Stage de recherche, FIP, M1

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ETATS NON GAUSSIENS DANS DES ENSEMBLES D’ATOMES

NON GAUSSIAN QUANTUM STATES IN ATOMIC ENSEMBLES

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Résumé

Sous la direction de Klemens Hammerer, j’ai travaillé sur un sujet théorique d’optique quantique. Il s’agissait d’étudier théoriquement et numériquement une expérience et sa faisabilité. Le but de l’expérience est de préparer le spin collectif d’un ensemble d’atomes dans un état du type chat de Schrödinger. Les “chats” sont des superpositions d’états “classiques” - les états cohérents. Ils sont particulièrement utiles pour l’étude de la décohérence. Le protocole expérimental proposé s’inspire de ce qui a déjà été fait pour la lumière. L’ensemble d’atomes, initialisé avant l’expérience, est excité par un premier pulse de lumière, puis comprimé (“squeezed”) selon un processus connu sous le nom de “quantum non demolition” (QND). Mon analyse de l’expérience proposée s’appuie sur plusieurs outils théoriques tels que la fonction caractéristique des états quantiques, sa transformée de Fourier (la fonction de Wigner) et le formalisme des états cohérents. J’ai également effectué des calculs numériques permettant d’optimiser la proximité entre l’état final et un ”chat” (fidélité). L’étude a en effet été menée en collaboration avec des expérimentateurs. Les résultats sont très encourageants: des “chats” de taille intéressante, croissant avec le nombre initial d’excitations, devraient pouvoir être effectivement produits et observés avec une fidélité élevée.

Abstract

With Klemens Hammerer as project advisor, I have been working on a theoretical research project in quantum optics. I studied theoretically and numerically an experiment and its feasibility. The goal of the experiment is to create a Schrödinger cat state (that is a superposition of “classical” states - the coherent states) of the collective spin of an atomic ensemble. Cat states are of particular interest for studying decoherence. The proposed experimental procedure relies on an analogy with what has been done in light. The atomic ensemble, initialized before the experiment, is excited by a first light pulse, and then squeezed through a process known as “quantum non demolition measurement” (QND). For the analysis of the experiment, I used different theoretical tools, as the characteristic function of quantum states, its Fourier transform (the Wigner function) and the formalism of coherent states. I also performed numerical calculations allowing to optimize the proximity between the final state and a “cat” (fidelity). The study has been conducted in collaboration with experimentalists. Finally, the results are very encouraging: “cats” of interesting size, increasing with the initial number of excitations, should effectively be produced and observed with a high fidelity.
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1 Introduction

Within Peter Zoller’s group at IQOQI (Institute for Quantum Optics and Quantum Information) in Innsbruck, I worked under the direction of Klemens Hammerer on a theoretical research project in quantum optics. I studied theoretically and numerically an experimental procedure and its feasibility.

The goal of the proposed experiment is to create a Schrödinger cat state (that is a superposition of “classical” states - the coherent states) of the collective spin of an atomic ensemble. This can be achieved in close analogy to what has been experimentally done in light by Philippe Grangier’s group in Orsay and Eugene Polzik’s group in Copenhagen (see [9] and [8]). A cat state can be seen as the closest quantum state to the cat of Schrödinger that is a coherent superposition of two distinguishable “classical” states: a dead cat and a still alive cat. Such states can be used as sensitive detectors of decoherence and are a key component for research at the boundary between quantum and classical worlds. Grangier’s and Polzik’s experiments allow to prepare a free-propagating light pulse in a cat state. However it would also be very useful to generate long-lived cat states, stored in quantum memories. The proposed experimental procedure that I studied thus aimed at producing cat states in atomic ensembles. Furthermore, with this procedure, it should be possible to increase the size of the created cat.

In order to explain what my research project consisted in, I need to give details about the proposed experiment. The collective atomic spin is initially prepared in its internal ground state (with all the spins pointing in the same direction). A first light pulse is sent into atoms in order to create an excitation (a spin-flip) in the ensemble, by Raman effect. The collective spin is then squeezed through a process known as “Quantum Non Demolition measurement” (QND: see [6]): a second light pulse is sent and one quadrature of the outgoing light is measured by homodyne detection. The whole procedure should generate a cat-like state in the atomic ensemble. Another possibility (that I study in the last section of the report) for the procedure would be to first squeeze the initialized collective spin and then create an excitation. Experiments in light indeed consisted in subtracting (or adding) a photon from a squeezed vacuum beam. The analogy between light and atoms is based on an identical formalism for the collective spin of an atomic ensemble and a light mode: both can be described similarly to the quantum harmonic oscillator (such systems are usually referred to as continuous variables systems). In this formalism, one can define the “quadratures” similarly to position and momentum for the harmonic oscillator. A squeezed state is then characterized by a reduced value for the variance of one quadrature. This implies (according to Heisenberg ‘s uncertainty relation) an enhanced variance of the complementary quadrature.
I performed a theoretical and numerical analysis of the proposed experiment. I used different theoretical tools, as the characteristic function of quantum states, its Fourier transform (the Wigner function, that is a quasi-probability distribution, but that takes negative values for cat states) and the formalism of coherent states. Similarly to what has been done for light, I calculated and plotted the overlap ("fidelity") between the generated atomic state and a cat. And I computed the Wigner function in order to observe its negativity. My theoretical calculations showed that the procedure effectively prepares the atomic ensemble in a squeezed excited state when the measured value of the light quadrature is zero. The initial idea was that a feedback in the atomic ensemble (a shift depending on the measured value) should have made the result independent of the measured quadrature value. But I found that, even with feedback, the generated state in the case of a non-zero measured value is a mixture of a squeezed excited state and squeezed vacuum. And this results in a strongly reduced fidelity. Therefore, we suggested to add a post-selection eliminating all the cases when the measured quadrature would be outside a narrow band around zero.

Working in collaboration with the experimentalist Eugene Polzik, we wanted to study the experiment under realistic conditions. We therefore introduced some additional parameters taking noise and experimental uncertainties into account. The results of my theoretical and numerical calculations (with post-selection), both in the ideal and noisy case, are very close to those that have been obtained for light, which is encouraging. Finally, thinking of previous theoretical works (see [2]), Eugene Polzik also suggested to study the possibilities of increasing the size of the generated cat by starting with more than one excitation in the atomic ensemble. I thus also performed the analysis in this case.

At the very beginning of my research project, I had the opportunity to attend a winter school school on quantum information in Obergurgl organized by Klemens Hammerer and Barbara Kraus (Innsbruck). Among the speakers, there were I.J. Cirac (Munich), A. Imamoglu (California), P. Grangier (Orsay) and A. Zeilinger (Vienna). It was a really good introduction to the current research in that field and to my research project in Innsbruck. Besides my research project, I also attended two courses at the University of Innsbruck, about cold atoms (given by Andrew Daley who works at IQOQI on Bose-Einstein condensates) and quantum optics (given by Klemens and Barbara Kraus who is also working in Innsbruck). I found these lectures very interesting and useful, and it helped me to understand the physics of the experiment that I was studying. Furthermore, the talks given by researchers at the group meeting contributed to give me an idea of the current research topics in quantum optics and quantum information.
2 Experiment and modeling

The goal of the experiment is to prepare the collective spin of an atomic ensemble in a Schrödinger cat state, that is a superposition of coherent states: $|\alpha\rangle - |\alpha\rangle$. For that purpose, laser pulses of particular polarization are sent into atoms. In this section, I will first present the physical system and the model that we chose for the interaction between light and atoms. I will then explain in details the steps of the proposed experimental procedure in its final form: the form that has been modified by taking into account the results of my analysis (that lead to add a post-selection). And I will finally present some theoretical tools that I used for the analysis.

2.1 A quantum model for atoms and light

In the experiment, we consider the collective spin of the atomic ensemble. The $N$ atoms are supposed to be effective two-level atoms. The two levels correspond to opposite spins $|\pm 1/2\rangle$. They are coupled by laser light through a process similar to Raman transitions. A Raman transition is a second order transition: the atom is intermediately excited to an upper level.

The atoms of the ensemble have a particular internal structure with a single electron (spin 1/2) outside a closed shell that can be in different hyperfine sublevels. The two relevant levels in the experiment are the sublevels with lowest energy ($J$): $|\pm 1/2\rangle = |J, \pm 1/2\rangle$, where $\pm 1/2$ is the spin projection along the $x$-axis.

All the atoms are initially pumped in $|g\rangle = |+1/2\rangle$. We will call “ground state” this state of the collective spin, where all the atomic spins of the ensemble are aligned (with projection $+1/2$ along $x$). A transition from $|g\rangle = |+1/2\rangle$ to $|e\rangle = |−1/2\rangle$ for one of the atoms (that corresponds to a spin-flip) will be called an excitation. We assume that $N$ is large and that only a few excitations are produced in the experiment. With these assumptions, one can use a formalism similar to the one associated to the quantum harmonic oscillator. One can indeed represent the collective spin by $N + 1$ “Dicke states”, $|n\rangle_A$, where $n$ is the number of atomic excitations, and thus introduce the creation and annihilation operators $a_A^\dagger$ and $a_A$ such that:

$|0\rangle_A = |g\rangle|g\rangle...|g\rangle$

$|1\rangle_A = \frac{1}{\sqrt{N}} (|e\rangle|g\rangle...|g\rangle + |g\rangle|e\rangle...|g\rangle + 2 |g\rangle|g\rangle...|e\rangle) = a_A^\dagger |0\rangle_A$

(Thanks to the geometrical configuration, one can write $|1\rangle_A$ as the symmetric sum of the different possible cases for one excitation)

In this model (Holstein-Primakoff), the collective spin operators (with

---

1. see appendix A.2
\( \omega_l \) is the laser frequency, \( \Delta \) the detuning and \( \Omega \) the 
Rabbi frequency for the transition \( |g\rangle \rightarrow |3\rangle \) driven by the 
laser. \( G \) is the coupling between \( |e\rangle \) and \( |3\rangle \). \( \sigma \) is the 
circular light polarization.

quantization along the \( x \)-axis) read

\[
\dot{J}_x = \frac{1}{2} \left( \sum_i |g\rangle_i \langle g| - |e\rangle_i \langle e| \right) = N/2 - a_A^* a_A
\]

\[
\begin{align*}
\dot{J}_- &= \sum_i |e\rangle_i \langle g| = \sqrt{N} a_A^* \\
\dot{J}_+ &= \sum_i |g\rangle_i \langle e| = \sqrt{N} a_A
\end{align*}
\]

\[
\begin{align*}
\dot{J}_y &= \frac{\sqrt{N} (a_A + a_A^*)}{2} = \sqrt{N/2} \dot{X}_A \\
\dot{J}_z &= -i \frac{\sqrt{N} (a_A - a_A^*)}{2} = \sqrt{N/2} \dot{P}_A
\end{align*}
\]

where the raising (respectively lowering) operator is \( \dot{J}_\pm = \dot{J}_y \pm \dot{J}_z \)

The two levels \( |g\rangle = |J, +1/2\rangle \) and \( |e\rangle = |J, -1/2\rangle \) are indirectly coupled 
by laser light, through Raman-like transitions. In the experiment, we use two 
types of coupling in order to induce two different interaction Hamiltonian.

The first one is a Raman coupling involving three atomic sublevels: the 
lower sublevels \( |g\rangle \) and \( |e\rangle \), and an upper sublevel \( |3\rangle \) (see figure A.1). The 
energy difference between the lower levels and \( |3\rangle \) is \( \omega_0 \approx 10^{14} \) Hz. Left circularly 
polarized laser light (\( \sigma_- \)) at frequency \( \omega_l \) drives the (optical) transition 
between \( |g\rangle \) and \( |3\rangle \) (with Rabbi frequency \( \Omega \)). The transition between \( |3\rangle \) 
and \( |e\rangle \) is also allowed by the selection rules, and is associated with emission 
or absorption of a right circularly polarized photon (\( \sigma_+ \)) at frequency \( \omega_l \). 
Assuming that the laser detuning \( \Delta = \omega_0 - \omega_l \) is large enough, one can make an 
adibatic elimination of the level \( |3\rangle \). One can thus derive an effective 
interaction Hamiltonian\(^2\) (for one atom) coupling only \( |g\rangle \) and \( |e\rangle \):

\[
H_{eff} = -\frac{G \Omega}{\Delta} \left( a_A^* |e\rangle \langle g| + a_l |g\rangle \langle e| \right)
\]

\(^2\)see appendix A.1
where $a^\dagger_l$ is the creation operator for the mode of the photons associated to $|3\rangle \rightarrow |e\rangle$ transitions (at frequency $\omega_l$ and $\sigma_+$ polarized), and $G$ is the single photon Rabbi frequency for these transitions. In that case, the effective Hamiltonian for the whole atomic ensemble is thus simply of the form:

$$H = H_{\text{tot}}^{\text{eff}} \propto a^\dagger_l a^\dagger_A + a_l a_A$$

The second possible coupling involves four sublevels: the two lower $|g\rangle = |J, +1/2\rangle$ and $|e\rangle = |J, -1/2\rangle$, and two upper $|g\rangle = |J', +1/2\rangle$ and $|e\rangle = |J', -1/2\rangle$ (see figure A.1). The two upper sublevels are associated to the same excited level $J'$, separated by an energy $\hbar \omega$. Circularly polarized light drives spin-dependant transitions between the levels $J$ and $J'$. The driving light in the experiment is a laser of coherent light at frequency $\omega_L$ propagating along the $z$-axis and linearly polarized along the $x$-axis. We need to consider two light modes, both at frequency $\omega_L$: the one polarized along $x$, the other along $y$. The $x$-polarized light is a coherent pulse that can be described classically. But the $y$-polarized mode initially doesn’t contain any photon, it must thus be described as a quantum field with operators $a^\dagger_y$ and $a_y$, respectively corresponding to the creation and annihilation of one photon in the $y$-polarized mode. With this model, when the laser detuning $\Delta = \omega - \omega_L$ is large enough, an effective interaction Hamiltonian\footnote{see appendix A.1} can again be derived by adiabatic elimination of the upper sublevels:

$$H \propto \hat{P}_y \hat{P}_A$$

where $\hat{P}_A = -i(a_A - a^\dagger_A)/\sqrt{2}$ and where $\hat{P}_y = -i(a_y - a^\dagger_y)/\sqrt{2}$ is the “momentum” operator for the $y$-polarized light mode.

In both cases, I will use the following notations for the effective interaction Hamiltonian and the corresponding evolution operator:

$$U = e^{-i\kappa H} \quad \text{with } \kappa \text{ the coupling strength}$$

and $H = a^\dagger_l a^\dagger_A + a_l a_A$ or $H = \hat{P}_y \hat{P}_A$ the (dimensionless) Hamiltonian.

$\kappa$ is proportional to $1/\Delta$ and to the interaction time.

### 2.2 Experimental procedure

#### 2.2.1 Description of one shot

The atomic ensemble is initially prepared in the ground state $|0\rangle_A = |g\rangle |g\rangle \ldots |g\rangle$: all the atomic spins have been aligned by optical pumping.
**First light pulse**

A first ($\sigma_-$ polarized) light pulse at frequency $\omega_l$ is sent into the atomic ensemble. As explained above, the effective interaction Hamiltonian in that case (three-level atoms) can be described by $H_1 = a_A^\dagger a_i + a_A a_l$, with evolution operator $U_1 = e^{-i\epsilon H_1}$. The coupling strength $\epsilon$ of the pulse is supposed to be small (it can easily be realized experimentally by making the pulse short enough). And as the $\sigma_+$ polarized light mode at frequency $\omega_l$ (associated to $a_l$) is initially in vacuum, the evolution of the state of the global system (atomic ensemble plus $\omega_l$-frequency light mode) can be written at the first order in $\epsilon$ as

$$|0\rangle_A|0\rangle_l \rightarrow |0\rangle_A|0\rangle_l - i\epsilon|1\rangle_A|1\rangle_l$$

We consider only the cases when a $\sigma_+$ polarized photon is detected (by getting a click of a photon detector after separation of the polarizations). The atomic state is thus projected in the excited state $|1\rangle_A = a_A^\dagger|0\rangle_A$.

**Second light pulse and QND measurement**

The second step of the experiment is a so-called quantum non-demolition (QND) measurement, whose purpose is to create squeezing.

A second pulse at frequency $\omega_L$ is thus sent, but with a different phase and configuration, so that the interaction Hamiltonian is $H = \hat{P}_A \hat{P}_y$ where $\hat{P}_A = \frac{a_A - a_A^\dagger}{\sqrt{2}}$ and $\hat{P}_y = \frac{a_y - a_y^\dagger}{\sqrt{2}}$. Before interaction, the $y$-polarized light mode is in vacuum, exactly as before the first pulse. But now the atomic ensemble is in an excited state. With a coupling strength $\kappa$, the evolution operator is $U = e^{-i\kappa H}$ so that the evolution of the system can be written:

$$|1\rangle_A|0\rangle_y \rightarrow U|1\rangle_A|0\rangle_y$$

And the outgoing $y$-polarized light mode is analyzed by homodyne detection$^4$. This measurement process provides the value of one quadrature of the quantized electromagnetic field $\hat{X}_y = \hat{X}_y \cos \theta + \hat{P}_y \sin \theta$. We choose here $\theta = 0$ in order to project the state of the system in an eigenstate of $\hat{X}_y$.

Thus, after measurement, the atomic state becomes

$$y \langle X|U|1\rangle_A|0\rangle_y$$

up to a normalization factor depending on the measured value $X$.

**Feedback**

Then, a feedback is performed, shifting the state of the atomic ensemble proportionally to the measured value $X$. This corresponds to applying the unitary transformation $V = e^{-igX\hat{X}_A}$ to the atomic state, where $g$ is the shift parameter:

$$|\psi_A\rangle \rightarrow V|\psi_A\rangle = e^{-igX\hat{X}_A}|\psi_A\rangle$$

$^4$see appendix A.3
2.2.2 Reconstruction of the atomic state and post-selection

Experimentally, one must be able to characterize the final atomic state in order to show that a cat state has been produced. The whole experiment thus consists on realizing many times the procedure described in 2.2.1 (one “shot”), and every time adding a measurement step: an additional light pulse is sent in atoms and one quadrature of the outgoing light is measured after interaction. This provides the probability distributions of different atomic quadratures and thus allows to reconstruct the atomic state: this process is known as quantum tomography.

Therefore, the atomic state at the end of the experiment is get theoretically by integration over all possible $X$ that can be measured for the $\hat{X}_y$-light quadrature after the second pulse. However, I have shown that only the near-zero values of $X$ (theoretically ) give a state very close to a cat state. Thus, we decided to add a post-selection to the procedure: a $X_{\text{max}}$ is chosen such that, every time a value $|X| > X_{\text{max}}$ is measured, no additional pulse is sent, but the atomic ensemble is directly pumped again in its ground state $|0\rangle_A$, ready for a new shot. The final density matrix of the atomic ensemble is thus

$$
\rho_{\text{final}} (X_{\text{max}}) = \frac{\int_{-X_{\text{max}}}^{X_{\text{max}}} \rho_X P(X) \, dX}{\int_{-X_{\text{max}}}^{X_{\text{max}}} P(X) \, dX}
$$

where $\rho_X = V|\psi_A\rangle\langle \psi_A|V^\dagger$ is the atomic state at the end of one shot, when a value $X$ has been measured, and $P(X)$ is the probability of measuring $X$.

