{T-t}\frac{1}{1+\frac{x^2}{8(T-t)|\log(T-t)|}} + \tilde{u},$$

where the remainder $\tilde{u}$ satisfies for $0 \leq j \leq J$ for some constant $C > 0$:

$$|\partial_y^j \tilde{u}| \leq \frac{C}{(T-t)|\log(T-t)|^{\frac14}} \frac{1}{\left(1+\frac{x^2}{8(T-t)|\log(T-t)|}\right)^{\frac34}} \frac{1}{\left(\sqrt{(T-t)|\log(T-t)|}+|x|\right)^j}.$$

For any $k \in \mathbb{N}$, $k \geq 2$, $a > 0$, there exists $T^* > 0$, such that for any $0 < T < T^*$ there exists a solution to (3.26) blowing up at time T with

$$u(t,y) = \frac{1}{T-t+ay^{2k}} + \tilde{u},$$

where the remainders $\tilde{\xi}$ satisfies for $j = 0, ..., J$ for some constant $C > 0$:

$$|\partial_y^j \tilde{u}| \le C \frac{1}{\left((T-t)^{\frac{1}{2k}} + |y|\right)^{2k+j}} \left((T-t)^{\frac{1}{2k}} + |y|\right)^{\frac{1}{2}}. \qquad (3.27)$$

We mention that the result of [85] was done for even solutions for simplicity, but that the proof adapts in a straightforward way to the unsymmetric case.

3.2.2 Singularity formation for the two dimensional Prandtl system around an axis

The two dimensional Prandtl system is

$$\begin{cases} u_t - u_{yy} + uu_x + vu_y = -p_x^E & (t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \qquad u|_{y\to\infty} = u^E, \end{cases} \qquad (3.28)$$

where $\overrightarrow{u} = (u, v)$ is the velocity field, u^E and p^E are the traces at the boundary of the tangential component of the underlying Eulerian velocity field and the pressure. They are global solutions to the Bernouilli equation $u_t^E + u^E u_x^E = -p_x^E$.

Prandtl in [364] introduced this model to describe the behaviour of a fluid close to a physical boundary for high Reynolds numbers. The leading order term in the expansion in the boundary layer solves (3.28), see for example [379, 380, 294] for more on the derivation of the system.

We already gave a detailed study of the inviscid Prandtl system (3.13) in Section 3.1.3, which is the system obtained from (3.28) by discarding the normal viscosity u_{yy} term. Unfortunately, the study of the inviscid and viscous Prandtl system are completely different. Indeed, the whole study performed in Section 3.1.3 was intimately linked to the ability of solving the inviscid system thanks to the method of characteristics, which know fails for the original viscous problem (3.28).

In fact, viscosity can have a destabilizing effect for the Prandtl system. In the special class of solution satisfying the monotonicity condition $u_y > 0$ (which are monotone shear flow-like), the well-posedness holds in Sobolev regularity [355, 309, 3] and global weak solutions also exist globally [418]. Note that the solutions we consider here do not satisfy the monotonicity assumption. In this case, the equation can be ill-posed in Sobolev regularity [187], and is only locally well-posed in the analytical setting [379, 287, 269], or Gevrey setting [131]. Similar instabilities prevent Prandtl's system from being a good approximation of the Navier-Stokes equations at high Reynolds number in certain cases [210]. Indeed, monotonicity and/or Gevrey regularity in the tangential x-variable are necessary to insure that this approximation holds. We refer to [380, 188] and the references therein.

From physicists' perspective, solutions of System (3.28) becoming singular in finite time should be equivalent to unsteady boundary layer separation, as was explained in Section 3.1.3. Van Dommelen and Shen [407] obtained the first reliable numerical result in this direction, and explained how the separation is linked to the formation of singularity, and we refer to [107, 121, 185, 235] for additional numerical results on the singularity formation.

The precise description of the formation of singularity is still an open problem. However, E and Engquist [412] (see also [270]) proved that blow-up can happen. For a survey of blow-up results we refer to [67]. They assumed the following symmetry assumption:

$$u \text{ and } u^E \text{ are odd with respect to } x.$$

Then, the trace of the tangential derivative on the normal axis

$$\xi(t,y) = -u_x(t,0,y), \tag{3.29}$$

obeys the following equation for $y \in [0,+\infty)$:

$$\begin{cases} \xi_t - \xi_{yy} - \xi^2 + \left(\int_0^y \xi\right)\xi_y = p^E_{xx}(t,0), \\ \xi(t,0) = 0, \\ \xi(0,y) = \xi_0(y). \end{cases} \tag{3.30}$$

The two nonlinear terms are also found in the Proudman-Johnson equation (3.23) (the Proudman-Johnson equation having an additional nonlocal term coming from the pressure). It also corresponds to the quadratic heat equation (3.25) to which one adds a nonlinear advection term. The reduced one dimensional problem (3.30) with a different domain and boundary conditions also appears in a special class of infinite energy solutions to the Navier-Stokes equations [202]. The authors proved the existence of a similar stable blow-up pattern as the one we describe here, but for a particular class of solutions requiring many special assumptions due to their approach based on maximum principle.

Our approach was inspired by the description of the so-called ODE blow-up for the semi-linear heat equation (3.25) as described in Section 3.2.1, see in particular [199, 34, 231, 320]. In comparison with the semilinear heat equation, the nonlocal advection term will induce two new effects: first, the singular point is ejected to infinity in finite time, and second the solution forms a plateau with a growing length. By discarding the diffusion and pressure terms in (3.30) we obtain the following equation

$$\xi_t - \xi^2 + \left(\int_0^y \xi\right)\xi_y = 0.$$

It admits an explicit backward self-similar solution

$$\xi(t,y) = \frac{1}{T-t}\cos^2\left(\frac{y-\mu\pi(T-t)^{-1/2}}{2\mu(T-t)^{-1/2}}\right)\mathbb{1}_{0\le y\le \frac{2\mu\pi}{(T-t)^{-1/2}}}$$

for any $\mu > 0$. We also showed it admits countably many other self-similar solutions [85]. Our first result was to show that Equation (3.30) admits a stable blow-up behaviour modelled on the above solution. The effect of the viscosity is asymptotically negligible, and this is a perturbed backward self-similar blow-up from the inviscid equation.

Theorem 3.12 (Stable blow-up for Equation (3.30), [85])**.** *Assume $p^E_{xx}(t,0) = 0$. There exists $\lambda_0^* \gg 1$ such that for all $\lambda_0 \geq \lambda_0^*$, an $\epsilon(\lambda_0) > 0$ exists with the following property. For an initial datum of the form:*

$$\xi_0(y) = \lambda_0^2 \cos^2\left(\frac{y-\lambda_0\pi}{2\lambda_0}\right) \mathbb{1}_{0\leq y\leq 2\lambda_0\pi} + \tilde{\xi}_0(y), \quad \text{with} \quad \|\tilde{\xi}_0\|_{L^1([0,+\infty))} \leq \epsilon(\lambda_0),$$

the unique solution to (3.30) blows up at some time $T > 0$ with:

$$\xi(t,y) = \lambda^2(t)\cos^2\left(\frac{y-y^*(t)}{2\lambda(t)\mu(t)}\right) \mathbb{1}_{-\pi\leq \frac{y-y^*}{\lambda\mu}\leq \pi} + \tilde{\xi},$$

where, for some $\mu_\infty > 0$:

$$\lambda(t) = \frac{1}{\sqrt{T-t}} + O((T-t)^{3/2}), \;\; \mu(t) = \mu_\infty + O((T-t)), \;\; y^*(t) = \frac{\mu_\infty \pi}{\sqrt{T-t}} + O((T-t)^{-1/4}),$$

and

$$\|\tilde{\xi}(t)\|_{L^\infty([0,\infty))} \leq (T-t)^{-1+\frac{1}{8}}.$$

The assumption $p^E_{xx}(t,0) = 0$ is not relevant. Indeed, since p^E is a regular function, the effect of the $p^E_{xx}(t,0) = 0$ term in (3.30) is completely negligible and one could easily extend the above result to the case of a nonzero pressure. It is to be noted that the scale in the normal direction is $(T-t)^{-1/2}$ for this class of blow-up solutions with symmetry, which is different than the scale $(T-t)^{-1/4}$ that appears in the generic singularity formation for non-symmetric solutions of the inviscid system as described in Theorem 3.9.

From the numerics in [212] it seems that for certain solutions with symmetries the blow-up indeed happens on the vertical axis. However, for other solutions, such as the singularity considered by Van Dommelen and Shen, the numerics show that another singularity appears before the one on the vertical axis. Our second result shows that for analytic solutions, if the solution of the reduced one dimensional problem blows up with the aforementioned stable blow-up pattern of Theorem 3.12, then the solution exists up to this blow-up time in a suitable causal neighborhood of the vertical axis with a universal lower bound on its local analyticity radius. This justifies that the one-dimensional profile constructed in Theorem 3.12 describes blowing up solutions for the two-dimensional Prandtl system (3.28)

In the sequel, we still restrict ourselves to solutions of (3.28) that are odd in x, with vanishing outer flow $u^E = p^E = 0$ (again, this second assumption is for simplicity only). We consider higher order derivatives restricted to the vertical axis and introduce for $i \geq 0$:

$$\xi_i(t,y) := \partial_x^{2i+1} u(t,0,y)$$

(hence $\xi = -\xi_0$ with this notation). They solve the following system for $i \geq 0$ and $y \in [0,+\infty)$:

$$\begin{cases} \partial_t \xi_i = \partial_{yy}\xi_i - \sum_{j=0}^i \binom{2i+1}{2j+1} \xi_j \xi_{i-j} + \sum_{j=0}^i \binom{2i+1}{2j} (\partial_y^{-1}\xi_j)\partial_y \xi_{i-j}, \\ \xi_i(t,0) = 0, \\ \xi_i(0,y) = \partial_x^{2i+1} u(0,0,y). \end{cases} \tag{3.31}$$

While a priori blow-up rates in Lebesgue and Sobolev spaces during singularity formation had been showed for various evolution equations, no result had been obtained regarding a priori bounds for the radius of analyticity of solutions during singularity formation. Our study of System (3.31) gave the first obtention of a (presumably sharp, we believe) universal lower bound for the radius of analyticity nearby the vertical axis.

Given a function $\tau \in \mathscr{C}^0([0,T],(0,\infty))$, we introduce the set $E_{T,\tau}$:

$$E_{T,\tau} := \{(t,x,y) \in [0,T) \times \mathbb{R} \times [0,\infty),\ |x| \le \tau(t)(T-t)^{7/4}\}.$$

Note that $\tau \ge \tau^* > 0$ for $\tau^* = \min_{[0,T]} \tau > 0$.

Theorem 3.13 (Lower bound for the radius of analyticity on the axis [85])**.** *Assume $p^E = u^E = 0$. Assume that $u_0 : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is odd in x, and analytic in x on the set $\{|x| < \delta\}$ for some $\delta > 0$, and satisfies the following hypotheses:*

(i) Analytic bound on the axis at initial time: *There exist $C_0, \tau_0 > 0$ such that for all $i \ge 0$, $\partial_x^{2i+1} u_0 \in C([0,\infty))$ with for all $y \ge 0$:*

$$|\partial_x^{2i+1} u_0(0,y)| \le C_0 \tau_0^{-2i-1}(2i+1)!\langle y\rangle^{-2}.$$

(ii) Stable blow-up behaviour on the axis: *the restriction of the tangential derivative u_x to the vertical axis blows up at some time T in a similar way as described in Theorem 3.12. Namely, there exist $T, \mu, \iota, C_0' > 0$ such that the solution ξ to (3.30) with initial datum $\xi_0(y) = -\partial_x u_0(0,y)$ blows up at time T with:*

$$\xi(t,y) = \frac{1}{T-t}\cos^2\left(\frac{y-\mu\pi(T-t)^{-\frac12}}{2\mu(T-t)^{-\frac12}}\right)\mathbb{1}_{-\pi \le \frac{y-\mu\pi(T-t)^{-1/2}}{\mu(T-t)^{-1/2}} \le \pi} + \tilde{\xi}(t,y)$$

where $\tilde\xi$ satisfies $(T-t)\|\tilde\xi\|_{L^\infty([0,\infty))} \to 0$ plus some other estimates that were showed in [85] to be true for an open set of solutions blowing up as in Theorem 3.12.

Then, there exists $\tau \in C^0([0,T],(0,+\infty))$ and a function $u \in C(E_{T,\tau})$ with $u \in C^\infty(E_{T,\tau} \cap \{t > 0\})$ such that:

(i) u is a classical solution to the Prandtl system (3.28) on $E_{T,\tau} \cap \{t > 0\}$ and $u = u_0$ on $E_{t,\tau} \cap \{t = 0\}$.

(ii) There exist $C_1, \tau_1 > 0$ such that for all $t \in [0,T)$ and $y \ge 0$:

$$|\partial_x^3 u(t,0,y)| \le C_1(T-t)^{-4}$$
$$|\partial_x^{2i+1} u(t,0,y)| \le C_1(T-t)^{-\frac72 i - \frac18}\tau_1^{-2i-1}(2i+1)! \qquad \textit{for } i \ge 2.$$

(iii) the set $E_{T,\tau}$ is causal in the sense that at its boundary:

$$|u_{|\{x=\pm\tau(t)(T-t)^{7/4}\}}| < |\frac{d}{dt}(\tau(t)(T-t)^{7/4})|.$$

The radius of analyticity of the solution is thus $\gtrsim (T-t)^{7/4}$ near the vertical axis. We believe that the exponent $7/4$ is optimal. This value comes from optimal bounds for the linearised dynamics induced by assumption (i), and from certain nonlinear bounds for what we identify as the worst terms. This value was critical for the analysis and reaching it required a delicate treatment.

3.2.3 Blow-up for the inviscid primitive equations

We consider the $2D$ inviscid primitive equations (PEs)

$$\begin{cases} u_t + u\,u_X + wu_Z + p_X = 0, \\ p_Z = 0, \\ u_X + w_Z = 0. \end{cases} \tag{3.32}$$

set on

$$\mathscr{D} = \big\{(X, Z) : -L \leq X \leq L, 0 \leq Z \leq 1\big\},$$

and with the following boundary conditions

$$\begin{aligned} &w(t, X, 0) = w(t, X, 1) = 0, \\ &\int_0^1 u(t, -L, Z)dZ = \int_0^1 u(t, L, Z)dZ = 0. \end{aligned} \tag{3.33}$$

We note that System (3.32) is very similar to the inviscid Prandtl system (3.13) for which we described singularity formation in Section 3.1.3. The key difference is that the pressure in the Prandtl system is a given function, while now for the inviscid primitive Equations (3.32) it depends on the solution. Indeed, differentiating (3.32) with respect to X, one obtains

$$u_{Xt} + u\,u_{XX} + u_X^2 + w_X u_Z + wu_{XZ} + p_{XX} = 0, \tag{3.34}$$

Then, integrating (3.34) with respect to Z over the interval $[0, 1]$, performing an integration by parts and using the conditions (3.33) enables us to solve for the pressure:

$$p_{XX} = -\int_0^1 (u^2)_{XX} dZ. \tag{3.35}$$

The fact that the pressure depends on the solution in a nonlocal way is a new difficulty, which prevents from using the characteristics as we did in Section 3.1.3 for the inviscid Prandtl system. Rather, we will try to implement a similar approach to singularity formation as the one we followed for the viscous Prandtl system (3.28) in Section 3.2.2.

The inviscid primitive equations (3.32) is derived as a formal asymptotic limit of the small aspect ratio (the ratio of the depth or the height to the horizontal length scale) from the Boussinesq system (see [7, 280]). With full viscosity, the global existence of strong solutions for the $3D$ PEs was firstly established in [48], and later in [233, 257, 271, 271].

This system is also called the hydrostatic Euler equations. Such system has a loss of horizontal derivative, making local well-posedness in Sobolev a hard problem for general initial data. Indeed, the linear and nonlinear ill-posedness in any Sobolev space have been established in [373] and in [221], respectively. On the other hand, by assuming either real analyticity or some special structures (local Rayleigh condition) on the initial data, one is able to establish the local well-posedness, see [31, 32, 194, 209, 267, 268, 308]. Moreover, it was proven that smooth solutions to the inviscid PEs, in the absence of rotation, can develop singularities in finite time. (cf. [47, 417]). Recently, it is shown in [238] that the results about ill-posedness and finite-time blowup can be extended to the case with rotation.

We consider from now on solutions u that are odd in X, i.e. $u(X) = -u(-X)$, and introduce the trace of their horizontal derivative on the central line:

$$a(t, Z) = -\partial_X u(t, 0, Z) \tag{3.36}$$

Differentiating (3.34) with respect to X, then injecting (3.35) and taking $X = 0$, one obtains the following closed evolution equation for a:

$$a_t - a^2 + \Big(\int_0^Z a(t, \widetilde{Z}) d\widetilde{Z} \Big) a_Z + 2 \int_0^1 a^2 dZ = 0, \tag{3.37a}$$

$$\int_0^1 a(t, Z) \, dZ = 0. \tag{3.37b}$$

Note that for solutions u having the form

$$u(t, X, Z) = -X a(t, Z), \text{ and thus, } u_X(t, X, Z) = -a(t, Z), \tag{3.38}$$

system (3.32) and the boundary condition (3.33) for u are equivalent to system (3.37) for a. We emphasize here that the term $2 \int_0^1 a^2 dZ$ comes from the pressure term.

We note that Equation (3.37) is precisely the Proudman-Johnson equation (3.23) that appeared after renormalizing solutions of a reduced equation for a special class of infinite energy solutions to the incompressible porous medium equation in Section 3.1.4. While our previous result of Theorem 3.10 relied on the stability of the family of steady states $\{\mu \cos\}_{\mu>0}$ to Equation (3.37) (which then corresponded to finite time blow-up solutions to the incompressible porous medium equations after a change a variables), we are here interested in singularity formation.

In [47], a family $(\psi_m)_{m>0}$ of initial data has been constructed for which the corresponding solution a to (3.37) blows up at time $t = 1$ with $a(t, Z) = (1-t)^{-1} \psi_m(Z)$. Lifting this result to a blowup for the original 2D system (3.32) is however non-trivial given the lack of well-posedness result in the class of regularity of the profiles ψ_m. In addition perturbation of these solutions seems challenging given their rigidity. On the other hand, Wong [417] has constructed explicit initial data to (3.32) that are analytic for which the corresponding solution will exhibit a singularity in finite time.

We obtained the existence and stability of a smooth blowup solution for (3.37), for which blow-up happens at the boundary $Z = 0$. It is a perturbed backward self-similar blow-up for which the nonlocal term from the pressure is negligible, which corresponds to the equation

$$a_t - a^2 + \Big(\int_0^Z a(t, \widetilde{Z}) d\widetilde{Z} \Big) a_Z = 0, \qquad z \in [0, \infty), \tag{3.39}$$

that we already encountered in the study of the Prandtl system in Section 3.2.2. The above equation admits the explicit self-similar blow-up solution

$$a(t, z) = \frac{1}{T-t} e^{-Z/\nu}$$

for any $\nu > 0$, which will be at the heart of the present blow-up dynamics. Note that actually the pressure term in (3.37) is non negligible when compared to the nonlinear term when computed on the above function, since

$$\left|\frac{1}{T-t}e^{-Z/\nu}\right|^2 \approx \frac{1}{(T-t)^2} \quad \text{and} \quad \int_0^1 \left|\frac{1}{T-t}|e^{-Z/\nu}\right|^2 dZ \approx \frac{\nu}{(T-t)^2}.$$

Thus, despite the fact that the pressure will be subleading in the present blow-up dynamics, it will nonetheless be non-negligible and its effect is to make the self-similar solution concentrate logarithmically close to the boundary $Z = 0$. This logarithmic effect shares similarities with that appearing in the stable ODE blow-up solution to the semilinear heat equation (3.2.1) as explained in Section 3.25.

Theorem 3.14 (Stable blow-up on a line [92])**.** *Consider the profile $\phi(z) = \phi_0(z) = e^{-z}$. Then there exist $\lambda_0^* > 0$ and $\delta > 0$ such that for all $0 < \lambda_0 \le \lambda_0^*$ a constant $\kappa > 0$ exists such that, if initially*

$$a_0(Z) = \frac{1}{\lambda_0}\phi\left(\frac{Z}{\nu_0}\right) + \tilde{a}_0(Z), \quad 0 \le Z \le 1,$$

with

$$\frac{2}{3\log(\lambda_0^{-1})} \le \nu_0 \le \frac{3}{2\log(\lambda_0^{-1})}, \quad \textit{and} \quad \|\tilde{a}_0\|_{C^2([0,1])} \le \kappa,$$

then there exists $T > 0$ and $C > 0$ such that the solution a to (3.37) with initial data $a(t=0) = a_0$ blows up at time $T > 0$ according to

$$a(t,Z) = \frac{1}{(T-t)}\phi\left(\frac{Z}{\nu(t)}\right) + \tilde{a}(t,Z) \qquad \textit{with} \quad \nu(t) = \frac{1}{|\log(T-t)|},$$

where $\|\tilde{a}(t,\cdot)\|_{L^\infty([0,1])} \le C(T-t)^{-1}|\log(T-t)|^{-\delta}$ for all $t \in [0,T)$.

Remark 3.15. Interestingly, we saw in the present Section and in Section 3.2.2 that under symmetry assumption, there exists a stable blow-up behaviour for both the inviscid primitive equations and the Prandtl system that involves the perturbation of a backward self-similar solution to the reduced Equation 3.39. It is to be noted that they are based on different backward self-similar solutions to Equation 3.39, namely the solution $\frac{1}{T-t}\cos^2\left(\frac{y-\mu\pi(T-t)^{-1/2}}{2\mu(T-t)^{-1/2}}\right)\mathbb{1}_{0\le y\le \frac{2\mu\pi}{(T-t)^{-1/2}}}$ for the Prandtl system, and the solution $\frac{1}{T-t}e^{-z}$ for the inviscid primitive equations.

On the one hand, the self-similar solution used for the Prandtl system cannot appear in the present inviscid primitive equations, because its expanding scale $(T-t)^{-1/2}$ is incompatible with the fact that Equation 3.37 is set on the bounded domain $[0,1]$. On the other hand, the self-similar solution used for the present inviscid primitive equations cannot appear in the Prandtl system, because it does not satisfy the Dirichlet condition at the boundary. Thus, despite the fact that singularity formation for Equations (3.30) and (3.37) involves the same reduced equation (3.39), their two stable blow-up behaviours are actually completely different due to the fact that the two equations are not set on the same domain and with the same boundary conditions.

Remark 3.16. We also constructed in [92] Theorem 1.3. non-smooth solutions with limited Hölder regularity that blow up like $a(t,Z) = \frac{1}{T-t}\phi_\beta\left(\frac{Z}{\nu(t)}\right) + \tilde{a}(t,Z)$ with $\|\tilde{a}(t)\|_{L^\infty([0,1])} \leq C(T-t)^{-1+\delta}$ and $\nu(t) = \tilde{\nu}_\infty(T-t)^\beta$. The profiles ϕ_β are no longer explicit, but they are stable from smoother perturbations. Stability of certain non-smooth blow-up patterns by smoother perturbations is a feature shared by hyperbolic equations, see for exemple [155, 261].

3.3 Type II blow-up

We have seen so far singular dynamics revolving around exact backward self-similar solutions, either for the original equation in Section 3.1 or for a reduced equation that keeps only the leading order terms of the dynamics in Section 3.2. We describe now singular solutions that do not involve exact backward self-similar solutions.

What can such other blow-up patterns look like? A singular solution that undergoes backward self-similar blow-up converges, after renormalizing at a constant rate along the corresponding symmetry group, to a backward self-similar profile. The soliton resolution, which we shall discuss more in Section 4.2, asserts that for singular solutions of general evolution partial differential equations, it is always possible to observe coherent structures after renormalizing along the invariance group of the equation. Renormalizing at a constant rate along the scaling group means, formally, to replace in the equation $\partial_t u$ by Λ, the corresponding scaling operator; this leads to the equation for backward self-similar profiles. A possibility for not having backward self-similar blow-up is that the time derivative term $\partial_t u$ is subleading, so that one observes a coherent structure by renormalizing with a rate that converges to zero; then by formally replacing $\partial_t u$ in the equation it by 0, this leads to the equation for stationary states. Another possibility is to renormalize at a constant rate but along another group of symmetry, for example the spatial translation group; then by formally replacing $\partial_t u$ by $v\cdot\nabla_x$, this leads to the equation for traveling wave solutions.

Concentration will in the present section indeed involve stationary states and traveling waves. One cannot determine the spatial scale at which they concentrate by dimensional analysis, so this corresponds again to self-similarity of the second kind. The scaling exponents will be computed thanks to an eigenvalue problem, or to a combination of dimensional analysis and PDE reduction, for the exemples below.

In reaction-diffusion equations, since for such blow-up patterns, $|\partial_t u|$ is asymptotically negligible with respect absolute value of the nonlinear terms modelling reaction effects, the size of u will not have the same asymptotic equivalent as $t\to T$ as for the reduced equation obtained by keeping only the $\partial_t u$ and these nonlinear terms. Blow-up rates, which compare the size of u with functions of $(T-t)^{-1}$, will then be different for such blow-up patterns and for backward self-similar singularity formation. This yields to an insightful a priori division of blow-up solutions depending on their blow-up rates, regarded as type I for the later on type II for the former, see Section 3.1. We will thus describe here type II blow-up patterns.

While the distinction between type I and type II blow-up is a well identified and unified notion for reaction-diffusion equations, this is not always the case for other classes of equations. Indeed, for equations with a scaling group that is of dimension greater or equal to two, it is unclear what type I blow-up rate should be unless one identifies a space in which blow-up rates of all potential backward self-similar solutions are the same, this is the case for the norm $\|\nabla u\|_{L^\infty}$ for the Burgers equation, see Section 2.1, and for the Euler equations (see e.g. [55]). Sometimes type II blow-up has been referring to solutions remaining bounded in a critical space [146] which is different than precising how a norm of u diverges as $t \to T$.

3.3.1 Type II blow-up for the Keller-Segel system

We start our discussion on type II blow-up by pursuing the study of the parabolic-elliptic Keller-Segel system

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u) \, , \\ 0 = \Delta \Phi_u + u, & \text{in } \mathbb{R}^d, \\ u(t=0) = u_0 \geq 0, \end{cases} \tag{3.40}$$

we started in Section 3.1.1.

Let us first consider the two dimensional case, which we explained in Section 3.1.1 is the mass critical case. The mass, as the Cauchy problem is well posed in $L^1(\mathbb{R}^2)$, appears as a key quantity to control the flow.

For small enough solutions, below the explicit threshold $\int_{\mathbb{R}^2} u_0 dx < 8\pi$, Dolbeault-Perthame announced in [133] that there is global existence of a solution for system (3.40) in a weak sense. This result was further completed and improved in [24], [22]. All such solutions are in fact global, and the asymptotic behavior is given by a unique self-similar profile of the system (see also [341] for radially symmetric results concerning self-similar behavior).

At the threshold $M = 8\pi$, the authors of [21] show the existence of global radially symmetric solutions to system (3.40) for initial data with finite or infinite second moment. In [23], Blanchet-Carrillo-Masmoudi proved that if such solutions to (3.40) have finite second momentum, then they concentrate in infinite time using the *free energy functional* introduced by Nagai-Senbai-Yoshida in [339]. Furthermore, they showed that the solution converges to a delta Dirac distribution at the center of mass.

The system (3.40) has a family of explicit stationary solutions of the form

$$\forall \lambda > 0, \; a \in \mathbb{R}^2, \quad U_{\lambda,a}(x) = \frac{1}{\lambda^2} U\left(\frac{x-a}{\lambda}\right) \quad \text{with} \quad U(x) = \frac{8}{(1+|x|^2)^2}. \tag{3.41}$$

These solutions have the threshold mass $M = 8\pi$ and infinite second moment. They play an important role in the description of concentration both in finite and infinite time. Ghoul-Masmoudi [195] construct concrete infinite time blowup solutions to (3.40) with threshold mass $M = 8\pi$ admitting the asymptotic dynamic as $t \to +\infty$,

$$u(x,t) \sim U_{\lambda(t)}(x) e^{-\frac{|x|^2}{2t}} \quad \text{with} \quad \lambda^2(t) \sim \frac{I}{\ln t} \quad \text{and} \quad I = \int_{\mathbb{R}^2} |x|^2 u_0(x) dx,$$

see also Davila-del Pino-Dolbeault-Musso-Wei [112] for an approach using gluing methods that leads to the same blowup rate.

Above the threshold $M > 8\pi$, concrete examples of finite time blowup solutions are constructed by Herrero-Velázquez in [232] (the scaling law found there is false but after correcting it the rest of the proof remains valid), with a further stability study in [409, 410, 410] and a different construction which proved radial stability was done by Raphäel-Schweyer [371]. The Keller-Segel system is an important model since, to the best of our knowledge, it is the first equation for which type II blow-up was constructed in [232].

Blow-up solutions were divided between type I and type II blow-up in Section 3.1.1, and type I blow-up was presented there. Importantly, it is known that in the two dimensional case any blowup solution of (3.40) is of type II (see Theorem 8.19 in [393] and Theorem 10 in [342] for such a statement).

