Stacks in groupoids as homotopy limits

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Abstract

This document is devoted to prove that a stack in groupoids can be viewed as an homotopy limit. It constitutes an oral exam for the course *Moduli problems, stacks and simplicial presheaves* given by Gabriele Vezzosi during spring 2014 at Université Paris 7 (Diderot).

Quickly recall what is classically a stack.

Definition 1 (Category fibered in groupoids)**.** Let $\mathcal{C}$ be a small category. The *category of categories fibered in groupoids over* $\mathcal{C}$, denoted $\mathsf{FibGrpd}(\mathcal{C})$, is the full subcategory of $\mathsf{Cat}_{/\mathcal{C}}$ whose objects are the functor $p\colon \mathcal{F} \to \mathcal{C}$ satisfying the following property :

FIB for all $X \in \operatorname{Ob}\mathcal{F}$, writing $X = p(X')$, the induced functor

$$p_{X'}\colon \mathcal{F}_{/X'} \to \mathcal{C}_{/X}$$

is an equivalence which is surjective on objects.

Definition 2 (Stack in groupoids)**.** Let $(\mathcal{C},\tau)$ be a Grothendieck site. A category fibered in groupoids $p\colon \mathcal{F} \to \mathcal{C}$ is a τ*-stack in groupoids* if there exists a choice f^* of pullback functor for any $f \in \operatorname{Mor}\mathcal{C}$ such that : for any object X of $\mathcal{C}$, and any τ-covering family $\{r_i\colon U_i \to X\}_{i\in I}$

ST1 (*morphisms glue*) for any $X', X'' \in \mathcal{F}_X$, the diagram

$$\prod_{i\in I} \operatorname{Hom}_{\mathcal{F}_{U_i}}(r_i^*(X'), r_i^*(X'')) \overset{\gamma_1}{\underset{\gamma_2}{\rightrightarrows}} \prod_{(i,j)\in I^2} \operatorname{Hom}_{\mathcal{F}_{U_{ij}}}(r_{ij}^*(X'), r_{ij}^*(X''))$$

has equalizer $\operatorname{Hom}_{\mathcal{F}_X}(X',X'')$, where the r_{ij} are the induced maps $U_{ij} = U_i \times_X U_j \to X$,

ST2 (*objects glue*) if we are given families $(X'_i \in \mathcal{F}_{U_i})_{i\in I}$ and

$$\Big(\alpha_{ij}\colon (U_{ij} \to U_i)^*(X'_i) \xrightarrow{\sim} (U_{ij} \to U_j)^*(X'_j)\Big)_{(i,j)\in I^2}$$

with $\alpha_{ii} = \mathbf{1}_{(U_{ii} \to U_i)^*(X'_i)}$ for all i, satisfying the cocycle condition

$$(U_{ijk} \to U_{jk})^*(\alpha_{jk}) \circ (U_{ijk} \to U_{ij})^*(\alpha_{ij}) = (U_{ijk} \to U_{ik})^*(\alpha_{ik}),$$

then there exists $X' \in \mathcal{F}_X$ and isomorphisms

$$\left(\beta_i : (U_i \to X)^*(X') \xrightarrow{\sim} X'_i\right)_{i \in I}$$

such that commutes

$$\begin{array}{ccc} (U_{ij} \to X)^*(X') & \xrightarrow{(U_{ij} \to U_i)^*(\beta_i)} & (U_{ij} \to U_i)^*(X'_i) \\ & \searrow^{(U_{ij} \to U_j)^*(\beta_j)} & \downarrow \alpha_{ij} \\ & & (U_{ij} \to U_j)^*(X'_j). \end{array}$$

In the course, we have seen that there is a functor of *global section*

$$\Gamma\colon \mathsf{FibGrpd}(\mathcal{C}) \to \mathrm{Pr}(\mathcal{C}, \mathsf{Grpd})$$

defined by $X \mapsto \mathrm{Hom}_{\mathsf{FibGrpd}(\mathcal{C})}(\mathcal{C}_{/X}, \mathcal{F})$ on objects and obviously on maps. Moreover, the Grothendieck construction gives rise to a functor $\underline{\mathrm{Fib}}$, left adjoint to Γ.

The ultimate goal of the document is to show theorem 3. We will use the following vocabulary : a presheaf F in groupoids on a Grothendieck site $(\mathcal{C}, \tau)$ has *τ-descent* if for any object X in $\mathcal{C}$ and any τ-cover $\{U_i \to X\}_{i \in I}$ of X, the canonical map

$$F(X) \to \mathrm{holim}_{\mathbf{\Delta}}\,(F(\mathscr{U}^\bullet))$$

is an isomorphism in $\mathbf{Ho}(\mathsf{Grpd})$, where $F(\mathscr{U}^\bullet)$ is the cosimplicial groupoid

$$[n] \mapsto \prod_{(i_0, \dots, i_n) \in I^{n+1}} F(U_{i_0} \times_X \cdots \times_X U_{i_n}).$$

Theorem 3 (Hollander)**.** *Let $(\mathcal{C}, \tau)$ be a site. A category fibered in groupoids $p\colon \mathcal{F} \to \mathcal{C}$ is a τ-stack in groupoids if and only if its global sections $\Gamma(p)$ has τ-descent.*

We recall here the proof sketched in the course and fill in the omitted detail.