2.3 Theoretical background

In this subsection, I will define precisely the states involved in the experiment, as cats and squeezed states. And I will present theoretical tools (in particular the characteristic function of a quantum state) that I will use for the analysis.

2.3.1 Characteristic function

In our model of the atomic ensemble and of light (similar to the quantum harmonic oscillator), a general physical state is described by a density matrix on the Hilbert space $\mathcal{H}(R^2)$ generated by the number states $\{|n\rangle\}$. From the operators $a$ and $a^\dagger$, one can define the quadratures $\hat{X}$ and $\hat{P}$ as the equivalent of position and momentum, such that $a = \frac{\hat{X} + i\hat{P}}{\sqrt{2}}$ and $a^\dagger = \frac{\hat{X} - i\hat{P}}{\sqrt{2}}$.

An equivalent representation of a quantum state with density matrix $\rho$ is given by its characteristic function that is defined as

$$
\chi_\rho (\vec{\tau}) = tr \{D(\vec{\tau})\rho\} \quad \text{where} \quad D(\vec{\tau}) = e^{-i(X\hat{P} - P\hat{X})} = e^{-i(\vec{\tau}^T \Omega \vec{\tau})}
$$
with \( \vec{r}^T = (X, P) \) , \( \hat{\vec{r}}^T = (\hat{X}, \hat{P}) \) and \( \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) the symplectic matrix.

\( D(\vec{r}) \) is called the displacement operator. It can indeed also be written in the complex plane as \( D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} \) where \( \alpha = \frac{X+iP}{\sqrt{2}} \) and by applying \( D(\alpha) \) to the operator \( a \), one translates the latter by a vector of affix \( \alpha \): \( D(\alpha) a D(\alpha)^\dagger = a + \alpha \).

The characteristic function completely characterizes the quantum state because the density matrix can be conversely expressed as :

\[
\rho = \int_C \frac{d^2 \alpha}{\pi} D^\dagger(\alpha) \chi_\rho(\alpha) = \int_{\mathbb{R}^2} \frac{d^2 \vec{r}}{2\pi} D^\dagger(\vec{r}) \chi_\rho(\vec{r})
\]

The states with a Gaussian characteristic function are called Gaussian states. As first example, one has for the ground number state \(|0\rangle\) (vacuum):

\[
\chi_{|0\rangle\langle0|}(\vec{r}) = \langle0|D|0\rangle = e^{-\frac{|\alpha|^2}{2}} = e^{-\frac{\vec{r}^T \vec{r}}{4}} = e^{-\frac{X^2+P^2}{4}}
\]

The definition of the characteristic function can be generalized to a multipartite system described by a tensor product of Hilbert spaces \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \ldots \otimes \mathcal{H}_p \):

\[
\chi_\rho(\vec{R}) = \text{tr} \left\{ D(\vec{R}) \rho \right\} = \text{tr}_{1,2,\ldots,p} \left\{ D(\vec{r}_1) D(\vec{r}_2) \ldots D(\vec{r}_p) \rho \right\}
\]

with \( D(\vec{R}) = e^{-i(\vec{R}^T J_p \vec{R})} \)

\[
\vec{R}^T = \begin{pmatrix} \vec{r}_1^T & \ldots & \vec{r}_p^T \end{pmatrix}
\]

where \( J_p = \begin{pmatrix} \Omega & 0 & \ldots & 0 \\ 0 & \Omega & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \Omega \end{pmatrix} \) is the generalized symplectic matrix.

When \( \rho \) is simply a product state \( \rho = \rho_1 \otimes \rho_2 \otimes \ldots \otimes \rho_p \), the state is said to be separable and its characteristic function factorizes:

\[
\chi_\rho(\vec{R}) = \text{tr}_1 \left\{ D(\vec{r}_1) \rho_1 \right\} \text{tr}_2 \left\{ D(\vec{r}_2) \rho_2 \right\} \ldots \text{tr}_p \left\{ D(\vec{r}_p) \rho_p \right\}
\]

\[
= \chi_{\rho_1}(\vec{r}_1) \chi_{\rho_2}(\vec{r}_2) \ldots \chi_{\rho_p}(\vec{r}_p)
\]

### 2.3.2 Coherent states

A coherent state \(|\alpha\rangle\) is a normalized eigenstate of the operator \( a \) for the eigenvalue \( \alpha \), that is \( a |\alpha\rangle = \alpha |\alpha\rangle \). \( \alpha \) can be any complex number. Many properties of the coherent states make them very close to classical states. First of all, \(|\alpha\rangle\) is unchanged by annihilation of a particle. Thus the probability to detect a second particle (photon, in the case of light) remains the same after the detection of a first one.
A coherent state can be decomposed in the number state basis $|n\rangle$:

$$|\alpha\rangle = e^{-|\alpha|^2} \sum_n \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$ 

Thus $\langle n|\alpha\rangle = e^{-|\alpha|^2} \frac{|\alpha|^n}{\sqrt{n!}}$ and $|\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!}$. This corresponds to a Poissonian distribution with mean value $<n> = |\alpha|^2$.

Moreover, under the harmonic Hamiltonian $H = \hbar \omega a^\dagger a$, a coherent state $|\alpha\rangle$ oscillates at frequency $\omega$: at time $t$, the state is proportional to $|\alpha e^{-i\omega t}\rangle$. It behaves like a classical particle in a harmonic potential.

It can easily be verified that $|\alpha\rangle = D(\alpha)|0\rangle$. $|\alpha\rangle$ is thus simply the vacuum translated by a vector $\alpha$ in the complex plane. Therefore, coherent states, as the vacuum, have minimum quantum fluctuations’ uncertainty, that is $\Delta X^2 \Delta P^2 = \frac{1}{4}$ after Heisenberg inequality. Furthermore, as for vacuum, this equality is achieved for $\Delta X^2 = \Delta P^2 = \frac{1}{2}$.

Finally, one can show the identity $\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle \alpha| = 1$. Thus, although they are not orthogonal to each other, the coherent states form a basis of $\mathcal{H}(R^2)$.

### 2.3.3 Schrödinger cat states

A Schrödinger cat is a superposition of two coherent states $|\alpha\rangle$ and $|-\alpha\rangle$ with opposite phases:

$$|\text{Cat}_\pm\rangle = N_\alpha \{ |\alpha\rangle \pm |-\alpha\rangle \}$$

where $N_\alpha$ is a normalization constant: $N_\alpha = \left[ 2 \left( 1 \pm e^{-2|\alpha|^2} \right) \right]^{-\frac{1}{2}}$. With the $-$ sign, the cat is a superposition of odd number states $|2n+1\rangle$ and is thus called an odd cat, whereas with the $+$ sign, it is a superposition of even number states $|2n\rangle$ and is called an even cat. In both cases, $|\alpha|^2 = \langle \alpha|a^\dagger a|\alpha\rangle$ is the mean number of excitations in the state $|\alpha\rangle$ (for light, it is the mean number of photons). We will thus refer to $|\alpha\rangle$ as the size of the cat.

In the case of light, for example, a laser can be described as being in a coherent state $|\alpha\rangle$ (associated to the laser mode): the associated classical electric field is proportional to $\alpha$. A cat of large size $|\alpha|$ is thus a superposition of two “classical” macroscopically distinct states. At the limit between quantum and classical worlds, such states can be used as sensitive detectors of decoherence. ref...???

### 2.3.4 Squeezed states

A squeezed state is characterized by a reduced variance for one quadrature: the variance of one quadrature is less than $\frac{1}{2}$. Heisenberg inequality implies that the variance of the conjugate quadrature is larger than $\frac{1}{2}$. For example, a state squeezed in $x$-direction verifies $\Delta X^2 < \frac{1}{2}$ and $\Delta P^2 > \frac{1}{2}$.
The squeezing operator is defined as

\[ S(r, \phi) = e^{\frac{r}{2}e^{-i\phi}a^2 - \frac{r}{2}e^{i\phi}a'^2} \]

with \( r \) and \( \phi \) real. A squeezed coherent state is thus simply \( |r, \phi, \alpha \rangle = S(r, \phi)D(\alpha)|0\rangle \). We will refer to \( \xi = e^{2r} \) as the squeezing factor.

For a squeezed coherent state, one has \( \Delta P^2 = \xi \) and \( \Delta X^2 = 1/\xi \).

As the squeezing operator creates excitations in pairs, a squeezed number state \( S|n\rangle \) is a superposition of number states of same parity as \( n \). Therefore, and as it has been observed in light, \( S|n\rangle \) should have a non-negligible overlap with a certain cat of same parity. We will see that cats of interesting size can effectively be produced by squeezing number states \( |n\rangle \) for \( n \geq 1 \).

The characteristic function of a squeezed state \( S(\xi)\rho S(\xi)^\dagger \) can be easily computed from the characteristic function of \( \rho \):

\[
\chi_{S(\xi)\rho S(\xi)^\dagger}(\vec{r}) = \text{tr} \left\{ D(\vec{r}) S(\xi)\rho S(\xi)^\dagger \right\} = \text{tr} \left\{ S^\dagger D(\vec{r}) S \rho \right\} = \text{tr} \left\{ D(\vec{r}_{sq}) \rho \right\} = \chi_{\rho}(\vec{r}_{sq})
\]

where \( \vec{r} = \left( \begin{array}{c} X \\ P \end{array} \right) \) and \( \vec{r}_{sq} = \left( \begin{array}{c} X \sqrt{\xi} \\ P/\sqrt{\xi} \end{array} \right) \)

For example, for a squeezed vacuum state one gets:

\[
\chi_{S(0)\rho S(0)^\dagger}(\vec{r}) = e^{-\frac{\vec{r}_{sq}^T \vec{r}_{sq}}{4}} = e^{-\frac{X^2}{4}e^{-\frac{P^2}{4\xi}}} = e^{-\frac{\vec{r}_{sq}^T \vec{r}}{4}} \quad \text{with} \quad \sigma = \left( \begin{array}{cc} \xi & 0 \\ 0 & \frac{1}{\xi} \end{array} \right)
\]

3 Approximating a cat state: ideal case

The experiment aims at generating a Schrödinger cat. Therefore we want to maximize the overlap \( \langle \text{Cat}|\rho_{\text{final}}|\text{Cat} \rangle \) between the atomic state \( \rho_{\text{final}} \) at the end of the experiment and a cat state \( |\text{Cat}\rangle \). This overlap is called the fidelity. In this section, I will calculate the characteristic function of the final atomic state, and I will use it in order to compute the fidelity and the negativity of the Wigner function, and thus to perform the analysis in ideal case (without taking noise into account).

3.1 Characteristic function of the atomic state

3.1.1 Characteristic function of a number state

After the first light pulse and the detection of a \( y \)-polarized photon, the atomic ensemble is (in first approximation) in the pure excited state \( |1\rangle_A \). We want to calculate its characteristic function \( \chi_{|1\rangle}(\vec{r}_A) = \langle 1|D(\vec{r}_A)|1 \rangle \).
As \( D^\dagger aD = (a + \alpha) \) and \( D^\dagger = D^{-1} \), one has \( aD = D(a + \alpha) \) and

\[
\langle 1 | D | 1 \rangle = \langle 0 | aD a^\dagger | 0 \rangle = \langle 0 | D | 0 \rangle + \alpha \langle 0 | D | 1 \rangle = e^{-|\alpha|^2/2} + \langle 0 | D | 1 \rangle
\]

One can decompose the number state \( |1 \rangle \) in the coherent state basis:

\[
|1 \rangle = \int_C \frac{d^2 \beta}{\pi} |\beta \rangle \langle \beta | 1 \rangle = \int C \frac{d^2 \beta}{\pi} e^{-|\beta|^2/2} \beta^* |\beta \rangle
\]

Thus, in the complex plane: \( \langle 0 | D(\alpha) | 1 \rangle = \int \frac{d^2 \beta}{\pi} \langle 0 | D(\alpha) D(\beta) | 0 \rangle e^{-|\beta|^2/2} \beta^* \)

Furthermore \( D(\alpha) D(\beta) = e^{i\alpha^* \beta - \frac{1}{2} |\alpha|^2 - \frac{1}{2} |\beta|^2} D(\alpha + \beta) \) and \( \langle 0 | D(\alpha + \beta) | 0 \rangle = e^{-|\alpha + \beta|^2/2} \).

Thus \( \langle 0 | D(\alpha) | 1 \rangle = \int \frac{d^2 \beta}{\pi} e^{i\alpha^* \beta - |\alpha|^2/2 - |\beta|^2/2} e^{-|\beta|^2/2} \beta^* \)

\[
= e^{-|\alpha|^2/2} \left[ \frac{\partial}{\partial \alpha} - \alpha^* \right] \int \frac{d^2 \gamma}{\pi} e^{\alpha^* \gamma} e^{-|\gamma|^2} \gamma^* = -\alpha^* e^{-|\alpha|^2/2}
\]

Therefore \( \chi_{|1 \rangle |1 \rangle}(\alpha) = \langle 1 | D(\alpha) | 1 \rangle = e^{-|\alpha|^2/2} (1 - |\alpha|^2) \)

and \( \chi_{|1 \rangle |(\vec{r}_A)} = e^{\frac{r_A^2}{4}} \left( 1 - \frac{\vec{r}_A^2}{2} \right) \) with \( \{ \alpha = \frac{X_A + iP_A}{\sqrt{2}}, \beta = \frac{(X_A, P_A)}{\sqrt{2}} \} \)

More generally, it can be shown that the characteristic function of a number state \( |n \rangle \) is a function of the \( n \)th Laguerre polynomial \( L_n \):

\( \chi_{|n \rangle |n \rangle}(\vec{r}_A) = \langle n | D(\alpha) | n \rangle = e^{-\frac{\vec{r}_A^T \vec{r}_A}{2}} L_n \left( \frac{\vec{r}_A^T \vec{r}_A}{2} \right) \)

### 3.1.2 Light-matter interaction

Light-matter interaction during the second light pulse is modeled by an evolution operator \( U = e^{-i\hbar H} \) with \( H = \hat{P}_A \hat{P}_y \). As \( [H, \hat{P}_y] = 0 = [H, \hat{P}_A], [H, \hat{X}_y] = -i\hat{P}_A \) and \( [H, \hat{X}_A] = -i\hat{P}_y \), we obtain:

\[
\begin{align*}
\{ U^\dagger \hat{P}_y U & = \hat{P}_y \\
U^\dagger \hat{X}_y U & = \hat{X}_y + \kappa \hat{P}_A \\
U^\dagger \hat{P}_A U & = \hat{P}_A \\
U^\dagger \hat{X}_A U & = \hat{X}_A + \kappa \hat{P}_y
\end{align*}
\]

that can be summed up by:

\[
U^\dagger \tilde{R} U = S^T \tilde{R} \quad \text{with} \quad \tilde{R}^T = \begin{pmatrix} \hat{X}_y \\ \hat{P}_y \\ \hat{X}_A \\ \hat{P}_A \end{pmatrix} \quad \text{and} \quad S^T = \begin{pmatrix} 1 & 0 & 0 & \kappa \\ 0 & 1 & 0 & 0 \\ 0 & \kappa & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\]

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If $\rho$ is the density matrix of the global system (atoms plus light) before interaction (in our case $\rho = |1\rangle_A \langle 1| \otimes |0\rangle_y \langle 0|$), the characteristic function of the state after interaction $U \rho U^\dagger$ is thus (with $U$ the unitary evolution operator: $U^\dagger U = 1$)

$$
\chi_{U \rho U^\dagger}(\vec{R}) = \text{tr} \left\{ e^{-i\vec{R} J \hat{R}} U \rho U^\dagger \right\} = \text{tr} \left\{ U^\dagger e^{-i\vec{R} J \hat{R}} \rho U \right\} = \text{tr} \left\{ e^{-i(\Sigma \vec{R}) J \hat{R}} \rho \right\} = \chi_{\rho}(\Sigma \vec{R})
$$

In the experiment, the state of the system before interaction is $\rho = |1\rangle_A \langle 1| \otimes |0\rangle_y \langle 0|$ with the characteristic function

$$
\chi_{\rho}(\vec{R}) = \chi_{|0\rangle\langle 0|}(\vec{r}_y) \chi_{|1\rangle\langle 1|}(\vec{r}_y) = e^{-\frac{\vec{r}_y^T \vec{r}_y}{4}} e^{-\frac{\vec{r}_A^T \vec{r}_A}{4}} \left( 1 - \frac{\vec{r}_y^T \vec{r}_y}{2} \right) = e^{-\frac{\vec{r}_A^T \vec{r}_A}{4}} \left( 1 - \frac{\vec{r}_y^T \vec{r}_y}{2} \right)
$$

Therefore, after interaction between light and matter during the second pulse, the characteristic function of the whole system becomes:

$$
\chi_{U \rho U^\dagger}(\vec{R}) = \chi_{\rho}(\Sigma \vec{R}) = e^{-\frac{\vec{r}_y^T \vec{r}_y}{4}} \left( 1 - \frac{(X_A - \kappa P_y)^2}{2} - \frac{P_A^2}{2} \right)
$$

### 3.1.3 Light measurement

After the second pulse, the $\hat{X}_y$ quadrature of light is measured by homodyne detection. Assuming that the value $X$ has been measured, the atomic density matrix $\rho_A$ after measurement is proportional to

$$
\rho_A \propto \langle X | U(\rho U^\dagger) | X \rangle_y = \int \frac{d^4 \vec{R}}{4\pi^2} \langle X | D(\vec{R}) | X \rangle_y \chi_{U \rho U^\dagger}(\vec{R}) = \int \frac{d^4 \vec{R}}{4\pi^2} \delta(X_y) e^{-iP_yX} D(\vec{r}_A) \chi_{\rho}(\Sigma \vec{R})
$$

Thus, after measurement:

$$
\chi_{\rho_A}(\vec{r}_A) \propto \text{tr}_A \left\{ D(\vec{r}_A) \ y \langle X | U \rho U^\dagger | X \rangle_y \right\} = \int \frac{d^2 \vec{r}_y}{2\pi} \delta(X_y) e^{-iP_yX} \chi_{\rho}(\Sigma \vec{R})
$$
3.1.4 Feedback

The third step is a feedback: the atomic state is shifted proportionally to the measured value $X$ for the $\hat{X}_y$-quadrature of the outgoing light:

$$\rho_A \rightarrow \rho_X = V \rho_A V^\dagger$$

with $V = e^{-igX_{\hat{X}A}}$ and where $\rho_A = |\psi_A\rangle\langle\psi_A|$ is the atomic state after light measurement.