Extending the previous works [232, 371], we obtained a refined expansion for the scale (proving the precise universal law (3.42)), the nonradial stability of the dynamics, the existence of unstable blow-up laws and removed the slightly supercritical mass restriction (i.e. that the mass had be be close to 8π). The solutions we construct are in the following function space

$$\mathscr{E} := \left\{ u : \mathbb{R}^2 \to \mathbb{R}, \quad \|u\|_{\mathscr{E}}^2 := \sum_{k=0}^{2} \int_{\mathbb{R}^2} \langle x \rangle^{\frac{3}{2}+2k} |\nabla^k u|^2 < \infty \right\}.$$

Theorem 3.17 (Stable type II blowup solution in the mass critical dimension [88])**.** *There exists a set $\mathcal{O} \subseteq \mathscr{E} \cap L^1(\mathbb{R}^2)$ of initial data u_0 such that the following holds for the associated solution to (3.40). It blows up in finite time $T = T(u_0) > 0$ according to the dynamic*

$$u(x,t) = \frac{1}{\lambda^2(t)} (U + \tilde{u}) \left(\frac{x - x^*(t)}{\lambda(t)} \right),$$

where

— (Precise law for the scale)

$$\lambda(t) = 2e^{-\frac{2+\gamma}{2}} \sqrt{T-t} e^{-\sqrt{\frac{|\ln T-t|}{2}}} (1 + o_{t\uparrow T}(1)) \tag{3.42}$$

where γ denotes the Euler-Macheroni constant.

— (Convergence of the blow-up point) *There exists $X = X(u_0) \in \mathbb{R}^2$ such that $x^*(t) \to X$ as $t \uparrow T$.*

— (Convergence to the stationary state profile)

$$\int_{\mathbb{R}^2} \left(\tilde{u}^2(t,y) + \langle y \rangle^2 |\nabla \tilde{u}(t,y)|^2 \right) dy \to 0 \quad \text{as } t \uparrow T.$$

— (Stability) *For any $u_0 \in \mathcal{O}$, there exists $\delta(u_0) > 0$ such that if $v_0 \in \mathscr{E} \cap L^1(\mathbb{R}^2)$ satisfies $\|v_0 - u_0\|_{\mathscr{E}} \le \delta(u_0)$ then $v_0 \in \mathcal{O}$ and the same conclusions hold true for the corresponding solution v.*

— (Continuity of the blow-up point and blow-up time) *For any fixed $u_0 \in \mathcal{O}$, one has $(T(v_0), X(v_0)) \to (T(u_0), X(u_0))$ as $\|v_0 - u_0\|_{\mathscr{E}} \to 0$.*

Remark 3.18. For any $\ell \in \mathbb{N}$ with $\ell \geq 2$, we also constructed other blow-up solutions, that are radial, and blow-up as in Theorem 3.17 but with $x^*(t) = 0$ and

$$\lambda(t) \sim C(u_0)(T-t)^{\frac{\ell}{2}} |\ln(T-t)|^{-\frac{\ell}{2(\ell-1)}}.$$

These solutions are however sign-changing. They are less relevant for modelling as u should be nonnegative since it is a density.

Singularity formation by the concentration of a stationary state or a standing wave has been shown to appear for numerous critical evolution equations. Numerous methods have been developped in the following pioneering works: the use of Fourier analysis in [362, 265, 266], the use of modulation and virial methods in [299, 315, 314, 376, 369, 304, 306, 305, 241], gluing methods [113].

The parabolic extension of the above modulation and virial methods was developped in [370, 372] and in particular applied to the Keller-Segel system in [371]. The method consists in constructing a suitable ansatz for the approximate solution using "tail dynamics" in which the accuracy of the approximate solution is obtained by requiring certain cancellations far away from the stationary states, and then showing its nonlinear stability by modulation, virial methods, and energy estimates.

We extended these techniques, and in particular replaced the tail dynamics by a more precise spectral analysis. This spectral approach was started in [219, 95]. Let us mention that this spectral approach can in fact be used to study also wave problems, in particular in the determination of the sharp blow-up rate for wave maps [255]. Assuming that the blow-up is radial for simplicity, in so-called self-similar parabolic variables

$$\tau = -\ln(T-t), \qquad z = \frac{x}{\sqrt{T-t}}, \qquad u(t,x) = \frac{1}{T-t} v(\tau, z),$$

the renormalized solution solves

$$\begin{cases} \partial_\tau v = \Delta v - \nabla \cdot (v \nabla \Phi_v) - \frac{1}{2}\Lambda v, \\ 0 = \Delta \Phi_v + v, \end{cases} \quad \text{in } \mathbb{R}^d,$$

where $\Lambda = z \cdot \nabla + 2$ is the scaling group generator. Assuming formally that v concentrates a stationary state,

$$v(\tau, z) = \frac{1}{\nu^2(\tau)} U\left(\frac{z}{\nu}\right) + \varepsilon(\tau, z) \tag{3.43}$$

then the linearized operator driving the evolution of ε is

$$\mathscr{L}_\nu(\varepsilon) = \Delta\varepsilon - \nabla \cdot (\varepsilon \nabla \Phi_{U_\nu}) - \nabla \cdot (U_\nu \nabla \Phi_\varepsilon) - \frac{1}{2}\Lambda\varepsilon,$$

where $-\Delta\Phi_\varepsilon = \varepsilon$. In the radial setting, the nonlocal operator $\mathscr{L}_\nu$ reduces to a local one in terms of the partial mass

$$m_f(\zeta) = \frac{1}{2\pi} \int_{|z|<\zeta} f(z) z dz,$$

as we have the relation

$$\mathscr{L}^z f(\zeta) = \frac{1}{\zeta}\partial_\zeta\Big(\mathscr{A}^\zeta m_f(\zeta)\Big),$$

where $\mathscr{A}^\zeta$ is the linear operator defined by

$$\mathscr{A}_\nu = \partial_\zeta^2 - \frac{1}{\zeta}\partial_\zeta + \frac{Q_\nu}{\zeta}\partial_\zeta + \frac{\partial_\zeta(Q_\nu)}{\zeta} - \frac{1}{2}\zeta\partial_\zeta \tag{3.44}$$

where $Q_\nu(\zeta) = \frac{4\zeta^2}{\zeta^2+\nu^2} = m_{U_\nu}$ is the partial mass of the rescaled stationary state U_ν. Hence, in the radial setting $\mathscr{L}_\nu$ and $\mathscr{A}_\nu$ share the same spectrum.

The spectral analysis of $\mathscr{A}_\nu$ had been done formally by Dejak, Lushnikov, Yu, Ovchinnikov and Sigal [117] via matched asymptotic expansions. Building on the works [219, 95], we obtained a detailed description of the spectrum.

Proposition 3.19 (Spectrum of the linearization around a concentrated ground state [89])**.** *The linear operator $\mathscr{A}_\nu : H^2(e^{-\zeta^2/4}/(\zeta U_\nu)) \to L^2(e^{-\zeta^2/4}/(\zeta U_\nu))$ is self-adjoint with compact resolvent. Moreover, given any $N \in \mathbb{N}$ and $0 < \delta \ll 1$, there exists a $\nu^* > 0$ such that the following holds for all $0 < \nu \le \nu^*$.*
(i) (Eigenvalues) *The first $N+1$ eigenvalues are given by*

$$\begin{aligned}
\lambda_{0,\nu} &= 1 + \frac{1}{2\ln\nu} - \frac{\ln 2 - \gamma - \ln\beta}{4|\ln\nu|^2} + O(\frac{1}{|\ln\nu|^3}),\\
\lambda_{1,\nu} &= \frac{1}{2\ln\nu} - \frac{\ln 2 - \gamma - 1 - \ln\beta}{4|\ln\nu|^2} + O(\frac{1}{|\ln\nu|^3}),\\
\lambda_{n,\nu} &= 1 - n + \frac{1}{2\ln\nu} + O(\frac{1}{|\ln\nu|^2}), \quad \text{for } n = 2, 3, \cdots, N,
\end{aligned}$$

where

$$\tilde\alpha_{n,\nu} = \frac{1}{2\ln\nu} + \bar\alpha_{n,\nu} \quad \text{with} \quad |\bar\alpha_{n,\nu}| + \left|\nu\partial_\nu\tilde\alpha_{n,\nu}\right| \lesssim \frac{1}{|\ln\nu|^2}.$$

In particular, we have the refinement of the first two eigenvalues with γ the Euler constant:

$$\left|\tilde\alpha_{n,\nu} - \frac{1}{2\ln\nu} - \frac{\ln 2 - \gamma - n - \ln\beta}{4|\ln\nu|^2}\right| \lesssim \frac{1}{|\ln\nu|^3}, \quad \text{for } n = 0, 1.$$

(ii) (Eigenfunctions) *There exist eigenfunctions $\phi_{n,\nu}$ satisfying very precise asymptotic expansions as $\nu \to 0$.*
(iii) (Spectral gap estimate) *For any $g \in H^1(e^{-\zeta^2/4}/(\zeta U_\nu))$ with $\langle g, \phi_{j,\nu}\rangle_{L^2(e^{-\zeta^2/4}/(\zeta U_\nu))} = 0$ for $0 \le j \le N$, one has*

$$\big\langle g, \mathscr{A}_\nu g\big\rangle_{L^2(e^{-\zeta^2/4}/(\zeta U_\nu))} \le \lambda_{N+1,\nu}\|g\|^2_{L^2(e^{-\zeta^2/4}/(\zeta U_\nu))}.$$

Proposition 3.19 explains why the scale in Theorem 3.17 can be infered formally (and then rigorously with extra work) from the spectrum of the linearized operator. Indeed, the partial mass of the perturbation ε in the decomposition (3.43) cannot be localized on the first eigenmode corresponding to the eigenvalue $\lambda_{0,\nu}$; this is an instability which corresponds to the instability of the solution by time translation, which does not affect the blow-up pattern and just changes the blow-up time. It should thus be localized (3.43) cannot be localized on the second eigenmode corresponding to the eigenvalue $\lambda_{1,\nu}$ since it is the one with the slowest decay; i.e. $m_\varepsilon(\tau,\zeta) = \alpha(\tau)\phi_{1,\nu}(\zeta)$ + subleading terms. One then has $\partial_\tau \alpha = \lambda_{1,\nu}\alpha$ at leading order. Matching the size of $\alpha\phi_{1,\nu}$ with the size of the stationary states then leads to the requirement $\alpha \approx \nu^2$, so that $\nu_\tau = \lambda_{1,\nu}\nu/2$, which gives $\nu \approx e^{-\sqrt{\tau/2}}$ and then $\lambda(t) \approx \sqrt{T-t}e^{-\sqrt{|\ln(T-t|/2}}$ for the scale in original variables.

Let us now consider type II blow-up dynamics in supercritical dimensions $d \geq 3$. We recommend [33] for a nice survey and numerical observations for singularity formation in three dimensions. Senba and Mizoguchi proved the existence of type II blow-up solutions concentrating a stationary state in large dimensions $d \geq 11$ in [333]. We will now describe another blow-up pattern, that of a collapsing sphere. This other Type II blowup solutions was formally predicted by Herrero-Medina-Velázquez [228] in the radially symmetric setting for $d = 3$.

For this blow-up pattern, A part of the mass of the solution is concentrated around a sphere that collapses to the origin. We refer to this pattern as a collapsing-sphere blow-up, in analogy with a similar blow-up that occurs for the nonlinear Schrödinger equation [318, 175, 174]. Our result is for spherically symmetric solutions for which we show the stability of the dynamics. We introduce the profile

$$W(\xi) = \frac{1}{8}\cosh^{-2}\left(\frac{\xi}{4}\right).$$

Theorem 3.20 (Existence and radial stability of a collapsing-sphere type II blowup solution in mass supercritical dimensions, [90])**.** *For any $d \geq 3$, there exists an open set of spherically symmetric functions $\mathcal{O} \subseteq L^\infty(\mathbb{R}^d)$ such that for any $u_0 \in \mathcal{O}$, the solution u to (3.40) with initial data $u(0) = u_0$ blows up with type II at time $T(u_0) > 0$ and can be decomposed as*

$$u(x,t) = \frac{M(t)}{R^{d-1}(t)\lambda(t)}\left[W\left(\frac{|x|-R(t)}{\lambda(t)}\right) + \tilde{u}(x,t)\right],$$

where

$$\lambda(t) = \frac{R(t)^{d-1}}{M(t)}, \quad M(t) = M_\infty\big(1+o_{t\uparrow T}(1)\big), \quad R(t) = c_d M_\infty^{\frac{1}{d}}(T-t)^{\frac{1}{d}}\big(1+o_{t\uparrow T}(1)\big),$$

with $c_d = (\frac{d}{2})^{\frac{1}{d}}$ and $M_\infty(u_0) > 0$, and

$$\|\tilde{u}(t)\|_{L^\infty(\mathbb{R}^d)} \to 0 \quad \text{as} \quad t \to T.$$

Moreover, the functions $T : u_0 \mapsto T(u_0)$ and $M_\infty : u_0 \mapsto M_\infty(u_0)$ are continuous on $\mathcal{O}$.

Remark 3.21. The collapsing ring is located at the distance $R(t)$ from the origin and has the width $\lambda(t)$. The total mass carried around the ring is $|\mathbb{S}^{d-1}|M(t)$.

The spatial scale in Theorem 3.20 can be obtained formally as follows. In a first approximation, the mass is concentrated on the sphere $u(t) \approx \delta_{|x|=R(t)}$ and the fact that R is far away with respect to the parabolic scale $R(t) \gg \sqrt{T-t}$ makes diffusion effects subleading, so $\partial_t u \approx -\nabla(u \cdot \nabla \Phi_u)$. In partial mass this is a Burgers-type equation with a shock at $|x| = R(t)$, and the Rankine Hugoniot condition gives $\dot{R}(t) \approx -R^{-d}$ so that R touches zero in finite time according to $R \approx (T-t)^{1/d}$. Then, in a second approximation the viscosity is not neglected, and the Burgers shock is smoothed by a Burgers traveling wave of width $\lambda(t)$. Dimensional analysis, by equaling the size of the ∂_r^2 term in the Laplacian and the nonlinearity $\nabla(u \cdot \nabla \Phi_u)$, then predicts $\lambda \approx R^{d-1} \approx (T-t)^{1-1/d}$.

3.3.2 Type II blow-up for the energy supercritical heat equation

We pursue our study of type II blow-up with the semilinear heat equation

$$(NLH) \quad \begin{cases} \partial_t u = \Delta u + |u|^{p-1}, \\ u(0) = u_0, \ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3.45}$$

where now Ω is either $\mathbb{R}^d$ or a smooth bounded open domain in $\mathbb{R}^d$, in which case we impose Dirichlet boundary conditions. We distinguished between type I and type II blow-up and described backward self-similar dynamics in Section 3.1.2, and we saw how the ODE blow-up was a pertubed backward self-similar dynamics in Section 3.2.1.

We will now describe the known type II blow-up dynamics, in which a radial stationary state Q, solving

$$\Delta Q + Q^p = 0, \tag{3.46}$$

is concentrated. The dynamics is thus similar to the type II blow-up solution of the Keller-Segel system of Theorem 3.17. The main difference is that the one of the critical Keller-Segel equation is stable, while all the solutions we will describe below are unstable, as it is expected that only the ODE blow-up is stable. Moreover, the possibility of such type II blow-up depends on the property of the stationary state, as dictated by the Joseph-Lundgren exponent below.

Since we know from Section 3.1.2 that in the energy subcritical case $p < p_c = 1 + 4/(d-2)$ all blow-up solutions are type I, let us first consider the energy critical case $p = p_c$. In $d = 4$, Schweyer constructed in [389] a radial type II blow-up solution, following the approach of critical blow-up problems of [314, 315, 369, 316, 370, 372]. In that case,

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x}{\lambda(t)}\right) + \tilde{u}(t,x), \quad \lambda(t) = \frac{C(u_0)(T-t)}{|\log T - t|^2}(1 + o(1))$$

where $\lim_{t\to T} \lambda^{2/(p-1)}(t)\|\tilde{u}(t)\|_{L^\infty(\mathbb{R}^4)} = 0$. These blow-up solutions were formally predicted by matched asymptotics in [176], and later their construction in dimensions three and five was done in [120, 119, 119], and a more complex combination of type II and ODE blow-up is involved in six dimensions [223]. In greater dimensions $d \geq 7$ such dynamics cannot occur near the stationary state due to our work [93], that we shall not describe in the present document.

In the radial energy supercritical case $p > \frac{d+2}{d-2}$ the Joseph-Lundgren exponent [249]

$$p_{JL} := \begin{cases} +\infty & \text{if } d \leq 10, \\ 1 + \frac{4}{d-4-2\sqrt{d-1}} & \text{if } d \geq 11, \end{cases} \tag{3.47}$$

dictates the existence of type II blow-up solutions. For $\frac{d+2}{d-2} < p < p_{JL}$, type II blow-up solutions do not exist [311, 331]. For $p > p_{JL}$ type II blow-up solutions are completely classified. In the unfortunately unpublished work [230] the authors predicted the existence of a countable family of solutions u_ℓ such that:

$$\|u(t)\|_{L^\infty} \sim C(u_n(0))(T-t)^{\frac{\ell}{\alpha(d,p)}\frac{2}{p-1}}, \quad \ell \in \mathbb{N}, \ \ell > \frac{\alpha}{2},$$

(α is defined in (3.49)). A rigorous proof was published later [327]. In the series of work [310, 312, 328, 330] any type II blow-up solution was proved to have one of the above blow-up rate (under some additional assumptions). At the Joseph-Lundgren exponent, type II was proved later to also occur [391, 336]. For a survey on type II blow-up and dynamics close to radial stationary states, we refer to [68].

Unfortunately, in the non-radial case, the maximum principle based techniques of [230, 327] cannot be used. It was left open whether such solutions existed for example in a bounded domain. We proved that these type II blow-up dynamics persist in the non-radial case, and can occur in a smooth, bounded and possibly non-symmetric domain Ω. We extended the techniques combining tail dynamics, modulation and energy methods for the energy supercritical Schrödinger equation [317] and our work [70] for the energy supercritical wave equation which we shall not discuss in this document, to the non-radial parabolic case. From [283], for $p > p_{JL}$ (defined in (3.47)) the radially decaying ground state Q admits the asymptotic:

$$Q(x) = \frac{c_\infty}{|x|^{\frac{2}{p-1}}} + \frac{a_1}{|x|^\gamma} + o(|x|^{-\gamma}) \ \text{ as } |x| \to +\infty, \ a_1 \neq 0,$$

with

$$\begin{aligned} c_\infty &:= \left[\frac{2}{p-1}\left(d-2-\frac{2}{p-1}\right)\right]^{\frac{1}{p-1}}, \\ \gamma &:= \frac{1}{2}(d-2-\sqrt{\triangle}), \ \triangle := (d-2)^2 - 4pc_\infty^{p-1} \ \ (\triangle > 0 \text{ iff } p > p_{JL}), \end{aligned} \tag{3.48}$$

and we define

$$\alpha := \gamma - \frac{2}{p-1}. \tag{3.49}$$

Theorem 3.22 (Existence of non radial type II blow-up for the energy supercritical heat equation [69])**.** *Let $d \geq 11$ and $p > p_{JL}$ be an odd integer, where p_{JL} is given by (3.47). Let $\Omega \subseteq \mathbb{R}^d$ be a smooth open bounded domain. For $x_0 \in \Omega$ let $\chi(x_0)$ be a smooth cut-off function around x_0 with support in Ω. Pick $\ell \in \mathbb{N}$ satisfying $2\ell > \alpha$. Then, there exists a large enough regularity exponent:*

$$s_+ = s_+(\ell) \in 2\mathbb{N}, \ s_+ \gg 1$$

such that under some non degeneracy conditions on p and d, there exists a solution u of (3.45) *with* $u_0 \in H^{s+}(\Omega)$ *(which can be chosen smooth and compactly supported) blowing up in finite time* $0 < T < +\infty$ *by concentration of the ground state at a point* $x_0' \in \Omega$ *with* $|x_0' - x_0| \leq \epsilon$*:*

$$u(t,x) = \chi_{x_0}(x) \frac{1}{\lambda(t)^{\frac{2}{p-1}}} Q\left(\frac{x - x_0'}{\lambda(t)}\right) + v \tag{3.50}$$

with: (i) Blow-up speed and asymptotic stability*:*

$$\lambda(t) = c'(u_0)(1 + o_{t\to T}(1))(T-t)^{\frac{\ell}{\alpha}}, \ \ \text{as } t \to T, \ c'(u_0) > 0,$$
$$\lim_{t\to T} \lambda(t)^{2/(p-1)} \|v(t)\|_{L^\infty} = 0.$$

(ii) Boundedness below scaling*:*

$$\limsup_{t\to T} \|u(t)\|_{H^s(\Omega)} < +\infty, \ \ \text{for all } 0 \leq s < s_c.$$

(iii) Asymptotic of the critical norm*:*

$$\|u(t)\|_{H^{s_c}(\Omega)} = c(d,p)\sqrt{\ell}\sqrt{|\log(T-t)|}(1 + o(1)), \ \ \text{as } t \to T, \ c(d,p) > 0.$$

3.4 Nonradial patterns

So far, all the blow-up behaviours described in Sections 3.1, 3.2 and 3.3 were spherically symmetric to leading order (apart for the models from fuild mechanics). The reason is twofold, spherical symmetry allows to reduce a d-dimensional problem to a "1-dimensional" one which eases the mathematical analysis, but more importantly for isotropic equations radial behaviours are expected to be stable and generic in some cases, such as the blow-up by concentration of a stable radial backward self-similar profile for supercritical focusing isotropic equations.

Truly non-radial blow-up patterns have only started to be described very recently. A first possibility for a non-radial pattern is that the scale at which the solution is concentrated depends on the spatial direction. Such phenomenon appears naturally for anisotropic equations, such as the backward self-similar solution of the inviscid Prandtl system seen in Section 3.1, and we will discuss more on anisotropic equations in Section 3.4.1 concerning the Burgers equation with transverse viscosity. It also appears for isotropic equations, as their solutions can have scales that depends on the spatial direction, and that this property persists during singularity formation. While this is of particular relevance to incompressible fluids that cannot admit spherically symmetric solutions such as the incompressible Euler Equations (see in particular the singularity formation results [154, 59, 60] and the review [134]), we will focus here our discussion on equations without transport effects in Section 3.4.2 which will be concerned with anisotropic type II blow-up. A second possibility is that around the singularity, the solution not only concentrates at one point, but at several points which collide at the blow-up time, which will be discussed in Section 3.4.3.

3.4.1 Singularity formation for the Burgers equation with transverse viscosity

The first equation we will consider for discussing non-radial blow-up patterns is the Burgers equation with transverse viscosity

$$\begin{cases} \partial_t u + u\partial_x u - \partial_{yy} u = 0, \quad (x,y) \in \mathbb{R}^2, \\ u_{t=0} = u_0, \end{cases} \tag{3.51}$$

for $u : [0,T) \times \mathbb{R}^2 \to \mathbb{R}$. This model is on the one hand a simplification of the two dimensional Prandtl system (3.28) studied in Section 3.2.2 by discarding the normal transport term $v\partial_y u$ and by removing the boundary and the boundary conditions, and on the other hand a complexification of the Burgers equation (2.1) by adding a transerse viscosity. Note that adding a streamwise viscosity results in the standard viscous Burgers equation (2.8) discussed in Section 2.1, which is very different from Equation (3.51) since in particular all solutions were global. Equation 3.51 is rather a mathematical toy model, without a clear physical application.

We aim at understanding precisely the consequence of the additional transverse viscosity effect, on a blow-up dynamics that it does not prevent. Equation (3.51) is clearly anisotropic, with the Burgers transport effect happening Moreover, this new effect changes the nature of the equation which is of a mixed hyperbolic/parabolic type. .

The existence of smooth enough solutions to (3.51) follows from classical arguments, and there holds the following blow-up criterion. The solution u blows up at time $T > 0$ if and only if

$$\limsup_{t\uparrow T} \|\partial_x u\|_{L^\infty(\mathbb{R}^2)} = +\infty.$$

The existence of global kinetic solutions $u \in L^\infty([0,\infty), L^1(\mathbb{R}^2))$ has been showed for such type of non-isotropic degenerate parabolic-hyperbolic equations in [58], following the framework of [286]. Before stating the main theorem, let us give the structure of the singularities of Burgers equation, and of the ones of the parabolic system encoding the effects of the transverse viscosity.

How do singular solutions to Equation (3.51) form? We know from Section 2.1 that the Burgers term alone produces shocks along the x direction by concentrating self-similar profiles. Without the viscosity term in (3.51), each line $\{y = \text{constant}\}$ would then undergo such shock formation, which would be described by having a self-similar profile along the x direction that concentrates at a y-dependent scale near a y-dependent point.

A prototype class of solutions for understanding the role of a y-dependent point of concentration are solutions of (3.51) of the form $u = V(t, x - \epsilon y)$. In this case the function V solves the viscous Burgers equation $V_t + VV_x - \epsilon^2 V_{xx} = 0$. Hence from Section 2.1 such solutions are global and no singularity form. This hints to the fact that in order to have a singular solution, the point for each y at which $\{x \mapsto u(t,x,y)\}$ is concentrated should actually be y independent to leading order. This leads to the following further simplifying symmetry assumption:

$$u \text{ is odd in } x.$$

For such solutions, the behaviour on the transverse axis $\{x = 0\}$ can be encoded by a closed system as follows. Indeed, assume $\partial_x^j u_0(0, y) = 0$ for all $y \in \mathbb{R}$ for $2 \leq j \leq 2i$ for some integer $i \in \mathbb{N}$ (possibly $i = 1$ in which case no assumption is made thanks to the oddness of u with respect to x). This remains true for later times and the trace of the derivatives

$$\xi(t, y) := -\partial_x u(t, 0, y) \text{ and } \zeta(t, y) = \partial_x^{2i+1} u(t, 0, y)$$

solve the parabolic system

$$\begin{cases} \xi_t - \xi^2 - \partial_{yy}\xi = 0, \\ \zeta_t - (2i+2)\xi\zeta - \partial_{yy}\zeta = 0, \end{cases} \qquad y \in \mathbb{R}. \tag{3.52}$$

Interestingly, the first equation above for ξ is the nonlinear heat equation we discussed extensively in Sections 3.1.2 and 3.2.1. In particular, in the present one-dimensional case only the ODE blow-up presented in Section 3.2.1 can happen, and ODE blow-up solutions are for exemple constructed in Theorem 3.11. When ξ undergoes ODE blow-up, the solution ζ to the second equation, which is a linearly forced heat equation, may also blow-up.

Proposition 3.23 ([85]). *Consider $i = 1$ in System 3.52. There exist solutions (ξ, ζ) to the system (3.52) such that ξ blows up with ODE blow-up as described in Theorem 3.11 according to the two possible cases, and ζ blows up with, for some parameter $b > 0$, in the first case*

$$\zeta(t, y) = \frac{b}{(T-t)^4} \frac{1}{\left(1 + \frac{y^2}{8(T-t)|\log(T-t)|}\right)^4} + \tilde{\zeta},$$

where the remainder $\tilde{\zeta}$ satisfies for $0 \leq j \leq J$ for some constant $C > 0$:

$$|\partial_y^j \tilde{\zeta}| \leq \frac{C}{(T-t)^4 |\log(T-t)|^{\frac{1}{4}}} \frac{1}{\left(1 + \frac{y^2}{(T-t)|\log(T-t)|}\right)^{3+\frac{3}{4}}} \frac{1}{\left(\sqrt{(T-t)|\log(T-t)|} + |y|\right)^j}.$$

and in the second case

$$\zeta = \frac{b}{(T - t + ay^{2k})^4} + \tilde{\zeta},$$

where the remainder $\tilde{\zeta}$ satisfy for $j = 0, ..., J$ for some constant $C > 0$:

$$|\partial_y^j \tilde{\zeta}| \leq C \frac{1}{\left((T-t)^{\frac{1}{2k}} + |y|\right)^{8k+j}} \left((T-t)^{\frac{1}{2k}} + |y|\right)^{\frac{1}{2}}.$$

Let us mention that the proof of the construction for ζ is only given in the second case in [85], but that the method adapts to show the construction in the first case as well. The above statement is for $i = 1$ but can be easily adapted (along with its proof) to any i.

Having constructed precisely the first and third order derivatives on the axis $\{x = 0\}$ thanks to the above Proposition, this determines a (t, y) dependent x scale $\lambda(t, y)$, given formally by $\lambda(t, y) = \sqrt{\frac{|\partial_x^3 u(t,0,y)|}{|\partial_x u(t,0,y)|}} = \sqrt{\frac{\zeta(t,y)}{\xi(t,y)}}$. This can be made rigorous by the following construction, the scale $\lambda(t, y)$ being that of a self-similar profile for the Burgers equation that concentrates in the x direction.

Theorem 3.24 (Blow-up solutions with logarithmic transverse scale [85]). *For any $i \in \mathbb{N}^*$ and $\mu > 0$, there exist solution to (3.51) blowing up at time T with*

$$u(t,x,y) = \mu\lambda^{\frac{1}{1+2i}}(t,y)\Psi_i\left(\frac{x}{\mu\lambda(t,y)}\right) + \tilde{u}(t,x,y)$$

where Ψ_i is the backward self-similar blow-up profile of the Burgers equation given by Proposition 2.1, and the transverse scale is

$$\lambda(t,y) = \left(T - t + \frac{y^2}{8|\log(T-t)|}\right)^{1+\frac{1}{2i}},$$

and one has the convergence in self-similar variables (X,Z)

$$(T-t)^{-\frac{1}{2i}}u\left((T-t)^{1+\frac{1}{2i}}X, \sqrt{(T-t)|\log(T-t)|}Z\right) \to \mu(1+Z^2/8)^{\frac{1}{2i}}\Psi_i\left(\frac{X}{\mu(1+Z^2/8)^{1+\frac{1}{2i}}}\right)$$

in C^1 on compacts sets and for some constants $C > 0$ the remainder satisfies

$$\|\partial_x \tilde{u}\|_{L^\infty} \le C(T-t)^{-1}|\log(T-t)|^{-\frac{1}{4}}.$$

Theorem 3.25 (Blow-up solutions with flat transverse scale [85]). *For any $k, i \in \mathbb{N}^*$, $k \ge 2$, $\nu, \mu > 0$, there exists $T^* > 0$, such that for any $0 < T < T^*$ there exists a solution u to (3.51) that is odd in x and even in y blowing up at time T with*

$$u(t,x,y) = \mu\lambda^{\frac{1}{1+2i}}(t,y)\Psi_i\left(\frac{x}{\mu\lambda(t,y)}\right) + \tilde{u}(t,x,y)$$

where $\lambda(t,y) = (T-t+(y/\nu)^{2k})^{1+\frac{1}{2i}}$ and one has the convergence in self-similar variables (X,Z)

$$(T-t)^{-\frac{1}{2i}}u\left((T-t)^{1+\frac{1}{2i}}X, (T-t)^{\frac{1}{2k}}Z\right) \to b^{-1}(1+aZ^{2k})^{\frac{1}{2}}\Psi_i\left(b\frac{X}{(1+aZ^{2k})^{\frac{3}{2}}}\right)$$

in C^1 on compact sets and for some constants $C, \eta > 0$ the remainder satisfies

$$\|\partial_x \tilde{u}\|_{L^\infty} \le C(T-t)^{-1+\eta}.$$

We see that here a wide range of different scaling laws in the x and y variables are possible. At the blow-up point $(x,y) = (0,0)$, in the first theorem the y-scale is $\sqrt{(T-t)|\log(T-t)|}$ and the x-scale is $\mu(T-t)^{1+1/(2i)}$ while in the second theorem the y-scale is $\nu(T-t)^{1-1/(2k)}$ and the x-scale is $\mu(T-t)^{1+1/(2i)}$.