As $V^\dagger \hat{X}_A V = \hat{X}_A$ and $V^\dagger \hat{P}_A V = \hat{P}_A - gX$, the feedback simply corresponds to a phase-shift in the characteristic function:

$$\chi_{\rho_X}(\vec{r}_A) = \chi_{V \rho_A V^\dagger}(\vec{r}_A) = \text{tr} \{ D(\vec{r}_A) V \rho_A V^\dagger \} = e^{igXX_A} \text{tr} \{ D(\vec{r}_A) \rho_A \} = e^{igXX_A} \chi_{\rho_A}(\vec{r}_A)$$

3.1.5 Atomic state at the end of the experiment

After calculation and normalization ($\chi_\rho(\vec{0}) = tr \rho = 1$) of the characteristic function$^5$, I got for the atomic state at the end of one shot:

$$\chi_{\rho_{X}}(\vec{r}_A) = \chi_{V \rho_A V^\dagger}(\vec{r}_A) = \text{tr} \{ D(\vec{r}_A) V \rho_A V^\dagger \}$$

$$= \text{tr} \left\{ e^{-i(\vec{r}_A^T \vec{A} V^\dagger \vec{A})} \rho_A \right\}$$

$$= e^{igXX_A} \text{tr} \{ D(\vec{r}_A) \rho_A \}$$

$$= e^{igXX_A} \chi_{\rho_A}(\vec{r}_A)$$

but in the general case, it is a mixture of $S|1\rangle_A$ and $S|0\rangle_A$. And the proportion of squeezed excited state $S|1\rangle_A$ in the atomic state after one shot decreases with $|X|$ as $\frac{1}{1+\kappa^2+2XX^2\kappa^2}$.

However, the post-selection allows to keep only the shots with a near-zero measured value $X$. Furthermore, the probability of measuring $X$

$$P(X) = tr_A \left\{ y \langle X|U \rho U^\dagger |X\rangle y \right\} = \frac{e^{-\frac{X^2}{1+\kappa^2}}}{\sqrt{\pi (1+\kappa^2)}} \left[ \frac{1+\kappa^2+2XX^2\kappa^2}{(1+\kappa^2)^2} \right]$$

is maximal around $X \approx 0$ and sinks quickly when $|X|$ increases. The post-selection can thus be realized, without rejecting too many results.

With the expression of the characteristic function, we can now calculate the Wigner function and the fidelity.

$^5$see appendix B.1 for the calculation
3.2 Wigner quasi-probability distribution

3.2.1 Definition and examples

The Wigner function of a state is defined as the Fourier transform of its characteristic function. For a state with density matrix $\rho$, the Wigner function in the complex plane is thus

$$W_\rho(\beta) = \int \frac{d^2\alpha}{\pi^2} \chi_\rho(\alpha) e^{-\alpha\beta^* + \alpha^*\beta}$$

Usually, the definition of the Wigner function in the $X-P$ plane is

$$W_\rho(\vec{r}) = \int \frac{dx dp}{4\pi^2} \chi_\rho(x, p) e^{i(px - xp)} = \frac{1}{2} W_\rho(\beta) \text{ where } \begin{cases} \vec{r} = (X, P) \\ \beta = (X + iP) / \sqrt{2} \end{cases}$$

The marginals of $W_\rho$ give the $X$ (resp. $P$) probability distribution in state $\rho$: $\langle X|\rho|X \rangle = Prob(X) = \int dP W_\rho(\vec{r})$ and $\langle P|\rho|P \rangle = Prob(P) = \int dX W_\rho(\vec{r})$. Therefore the Wigner function is considered as a quasi-probability distribution. However, for some quantum states (as cat states), $W$ takes negative values. For such non-classical states, one defines the negativity as $W(\vec{0})$.

Figure 3: Negativity $W(\vec{0})$ of the Wigner function for (a) a cat with $\alpha = 1.3$ (coherent superposition of $|\alpha\rangle$ and $|-\alpha\rangle$), (b) an equal mixture (incoherent superposition) of $|\alpha\rangle$ and $|-\alpha\rangle$, and (c) a squeezed excited state $S(\xi)|1\rangle$ with squeezing $\xi = 1/3$
As first example, the Wigner function of $|0\rangle$ is Gaussian and thus positive:

$$W_{|0\rangle\langle 0|}(\beta) = \frac{2}{\pi} e^{-2|\beta|^2} \text{ or } W_{|0\rangle\langle 0|}(\vec{r}) = \frac{1}{\pi} e^{-X^2-P^2}$$

The Wigner function of $|1\rangle$ can also be calculated:

$$W_{|1\rangle\langle 1|}(\beta) = \frac{2}{\pi} e^{-2|\beta|^2} (4|\beta|^2 - 1) \text{ or } W_{|1\rangle\langle 1|}(\vec{r}) = \frac{1}{\pi} e^{-X^2-P^2} (2X^2 + 2P^2 - 1)$$

$W_{|1\rangle\langle 1|}$ takes negative values. Its negativity is $W_{|1\rangle\langle 1|}(\vec{0}) = -\frac{1}{\pi}$.

### 3.2.2 Wigner function at the end of one shot

From the characteristic function of the atomic state at the end of one shot (see 3.1.5), I could calculate the Wigner function at the end of one shot:

$$W_X(\vec{r}_a) = \int \frac{dx dp}{4\pi^2} \chi_{\rho_X}(x,p) e^{i(Pa x - X a p)}.$$

When $X = 0$, as we already showed, the atomic state at the end of one shot is the squeezed excited state $S(\xi = \frac{1}{1+\kappa^2}) \ket{1}_A$. Its Wigner function reduces to

$$W_X(\vec{r}_a) = \frac{1}{\pi} e^{-\frac{X^2}{1+\kappa^2}-P_a^2(1+\kappa^2)} \left( 2P_a^2(1+\kappa^2) + \frac{2X_a^2}{1+\kappa^2} - 1 \right)$$

with a negativity $W_A(\vec{0}) = -\frac{1}{\pi}$, the same as for $|1\rangle$. However, thanks to the squeezing, the shape of the Wigner function of a squeezed excited state $S|1\rangle$ is very close to the one of a Schrödinger cat, as we see in figure 3.

I will discuss the results for the general case and also with post-selection in the next section, because the conclusions are the same for the negativity of the Wigner function ("Wigner negativity") and the fidelity, as we will see.

### 3.3 Fidelity: overlap with a cat

#### 3.3.1 Analytical expression of the fidelity

We want to maximize the fidelity, that is the overlap $\langle \text{Cat} | \rho_{\text{final}} | \text{Cat} \rangle$ between the final atomic state $\rho_{\text{final}}$ at the end of the experiment and an odd cat state $|\text{Cat}\rangle = N_\alpha \{ |\alpha\rangle - | - \alpha\rangle \}$ with $N_\alpha = \left[ 2(1 - e^{-2|\alpha|^2}) \right]^{-\frac{1}{2}}$. As the final state of the experiment is squeezed in the $P$ direction ($\xi = \frac{1}{1+\kappa^2}$), we must take $\alpha$ real if we want a significant overlap between our state and the cat. The

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6see appendix B.2 for the final expression of the Wigner function
Figure 4: Experiment in the ideal case, at the end of one shot, for different values of the measured light quadrature $X$: (a) fidelity (maximal overlap between the final atomic state and a cat), (b) size of the closest cat, and (c) Wigner negativity plotted against the coupling strength $\kappa$. The size of the cat increases with $\kappa$. But the fidelity and the absolute value of the negativity decrease when $\kappa$ increases, and sink quickly when $|X|$ gets larger.

Figure 5: Experiment in the ideal case, with post-selection, for different coupling strengths $\kappa$: (a) fidelity, (b) size of the closest cat, and (c) Wigner negativity plotted against the proportion of shots that are kept. The fidelity and the absolute value of the negativity decrease when the proportion increases, but are still high even for a proportion of 0.1 to 0.15 (10% to 15%). And the size of the cat does not really vary with the proportion.
fidelity can be calculated from the characteristic function of the states:

$$\langle \text{Cat}|\rho|\text{Cat} \rangle = \text{tr} \{ \rho |\text{Cat}\rangle \langle \text{Cat}| \}$$

$$= \int \frac{d^2 \vec{r}}{2\pi} \text{tr} \{ D(\vec{r}) |\text{Cat}\rangle \langle \text{Cat}| \} \chi_{\rho}(\vec{r})$$

$$= \int \frac{d^2 \vec{r}}{2\pi} \chi_{\rho}(\vec{r}) \chi_{\text{Cat}}(-\vec{r})$$

The characteristic function of a cat $\chi_{\text{Cat}}(\beta) = \text{tr} \{ |\text{Cat}\rangle \langle \text{Cat}| D(\beta) \}$ can easily be calculated in the complex plane. For an odd cat $|\text{Cat}\rangle$, I got

$$\chi_{\text{Cat}}(\beta) = N_\alpha^2 \left[ \langle \alpha|D|\alpha \rangle + \langle -\alpha|D| - \alpha \rangle - \langle -\alpha|D|\alpha \rangle - \langle \alpha|D| - \alpha \rangle \right]$$

$$= e^{-|\beta|^2} \left[ e^{\beta \alpha^* - \alpha \beta^*} + e^{-\beta \alpha^* + \alpha \beta^*} - e^{-2|\alpha|^2} \left( e^{\beta \alpha^* - \alpha \beta^*} + e^{\beta \alpha^* + \alpha \beta^*} \right) \right] / \left[ 2(1 - e^{-2|\alpha|^2}) \right]$$

with $\beta = \frac{X + iP}{\sqrt{2}}$ and $\vec{r}^T = (X, P)$.

### 3.3.2 Fidelity and Wigner negativity at the end of one shot

With Mathematica, I could compute analytically the expression of the fidelity $F(\alpha, \kappa, g, X) = \langle \text{Cat}|\rho_X|\text{Cat} \rangle$ at the end of one shot, and thus maximize this fidelity with respect to the size $|\alpha|$ of the cat. I found that the maximal fidelity as well as the absolute value of the Wigner negativity decrease with the interaction coupling strength $\kappa$ (that characterizes the squeezing of the final state: $\xi = \frac{1}{1+\kappa^2}$), whereas the size of the closest cat $|\alpha_{opt}|$ increases with $\kappa$. As the results did not really vary with the feedback $g$, I chose to fix it in all figures: $g = 0.1$.

As figure 4 (a) shows, for $X = 0$, I found a theoretical maximal fidelity very close to 1 ($F > 0.99$), so far $\kappa$ does not exceed 1. For $\kappa \approx 1$, the final state of the experiment can thus theoretically be approximated by a cat of size $|\alpha_{opt}| \approx 1$ with a very high fidelity ($F > 0.99$), as figure 4 (b) shows.

But the fidelity and the absolute value of the Wigner negativity sink very quickly when $|X|$ increases, as figure 4 (a) and (c) show. That is the reason why we decided to add a post-selection.
Figure 6: Experiment in the ideal case, with post-selection, for different post-selection bandwidths $X_{\text{max}} = 0.2$, compared to the case of one shot with a measured light quadrature $X = 0$ (red curve): (a) fidelity and Wigner negativity plotted against the coupling strength $\kappa$. For a small enough bandwidth ($X_{\text{max}} \leq 0.2$), the results are almost as good as for one shot with $X = 0$, even for quite large $\kappa$ ($\kappa \approx 1$).

### 3.3.3 Fidelity and Wigner negativity at the end of the experiment

The fidelity and Wigner function after post-selection\(^7\) (that correspond to what should be measured in the experiment) are expressed by

$$F_{\text{final}}(\alpha, \kappa, g, X_{\text{max}}) = \langle \text{Cat}|\rho_{\text{final}}|\text{Cat} \rangle = \frac{\int_{-X_{\text{max}}}^{X_{\text{max}}} F(\alpha, \kappa, g, X)P(X)dX}{R(X_{\text{max}})}$$

$$W_{\text{final}}(\vec{r}_a) = \int \frac{dx dp}{4\pi^2} \chi_{\rho_{\text{final}}}(x, p) e^{i(P_{ax} - Xa)p} = \frac{\int_{-X_{\text{max}}}^{X_{\text{max}}} W_X(\vec{r}_a)P(X)dX}{R(X_{\text{max}})}$$

where $R(X_{\text{max}}) = \int_{-X_{\text{max}}}^{X_{\text{max}}} P(X) dX$ is the proportion of shots that are kept. $R$ increases almost linearly with $X_{\text{max}}$ for small $X_{\text{max}}$. I could calculate exactly the Wigner function with Mathematica. But, for computing the final fidelity, I had to make an approximation: I expanded the integral assuming $X_{\text{max}} \approx 0$. We will indeed choose $X_{\text{max}}$ small because both the fidelity and the probability decrease quickly when $|X|$ increases.

For a post-selection with small bandwidth $X_{\text{max}} \leq 0.2$, I got almost the same interesting results as for the case of one shot with a measured light quadrature $X = 0$, as figure 3.3.3 shows: the fidelity and the absolute value of the Wigner negativity are still high even when $\kappa$ increases. Furthermore, I found that the size of the closest cat $|\alpha_{\text{opt}}|$ does not significantly vary with $X_{\text{max}}$ for $X_{\text{max}} \leq 0.4$.

Finally, the results must be compared to the proportion of kept shots. A small proportion would indeed reduce the feasibility of the experiment. How-

\(^7\)see appendix B.3
ever, figure 5 shows that, for $\kappa \approx 1$, one can theoretically hope to generate a cat of size $|\alpha| \approx 1$ with a high fidelity $F \geq 0.96$, (and with a significant Wigner negativity) even by still keeping a proportion $R(X_{\text{max}}) > 0.15$ (this means to keep more than 15% of the shots), which is very encouraging.

4 Noise and experimental uncertainties

All the calculations until now have been done in the ideal case. However, in order to study the feasibility of the experiment, we need to perform the analysis under realistic conditions. Therefore we add in this section corrections taking into account noise and experimental uncertainties.

4.1 Parameters characterizing noise

Firstly, with a small probability (as $\epsilon \ll 1$), more than one atom can be excited by the first pulse: $|0\rangle_A|0\rangle_l \rightarrow |0\rangle_A|0\rangle_l - i\epsilon|1\rangle_A|1\rangle_l - \epsilon^2|2\rangle_A|2\rangle_l + ...$ As unitary operations can easily be computed in terms of characteristic function, I directly used the formalism of the evolution operator $U_1 = e^{-i\epsilon H_1}$ with $H_1 = a_A^\dagger a_l^\dagger + a_A a_l$ (as described in 2.2.1):

$$\rho_{\text{in}} = |0\rangle_A\langle 0| \otimes |0\rangle_y\langle 0| \rightarrow U_1 \rho_{\text{in}} U_1^{\dagger} = e^{-i\epsilon H_1} \rho_{\text{in}} e^{i\epsilon H_1} \quad (1)$$

Secondly, uncertainties arise from errors in photon counting. A click of an ideal photon detector would mean that at least one photon has been emitted by atoms. The state of the system would thus be projected in the subspace orthogonal to $|0\rangle_l$:

$$U_1 \rho_{\text{in}} U_1^{\dagger} \rightarrow P_l U_1 \rho_{\text{in}} U_1^{\dagger} P_l \quad \text{with} \quad P_l = 1 - |0\rangle_l\langle 0|$$

However, a detector produces sometimes a click even if no photon has reached it. This error of photon counting is known as dark counts. Therefore we introduce a second parameter: the detection efficiency $\eta$. Its experimental value is about $\eta = 0.8$ for good detectors. Thus, after a click of the detector, the atomic density matrix can be written as

$$\rho_{\text{at}} = \eta \frac{\text{tr}_l \left\{ P_l U_1 \rho_{\text{in}} U_1^{\dagger} P_l \right\}}{\text{tr}_{A,l} \left\{ P_l U_1 \rho_{\text{in}} U_1^{\dagger} P_l \right\}} + (1 - \eta) \frac{|0\rangle_A \langle 0|}{(2)$$

Thirdly, we must take into account spontaneous decay in our description of the second pulse. Spontaneous emission leads to $|0\rangle_A\langle 0| \rho_{\text{at}} |0\rangle_A\langle 0|$. As
a simple model, we introduce a small parameter $\zeta$ ($\zeta \ll 1$) such that the evolution of the system is now described by

$$\rho = \rho_{at} \otimes |0\rangle_y \langle 0| \rightarrow (1 - \zeta) U \rho U^\dagger + \zeta |0\rangle_A \langle 0| \otimes |0\rangle_y \langle 0| \quad (3)$$

We have thus modeled noise with three additional parameters: $\epsilon$ for possible supplementary excitations during the first pulse, $\eta$ as detection efficiency for photon counting, and $\zeta$ for spontaneous emission.

4.2 Corrected characteristic function of the atomic state at the end of one shot

We want to observe the behavior of the fidelity and Wigner function with respect to the noise parameters we introduced above. Therefore, we need to calculate previously the corrected characteristic function of the final state.

First light pulse

The characteristic function of the state $U_1 \rho_{in} U_1^\dagger$ (after the first pulse) can be calculated similarly to what we did in 3.1.2 for the second pulse, but with $U_1 = e^{-i H_1}$ and $H_1 = a_A^\dagger a^\dagger + a_A a_l$. I found

$$\chi_{U_1 \rho_{in} U_1^\dagger}(\vec{R}) = \chi_{\rho_{in}}(\Sigma_\epsilon \vec{R}) \quad \text{where} \quad \Sigma_\epsilon = JS_\epsilon J^T = S_{-\epsilon} \quad \text{with} \quad \left\{ \begin{array}{l} J = J_2 = \Omega \oplus \Omega \\ U_1^\dagger \vec{R} U_1 = S^T_\epsilon \vec{R} \end{array} \right.$$  

where $\vec{R} = \left( \begin{array}{c} \hat{X}_y \\ \hat{P}_y \\ \hat{X}_A \\ \hat{P}_A \end{array} \right)$ and $S_\epsilon = \left( \begin{array}{cccc} \cosh \epsilon & 0 & 0 & -\sinh \epsilon \\ 0 & \cosh \epsilon & -\sinh \epsilon & 0 \\ 0 & -\sinh \epsilon & \cosh \epsilon & 0 \\ -\sinh \epsilon & 0 & 0 & \cosh \epsilon \end{array} \right)$

Therefore, with the initial state $\rho_{in} = |0\rangle_A \langle 0| \otimes |0\rangle_l \langle 0|$, one has:

$$\chi_{U_1 \rho_{in} U_1^\dagger}(\vec{R}) = \exp \left[ -\frac{\vec{R}^T \Sigma_\epsilon \vec{R}}{4} \right] = \exp \left[ -\frac{\vec{R}^T \Sigma_{2_\epsilon} \vec{R}}{4} \right]$$

From (2), and by using the reverse formula $\rho = \int \frac{d^2 \vec{x}}{2\pi} D(\vec{x}) \chi_{\rho}(\vec{x})$ and the expression of $\chi_{U_1 \rho_{in} U_1^\dagger}$, I found for the atomic state after a click$^8$:

$$\chi_{\rho_{at}}(\vec{r}_A) = \frac{\eta}{\tanh^2 \epsilon} e^{-\frac{(x_A^2 + p_A^2) \cosh(2\epsilon)}{4}} + (1 - \frac{\eta}{\tanh^2 \epsilon}) e^{-\frac{x_A^2 + p_A^2}{4}}$$

$^8$see appendix C.1

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Figures 7 and 8: Experiment in the noisy case, with post-selection (of bandwidth $X_{\text{max}} = 0.2$), for an efficiency rate $\eta = 0.8$ (for photon counting): behavior of the results with respect to the coupling strength $\epsilon$ of the first pulse (additional possible excitations) and $\zeta$ (spontaneous decay).