3.4.2 Anisotropic blow-up for the semilinear heat equation

We pursue the analysis of blow-up solutions to the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u_{|t=0} = u_0, \end{cases} \tag{3.53}$$

for which we consider now non-radial blow-ups. Blow-up solutions were divided between type I and type II in Section 3.1.2 which focused on type I blow-up solutions, and where the definition of blow-up points was given. The blow-up set of a solution is defined as the set of all blow-up points. A particular type of type I blow-up is the ODE blow-up which was studied in Section 3.2.1, and Section 3.3.2 presented known examples of type II blow-up solutions. For the exemples of known solutions presented in all these Sections 3.1.2, 3.2.1, 3.3.2, the blow-up set consists of an isolated blow-up point, where the singularity is radial to leading order.

A first possibility for having a non-radial singularity is that the blow-up set is a submanifold of $\mathbb{R}^d$ instead of an isolated point. Up to now, all known examples of such singular solutions remain highly symmetric, and the blow-up set is either a sphere in $\mathbb{R}^n$, or a sphere in $\mathbb{R}^d \subseteq \mathbb{R}^n$ for $d < n$. We refer to [135] for a solution blowing up on the sphere for the semilinear heat equation (3.53), to [237, 44] for a solution blowing up on a ring for the Keller-Segel system (3.1), and to [368] for the first existence of a solution blowing up on a sphere which was done for the nonlinear Schrödinger equation. A long-standing open problem in the field of singularity formation is to construct a solution that blows up along a submanifold that is not a sphere in the same or a lower dimension, for an equation with infinite speed of propagation (for the wave equation which has finite speed of propagation, this was done in [254, 53]).

If solutions of the semilinear heat equation blowing up on a general set exist, their blow-up set has to be regular. Indeed, under some assumptions its Hausdorff dimension is $d-1$ [408], and it is C^2 [419], and we refer to [324, 197] for more recent results.

From the exemple described for the Burgers equation with transverse viscosity in Section 3.4.1, when a solution blows up on a general blow-up set of dimension $n-d$, the intuition is that it should concentrate a d dimensional blow-up profile along the transverse directions, whose scale and other modulation parameters depend on the point of the blow-up set. Thus, the prototypical exemple it the following canonical situation: for a space dimension

$$n = d + 1,$$

any d dimensional blow-up solution $U(t, x_1, ..., x_d)$ can be lifted to a n-dimensional blow-up solution by choosing $u(t, x_1, ..., x_{d+1}) = U(t, x_1, ..., x_d)$. However, in such case the solution is of infinite energy and is not localized. So the natural question is: can such blow-up solution be localized along the z direction? How does the solution reconnect from a singular behaviour at $z = 0$ to a more regular behaviour for larger values of $|z|$?

We constructed an exemple of such a solution, by lifting and localizing a lower d-dimensional type II blow-up as described in Theorem 3.22. After renormalization, the solutions are very elongated along the e_{d+1} direction, and eventually reconnect $U(t, x)$ to a decreasing profile in the z direction. The associated reconnection profile is *universal and rigid*. This eventually produces a *blow-up at an isolated point and not a line singularity* but with an elongated pancake like profile in self similar variables and new blow up rates.

Theorem 3.26 (Existence and finite codimensional stability of anistropic type II blow-up [95])**.** *Let*

$\alpha = \alpha(d,p)$, $\Delta = \Delta(d,p)$ be the super critical numbers given by (3.49)-(3.48), and assume:

$$d \geq 11, \ \ p \geq \max\{3, p_{JL}(d)\}, \ \ \sqrt{\Delta} > 2, \ \ \alpha \notin 2\mathbb{N}$$

where the Joseph-Lundgren exponent was defined in (3.47). Pick

$$\ell \in \mathbb{N}^* \ \ \textit{with} \ \ \ell > \frac{\alpha}{2}.$$

Then there exists a finite codimensional set of initial data $u_0 \in \mathscr{C}_c^\infty(\mathbb{R}^{d+1}, \mathbb{R})$ with cylindrical symmetry, i.e. $u_0(x,z) = u_0(|x|,z)$ such that the corresponding solution $u(t,x)$ to (3.53) with $n = d+1$ blows up in finite time $0 < T < +\infty$ with the following asymptotics. The solution admits on $[0,T)$ in self similar variables

$$r = \frac{|(x_1,...,x_d)|}{\sqrt{T-t}}, \ \ z = \frac{x_{d+1}}{\sqrt{T-t}}, \ \ u(t,x_1,...,x_d,x_{d+1}) = \frac{1}{(T-t)^{\frac{1}{p-1}}} v(t,r,z)$$

a decomposition

$$v(t,r,z) = \frac{1}{D(t,z)^{\frac{2}{p-1}}} Q\left(\frac{r}{D(t,z)}\right) + V(t,r,z)$$

where Q denotes the d-dimensional radially symmetric stationnary state (3.46), and with the following sharp description:

1. Computation of the reconnection*: there holds*

$$D(t,z) = \sqrt{b(t)}(1 + a(t)P_{2\ell}(z))^{\frac{1}{\alpha}}$$

where $P_{2\ell}(z)$ is the 2ℓ-th one dimensional Legendre polynomial given by

$$P_m(z) = c_m \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-2k)!}(-1)^k z^{m-2k},$$

and $(a,b) \in \mathscr{C}^1([0,T), \mathbb{R}_+^*)$ with the sharp asymptotics near blow up time:

$$b(t) = b^*(1 + o_{t\to T}(1))(T-t)^{\frac{2\ell-\alpha}{\alpha}}, \ \ 0 < b^*(u_0),$$
$$a(t) = a^*(1 + o_{t\to T}(1)), \ \ 0 < a^*(u_0) \ll 1.$$

2. Stability and blow up speed*:*

$$\lim_{t\to T} b^{\frac{1}{p-1}} \|V(t,\cdot)\|_{L^\infty} = 0$$

and

$$\|u(t,\cdot)\|_{L^\infty} = \frac{c(u_0)(1+o(1))}{(T-t)^{\frac{2}{p-1}\frac{\ell}{\alpha}}}, \ \ c(u_0) > 0.$$

3. Isolatedness of the blow-up point*: The origin is the only blow-up point.*

A related question is to construct such a blow-up dynamics, where the d-dimensional blow-up is a type I non-ODE blow-up as described in Theorem (3.5). This was achieved in [319], and the transverse scale is

$$D(t,z) \sim \sqrt{1+b(t)z^2}, \ \ b(t) \sim \sqrt{\log(T-t)}$$

The case of anisotropic ODE blow-up at an isolated point is more understood, and we refer to [325] for references, and where a cross-shaped profile is obtained.

The blow-up solution of Theorem 3.26 was constructed using spectral analysis, thanks to a detailed description of the linearized operator in self-similar parabolic variables as was obtained in Proposition 3.19 for the mass critical Keller-Segel equation. This shows that a general spectral approach is possible for both type II blow-up in critical or supercritical equations, but while in both cases we face a singularly perturbed spectral problem, in the critical case of Proposition 3.19 the perturbed spectrum really differs from the unperturbed spectrum, see [89].

We finally remark that the method to prove Theorem 3.26 clearly designs an iteration process for this dimensional reduction procedure.

3.4.3 Non-radial collision of collapsing stationary states for the Keller-Segel system

Let us consider again the Keller-Segel equation in two dimensions

$$\begin{cases} \partial_t u = \Delta u - \nabla \cdot (u \nabla \Phi_u), \\ 0 = \Delta \Phi_u + u, \\ u(t=0) = u_0 \geq 0, \end{cases} \quad \text{in } \mathbb{R}^2. \tag{3.54}$$

We explained in Section 3.1.1 the difference between type I and type II blow-up and studied type I blow-up in mass supercritical dimensions $d \geq 3$. We then saw in Section 3.3.1 that in the mass critical two dimensional case only type II blow-up solutions exist, and gave the exemple of a collapsing stationary state in Theorem 3.17. This dynamics was radial to leading order around the blow-up point.

What can non-radial singular dynamics of (3.54) look like? The blow-up set $\mathcal{S}$ of any singular solution has to consist of finitely many points $\#\mathcal{S} < \infty$, as was shown by Suzuki [402] (Theorem 3.3), and he moreover showed the following mass quantization,

$$u(x,T) \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0} dx + f(x)dx, \quad m(x_0) \in 8\pi\mathbb{N}, \tag{3.55}$$

where the above weak convergence is in L^1, and where $f \in L^1(\mathbb{R}^2)$. We recall that 8π is the mass of the radial stationary state of the equation, see Section 3.3.1. Solutions of Theorem 3.17 provide an example of blow-up solutions where in (3.55) we have $\mathcal{S}\{x_0\}$ and $m(x_0) = 8\pi$. Recently, a solution blowing up at k different points, with the same dynamics as in Theorem 3.17 around each point, was constructed in [43], providing exemples of blow-up solutions where in (3.55) we have $\mathcal{S}\{x_1, ..., x_k\}$ and $m(x_i) = 8\pi$ for $i = 1, ..., k$.

Whether blow-up solutions with $m(x_0) = 16\pi$ in (3.55) existed was an important open question that we answered positively recently. What could the dynamics in this case look like? Due to the finiteness of the blow-up set, the fact that the equation (3.54) is global in one dimension, and the conservation of mass $\int u(t)dx = \int u_0 dx$, an anisotropic behaviour similar to that of Section 3.4.2 seems unlikely. Rather, recalling that the single mass blow-up solutions of Theorem 3.17 were concentrated around a point $x^*(t)$ that converges $x^*(t) \to x^*(T)$ as $t \to T$, a possibility would be that two such solutions that are concentrated around $x_+(t)$ and $x_-(t)$ would collapse at the same point $x_\pm(t) \to x * (T)$ as $t \to T$. It is expected to be impossible that $x_+ = x_-$, because in this case the most concentrated stationary state would collapse first in finite time, resulting in a single 8π blow-up. This is at least true indeed impossible in the radial case, as Mizoguchi proved that all radial blow-up solutions to (3.54) blow up as described in Theorem 3.17, see [332].

Hence, this collision has to be genuinely non-radial $x_+ \neq x_-$. A formal approximate solution was given in [392]. It was only obtained through formal matched asymptotic expansions, and we were able to give a rigorous construction by a different approach. We introduce the following function space

$$\mathscr{E} := \left\{ u : \mathbb{R}^2 \to \mathbb{R}, \quad \|u\|_{\mathscr{E}}^2 := \sum_{k=0}^{2} \int_{\mathbb{R}^2} \langle x \rangle^{\frac{3}{2}+2k} |\nabla^k u|^2 < \infty \right\}.$$

Theorem 3.27 (A blow-up solution in the mass critical case with multiple collapses [91])**.** *There exists a smooth initial data $u_0 \in L^1 \cap \mathscr{E}$ with $u_0 \geq 0$ such that the corresponding solution u to* (3.54) *blows up in finite time $T = T(u_0) > 0$ and admits the decomposition*

$$u(x,t) = \frac{1}{\lambda^2(t)} U\Big(\frac{x - x_0(t)}{\lambda(t)}\Big) + \frac{1}{\lambda^2(t)} U\Big(\frac{x + x_0(t)}{\lambda(t)}\Big) + \tilde{u}(x,t),$$

where as $t \to T$,

$$\lambda(t) = \sqrt{T-t}\, e^{-\sqrt{\gamma_1 |\ln(T-t)|}} C_1(t), \qquad \frac{x_0(t)}{\sqrt{T-t}} \to \big(2, 0\big), \qquad \lambda^2(t) \|\tilde{u}(t)\|_{L^\infty} \to 0, \tag{3.56}$$

for $\gamma_1 > 0$ satisfying $\gamma_1 \neq 1/2$ and $C_1(t)$ that satisfies $c_* \leq C_1(t) \leq c^*$ for all $t \in [0,T)$ for some constants $c^* > c_* > 0$. Moreover, we have

$$\lim_{\delta \downarrow 0} \lim_{t \uparrow T} \int_{|x|<\delta} u(t,x) dx = 16\pi.$$

Interestingly, the scale (3.56) differs from the one of the solutions of Theorem 3.17 that concentrate a single stationnary state. Hence, the collision of the two stationary states in Theorem 3.27 produces a new blow-up rate in the non-radial case.

Only a few examples of non-radial collisions were known before Theorem 3.27. Solutions concentrating several backward self-similar solutions to the one-dimensional focusing semilinear wave equation were constructed in [106], producing a corner-shaped singularity. A pyramid-shaped singularity was constructed in two dimensions in [323]. Then, the only known example for a critical equation was the collision of solitons for the mass critical Schrödinger equation [307]. However, two specificities made the construction of [307] very different: first a symmetry called the pseudo-conformal symmetry allowed to make the construction of such blow-up solution equivalent to the construction of a global in time multisoliton solution, and second by time reversal such solution could be constructed from $t = +\infty$ allowing to prescribe a vanishing radiation.

In order to construct the solution of Theorem 3.27, we had to give a completely different proof. Namely, we extended the radial spectral analysis of Proposition 3.19 the non-radial case around two solitons in order to produce an approximate blow-up solution, and then we obtained its stability by introducing a matched scalar product which matched the scalar products associated to the linearized dynamics near each stationary states with the scalar product associated to the linearized dynamics away in the parabolic neighbourhood from the singularity.

Chapter 4

Dynamics around solitons for dispersive equations

The previous Section 3 was concerned with singular solutions for equations with advection, reaction or diffusion terms. In the present section we will consider mostly global in time solutions $T = \infty$, for dispersive equations of Schrödinger or wave type. Similar solutions as in Section 3 arise: stationary states, traveling waves, etc. However, their stability analysis is more delicate, since on the one hand such equations do not have such a strong stabilizing effect like the smoothing effect of parabolic equations, and that dissipation and comparison principle-based estimates are not available like for transport or parabolic equations. Similarly, less tools are available for the description of general solutions in the large, since global Lyapunov functionals based on energy dissipation or intersection numbers are not available.

The first section 4.1 will be concerned with asymptotic stability of traveling waves and equilibria in low dimensions and possibly with non-trivial behaviour at spatial infinity. The second section 4.2 will address the description of unprepared initial data. It will first motivate the soliton resolution conjecture based on the examples seen in this manuscript and on a historical perspective, which actually encompasses both equations with advection, reaction or diffusion terms as seen in Section 4 and dispersive equations as in the present section. It will then focus on such a soliton resolution result for the energy critical wave equation, especially on the six dimensional case on which we proved the continuous version of this conjecture.

4.1 Stability of steady states and traveling waves

In this section we address the global in time asymptotic stability of certain steady states and traveling waves for Schrödinger-type equations. The linear stability relies on a decay produced by the dispersive properties of the linearized flow. This linear decay can affected by three features: first the dimension in which the equation is consider since the higher the dimension the greater linear waves decay (since in a sense there is "more" spatial directions in which they can spread), second the behaviour at spatial infinity of the soliton since when non-zero it can change the group velocity of linear waves and alter their decay, third the presence of internal modes which are eigenfunctions of the linearized operator and prevent linear decay. Due to these facts, some problems of asymptotic stability in low dimensions, with non-zero background at spatial infinity, and internal modes, remain open to this day, see for exemple the survey [259]. We shall only present problems involving the first two features.

4.1.1 Traveling waves for the one dimensional nonlinear Schrödinger equation

We consider the one dimensional nonlinear Schrödinger equation

$$\begin{cases} \partial_t u - \partial_x^2 u - F'(|u|^2)u = 0, \\ u(t=0) = u_0, \end{cases} \qquad x \in \mathbb{R}, \tag{4.1}$$

where F' is the derivative of a given function F that will only be assumed to be smooth and to have a non-degenerate local minimum at zero.

Standing waves of the type

$$u(t) = e^{-it\omega}\Phi_\omega$$

are given by solutions of

$$\partial_x^2 \Phi_\omega - \omega \Phi_\omega + F'(\Phi_\omega^2)\Phi_\omega = 0.$$

Under our assumptions on F, there exists a unique solution of the above equation on an interval $\omega \in (0, \omega^*)$, for some $\omega^* > 0$. Furthermore, Φ_ω is even, positive, decreasing on $x > 0$, and exponentially decreasing at infinity, along with its derivatives (see [19]). Such standing waves are often called ground states, while sign changing solutions are called excited states. Equation (4.1) has Galilean, phase and translation symmetries, namely for $p, \gamma, y \in \mathbb{R}$,

$$e^{i(px+p^2t+\gamma)}u(t, x + 2pt - y)$$

is again a solution. In particular, standing waves generate the family of traveling waves

$$u(t, x) = e^{i(px+(p^2-\omega)t+\gamma)}\Phi_\omega(x + 2pt - y). \tag{4.2}$$

Under which conditions are these traveling waves stable? Let us first investigate linear stability. Recasting (4.1) as a vector equation for $(u, \bar{u})^\top$, the linearized operator around the soliton Φ is

$$\mathscr{H}_\omega = \begin{pmatrix} -\partial_x^2 + \omega & 0 \\ 0 & \partial_x^2 - \omega \end{pmatrix} - \begin{pmatrix} V_+ & V_- \\ -V_- & -V_+ \end{pmatrix}$$

where $V_- = F''(\Phi^2)\Phi^2$ and $V_+ = F'(\Phi^2) + V_-$. The spectral properties of $\mathscr{H}_\omega$ have been studied in [211, 413, 45, 156, 108, 56, 264] and references therein.

Under the current assumptions, we have that up to spatial translation Φ is even, positive, with $\Phi' < 0$ on $(0, \infty)$, and decays exponentially fast. The essential spectrum of $\mathscr{H}$ equals $(-\infty, -\omega] \cup$

$[\omega, \infty)$, while the rest of the spectrum consists of finitely many eigenvalues of finite algebraic multiplicity inside $\mathbb{R} \cup i\mathbb{R}$. Under the additional condition of orbital stability (4.3) we shall explain shortly, eigenvalues are in fact necessarily real. Eigenvalues inside $(-\infty, -\omega] \cup [\omega, \infty)$ are termed as embedded eigenvalues. We recently proved they in fact cannot exist for ground state traveling waves.

Theorem 4.1 ([83]). *Under the present assumptions, there is no* $\lambda \in (-\infty, -\omega] \cup [\omega, \infty)$ *and* $\varphi \in L^2 \backslash \{0\}$ *such that* $\mathscr{H}_\omega \varphi = \lambda \varphi$.

Hence, in the present case eigenvalues can only be located in $(-\omega, \omega)$. 0 is always an eigenvalue as the kernel of $\mathscr{H}$ is never null, since the set of solitons (4.2) is of dimension four. Indeed, by differentiating with respect to each parameter, we obtain four functions

$$\Xi_0 = \begin{pmatrix} \Phi \\ -\Phi \end{pmatrix}, \quad \Xi_1 = \begin{pmatrix} \partial_\omega \Phi \\ \partial_\omega \Phi \end{pmatrix}, \quad \Xi_2 = \begin{pmatrix} \partial_x \Phi \\ \partial_x \Phi \end{pmatrix}, \quad \Xi_3 = \begin{pmatrix} x\Phi \\ -x\Phi \end{pmatrix}$$

that are always elements of the generalised kernel as

$$\mathscr{H}\Xi_0 = 0, \quad \mathscr{H}\Xi_1 = -\Xi_0, \quad \mathscr{H}\Xi_2 = 0, \quad \mathscr{H}\Xi_3 = -2\Xi_2.$$

Other nonzero eigenvalues in $(-\omega, 0) \cup (0, \omega)$ are called *internal modes*. They are a natural obstruction to linear asymptotic stability since they generate periodic modes that do not decay over time. Finally, the tip of the essential spectrum might exhibit *edge resonances*; they are absent if

$$\text{there is no nonzero bounded solution } f \text{ to } \mathscr{H} f = \omega f.$$

The absence of edge resonance favours local linear decay, while their presence induces a very slow local decay in which case nonlinear stability remains a challenging open problem.

A first notion of nonlinear stability is that of orbital stability, which means that the solution remains close to the full set of traveling waves. By the general theory of Grillakis-Shatah-Strauss [212, 213], see also [52, 396, 413, 414], it is completely understood: it holds if and only if

$$c_\omega = \frac{d}{d\omega} \int |\Phi_\omega|^2 \, dx > 0 \tag{4.3}$$

(leaving aside the limiting case $c_\omega = 0$). For homogeneous power nonlinearities, more can be said when $c_\omega < 0$: namely, there exists initial states arbitrarily close to Φ_ω which lead to finite-time blow up [18].

A second notion of nonlinear stability is that of asymptotic stability, which requires that the solution eventually converges to a fixed traveling wave. Asymptotic stability was first obtained by Buslaev and Perelman [45] for functions F vanishing to order ≥ 5 at the origin, in the absence of a resonance at the edge of the essential spectrum, and in the absence of internal modes. The next important development was due to Krieger and Schlag [264], who were able to construct finite-codimension stable manifolds around the soliton in the monic supercritical case $F(x) = |x|^p, p > 3$. This improvement relied on the use of Strichartz estimates and sharper dispersive estimates; and an additional difficulty occurs because of unstable modes. In the case of *small* solitons arising from an exterior potential, asymptotic stability was obtained by Mizumachi [334] and Chen [57]. Recently, Martel was able via virial methods to prove asymptotic stability (more precisely, what is now termed local asymptotic stability) for $F(x) = x^2 - x^3$ and $F(x) = x^2 + x^3$, handling in the later case the presence of internal modes [297, 298]. In the case $F(z) = |z|^2$, asymptotic stability can be proved by taking advantage of the completely integrable structure: this was achieved by Cuccagna and Pelinovsky [109]. The stability of traveling waves for the one dimensional Schrödinger equation is linked to similar stability problems for other one dimensional dispersive equations, and we refer to the recent review [189] for a detailed discussion of the mathematical connexion.

In the obtention of the asymptotic stability, is it possible to refine the asymptotic expansion and to give a leading order description of the remainder? It is instructive to consider first the case of the stability of the 0 solution. When the non-linearity is cubic $F(x) = cx^2 + O(|x|^3)$, the slow decay of linear solutions renders nonlinear effects non-perturbative, and small localized solutions undergo modified scattering

$$u(t) \sim \mathscr{F}^{-1}\left(e^{i\left(\xi^2 t - \frac{L}{2}|\widehat{f}|^2 \ln t\right)}\widehat{f}\right) \qquad \text{as } t \to \infty,$$

for a profile f and $L = F''(0)$, see [226, 239, 285, 337]. Under the absence of internal modes and of edge resonance, we were able to show asymptotic stability, and that the radiation remainder undergoes modified scattering.

Theorem 4.2 ([80]). *Assume that for* $\omega \in (0, \omega^*)$*:*

- *The orbital stability condition* $c_\omega = \frac{d}{d\omega}\int |\Phi_\omega|^2\, dx > 0$ *is satisfied.*
- *The only eigenvalue of* $\mathscr{H}$ *is* 0*, i.e. there are no internal modes.*
- $\mathscr{H}$ *does not have an edge resonance.*

Then, for each $\omega_0 \in (0, \omega^*)$*, there exists* $\epsilon_0 > 0$ *such that if the data*

$$v_0 = \Phi_{\omega_0} + u_0$$

are sufficiently close to the soliton:

$$\|u_0\|_{H^1(\mathbb{R})} + \|\langle x \rangle u_0\|_{L^2(\mathbb{R})} = \epsilon < \epsilon_0,$$

then the solution to (4.1) *is global and*

(i) Asymptotic stability. *The solution* v *can be written as*

$$v(t,x) = e^{i(px+\gamma)}\Phi_\omega(x-y) + u(t,x)$$

where the parameters $\omega(t), \gamma(t), p(t), y(t) \in \mathbb{R}$ *enjoy the asymptotics*

$$\begin{aligned}
\omega(t) &= \underline{\omega} + O(\epsilon\langle t\rangle^{-1-\nu}), & p(t) &= \underline{p} + O(\epsilon\langle t\rangle^{-1-\nu}) \\
\gamma(t) &= (\underline{p}^2 - \underline{\omega})t + \underline{\gamma} + O(\epsilon\langle t\rangle^{-\nu}), & y(t) &= -2\underline{p}t + \underline{y} + O(\epsilon\langle t\rangle^{-\nu}),
\end{aligned}$$

for constants $\underline{\omega}, \underline{\gamma}, \underline{p}, \underline{y}$ and some $\nu > 0$, and where the radiation u satisfies $\|u\|_{H^1} \lesssim \epsilon$ and disperses:

$$\|u(t,\cdot)\|_{L^\infty} \lesssim \epsilon \langle t \rangle^{-1/2}.$$

(ii) Modified scattering for the radiation. *There exists a profile function f with $\|f\|_{H^1} + \|\widehat{f}\|_{L^\infty} \lesssim \epsilon$ such that*

$$u = \mathscr{F}^{-1}\left(e^{i\left(\xi^2 t - \frac{L}{2}|\widehat{f}|^2 \ln t\right)} \widehat{f}\right) + \widetilde{u},$$

with

$$\|\widetilde{u}(t,\cdot)\|_{L^2} \to 0 \qquad \text{as } t \to \infty.$$

(iii) Continuity. *The dependence of the asymptotic parameters and profile function on the initial data $u_0 \mapsto (\omega, \gamma, p, y, f)$ is continuous from $H^1 \cap L^{2,1}$ into $\mathbb{R}^4 \times H^1 \cap \mathscr{F}^{-1} L^\infty$.*

When are the spectral assumptions of the above theorem satisfied? They are for instance verified in the case of the cubic-quintic nonlinear Schrödinger equation, corresponding to $F(x) = x^3 - x^5$, see [361]. Rialland proved the spectral assumptions of our Theorem hold for more general nonlinearities which have a focusing cubic term and a defocusing higher order term [374], and proved local asymptotic stability by means of a virial method. The spectral analysis which is needed to understand whether the conditions hold or not is quite involved, which explains that this is the only known case; but the expectation is that many other examples exist.

The key novelty in the proof of Theorem 4.2 was to combine the use of the distorted Fourier transform (that we will present in the next Section 4.1.2) initiated in [45, 264] with modulation techniques and space-time resonance analysis akin to [191, 193].

4.1.2 Vortices of the Gross-Pitaevskii equation

We have studied asymptotic stability problems for the nonlinear Schrödinger equation in one dimension in the previous Section 4.1.1. Does this problem become simpler in two dimensions? The larger the dimension, the stronger linear waves decay as there are more directions to spread. However, in two dimensions the linear Schrödinger decay ($\lesssim \langle t^{-1} \rangle$) is still not strong enough to ensure all quadratic nonlinearities are negligible (since $t \mapsto \langle t \rangle^{-1}$ is not integrable) see [192], and this is even worse if the solution does not converge to the rest state 0 at infinity, but to another "boundary" condition because this may reduce the group velocity at low frequencies.

This will be the case in the present situation, where we consider the nonlinear Schrödinger equation (4.1) in two dimensions, for a cubic nonlinearity, which after a fixed gauge transform is equivalent to the Gross-Pitaevskii equation

$$\begin{cases} i\partial_t u + \Delta u + (1 - |u|^2)u = 0, \\ u_{|t=0} = u_0, \end{cases} \qquad x \in \mathbb{R}^2 \tag{4.4}$$

with the boundary condition that $|u(x)| \to 1$ as $|x| \to \infty$. This equation appears in the study of Bose-Einstein condensates, superfluidity and superconductivity (see for instance [1],[203],[256],[345],[363]).

Adopting radial coordinates (r, θ) such that $(x_1, x_2) = r(\cos\theta, \sin\theta)$, we focus on solutions whose expansion in Fourier modes only contain the first harmonic in the angular coordinate,

$$u(t,x) = w(t,r)e^{i\frac{x}{|x|}}$$

The equation becomes

$$i\partial_t w + \partial_r^2 w + \frac{1}{r}\partial_r w - \frac{1}{r^2}w + (1-|w|^2)w = 0.$$

There exists a unique stationary solution $\rho(r)$ which satisfies

$$\partial_r^2\rho + \frac{1}{r}\partial_r\rho - \frac{1}{r^2}\rho + (1-\rho^2)\rho = 0,$$

is smooth, bounded and nonnegative with limit 1 as $r \to \infty$. The function

$$u(t,x) = \rho(|x|)e^{i\frac{x}{|x|}} \tag{4.5}$$

is the only stationnary solution of the Gross-Pitaevskii equation with degree 1 at $+\infty$, see [326].

The solution $\rho(|x|)e^{i\frac{x}{|x|}}$ is called a vortex. Its orbital stability has been proven in a metric space in [207]. In dimension 1, asymptotic stability of the black soliton, which plays a similar role to the vortex in dimension 2, has been shown in [208]. Higher degrees vortices exist, but they are expected to be unstable although radially stable, see [357].

As far as asymptotic stability in dimension 2 goes, the only result seems to be the exclusion of exponentially growing modes for the linearized problem [118], [415]. For the related question of the asymptotic stability of 1 in the Gross-Pitaevskii equation, a precise analysis of nonlinear interactions led to important progress [214, 215, 216, 217].

We obtained linear asymptotic stability in a recent work [84] that we now describe. It is based on the implementation of the distorted Fourier transform which, in the one-dimensional case, was crucial in the proof of Theorem 4.2 regarding the stability of traveling waves. Around the same time as our work appeared, Lührmann, Schlag and Shahshahani [291] proved L^2 bounds for the linear evolution by implementing the distorted Fourier transform at low frequencies. The two implementations in [84] and [291] are different and complementary, the former proposing a direct proof of the invertibility of the Fourier transform via functional and Fourier analysis, while the later relies on a full analysis of Stone's formula.

Linearizing in the first spherical class around $\rho(|x|)e^{i\frac{x}{|x|}}$ gives the equation

$$i\partial_t w + \mathscr{L}w = 0 \qquad \text{with} \qquad \mathscr{L}w = \partial_r^2 w + \frac{1}{r}\partial_r w - \frac{1}{r^2}w + w - 2\rho^2 w - \rho^2\overline{w}.$$

In order to make this operator complex-linear, we can change the unknown function to $(w, \bar{w})$ which leads to the equivalent equation

$$i\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \mathscr{H}\begin{pmatrix} u \\ v \end{pmatrix}, \qquad \mathscr{H} = \mathscr{H}_0 + V,$$

where the matrix operators $\mathscr{H}_0$ and V are given by

$$\mathscr{H}_0 = \begin{pmatrix} -\partial_r^2 - \frac{1}{r}\partial_r + \frac{1}{r^2} + 1 & 1 \\ -1 & \partial_r^2 + \frac{1}{r}\partial_r - \frac{1}{r^2} - 1 \end{pmatrix},$$

$$V = (\rho^2 - 1)\begin{pmatrix} 2 & 1 \\ -1 & -2 \end{pmatrix}.$$

While this is a one-dimensional differential operator, it combines many difficulties

— It is a matrix rather than a scalar operator.

— The constant-coefficient part of the operator is not diagonal; it actually gives the dispersion relation $\tau = |\xi|\sqrt{2+\xi^2}$ which interpolates between the wave and Schrödinger cases.

— The potential part of the operator has the critical (scale-invariant) decay $\frac{1}{r^2}$.

— There is a resonance in L^∞ ("s-wave" in the terminology of two-dimensional Schrödinger operators) at zero energy.

— Besides the aforementioned resonance (kernel of $\mathscr{H}$), the kernel of $\mathscr{H}^2$ contains a further resonant state. Thus, geometric and algebraic multiplicities differ (in a generalized sense, we are dealing with resonances instead of eigenfunctions).

In order to prove decay estimates for Schrödinger operators (scalar and matrix) in dimension one, a classical approach is to express the spectral projectors (known in this context as the distorted Fourier transform) through the generalized eigenfunctions, which are the solutions of ODEs [46, 205, 264]. This approach is appealing since it gives explicit formulas, but it can be heavy-handed. It generalizes well to higher dimensions provided one can separate variables by exploiting the symmetries of the problem. For this, we refer in particular to [358] which deals with the linear stability of vortices in the Ginzburg-Landau equation, which is the relativistic equivalent of the Gross-Pitaevskii equation studied here, see also [387, 388]. All these references address the presence of resonances and critically decaying potentials, which are also a feature of the problem studied in the present paper.

An alternative route to decay estimates proceeds by expanding the resolvent, which applies without symmetry assumptions but gives a less precise approach, see in particular the reviews [385, 386] and [250, 375, 205].

Because of the various difficulties which are present in the problem at hand, it is not clear whether the general approach can succeed. Therefore, we resorted to separation of variables and the construction of the distorted Fourier transform to prove decay. We furthermore focused on a single angular harmonic, which is natural since the corresponding symmetry of the solution is preserved by the nonlinear evolution.

Theorem 4.3 (Decay estimates for localized initial data [84])**.** *Let $\epsilon > 0$. The following hold true*

(i) Large time decay estimates. *For $\phi \in L^{2,1+\epsilon}$, we have for $t \geqslant 2$,*

$$\|e^{it\mathscr{H}}\phi\|_{L^\infty} \lesssim \frac{1}{t^{2/3}}\|\phi\|_{L^{2,1+\epsilon}} \qquad \textit{and} \qquad \|e^{it\mathscr{H}}\phi\|_{L^2} \lesssim \sqrt{\ln t}\,\|\phi\|_{L^{2,1+\epsilon}}.$$

(ii) Improved large time estimates. *If furthermore $\phi \in L^{2,2+\epsilon}$ and $\langle \phi, (\rho, \rho)^\top \rangle_{L^2(rdr)} = 0$, then*

$$\|e^{it\mathscr{H}}\phi\|_{L^\infty} \lesssim \frac{1}{t}\|\phi\|_{L^{2,2+\epsilon}} \qquad \text{and} \qquad \|e^{it\mathscr{H}}\phi\|_{L^2} \lesssim \|\phi\|_{L^{2,1+\epsilon}}.$$

(iii) General large time estimates. *The above estimates are consequence of the more general estimates for $\phi \in L^{2,2+\epsilon}$ and $t \geqslant 2$:*

$$\|e^{it\mathscr{H}}\phi\|_{L^\infty} \lesssim \frac{\|\phi\|_{L^{2,2+\epsilon}}}{t} + \frac{|\langle \phi, (\rho, \rho)^\top \rangle_{L^2(rdr)}|}{t^{2/3}}$$

and for $\phi \in L^{2,1+\epsilon}$

$$\|e^{it\mathscr{H}}\phi\|_{L^2} \lesssim \|\phi\|_{L^{2,1+\epsilon}} + |\langle \phi, (\rho, \rho)^\top \rangle_{L^2(rdr)}|\sqrt{\ln t}.$$

(iv) Short time dispersive estimate. *If $\phi \in L^{2,1+\epsilon}$, then for $0 < t < 2$ we have $\|e^{it\mathscr{H}}\phi\|_{L^\infty} \lesssim t^{-1}\|\phi\|_{L^{2,1+\epsilon}}$.*

Remark 4.4. — We believe that the estimate in the endpoint case $\epsilon = 0$ might not be true, since the two dimensional embeddings $L^{2,1+\epsilon} \subseteq L^1$ and $H^{1+\epsilon} \subseteq L^\infty$ are only true if $\epsilon > 0$. We stated decay estimates in weighted L^2 spaces, but our proof can be adapted to provide estimates on other function spaces.

— We believe that these estimates are sharp, since they coincide with the estimates for the linearization over the flat background 1.

— In particular, the growth of the L^2 norm is sharp, in that if $\langle \phi, (\rho, \rho)^\top \rangle_{L^2(rdr)} \neq 0$, then there holds $\|e^{it\mathscr{H}}\phi\|_{L^2} \gtrsim \sqrt{1 + \ln\langle t\rangle}$, see [84]. This shows that the vortex is not linearly stable in H^1 (as is the constant solution, see [186]). On the other hand, it is nonlinearly orbitally stable in the energy space [207]. While there is no contradiction since the energy space does not embed in H^1, this indicates that the nonlinear stability is sensitive to the topology in which it is measured. The unboundedness of the linear group in L^2 as $t \to \infty$ was obtained in [291].

The proof of Theorem 4.3 was based on a precise spectral analysis. We showed that the linearized operator has no eigenmodes, that its essential spectrum is $\mathbb{R}$, and that for any $\lambda \in \mathbb{R}$ there exists a unique (up to multiplication by a scalar) generalized eigenfunction, i.e. a smooth and bounded solution $\psi(\lambda)$ to $\mathscr{H}\psi(\lambda) = \lambda\psi(\lambda)$.

While λ corresponds to the time frequency in the equation, it will often be convenient to use the parameter ξ, which gives the space frequency at $+\infty$ of generalized eigenfunctions and is defined as

$$\xi = (\operatorname{sign}\lambda)\sqrt{\langle\lambda\rangle - 1} \quad \Longleftrightarrow \quad \lambda = \xi\sqrt{\xi^2 + 2}$$

The key to obtain the dispersive estimates of Theorem 4.3 was to obtain a sharp description of the generalized eigenfunctions. We state them for $\xi \geq 0$ and they imply an analogue decription for $\xi \leq 0$ by symmetry. We introduce the unit vector

$$e(\xi) = \frac{1}{\sqrt{2(1+\xi^2)(1+\xi^2+\xi\sqrt{2+\xi^2})}}\begin{pmatrix} 1+\xi^2+\xi\sqrt{2+\xi^2} \\ -1 \end{pmatrix}.$$

Theorem 4.5 (Description of generalized eigenfunctions [84]). *(i) (Decomposition into singular and regular parts at low frequency) If $0 \leqslant \xi \leqslant 2$,*

$$\psi(\xi, r) = \psi_\flat^S(\xi, r) + \psi_\flat^R(\xi, r),$$

where

$$\psi_\flat^S(\xi, r) = \left(\sqrt{\frac{\pi}{2}} b_\flat(\xi)((\rho(r) - 1)\chi(\xi r) + J_0(\xi r)) + c_\flat(\xi) \frac{\sin\left(\xi r - \frac{\pi}{4}\right)}{\sqrt{\xi r}}(1 - \chi(\xi r))\right) e(\xi)$$

with $b_\flat^2 + c_\flat^2 = 1$ and, for any $j \in \mathbb{N}$, $|\partial_\xi^j (b_\flat - 1)| + |\partial_\xi^j c_\flat| \lesssim_j \xi^{2-j} \ln^2(\xi)$, and $\psi_\flat^R$ is a lower order remainder with precise estimates both as $\xi \to 0$ and $r \to \infty$.

(ii) (Decomposition into singular and regular parts at high frequency) If $\xi \geqslant \frac{1}{2}$,

$$\psi(\xi, r) = \psi_\sharp^S(\xi, r) + \psi_\sharp^R(\xi, r),$$

with

$$\psi_\sharp^S(\xi, r) = \left(a_\sharp(\xi) J_1(\xi r)\chi(r) + \frac{b_\sharp(\xi)\cos(\xi r) + c_\sharp(\xi)\sin(\xi r)}{\sqrt{\xi r}}(1 - \chi(r))\right) e(\xi)$$

with $b_\sharp^2 + c_\sharp^2 = 1$ and $|\partial_\xi^k (a_\sharp - 1)| + |\partial_\xi^k (b_\sharp(\xi) + \frac{1}{\sqrt{2}}| + |\partial_\xi^k (c_\sharp(\xi) - \frac{1}{\sqrt{2}})| \lesssim_k \frac{1}{\xi^{1+k}}$ if $k \in \mathbb{N}$ and $\psi_\sharp^R$ is a lower order remainder with precise estimates both as $\xi \to \infty$ and $r \to \infty$.

With the help of the generalized eigenfunctions ψ defined above, we can define a distorted Fourier transform, which will diagonalize the operator $\mathscr{H}$. Define the distorted Fourier transform, mapping functions from $\mathbb{R}_+$ to $\mathbb{C}^2$ to functions from $\mathbb{R}$ to $\mathbb{C}$:

$$\widetilde{\mathscr{F}}(\phi)(\xi) = \int_0^\infty \psi(\xi, r) \cdot \sigma_3 \phi(r) r dr, \quad \xi \in \mathbb{R}.$$

where σ_3 is the Pauli matrix defined by

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and its inverse

$$\widetilde{\mathscr{F}}^{-1}(\zeta)(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \zeta(\xi)\psi(\xi, r)\lambda'(\xi)\operatorname{sign}(\xi) d\xi, \quad r \in \mathbb{R}_+.$$

Theorem 4.6 (Existence of the distorted Fourier transform [84]). *The distorted Fourier transform and its inverse satisfy*

$$\widetilde{\mathscr{F}}\widetilde{\mathscr{F}}^{-1} = \mathrm{Id}, \qquad \widetilde{\mathscr{F}}^{-1}\widetilde{\mathscr{F}} = \mathrm{Id}.$$

Furthermore, the group can be expressed as

$$e^{it\mathscr{H}} = \widetilde{\mathscr{F}}^{-1} e^{it\lambda} \widetilde{\mathscr{F}}.$$

All these equalities hold true on spaces of smooth, rapidly decaying functions.

The key novelty in the proof of the above Theorem was that we did not rely on the application of the resolvent of $\mathscr{H}$ for complex values of λ, but proved it directly by combining functional and Fourier analysis with the previous description of the spectrum of $\mathscr{H}$.

4.1.3 Homogeneous steady states of the Hartree equation

The previous Section 4.1.2 was devoted to the linear stability of the vortex 4.5. This vortex behaves asymptotically as $e^{i\frac{x}{|x|}}$ as $|x| \to \infty$. This produces a non-zero background of modulus one a spatial infinity. There is a simpler steady state, which is the constant function 1, which also produces a non-zero background of modulus one, and whose asymptotic stability under general perturbations remains open in low dimensions [215, 216, 217, 214]. Such steady states are not localized. In this subsection we will consider steady states which are not localized for the Hartree equation, which is a system of coupled Schrödinger equations.

The exact equation (4.7) that will be at stake here is a reformulation of the commonly used Hartree equation for density matrices

$$i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma]. \tag{4.6}$$

Above, γ is a time dependent bounded operator on $L^2(\mathbb{R}^3)$ with integral kernel $\tilde{\gamma}(x,y)$, and $\rho_\gamma(x) = \tilde{\gamma}(x,x)$ is the density of particles, that is the diagonal of $\tilde{\gamma}$. An infinite number of particles can then be modelled by a solution of (4.6) which is not of finite trace (the trace of the operator being, by the derivation of the model, the number of particles). Solutions of (4.6) with an infinite number of particles were studied previously in [29, 30, 54, 420] for exemple, and more recently in [61, 62, 277, 278]. The derivation of equation (4.6) from many body quantum mechanics has been treated in [9, 10, 16, 17, 153, 181].

In [114], de Suzzoni proposed the following equation

$$\begin{cases} i\partial_t X = -\Delta X + (w * \mathbb{E}(|X|^2))X, \\ X(t=0) = X_0, \end{cases} \qquad x \in \mathbb{R}^3, \tag{4.7}$$

as an alternative equation to (4.6). $X : \Omega \times \mathbb{R}_t \times \mathbb{R}^3_x \to \mathbb{C}$ is a random field defined over a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ and w is an even pair interaction potential. For the transformation that maps Equation 4.6 to Equation (4.7), we refer to [72, 114, 115]. Even if Equation (4.7) is for a random field, this is not similar to an equation for random initial data, or to an equation with noise, rather it is a system of coupled Schrödinger Equations like (4.6) but for which the framework of random fields allows to give another viewpoint.

Equation (4.7) admits the following phase invariance: if $X(\omega,t,x)$ is a solution to (4.7) then so is

$$e^{ia(\omega)}X(t,x,\omega) \quad \text{for all measurable } a : \Omega \to \mathbb{R}. \tag{4.8}$$

Equation (4.7) admits particular unlocalized steady states, whose stability will be the focus of this subsection. We assume that on Ω is defined a Wiener process W of dimension 3 (a white noise on $\mathbb{R}^3$), namely $(dW(\xi))_{\xi\in\mathbb{R}^3}$ is a family of infinitesimal independent Gaussian fields with values in $\mathbb{C}$, such that for all $\xi, \eta \in \mathbb{R}^3$

$$\mathbb{E}(dW(\xi)\overline{dW(\eta)}) = \delta(\xi - \eta)d\xi d\eta.$$

We refer to the appendix of [73] for some basic results and references on Wiener integration used in this article. Consider for $f \in L^2(\mathbb{R}^3, \mathbb{C})$ and $m = \int_{\mathbb{R}^3} w \cdot \int_{\mathbb{R}^3} |f|^2 \in \mathbb{R}$ (where $\int_{\mathbb{R}^3} w$ denotes the total mass of w) the random field:

$$\begin{array}{rlcl} Y_f : & \Omega \times \mathbb{R} \times \mathbb{R}^3 & \to & \mathbb{C} \\ & (\omega, t, x) & \mapsto & \int_{\mathbb{R}^3} f(\xi) e^{-it(m+\xi^2)+i\xi \cdot x} dW(\xi)(\omega) \end{array} . \tag{4.9}$$

For every (t, x), $Y_f(t, x)$ is a centred Gaussian variable with constant variance $\mathbb{E}(|Y_f|^2(t, x)) = \int_{\mathbb{R}^3} |f|^2$. If for $k \in \{0, 1, ...\}$ and $s > k$, $\int_{\mathbb{R}^3} |f(\xi)|^2 \langle \xi \rangle^{2s} d\xi < \infty$, then for almost every ω, $Y_f(\omega)$ is a continuous function with subpolynomial growth at infinity on $\mathbb{R}^{1+3}$, and with continuous $\partial_t^\alpha \nabla_x^\beta Y_f$ derivatives for $2\alpha + \beta \le k$. For $s > 2$ in particular, almost surely, the identity

$$i\partial_t Y_f = \int f(\xi)(m + \xi^2) e^{-it(m+\xi^2)+ix\cdot\xi} dW(\xi) = (m - \Delta) Y_f$$

holds in a classical sense everywhere on $\mathbb{R} \times \mathbb{R}^3$, showing that Y_f is a solution to (4.7). Assuming solely $s > 0$, we still have that Y_f is a weak solution to (4.7) almost surely.

The field Y_f is a Gaussian field whose law is invariant under space translations, which makes it non-localised, and time translations, which suggests the denomination "equilibrium" even though Y_f is not a invariant state. Its stability seems therefore delicate, in particular in low dimensions, as it generates a non-localized homogeneous background.

In the seminal work [277], the authors show the stability of the above equilibria for the Equation (4.6) for density matrices in dimension 2. Important tools are dispersive estimates for orthonormal systems [177, 178]. This work has been extended to higher dimension in [62]. Note that in higher dimension, some structural hypothesis is made on the interaction potential w, in particular, in dimension 3, it imposes $\hat{w}(0) = 0$, to solve some technical difficulties about a singularity in low frequencies of the equation that we will identify precisely in the sequel. The stability result corresponds to a scattering property in the vicinity of these equilibria: any small and localized perturbation evolves asymptotically into a linear wave which disperses. If the interaction potential is the Coulomb one, its slow decrease at spatial infinity changes the decay of linear waves and plasmons appear [348].

We mention equally [61, 278] about problems of global well-posedness for the equation on density matrices. More recently, the stability of the equilibria (4.9) for (4.6) has been studied in the semiclassical limit towards the Vlasov equation in [279]. Other similar effective equations for large fermionic systems appear in density functional theory, and the stability of the 0 solution was studied in [366].

In [72], we proved the asymptotic stability of equilibria for (4.7) in dimension higher than 4 without the structural hypothesis on the potential of interaction at low frequencies of [62]. We were able in the work [73]that we will describe below to obtain asymptotic stability in low dimensions 2 and 3 and to lower the assumptions on the interaction potential w, namely under the mild assumption for w that:

$$w \text{ is a finite Borel measure on } \mathbb{R}^3. \tag{4.10}$$

This is the case if, for example, $w \in L^1(\mathbb{R}^3)$, or, if $w = \pm\delta$ is the Dirac mass, in which case (4.7) is a variant of the nonlinear Schrödinger equation

$$i\partial_t X = -\Delta X \pm \mathbb{E}(|X|^2)X.$$

We adopted a strategy similar to [214, 216, 217, 277], which is to treat differently the first Picard iteration of the solution than the rest of the solution. In [214, 216, 217], this was done through a normal form to remove the difficult quadratic part of the equation, and in [277], this was done through a complete expansion of the solution into Picard interactions. Random cancellations and homogeneous Strichartz estimates allow us to close the argument.

We state here our result in dimension 3. An analogous result in dimension 2 is given in [73]. In what follows, we write $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and, given $a \in \mathbb{R}$, $(a)_+ = \max(a,0)$ and $(a)_- = \max(-a,0)$ the nonnegative and nonpositive parts of s. We write with an abuse of notation $f(\xi) = f(r)$ with $r = |\xi|$, if f has spherical symmetry. The space $L^2_\omega H^s_x$ is the set of measurable functions $Z : \Omega \times \mathbb{R}^d \to \mathbb{C}$ such that $Z(\omega,\cdot) \in H^s(\mathbb{R}^3)$ almost surely and

$$\int_{\mathbb{R}^d\times\Omega} \langle\xi\rangle^{2s}|\hat{Z}(\omega,\xi)|^2 d\xi d\omega < +\infty.$$

We introduce the notation for the solutions to $iu_t = (-\Delta+m)u + VY$ for $V \in L^1_{\text{loc}}(\mathbb{R}, L^2(\mathbb{R}^3))$:

$$S(t) = e^{-it(m-\Delta)} \quad , \quad W_V(Y)(t) = -i\int_0^t S(t-\tau)(V(\tau)Y(\tau))d\tau. \tag{4.11}$$

Theorem 4.7 (Stability of homogeneous equilibria of the Hartree equation in its formulation for random fields [73])**.** *We denote by h the Fourier transform of $|f|^2$ on $\mathbb{R}^3$. Assume the momentum distribution function f satisfies :*

- *$f > 0$ is a bounded C^1 radial function on $\mathbb{R}^3$, with $\partial_r f < 0$,*
- *$\int_{\mathbb{R}^3}\langle\xi\rangle f^2(\xi)d\xi < \infty$ and $\int_{\mathbb{R}^3}|\xi|^{-1}|f(\xi)\nabla f(\xi)|d\xi < \infty$,*
- *$\int_0^\infty (1+r)|h|(r)dr < \infty$ and $\int_0^\infty \left(\frac{|h'|(r)}{r} + |h''|(r)\right)dr < \infty$ where the derivatives h' and h'' are defined in the sense of distributions,*

and that w satisfies (4.10) and[1] *(where below ϵ_h is a constant depending on h given in [73])*

$$\|(\hat{w})_-\|_{L^\infty}\left(\int_0^\infty r|h(r)|dr\right) < 2 \quad \text{and} \quad \hat{w}(0)_+\epsilon_h < 1.$$

Then there exists $\delta > 0$ such that for all $Z_0 \in L^2_\omega H^{1/2}_x \cap L^{3/2}_x L^2_\omega$ with $\|Z_0\|_{L^2_\omega H^{1/2}_x \cap L^{3/2}_\omega L^2_x} \le \delta$ the following holds true. The Cauchy problem (4.7) with initial datum $Y_f(t=0)+Z_0$ is globally well-posed in $Y_f + \mathscr{C}(\mathbb{R}, L^2_\omega H^{1/2}_x)$, and what is more, there exist $Z_\pm \in L^2_\omega H^{1/2}_x$ and $V \in L^2_t H^{1/2}_x \cap L^{5/2}_{t,x}$ such that

$$X(t) = Y_f(t) + W_V(Y)(t) + S(t)Z_\pm + o_{L^2_\omega H^{1/2}_x}(1) \qquad \text{as } t \to \pm\infty.$$

1. Note that since w is an even finite Borel measure, its Fourier transform is continuous and real, so that w_+ and w_- are well-defined.

For the third term above, there exists $\tilde Z_\pm \in L^3_x L^2_\omega$ with $S(t)\tilde Z_\pm \in C(\mathbb{R}, L^3_x L^2_\omega)$ such that

$$W_V(Y) = S(t)\left(\tilde Z_\pm + o_{L^3_x L^2_\omega}(1)\right) = S(t)\tilde Z_\pm + o_{L^3_x L^2_\omega}(1) \qquad \text{as } t \to \pm\infty.$$

Relating the framework of random fields to that of density matrices, from the above Theorem 4.7 one obtains a scattering result for the operator γ whose kernel is the correlation function of X:

$$\gamma = \mathbb{E}(|X\rangle\langle X|) := u \mapsto \Big(x \mapsto \mathbb{E}(X(x)\langle X, u\rangle_{L^2(\mathbb{R}^d)})\Big),$$

with respect to the one associated to the equilibrium Y_f:

$$\gamma_f = \mathbb{E}(|Y_f\rangle\langle Y_f|),$$

which is the Fourier multiplier by $|f|^2(\xi)$. This convergence holds in Schatten-Sobolev spaces (where below $\mathcal{S}^p$ is the standard Schatten space p-norm for operators on $L^2(\mathbb{R}^3)$ and $\alpha \in \mathbb{R}$):

$$\|\gamma\|_{\mathcal{S}^{\alpha,p}} = \|\langle\nabla\rangle^\alpha \gamma \langle\nabla\rangle^\alpha\|_{\mathcal{S}^p}.$$

Corollary 4.8 (Stability of homogeneous equilibria of the Hartree equation in its formulation for density matrices [73])**.** *Under the hypotheses of Theorem 4.7, defining the operators on $L^2(\mathbb{R}^3)$:*

$$W_{V,\pm} : u \mapsto -i\int_0^{\pm\infty} S(-\tau)(V(\tau)S(\tau)u)d\tau,$$

and

$$\gamma_\pm = \mathbb{E}\left(|W_{V,\pm}Y_0 + Z_\pm\rangle\langle Y_0| + |Y_0\rangle\langle W_{V,\pm}Y_0 + Z_\pm| + |W_{V,\pm}Y_0 + Z_\pm\rangle\langle W_{V,\pm}Y_0 + Z_\pm|\right)$$

there holds for any $\epsilon > 0$ that $\gamma_\pm \in \mathcal{S}^{1/2,4+\epsilon}$ and the convergence:

$$\gamma = \gamma_f + e^{i\Delta t}\gamma_\pm e^{-i\Delta t} + o_{\mathcal{S}^{\frac12,4+\epsilon}}(1) \ \text{ as } t \to \pm\infty.$$

Remark 4.9. The conditions on f are satisfied by thermodynamical equilibria for bosonic or fermionic gases at a positive temperature T, and the Bessel potential distribution:

$$|f(\xi)|^2 = \frac{1}{e^{\frac{|\xi|^2-\mu}{T}} - 1}, \ \mu < 0, \quad \text{or} \quad |f(\xi)|^2 = \frac{1}{e^{\frac{|\xi|^2-\mu}{T}} + 1}, \ \mu \in \mathbb{R}, \quad \text{or} \quad |f(\xi)|^2 = \langle\xi\rangle^{-\alpha}, \ \alpha > 4,$$

respectively, but it is not the case of the fermionic gases at zero temperature $|f(\xi)|^2 = \mathbf{1}_{|\xi|^2\le\mu}$ for $\mu > 0$. The stability of fermionic gases at zero temperature was obtain later by Hadama in [218]. Given an f satisfying the hypotheses, interaction potentials satisfying the requirements are for example any Borel measure with total mass c (for example $w = \pm c\delta$) or $w \in L^1$ with $\|w\|_{L^1} = c$ for $c \le 2(\int_0^\infty r|h|)^{-1}$.

Remark 4.10. Theorem 4.7 has a direct consequence for Equation (4.6) on density matrices. It implies scattering for (4.6) near γ_f for all perturbations in $\mathcal{S}^{1/2,1}$ (with a finite number of particles). Indeed, the density matrix associated to Y_0+Z_0 is $\gamma_f+\gamma'$, where $\gamma' = \gamma_1'+\gamma_2'$ with $\gamma_1' = \mathbb{E}|Z_0\rangle\langle Z_0|$

and $\gamma_2' = \mathbb{E}(|Z_0\rangle\langle Y_0| + |Y_0\rangle\langle Z_0|)$. By taking $Z_0 \in L^2_\omega H^{1/2}$ independent in probability of Y we have $\gamma_2' = 0$, and that the operator γ_1' can be any non-negative operator in $\mathcal{S}^{1/2,1}$. Theorem 4.7 implies also scattering for (4.6) near γ_f for perturbations in a subset of $\mathcal{S}^{1/2,2}$ (infinite number of particles). This is obtained by taking Z_0 not independent of Y_0, so that $\gamma_2' \in \mathcal{S}^{1/2,2}$. The appearance of Y in γ_2' has a regularising effect. Hence, the operators that can be written as $\gamma_1' + \gamma_2'$ for $Z_0 \in L^2_\omega H^{1/2}$ are a subset of $\mathcal{S}^{1/2,2}$ with higher regularity that we did not try to characterize.

Stability results can also be obtained for different nonlinearities. For a higher order quintic nonlinearity in (4.7), analogous homogeneous equilibria exist and their stability was obtained in [295]. For Fermions, a physically relevant equation is the Hartree-Fock equation which contains an additional nonlinear exchange term with respect to (4.7). We obtained the existence and stability of homogeneous equilibria with some smallness condition in [71], and the scattering of small solutions in the one dimensional case was obtained in [296] (see references therein for higher dimensions).

4.2 Soliton resolution

4.2.1 The soliton resolution conjecture

The soliton resolution conjecture is a conjecture on the asymptotic behaviour of solutions of certain nonlinear evolution equations close to their maximal time of existence. It does not have a clean statement in the literature, due to the fact that there is no consensus on the definition of "soliton" involved in this conjecture, and to the fact that its formulation depends on the equation at stake. We will try to formulate our own general informal statement of the conjecture below, which will probably not be the same one as in other instances where this conjecture is mentioned. For recent surveys on soliton resolution, see [79, 242].