Figure 7: (a) Fidelity, (b) size of the closest cat, and (c) Wigner negativity plotted against the coupling strength $\kappa$, for $\epsilon = 0.1$ and for different $\zeta$. Compared to the ideal case, the fidelity and the absolute value of the negativity are strongly reduced because of the value of $\eta$. They are also significantly affected by $\zeta$. But the noise does not change the size of the cat.

Figure 8: (a) Fidelity, (b) size of the closest cat, and (c) Wigner negativity plotted against the coupling strength $\kappa$, for $\zeta = 0.15$ and for different $\epsilon$. The fidelity, the negativity and the size of the cat do not significantly vary with $\epsilon$. 
Second light pulse, light measurement and feedback

At the beginning of the second light pulse, the whole system is in the state $\rho = \rho_{at} \otimes |0\rangle_y \langle 0|$. Its characteristic function is thus

$$\chi_\rho(\vec{R}) = \chi_{\rho_{at} \otimes |0\rangle_y \langle 0|} (\vec{r}_y) = \chi_{\rho_{at}} (\vec{r}_A) e^{-\vec{r}_y^T \vec{r}_y}$$

By taking into account spontaneous emission (see (3) in 4.1), one has thus for the whole system after interaction:

$$\chi_{after} = (1 - \zeta) \chi_{U \rho U\dagger} (\vec{R}) + \zeta \chi_{|0,0\rangle_0 \langle 0|} (\vec{R}) = (1 - \zeta) \chi_\rho (S\vec{R}) + \zeta e^{-\vec{r}_y^T \vec{r}_y}$$

I performed calculations for light measurement and feedback exactly as in the ideal case (see 3.1.3 and 3.1.4). I thus got an exact expression for the atomic characteristic function $\chi_{\rho_X} (\vec{r}_A)$ at the end of one shot\(^9\). And, from $\chi_{\rho_X}$, I could make numerically the analysis in the noisy case.

### 4.3 Wigner function and fidelity in the noisy case

I calculated from the characteristic function and with Mathematica the fidelity and Wigner function after one shot. Their behavior with respect to $X$ is the same as in the ideal case, and the post-selection allows to get almost as good results as for $X = 0$. I will thus present only the results after post-selection (for a small bandwidth $X_{max} \leq 0.2$).

All the plots presented here are realized with $\eta = 0.8$, that is the value experimentally observed for good photon detectors.

Figure 7 shows that the fidelity and the negativity of the Wigner function are strongly reduced by $\eta$, and are also significantly affected by the spontaneous emission rate $\zeta$. Figure 8 shows that the effect arising from $\epsilon$ (corresponding to the possible emission of additional photons during the first pulse), is much lesser. Furthermore, as $\epsilon$ is proportional to the interaction time, it can be made very small experimentally.

Finally, the size of the closest cat is not affected by noise. Moreover, although noise seems a strong limitation, these results are very similar to those obtained by Polzik and Grangier’s group for light (see [9] and [8]), and are thus still encouraging.

### 5 Increasing the size of the cat

Cats of bigger size can be produced by increasing the initial number of excitations in the atomic ensemble.

\(^9\)see appendix C.2
Figure 9: Experiment starting with $n$ excitations, in the ideal case, for a post-selection bandwidth $X_{\text{max}} = 0.15$: (a) fidelity and (b) size of the closest cat plotted against the coupling strength $\kappa$, for different $n$. The fidelity presents a peak in $\kappa \approx 1$, for $n > 1$, with a high maximal value for every $n$. And for $\kappa \approx 1$, the size of the cat significantly increases with $n$.

5.1 $n$ excitations in ideal case

We assume here that $n$ atoms are excited by light during the first pulse: this is ensured by the detection of $n$ emitted photons, after a longer pulse. In first approximation, the atomic state after the first pulse is thus $|n\rangle_A$.

At the end of one shot, the atomic state is thus simply

$$|\psi_n\rangle = \frac{1}{\sqrt{P_n(X)}} e^{-igX\hat{X}_A} \langle X|U|n\rangle_A |0\rangle = \frac{1}{\sqrt{P_n(X)}} |\psi_f\rangle_A$$

where $U = e^{-i\kappa H} = e^{-i\kappa \hat{P}_A \hat{P}_y}$ and $P_n(X) = tr_A \{\langle y|X|U|n,0\rangle_A \langle n,0|U^\dagger|X\rangle_y\}$ is the probability of measuring $X$ for light (after the second pulse).

As general calculations for $n \in \mathbb{N}$ are difficult, we adopt a new method. Without calculating the characteristic function, we directly compute the fidelity in the quadrature-representations: we decompose the states in the eigenstates basis of one quadrature, using $\int dp|p\rangle = 1$ for $\hat{P}$ and $\int dx|x\rangle = 1$ for $\hat{X}$. In the $p$-representation, the $n^{th}$ number state is a function of the $n^{th}$ Hermite polynomial: $\langle p|n\rangle = \frac{H_n(p) e^{-p^2/2}}{\sqrt{n!}}$. As $\langle X|p_y\rangle = e^{ip_y X}$, one finds:

$$|\psi_n\rangle \propto |\psi_f\rangle_A = \int dp_A dp_y \langle X|p_y\rangle \langle p_y|0\rangle e^{-i\kappa \hat{P}_A \hat{P}_y} \langle p_A|n\rangle e^{-igX\hat{X}_A} |p_A\rangle$$

$$= e^{-\frac{\kappa^2}{2}} \int \frac{dp_A dx_A}{\sqrt{\pi} n! 2^n} e^{-(1+\kappa^2)p_A^2/2} e^{\kappa X p_A} H_n(p_A) e^{-igX x_A} \frac{e^{i\kappa p_A x_A}}{\sqrt{2\pi}} |x_A\rangle$$
We want to calculate the overlap $|\langle \text{Cat}|\psi_n \rangle|^2$ where $|\text{Cat}\rangle$ is a Schrödinger cat state of same parity as $n$: $|\text{Cat}_\pm\rangle = N_\pm \{(\alpha) \pm | -\alpha \rangle\}$ with $|\alpha\rangle$ a coherent state and $N_\pm = \left[2(1 \pm e^{-2|\alpha|^2})\right]^{-\frac{1}{2}}$. As for $n = 1$, $\alpha$ must be real.

As $\langle \alpha| x \rangle = e^{-\frac{(x-\alpha \sqrt{2})^2}{2\pi^{1/4}}}$ for $\alpha$ real, and by using some properties of the Hermite polynomials, I finally found

$$\langle \alpha|\psi_n \rangle = e^{-\frac{X^2}{2}} e^{-\alpha^2} e^{-\frac{\alpha \sqrt{2} - i \alpha X^2}{2}} \frac{\kappa^n e^{\frac{\kappa X + g X + i \alpha \sqrt{2}}{2(2 + \kappa^2)}}}{\sqrt{P_n(X) \sqrt{n!}} 2^{n-1}} H_n \left(\frac{\kappa X + g X + i \alpha \sqrt{2}}{\kappa \sqrt{2 + \kappa^2}}\right)$$

As $\langle -\alpha|\psi_n \rangle_A = \langle \alpha|\psi_n \rangle_A^*$, one gets easily an expression for the fidelity.

Exactly as in the noisy case (see 4.3), I directly present the results with post-selection. Figure 5.1 (a) shows a peak for the fidelity in $\kappa \approx 1$, for every $n > 1$. Furthermore, for $\kappa \approx 1$, the fidelity is very high for every $n$ and the size of the closest cat significantly increases with $n$, as one can see in figure 5.1 (b). As for $n = 1$, I found that the size $|\alpha_{\text{opt}}|$ of the closest cat does not vary with the proportion of results that are kept.

The procedure for atomic ensembles thus allows to generate cats of larger size with a high theoretical fidelity, which could not be made in light. Moreover, although I only made the calculations in the ideal case because of their complexity, we expect that the conclusions in the noisy case would be very similar, but with a reduced fidelity.

### 5.2 $n$ excitations in ideal case, inverted procedure

In the proposed experiment, the atomic excitations are created before the squeezing. But, as for a squeezing factor $\xi = e^{2r}$ one has $a^\dagger S(\xi)|0\rangle = \cosh r \ S(\xi)a^\dagger|0\rangle$, one should also be able to generate cat-like states by squeezing before exciting. We thus study here an inverted procedure where the first step squeezes the atomic state, and the second produces $n$ excitations.

Moreover, we consider for the interaction during the second step (exciting) the general Hamiltonian $H_\psi = \cos \psi \ X_A \hat{X}_l + \sin \psi \ P_A \hat{P}_l$. The most general form of Hamiltonian quadratic in $\vec{r}_A$ and $\vec{r}_l$ can indeed be written as $\vec{r}_A^T M \vec{r}_l$ with $M$ a $4 \times 4$ real matrix. And $M$ can be diagonalized by applying unitary transformations to $\vec{r}_A$ and $\vec{r}_l$, so that the Hamiltonian takes the form described above.

The first step now produces a squeezed vacuum state: $\rho_A = |\xi\rangle_A \langle \xi|$. The second step consists in sending $n$ successive light pulses and detecting every time a photon (as for the first pulse in the direct procedure, see 2.2.1).

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10 see appendix D.1
Figures 10 and 11: Inverted experimental procedure (squeezing before exciting) with $n$ excitations, in the ideal case, with an interaction Hamiltonian $H_\psi = \cos \psi \hat{X}_A \hat{X}_l + \sin \psi \hat{P}_A \hat{P}_l$ ($t = \tan \psi$)

$n$ atomic excitations are thus created in the atomic ensemble: $\rho_A \longrightarrow \mathcal{E}^n(\rho_A)$ with $\mathcal{E}$ such that $\mathcal{E}(\rho) \propto \text{tr}_1 \{ |1\rangle_i \langle 1| e^{-i\epsilon H_\psi} \rho \otimes |0\rangle_i \langle 0| e^{i\epsilon H_\psi} \}$. Thus

\[
\mathcal{E}^n(\rho_A) \propto \text{tr}_{1,2,\ldots,n} \left\{ P_n \ldots P_1 \underbrace{U_n A \ldots U_1 A}_U \rho_{LA} |0\rangle_{n} \langle 0| \ldots |0\rangle_1 \langle 0| U_A \ldots U_n A}^{\rho}_{\rho_{LA}} \right\}
\]

where the index $i$ ($1 \leq i \leq n$) refers to the $i^{th}$ light pulse, $P_i = |1\rangle_i \langle 1|$ is the projector associated to the detection of one photon after the $i^{th}$ light pulse, and $U_{iA} = e^{-i\epsilon H_\psi A} = \exp \left[ -i\epsilon \left( \cos \psi \hat{X}_A \hat{X}_i + \sin \psi \hat{P}_A \hat{P}_i \right) \right]$ where $\hat{X}_i$ and $\hat{P}_i$
are the light quadratures for the $i^{th}$ pulse. Finally:

$$\chi(\vec{r}_A) = tr_A \{ \mathcal{E}^n(\rho_A) \, D_A(\vec{r}_A) \} \propto tr_{LA} \{ P_n \ldots P_1 \, U \rho_{LA} U^\dagger \, D_A(\vec{r}_A) \}$$

$$\propto \int d\vec{R}_L \, \chi_{U_{PLA}U^\dagger}(\vec{R}) \prod_i \chi|1\rangle\langle 1| (\vec{r}_i)$$

$$\propto \int d\vec{R}_L \, \chi_{U_{PLA}U^\dagger}(\vec{R}) \, e^{-\vec{R}_L^T \vec{r}_A \frac{4}{\pi}} \prod_i \left( 1 - \frac{\vec{r}_i^T \vec{r}_i}{2} \right)$$

$\vec{R}$ is the $2(n+1)$ dimension vector associated to the global product Hilbert space for light and atoms $\mathcal{H}_1 \otimes \ldots \mathcal{H}_n \otimes \mathcal{H}_A$: $\vec{R}^T = (\vec{r}_1^T, \ldots \vec{r}_n^T, \vec{r}_A^T) = (\vec{R}_L^T, \vec{r}_A^T)$ with $\vec{r}_i^T = (X_i, P_i)$. As in 3.1.2, I expressed $\chi U_{\rho LA}U^\dagger$ as:

$$\chi_U(\vec{R}) = \chi_\rho \left( J S J^T \vec{R} \right) \text{ with } U^\dagger \vec{R} = S^T \vec{R} \text{ and } J = J_{n+1} = \Omega_1 \oplus \ldots \oplus \Omega_n \oplus \Omega_A$$

with $\chi_{\rho LA}(\vec{R}) = \chi_{\omega L \otimes |\xi\rangle A \langle \xi|} = e^{-\vec{R}_L^T \vec{r}_A \frac{4}{\pi}} e^{-\vec{r}_T \vec{r}_A \frac{4}{\pi}}$ where $\sigma = \left( \begin{array} {c} \xi \\ 0 \\ 1/\xi \end{array} \right)$.

I noticed that $-\vec{r}_i^T \vec{r}_i \frac{4}{2} e^{-\vec{r}_i^T \vec{r}_i \frac{4}{2}} = \frac{\partial}{\partial x_i} \left[ e^{-x_i^2 \frac{4}{2}} \right] \bigg|_{x_i=1}$, so that I could write:

$$\chi(\vec{r}_A) \propto \prod_i \left( 1 + \frac{\partial}{\partial x_i} \left[ \int d\vec{R}_L \, e^{-\vec{r}_L^T Q_x \vec{R} \frac{4}{4}} \right] \right) \bigg|_{x_i=1, \forall i}$$

with $Q_x = Q(x_1, \ldots, x_n) = Q_x^T$ a $2(n+1)$ square matrix. I diagonalized the restriction of $Q_x$ to the light subspace ($\mathcal{H}_1 \otimes \ldots \mathcal{H}_n$) by applying a unitary transformation to the integration variable $\vec{R}_L$: I found that only two modes effectively couple to atoms. I could thus calculate the fidelity from $\chi(\vec{r}_A)$.

Figure 10 (b) shows that the size of the closest cat decreases with $t = \tan \psi$. In figure 10 (a), one sees that the fidelity is very high in band around $t \geq 0$. For $t = 0$, cats of interesting size, increasing with $n$, can be produced. Figure 11 shows that the size of the cat (b) increases with the squeezing $\xi$, whereas the fidelity (a) decreases slowly. For $\xi = 2$, that must be compared to $\kappa = 1$ in the direct procedure (here $1 + \kappa^2 = \xi$), the results are the same for both procedures. But in the inverted procedure, one can increase the squeezing and produce (for every $n$) larger cats with a still very high fidelity.

6 Conclusion

My analysis showed that one should be able to generate cat states in atomic ensembles with a high theoretical fidelity. And although noise reduces the

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11 see appendix D.3 for the whole calculations in the case of the inverted procedure
fidelity, my results under realistic conditions are very similar to those obtained in the experiments with light, which is encouraging. Finally, I also showed in the ideal case that the procedure with atoms allows to increase the size of the cat, by starting with more excitations, or, even better, by squeezing before creating several excitations. These possibilities arising with atomic ensembles are really promising. It would then also be interesting to perform the analysis in a more general case, by varying the expression of the Hamiltonian at every pulse, and optimizing the experimental procedure in order to generate very large cats with a high fidelity.
A Appendix: Quantum model for atoms and light

A.1 Effective interaction Hamiltonian

Raman effect for a three level atom We consider an atom with three levels: two lower degenerate levels $|g\rangle$ and $|e\rangle$, and one upper level $|3\rangle$. The energy levels are represented in figure A.1. A laser at frequency $\omega_l$ drives transitions between $|g\rangle$ and $|3\rangle$ (with Rabbi frequency $\Omega$). The transition between $|3\rangle$ and $|e\rangle$ is also allowed by the selection rules, and is associated with emission or absorption of a photon at frequency $\omega_l$ and whose polarization is imposed by the configuration (the photon detector detects forward scattered light): the atom is in a virtual optical resonator at frequency $\omega_l$. The creation operator for the photons associated to $|3\rangle \rightarrow |e\rangle$ transitions will be written $a^\dagger$ and the whole system will be represented by the states $|j, n\rangle$ where $j \in \{g, e, 3\}$ is the atomic level and $n$ is the number of photon in the light mode defined by $a$.

Hamiltonian of the system is:

$$H = \hbar \omega_0 |3\rangle\langle 3| + \hbar \omega_l a^\dagger a + \hbar G (a|3\rangle\langle e| + a^\dagger|e\rangle\langle 3|) + \hbar \Omega (|3\rangle\langle g| e^{-i\omega_l t} + |g\rangle\langle 3| e^{i\omega_l t})$$
Let us apply the unitary transformation
\[ U = e^{-i\omega_l (|3\rangle\langle 3| + a\dagger a)} \] such that the wave-function \( |\psi\rangle \) becomes \( |\psi_I\rangle = U|\psi\rangle \) in the rotating frame, and the new Hamiltonian is:
\[ H_I = i\hbar \dot{U}^\dagger U + U^\dagger H U \] (so that: the Schrödinger equation
\[ i\hbar \partial_t |\psi\rangle = H |\psi\rangle \] is equivalent to \( i\hbar \partial_t |\psi_I\rangle = H_I |\psi_I\rangle \)). Thus
\[ H_I = \hbar \Delta |3\rangle\langle 3| + \hbar G (|e\rangle\langle e| + a\dagger|e\rangle\langle 3|) + \hbar \Omega (|3\rangle\langle g| + |g\rangle\langle 3|) \]
where \( \Delta = \omega_0 - \omega_l \) is the laser detuning.

The Hamiltonian couples only the states \( |e, n+1\rangle \), \( |g, n\rangle \), and \( |3, n\rangle \). It is indeed easy to prove that: \([H, \mathbb{P}_n] = 0\) where \( \mathbb{P}_n \) is the projector on the subspace \( \mathcal{E}_n \) generated by \( |e, n+1\rangle \), \( |g, n\rangle \), and \( |3, n\rangle \).