Let us start with physics and completely integrable equations, where a precise sense to what a soliton is and to what soliton resolution is can be given. We refer to the introduction of [346] for a historical perspective. In physics, a soliton is a localized nonlinear traveling wave of a dispersive equation, and that has elastic collisions with other solitons (i.e. after a collision the two solitons emerged unaltered, with the same speeds, but possibly with a shift with respect to the positions they would have had without the collision) or linear radiations. The development of the mathematical theory of solitons culminated in the sixties and seventies, especially in the role they played in the dynamics of the Korteweg-de Vries (KdV) equation, the modified KdV equation, the sine-Gordon equation and the one-dimensional cubic nonlinear Schrödinger equation. Solitons were closely related to the fact that these equations were found to be completely integrable, and this lead to a more precise definition of a soliton, involving its identification with certain of the scattering data of an eigenvalue problem. A special transformation which allowed to compute the solutions of the Cauchy problems for these equations, called the inverse scattering method, see [2]. This allowed to compute the asymptotic behaviour as $t \to \pm\infty$ of solutions, and it showed that for the KdV equation all solutions emerging from generic localized initial data would eventually decompose as a sum of distinct traveling waves plus a linear radiative wave [151, 272]. For this equation, the soliton resolution has its clearest definition: the fact that the asymptotic behaviour as $t \to \pm\infty$ of arbitrary solutions is to decompose into solitons decoupled by their position plus a lower order linear remainder. For the other integrable equations that are the modified KdV equation, the one dimensional cubic Schrödinger equation and the sine-Gordon equation however, there are other particular solutions that are called breathers, which are localized traveling waves or standing waves whose shape is periodic with time. They may emerge in the asymptotic behaviour of general solutions, in addition to solitons and linear radiation. Thus, the common feature of these equations is that arbitrary solutions decompose asymptotically as $t \to \infty$ as the sum of solitons and breathers decoupled by their position and of a linear radiation, which can be interpreted as a simplification into asymptotic states at $t = \pm\infty$ for the system.

Such phenomenon is called scattering and occurs in various instances in physics: given the asymptotic states of the system at $t = -\infty$, the dynamics of the equation for $t \in (-\infty, \infty)$ describes how they interact, and the asymptotic states of the system at $t = \infty$ gives the result of their interactions. For a given equation, this requires that it has the property that the dynamics simplifies as $t \to \pm\infty$ into such asymptotic states. In linear classical and quantum field theories, this property is called asymptotic completeness and has been proved for various models, see [130] for instance.

For many nonlinear non-integrable equations, a conjecture is that for general solutions such simplification into asymptotic states as $t \to \infty$ also occurs, and that it also holds as $t \to T < \infty$ for finite time blow-up solutions. It could have been termed as "asymptotic completeness conjecture" as well, as is mentioned in Soffer's presentation at the international congress of mathematicians [397], but this conjecture now goes by the name of the soliton resolution conjecture, perhaps due to the fact that the name "soliton" reflects well a particle-like behaviour, and due to the influence of Tao [403] and Duyckaerts-Kenig-Merle (see e.g. a presentation of their works in [79]). However, as seen above, the asymptotic states may not only be solitons as they are precisely defined for integrable equations, but more general coherent structures.

Let us now give a more precise formulation of the soliton resolution conjecture. We have seen that it requires to give a sense to the name "soliton" that is more general than the precise sense of "soliton" in integrable equations. For this, we remark that all the equations mentioned in the present document have a group of invariances (space and time translations, scaling renormalization, phase shift etc.), and we can list all the nonlinear coherent structures mentioned in the present document: backward self-similar solutions (also called shrinkers), forward self-similar solutions (also called expanders), stationary states, traveling waves, standing waves, and breathers. The common feature of all these solutions is the following: they are particular solutions that are invariant under the action of a one-dimensional subgroup of the group of symmetries; stationary states are invariant by time translations, traveling waves by a combination of space and time translations, backward and forward self-similar solutions by a scaling group, standing waves of the Schrödinger equation by a combination of time translation and phase shift, etc. In all relevant examples, this one-dimensional subgroup involves the time variable, and these particular solutions are at time t obtained by a fixed transformation of the symmetry group applied to these particular solutions at time 0. This means that all these solutions are "self-similar" in the sense physicists give to self-similarity [11]. We note that the word "self-similar" in the mathematical literature on partial differential equations usually refers to the case where the particular solution is obtained at time t by a scaling transformation applied to the particular solution at time 0, so that here "self-similarity" is understood in a broader sense. When the one-dimensional group is not continuous but isomorphic to $\mathbb{Z}$, the particular solution is a discretely self-similar solution: this is the case for the breathers, and for discretely backward self-similar solutions (which exist for example for the Burgers equation [85]). This leads to the following generalized definition of a soliton: a soliton is a solution that is invariant by the action of a one-dimensional subgroup of the group of symmetries that involves a transformation of the time variable.

Given this generalized definition of a soliton, which solitons should we expect to appear asymptotically in time? We have seen in Section 3.2 that during singularity formation, certain terms in the equation can be subleading, and that the particular solution that appears is a backward self-similar solution for the reduced equation obtained from the original equation by removing the subleading terms. We have seen in Section 3.4 that during singularity formation, the solution may become independent of a spatial direction to leading order, and that the particular solution that appears is a backward self-similar solution or a stationary state of the equation but in lower dimensions. Combining, we see that the leading order dynamics may simplify to the one of a reduced equation, and possibly in lower dimensions.

On which domain and for which data should the soliton resolution hold? In the case of the Schrödinger equation on a bounded domain for exemple, wave packets may travel and meet infinitely many times after reflecting on the boundary, what would prevent a relaxation mechanism such as the soliton resolution. Similarly, if the equation is set on the whole space $\mathbb{R}^n$, but the solution is not localized, the various regions of space could interact indefinitely what would prevent solitons to appear. Thus, at least two clear set-up appear: global solutions on the whole space $\mathbb{R}^n$ emerging from localized initial data, or finite time blow-up solutions on any domain and for any kind of data (for local enough equations, since near the singularity the dynamics is unaltered from the values of the solutions elsewhere and resembles to leading order the dynamics on the whole space). In addition, there might be very special solutions for which the trajectory is chaotic, but they are expected to be highly unstable, in which case the soliton resolution still holds by restricting to a generic set of initial data.

This brings us to the following statement of the soliton resolution conjecture, which may not be the most general one, but in which the key features appear.

Soliton resolution conjecture for a given evolution PDE: Consider this equation on any domain for a generic finite time blow-up solution $T < \infty$, or on $\mathbb{R}^n$ (or a suitably non-trapping domain) for a generic global in time solution $T = \infty$. Then in the finite time blow-up case the solution decomposes as $t \to T$ near each singular point as a sum of decoupled solitons (in the generalized sense given above), which may be solitons of the original equation, or of a reduced equation obtained by discarding certain terms or lowering the dimension; in the global in time case the solution decomposes as $t \to \infty$ as a sum of solitons plus a lower order radiation.

Stated as above, apart from the case of completely integrable equations, the soliton resolution conjecture has only been established for very few non-integrable equations. We start by describing works on parabolic equations. In the one dimensional case, for the viscous Burgers equation and the Fischer-KPP equation, the convergence towards a forward self-similar solution or a traveling wave has been known for a long time, we refer to Section 2.1 for the former equation and to [220] and references therein for the later. For monotone solutions of the steady two-dimensional Prandtl system, which can be cast as a one dimensional parabolic equation, Serrin showed the convergence to a forward self-similar solution [395]. For the semilinear heat equation, Giga and Kohn [199, 198, 200] proved the convergence of all blow-up solutions to the ODE backward self-similar solution in the energy subcritical case, see Section 3.2.1, Matano and Merle proved the convergence to backward self-similar solutions are stationary states for type I or type II blow-up in the energy supercritical case [312], and recently Lawrie and Aryan showed that in the energy critical case singular solutions with bounded energy norm concentrate stationary states [274, 5]. For the energy critical harmonic map heat flow between Riemanian manifolds, the soliton resolution was obtained in the pioneering work of Struwe [401] and in subsequent works for which we refer to the recent work of Jendrej, Lawrie and Schlag [246].

We now describe works on hyperbolic equations, for which soliton resolution has only been obtained for very few equations. We presented in Section 2.1 the soliton resolution for the inviscid Burgers equation, and we have to stress that we were surprised that our result stated in Proposition 2.2 of convergence of blow-up solutions to backward self-similar solutions did not appear in the literature before, as it gives the simplest yet insightful exemple of soliton resolution for singularity formation. Our result of Theorem 3.9 shows soliton resolution for generic solutions of the inviscid two-dimensional Prandtl system, and we are not aware of other soliton resolution results for non-linear transport equations in two or higher dimensions. Much of the work has been done on the semilinear wave equation

$$\partial_t^2 u - \Delta u = |u|^{p-1} u$$

for $p > 1$ a real number, on which we will focus in the next subsection. Merle and Zaag proved any blow-up solution concentrates backward self-similar solutions [321, 322] and references therein. Then the soliton resolution has been proved in the energy critical case $p = 1 + 4/(N-2)$ in dimensions $N \geq 3$ for solutions that are bounded in the energy space and radial, and in the non-radial case it has been showed for such solutions along a sequence of times, as we shall describe in the next section. A related model is the energy critical wave maps equation, for which the soliton resolution has been obtained by Duyckaerts, Kenig, Martel and Merle [138], and Jendrej and Lawrie [245], with earlier works such as [102, 102, 247, 101]. For the damped Klein-Gordon equation, soliton resolution has been obtained in the radial case by Burq, Raugel and Schlag in [40] and in the non-radial case along a sequence of times in [171].

4.2.2 Soliton resolution for the energy critical wave equation

In this subsection we will describe the soliton resolution for the energy critical semilinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = |u|^{\frac{4}{N-2}} u, \\ (u(0), \partial_t u(0)) = (u_0, u_1), \end{cases} \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N \tag{4.12}$$

in general space dimension $N \geq 3$. In the radial case, we will see that solutions that remain bounded in resolve into the rescaled stationary state $W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{1-\frac{N}{2}}$. We shall focus mostly on our work on the six dimensional case

$$\partial_t^2 u - \Delta u = |u| u, \tag{4.13}$$

together with a similar problem

$$\partial_t^2 u - \Delta u = u^2. \tag{4.14}$$

Introducing the vector notation $\vec{u} = (u, \partial_t u)$ we shall consider radial solutions in the energy space $\vec{u}(0) \in \mathscr{H} = \dot{H}(\mathbb{R}^6) \times L^2(\mathbb{R}^6)$. The ground state in six dimensions is

$$W(x) = \frac{1}{(1 + \frac{|x|^2}{24})^2}$$

which solves $-\Delta W = W^2$. Equations (4.13) and (4.14) admit the following scaling symmetry: if u is a solution then so is

$$u_{(\lambda)}(t,x) = \frac{1}{\lambda^2} u\left(\frac{t}{\lambda}, \frac{x}{\lambda}\right) \tag{4.15}$$

In proving soliton resolution in the context of dispersive equations and of energy critical nonlinearities, the energy critical semilinear wave equation (4.12), was the first to be considered using some decoupling related to the finite speed of propagation. First, results for data close to the ground state were obtained (see [139], [141], [262], [263]). Then, the soliton resolution for sequences of times $t_n \to T$ in the radial case, for solutions which are bounded in the energy norm, was proved by [140] in 3 dimensions, [377] in all other odd dimensions, in [103] in 4 dimensions and in [247] in 6 dimensions. The work [137] proved the decomposition for sequences of times, in the nonradial case, for solutions which are bounded in the energy norm, in dimensions 3, 4 and 5.

To consider the full problem (proving the decomposition for all times) one has to understand the collision of solitons and prove that all collisions produce some radiation, which limits their number by energy considerations. This is the approach introduced by Duyckaerts, Kenig and Merle in [142] and fully developed by them in [145, 144, 143]. More precisely, the natural object to consider is a pure multisoliton in both time directions, which is a solution that is, asymptotically as $t \to +\infty$ and as $t \to -\infty$, a sum of decoupled solitons without radiation (i.e. the radiation term is zero). For non-integrable equations such as (4.14), (4.12), it is expected that collisions are inelastic and should always generate some radiation (see e.g. [300, 301, 302] in the context of generalized Korteweg-de Vries equations and also [303] for (4.12) with $N = 5$), ruling out the existence of such an object.

To deal with this problem and using fully the finite speed of propagation, Duyckaerts, Kenig and Merle have introduced the concept of *solutions of* (4.12) *that are non-radiative for* $|x| > R + |t|$. By definition, these are solutions of (4.12), defined for $|x| > R + |t|$, and such that

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>R+|t|} \left(|\nabla u(t,x)|^2 + (\partial_t u(t,x))^2\right) dx = 0. \tag{4.16}$$

The usefulness of this concept is that, using finite speed of propagation, it can be applied by first studying solutions in the exterior of a wave cone $\{|x| > R + |t|\}$, for large R, thus restricting to small solutions, that are close to solutions of the linear wave equation. We were able to classify all solutions that are non-radiative in the sense of (4.16) for $R > 0$, and to show they belong to a finite dimensional manifold of initial data, this will be presented in the next Section 4.2.3. Thus, if a pure multisoliton were to exist, away from the origin it would be given by a function of this finite dimensional set, whose expansion at spatial infinity is explicitly computable.

Solutions which are radiative for $|x| > R + |t|$ send to null infinity some energy that channels at the speed of light $\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>R+|t|} \left(|\nabla u(t,x)|^2 + (\partial_t u(t,x))^2\right) dx > 0$. It is useful to estimate the fraction of the initial energy channels to infinity. For free waves, this amounts to the study lower bounds of the form

$$C \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{t,x} u_L(t,x)|^2 dx \geq \int_{|x|>R} (u_1(x))^2 + |\nabla u_0(x)|^2 dx \tag{4.17}$$

for radial solutions of the linear wave equation

$$\begin{cases} \partial_t^2 u_L - \Delta u_L = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ \vec{u}_{\restriction t=0} = (u_0, u_1) \end{cases} \tag{4.18}$$

for initial data in the restricted energy space $\mathscr{H}_R = \{(u_0,u_1),\ \int_{|x>R|} |\nabla u_0|^2 + u_1^2 < \infty\}$. We will discuss this estimate in details in the next Section 4.2.3.

The validity of the linear estimate (4.17) depends strongly on the dimension N (its size and the oddness/evenness), and is intimately related to the classification of non-radiative solutions of the nonlinear problem satisfying (4.16). The summary is the following:

- *Odd space dimensions:*

For N odd and $R > 0$, (4.17) holds for all radial data in an $\frac{N-1}{2}$ co-dimensional subspace of $\mathscr{H}_R$. The space of non-radiative solutions to the nonlinear problem (4.12), i.e. that satisfy (4.16), is then a deformation of a $\frac{N-1}{2}$ subspace, as we proved in Theorem 4.14 which we will discuss in the next subsection. In particular, this subspace always contains the ground state $W = \left(1 + r^2/3\right)^{-1/2}$. The dimension $N = 3$ was first considered due to the exceptional property that this subspace is only spanned by rescaled ground states (this was already proved in [142]), and this lead to the proof of the soliton resolution in [142].

In higher dimensions $N \geq 5$, such a strong rigidity result for (4.12) as in space dimension 3 fails. The proof of the soliton resolution in this case is more involved: it combines asymptotic estimates on non-radiative solutions of (4.14) deduced from (4.17) (see Theorem 4.14) with a careful study of the modulation equations close to a multisoliton for non-radiative solutions, which gives enough parameters to deal with the large dimension of the counter examples at infinity. Using a gain of decay in space related to the non-radiative property, the last three authors were reduced to study a finite dimensional dynamics of the scaling parameters of the solitons and were able to prove the soliton resolution for all radial solutions of (4.14) that are bounded in the energy space, for all times [145, 144, 143].

- *Even space dimensions*:

The estimate (4.17) is not valid in its full generality, even when $R = 0$. In even space dimensions, up to now, no lower bound of the form (4.17) has been known and counter examples are established in [104] and recently in [75]. Nevertheless, (4.17) holds in a finite codimension space (at least in the radial case) for initial data of the form $(u_0, 0)$ when N has a congruence to 0 modulo 4, or $(0, u_1)$ when N has a congruence to 2 modulo 4 (see [104], [138], [282]). In each dimension, for the other case (initial data of the form $(0, u_1)$ or $(u_0, 0)$ respectively), one can see this failure as a consequence of the existence of an explicit singular resonant non-radiative solution of (4.18), that fails to be in the energy space by a logarithmic factor. A weaker estimate than (4.17) for this other case, with a logarithmic loss, is given in [75].

The four dimensional case was first treated by the last three authors and Martel in [138]. This case turns out to be the critical case for the exceptional property mentioned for $N = 3$ above: for $R > 0$, the solitons are the only radial non-radiative solutions in the region $\{|x| > R + |t|\}$. This property is proved in [138] by a delicate analysis based on the separate study of the projections $u_\pm(t) = \frac{1}{2}(u(t) \pm u(-t))$ of the solution u on the vector space of odd (respectively even) in time functions, noticing that the equations satisfied by $u_\pm$ are decoupled at first order. The soliton resolution for all radial solutions of (4.12) with $N = 4$ and also the $k = 1$ equivariant wave maps follows.

The six dimensional case (4.14)-(4.13) is the one that we will now describe. We were able to classify non-radiative solutions away from the origin in [76], which involve different solutions other than the sole ground state. We showed that the estimate (4.17) holds with a logarithmic loss in [75]. This was still enough to be able to implement the strategy of [145] for showing the continuous soliton resolution, by reducing the proof to study a finite dimensional dynamics of the scaling parameters of the solitons in a pure multisoliton. As a by-product of our methods, this can also be applied to give the corresponding soliton resolution and rigidity result for the equivariant energy critical wave map ($k = 2$) and the energy critical radial Yang-Mills equations.

In higher even dimensions $N \geq 8$, the logarithmically weakened channel of energy estimate, analogue to (4.17), still holds true. It was proved in [75] for $N = 8$ and recently extended to all other dimensions in [234]. However, the classification of non-radiative solutions away from the origin has not been proved yet, because of the technical fact that the nonlinearity $u \mapsto |u|^{\frac{4}{N-2}} u$ is no more twice differentiable at the origin. We note that it is still possible to classify non-radiative solutions in all large even dimensions for analytic nonlinearities [76]. The soliton resolution was then obtained by Jendrej and Lawrie [244] with a different strategy, in which the mechanism for proving the inelastic collision of solitons is not through a rigidity theorem (say in the style of Theorem 4.11), but through the use of modulation equations (introduced by these authors in a similar context in their work [243] on "two-bubble dynamics for threshold solutions") combined with a delicate "no return analysis", in the neighborhood of a multisoliton. This was an adaptation of their proof of the soliton resolution they obtained before for the energy critical wave maps equation under equivariant symmetry [245].

We now describe the results we obtained in the six dimensional case. We obtained the following two theorems for radial solutions to Equations (4.13) and (4.14). We introduce the set of radial nonzero stationary solutions:

$$\mathscr{W} = \left| \begin{array}{ll} \{(\iota W_{(\lambda)}, 0),\ (\iota,\lambda) \in \{-1,+1\} \times (0,\infty)\} & \text{for Equation (4.13)}, \\ \{(W_{(\lambda)}, 0),\ \lambda \in (0,\infty)\} & \text{for Equation (4.14)}. \end{array} \right.$$

Theorem 4.11 (A Liouville theorem for non-radiative $6D$ radial critical waves [77])**.** *Assume that u is a spherically symmetric solution of (4.13) (respectively, of (4.14)) that is global in time and bounded in energy norm:*

$$\sup_{t\in\mathbb{R}} \int_{\mathbb{R}^6} \left((\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2\right) dx < \infty,$$

and whose initial data $(u_0, u_1) \in \mathscr{H}$ is not a stationary solution of (4.13) (respectively, of (4.14)) in the sense that $(u_0,u_1) \notin \mathscr{W} \cup \{(0,0)\}$. Then there exists $R_0, \eta_0 > 0$ and $t_0 \in \mathbb{R}$ such that the following holds for all $t > t_0$ or for all $t < t_0$:

$$\int_{|x|>R_0+|t-t_0|} ((\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2) dx \geq \eta_0.$$

In other words, the stationary solutions are the only solutions that are non-radiative for $|x| > R_0 + |t-t_0|$ for any $R_0 > 0$ and $t_0 \in \mathbb{R}$.

Note that Theorem 4.11 implies the fact that the collision of two or more solitons emits some radiation, and thus that there is no pure multisoliton solution of either Equation (4.13) or Equation (4.14) in the radial case. As a consequence of the rigidity Theorem 4.11 and its proof, we obtain the soliton resolution for these equations.

Theorem 4.12 (Soliton resolution for radial $6D$ critical waves)**.** *Let u be a radial solution of (4.13) (respectively, of (4.14)) and T_+ be its maximal time of existence. Assume*

$$\limsup_{t\uparrow T_+} \int_{\mathbb{R}^6} \left((\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2\right) dx < \infty.$$

Then if $T_+ < \infty$, there exist $(v_0, v_1) \in \mathscr{H}$, an integer $J \in \mathbb{N}\setminus\{0\}$, and for each $j \in \{1,\ldots,J\}$, a positive function $\lambda_j(t)$ defined for t close to T_+ such that

$$0 < \lambda_J(t) \ll \ldots \ll \lambda_1(t) \ll (T_+ - t), \quad \text{as } t \to T_+,$$

and signs $(\iota_j)_{1\leq j\leq J}\in\{-1,+1\}^J$ *(respectively, the signs are* $(\iota_j)_{1\leq j\leq J}\equiv(1,....,1)$ *by convention for Equation* (4.14)*), such that*

$$\left\|(u(t),\partial_t u(t))-\left(v_0+\sum_{j=1}^J\frac{\iota_j}{\lambda_j^2(t)}W\left(\frac{x}{\lambda_j(t)}\right),v_1\right)\right\|_{\mathscr{H}}\underset{t\to T_+}{\longrightarrow}0.$$

If $T_+=+\infty$*, there exists a solution* v_L *of the linear wave equation* (4.18)*, an integer* $J\in\mathbb{N}$*, and for each* $j\in\{1,\dots,J\}$*, a positive function* $\lambda_j(t)$ *defined for large* t *such that*

$$0<\lambda_J(t)\ll\ldots\ll\lambda_1(t)\ll t,\quad \text{as } t\to+\infty$$

and signs $(\iota_j)_{1\leq j\leq J}\in\{-1,+1\}^J$ *(respectively, the signs are* $(\iota_j)_{1\leq j\leq J}\equiv(1,....,1)$ *by convention for Equation* (4.14)*), such that*

$$\left\|(u(t),\partial_t u(t))-\left(v_L(t)+\sum_{j=1}^J\frac{\iota_j}{\lambda_j^2(t)}W\left(\frac{x}{\lambda_j(t)}\right),\partial_t v_L(t)\right)\right\|_{\mathscr{H}}\underset{t\to+\infty}{\longrightarrow}0.$$

4.2.3 Energy radiation for wave equations

The purpose of this subsection is to explain more the classification of non-radiative solutions satisfying (4.16), as well as the complementary channel of energy estimate (4.17). Note that by finite speed of propagation, the solution in $\{|x|>R+|t|\}$ can be determined from the initial data on $\{|x|>R\}$ only, and is independent of the values of the solution in $\{|x|<R+|t|\}$.

Let us start with free waves, that are solutions to the linear wave equation

$$\begin{cases}\partial_t^2 u-\Delta u=0,\\ \vec{u}(0)=(u_0,u_1),\end{cases}\tag{4.19}$$

and let us investigate the validity of an estimate of the form

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|>R+|t|}|\nabla_{t,x}u(t,x)|^2dx\geq c_0\sum_{\pm}\int_{|x|>R}|\nabla_{t,x}u(0,x)|^2dx,\tag{4.20}$$

for some $c_0>0$. Non-radiative solutions, which are those satisfying

$$\sum_{\pm}\lim_{t\to\pm\infty}\int_{|x|>R+|t|}\left(|\nabla u(t,x)|^2+(\partial_t u(t,x))^2\right)dx=0\tag{4.21}$$

are clearly counter-examples to the estimate (4.20).

The so-called channel of energy estimate (4.20) is an observability estimate, in which one can measure the energy of the initial data in the set $\{|x|>R\}$ by measuring the outgoing energy that channels to null infinity in the set $\{|x|>R+|t|\}$ as $t\to\pm\infty$. The space of non-radiative solutions to the linear wave equation (4.19) is explicit as we shall see shortly, and finite dimensional with a dimension that grows with the spatial one N, and a clear difference appears between high dimensions, and low ones where rigidity occurs. Namely, for $N = 3$ [142] proves that the only non-radiative solutions of (4.19) (i.e. the solutions that satisfy (4.21)) are $(r^{-1}, 0)$ which corresponds to the tail of the stationary state (4.12). For higher odd dimensions $N \geq 5$ other non-radiative solutions exist as we will see below. Even dimensions $N \geq 4$ are more degenerate, since the estimate (4.20) for solutions to (4.19) only holds for one component of the initial data (u_0, u_1) and fails for the other. Nevertheless, in $N = 4$ dimensions [138] could still prove that the only non-radiative solutions are $(r^{-2}, 0)$ which are again the tail of the stationary state. For higher dimensions $N \geq 6$ other non-radiative solutions exist, as we will see below. It was left open to describe non-linear non-radiative waves in all high dimensions $N \geq 5$.

For the linear wave equation (4.19), non-radiative solutions are classified in all dimensions. There exist explicit polynomials $(p_m)_{0\leq m<\frac{N}{2}-1}$ of degree m, with p_m even or odd if m is even or odd, such that for each $0 \leq m < \frac{N}{2} - 1$, the function

$$\phi_m(t,x) = \frac{1}{|x|^{N-2-m}} p_m\left(\frac{t}{|x|}\right)$$

satisfies $\partial_t^2 \phi_m - \Delta \phi_m = 0$ for $|x| \neq 0$, and is non-radiative for any $R \in \mathbb{R}$ in the sense that it satisfies (4.16) (the integral being well-defined if $|t| > -R$). These properties are also satisfied by linear combinations of such functions, that is, for any $\boldsymbol{c} = (c_m)_{0\leq m<\frac{N}{2}-1} \in \mathbb{R}^{\lfloor \frac{N-1}{2} \rfloor}$, for the function $a_F = a_F[\boldsymbol{c}]$ defined by

$$a_F(t,x) = \sum_{0\leq m<\frac{N}{2}-1} c_m \phi_m(t,x). \tag{4.22}$$

Theorem 4.13 (Classification of non-radiative linear waves, see [253], [138], [282]). *Let $N \geq 2$ and $R > 0$. A radial solution u to (4.19) is non-radiative for $|x| > R + |t|$ if and only if there exists $\boldsymbol{c} \in \mathbb{R}^{\lfloor \frac{N-1}{2} \rfloor}$ such that $u(t,x) = a_F[\boldsymbol{c}](t,x)$ for all $|x| > R + |t|$.*

Note that for $N = 2$, the conclusion of Classification 4.13 has to be interpreted in the sense that $u(t,x) = 0$ for all $|x| > R + |t|$. In [253], [138] and [282] a much stronger statement is actually showed: that on the orthogonal of these non-radiative solutions, a channels of energy estimate (4.20) holds true (only for half of the data in even dimensions, see also [104]).

Much of the difficulty in even dimensions is linked to the existence of a resonance

$$\phi_{\frac{N}{2}-1}(t,x) = \frac{1}{|x|^{\frac{N}{2}-1}} p_{\frac{N}{2}-1}\left(\frac{t}{|x|}\right),$$

for $p_{\frac{N}{2}-1}$ a polynomial of degree $\frac{N}{2} - 1$ that has the same parity as $\frac{N}{2} - 1$. It solves $\partial_t^2 \phi_{\frac{N}{2}-1} - \Delta \phi_{\frac{N}{2}-1} = 0$ for $|x| \neq 0$ but critically fails to belong to the energy space.

We extended the classification of Theorem 4.13 to non-linear wave equations in $N \geq 3$ dimensions,

$$\begin{cases} \partial_t^2 u - \Delta u = \varphi(|x|, u), \\ \vec{u}(0) = (u_0, u_1), \end{cases} \tag{4.23}$$

with analytic energy critical nonlinearities

$$\varphi(|x|,u)=\sum_{k\geq 2}^{\infty}\varphi_k|x|^{(k-1)(\frac{N}{2}-1)-2}u^k, \tag{4.24}$$

where φ_k are coefficients such that $\lim_{k\to\infty}\tau^k\varphi_k=0$ for all $\tau>0$, or energy critical power nonlinearities in $N=4,6$ or N odd dimensions:

$$\varphi(|x|,u)=|u|^{\frac{4}{N-2}}u. \tag{4.25}$$

Nonlinearities of the form (4.24) and (4.25) include the aforementioned equations (4.25) as well as the energy critical equivariant wave maps and Yang Mills equations after a suitable dimensional reduction for the last two.