One can thus consider the Hamiltonian as the direct sum of the restrictions \( H_n \) of \( H \) to \( \mathcal{E}_n \): \( H = \bigoplus_n H_n = \bigoplus_n \mathbb{P}_n H \mathbb{P}_n \). Let us consider a wave-function solution \( |\psi_n\rangle \) of the Schrödinger equation for \( H_n \) in \( \mathcal{E}_n \):
\[ \partial_t |\psi_n\rangle = -i\hbar H_n|\psi_n\rangle \quad \text{with} \quad |\psi_n(t)\rangle = \alpha(t)|e, n+1\rangle + \beta(t)|g, n\rangle + \gamma(t)|3, n\rangle \]

The Schrödinger equation for \( H_n \) in \( \mathcal{E}_n \) is equivalent to:
\[
\begin{cases}
\dot{\alpha} = -i G\sqrt{n+1}\gamma \\
\dot{\beta} = -i \Omega \gamma \\
\dot{\gamma} = -i \Delta \gamma - i G\sqrt{n+1} \alpha - i \Omega \beta
\end{cases}
\] (4)

But the detuning \( \Delta \) is supposed to be large enough compared to \( g \) and \( \Delta \) so that one can perform an adiabatic elimination: \( \gamma \) oscillates very fast compared to \( \alpha \) and \( \beta \), and thus follows adiabatically their evolution. One can hence set \( \dot{\gamma} \) to zero in the last equation:
\[ \Delta \gamma + G\sqrt{n+1} \alpha + \Omega \beta = 0 \]

Thus:
\[ \gamma(t) = -\frac{G}{\Delta}\sqrt{n+1} \alpha(t) - \frac{\Omega}{\Delta} \beta(t) \] (5)

When the atom is initially in one of the lower levels (\(|g\rangle \) or \(|e\rangle \)), this approximation allows to eliminate the upper level \( |3\rangle \) by writing an effective Hamiltonian that couples only \( |g\rangle \) and \( |e\rangle \). This effective Hamiltonian can be directly read off by replacing \( \gamma \) by its expression (5) in the equations (4) for \( \dot{\alpha} \) and \( \dot{\beta} \):
\[
H_{neff} = -\frac{G^2(n+1)}{\Delta} |e, n+1\rangle\langle e, n+1| - \frac{\Omega^2}{\Delta} |g, n\rangle\langle g, n| - \frac{G\Omega\sqrt{n+1}}{\Delta} \left( |e, n+1\rangle\langle g, n| + |g, n\rangle\langle e, n+1| \right)
\]
Therefore, one gets an effective Hamiltonian $H_{\text{eff}} = \oplus_n H_{\text{neff}}$ for the whole system:

$$H_{\text{eff}} = -\frac{G^2}{\Delta} a^\dagger a|e\rangle\langle e| - \frac{\Omega^2}{\Delta}|g\rangle\langle g| - \frac{G\Omega}{\Delta} (a^\dagger|e\rangle\langle g| + a|g\rangle\langle e|)$$

The interaction Hamiltonian is thus simply:

$$H_{\text{eff,int}} = -\frac{G\Omega}{\Delta} (a^\dagger|e\rangle\langle g| + a|g\rangle\langle e|)$$

**Four-level atom**

![Energy levels of a four level-atom.](image)

As explained in 2.1, every atom of the ensemble has a single electron outside a closed shell that can be either in its ground state $|J\rangle$ or in an excited state $|J'\rangle$, both with spin $1/2$. Circularly polarized light (propagating along the $z$-axis) drives spin dependent optical dipole transitions between $|J\rangle$ and $|J'\rangle$ such that the interaction Hamiltonian between the light and one atom can be written:

$$H_{\text{int}} = G \left[ a_+ |J', +z\rangle\langle J, -z| + a_- |J', -z\rangle\langle J, +z| \right] + h.c.$$  

where $g$ is a coupling constant, $a_+$ (resp. $a_-$) is the annihilation operator associated to the quantized right (resp. left) circularly polarized light mode. And the driven transitions depend on the spin projection along the $z$-axis: $|\pm_z\rangle = |s_z = \pm \rangle$.

When the driving laser is detuned from the transition frequency of a value $\Delta$ (detuning), a second order effective Hamiltonian can be calculated for the lower levels ($|J, \pm_z\rangle$):

$$H_{\text{eff}} = -\frac{G^2}{\Delta} \left[ a_+^\dagger a_+ |J, -z\rangle\langle J, -z| + a_-^\dagger a_- |J, +z\rangle\langle J, +z| \right]$$
But the circular polarizations can be decomposed in the basis of linear polarizations $a_x$ and $a_y$ (as light propagates along $z$):

$$a_+ = \frac{a_x + i a_y}{\sqrt{2}} \quad \text{and} \quad a_- = \frac{a_x - i a_y}{\sqrt{2}}$$

Thus, by simply writing $|J,+z\rangle = |+z\rangle$ and $|J,-z\rangle = |-z\rangle$, one has:

$$H_{\text{eff}} = -\frac{G^2}{\Delta} \left( a_x a_x + a_y a_y \right) \left[ |+z\rangle \langle +z| + |-z\rangle \langle -z| \right] - ig^2 \left( a_x a_y^\dagger - a_y a_x^\dagger \right) \left[ |-z\rangle \langle -z| + |+z\rangle \langle +z| \right]$$

$1 = |+z\rangle \langle +z| + |-z\rangle \langle -z|$ is the identity in the subspace $|J,\pm\rangle$: hence, the first term in the Hamiltonian does not affect the internal dynamics of this subspace (it is only a global energy shift). But $\hat{J}_z = |-z\rangle \langle -z| - |+z\rangle \langle +z|$ is the spin angular momentum operator along $z$.

Moreover, if the driving light is a laser polarized along $x$, the $x$ polarization is described by a coherent state $|\alpha\rangle$, such that $a_x |\alpha\rangle = \alpha |\alpha\rangle$. Thus, finally, for $\alpha$ real:

$$H_{\text{eff}} \propto \frac{G^2}{\Delta} \left( a_y^\dagger a_y - a_y a_y^\dagger \right) \hat{J}_z \propto \frac{G^2}{\Delta} \hat{P}_y \hat{J}_z$$

In the experiment, a magnetic field applied along $x$ breaks the spin degeneracy through Zeeman effect, and all the atoms are initially pumped in $|+x\rangle$. In the Holstein-Primakoff approximation with spin quantization along $x$, the global atomic spin operator $\sum_{at} \hat{J}_{z,at}$ is proportional to $\hat{P}_A$. Hence, the effective interaction Hamiltonian between light and the whole atomic ensemble is proportional to $\hat{P}_y \hat{P}_A$.

Furthermore, if the laser coherent state is chosen to be imaginary ($\alpha \in i\mathbb{R}$), one gets:

$$a_x a_y^\dagger - a_y a_x^\dagger \propto \hat{X}_y$$

And if the magnetic field is applied along $y$, the global spin operator $\sum_{at} \hat{J}_{z,at}$ is proportional to $\hat{X}_A$ (in the Holstein-Primakoff approximation with spin quantization along $y$). In this case, the effective Hamiltonian for the system is proportional to $\hat{X}_y \hat{X}_A$.

The choice of the light phase and of the geometrical configuration thus allows to induce an interaction with a more general Hamiltonian proportional to $(\cos \psi_y \hat{P}_y + \sin \psi_y \hat{X}_y) (\cos \psi_A \hat{P}_A + \sin \psi_A \hat{X}_A)$.

### A.2 Holstein-Primakoff approximation

The atomic ensemble is modeled by $N + 1$ states $|n\rangle$ ($0 \leq n \leq N$), where $n$ is the number of excitations (spin flips). In the ground state $|0\rangle = \ldots$
$|+x, +x, \ldots, +x\rangle$, all the spins are pointing in the same direction (along $x$). One can define the collective raising operator $\hat{J}_+$ and lowering operator $\hat{J}_- = \hat{J}_+^{\dagger}$ such that

$$\begin{align*}
\hat{J}_+ |n\rangle &\propto |n+1\rangle \\
\hat{J}_+ |N\rangle &= 0 \\
\hat{J}_- |0\rangle &= 0
\end{align*}$$

and $[\hat{J}_+, \hat{J}_-] = \hat{J}_x$ where $\hat{J}_x$ is the collective spin operator along $x$: $\hat{J}_x |n\rangle = \frac{N-n}{2} |n\rangle$.

One can introduce a bosonic operator $a_A$ ($[a_A, a_A^{\dagger}] = 1$) associated to the annihilation of one excitation such that:

$$\begin{align*}
\hat{J}_+ &= a_A^{\dagger} \sqrt{1 - \frac{a_A^{\dagger} a_A}{N}} \sqrt{N} \\
\hat{J}_- &= \sqrt{1 - \frac{a_A^{\dagger} a_A}{N}} a_A \sqrt{N}
\end{align*}$$

One gets indeed: $[\hat{J}_+, \hat{J}_-] = N - a_A^{\dagger} a_A = \hat{J}_x$.

If $N$ is large enough and if only a few excitations are produced in the experiment ($< a_A^{\dagger} a_A \ll N$), one can simply write:

$$\hat{J}_x \approx N \quad \text{and} \quad \begin{cases} 
\hat{J}_+ \approx a_A^{\dagger} \sqrt{N} \\
\hat{J}_- \approx a_A \sqrt{N}
\end{cases}$$

Thus, in this model:

$$\hat{P}_A = \frac{b - b^{\dagger}}{i \sqrt{2}} = \frac{\hat{J}_- - \hat{J}_+}{i \sqrt{2N}} = \hat{J}_z / \sqrt{N}$$

$$\hat{X}_A = \frac{b + b^{\dagger}}{\sqrt{2}} = \frac{\hat{J}_- + \hat{J}_+}{\sqrt{2N}} = \hat{J}_y / \sqrt{N}$$

### A.3 Homodyne detection

Homodyne detection is a light measurement process based on interferences between the analyzed beam and a strong coherent light beam. It provides the expectation value of one light quadrature.

A simple explanation of this measurement procedure can be given by assuming that incoming light is restricted to one single mode (represented by an annihilation operator $a_j$). The analyzed light beam ($a_1$) interacts with a reference beam ($a_2$) through a beam-splitter (see figure A.3). The reference beam is called the “local oscillator”. In the experiment, in the case of four-level atoms, for example, this beam is the $x$-polarized light mode at frequency...
Figure 15: Scheme of a homodyne detection measurement: the analyzed beam \( (a_1) \) and a strong coherent reference beam \( (a_2) \) are mixed by a 50-50 beam-splitter (BS). The difference of the outgoing (measured) intensities \( I_1 - I_2 \) is proportional to one quadrature of the analyzed beam.

The annihilation operators for the light modes after the beam-splitter are:

\[
\begin{align*}
b_1 &= (a_1 + ia_2)/\sqrt{2} \\
b_2 &= (a_1 - ia_2)/\sqrt{2}
\end{align*}
\]

The intensity of light is then measured in each branch:

\[
\begin{align*}
I_1 &\propto <b_1^\dagger b_1> \\
I_2 &\propto <b_2^\dagger b_2>
\end{align*}
\]

And the difference of these intensities is computed:

\[
I_1 - I_2 \propto <b_1^\dagger b_1> - <b_2^\dagger b_2> = i <a_1^\dagger a_2 - a_2^\dagger a_1>
\]

When the local oscillator \( (a_2) \) is in a coherent state \( \alpha = |\alpha|e^{-i\theta} \), the intensity difference is proportional to the expectation value of the quadrature \( \hat{X}_{1,\theta+\pi/2} \) of the analyzed beam \( (a_1) \):

\[
I_1 - I_2 \propto i <\alpha^* a_2 - \alpha a_2^\dagger>
\]

\[
\propto <e^{-i\theta} a_2 - e^{i\theta} a_2^\dagger>
\]

\[
\propto <e^{-i(\theta+\pi/2)} a_2 + e^{i(\theta+\pi/2)} a_2^\dagger>
\]

\[
= <\cos(\theta + \pi/2)\hat{X}_1 + \sin(\theta + \pi/2)\hat{P}_1>
\]

\[
= <\hat{X}_{1,\theta+\pi/2}>
\]
B Appendix: Calculations in the ideal case

B.1 Atomic characteristic function at the end of one shot

After interaction during the second light pulse the characteristic function of the whole system (atoms plus $y$-polarized light) is (see 3.1.2)

$$\chi_{U\rho U^{\dagger}}(\bar{R}) = \chi_{\rho}(\Sigma \bar{R}) = e^{-\frac{(\Sigma \bar{R})^T \Sigma \bar{R}}{4}} \left(1 - \frac{(X_A - \kappa P_y)^2}{2} - \frac{P_A^2}{2}\right)$$

And after light measurement, the characteristic function of the atomic ensemble verifies (see 3.1.3)

$$\chi_{\rho A}(\bar{r}_A) \propto T(\bar{r}_A) = tr_A \left\{ D(\bar{r}_A) \langle X|U \rho U^{\dagger}|X \rangle_y \right\} = \int \frac{d^2 \bar{r}_y}{2\pi} \delta(X_y) \ e^{-iP_yX} \ \chi_{\rho}(\Sigma \bar{R})$$

Thus:

$$T(\bar{r}_A) = \int \frac{dP_y}{2\pi} \left\{ \left(1 - \frac{(X_A - \kappa P_y)^2 + P_A^2}{2}\right) e^{-iP_yX} \ \exp \left[ -\frac{P_y^2 + (1 + \kappa^2)P_A^2 + (X_A - \kappa P_y)^2}{4}\right] \right\}$$

where $a = 1 - \frac{X_A^2 + P_A^2}{2}$, $b = \kappa X_A$ and $c = -\frac{\kappa^2}{2}$.

One can define a new variable $P'_y = P_y + \frac{2iX - \kappa X_A}{1 + \kappa^2}$.

And, as

$$\int \frac{dP'_y}{2\pi} \ e^{-\frac{(1 + \kappa^2)P'^2_y}{4}} = \frac{1}{\sqrt{\pi(1 + \kappa^2)}}$$

$$\int \frac{dP'_y}{2\pi} P'_y \ e^{-\frac{(1 + \kappa^2)P'^2_y}{4}} = 0$$

$$\int \frac{dP'_y}{2\pi} P'^2_y \ e^{-\frac{(1 + \kappa^2)P'^2_y}{4}} = \frac{2}{\sqrt{\pi(1 + \kappa^2)^{3/2}}}$$

one gets:

$$T(\bar{r}_A) = \frac{e^{-\frac{X^2}{1 + \kappa^2}}}{\sqrt{\pi(1 + \kappa^2)}} e^{-\frac{(1 + \kappa^2)P_A^2}{4}} e^{-\frac{X_A^2}{4(1 + \kappa^2)}} \left[ \frac{1 + \kappa^2 + 2X^2\kappa^2}{(1 + \kappa^2)^2} - \frac{P_A^2}{2} - \frac{X_A^2}{2(1 + \kappa^2)} - \frac{2iXX_A\kappa}{(1 + \kappa^2)^2} \right]$$
But the probability of measuring $X$ is

$$P(X) = \text{tr}_A \left\{ y \langle X | U \rho U^\dagger | X \rangle_y \right\} = T(\vec{0})$$

Thus

$$P(X) = T(\vec{0}) = \frac{e^{-\frac{X^2}{2(1 + \kappa^2)}}}{\sqrt{\pi(1 + \kappa^2)}} \left[ \frac{1 + \kappa^2 + 2X^2\kappa^2}{(1 + \kappa^2)^2} \right]$$

(6)

The probability, maximal for $X = 0$, quickly decreases when $|X|$ increases, as figure 16 shows.

Moreover $\chi_{\rho_A}(\vec{r}_A) = T(\vec{r}_A)/T(\vec{0}) = T(\vec{r}_A)/P(X)$ (so that $\chi_{\rho_A}(\vec{0})) = tr\rho_A = 1$). Therefore

$$\chi_{\rho_A}(\vec{r}_A) = \exp \left[ -\frac{P_A^2(1 + \kappa^2)}{4} - \frac{X_A^2}{4(1 + \kappa^2)} \right] \exp \left[ -\frac{iXX_A\kappa}{1 + \kappa^2} \right] *$$

$$\left[ \frac{1 + \kappa^2}{1 + \kappa^2 + 2X^2\kappa^2} \left( 1 - \frac{P_A^2(1 + \kappa^2)}{2} - \frac{X_A^2}{2(1 + \kappa^2)} \right) + \frac{2X^2\kappa^2 - 2iXX_A\kappa}{1 + \kappa^2 + 2X^2\kappa^2} \right]$$

And as the feedback simply corresponds to a phase shift

$$\chi_{\rho_X}(\vec{r}_A) = e^{igXX_A} \chi_{\rho_A}(\vec{r}_A)$$

(see 3.1.4), one gets finally for the atomic state at the end of one shot:

$$\chi_{\rho_X}(\vec{r}_A) = \exp \left[ -\frac{P_A^2(1 + \kappa^2)}{4} - \frac{X_A^2}{4(1 + \kappa^2)} \right] \exp \left[ -\frac{iXX_A\kappa}{1 + \kappa^2} + igXX_A \right] *$$

$$\left[ \frac{1 + \kappa^2}{1 + \kappa^2 + 2X^2\kappa^2} \left( 1 - \frac{P_A^2(1 + \kappa^2)}{2} - \frac{X_A^2}{2(1 + \kappa^2)} \right) + \frac{2X^2\kappa^2 - 2iXX_A\kappa}{1 + \kappa^2 + 2X^2\kappa^2} \right]$$

(7)
or equivalently:
\[
\chi_{\rho X}(\vec{r}_A) = \left[ \frac{1 + \kappa^2}{1 + \kappa^2 + 2X^2\kappa^2} \chi_{|1\rangle\langle 1|}(\vec{r}_S^q) + \frac{2X^2\kappa^2 - 2iXAX_A\kappa}{1 + \kappa^2 + 2X^2\kappa^2} \chi_{|0\rangle\langle 0|}(\vec{r}_S^q) \right] e^{i\lambda XX_A}
\]

with \(\lambda = g - \kappa/(1 + \kappa^2)\) and \(\vec{r}_S^q = (X_A/\sqrt{1 + \kappa^2}, P_A\sqrt{1 + \kappa^2})\) and where

\[
\chi_{|0\rangle\langle 0|}(\vec{r}_S^q) = e^{-\frac{\vec{r}_S^q\vec{r}_S^q}{4}} = \exp \left[ -\frac{P_A^2(1 + \kappa^2)}{4} - \frac{X_A^2}{4(1 + \kappa^2)} \right]
\]

and

\[
\chi_{|1\rangle\langle 1|}(\vec{r}_S^q) = e^{-\frac{\vec{r}_S^q\vec{r}_S^q}{4}} \left( 1 - \frac{\vec{r}_S^q\vec{r}_S^q}{2} \right) = \exp \left[ -\frac{P_A^2(1 + \kappa^2)}{4} - \frac{X_A^2}{4(1 + \kappa^2)} \right] \left( 1 - \frac{P_A^2(1 + \kappa^2)}{2} - \frac{X_A^2}{2(1 + \kappa^2)} \right)
\]

### B.2 Wigner function at the end of one shot

The atomic Wigner function at the end of one shot (see 3.2.2)

\[
W_X(\vec{r}_a) = \int \frac{dx dp}{4\pi^2} \chi_{\rho X}(x, p) e^{i(P_x-x)\vec{r}_a} dP_x dP_a
\]

can be calculated from the characteristic function \(\chi_{\rho X}(\vec{r}_A)\) (at the end of one shot). As the calculation is very similar what I did above (??) for the characteristic function \(\chi_{\rho X}\). I found

\[
W_X(\vec{r}_a) = \frac{1}{\pi} \exp \left( -\frac{X_a^2}{1 + \kappa^2} - (P_a + \lambda X)^2(1 + \kappa^2) \right) \star \\
\left[ \frac{1 + \kappa^2}{1 + \kappa^2 + 2X^2\kappa^2} \left( 2(P_a + \lambda X)^2(1 + \kappa^2) + 2 \frac{X_a^2}{1 + \kappa^2} - 1 \right) + \frac{2X^2\kappa^2}{1 + \kappa^2 + 2X^2\kappa^2} \right.
\left. + \frac{4X \kappa(1 + \kappa^2)(P_a + \lambda X)}{1 + \kappa^2 + 2X^2\kappa^2} \right]
\]

(8)

with \(\lambda = g - \frac{\kappa}{1 + \kappa^2}\).

When \(X = 0\), the atomic state at the end of one shot is simply the squeezed excited state \(S(\xi = \frac{1}{1 + \kappa^2}) |1\rangle_A\). Its Wigner function reduces to

\[
W_0(\vec{r}_a) = \frac{1}{\pi} e^{-\frac{X_a^2}{1 + \kappa^2} - P_a^2(1 + \kappa^2)} \left( 2P_a^2(1 + \kappa^2) + \frac{2X_a^2}{1 + \kappa^2} - 1 \right)
\]

with a negativity \(W_A(0) = -\frac{1}{\pi}\). 

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Figure 17: (a) Fidelity and (b) Wigner negativity plotted against the measured value $X$, in ideal case, for different coupling strengths $\kappa$. Both the fidelity and the absolute value of the Wigner negativity sink when $|X|$ increases.

Figure 18: (a) Fidelity and (b) Wigner negativity plotted against the size of the closest cat, in ideal case, for different measured values of $X$ (parametric plot with parameter the coupling strength $\kappa$). Both the fidelity and the absolute value of the Wigner negativity decrease when $|X|$ increases.

B.3 Post-selection

After a post-selection with bandwidth $X_{\text{max}}$, the atomic state at the end of the experiment is:

$$\rho_{\text{final}}(X_{\text{max}}) = \frac{\int_{-X_{\text{max}}}^{X_{\text{max}}} \rho_X P(X) \, dX}{\int_{-X_{\text{max}}}^{X_{\text{max}}} P(X) \, dX}$$

where $\rho_X = V |\psi_A \rangle \langle \psi_A| V^\dagger$ is the atomic state at the end of one shot, when a value $X$ has been measured, and $P(X)$ is the probability of measuring $X$.

One can also calculate the proportion of shots that are kept in a case of
a post-selection with bandwidth $X_{\text{max}}$:

$$R(X_{\text{max}}) = \int_{-X_{\text{max}}}^{X_{\text{max}}} P(X) dX$$

$$= -2 \ e^{-\frac{2 X_{\text{max}}^2 \ \kappa^2}{1+\kappa^2}} \ X_{\text{max}} \ \kappa^2 + \text{Erf} \left( \frac{X_{\text{max}}}{\sqrt{1+\kappa^2}} \right)$$

where $\text{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_{0}^{z} e^{-t^2} dt$. Figure 19 (b) shows that the proportion $R$ increases almost linearly with the bandwidth $X_{\text{max}}$ for small $X_{\text{max}}$.

I calculated the fidelity (resp. Wigner function) at the end of the experiment directly from the fidelity (resp. Wigner function) at the end of one shot:

$$F_{\text{final}}(\alpha, \kappa, g, X_{\text{max}}) = \langle \text{Cat} | \rho_{\text{final}} | \text{Cat} \rangle$$

$$= \langle \text{Cat} | \int_{-X_{\text{max}}}^{X_{\text{max}}} \rho_X P(X) dX | \text{Cat} \rangle$$

$$= \int_{-X_{\text{max}}}^{X_{\text{max}}} \langle \text{Cat} | \rho_X | \text{Cat} \rangle P(X) dX$$

$$= \int_{-X_{\text{max}}}^{X_{\text{max}}} F(\alpha, \kappa, g, X) P(X) dX$$

And, as the calculations were very difficult, even with Mathematica, I
expanded the fidelity assuming that $X_{\text{max}}$ is small:

$$
F_{\text{final}}(\alpha, \kappa, g, X_{\text{max}}) = \frac{\int_{-X_{\text{max}}}^{X_{\text{max}}} F(\alpha, \kappa, g, X) P(X) \, dX}{R(X_{\text{max}})} \approx \frac{1}{R(X_{\text{max}})} \left\{ 2X_{\text{max}} F(\alpha, \kappa, g, 0) P(0) + \frac{X_{\text{max}}^3}{3} \frac{\partial^2}{\partial X^2} [F(\alpha, \kappa, g, X) P(X)] \bigg|_{X=0} \right\}
$$

We needed indeed to consider only small post-selection bandwidths because both the fidelity and the probability decrease quickly with $|X|$.

C Appendix: Calculations in the noisy case

C.1 First light pulse

**Light-matter interaction: matrix $S_\epsilon$**

We want to establish the evolution of the operators associated to light and atoms (during the first pulse) in order to determine the matrix $S_\epsilon$ such that:

$$
U_1^\dagger \hat{R} U_1 = S_\epsilon^T \hat{R} \quad \text{with} \quad U_1 = e^{-i\epsilon H_1} \quad \text{where} \quad H_1 = a_\dagger A a_\dagger + a_A a_l
$$

The commutation relation $[H_1, a_A] = [a_\dagger A, a_A]a_\dagger = -a_\dagger$ implies:

$$
U_1^\dagger a_A U_1 = e^{i\epsilon H_1} a_A e^{-i\epsilon H_1} = a_A + [i\epsilon H_1, a_A] + \frac{1}{2!} [i\epsilon H_1, [i\epsilon H_1, a_A]] + \frac{1}{3!} [i\epsilon H_1, [i\epsilon H_1, [i\epsilon H_1, a_A]]] + ... = a_A - i\epsilon a_\dagger + \frac{(i\epsilon)^2}{2} a_A + \frac{(i\epsilon)^3}{3!} a_\dagger + ... = a_A \cosh \epsilon - i\epsilon a_l \sinh \epsilon
$$

By the same way, one can prove:

$$
\begin{align*}
U_1^\dagger a_\dagger A U_1 &= a_\dagger A \cosh \epsilon - i\epsilon a_\dagger \sinh \epsilon \\
U_1^\dagger a_\dagger A U_1 &= a_\dagger A \cosh \epsilon + i\epsilon a_l \sinh \epsilon \\
U_1^\dagger a_l U_1 &= a_l \cosh \epsilon - i\epsilon a_\dagger A \sinh \epsilon \\
U_1^\dagger a_\dagger l U_1 &= a_\dagger l \cosh \epsilon + i\epsilon a_\dagger A \sinh \epsilon
\end{align*}
$$

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Figure 20: Noisy case with efficiency rate $\eta = 0.8$, coupling strength (first pulse) $\epsilon = 0.1$ and decay factor $\zeta = 0.15$: (a) Fidelity, (b) size of the closest cat and (c) Wigner negativity plotted against the measured value $X$, for different coupling strengths $\kappa$ (second pulse). As in the ideal case, the best results are got for $X = 0$ and are quickly compromised when $|X|$ increases. But even for $X = 0$, the results are strongly limited by noise.

Figure 21: Noisy case with efficiency rate $\eta = 0.8$, coupling strength (first pulse) $\epsilon = 0.05$ and decay factor $\zeta = 0.1$: (a) Fidelity and (b) Wigner negativity plotted against the coupling strength $\kappa$ (second pulse), for different measured $X$. As in the ideal case, the fidelity and the absolute value of the negativity decrease when $\kappa$ increases. But all the values are reduced by noise.
And as \( \hat{X} = \frac{a + a^\dagger}{\sqrt{2}} \) and \( \hat{P} = \frac{a - a^\dagger}{i\sqrt{2}} \), one deduces:

\[
U_1^\dagger \hat{X} A U_1 = \hat{X}_A \cosh \epsilon - \hat{P}_l \sinh \epsilon \\
U_1^\dagger \hat{P} A U_1 = \hat{P}_A \cosh \epsilon - \hat{X}_l \sinh \epsilon \\
U_1^\dagger \hat{X}_l U_1 = \hat{X}_l \cosh \epsilon - \hat{P}_A \sinh \epsilon \\
U_1^\dagger \hat{P}_l U_1 = \hat{P}_l \cosh \epsilon - \hat{X}_A \sinh \epsilon
\]

or equivalently:

\[
U_1^\dagger \hat{R} U_1 = S^T \hat{R} \quad \text{with} \quad S = \begin{pmatrix}
\cosh \epsilon & 0 & 0 & -\sinh \epsilon \\
0 & \cosh \epsilon & -\sinh \epsilon & 0 \\
0 & -\sinh \epsilon & \cosh \epsilon & 0 \\
-\sinh \epsilon & 0 & 0 & \cosh \epsilon
\end{pmatrix}
\] (11)

Detection of one photon

As explained in 4.1, the detection of one photon by a non-perfect detector projects the atomic state in:

\[
\rho_{at} = \eta \frac{tr_A \left\{ P_l U_1 \rho_{in} U_1^\dagger P_l \right\}}{tr_A \left\{ P_l U_1 \rho_{in} U_1^\dagger P_l \right\}} + (1 - \eta) |0\rangle_A \langle 0| \quad (12)
\]

with \( P_l = 1 - |0\rangle_l \langle 0| \).

The corresponding characteristic function is:

\[
\chi_{\rho_{at}}(\vec{r}_A) = \eta \frac{tr_{A,l} \left\{ D_A(\vec{r}_A) P_l U_1 \rho_{in} U_1^\dagger P_l \right\}}{tr_{A,l} \left\{ P_l U_1 \rho_{in} U_1^\dagger P_l \right\}} + (1 - \eta) tr_{A,l} \left\{ D_A(\vec{r}_A) |0\rangle_A \langle 0| \right\}
\]

But, as \( P_l = 1 - |0\rangle_l \langle 0| \), one has

\[
tr_{A,l} \left\{ D_A(\vec{r}_A) P_l U_1 \rho_{in} U_1^\dagger P_l \right\} = tr_{A,l} \left\{ D_A(\vec{r}_A) U_1 \rho_{in} U_1^\dagger \right\} - tr_{A,l} \left\{ D_A(\vec{r}_A)_l |0\rangle U_1 \rho_{in} U_1^\dagger |0\rangle_l \right\}
\]
with \( tr_{A,l} \{ D_A(\vec{r}_A) \ U_1 \rho_{in} \ U_1^\dagger \} = \chi_{U_1 \rho_{in} \ U_1^\dagger} \left( \frac{\vec{r}_l = \vec{0}}{\vec{r}_A} \right) = \exp \left[ -\frac{(X_A^2 + P_A^2) \cosh(2\epsilon)}{4} \right] \)

and, as \( U_1 \rho_{in} \ U_1^\dagger = \int \frac{d^4 \vec{r}_A'}{(2\pi)^2} \ D_1^\dagger(\vec{R}') \ x_{U_1 \rho_{in} \ U_1^\dagger}(\vec{R}') \) and \( tr_A \{ D^\dagger(\vec{r}_A')D_A(\vec{r}_A) \} = 2\pi \delta (\vec{r}_A - \vec{r}_A') \), one obtains

\[
tr_A \left\{ D_A(\vec{r}_A)|0\rangle(U_1 \rho_{in} U_1^\dagger)|0\rangle \right\} = \int \frac{d^4 \vec{R}'}{(2\pi)^2} tr_A \left\{ |0\rangle D^\dagger(\vec{R}') D_A(\vec{r}_A) |0\rangle \right\} \chi_{U_1 \rho_{in} \ U_1^\dagger}(\vec{R}')
\]

\[
= \int \frac{d^2 \vec{r}_A d^2 \vec{r}_l}{2\pi} \ e^{-\frac{\vec{r}_1^2}{2}} \ e^{-\frac{\vec{r}_2^2}{2}} e^{\frac{(X_A^2 + P_A^2)^2}{4}} \ \frac{(X_A \ P_A \ X_l) \ sinh(2\epsilon)}{\cosh^2 \epsilon}
\]

But

\[
tr_{A,l} \left\{ P_l U_1 \rho_{in} U_1^\dagger P_l \right\} = tr_{A,l} \left\{ D_A(\vec{0}) \ P_l U_1 \rho_{in} U_1^\dagger P_l \right\} = 1 - \frac{1}{\cosh^2 \epsilon} = \tanh^2 \epsilon
\]

Thus, the normalized atomic characteristic function after light measurement is:

\[
\chi_{\text{pat}}(\vec{r}_A) = \frac{\eta}{\tanh^2 \epsilon} \left[ e^{-\frac{(X_A^2 + P_A^2)^2}{4}} \ - \frac{1}{\cosh^2 \epsilon} e^{-\frac{X_A^2 + P_A^2}{4}} \right] + (1 - \eta) e^{-\frac{X_A^2 + P_A^2}{4}}
\]

\[
= \eta \left[ \frac{1}{\tanh^2 \epsilon} e^{-\frac{(X_A^2 + P_A^2)^2}{4}} \ - \frac{1}{\sinh^2 \epsilon} e^{-\frac{X_A^2 + P_A^2}{4}} \right] + (1 - \eta) e^{-\frac{X_A^2 + P_A^2}{4}}
\]

\[
= \eta \frac{1}{\tanh^2 \epsilon} e^{-\frac{(X_A^2 + P_A^2)^2}{4}} \ + (1 - \eta) e^{-\frac{X_A^2 + P_A^2}{4}}
\]

### C.2 Light quadrature measurement following the second pulse

The measurement of the \( \hat{X}_y \)-quadrature of light projects the atomic state in a state with density matrix proportional to (1 - \( \zeta \)) \( y \langle X | U \rho U^\dagger | X \rangle y + \zeta \langle 0 \rangle_A \langle 0 \rangle | y \rangle \langle X | 0 \rangle \ y \rangle ^2 \) (where \( U \) is the evolution operator describing the interaction during the second pulse) with characteristic function

(1 - \( \zeta \)) \( tr \left\{ D(\vec{r}_A) y \langle X | U \rho U^\dagger | X \rangle y \right\} + \zeta \ tr \left\{ D(\vec{r}_A) | 0 \rangle_A \langle 0 \rangle \right\} \ y \langle X | 0 \rangle \ y \rangle ^2. \)
Figure 23: Noisy case with efficiency rate $\eta = 0.8$, for different coupling strengths (first pulse) $\epsilon$ and different decay factors $\zeta$, but with fixed coupling strength $\kappa = 0.8$ (second pulse) and with a post-selection with bandwidth $X_{max} = 0.2$: (a) fidelity plotted against $\zeta$ for different $\epsilon$. (b) Fidelity plotted against $\epsilon$ for different $\zeta$. The fidelity is reduced when the noise increases. But the effect of $\zeta$ is more important than the effect of $\epsilon$.

But, exactly as in the ideal case (see 3.1.3), one has:

$$y\langle X | U \rho U^\dagger | X \rangle_y = \int \frac{d^4 \vec{R}}{4\pi^2} y\langle X | D^\dagger(\vec{R}) | X \rangle_y \chi_U \rho_U(\vec{R})$$

$$= \int \frac{d^4 \vec{R}}{4\pi^2} \delta(X_y) e^{- iP_yX} D^\dagger(\vec{r}_A) \chi_{\rho}(S\vec{R})$$

so that
function of the atomic ensemble at the end of one shot is then:

\[ \text{tr} \left\{ D(\vec{r}_A) \right\} y \langle X | \rho \rho^\dagger | X \rangle y \right\} = \int \frac{d^2 \vec{r}_L}{2\pi} \delta(X_y) e^{-iP_y X} \chi_{\rho}(S \vec{R}) \]

\[ = \eta \frac{1}{\text{tanh}^2 \epsilon} \int \frac{dP_y}{2\pi} e^{-iP_y X} e^{-\frac{(X - \kappa P_y)^2 + P_y^2}{4} \cosh(2\epsilon)} e^{-\frac{(X_A - \kappa P_y)^2 + P_y^2}{4} \cosh(2\epsilon)} \]

\[ + \left( 1 - \eta \frac{1}{\text{tanh}^2 \epsilon} \right) \int \frac{dP_y}{2\pi} e^{-iP_y X} e^{-\frac{(X - \kappa P_y)^2 + P_y^2}{4} \cosh(2\epsilon)} e^{-\frac{(X_A - \kappa P_y)^2 + P_y^2}{4} \cosh(2\epsilon)} \]

\[ = \eta \frac{1}{\text{tanh}^2 \epsilon} e^{-\frac{X^2}{4} \cosh(2\epsilon)} e^{-\frac{\theta e}{4(1 + \kappa^2) \cosh(2\epsilon)}} e^{-\frac{(X_A - 2X)^2}{4(1 + \kappa^2)} \cosh(2\epsilon)} \]

\[ + \left( 1 - \eta \frac{1}{\text{tanh}^2 \epsilon} \right) e^{-\frac{X^2}{4(1 + \kappa^2)}} \]

Furthermore:

\[ \text{tr} \left\{ D(\vec{r}_A) |0 \rangle_A \langle 0 | \right\} \right| y \langle X | 0 \rangle_y \right|^2 = \chi_{0 \rangle_A |0 \rangle} \right| y \langle X | 0 \rangle_y \right|^2 = e^{-\frac{X^2 + P_y^2}{4 \sqrt{\pi}} e^{-X^2}} \]

Thus, finally, after measurement and feedback (that corresponds simply to a phase-shift \( e^{i\theta X_A} \) applied to the characteristic function), the characteristic function of the atomic ensemble at the end of one shot is then:

\[ \chi_{\rho_X}(\vec{r}_A) = \frac{e^{i\theta X_A}}{\sqrt{\pi}} Pb(X) \left\{ \zeta e^{-\frac{X^2 + P_y^2}{4} e^{-X^2}} \right\} \]

\[ + (1 - \zeta) \left( 1 - \frac{\eta}{\text{tanh}^2 \epsilon} \right) e^{-\frac{X^2}{4(1 + \kappa^2)} \cosh(2\epsilon)} e^{-\frac{\theta e}{4(1 + \kappa^2) \cosh(2\epsilon)}} e^{-\frac{(X_A - 2X)^2}{4(1 + \kappa^2)} \cosh(2\epsilon)} \]

\[ + \left( 1 - \frac{\eta}{\text{tanh}^2 \epsilon} \right) \]

with \( Pb(X) \) the probability of measuring \( X \) for the \( \hat{X}_y \)-light quadrature. by homodyne detection (after the second pulse):

\[ Pb(X) = \frac{1}{\sqrt{\pi}} \left\{ (1 - \zeta) \frac{\eta}{\text{tanh}^2 \epsilon} e^{-\frac{X^2}{\cosh(2\epsilon)}} + (1 - \zeta) (1 - \frac{\eta}{\text{tanh}^2 \epsilon} \sqrt{1 + \kappa^2 \cosh(2\epsilon)} \right\} + \zeta e^{-X^2} \]
D Appendix: Increasing the size of the cat

D.1 \( n \) excitations: direct procedure

Overlap between the final atomic state and a cat

Figure 24: \( n \) excitations in ideal case, with post-selection: (a) fidelity plotted against the proportion of shots that are kept, for different \( n \) (parametric plot with parameter the coupling strength \( \kappa \)). (b) Proportion of shots that are kept plotted against the post-selection bandwidth \( X_{\text{max}} \) for different \( n \). The fidelity decreases with the proportion. The proportion increases almost linearly with \( X_{\text{max}} \).