The result we are going to state concerns solutions with small energy in the exterior cone $|x|>R+|t|$. We were then able to extend them in order to describe all non-radiative solutions without smallness assumption, which we shall only mention in Remark 4.15. We introduce for $\boldsymbol{c}\in\mathbb{R}^{\lfloor\frac{N-1}{2}\rfloor}$ and $R>0$ the norm $|\boldsymbol{c}|_R=\left|(R^{-\frac{N}{2}+1+m}c_m)_{0\leq m<\frac{N}{2}-1}\right|$ where $|\cdot|$ denotes the usual Euclidean norm. This is the energy of non-radiative free waves for $|x|>R+|t|$ as:

$$\delta|\boldsymbol{c}|_R^2\leq\sup_{t\in\mathbb{R}}\int_{|x|>R+|t|}|\nabla_{t,x}a_F[\boldsymbol{c}](t,x)|^2dx\leq\frac{1}{\delta}|\boldsymbol{c}|_R^2$$

where $\delta>0$ is independent of R. For $R>0$, we introduce for functions defined for $|x|>R$:

$$\|u\|_{L^2_R}^2=\int_{|x|\geq R}|u|^2dx,\qquad \|u\|_{\dot H^1_R}=\|\nabla u\|_{L^2_R}$$

and denote by L^2_R and $\dot H^1_R$ their associated Hilbert spaces (where in addition we require that $\lim_{|x|\to\infty}u(x)=0$ for $u\in\dot H^1_R$). We define

$$\Pi_{\dot H^1,R}=\Pi_{\dot H^1_R}\left(\text{Span}\left(\left(\frac{1}{|x|^{N-2-2l}}\right)_{0\leq l\leq\lfloor\frac{N-3}{4}\rfloor}\right)\right),\qquad \Pi^\perp_{\dot H^1,R}=1-\Pi_{\dot H^1,R}$$

$$\Pi_{L^2,R}=\Pi_{L^2_R}\left(\text{Span}\left(\left(\frac{1}{|x|^{N-2-2l}}\right)_{0\leq l\leq\lfloor\frac{N-5}{4}\rfloor}\right)\right),\qquad \Pi^\perp_{L^2,R}=1-\Pi_{L^2,R}$$

where $\Pi_H(E)$ is the orthogonal projection in the Hilbert space H onto the closed set E, and

$$\Pi_{\mathscr{H},R}(u_0,u_1)=(\Pi_{\dot H^1,R}u_0,\Pi_{L^2,R}u_1),\qquad \Pi^\perp_{\mathscr{H},R}(u_0,u_1)=(\Pi^\perp_{\dot H^1,R}u_0,\Pi^\perp_{L^2,R}u_1).$$

We remark by the classification Theorem 4.13 above that such spaces characterise the initial data of free non-radiative waves as a solution u to (4.19) with $(u_0,u_1)\in\mathscr{H}$ satisfies $u(t,x)=a(t,x)$ for all $|x|>R+|t|$ if and only if $(u_0,u_1)=\Pi_{\mathscr{H},R}(u_0,u_1)$. We were able to show that the set of non-radiative solutions is a small deformation of the set of explicit non-radiative linear waves (4.22).

Theorem 4.14 (Classification of non-radiative nonlinear waves away from the origin [76]). *Assume either $N \geq 4$ and φ of the form (4.24), or $N \geq 3$ is odd, or $N \in \{4,6\}$, and $\varphi(u) = |u|^{\frac{4}{N-2}}u$, and let $\delta = 1$ or $\delta = \frac{4}{N-2}$ in the former or latter cases respectively. Then there exist $\epsilon, \epsilon' > 0$ such that the following holds true for any $R > 0$.*

1. Existence. *Assume that $\boldsymbol{c} \in \mathbb{R}^{\lfloor \frac{N-1}{2} \rfloor}$ satisfies $|\boldsymbol{c}|_R \leq \epsilon$. Then there exists a unique solution $a = a[\boldsymbol{c}]$ to (4.23) that is non-radiative for $|x| > R + |t|$ such that*

$$\Pi_{\mathscr{H}_R} \vec{a}(0) = \vec{a}_F[\mathbf{c}](0) \qquad \textit{and} \qquad \|\Pi^{\perp}_{\mathscr{H}_R} \vec{a}(0)\|_{\mathscr{H}_R} \lesssim |\boldsymbol{c}|_R^{1+\delta}.$$

2. Uniqueness. *Conversely, if u is any solution to (4.23) that is non-radiative for $|x| > R + |t|$ with $\int_{|x|>R} |\nabla_{t,x} u(0)|^2 dx \leq \epsilon'^2$, then there exists $\boldsymbol{c} \in \mathbb{R}^{\lfloor \frac{N-1}{2} \rfloor}$ with $|\boldsymbol{c}|_R \leq \epsilon$ such that for $a[\boldsymbol{c}]$ described in (1) above:*

$$\forall |x| > R + |t|, \qquad u(t,x) = a[\mathbf{c}](t,x).$$

The result of Theorem 4.14 was already proved in the low dimensions $N = 3, 4$ in [142] and [138].

Remark 4.15. — We proved in [76] better estimates actually than what is written in (1), in the form of optimal weighted estimates.

— While Theorem 4.14 is a small data result, we proved in [76] that the solutions it describe, which are defined for $|x| > R + |t|$ for some $R > 0$ large enough depending on $\boldsymbol{c}$, can be extended to a unique maximal set $|x| > R^*[\mathbf{c}] + |t|$, and give the whole set of non-radiative solutions.

The first main novelty in the proof of Theorem 4.14 lied in the introduction of a fixed point scheme between radiation fields and initial data. This is enough for the proof of Theorem 4.14 in the non-degenerate case of odd dimensions. The main difficulty resides in the treatment of degenerate directions in even dimensions. First, we showed a regularity gain of a derivative for u_0 and u_1. Then, a resonant direction needs to be ruled out. We did so by approximating with solutions of the form $r^{-d/2+1}\phi(t/r)$, which introduces an elliptic operator with a resonance. Careful computations show that depending on the nonlinearity, quadratic effects either do not see the resonance, or lead to an energy increase between nearby dyadic regions. In both cases, an excitation of the resonance is prohibited since it is shown to lead to infinite energy data.

Thanks to Theorems 4.13 and 4.14, we are able to describe all non-radiative solutions. We now turn to radiative solutions, and we want to quantify, through an estimate of the form (4.20), how much of their initial energy will be emitted in the zone $|x| > R + |t|$ as $t \to \pm\infty$.

For the free wave equation (4.19), such estimates for $R = 0$ or $R > 0$ were investigated in [139, 141, 104, 253, 138, 282]. Let us consider the case $R > 0$. In odd dimensions the energy that is radiated controls the initial energy of the orthogonal projection of the data to the set of non-radiative solutions:

$$\|\Pi^{\perp}_{\mathscr{H},R}(u_0,u_1)\|^2_{\mathscr{H}_R} \lesssim \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{t,x} u(t,x)|^2 dx \qquad \text{if } N \geq 3 \text{ is odd.} \tag{4.26}$$

However, it only controls half of it in even dimensions [104]:

$$\|\Pi^{\perp}_{\dot H^1,R} u_0\|^2_{\dot H^1_R} \lesssim \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{t,x} u(t,x)|^2 dx \quad \text{if } N \equiv 4 \text{mod} 4, \tag{4.27}$$

$$\|\Pi^{\perp}_{L^2,R} u_1\|^2_{L^2_R} \lesssim \sum_{\pm} \lim_{t\to\infty} \int_{|x|>R+|t|} |\nabla_{t,x} u(t,x)|^2 dx \quad \text{if } N \equiv 6 \text{mod} 4. \tag{4.28}$$

while the full estimate (4.26) is known to fail [104]. Other recent results on the asymptotic behaviour of linear waves can be found in [122, 105, 281].

As explained in the previous Section 4.2.2, in proving continuous in time soliton resolution one has to study properties of multisoliton solutions. Hence, rather than the free wave equation (4.19), we have to consider the linearized evolution of (4.12) around a stationary state

$$\begin{cases} \partial_t^2 u_L - \Delta u_L + V u_L = 0 \\ \vec u_{L\restriction t=0} = (u_0, u_1), \end{cases} \tag{4.29}$$

where $(u_0,u_1) \in \mathscr{H} = \dot H^1 \times L^2(\mathbb{R}^N)$ and

$$V = -\frac{N+2}{N-2} W^{\frac{4}{N-2}}, \qquad W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N-2}{2}}.$$

For the linearised wave equation (4.29), two natural counter examples to estimates like (4.26) and (4.27)-(4.28) are ΛW and $t\Lambda W$ (for $N \geq 5$), as they are non-radiative i.e. satisfy (4.21) for all $R \in \mathbb{R}$ from an explicit computation. Here $\Lambda W = x.\nabla W + \frac{N-2}{2} W$ is in the radial kernel of $-\Delta + V$. For odd dimensions, the strong estimates (4.26) and the analogue estimate for $R=0$ in which the projection operators are removed, allowed the authors of [143] to obtain an estimate for Equation (4.29) for $R=0$ in the radial case:

$$\|\tilde\Pi^{\perp}_{\dot H^1} u_0\|^2_{\dot H^1} + \|\tilde\Pi^{\perp}_{L^2} u_1\|^2_{L^2} \lesssim \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>|t|} |\nabla_{t,x} u(t,x)|^2 dx \qquad \text{if } N \geq 3 \text{ is odd.} \tag{4.30}$$

Above, we used the projectors

$$\tilde\Pi^{\perp}_{\dot H^1} = \Pi_{\dot H^1}(\text{Span}(\Lambda W))^{\perp}, \qquad \tilde\Pi^{\perp}_{L^2} = \Pi_{L^2}(\text{Span}(\Lambda W))^{\perp}.$$

We were able to extend such estimate to the even dimensional case. We focused on six and eight dimensions, and an extension of our method allowed Hip [234] to prove analogue estimates in all even dimensions. Since only (4.27)-(4.28) are valid in even dimensions, and that the analogue of the estimate (4.26) is known to fail with an explicit counter-example, the estimate (4.30) was doomed to fail in even dimensions. We found a weaker version of (4.30) involving the space $Z_\alpha(\mathbb{R}^N)$ associated to the following dyadic norm for $\alpha \in \mathbb{R}$:

$$\|f\|_{Z_\alpha} = \sup_{R>0} \frac{R^{-\frac{N}{2}-\alpha}}{\langle \log R \rangle} \left(\int_{R<|x|<2R} f^2 dx \right)^{\frac12}.$$

Note that, $L^2(\mathbb{R}^6) \subseteq Z_{-3}(\mathbb{R}^6)$ with $\|f\|_{Z_{-3}(\mathbb{R}^6)} \lesssim \|f\|_{L^2(\mathbb{R}^6)}$, so that $Z_{-3}(\mathbb{R}^6)$ is a logarithmic weakening at 0 and ∞ of $L^2(\mathbb{R}^6)$.

Theorem 4.16 (Channels of energy around the ground state [75]). *Assume $N = 6$. There exists $C > 0$ such that any radial solution u_L of (4.29) satisfies:*

$$\left\|\tilde{\Pi}^{\perp}_{L^2} u_1\right\|^2_{L^2} + \left\|\nabla\tilde{\Pi}^{\perp}_{\dot{H}^1} u_0\right\|^2_{Z_{-3}} \leq C \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>|t|} |\nabla_{t,x} u(t,x)|^2 dx. \tag{4.31}$$

Remark 4.17. — We obtained the analogue estimate for $N = 8$ in [75].

— We extended in [75] the estimate (4.31) for the linearized evolution around one stationary state to an analogue estimate close to a multi-soliton consisting of $J \geq 1$ radial stationary states decoupled by scaling.

— We showed that the stronger estimates (4.30) fail in the six dimensional case, hence the need for a weakening such as the use of the Z norm.

Remark 4.18. An analog of the bound on u_0 in (4.31) for solutions of the free wave equation (4.19) is also valid. Indeed, let Φ be a smooth radial function with compact support included in $\{r > 0\}$ such that $\int \frac{1}{r^4}\Phi dx \neq 0$. Then there exists a constant $C > 0$ such that for all solution u_F of (4.19) with initial data (u_0, u_1),

$$\int \nabla u_0 \cdot \nabla\Phi \, dx = 0 \Longrightarrow \left\|u_0\right\|^2_{Z_{-2}} \lesssim \left\|\nabla u_0\right\|^2_{Z_{-3}} \leq C \sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>|t|} |\nabla_{t,x} u(t,x)|^2 dx, \tag{4.32}$$

with a proof that is similar to the proof of (4.31). Note that this does not imply the estimate on the projection: $\left\|\Pi_{\dot{H}^1}\left(\{\Phi\}^{\perp}\right)u_0\right\|_{Z_{-2}} \lesssim \sqrt{E_{\text{out}}}$, which is false.

Chapter 5

Weak wave turbulence

5.1 Kinetic limit in the isotropic case

5.1.1 Set up for weakly turbulent solutions to the nonlinear Schrödinger equation

Random initial data for weak wave turbulence

Let us first present the problem of the derivation of the so-called kinetic wave equation as an effective equation to describe weakly turbulent solutions to some nonlinear dispersive equations. We consider the main problem for this issue, which is the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = \lambda^2 |u|^2 u \\ u(t=0) = u_0 \end{cases} \tag{5.1}$$

set on the torus: $x \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d)$, where $d \geq 2$, and with initial data of the form

$$u_0(x) = \epsilon^{d/2} \sum_{k\in\mathbb{Z}^d} A(\epsilon k) G(k) e^{ik\cdot x}, \tag{5.2}$$

where $A \in \mathscr{C}_0^\infty(\mathbb{R}^d, \mathbb{R})$ and $(G(k))_{k\in\mathbb{Z}^d}$ are independent standard centred complex Gaussians. The normalization is such that on average $\|u_0\|_{L^2} \sim 1$, and, by Khinchine's inequality, $\|u_0\|_{L^p} \sim_p 1$ on average as well, for any $p < \infty$. Initial data of the form (5.2) represent a solution that initially satisfies the RPA assumption: random phases and amplitudes for the Fourier coefficients (see for exemple the book of Nazarenko [343]). Such an assumption is justified, from a physical point of view, by the fact that the effects of the linear Schrödinger evolution, together with weak nonlinear coupling between Fourier coefficients, and possibly some randomness inherited from a random forcing (i.e. some random wind at the see surface), should randomize the Fourier coefficient in such a way. It will be clear however from the discussion below, explaining the appearance of a kinetic equation with a time arrow while the equation (5.1) has time reversal symmetry, that the form (5.2) cannot be propagated for large times, as this would contradict Loschmidt's paradox. A satisfactory form of the initial data, more general than (5.2), is then yet to be found, even though the recent breakthrough work of Deng-Hani [126] provides an answer (via a specific form for the correlations between Fourier coefficients in terms of Feynman diagrams for their cumulants).

Relevant time scales, parameters range

There are four relevant times for the present problem:

— $T_{lin} = \epsilon^2$: characteristic time scale for the linear part of the equation

— $T_{nonlin} = \dfrac{1}{\lambda^2}$: characteristic time-scale for the nonlinear part of the equation (given that we expect $\|u\|_{L^\infty} \sim 1$ - up to logarithmic losses).

— $T_{finite-box} = 1$: time at which the resonance modulus $(k, \ell, m, n) \mapsto |k|^2 - |\ell|^2 + |m|^2 - |n|^2$ is no-longer equidistributed at scales $\lesssim t^{-1}$. Since $|k|^2 - |\ell|^2 + |m|^2 - |n|^2$ can only take integer values for $k, \ell, m, n \in \mathbb{Z}^d$, one sees that there is distinction between the cases $t \ll 1$ for which the nearly resonant set $\{||k|^2 - |\ell|^2 + |m|^2 - |n|^2|\} \lesssim t^{-1}$ contains many values for the resonance modulus $|k|^2 - |\ell|^2 + |m|^2 - |n|^2$ and the case $t > 1$ for which it only contains the value $|k|^2 - |\ell|^2 + |m|^2 - |n|^2 = 0$. When $t \gtrsim 1$ the exact geometry of the domain in (5.1) matters, which is called the finite-box effect.

— $T_{kin} = \dfrac{1}{\epsilon^2 \lambda^4}$: characteristic time scale for the kinetic wave equation. We shall derive it in the next subsection.

We will consider the regime where

$$T_{lin} \ll T_{kin} \iff T_{nonlin} \ll T_{kin} \iff T_{lin} \ll T_{nonlin} \iff \lambda\epsilon \ll 1,$$

which means that the regime we are considering is weakly nonlinear. In this regime,

$$T_{lin} \ll T_{nonlin} \ll T_{kin}.$$

We choose a power type relation between the strength of the nonlinearity and the frequency

$$\lambda = \epsilon^{-\gamma} \qquad \text{with} \qquad 0 < \gamma < \frac{1}{2} \tag{5.3}$$

so that:

$$T_{nonlin} < 1 < T_{kin}.$$

Finally, let us consider resonances. The resonance modulus is classically given by

$$\Omega(k, \ell, m, n) = |k|^2 - |\ell|^2 + |m|^2 - |n|^2.$$

Since $k, \ell, m, n \in \mathbb{Z}^d$, Ω takes integer values; and since $|k|, |\ell|, |m|, |n| \lesssim \frac{1}{\epsilon}$, it satisfies $|\Omega| \lesssim \frac{1}{\epsilon^2}$. This means that only time scales T such that

$$\epsilon^2 \ll T \ll 1$$

are susceptible to yield the kinetic wave equation in an asymptotic regime. Indeed, if $T \lesssim \epsilon^2$, resonances are hardly playing any role; while if $T \gtrsim 1$, the resonance moduli Ω cannot be equidistributed modulo $\frac{1}{T}$, which prevents from taking the discrete to continuous limit in frequency.

The kinetic wave equation and formal derivation of T_{kin}

Heuristic derivations, to which we will come back shortly, show that, as $\epsilon\lambda \to 0$ (weakly non-linear regime) and $\epsilon \to 0$ (high frequency limit), and $T_{kin} \ll 1$ with T_{kin} given below (avoidance of finite-box effects), scaling properly the square modulus of the Fourier coefficients of u, they satisfy on average (with $\mathbb{E}$ the expectation)

$$\epsilon^{-d}\mathbb{E}\left|\hat{u}(\lfloor \epsilon k \rfloor)\left(T_{kin}t'\right)\right|^2 \longrightarrow \rho(t',k), \qquad \text{for } T_{kin} = \frac{1}{\epsilon^2\lambda^4} \tag{5.4}$$

where ρ solves the kinetic wave equation

$$\begin{cases} \partial_{t'}\rho(t',k) = \mathscr{C}[\rho](k), \\ \rho(0,k) = |A(k)|^2, \end{cases} \tag{5.5}$$

with the collision operator given by

$$\begin{aligned} \mathscr{C}[\rho](k) = c_0 \int_{(\mathbb{R}^d)^3} & \delta(k+\ell-m-n)\delta(|k|^2+|\ell|^2-|m|^2-|n|^2) \\ & \rho(k)\rho(\ell)\rho(m)\rho(n)\left[\frac{1}{\rho(k)}+\frac{1}{\rho(\ell)}-\frac{1}{\rho(m)}-\frac{1}{\rho(n)}\right] d\ell\, dm\, dn. \end{aligned} \tag{5.6}$$

and $c_0 = 2^{2-2d}\pi^{1-2d}$.

Note that, for quick formal computations to derive the kinetic time and the restriction $T_{kin} \ll 1$, one can consider the first nonlinear correction to the linear dynamics obtained by Duhamel formula:

$$\begin{aligned} \hat{u}(k) \approx\ & e^{-it|k|^2}\epsilon^{\frac{d}{2}}A(\epsilon k)G(k) \\ & - ie^{-it|k|^2}\epsilon^{\frac{3}{2}d}\lambda^2 \sum_{\ell+m+n=k}\int_0^t e^{is(|k|^2-|\ell|^2+|m|^2-|n|^2)}ds A(\epsilon\ell)A(-\epsilon m)A(\epsilon n)G(\ell)\overline{G(-m)}G(n). \end{aligned} \tag{5.7}$$

On the nearly resonant set $||k|^2-|\ell|^2+|m|^2-|n|^2| \lesssim t^{-1}$ the time integral is the biggest, of order t. Assuming $t \ll 1$, there are $\epsilon^{2-2d}t^{-1}$ integer points of size $\lesssim \epsilon^{-1}$ in this set. The sum of gaussians restricted to this set is performed over $\epsilon^{2-2d}t^{-1}$ integer points, hence of order $\epsilon^{1-d}t^{-1/2}$ from square root cancellation. The first nonlinear correction is thus of order $\epsilon^{d/2}(\lambda^2\epsilon\sqrt{t})$ and compares with the initial datum of order $\epsilon^{d/2}$ for $t \sim T_{kin}$.

When $t > 1$, this computation fails and so does the derivation of the kinetic wave equation for (5.1). For exemple, a divergence result for $t \gtrsim (\log\epsilon)^c$ is given Proposition 5.4 (even though it is not an exact result saying that the kinetic limit fails, it is believed that it fails and this could be used to show it). If $t > 1$ but that the equation is no-longer set on the flat torus $\mathbb{T}^d$ but on a generic torus, we expect the kinetic limit to hold for a certain range of times, as we will discuss in the next Section 5.2.

We finally mention that for more general initial data with Gaussian iid Fourier modes (i.e. $A(\epsilon k)$ replaced by a more general $f(k)$ in (5.2) with algebraic decay), Deng-Nahmod-Yue derived the notion of "probabilistic scaling" in [129] to predict Sobolev spaces in which the Cauchy problem is locally well-posed. This is to random initial data Cauchy problems the analogue of the computation of critical Sobolev exponents for deterministic Cauchy problems.

Rescaling to a large torus

In the problem formulated above, the equation is set on a torus of size 1, and u_0 has size ~ 1 in any L^p (neglecting logarithmic factors if $p = \infty$), and varies on a typical scale $\sim \epsilon$. For the reader's convenience, we show how it can be rescaled to fit the setup adopted in [170, 35, 36] or by Deng-Hani in their series of work starting with [123] to which we will come back.

We now let $\epsilon = L^{-1}$, and rescale u by setting

$$u'(t', x') = u'\left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}\right) = \epsilon^{d/2} u\,(t, x)\,.$$

The equation solved by u' is now

$$i\partial_{t'} u' + \Delta_{x'} u' = (\lambda')^2 |u'|^2 u' \qquad \text{with} \qquad \lambda' = \lambda \epsilon^{1-\frac{d}{2}}. \tag{5.8}$$

In this new setting, the domain has size L, u' is of size $\sim L^{\frac{d}{p}-\frac{d}{2}}$ in L^p, and varies on a typical scale ~ 1. These orders of magnitude coincide with the framework adopted in [36].

To convert results from (5.1) to (5.8), observe that the time scale t_0 for (5.1) corresponds to $t'_0 = \frac{t_0}{\epsilon^2}$ for (5.8). In particular, the kinetic time scale $T_{kin} = \frac{1}{\lambda^4 \epsilon^2}$ for (5.1) corresponds to the kinetic time scale $T'_{kin} = \dfrac{1}{(\lambda')^4 \epsilon^{2d}} = \dfrac{L^{2d}}{(\lambda')^4}$ for (5.8).

5.1.2 Derivation of the kinetic limit

Background prior to our first work on the derivation of the kinetic wave equation

The kinetic wave equation was first introduced by Peierls [360] in his work on solid state physics, and independently much later by Hasselmann [224, 225] who worked on water waves. Later, Zakharov and collaborators [425, 424] revisited the topic and provided a broad framework applying to various Hamiltonian systems satisfying weak nonlinearity, high frequency, phase randomness assumptions. Nowadays, the kinetic theory of waves, known as wave turbulence theory, is fundamental to the study of nonlinear waves, having applications e.g. in plasma theory [111], oceanography [240, 356] and crystal thermodynamics [399]. For an introduction to this broad research field and its applications, see e.g. Nazarenko [343], Newell-Rumpf [347].

As far as rigorous mathematics go, a fundamental work, building up on [159], is due to Lukkarinen and Spohn [293], who reach the kinetic time scale for the correlations in a system at statistical equilibrium, obtaining a linearized version of the kinetic wave equation (see also [169] for another derivation of the linearized system). More heuristic considerations by the same authors on this question can be found in [399] and [292]. We are indebted to them for several ideas in the present work, around Feynman diagrams and their estimation.

The question of the derivation of (5.5) for random data out of statistical equilibrium was first tackled in [36], which is the source of a several ideas which we extend here (construction of an approximate solution and control of the remainder). Compared to this work, we introduced a number of novel ideas: use of Bourgain spaces, finer analysis of the expansion through Feynman diagrams, and estimate of the average operator norm of the linearized operator. This allows us to get arbitrarily close to the kinetic time scale.

Another appearance of (5.5) from (5.1) is when applying a random forcing to the equation and a dissipation [424]. Recently, Dymov and Kuksin [149, 150] studied the truncated series expansion for this model. They showed how in a weakly nonlinear context with non-vanishing dissipation the three first terms of the expansion were agreeing with (5.5), studying also higher order terms but without controlling the full solution.

Derivation of related collisional kinetic models

The kinetic wave equation is to phonons, or linear waves, what the Boltzmann equation is to classical particles. The derivation of the Boltzmann equation was put on a rigorous mathematical foundation with the foundational work of Lanford [273] its more recent clarification by Gallagher-Saint-Raymond-Texier [183], and the even more recent work of Deng-Hani-Ma [127] who justified the derivation for all times for which the solution of the Boltzmann equation exists. A few articles deal with the derivation of kinetic models for quantum particles [13, 14, 15]; this question is closely related to the derivation of the kinetic wave equation, but is harder, since (5.1) can be thought of as an intermediary step between a quantum mechanical model with a large number of particles, and kinetic theory.

Another strand of research focuses on linear dispersive models with random potential, from which one can derive the linear Boltzmann equation on a short time scale [398], and the heat equation on a longer time scale [158, 157].

Finally, let us mention the possibility of deriving Hamiltonian models for NLS with deterministic data in the infinite volume, or big box, limit [170, 35].

Our result on the onset of the kinetic wave equation

We obtained, when the kinetic time is close to 1, the existence of a solution over the nontrivial time interval $[0, \epsilon^{0+}T_{kin}]$ which missed the kinetic time by an arbitrarily small algebraic loss. We also obtained the validity of the approximation by the kinetic wave equation on this time interval. But since this time is shorter than the kinetic time scale, this means that we obtained the validity of the Taylor expansion of the kinetic wave equation, which is why it is rather a justification of the onset of the kinetic wave equation. The existence of (5.1) up to time 1 (on the complement of an exceptional set) is non trivial, and given by the first part of the Theorem; as for the local well-posedness of (5.5), it has been established for instance in [190]. We refer to [166] for global weak solutions and other qualitative results for (5.5).

Since $T_{kin} = \epsilon^{-2+4\gamma}$, as γ approaches $1/2$, the time interval $[0, 1]$ gets close, up to an arbitrarily small polynomial loss, from the kinetic time. Note that T_{kin} of order 1 means $\lambda^4\epsilon^2$ of order 1.

Theorem 5.1 (Derivation of the kinetic wave equation [82] up to arbitrarily small power loss)**.** *Pick* *any* $0 < \gamma < \frac{1}{2}$ *and* $\eta, \mu > 0$. *Then there exist* $0 < \epsilon^* \leq 1$ *and* $\nu > 0$ *depending on* γ, μ *and* η *such that for all* $0 < \epsilon \leq \epsilon^*$, *a set* E *of measure* $\mathbb{P}(E) > 1 - \epsilon^\mu$ *exists such that*

— Existence of solution: *On* E, *the equation* (5.1) *with initial datum* (5.2) *admits a solution* $u \in \mathscr{C}^\infty([0,1] \times \mathbb{T}^d)$.

— Validity of the kinetic wave equation: *Furthermore, denoting* $\mathscr{A} = |A|^2$, *for any* $\lambda^{-2} \leq t \leq \epsilon^\eta$*:*

$$\sum_{k \in \mathbb{Z}^d} \left| \mathbb{E}\left[\mathbb{1}_E \left[|\widehat{u}(k,t)|^2 - \epsilon^d \mathscr{A}(\epsilon k) - \epsilon^d \frac{t}{T_{kin}} \mathscr{C}(\mathscr{A})(\epsilon k) \right] \right] \right| \lesssim \frac{\epsilon^\nu t}{T_{kin}}, \tag{5.9}$$

where $\mathscr{C}$ *is the collision operator defined in* (5.6).

Let us remark that many assumptions could be relaxed. Among them, randomization through Gaussians is convenient since it allows the use of Wick's formula, but other randomizations should also be possible, the function A was taken in $\mathscr{C}_0^\infty$ to simplify the proof but much milder hypotheses should suffice, thetorus was chosen to be rational, but our proof applies verbatim to irrational tori - thanks to the work of Bourgain and Demeter [28], which gives Strichartz estimates for them. It should be possible to make the size of the exceptional set exponentially small in ϵ. Finally, as should be expected, whether the equation (5.1) is focusing or defocusing does not change anything in our argument.

Remark 5.2. *Bounds on the solution* Since the regime considered in the theorem is supercritical (in the classical, deterministic sense) for $d \geq 5$, the mere existence of a smooth solution on a unit time interval is not trivial. The proof of the theorem actually gives quantitative bounds: namely, for some $s > \frac{d}{2} - 1$ and $b > \frac{1}{2}$, the norm of the solution in a suitably scaled Bourgain space is $O(\epsilon^{-\mu})$.

Remark 5.3. *On the range of parameters*:

(i) *Restriction to small times* $t \lesssim 1$. As explained previously, one needs $t \ll 1$ in (5.9) to obtain that the right hand side is of lower order compared to the size of the correction $\frac{t}{T_{kin}}\mathscr{C}(\mathscr{A})$ in the left hand side. For a similar reason of equidistribution for the dispersion relation of the Schrödinger equation, we cannot control the solution on a time interval $[0,T]$ with $T \gg 1$.

(ii) *Restriction to large kinetic times* $T_{kin} \gg 1$. In the case $T_{kin} \lesssim 1$, the bounds we obtain for the approximate solution are powers of $\frac{1}{T_{kin}}$ due to degenerate low frequencies effects. This prevents both the convergence of our series expansion and its linear stability, so that we cannot cover this case. This explains the restriction $\gamma < \frac{1}{2}$. This is where our result is non-optimal since the kinetic time scales to reach are $T_{kin} \ll 1$. The low frequencies effects we encountered were justified rigorously to lead to divergent Feynman diagrams in the regime $T_{kin} \ll 1$ in our work [81] that followed. However, this divergence was linked to a specific graph property of the Feynman diagrams, and when summing over all such Feynman diagrams a cancellation appeared; this was proved by the analysis of chains in the work Deng-Hani [125].

(iii) *Unit time interval for existence*. We need our time interval for existence to avoid equidistribution problems, and not to exceed the kinetic time. The first condition requires to consider small times $t \lesssim 1$ from (i) above, for which the second condition is always fulfilled from (ii) above. Hence the time interval $[0, 1]$.