In order to calculate the fidelity in the case of \( n \) initial excitations, I use the \( p \) and \( x \) representations as explained in ???. The atomic state at the end of one shot is

\[
|\psi_n\rangle = \frac{1}{\sqrt{P_n(X)}} e^{-igX\hat{X}_A} y \langle X | U | n \rangle_A |0\rangle_y = \frac{1}{\sqrt{P_n(X)}} |\psi_f\rangle_A
\]

where \( U = e^{-i\kappa H} = e^{-i\kappa P_A\hat{P}_y} \) and \( P_n(X) = \text{tr}_A \{ y \langle X | U | n \rangle_A |0\rangle_y \langle n, 0| U^\dagger | X \rangle_y \} \) is the probability of measuring \( X \) for light (after the second pulse). But, in the \( p \)-representation, the \( n^{\text{th}} \) number state is a function of the \( n^{\text{th}} \) Hermite
polynomial: \( \langle p|n \rangle = \frac{H_n(p)}{\pi^{1/4} \sqrt{2^{n!}}} e^{\frac{-p^2}{2}} \). Furthermore: \( \langle X|p_y \rangle = e^{ip_y X} \). Thus

\[
\langle \psi_n \rangle \propto \langle \psi_f \rangle = \int dp_A dp_y \langle X|p_y \rangle \langle p_y|0 \rangle e^{-ikp_A p_y} \langle p_A|n \rangle e^{-igX A} |p_A\rangle
\]

\[
= \int dp_A dp_y \frac{e^{ip_y X} e^{-\frac{p^2}{2}} e^{-ikp_A p_y}}{\pi^{1/4} \sqrt{2^{n!}}} H_n(p_A) e^{-\frac{p^2}{2}} e^{-igX X_A |p_A\rangle}
\]

\[
= \frac{d p_A}{\sqrt{\pi} \sqrt{n! 2^n}} e^{-\frac{(X - \kappa p_A)^2}{2}} H_n(p_A) e^{-\frac{p^2}{2}} e^{-igX X_A |p_A\rangle}
\]

\[
= \int \frac{dp_A}{\sqrt{\pi} \sqrt{n! 2^n}} e^{-\frac{(1 + \kappa^2) p_A^2}{2}} e^{\kappa X p_A} e^{-\frac{X^2}{2}} H_n(p_A) e^{-igX X_A |p_A\rangle}
\]

\[
= e^{-\frac{X^2}{2}} \int \frac{dp_A d x_A}{\sqrt{\pi} \sqrt{n! 2^n}} e^{\frac{(1 + \kappa^2) p_A^2}{2}} e^{\kappa X p_A} H_n(p_A) e^{-igX x_A} \frac{e^{ip_A x_A}}{\sqrt{2\pi}} |x_A\rangle
\]

Figure 25: Experiment starting with \( n \) excitations, in the ideal case, for a post-selection bandwidth \( X_{max} = 0.15 \): (a) fidelity and (b) size of the closest cat plotted against the coupling strength \( \kappa \), for different \( n \). The fidelity presents a peak in \( \kappa \approx 1 \), for \( n > 1 \), with a high maximal value for every \( n \). And for \( \kappa \approx 1 \), the size of the cat significantly increases with \( n \).

Let us calculate know the overlap between \( |\psi_n\rangle \) and a coherent state \( |\alpha\rangle \) (with \( \alpha \) real).
As \( \langle \alpha | x \rangle = e^{-\frac{(x-\alpha \sqrt{2})^2}{\pi^{1/4}}} \) for \( \alpha \) real, one gets:

\[
\langle \alpha | \psi_n \rangle = e^{-\frac{X^2}{2}} \frac{1}{\sqrt{P_n(X)}} \int \frac{dp}{\sqrt{\pi n!}} \frac{e^{-\frac{(1+\alpha^2)p^2}{2}}}{2^{n}} H_n(p) \int \frac{dx}{\sqrt{2\pi^{1/4}}} e^{-igXx} e^{ipx} e^{-\frac{(x-\alpha \sqrt{2})^2}{2}}
\]

\[
= e^{-\frac{X^2}{2}} \frac{1}{\sqrt{P_n(X)}} \int \frac{dp}{\pi^{3/4} n! 2^n} e^{-\frac{(1+\alpha^2)p^2}{2}} e^{ipx} H_n(p) e^{-\alpha^2} e^{i(p-igX+\alpha \sqrt{2})x}
\]

\[
= e^{-\frac{X^2}{2}} e^{-\alpha^2} \frac{1}{\sqrt{P_n(X)}} \int \frac{dp}{\pi^{3/4} n! 2^n} e^{-\frac{(2+\alpha^2)p^2}{2}} e^{ipx} H_n(p) e^{i(p+\alpha \sqrt{2}-igX)}
\]

The last integral can be calculated by defining a new variable \( u = \frac{pA}{\cos \beta} \) where \( \cos^2 \beta = \frac{2}{2+\kappa^2} \). One has indeed:

\[
H_n(u \cos \beta) = n! \sum_{0 \leq k \leq n, k \equiv n[2]} \frac{\cos k \beta \sin^{n-k} \beta}{k! ((n-k)!)} (-1)^{\frac{n-k}{2}} H_k(u)
\]

and

\[
\int H_k(u) e^{-\frac{(u-z)^2}{2}} du = (2z)^k \sqrt{\pi}
\]

so that

\[
\int H_n(u \cos \beta) e^{-\frac{(u-z)^2}{2}} du = n! \sum_{0 \leq k \leq n, k \equiv n[2]} \frac{\cos k \beta \sin^{n-k} \beta}{k! ((n-k)!)} (-1)^{\frac{n-k}{2}} \int H_k(u) e^{-\frac{(u-z)^2}{2}} du du
\]

\[
= n! \sum_{0 \leq k \leq n, k \equiv n[2]} \frac{\cos k \beta \sin^{n-k} \beta}{k! ((n-k)!)} (-1)^{\frac{n-k}{2}} (2z)^k \sqrt{\pi}
\]

\[
= n! \sum_{0 \leq j \leq \frac{n}{2}} \frac{\cos^{n-2j} \beta \sin^{2j} \beta}{(n-2j)! j!} (-1)^j (2z)^{n-2j} \sqrt{\pi}
\]

\[
= \sin^n \beta H_n \left( \frac{\cos \beta}{\sin \beta} \right) \sqrt{\pi}
\]

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Thus:

\[
\langle \alpha | \psi_n \rangle = \frac{e^{-\frac{X^2}{2}} e^{-\alpha^2} e^{(\alpha \sqrt{2}-i\alpha X)^2}}{\sqrt{P_n(X)}} \int \frac{dp_A}{\pi^{3/4} \sqrt{n!} \sqrt{2^n}} e^{-\frac{(2+\kappa^2)p_A^2}{4}} e^{\kappa X p_A} H_n(p_A) e^{i p_A (\alpha \sqrt{2}-i\alpha X)}
\]

\[
= \frac{e^{-\frac{X^2}{2}} e^{-\alpha^2} e^{(\alpha \sqrt{2}-i\alpha X)^2}}{\pi^{3/4} \sqrt{P_n(X)}} \int du e^{-u^2} e^{u \cos \beta [\kappa X + gX + i\alpha \sqrt{2}]} H_n(u \cos \beta) \cos \beta
\]

\[
= \frac{e^{-\frac{X^2}{2}} e^{-\alpha^2} e^{(\alpha \sqrt{2}-i\alpha X)^2}}{\pi^{3/4} \sqrt{P_n(X)}} \cos \beta \int du e^{-\left(u - \frac{\cos \beta [\kappa X + gX + i\alpha \sqrt{2}]}{4}\right)^2} H_n(u \cos \beta) e^{\cos^2 \beta \frac{[\kappa X + gX + i\alpha \sqrt{2}]}{4}^2}
\]

But \(\cos^2 \beta = \frac{2}{2+\kappa^2}\) so that \(\sin^2 \beta = \frac{\kappa^2}{2+\kappa^2}\) and

\[
\langle \alpha | \psi_n \rangle = \frac{e^{-\frac{X^2}{2}} e^{-\alpha^2} e^{(\alpha \sqrt{2}-i\alpha X)^2}}{\pi^{1/4} \sqrt{P_n(X)}} \frac{\kappa^n e^{\frac{[\kappa X + gX + i\alpha \sqrt{2}]}{2(2+\kappa^2)}}}{\sqrt{n!} 2^{n-1}} H_n \left( \frac{\kappa X + gX + i\alpha \sqrt{2}}{2\sin \beta} \right) e^{\cos^2 \beta \frac{[\kappa X + gX + i\alpha \sqrt{2}]}{4}^2}
\]

Noticing that \(\langle -\alpha | \psi_n \rangle_A = \langle \alpha | \psi_n \rangle_A\), one gets easily an expression for the fidelity. Thus, for \(n\) even:

\[
|\langle \text{Cat}_+ | \psi_n \rangle|^2 = |N_\alpha^+|^2 \left| \langle \alpha | \psi_n \rangle + \langle -\alpha | \psi_n \rangle \right|^2
= |N_\alpha^+|^2 \left( \langle \alpha | \psi_n \rangle + \langle -\alpha | \psi_n \rangle \right)^2
\]

and for \(n\) odd:

\[
|\langle \text{Cat}_- | \psi_n \rangle|^2 = |N_\alpha^-|^2 \left| \langle \alpha | \psi_n \rangle - \langle -\alpha | \psi_n \rangle \right|^2
= -|N_\alpha^-|^2 \left( \langle \alpha | \psi_n \rangle - \langle -\alpha | \psi_n \rangle \right)^2
\]

Moreover, the probability \(P_n(X)\) of measuring \(X\) is expressed as an integral that can be numerically computed:

\[
P_n(X) = \frac{e^{-x^2}}{\pi^{2n! n!}} \int dp e^{-(1+\kappa^2)p^2} e^{2nXp} H_n(p)^2
\]

**D.2 n-photons subtracted squeezed vacuum**

By subtracting \(n\) photons from a squeezed vacuum beam, one gets a state proportional to \((a^\dagger)^n S(\xi) |0\rangle\) where \(S(\xi)\) is the squeezing operator with squeezing parameter \(\xi\). Our aim is to compare this state to a Schrödinger cat state
Figure 26: Case of a $n$-photon subtracted squeezed vacuum: (a) Size of the closest Schrödinger cat and (b) fidelity (overlap) plotted against the squeezing parameter and for different values of $n$. As expected, the size of the cat increases with $n$ for a fixed squeezing parameter. For a large enough squeezing parameter, the fidelity is still high when $n$ increases. And it seems even better: for $n > 1$, the fidelity increases with $n$ (at least for small values of $n$).

\[ |\text{Cat} \rangle = \frac{\alpha \pm |\alpha|}{\sqrt{2(1 \pm e^{-|\alpha|^2})}} \] (with $+$ for an even value of $n$ and $-$ for an odd value of $n$).

Plotting for small $n$ the maximal overlap $N(n, \xi) \langle \text{Cat} | (a_+^\dagger)^n S(\xi) | 0 \rangle$ (where $N(n, \xi)$ is the normalization factor for the atomic state) and the associated value of $\alpha$ showed, as we can see in figure 26 (a), that the size $|\alpha|$ of the cat increases with the number $n$ of excitations. And figure 26(b) shows that, when the squeezing parameter $\xi$ is large enough, such larger cats can be produced with a fidelity as high as for one excitation.

D.3 $n$ excitations, inverted procedure

Interaction between light and atoms

The interaction between the light and the atoms during the $i^{th}$ pulse (that creates an $i^{th}$ excitation in the atomic ensemble) is described by the evolution operator $U_{iA} = e^{-i\epsilon H_{iA}^\dagger} = \exp \left[ -i\epsilon \left( \cos \psi \hat{X}_A \hat{X}_i + \sin \psi \hat{P}_A \hat{P}_i \right) \right]$. Exactly as in 3.1.2, $U_{iA}$ can be represented by a $4 \times 4$ matrix $S_i$ such that:

\[ U_{iA}^\dagger \hat{R}_{iA} U_{iA} = S_i^T \hat{R}_{iA} \quad \text{where} \quad \hat{R}_{iA}^T = (\hat{r}_i^T, \hat{r}_A^T) \]

$S_i$ can be extended to a $2(n+1) \times 2(n+1)$ matrix $S_i \oplus_{j \neq i} 1_j$ that I will simply write also $S_i$, such that:

\[ U_{iA}^\dagger \hat{R} U_{iA} = S_i^T \hat{R} \quad \text{(14)} \]
where $\vec{R}$ is the $2(n+1)$ dimension vector associated to the global product Hilbert space for light and atoms $\mathcal{H}_1 \otimes \ldots \mathcal{H}_n \otimes \mathcal{H}_A = \mathcal{H}_L \otimes \mathcal{H}_A$. That is

$$\vec{R}^T = (\vec{r}_1^T, \ldots, \vec{r}_n^T, \vec{r}_A^T) = (\vec{R}_L^T, \vec{r}_A^T)$$

with $\vec{r}_i = \begin{pmatrix} x_i \\ p_i \end{pmatrix}$. Thus:

$$\chi_{U_{1A} \rho} U_{1A}^\dagger (\vec{R}) = \chi_\rho \left( J S_i J^T \vec{R} \right) \quad \text{with} \quad J = J_{n+1} = \bigoplus_{n+1}^{n+1}$$

so that for the interaction with the $n$ pulses, one can write:

$$\chi_{U_{1A} \rho} U_{1A}^\dagger (\vec{R}) = \chi_\rho \left( J S J^T \vec{R} \right) \quad \text{with} \quad S = S_1 S_2 \ldots S_n \quad \text{and} \quad U = U_{nA} \ldots U_{1A}$$

(16)

Figure 27: Inverted experimental procedure (squeezing before exciting) with $n$ excitations, in the ideal case, with an interaction Hamiltonian $H_\psi = \cos \psi \hat{X}_A \hat{X}_L + \sin \psi \hat{P}_A \hat{P}_L \quad (t = \tan \psi)$: (a) Fidelity maximized with respect to $|\alpha|$ (the size of the cat) and $t$, (b) size of the closest cat, and (c) optimal value of $t$ plotted against the squeezing $\xi$. For every $n$, the fidelity decreases slightly with $\xi$, but the size of the cat increases significantly. For $2 \lesssim \xi \lesssim 3$ the best results are obtained for $t \approx 0$.

Expression of the matrix $S_j$

Theorem

$$S_j = \begin{pmatrix} \alpha 1 & A_{A_j} \\ A_{jA} & \alpha 1_A \end{pmatrix} \quad \text{where} \quad A_{jA} = A_{A_j} = A = \begin{pmatrix} 0 & -\gamma \\ \beta & 0 \end{pmatrix}$$

(17)
with
\[
\begin{align*}
\alpha &= \cos(\epsilon \chi) \\
\beta &= \sin(\epsilon \chi) \eta_\psi \sqrt{t} & \text{if } t \geq 0 \quad \text{and} \quad \gamma &= \sin(\epsilon \chi) \delta_\psi / \sqrt{t} \\
\end{align*}
\]
where
\[
\begin{align*}
t &= \tan \psi \quad \text{and} \quad \chi &= \sqrt{\sin \psi \cos \psi}
\end{align*}
\]
and \( \eta_\psi = \sin \psi / |\sin \psi| \) is the sign of \( \sin \psi \) and \( \delta_\psi = \cos \psi / |\cos \psi| \) the sign of \( \cos \psi \).

Remark: For every \( \psi \), one has
\[
\alpha^2 + \beta \gamma = 1 \quad \text{and} \quad t = \tan \psi = \beta / \gamma
\]

Proof: As \([X, P] = i\) for atoms (resp. for light), the interaction Hamiltonian (during the \( j \)th pulse) \( H = H_j^A \) satisfies
\[
\begin{align*}
H &= \hat{X}_j \cos \psi + \hat{P}_j \sin \psi
\end{align*}
\]
verifies the following commutation relations:
\[
\begin{align*}
[H, \hat{X}_j] &= -i \hat{P}_j \sin \psi \\
[H, \hat{P}_j] &= i \hat{X}_j \cos \psi
\end{align*}
\]
Thus
\[
\begin{align*}
e^{i\epsilon H} \hat{X}_A e^{-i\epsilon H} &= \hat{X}_A + [i \epsilon H, \hat{X}_A] + \frac{1}{2!} [i \epsilon H, [i \epsilon H, \hat{X}_A]] + \frac{1}{3!} [i \epsilon H, [i \epsilon H, [i \epsilon H, \hat{X}_A]]] + \ldots \\
&= \hat{X}_A + \epsilon \hat{P}_j \sin \psi - \frac{\epsilon^2}{2} \hat{X}_A \sin \psi \cos \psi - \frac{\epsilon^3}{3!} \hat{P}_j \sin^2 \psi \cos \psi + \ldots \\
&= \hat{X}_A \left( 1 - \frac{\epsilon^2}{2} \sin \psi \cos \psi + \ldots \right) + \hat{P}_j \sin \psi \left( \epsilon - \frac{\epsilon^3}{3!} \sin \psi \cos \psi + \ldots \right) \\
&= \hat{X}_A \left( 1 - \frac{\epsilon^2 \chi^2}{2} \eta_\psi \delta_\psi + \ldots \right) + \hat{P}_j \eta_\psi \sqrt{|t|} \left( \epsilon \chi - \frac{\epsilon^3 \chi^3}{3!} \eta_\psi \delta_\psi + \ldots \right)
\end{align*}
\]
where \( \chi = \sqrt{\sin \psi \cos \psi} \) and \( t = \tan \psi \)
and where \( \eta_\psi \) is the sign of \( \sin \psi \) and \( \delta_\psi \) the one of \( \cos \psi \), so that \( \eta_\psi \delta_\psi \) is the sign of \( t = \tan \psi \). Thus
\[
\begin{align*}
e^{i\epsilon H} \hat{X}_A e^{-i\epsilon H} &= \alpha \hat{X}_A + \beta \hat{P}_j
\end{align*}
\]
with
\[
\begin{align*}
\alpha &= \begin{cases} 
\cos (\epsilon \chi) & \text{if } t \geq 0 \\
\cosh (\epsilon \chi) & \text{if } t < 0
\end{cases} \\
\beta &= \begin{cases} 
\eta_\psi \sqrt{|t|} \sin (\epsilon \chi) & \text{if } t \geq 0 \\
\eta_\psi \sqrt{|t|} \sinh (\epsilon \chi) & \text{if } t < 0
\end{cases}
\end{align*}
\]
In the same way, one shows:
\[
\begin{align*}
e^{i\epsilon H} \hat{P}_A e^{-i\epsilon H} &= \alpha \hat{P}_A - \gamma \hat{X}_j
\end{align*}
\]
with \[ \gamma = \begin{cases} \delta_{\psi}/\sqrt{|t|} \sin(\epsilon \chi) & \text{if } t \geq 0 \\ \delta_{\psi}/\sqrt{|t|} \sinh(\epsilon \chi) & \text{if } t < 0 \end{cases} \]

And, as \( H \) is symmetric in light \((j)\) and atoms \((A)\), one gets also the same relations when \( \tilde{X}_A \) and \( \tilde{X}_j \) (resp. \( \tilde{P}_A \) and \( \tilde{P}_j \)) are reversed.