(iv) *Nontrivial nonlinear effects*. The condition $\gamma > 0$ gives $T_{nonlin} = \lambda^{-2} \leq 1$, so that the time interval $[0, 1]$ exceeds the nonlinear time and the existence result is nontrivial. The condition $\lambda^{-2} \leq t \leq 1$ in (5.9) is due to the following. Before the nonlinear time λ^{-2}, the nonlinear effects did not kick in, so that the kinetic wave equation is irrelevant on $[0, \lambda^{-2}]$.

(v) *Validity of (KWE), almost up to the kinetic time*. The above shows that the prediction (5.4) is satisfied on time scales $\ll 1$; namely, the expectation of $\epsilon^{-d}|\widehat{u}_k|^2$ and the solution of (5.5) are as close as expected. In particular, we are able to treat kinetic time scale of order $\epsilon^{-\mu}$ for $\mu > 0$ as close as we want to 0 (that is, for $0 < \frac{1}{2} - \gamma \ll 1$). This shows how $T_{kin} = 1$ is attainable with arbitrarily small polynomial loss.

Nonlinear dispersive equations with random data

The main difficulty in proving Theorem 5.1 is to control a solution to (5.1) with random data of typical frequency ϵ^{-1} and amplitude λ, in the regime (5.3). But other regimes are of interest, and have been considered previously: at fixed frequency size and vanishing amplitude ($\epsilon = 1$, $\lambda \to 0$), correlations of the solutions were studied in [116]. Replacing $A(\epsilon k) = c_k$ in (5.2) by $\epsilon = 1 = \lambda$, Burq and Tzvetkov obtained the existence of solutions below the critical regularity for c_k [41, 42] (see [66, 340] for the Schrödinger equation). This result exploits the fact that randomizing Fourier series yields improved L^p bounds (see [359] for the historical work on the torus and [6, 39] for other Riemannian manifolds), which we also use here. Other measures than the ones generated by pushing forward $\sum c_k G(k) e^{ik.x}$ can be considered as well, as the Gibbs measure which was showed to be invariant in low dimensions [25, 26, 128], and interpolation between Gibbs and white noise [352]. Since we are here out of equilibrium, controlling the flow requires uniform bounds as $\epsilon \to 0$ instead of invariant measures properties.

Full derivation of the kinetic wave equation by Deng-Hani

At the same time we released our work [82] presented in this section, a similar result was obtained independently by Deng and Hani [123]. In an impressive series of works that followed, Deng and Hani reached the kinetic timescale in the particular regime $T_{kin} = \delta \ll 1$ in [125], then they reached it for all the regimes $T_{kin} \ll 1$ in [124]. Finally, they were able to show recently the validity of the kinetic equation for all large kinetic times for which the kinetic solution exists [126]. Their methods enabled them to show the validity of the Boltzmann equation as an approximation for hard spheres dynamics for all large kinetic times for which the kinetic solution exists in [127].

5.1.3 Finite-box effects for large time scales

To explain a result we obtained in [81] on the most probable failure of the kinetic limit when the formal kinetic time scale T_{kin} is too large $T_{kin} \gtrsim 1$, we now introduce the Dyson series used to obtain an asymptotic expansion for the solution to (5.1)-(5.2) in terms of Feynman diagrams.

To study a solution to (5.24), we apply first Wick renormalisation for the phase: $u = e^{-it\lambda^2 \frac{2}{(2\pi)^d}\|u_0\|_{L^2}^2} v$. This is a usual transformation that cancels the leading order nonlinear effect, as $\mathbb{P}_\nu |u|^2 u \approx 2(2\pi)^{-d}\|u_0\|_{L^2}^2 u$ in some appropriate sense, and which does not alter relevant statistics. The equation for v is

$$i\partial_t v = -\Delta v + \lambda^2 \mathbb{P}_\nu \left(|v|^2 - \frac{2}{(2\pi)^d}\|v\|_{L^2}^2 \right) v.$$

Since linear dynamics acts at a shorter time scale than nonlinear effects (by the weak nonlinearity assumption), it is customary to find an approximate solution by iterating Duhamel's formula, giving the so-called Dyson series expansion. We define for this purpose the truncation operator

$$\mathscr{F}[P(a,b,c)](k) = \frac{1}{(2\pi)^d} \sum_{k_1+k_2+k_3=k} \widehat{a}(k_1)\widehat{b}(k_2)\widehat{c}(k_3)(1-\delta(k_1+k_2)-\delta(k_2+k_3)).$$

This gives the decomposition of the product

$$abc = P(a,b,c) + \frac{1}{(2\pi)^d}\langle \overline{a}, b\rangle c + \frac{1}{(2\pi)^d} a \langle \overline{b}, c\rangle.$$

A quasi-solution of (5.1)-(5.2) is then a series expansion

$$u^{app} = e^{-it\lambda^2 \frac{2}{(2\pi)^d}\|u_0\|_{L^2}^2} v^{app} \qquad \text{with} \qquad v^{app} = \sum_{n=0}^{N} u^n. \tag{5.10}$$

The terms are defined through the following iterative resolution scheme

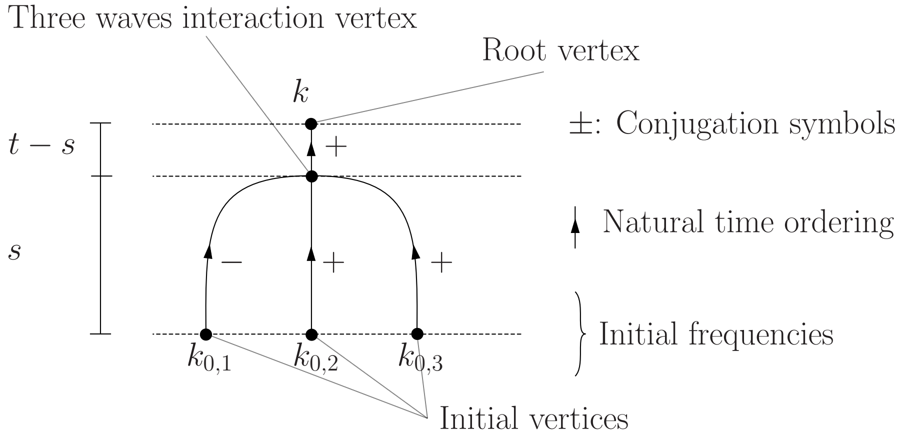
$$u^0 = e^{it\Delta}u_0 \qquad \text{and if } n \geq 1, \begin{cases} i\partial_t u^n + \Delta u^n = \lambda^2 \displaystyle\sum_{i+j+k=n-1} P(u^i, \overline{u^j}, u^k), \\ u^n(0) = 0, \end{cases} . \tag{5.11}$$

It is useful to give a formula for u^n with terms that are encoded by Feynman interaction diagrams:

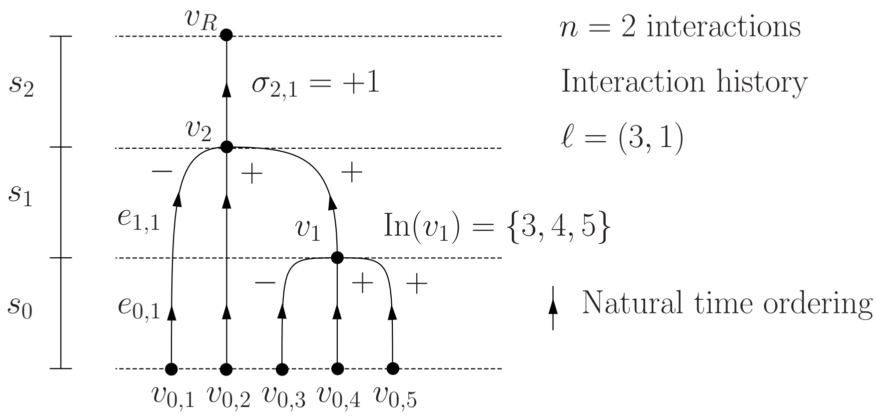
$$u^n = \sum_{G \in \mathscr{G}(n)} u_G, \tag{5.12}$$

see [81] for more details. Above, $\mathscr{G}(n)$ is the set of interaction diagrams of depth n. Formal derivations of the kinetic wave equation (see for example [343, 425]) usually stop at the second iteration $u \approx u^0 + u^1 + u^2$.

It is a graphical way to represent the resolution of the above scheme by a ternary tree, which is a tree made of interaction vertices (representing an iteration of the nonlinearity in Duhamel's formula) that each have three children (because the equation is cubic). At the bottom of the tree are the initial vertices, which represent the initial datum. For example, the first iteration of Duhamel's formula corresponds to the second term in the right-hand side of (5.7) and can be represented by the following Feynman diagram of depth 1:



A second iteration would produce Feynman diagrams of depth 2, for example the following one:



The Dyson series is always truncated, but in order for the kinetic limit to be justified, it seems necessary to have polynomial bounds making the radius of convergence be of order T_{kin}. Rough formal estimates lead to the guess that

$$u^n = O\left(\frac{t}{T_{kin}}\right)^{\frac{n}{2}} \tag{5.13}$$

in an appropriate topology. This was first proved in our work and that of Deng-Hani [82, 123] for $t \ll T_{kin} \sim 1$, and finally proved for all $t \lesssim T_{kin} \lesssim 1$ by Deng-Hani in [124]. Our aim here is to investigate the regime $T_{kin} \gtrsim 1$.

To understand the fundamental difference between these two regimes, consider the so-called four waves interaction between frequencies ξ_1, ξ_2, ξ_3 in (NLS), which generates frequency $\xi_4 = \xi_1 - \xi_2 + \xi_3$. This interaction can have a significant impact on the dynamics, on a time scale T, if the modulus of resonance $|k|^2 - |\ell|^2 + |n|^2 - |m|^2$ is such that

$$\left||k|^2 - |\ell|^2 + |n|^2 - |m|^2\right| \lesssim \frac{1}{T}.$$

In the kinetic limit that we consider, it is thus clear that the distribution properties of the resonance modulus on a scale $\sim \frac{1}{T}$ will be a decisive factor. As a simple heuristic argument shows that one expects equidistribution if $T \ll 1$, but not for $T \gg 1$ - indeed, it is obvious the resonance modulus takes values in the integers, hence it cannot be equidistributed on a scale less than 1.

We showed that the estimate $\|u_G\|_{L^2} \lesssim (t/T_{kin})^{\text{depth(G)}/2}$ fails if $T_{kin} \gg 1$ and $t \gg 1$ for the standard Laplacian (dispersion relation $H = \mathrm{Id}$). We will, for simplicity, assume that A takes the form

$$A(k) = \chi(|k|) \text{ is a smooth non-negative cut-off: } \chi(r) = 1 \text{ for } r \leq 1 \text{ and } \chi(r) = 0 \text{ for } r \geq 2. \tag{5.14}$$

Proposition 5.4. *Assume $d \geq 2$, $H = \mathrm{Id}$, and (5.14). For all $n \geq 1$, three constants $0 < c(n) < C(n)$ and $C'(n) > 0$ exist such that for any interaction diagram G of depth n, for all $t \geq \langle \log \epsilon \rangle^{C'(n)}$ there holds for ϵ small enough for $d \geq 3$:*

$$c(n)\left(\frac{t^2}{T_{kin}}\right)^n \leq \mathbb{E}\|u_G^n\|_{L^2}^2 \leq C(n)\left(\frac{t^2}{T_{kin}}\right)^n.$$

For $d = 2$ the same result holds with the estimate:

$$c(n)\left(\frac{t^2}{T_{kin}}\right)^n \langle \log \epsilon \rangle^n \leq \mathbb{E}\|u_G^n\|_{L^2}^2 \leq C(n)\left(\frac{t^2}{T_{kin}}\right)^n \langle \log \epsilon \rangle^n.$$

This does not prove that (5.13) actually fails, since u^n is the sum over all Feynman diagrams of depth n so that cancellations could occur. However, the proof shows that only exact resonances $\left||k|^2 - |\ell|^2 + |n|^2 - |m|^2\right| = 0$ contribute to the dynamics, what is very different from the kinetic regime where other quasi-resonances contribute to the effective dynamics. This mechanism for the failure of kinetic approximation already noticed in the physics literature, and is known as finite-box effect, see the discussion in Section 6.5 of [343]. We will see in the next Section 5.2 that for a generic quadratic dispersion relation instead of the Laplacian, the modulus of resonance becomes equidistributed at smaller scales and that the truncated Dyson series converge for $1 < T_{kin} \lesssim \epsilon^{\alpha(d)}$ for some dimension-dependent parameter $\alpha(d)$.

Such a divergence from the kinetic equation scaling $(\frac{t^2}{T_{kin}})^{\frac{n}{2}}$ is also expected for diagonal dispersion relations A (i.e. rectangular tori, with different bounds though). This could easily be investigated with the same techniques. The above proposition is proved for A a cut-off solely for the sake of clarity and would adapt to other functions. Deng and Hani [123], and Deng, Nahmod and Yue [129] proved existence up to time $T \sim \sqrt{T_{kin}}$ for $T_{kin} \gg 1$. Thus, Proposition 5.4 suggests that energy transfer between Fourier modes indeed starts at time $\sqrt{T_{kin}}$ and that their result is optimal.

5.2 Kinetic limit in anisotropic or spatially inhomogeneous cases

5.2.1 The spatially inhomogeneous case for quadratic nonlinearities

What happens when the equation (5.1) is set on the whole space $\mathbb{R}^d$ instead of flat torus $\mathbb{T}^d$? The initial data (5.2) represents a random Gaussian field that is isotropic. We will now consider fields that are anisotropic and spatially inhomogeneous. The initial data will be chosen to be a random Gaussian field

$$u(t=0,x) = u_0(x) = \int a(x,\xi) e^{i\frac{\xi}{\epsilon}\cdot x} dW(\xi) \tag{5.15}$$

where $\hat{a} \in \mathscr{C}_0^\infty(\mathbb{R}^{2d})$ and dW is a Wiener integral. Equivalently, u_0 can be characterized by its covariance

$$\mathbb{E}\left[\overline{u_0(x)} u_0(x')\right] = \int \overline{a(x,\xi)} a(x',\xi) e^{-i\frac{\xi}{\epsilon}\cdot(x-x')}\, d\xi.$$

We will come back to this definition later, suffice it to say for the time being that this Gaussian field exhibits random behavior at scale $\sim \epsilon$, with an envelope at a scale ~ 1. More precisely,

$$\text{as } \epsilon \to 0, \qquad \mathbb{E}\left[\overline{u_0(x)} u_0(x')\right] = F\left(\frac{x+x'}{2}, \frac{x-x'}{\epsilon}\right) + O(\epsilon),$$

where F is a smooth, decaying function. It is convenient at this point to introduce the rescaled Wigner transform

$$W^\epsilon[u](x,v) = \frac{1}{(2\pi)^{d/2}} \epsilon^{-d} \mathbb{E} \int \overline{u(x+\frac{z}{2})} u(x-\frac{z}{2}) e^{i\frac{v}{\epsilon}\cdot z}\, dz.$$

Roughly speaking, it provides a measure of the amount of energy of u (in L^2) localized in phase space at position x and frequency v/ϵ. In particular, it is such that

$$\text{as } \epsilon \to 0, \qquad W^\epsilon[u_0](x,v) \to |a(x,v)|^2 =: \rho_0(x,v). \tag{5.16}$$

Without nonlinear terms, the free Schrödinger evolution $i\partial_t u + \Delta u = 0$ is such that $W^\epsilon[u]$ solves the free transport equation $\partial_t W^\epsilon[u] + \frac{2v}{\epsilon} \cdot \nabla_x W^\epsilon[u] = 0$. This hints to the fact that when considering weakly turbulent solutions that are spatially inhomogeneous, waves will travel at the speed given by the group velocity, while interacting nonlinearly via the kinetic collision integral (5.6).

Our goal is thus to derive an inhomogeneous (transport) kinetic wave equation. This is achieved by considering data given by (5.15) whose spatial correlation exhibit a two-scale structure. The inhomogeneous kinetic wave equation approximates the average Wigner transform of the solution as the number of interacting waves goes to infinity and the strength of the nonlinearity goes to zero. We also provided examples of equations for which the kinetic limit might not hold.

Instead of the cubic Schrödinger equation, we consider the following nonlinear Schrödinger equations for complex fields in $\mathbb{R}^d$ with quadratic nonlinearities [1]:

$$i\partial_t u + \omega(D)u = \lambda M(Mu + M\overline{u})^2, \tag{5.17}$$

where we recall the notation for the Fourier multiplier p,

$$\widehat{p(D)f} = p(\xi)\widehat{f}(\xi)$$

and where

1. This also includes equations of the form $i\partial_t u + \omega(D)u = \lambda M(u+\overline{u})^2$ by a change of variables.

— $\omega(\xi) = \omega_0 + \dfrac{|\xi|^2}{2}$, with $\omega_0 = 0$ or ϵ^{-2}, is the dispersion relation,

— $M = m(\epsilon D)$, where m is a smooth, bounded, real valued even function,

— $\lambda > 0$ encodes the size of nonlinear effects.

(the scaling laws for the dispersion relation and the multiplier are natural in the limit we will be considering). The above generalized dispersion relation ω and quadratic nonlinearity with multiplier M are introduced because the most natural equation $i\partial_t u + \Delta u = (u + \bar{u})^2$ actually yields to the appearance of a singularity, which we will discuss in details in the next Section 5.2.2, so that the kinetic limit may fail.

This equation derives from the Hamiltonian

$$\mathscr{H}(u) = \int \frac{1}{2}|\sqrt{\omega(D)}u|^2 + \frac{8\lambda}{3}(\mathrm{Re} Mu)^3.$$

As we will see, the value of ω and m at zero will be key for the validity of the kinetic wave equation.

In many situations of physical interest, the leading nonlinear term is quadratic: for instance, this is the case for long-wave perturbations of the acoustic type (which can exist in most media), or interaction of three-wave packets in media with a decay dispersion law. These models have extremely wide applications, ranging from solid state physics to hydrodynamics, plasma physics etc. Recently, under the assumption of multiplicative noise, Staffilani and Tran [400] reached the kinetic timescale for the Zakharov-Kuznetsov (ZK) equation. In the absence of noise, the result of [400] is conditional.

It is a convenient model for our purposes: on the one hand, it retains all the difficulties related to the derivation of a kinetic wave equation, from a quadratic equation, in the inhomogeneous case; and on the other hand, it avoids further technicalities related to specific equations of physical interest (quasilinearity of the equations, singularity of the dispersion relations, vectorial nature of the unknown...).

Our aim is to show that

$$\text{as } \epsilon \to 0, \qquad W^\epsilon[u\,(t)](x,v) \to \rho(t,x,v),$$

where ρ solves the kinetic wave equation

$$\begin{cases} \partial_t \rho + \dfrac{1}{\epsilon} v \cdot \nabla_x \rho = \dfrac{8\pi}{T_{kin}} \mathscr{C}[\rho(x)] \\ \rho(t=0) = \rho_0. \end{cases} \qquad \text{where } T_{kin} = \frac{1}{\lambda^2 \epsilon^2} \tag{5.18}$$

The collision operator $\mathscr{C}$ is given by

$$\begin{aligned} \mathscr{C}[\rho](t,x,v) = & m^2 \int \Bigg[\delta(\Sigma_-)\delta(\Omega_-) m_1^2 m_2^2 \rho\rho_1\rho_2 \left(\frac{1}{\rho} - \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \\ & + 2\delta(\Sigma_+)\delta(\Omega_+) m_1^2 m_2^2 \rho\rho_1\rho_2 \left(\frac{1}{\rho} + \frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \Bigg] dv_1\, dv_2, \end{aligned} \tag{5.19}$$

$$\begin{cases}\Sigma_- = v - v_1 - v_2 \\ \Sigma_+ = v + v_1 - v_2\end{cases} \quad \begin{cases}\Omega_- = \omega(v) - \omega(v_1) - \omega(v_2) \\ \Omega_+ = \omega(v) + \omega(v_1) - \omega(v_2),\end{cases} \quad \begin{cases}\rho = \rho(v) \\ \rho_i = \rho(v_i)\end{cases} \quad \begin{cases}m = m(v) \\ m_i = m(v_i),\end{cases}$$

for $i \in \{1, 2\}$. This equation displays two (singular) time scales:

— ϵ, the transport time scale, since $\frac{1}{\epsilon}$ is the group velocity for solutions of the linear Schrödinger equation localized at frequency $\sim \frac{1}{\epsilon}$. In other words, ϵ is the time over which such solutions travel a distance ~ 1, which implies that, for $t \gg \epsilon$, one expects the solution to spread and nonlinear interactions to be damped.

— T_{kin}, the characteristic time scale for the mixing in frequency space occuring through the collision operator $\mathscr{C}$. Notice the dependence in λ^2 - as opposed to λ appearing in front of the nonlinearity of (5.17) - which is characteristic of square-root cancellations caused by randomness.

— Of particular relevance is of course the regime where both time scales agree, $T_{kin} = \epsilon$, or in other words $\lambda = \epsilon^{-3/2}$.

Other important time scales are ϵ^2 the linear time-scale, and λ^{-1} the nonlinear time-scale, after which nonlinear effects become relevant. However, the limitation $T_{kin} \ll 1$ is no longer relevant, since the equidistribution problem for Fourier modes over the lattice $\mathbb{Z}^d$ for $T_{kin} \gtrsim 1$ mentioned in Section 5.1.3 does not occur for Fourier modes over the continuum $\mathbb{R}^d$.

Most of the rigorous results on the derivation of the kinetic wave equation were obtained in the spatially homogeneous and isotropic case we discussed in the previous Section 5.1. Regarding the spatially inhomogeneous (KWE) and its connection to nonlinear waves, Spohn [399] discusses the emergence of a kinetic wave equation, which he calls phonon Boltzmann equation. However, to the best our knowledge, there are no rigorous results justifying a derivation of an inhomogeneous kinetic wave equation from dispersive dynamics. Recently, [222] justified the kinetic limit for the Wick ordered nonlinear Schrödinger equation, which is a simplified model where nonlinear interactions add-up but do not interact.

We now state the main result of our paper [4], regarding the well-posedness of equation (5.17) and its approximation by the corresponding kinetic wave equation.

Theorem 5.5. *Let $a \in \mathscr{C}_0^\infty(\mathbb{R}^{2d})$ and $\nu > 0$. Consider Equation (5.17) with initial data (5.15) and*

— *either $\omega_0 = \epsilon^{-2}$*

— *or $\omega_0 = 0$ and $m(0) = 0$.*

Then there exist $\epsilon^* > 0$ and $\kappa > 0$ such that for any $0 < \epsilon < \epsilon^*$ and for any $0 < T < \min\{\epsilon, \epsilon^\nu T_{kin}\}$, there exists a set E of probability $\mathbb{P}(E) > 1 - \epsilon^\kappa$, such that on E, there exists a unique solution u to (5.17) in $[0, T]$.

Moreover, the solution u is approximated by the solution ρ of the corresponding kinetic wave equation in the following sense:

For any $t \in [0, T]$ and $\xi \in \mathbb{R}^d$, there holds:

$$\int_{\mathbb{R}^d} |\widehat{\rho}(t, \xi, v) - \widehat{W}^\epsilon_E[u](t, \xi, v)| \, dv \lesssim \epsilon^\nu \left(\frac{T}{T_{kin}}\right),$$

where

$$W_E^\epsilon[u](x,v) = \frac{1}{(2\pi)^d}\mathbb{E}\left[\mathbb{1}_E \int_{\mathbb{R}^d} \overline{u\left(x+\frac{\epsilon y}{2}\right)} u\left(x-\frac{\epsilon y}{2}\right) e^{iv\cdot y}\, dy\right],$$

E is the exceptional set of existence obtained above and ρ solves (5.18) *with initial data* (5.16).

We could have studied the quadratic equation (5.17) on the torus $\mathbb{T}^d$ as in the previous Section 5.1. This is a configuration where there is no rigorous result yet, and Theorem 5.5 and its proof have a clear homogeneous counterpart for the Fourier modes of the solution if the equation (5.17) is set on the torus instead of $\mathbb{R}^d$.

Remark 5.6. What ranges of ϵ and λ are relevant in the previous theorem? First, the approximation is accurate in the limit $\epsilon \to 0$; second, in order to approach the kinetic time scale T_{kin} up to a small power of ϵ, the above theorem requires $\epsilon < T_{kin}$, or in other words $\lambda > \epsilon^{-3/2}$. Physically, this means that the kinetic time scale should be smaller than the kinetic time scale; otherwise, dispersive decay prevents nonlinear interaction from having a sizable effect.

5.2.2 Possible failure of kinetic limit without taming at low frequencies

We believe that the kinetic wave equation might fail to describe solutions to

$$i\partial_t u + \Delta u = (u+\bar{u})^2 \tag{5.20}$$

on the time scale T_{kin}, due to a low frequency inflation. Note that the kinetic equation (5.19) is not even well defined, as the mass of the unit ball for the measure $\delta(\Sigma_+)\delta(\Omega_+)dv_1dv_2 = \delta(v+v_1-v_2)\delta(2v.(v-v_2))$ diverges as $v \to 0$. This issue was already raised by Spohn, see Section 6 in [399] for a discussion, where an hypothesis for the non-vanishing of $\omega(0)$ that is analogue to the present one in Theorem 5.5 is assumed. Hence, our convergence result of Section 5.2.1 would be sharp in the sense that at the origin in Fourier, either a cancellation of nonlinear effects $m(0) = 0$, or a lack of resonance due to a non-zero dispersion relation $\omega_0 = \epsilon^{-2}$, $c_0 > 0$, would be needed to ensure the validity of the kinetic description.

We recall (see Section 5.1.3) that the Dyson series (5.10)-(5.11)-(5.12) can be represented as a sum over Feynman interaction diagrams. Their L^2 norm can be represented as a sum over paired graphs:

$$u^n = \sum_{G\in\mathscr{G}_n} u_G, \qquad \mathbb{E}\,\|u^n(t)\|^2_{L^2(\mathbb{R}^d)} = \sum_{G'\in\mathscr{G}_n^p} \mathscr{F}_t(G') \quad \text{for all } t\in\mathbb{R}. \tag{5.21}$$

Our second result is that the second series above is not absolutely convergent on the kinetic time scale. This itself does not imply the divergence of $\mathbb{E}\|u^n(t)\|^2_{L^2(\mathbb{R}^d)}$ as cancellations could occur, see Remark 5.9.

Proposition 5.7 (Diverging Feynman diagrams for quadratic nonlinearities with no taming effect at low frequencies [4])**.** *For all $d \geq 2$, there exists a Schwartz function $a \in \mathscr{S}(\mathbb{R}^{2d})$ such that, for any $\kappa > 0$, the following holds true for initial data of the form* (5.15) *in the range:*

$$\epsilon^{2-\kappa} \leq t \leq \epsilon^{1+\kappa}.$$

There exists $n^*(d,\kappa)$, such that for all $n \geq n^*$, there exists a paired graph $G^* \in \mathscr{G}_{2n}^p$ as defined in for equation (5.20), two constants $C, C' > 0$ and $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$:

$$C(\lambda t)^{4n}\epsilon^{2d}t^{-d} \leq \mathscr{F}_t(G^*) \leq C'(\lambda t)^{4n}\epsilon^{2d}t^{-d}. \tag{5.22}$$

Remark 5.8. The kinetic equation can a priori only be reached provided that its time scale T_{kin} is shorter than the transport time scale ϵ and that the regime is weakly nonlinear $\epsilon^2 \ll \lambda^{-1}$. The sum of the absolute values of the terms in the second series in (5.21) diverges at a time before T_{kin}, since the nonlinear time scale λ^{-1} at which the estimate (5.22) becomes singular is shorter than T_{kin}.

Remark 5.9. We believe that the first series in (5.21) does not either converge on the kinetic time scale, that is, $\mathbb{E}\|u_G(t)\|^2_{L^2(\mathbb{R}^d)}$ diverges as (5.22) for some $G \in \mathscr{G}_n$. In [81] the last two authors were able to show such result, for a similar counter-example graph for a cubic nonlinearity for a different time scale. The proof showed no cancellation occurred from other pairings for the same interaction diagram G. We believe the same strategy could be applied here. This would not imply the actual divergence of u^n, but would indicate that cancellations with another interaction diagram G' are required. Such cancellations were shown to exist by Deng-Hani [125] but for a quite different equation and domain: for the cubic (NLS) on the torus (5.1). At the moment we thus cannot make any conjecture regarding whether the kinetic limit for (5.20) might hold true or not, and believe it is an interesting open problem.

5.2.3 Longer time scales for the derivation of the kinetic limit on generic tori

We now want to study how the geometry of the problem, or the dispersion relation, affects the possibility of a kinetic limit for weakly turbulent solutions to the cubic nonlinear Schrödinger equation. We consider

$$\begin{cases} i\partial_t u - \Delta_H u = \lambda^2 \mathbb{P}_\nu |u|^2 u, \\ u(t=0) = u_0, \end{cases} \qquad x \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z}^d), \tag{NLS}$$

where $\mathbb{P}_\nu$ is the frequency cut-off projector $\widehat{\mathbb{P}_\nu u}(\xi) = \mathbb{1}(|\xi| \leq \nu)\hat{u}(\xi)$. We will consider solutions such that $\mathbb{P}_\nu u = u$, which is propagated by the flow, in which case the Hamiltonian, for $\lambda > 0$:

$$\frac{1}{2}\sum_{k\in\mathbb{Z}^d} |k|_H^2 |\hat{u}(k)|^2 - \frac{\lambda^2}{4}\int_{x\in\mathbb{T}^d} |u(x)|^4\, dx,$$

is conserved. We shall consider general linear Hamiltonians with quadratic dispersion relation:

$$\widehat{-\Delta_H u}(k) = |k|_H^2 \hat{u}(k), \qquad |k|_H^2 = \sum_{i,j=1}^d h_{i,j}k_ik_j = Hk\cdot k, \tag{5.23}$$

where the matrix $H = (h_{ij})_{1\leq i,j\leq d} \in \mathbb{R}^{d\times d}$ is symmetric positive definite. Note that $H = \mathrm{Id}$ corresponds to the standard Laplacian. We will see that "generic" dispersion relations are better behaved compared to the standard case $H = \mathrm{Id}$.

These more general dispersion relations can also arise from natural geometric considerations: namely, if one considers the nonlinear Schrödinger equation with standard Laplacian

$$\begin{cases} i\partial_t u - \Delta u = \lambda^2 |u|^2 u, \\ u(t=0) = u_0, \end{cases} \tag{5.24}$$

set on the quotient of $\mathbb{R}^d$ by a lattice. Mapping the equation to the standard torus, it is equivalent to consider (NLS) set on the standard torus $\mathbb{T}^d$ where $H = \mathrm{Id}$ for the standard torus, H is diagonal for a rectangular torus, and H is a general symmetric matrix for a general torus.

As in Section 5.1.1, the initial data is chosen to be a Gaussian field of the form

$$u_0(x,\omega) = \frac{\epsilon^{d/2}}{(2\pi)^{d/2}} \sum_{k\in\mathbb{Z}^d} A(\epsilon k) G_k(\omega) e^{ik\cdot x}, \tag{5.25}$$

where $A \in \mathscr{C}_0^\infty(\mathbb{R}^d, [0,\infty))$ and $(G_k)_{k\in\mathbb{Z}^d}$ are independent standard centred complex Gaussians defined on a probability space Ω. The cut-off frequency is chosen to be much larger than that of the data $\nu = \epsilon^{-1-\theta} \gg \epsilon^{-1}$ for some $\theta > 0$. The same heuristic derivations show that as $\epsilon\lambda \to 0$ (weakly nonlinear regime) and $\epsilon \to 0$ (high frequency limit), scaling properly the expectation (denoted $\mathbb{E}$) of the square modulus of the Fourier coefficients of u, it satisfies

$$\epsilon^{-d}\mathbb{E}\left|\widehat{u}(\lfloor \epsilon k \rfloor)\,(T_{kin} t)\right|^2 \longrightarrow \rho(t,k), \qquad \text{for } T_{kin} = \frac{1}{\epsilon^2\lambda^4}$$

where ρ solves the kinetic wave equation (5.5) where the collision operator is given by (5.6). However, in comparison with Section 5.1.1, the restriction $T_{kin} \gtrsim 1$ seems no longer justified; we recall it was made in order to have resonance moduli $|k|^2 - |l|^2 + |m|^2 - |n|^2$ to be equidistributed at scale $\lesssim T_{kin}^{-1}$, which was impossible when $T_{kin} \gtrsim 1$ as it only takes integer values. Now resonance moduli $|k|_H^2 - |l|_H^2 + |m|_H^2 - |n|_H^2$ may be equidistributed at much better scales.

The Dyson series

Similarly to Section 5.1.3, we explain here the Dyson series associated to Equation (NLS) with initial data of the form (5.25), and refer to this Section for illustrations. To study a solution to (5.24), we apply first Wick renormalisation for the phase: $u = e^{-it\lambda^2 \frac{2}{(2\pi)^d}\|u_0\|_{L^2}^2} v$. This is a usual transformation that cancels the leading order nonlinear effect, as $\mathbb{P}_\nu |u|^2 u \approx 2(2\pi)^{-d}\|u_0\|_{L^2}^2 u$ in some appropriate sense, and which does not alter relevant statistics. The equation for v is

$$i\partial_t v = -\Delta v + \lambda^2 \mathbb{P}_\nu \left(|v|^2 - \frac{2}{(2\pi)^d}\|v\|_{L^2}^2 \right) v.$$

Since linear dynamics acts at a shorter time scale than nonlinear effects (by the weak nonlinearity assumption), it is customary to find an approximate solution by iterating Duhamel's formula, giving the so-called Dyson series expansion. We define for this purpose the truncation operator

$$\mathscr{F}[P(a,b,c)](k) = \frac{1}{(2\pi)^d} \sum_{k_1+k_2+k_3=k} \widehat{a}(k_1)\widehat{b}(k_2)\widehat{c}(k_3)(1 - \delta(k_1+k_2) - \delta(k_2+k_3)).$$

This gives the decomposition of the product

$$abc = P(a,b,c) + \frac{1}{(2\pi)^d}\langle \overline{a}, b\rangle c + \frac{1}{(2\pi)^d} a\langle \overline{b}, c\rangle.$$

A quasi-solution is then a series expansion

$$u^{app} = e^{-it\lambda^2 \frac{2}{(2\pi)^d}\|u_0\|_{L^2}^2} v^{app} \qquad \text{with} \qquad v^{app} = \sum_{n=0}^{N} u^n.$$

The terms are defined through the following iterative resolution scheme

$$u^0 = e^{it\Delta}u_0 \qquad \text{and if } n \geq 1, \begin{cases} i\partial_t u^n + \Delta u^n = \lambda^2 \displaystyle\sum_{i+j+k=n-1} P(u^i, \overline{u^j}, u^k), \\ u^n(0) = 0, \end{cases} .$$

It is useful to give a formula for u^n with terms that are encoded by Feynman interaction diagrams:

$$u^n = \sum_{G \in \mathscr{G}(n)} u_G,$$

see [81] for more details. Above, $\mathscr{G}(n)$ is the set of interaction diagrams of depth n.

Our aim was to investigate the convergence properties of the Dyson series. In order for the kinetic limit to be justified, it seems necessary to have polynomial bounds making the radius of convergence be of order T_{kin}. Rough formal estimates lead to the guess that

$$u^n = O\left(\frac{t}{T_{kin}}\right)^{\frac{n}{2}} \tag{5.26}$$

To understand the fundamental difference between the regimes $t \ll 1$ and $t \gtrsim 1$, consider the so-called four waves interaction between frequencies ξ_1, ξ_2, ξ_3 in (NLS), which generates frequency $\xi_4 = \xi_1 - \xi_2 + \xi_3$. This interaction can have a significant impact on the dynamics, on a time scale T, if the modulus of resonance

$$\Omega_H(\xi_1, \xi_2, \xi_3, \xi_4) = |\xi_1|_H^2 - |\xi_2|_H^2 + |\xi_3|_H^2 - |\xi_4|_H^2$$

is such that

$$|\Omega_H(\xi_1, \xi_2, \xi_3, \xi_4)| \lesssim \frac{1}{T}.$$

In the kinetic limit that we consider, it is thus clear that the distribution properties of Ω_H on a scale $\sim \frac{1}{T}$ will be a decisive factor. As a simple heuristic argument shows, any choice of H is expected to give equidistribution if $T \ll 1$, but not for $T \gg 1$ - indeed, it is obvious Ω_{Id} takes values in the integers, hence it cannot be equidistributed on a scale less than 1.

Convergence for $T_{kin} \gg 1$, H generic

We proved the existence of a solution with suitable estimates up to times arbitrarily close to the kinetic time, in the large kinetic time regime. It is obtained by proving first that the estimate (5.26) holds.

Theorem 5.10 (Longer time scales for the possible [81]). *Consider for $d \geq 3$ the equation (NLS) with data (5.25), in the regime where*

$$T_{kin} \geq \epsilon^{-\iota} \tag{5.27}$$

for any fixed $0 < \iota \ll 1$. For almost all (in the Lebesgue sense) symmetric matrices $H \in \mathbb{R}^{d\times d}$ whose distance to the identity matrix is small, the following holds true.

For any $\kappa, \beta, \beta' > 0$, there exists $\theta(\beta') > 0$, and for $s \geq 0$ and $N \in \mathbb{N}$, there exist $\epsilon^*, \mu > 0$, such that for a cut-off frequency $\nu = \epsilon^{-1-\theta}$, for any $0 < \epsilon < \epsilon^*$, there exists an exceptional set of size ϵ^μ over the complement of which, for $T = \min(\epsilon^\beta T_{kin}, \epsilon^{2-d+\beta'})$:

— *The iterates u^n enjoy the bound for $0 \leq n \leq N$ and any $m \geq 0$:*

$$\|u^n\|_{C_t([0,T],H^m(\mathbb{T}^d))} \lesssim_{H,N,\kappa} \epsilon^{-m-\kappa} \left(\frac{T}{T_{kin}}\right)^{n/2},$$

where $H^m(\mathbb{T}^d)$ stands for the usual Sobolev space.

— *There exists a unique smooth solution u on the time interval $[0,T]$, with:*

$$u = e^{i\lambda^2\omega(t)} \sum_{n=0}^{N} u^n + \tilde{u}, \qquad \omega(t) = \frac{2}{(2\pi)^d}\|u_0\|^2_{L^2(\mathbb{T}^d)} t,$$

where the remainder satisfies

$$\|\tilde{u}\|_{C_t([0,T],H^s(\mathbb{T}^d))} \lesssim_{H,N,\kappa} \epsilon^{-s-\kappa} \left(\frac{T}{T_{kin}}\right)^{(N+1)/2}.$$

In comparison with the Laplacian's dispersion relation $|k|_{\mathrm{Id}} = \sum_1^d |k_i|^2$, new number theoretical results are needed. They have to do with the distribution properties of the functions

$$H\xi_0 \cdot \xi, \qquad |\xi|_H^2, \qquad \text{and} \qquad H\xi \cdot \eta$$

(which are functions on $\mathbb{R}^d$, $\mathbb{R}^d$, and $\mathbb{R}^{2d}$ respectively) over integers. More precisely, we prove in [81] that for any $\kappa > 0$, there exists $\theta > 0$ such that the following estimates hold for $L^{2-d+\kappa} < \delta < 1$, uniformly in $\zeta, \zeta' \in \mathbb{Z}^d$, $a \in \mathbb{R}$, and generically in H

$$\#\{\xi \in \mathbb{Z}^d,\ |\xi| \leq L,\ |H\zeta \cdot \xi| < \delta\} \lesssim_{H,\kappa} L^{d-1}\sqrt{\delta} \qquad \text{if } 1 \leq |\zeta| \leq \epsilon^{-1-\theta}, \tag{5.28}$$

$$\#\{\eta \in \mathbb{Z}^d,\ |\eta| < L,\ ||\eta+\zeta|_H^2 - a| < \delta\} \lesssim_{H,\kappa} L^{d-1}\sqrt{\delta}, \tag{5.29}$$

$$\#\left\{\eta \in \mathbb{Z}^d, \xi \in \mathbb{Z}^d \text{ with } |\eta|, |\xi| \leq L, \quad |H(\xi+\zeta)\cdot(\eta+\zeta') - a| \leq \delta\right\} \lesssim_{H,\kappa} L^{2d-2}\delta. \tag{5.30}$$

The parameters L and δ should be thought of as ϵ^{-1} and T^{-1} respectively. The above estimates allow to control resonant interactions in the kinetic limit under consideration. They are proved using tools of analytic number theory: geometry of numbers and the circle method, combined with averaging over the matrix H.

The first estimate in the three number theoretic estimates displayed above is the limitation that is responsible for the introduction of the cut-off frequency projector $\mathbb{P}_\nu$. We believe it might not hold uniformly in $\zeta \in \mathbb{Z}^d$. We nonetheless believe the result of Theorem 5.10 to be valid without such projection for compactly supported data A. This could be proved by controlling the error in analytic norms, but such technical developments are outside the scope of this article.

The above theorem provides the desired control of (NLS) almost up to the kinetic time scale. The natural next step would be to prove that the dynamics, on this time scale, are approximated by the kinetic wave equation (KWE). Following the proof in [82], the only missing item is a quantitative equidistribution result for the function $H\xi \cdot \eta$, extending the result of Sarnak [381] to include error terms as in the result of Bourgain [27] for generic diagonal forms.

We believe that by applying the ideas of Deng-Hani [124] who proved the validity of the kinetic limit for $T_{kin} \ll 1$, the restriction $T_{kin} \gtrsim \epsilon^{-\iota}$ in Theorem 5.10 can be removed. Thus, the kinetic limit for generic dispersion relation should be valid for all $T_{kin} \lesssim \epsilon^{-\alpha(d)}$ for some optimal coefficient $\alpha(d)$ to be determined (Theorem 5.10 shows $\alpha(d)$ is at least $d-2-$). This is a much large possible time scale than the one for the standard Laplacian $T_{kin} \ll 1$ as explained in Section 5.1.

5.3 Spectra and cascades

We are now interested in the asymptotic behaviour of solutions of the 4-wave isotropic kinetic wave equation

$$\begin{cases} \partial_t f(t,\omega) = \mathscr{C}(f)(t,\omega), \\ f(t=0) = f_0, \end{cases} \qquad \omega \in [0,\infty), \tag{5.31}$$

where the collision operator is

$$\mathscr{C}(f)(\omega_1) = \iint_{\omega_2,\omega_3,\omega_4 \geq 0} W[(f(\omega_1)+f(\omega_2))f(\omega_3)f(\omega_4) - (f(\omega_3)+f(\omega_4))f(\omega_1)f(\omega_2)]\, d\omega_3\, d\omega_4, \tag{5.32}$$

$$\text{with} \quad \omega_2 = \omega_3 + \omega_4 - \omega_1, \qquad \text{and} \qquad W = \frac{\min(\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4})}{\sqrt{\omega_1}}.$$

Here, in comparison with (5.6) the distribution f is expressed in terms of the dispersion relation $\omega(k) = |k|^2$, where k is the momentum.

The above collision operator is reminiscent of the collision operator in the Boltzmann equation, with the notable difference that the above operator is cubic instead of quadratic. Like the Boltzmann collision operator, the evolution preserves formally the mass and energy given by

$$M(f) = \int_0^\infty f(t,\omega)\,\omega^{1/2}\, d\omega \qquad \text{and} \qquad E(f) = \int_0^\infty f(t,\omega)\,\omega^{3/2}\, d\omega. \tag{5.33}$$

The conservation of mass is inherited from the nonlinear Schrödinger equation from which the kinetic equation is derived. The conservation of energy as well, since it is derived in a weakly nonlinear regime where the nonlinear contribution in the energy is negligible. As an analogue of the Boltzmann H-theorem, the entropy

$$S(f) = \int_0^\infty \log(f)\, \omega^{1/2}\, d\omega$$

is formally increasing as

$$\frac{d}{dt}S(f) = \frac{1}{4}\iint \min(\sqrt{\omega_1}, \sqrt{\omega_2}, \sqrt{\omega_3}, \sqrt{\omega_4}) f_1 f_2 f_3 f_4 \left(\frac{1}{f_1}+\frac{1}{f_2}-\frac{1}{f_3}-\frac{1}{f_4}\right)^2 d\omega_1\, d\omega_3\, d\omega_4$$

where we introduced the shorthand notation $f_i = f(\omega_i)$. Note however that this entropy is not well-defined, as this would require $f(\omega) \to 1$ as $\omega \to \infty$, which is not a physically relevant boundary condition at infinity, and which makes the collision operator ill-defined. Hence it is only a formal entropy.

Turning to stationary solutions, there exist formally the following ones:

— Rayleigh-Jeans solutions

$$f(k) = \frac{1}{\alpha + \beta\omega}$$

with $\alpha, \beta > 0$ are such that the collision integral vanishes formally $\mathscr{C}(f) = 0$ because $(f_1 + f_2)f_3 f_4 - (f_3 + f_4)f_1 f_2 = 0$. Identifying ω with a particle label, and then $f(\omega)$ as the mass of particles of label ω at equilibrium f and $f(\omega)\omega$ as the energy of particles of label ω at equilibrium f, we see that f above is such that $(\alpha + \beta\omega)f(\omega) = 1$ is the same for all ω, i.e. this is a statistical equilibrium giving equipartition among all the ω modes of this linear combination of mass and energy. A first problem for the Rayleigh-Jeans equilibria is that they have infinite total mass and total energy $M(f) = E(f) = \infty$. A second problem, which is worst, is that the two underlying gain and loss integrals in the collision operator (5.32) (those with + and - signs) are not convergent for such equilibria (see for example [74] for a sharp criterion for convergence), due to a divergence as $\omega \to \infty$. Balk and Zakharov in [8] term stationary solutions for which the gain and loss integrals are convergent as *local* and argue that physically relevant stationary solutions have to be local. If the problem is set on a finite frequency box by introducing a truncation in the collision kernel, then the Rayleigh-Jeans equilibria are stable [313]. However, the present kinetic wave equation (5.31) is set on the whole frequency space $\omega \in (0, \infty)$, for which Rayleigh-Jeans equilibria are unstable in the sense that either by truncating them or by studying the linearized dynamics around them for the untruncated problem one observes the formation of a Dirac delta of mass at $\omega = 0$ [161]. Thus, we believe the Rayleigh-Jeans equilibria of (5.31) are only formal stationary solutions and they should not arise in the dynamics of general solutions.

— More interesting are the Kolmogorov-Zakharov (KZ) solutions which correspond to out-of-equilibrium dynamics. They are given by

$$f(\omega) = \omega^{-7/6} \qquad \text{and} \qquad f(\omega) = \omega^{-3/2}.$$

One way to compute that they are stationary solutions is to use the following conformal change of variables due to Zakharov [423]: if one looks for an equilibrium of the form $f(\omega) = \omega^\alpha$ then

$$\mathscr{C}(f) = \int_{\Delta_1} W\omega_1^\alpha\omega_2^\alpha\omega_3^\alpha\omega_4^\alpha \left[\omega_1^{-\alpha} + \omega_2^{-\alpha} - \omega_3^{-\alpha} - \omega_4^{-\alpha}\right] \left[1 + \left(\frac{\omega_1}{\omega_2}\right)^{3\alpha+\frac{7}{2}} - \left(\frac{\omega_1}{\omega_3}\right)^{3\alpha+\frac{7}{2}} - \left(\frac{\omega_1}{\omega_4}\right)^{3\alpha+\frac{7}{2}}\right] d\omega_3\, d\omega_4$$

where $\Delta_1 = \{(\omega_3, \omega_4)$ such that $\omega_3 + \omega_4 \geq \omega_1,\ 0 \leq \omega_3 \leq \omega_1,\ 0 \leq \omega_4 \leq \omega_1\}$. Since $\omega_1 + \omega_2 = \omega_3 + \omega_4$, we then see that the second term in the integral vanishes if $\alpha \in \{-\frac{3}{2}, -\frac{7}{6}\}$. The KZ spectrum $\omega^{-7/6}$ is a well-defined solution because the collision integral (5.32) converges absolutely, while $\omega^{-3/2}$ is only a formal solution as the collision integral is not absolutely converging.

We will thus focus on $\omega^{-7/6}$, the only well-defined equilibrium among the above ones. It can be rescaled, and for any mass flux $j_M > 0$, the scaled Kolmogorov-Zakharov spectrum

$$\mathscr{f}_{j_M} = \left(\frac{j_M}{j_M^*}\right)^{1/3} \frac{1}{\omega^{7/6}} \tag{5.34}$$

is also a stationary solution to (5.31), where $j_M^* > 0$ is some normalization constant given for example in [74]. It corresponds to a constant flux of mass (particles) from infinite to zero frequency in the following sense. First, one has the flux of mass identity

$$\sqrt{\omega}\mathscr{C}(f) = -\partial_\omega(J_M(f)) \tag{5.35}$$

where J_M is a flux whose orientation is from 0 to ∞, given by

$$\begin{aligned}
J_M(f) &= J_1(f) + J_2(f) - J_3(f) - J_4(f),\\
J_1(f)(\omega) &= \int_{\substack{\omega_1>0\\ 0<\omega_2<\omega\\ \omega<\omega_3<\omega_1+\omega_2}} \sqrt{\omega_1} W f_1 f_2 f_3\, d\omega_1\, d\omega_2\, d\omega_3,\\
J_2(f)(\omega) &= \int_{\substack{0<\omega_1<\omega\\ \omega_2>\omega-\omega_1>0\\ 0<\omega_3<\omega_1+\omega_2-\omega}} \sqrt{\omega_1} W f_1 f_2 f_3\, d\omega_1\, d\omega_2\, d\omega_3,\\
J_3(f)(\omega) &= \int_{\substack{\omega_1>0\\ \omega_2>\omega\\ 0<\omega_3<\omega}} \sqrt{\omega_1} W f_1 f_2 f_3\, d\omega_1\, d\omega_2\, d\omega_3,\\
J_4(f)(\omega) &= \int_{\substack{\omega_1>\omega\\ 0<\omega_2\\ \omega_1+\omega_2-\omega<\omega_3<\omega_1+\omega_2}} \sqrt{\omega_1} W f_1 f_2 f_3\, d\omega_1\, d\omega_2\, d\omega_3.
\end{aligned}$$

Second, one has the energy flux identity

$$\omega^{3/2}\mathscr{C}(f) = -\partial_\omega(J_E(f))$$

where

$$J_E(f)(\omega) = \omega J_M(\omega) - \int_0^\omega J_M(\tilde{\omega})\, d\tilde{\omega}. \tag{5.36}$$

For the equilibrium $\mathscr{f}_{j_M}$ given by (5.34) the time variations of the mass and energy densities present an inverse cascade of mass with a source at ∞ and a sink at 0 of j_M mass per unit of time, and no cascade of energy:

$$- \omega^{1/2}\mathscr{C}(\mathscr{f}_{j_M}) = j_M\delta_\infty - j_M\delta_0,$$
$$- \omega^{3/2}\mathscr{C}(\mathscr{f}_{j_M}) = 0,$$

where the above identities are understood as (5.35) and (5.36) in the weak sense. Formally, the other (KZ) spectrum $\omega^{-3/2}$ corresponds to a constant flux of energy from 0 to ∞ in frequency.

We consider the problem of the stability of the mass cascade (KZ) spectrum $\mathscr{f}_{j_m}$ by the injection of some mass and energy around a fixed frequency in the system, which is modelled by the forced kinetic equation

$$\partial_t f = \mathscr{C}(f) + \phi \tag{5.37}$$

with forcing ϕ. We expect that steady solutions of (5.37) will connect the solution $\omega^{-7/6}$ corresponding to an inverse cascade of mass, with the solution $\omega^{-3/2}$ corresponding to a (direct) energy cascade. It is tempting to propose the following scenario for the forced kinetic wave equation; we choose for simplicity in this discussion the force ϕ to be smooth and compactly supported on $(0, \infty)$.

Static ideal scenario: The equation $-\mathscr{C}(f) = \phi$ admits stationary solutions f with tails $\omega^{-7/6}$ and $\omega^{-3/2}$, as $\omega \to 0$ and ∞ respectively. These tails correspond to outgoing fluxes of mass and energy which equilibrate the input of these quantities through ϕ.

Dynamic ideal scenario: Solutions of the dynamical problem $\partial_t f = \mathscr{C}(f) + \phi$ with arbitrary data converge to one such stationary solution.

In order to show the above scenarii, a first step is to study the stability of the $\omega^{-7/6}$ KZ spectrum. Some stability results for the present kinetic wave equation (5.31) can be inferred from results on Uhling-Uhlenbeck equation (also called Boltzmann-Bose-Einstein equation for hard spheres, or bosonic Nordheim equation). Both equations share the same cubic terms, but differ in lower order (quadratic) terms; their dynamics have many common features. The basic theory of the Uhling-Uhlenbeck equation was established in [288, 289, 290, 166]: existence of weak solutions, development of singularities (Bose-Einstein condensation), and large-time behavior. Escobedo-Mischler-Velazquez [163, 164, 160] further studied singular solutions corresponding to the KZ solution $\mathscr{f}_M$. They investigated linear stability (following [8]), and local well-posedness in their neighborhood; see [165] for RJ solutions. These articles will provide the foundation of the present work.

Importantly, [166] focused specifically on the kinetic wave equation (5.31) we are currently discussing, proving the existence of weak solutions, Bose-Einstein condensation (finite-time singularity formation), and characterizing large-time dynamics. The asymptotic stability of the (KZ) solution $\mathscr{f}_{j_M}$ remains however still open. We were able to show its stability for the time-independent problem. We partially validated the static scenario by proving the existence of stationary solutions of (5.37) in a neighborhood of $\omega^{-7/6}$ (mass cascade). Our main result showed that this solution is stable if one perturbs it by adding a localized source term, and that this also triggers a direct cascade of energy towards ∞.

We formulate the result in terms of the weighted L^∞ norm

$$\|f\|_{\alpha,\beta} = \sup_{0<\omega<1} \omega^\alpha |f(\omega)| + \sup_{\omega>1} \omega^\beta |f(\omega)|$$

and denote $E_{\alpha,\beta}(\mathbb{R}_+)$ the corresponding Banach space.

Theorem 5.11 (Stability of the Kolmogorov-Zakharov mass cascade spectrum [74]). *For all* $0 < \delta < 1/12$ *there exists* $\epsilon > 0$ *such that the following holds true. For all* $j_M^\infty > 0$ *and* $\phi \in L^\infty_{loc}(\mathbb{R}_+)$ *satisfying*

$$\|\phi\|_{3/2-\delta,3/2+\delta} \leq \epsilon j_M^\infty, \tag{5.38}$$

there exists a stationnary solution $f = f_{j_M^\infty} + g$ *to the forced kinetic wave equation* (5.37),

$$0 = \mathscr{C}(f)(\omega) + \phi(\omega), \qquad \forall \omega > 0$$

with

$$f \geq 0 \qquad \textit{and} \qquad \|g\|_{7/6,7/6+\delta} \lesssim \epsilon j_M^\infty.$$

Furthermore, it satisfies:

1. Stability estimates. *for all* $\omega > 0$,

$$f(\omega) = \begin{cases} \left(\frac{j_M^0}{j_M^*}\right)^{1/3} \omega^{-7/6} \left(1 + O(\epsilon\omega^\delta)\right) & \textit{for } \omega \leq 1, \\ \left(\frac{j_M^\infty}{j_M^*}\right)^{1/3} \omega^{-7/6} \left(1 + O(\epsilon\omega^{-\delta})\right) & \textit{for } \omega > 1, \end{cases}$$

with the mass balance

$$j_M^0 = j_M^\infty + \int_0^\infty \omega^{1/2} \phi \, d\omega.$$

2. Mass and energy cascades. *In the weak sense of* (5.35) *and* (5.36) *we have,*

$$-\omega^{1/2}\mathscr{C}(f) = \omega^{1/2}\phi + j_M^\infty \delta_\infty - j_M^0 \delta_0 \tag{5.39}$$

$$-\omega^{3/2}\mathscr{C}(f) = \omega^{3/2}\phi - \left(\int_0^\infty \tilde{\omega}^{3/2} \phi \, d\tilde{\omega}\right) \delta_\infty, \tag{5.40}$$

where the second identity holds provided $\int \tilde{\omega}^{3/2}\phi = \lim_{R\to\infty} \int_0^R \tilde{\omega}^{3/2}\phi$ *exists. Its mass and energy fluxes satisfy:*

$$J_M(f)(\omega) = -j_M^0 + \int_0^\omega \tilde{\omega}^{1/2} \phi(\tilde{\omega}) \, d\tilde{\omega} \quad \textit{and} \quad J_E(f)(\omega) = \int_0^\omega \tilde{\omega}^{3/2} \phi(\tilde{\omega}) \, d\tilde{\omega}.$$

Remark 5.12. — *Cascades*. The identity (5.39) shows that there is a source of j_M^∞ mass of particles at ∞ per unit of time, which, together with the additional mass added by the forcing ϕ, are dissipated in a sink of $j_M^0 = j_M^\infty + \int \sqrt{\omega}\phi \, d\omega$ mass at 0. The identity (5.40) shows that there is no source of energy at 0 and a sink of energy of $\int \omega^{3/2}\phi \, d\omega$ at ∞. Therefore, this solution presents an indirect cascade of mass as well as a direct cascade of energy.

— *Optimality*. The condition (5.38) for some $\delta > 0$ is optimal since for $\delta = 0$ the forcing ϕ would have infinite mass. The upper bound $\delta < 1/12$ allows for the shortest proof, avoiding to track logarithmic losses appearing in it; we did not try to optimise the range of δ.

The obvious discrepancy between the solution constructed in the above theorem and the one guessed in the static ideal scenario is in the behavior at ∞: $\omega^{-7/6}$ versus $\omega^{-3/2}$. This is related to the existence of a source of mass at ∞; it is needed since we are perturbing the KZ solution, but the mass cascading from ∞ results in the tail $\omega^{-7/6}$ which masks the energy cascading to ∞ with the power law $\omega^{-3/2}$.

We end this Section by mentioning other instances where cascades appear. The classical theory of two-dimensional hydrodynamic turbulence [260, 180] features a dual cascade: kinetic energy is transported to smaller frequencies, and enstrophy to higher frequencies. This behavior is different from three-dimensional hydrodynamic turbulence, and is explained by the two conservation laws associated to the two-dimensional Euler equation: energy and enstrophy.

Wave turbulence follows a similar pattern: when the microscopic model has two conserved quantities, it is expected that the associated wave kinetic equation will exhibit a dual cascade [425, 343]. This is typically the case for problems whose Hamiltonian only contain even terms, for instance the MMT toy model [421], elasticity [136], gravitational waves [184], or the nonlinear Schrödinger equation [423, 148], which is the focus of the present article.

Still in the class of problems exhibiting dual cascades, the (gravity) water-wave equation has played a very important role in the development of kinetic wave theory, see [344] for a review. The existence of stationary solutions of the associated kinetic wave equation in the presence of forcing and damping is discussed in [258], while [422] focuses on the tails (cascades) at low and high frequency in the absence of forcing and damping.

As a side remark, let us mention the Smoluchowski coagulation equation [411, 162], which describes the evolution of dust particles that can stick together. There the appearance of a flux towards infinity is well-understood and known as *gelation*. However, the corresponding collision integral is just quadratic and the evolution only supports one cascade. We refer to [173, 172] and references therein for the existence of self-similar solutions and of steady states with a cascade modelling an injection of particles.

Chapter 6

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