These relations can thus be summed up in a single formula:

\[
e^{i\epsilon H} \tilde{R}_{jA} e^{-i\epsilon H} = S_j^T \tilde{R}_{jA} \quad \text{with} \quad S_j^T = \begin{pmatrix} \alpha & 0 & 0 & \beta \\ 0 & \alpha & -\gamma & 0 \\ 0 & \beta & \alpha & 0 \\ -\gamma & 0 & 0 & \alpha \end{pmatrix}
\]

Thus, finally, \( S_j \) can be expressed as in 17.

**Expression of the matrix \( S \)**

\( S = S_1 \ldots S_n \) is calculated as the product of the \( 2(n+1) \times 2(n+1) \) matrices \( S_j \), where \( S_j \) denotes the extension \( S_j \oplus_{k \neq j} 1_k \) of the matrix \( S_j \) calculated above.

I found

\[
S = \begin{pmatrix} \alpha \mathbf{1} & A^2 & \alpha A^2 & \ldots & \alpha^{n-2} A^2 & \alpha^{n-1} A \\ 0 & \alpha \mathbf{1} & A^2 & \ldots & \alpha^{n-3} A^2 & \alpha^{n-2} A \\ 0 & 0 & \alpha \mathbf{1} & \ldots & \alpha^{n-4} A^2 & \alpha^{n-3} A \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \alpha \mathbf{1} & A \\ A & \alpha A & \alpha^2 A & \ldots & \alpha^{n-1} A & \alpha^n \mathbf{1} \end{pmatrix}
\]  
\text{(18)}

with \( A = \begin{pmatrix} 0 & -\gamma \\ \beta & 0 \end{pmatrix} \) and \( A^2 = \begin{pmatrix} -\beta \gamma & 0 \\ 0 & -\beta \gamma \end{pmatrix} = -\beta \gamma \mathbf{1} \)

We define \( \theta = t/\xi - 1 \) and \( \lambda = \xi/t - 1 \). Finally, we define \( s \) as

\[
\begin{cases} s = \sin(\epsilon \chi) & \text{when } t \geq 0 \\ s = \sinh(\epsilon \chi) & \text{when } t < 0 \end{cases}
\]

The interaction strength \( \epsilon \) is supposed to be small (as in the first experiment) in order to produce excitations one by one. We will take \( s \) as small parameter of the problem.

**Expression of the atomic characteristic function**

For the calculation of the characteristic function, I did not need to distinguish the different cases described above (according to the sign of \( t = \tan \psi \) and \( \cos \psi \)). I only used the relations verified by the parameters for all \( \psi \):

\[ \alpha^2 + \beta \gamma = 1 \quad \text{and} \quad t = \tan \psi = \beta/\gamma \]
Let us define \( \theta = t/\xi - 1 \), \( \lambda = \xi/t - 1 \) and

\[
s = \begin{cases} 
\sin(\epsilon \chi) & \text{when } t \geq 0 \\
\sinh(\epsilon \chi) & \text{when } t < 0 
\end{cases}
\]

The interaction strength \( \epsilon \) is supposed to be small (as in the first experiment) in order to produce excitations one by one. As \( \epsilon \) only appears through \( s \), I chose \( s \) as small parameter of the problem.

We saw in 5.2 that the atomic characteristic function at the end of the experiment is expressed as an integral:

\[
\chi(\vec{r}_A) \propto \int d\vec{R}_L \, \chi_{U\rho_{LA}U^\dagger}(\vec{R}) \, e^{-\frac{\vec{R}_L^T \vec{R}_L}{4}} \prod_i \left( 1 - \frac{\vec{r}_i^T \vec{r}_i}{2} \right)
\]

with

\[
\chi_{U\rho_{LA}U^\dagger}(\vec{R}) = \chi_{PLA} \left( JSJ^T \vec{R} \right)
\]

and (as \( \rho_{LA} = |0\rangle_1 |0\rangle_n \otimes |\xi\rangle_A |\xi\rangle \))

\[
\chi_{PLA}(\vec{R}) = e^{-\frac{\vec{R}_L^T \vec{R}_L}{4}} e^{-\frac{\vec{r}_A^T \sigma_A}{4}} = e^{-\frac{\vec{R}^T (\Omega L \oplus \sigma_A) J^T \vec{R}}{4}}
\]

with \( \sigma = \begin{pmatrix} \xi & 0 \\ 0 & 1/\xi \end{pmatrix} \) and \( \sigma_A = \Omega^T \sigma \Omega = \begin{pmatrix} 1/\xi & 0 \\ 0 & \xi \end{pmatrix} \).

Therefore, one gets

\[
\chi(\vec{r}_A) \propto \int d\vec{R}_L \, e^{-\frac{\vec{R}_L^T \vec{R}_L}{4}} e^{-\frac{\vec{r}_A^T \sigma_A}{4}} \prod_i \left( 1 - \frac{\vec{r}_i^T \vec{r}_i}{2} \right)
\]

with \( Q = S^T (1_L \oplus \sigma_A) S - 1_L \). \( Q^T = Q \). I considered all the matrices as block-matrices made of \( n+1 \) (or \( n \), for light only) \( 2 \times 2 \)-blocks. Let us write

\[
Q = \begin{pmatrix} Q_L & Q_{LA} \\ Q_{AL} & Q_{AA} \end{pmatrix}
\]

with \( Q_L = \{ Q_{ij} \}_{i,j} \) the \( 2n \times 2n \)-submatrix associated to light and \( Q_{LA} = Q_{AL}^T \) the \( 2n \times 2 \)-submatrix corresponding to the coupling between light and atoms, and \( Q_{AA} \) the \( 2 \times 2 \)-submatrix associated to atoms.

I calculated \( Q \) from the above expression of \( S \). I found

\[
Q_{ij} = \alpha^{i+j-2} \beta \gamma D \text{ and } Q_{iA} = \alpha^{n+i-1} \begin{pmatrix} 0 & \gamma \theta \\ -\beta \lambda & 0 \end{pmatrix}
\]

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and \( Q_{AA} = \begin{pmatrix} t (1 + \lambda \alpha^{2n}) & 0 \\ 0 & \frac{1}{t} (1 + \theta \alpha^{2n}) \end{pmatrix} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix} \)

with \( D = \begin{pmatrix} \theta & 0 \\ 0 & \lambda \end{pmatrix} \) where \( \theta = t / \xi - 1 \) and \( \lambda = \xi / t - 1 \)

I noticed that

\[
-\frac{\vec{r}_i^T \vec{r}_i}{2} e^{-\frac{\vec{r}_i^T \vec{r}_i}{2}} = \frac{\partial}{\partial x_i} \left[ e^{-\frac{\vec{r}_i^T \vec{r}_i}{2}} \right] \bigg|_{x_i=1}
\]

so that I could express the characteristic function as a derivative:

\[
\chi(\vec{r}_A) \propto \prod_i (1 + \frac{\partial}{\partial x_i}) \left[ \int d\vec{R}_L \ e^{-\frac{\vec{g}^T J Q \vec{r}^T \vec{R}}{4}} e^{-\frac{\sum_j x_j \vec{r}_j^T \vec{r}_j}{2}} \right] \bigg|_{x_i=1,\forall i}
\]

Let us define new variables \( \vec{r}_i' = \sqrt{x_i} \vec{r}_i \). Thus

\[
\chi(\vec{r}_A) \propto \prod_i (1 + \frac{\partial}{\partial x_i}) \left[ \frac{1}{\prod_k x_k} \int d\vec{R}_L \ e^{-\frac{\vec{g}^T J Q \vec{r}'^T \vec{R}}{4}} e^{-\frac{\sum_j x_j \vec{r}_j'^T \vec{r}_j'}{2}} \right] \bigg|_{x_i=1,\forall i}
\]

with

\[
(Q_x)_{ij} = Q_{ij} / \sqrt{x_i x_j}, (Q_x)_{iA} = Q_{iA} / \sqrt{x_i} \text{ and } (Q_x)_{AA} = Q_{AA}
\]

and thus \( Q_x^T = Q_x \).

Therefore \( (Q_x)_{ij} = \alpha^{j-i} \beta \gamma D / \sqrt{x_i x_j} = \tau_i \tau_j \beta \gamma D \) with \( \tau_i = \alpha^{i-1} / \sqrt{x_i} \).

The symmetric submatrix \( (Q_x)_L = (Q_x)_{ij} \bigg|_{1 \leq i, j \leq n} \) can be diagonalized with a unitary transformation (applied as change of variables in the integral over the light components). Only two eigenvalues are nonzero, which corresponds to an effective coupling between the atomic ensemble and one light mode (each mode is represented by two quadratures):

\[
W^T (Q_x)_L W = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu \beta \gamma D \end{pmatrix} \quad \text{with} \quad \nu = \tau_1^2 + \cdots + \tau_n^2 \quad \text{and} \quad D = \begin{pmatrix} \theta & 0 \\ 0 & \lambda \end{pmatrix}
\]

The eigenvectors \( G_1 \) and \( G_2 \) corresponding to the eigenvalues \( \nu \beta \gamma \theta \) and \( \nu \beta \gamma \lambda \) are respectively \( G_1^T = \frac{1}{\sqrt{v}} (\tau_1, 0, \tau_2, 0, \ldots, \tau_n, 0) \) and \( G_2^T = \frac{1}{\sqrt{v}} (0, \tau_1, 0, \tau_2, \ldots, 0, \tau_n) \), so that \( W \) can be written as \( W = (W_1, \ldots, W_{2n}, G_1, G_2) \) where the \( W_k \) form an orthonormal basis of the subspace orthogonal to \( G_1 \) and \( G_2 \). Thus, for all \( k \leq 2n \) and \( i \in \{1, 2\} \):

\[
W_k^T G_i^T = 0
\]
I thus chose new integration variables $\vec{R}_L' = W \vec{R}_L$. This is equivalent to replacing $Q_x$ by the new matrix $Z_x = V_{LA}^T Q_x V_{LA} = \begin{pmatrix} W^T (Q_x)_{VL} W & W^T (Q_x)_{LA} \end{pmatrix}$ where $V_{LA} = W \oplus 1_A$ (and as $(Q_x)_{AA} = Q_{AA}$).

But

$$(Q_{x})_{iA} = Q_{iA} \sqrt{x_i} = \frac{\alpha^{n+i-1}}{\sqrt{x_i}} \begin{pmatrix} 0 & \gamma \theta \\ -\beta \lambda & 0 \end{pmatrix} = \alpha^n \tau_i \begin{pmatrix} 0 & \gamma \theta \\ -\beta \lambda & 0 \end{pmatrix}$$

and thus the two vectors building $(Q_x)_{LA}$ belong to the subspace generated by the eigenvectors $G_1$ and $G_2$, and are hence orthogonal to $W_k$ for all $k \leq 2n$:

$$(W_1, ..., W_{2n})^T (Q_x)_{LA} = 0 \text{ or equivalently } (Q_x)_{AL}^T (W_1, ..., W_n) = 0$$

Therefore, the $2(n+1) \times 2(n+1)$-matrix $Z_x$ reduces to $Z_x = V_{LA}^T Q_x V_{LA} = \begin{pmatrix} 0 & 0 \\ 0 & Y_x \end{pmatrix}$ where $Y_x$ is a $4 \times 4$-matrix (for one light mode and atoms):

$$Y_x = \begin{pmatrix} v \beta \gamma D (Y_x)_{LA} \\ (Y_x)_{LA} \end{pmatrix}$$

with $D = \begin{pmatrix} \theta & 0 \\ 0 & \lambda \end{pmatrix}$ and $Q_{AA} = \begin{pmatrix} e & 0 \\ 0 & f \end{pmatrix}$

and with a coupling submatrix

$$(Y_x)_{LA} = \begin{pmatrix} G_1^T \\ G_2^T \end{pmatrix} (Q_x)_{LA} = \frac{1}{\sqrt{v}} \begin{pmatrix} 0 & \tau_1 & 0 & \tau_2 & 0 & \cdots & 0 & \tau_n \\ 0 & \tau_1 & 0 & \tau_2 & 0 & \cdots & 0 & \tau_n \end{pmatrix} (Q_x)_{LA}$$

so that

$$(Y_x)_{iA} = \frac{\alpha^n}{\sqrt{v}} \sum_k \tau_k^2 1 \begin{pmatrix} 0 & \gamma \theta \\ -\beta \lambda & 0 \end{pmatrix} = \alpha^n \sqrt{v} \begin{pmatrix} 0 & \gamma \theta \\ -\beta \lambda & 0 \end{pmatrix}$$

Let us write $\vec{R}_{nA}^T = (\vec{r}_{nA}^T, \vec{r}_A^T)$. Thus, finally:

$$\chi(\vec{r}_A) \propto \prod_i \left( 1 + \frac{\partial}{\partial x_i} \right) \left[ \frac{1}{\prod_k x_k} \int d^2 \vec{r}_L e^{-\frac{\vec{R}_{nA}^T J_x J_x^T \vec{R}_{nA}}{4}} e^{-\frac{\vec{R}_L^T J_x J_x^T \vec{R}_L}{2}} \right]_{x_i = 1, \forall i}$$

$$\propto \prod_i \left( 1 + \frac{\partial}{\partial x_i} \right) \left[ \frac{1}{\prod_k x_k} \int d^2 \vec{r}_n \exp \left[ -\frac{\vec{R}_{nA}^T J_x J_x^T \vec{R}_{nA}}{4} \right] e^{-\frac{\vec{r}_n^2}{2}} \right]_{x_i = 1, \forall i}$$

with

$$\exp \left[ -\frac{\vec{R}_{nA}^T J_x J_x^T \vec{R}_{nA}}{4} \right] = e^{-\frac{\epsilon \rho^2_3 + \epsilon \lambda^2 \rho^2_4}{4}} e^{-\frac{\psi \gamma \theta \rho^2_3 + \psi \lambda \rho^2_4}{4}} e^{-\frac{\alpha^n \sqrt{\gamma \theta} \rho^2_3 + \beta \lambda \rho^2_4}{2}}$$

(this is deduced directly from the expression that I derived above for $Y_x$)
Thus, after calculation of the Gaussian integral, I found:

\[ \chi(\vec{r}_A) = N_\chi e^{-\frac{\beta\alpha^2+\lambda^2}{4}} \prod_i \left( 1 + \frac{\partial}{\partial x_i} \right) \left[ \frac{1}{\prod_k x_k} \frac{e^{\frac{A_x X^2 + B_x X^2}{2}}}{\sqrt{1 + v\beta\gamma\lambda} \sqrt{1 + v\beta\gamma\theta}} \right] \bigg|_{x_i=1,\forall i} \]

(19)

with \( A_x = \frac{\nu\alpha^2\beta\lambda^2}{1+\beta\gamma\lambda} \) and \( B_x = \frac{\nu\alpha^2\beta^2\gamma\lambda}{1+\beta\gamma\lambda} \) and \( v = \sum_i a_i \frac{\partial^{2(1-i)}}{x_i} \).

The normalization factor \( N_\chi \) of the characteristic function is simply calculated by taking the above expression for \( \vec{r}_A = 0 \) (so that \( \chi(\vec{0}) = 1 \));

\[ N^{-1}_\chi = \prod_i \left( 1 + \frac{\partial}{\partial x_i} \right) \left[ \frac{1}{\prod_k x_k} \frac{1}{\sqrt{1 + v\beta\gamma\lambda} \sqrt{1 + v\beta\gamma\theta}} \right] \bigg|_{x_i=1,\forall i} \]

Fidelity

I computed the fidelity from the atomic characteristic function \( \chi(\vec{r}_A) \) calculated in the previous section:

\[ F_\alpha = \langle Cat|\rho_{final}|Cat \rangle = \int \frac{d^2\vec{r}_A}{2\pi} \chi(\vec{r}_A) \chi_{Cat}(\vec{0}) \]

with

\[ \chi_{Cat}(\beta) = N_\alpha^2 e^{-\frac{\beta^2}{4}} \left[ e^{\beta\alpha^* - \alpha\beta^*} + e^{-\beta\alpha^* + \alpha\beta^*} - e^{-2|\alpha|^2} \left( e^{-\beta\alpha^* - \alpha\beta^*} + e^{\beta\alpha^* + \alpha\beta^*} \right) \right] \]

where \( \beta = \frac{X_A + iP_A}{\sqrt{2}} \) and \( N_\alpha^2 = 1/ \left[ 2(1 - e^{-2|\alpha|^2}) \right] \) When \( \alpha \) is real, the characteristic function of the cat is simply

\[ \chi_{Cat}(\vec{r}_A) = e^{-\frac{X_A^2 + P_A^2}{4}} \left[ e^{iP_A\sqrt{2}} + e^{-iP_A\sqrt{2}} - e^{-2\alpha^*} \left( e^{\alpha X_A \sqrt{2}} + e^{-\alpha X_A \sqrt{2}} \right) \right] N_\alpha^2 \]

The fidelity is thus a linear superposition of integrals of the type

\[ I = \int \frac{d^2\vec{r}_A}{2\pi} \chi(\vec{r}_A) e^{-\vec{r}_A^T C A \vec{r}_A} e^{T A T} e^{T A T} = \int \frac{d^2\vec{r}_A}{2\pi} \chi(\vec{r}_A) e^{-\vec{r}_A^T C A \vec{r}_A} e^{2T A T} \]

By using the expression of \( \chi(\vec{r}_A) \), one gets:

\[ I = \frac{N_\chi}{2\pi} \prod_i \left( 1 + \frac{\partial}{\partial x_i} \right) \left[ \int \frac{d^2\vec{r}_A}{\prod_k x_k} \frac{e^{-\frac{\beta\alpha^2+\lambda^2}{4}} e^{\frac{A_x X^2 + B_x X^2}{2}}}{\sqrt{1 + v\beta\gamma\lambda} \sqrt{1 + v\beta\gamma\theta}} \right] \bigg|_{x_i=1,\forall i} \]

(20)

\[ = \frac{N_\chi}{2\pi} \prod_i \left( 1 + \frac{\partial}{\partial x_i} \right) \left[ \frac{1}{\prod_k x_k} \frac{\int d^2\vec{r}_A e^{-\vec{r}_A^T M x \vec{r}_A} e^{2T A T}}{\sqrt{1 + v\beta\gamma\lambda} \sqrt{1 + v\beta\gamma\theta}} \right] \bigg|_{x_i=1,\forall i} \]

(21)

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where

\[ M_x = C + \left( \frac{f + B_x}{4} \quad 0 \quad e^{A_x} \right) \]

so that finally

\[
I = \frac{N_x}{2\pi} \prod_i (1 + \frac{\partial}{\partial x_i}) \left[ \frac{1}{\prod_k x_k} \frac{e^{T M_x^{-1} T}}{\sqrt{1 + v\beta \gamma \lambda} \sqrt{1 + v\beta \gamma \theta} \sqrt{\det M_x}} \right]_{x_i = 1, \forall i}
\]
References