First, we compute $\mathrm{holim}_{\mathbf{\Delta}}\,(\mathcal{G}^\bullet)$ for any cosimplicial groupoid $\mathcal{G}^\bullet$. In order to do so, we begin by defining the totalization of a cosimplicial simplicial set. Recall that $\Delta[\bullet]$ is the cosimplicial simplicial set

$$[n] \to \Delta[n] = \mathrm{Hom}_{\mathbf{\Delta}}(-, [n]),$$

and that $\mathrm{sSet}^{\mathbf{\Delta}}$ is enriched over sSet as

$$\left(\underline{\mathrm{Hom}}_{\mathrm{sSet}^{\mathbf{\Delta}}}(Y^\bullet, X^\bullet)\right)_n = \mathrm{Hom}_{\mathrm{sSet}^{\mathbf{\Delta}}}(Y^\bullet \times \Delta[n], X^\bullet) \quad \forall n \in \mathbb{N}.$$

Definition 4 (Totalization)**.** Let $X^\bullet \in \mathrm{sSet}^{\mathbf{\Delta}}$. The totalization of $X^\bullet$ is defined as the simplicial set

$$\mathrm{Tot}(X^\bullet) = \underline{\mathrm{Hom}}_{\mathrm{sSet}^{\mathbf{\Delta}}}(\Delta[\bullet], X^\bullet).$$

Now define the maps

$$\prod_{n\geq 0}(X^n)^{\Delta[n]} \overset{a}{\underset{b}{\rightrightarrows}} \prod_{[m]\to[n]}(X^n)^{\Delta[m]}$$

as those induced by the maps of simplicial sets $a_\vartheta, b_\vartheta : \prod_{k\geq 0}(X^k)^{\Delta[k]} \to (X^n)^{\Delta[m]}$ for any $\vartheta \in \mathrm{Hom}_{\mathbf{\Delta}}([m],[n])$ defined in degree $r \in \mathbb{N}$ as follow : by Yoneda's lemma and adjunction, $((X^k)^{\Delta[\ell]})_r$ is naturally isomorphic to $\mathrm{Hom}_{\mathsf{sSet}}(\Delta[\ell] \times \Delta[r], X^k)$; plus the products of simplicial sets are computed pointwise ; then define for an element $(\varphi_{k,r} : \Delta[k] \times \Delta[r] \to X^k)_{k\geq 0}$ of $(\prod_{k\geq 0}(X^k)^{\Delta[k]})_r$,

$$a_{\vartheta,r}\left((\varphi_{k,r})_{k\geq 0}\right) = \quad \Delta[m] \times \Delta[r] \xrightarrow{\Delta[\vartheta]\times \mathbf{1}_{\Delta[r]}} \Delta[n] \times \Delta[r] \xrightarrow{\varphi_{n,r}} X^n,$$

$$b_{\vartheta,r}\left((\varphi_{k,r})_{k\geq 0}\right) = \quad \Delta[m] \times \Delta[r] \xrightarrow{\varphi_{m,r}} X^m \xrightarrow{X^\bullet(\vartheta)} X^n.$$

Proposition 5. *The diagram*

$$\mathrm{Tot}(X^\bullet) \longrightarrow \prod_{n\geq 0}(X^n)^{\Delta[n]} \overset{a}{\underset{b}{\rightrightarrows}} \prod_{[m]\to[n]}(X^n)^{\Delta[m]}$$

is an equalizer in sSet.

Proof. Limits are computed degreewise in sSet. So it is enough to check that for every degree $r \in \mathbb{N}$, the diagram

$$\mathrm{Tot}(X^\bullet)_r \longrightarrow \prod_{n\geq 0}\left((X^n)^{\Delta[n]}\right)_r \overset{(a_{\vartheta,r})_\vartheta}{\underset{(b_{\vartheta,r})_\vartheta}{\rightrightarrows}} \prod_{[m]\to[n]}\left((X^n)^{\Delta[m]}\right)_r$$

is an equalizer in Set. Equalizers of sets are easy to compute : the equalizer of

$$\prod_{n\geq 0}\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n] \times \Delta[r], X^n) \overset{(a_{\vartheta,r})_\vartheta}{\underset{(b_{\vartheta,r})_\vartheta}{\rightrightarrows}} \prod_{n\geq 0}\mathrm{Hom}_{\mathsf{sSet}}(\Delta[m] \times \Delta[r], X^n).$$

is the subset of the domain on which the two arrows agree. That is, it is the subset with elements those $(\varphi_{n,r})_{n\geq 0}$ such that $X^\bullet(\vartheta) \circ \varphi_{m,r} = \varphi_{n,r} \circ (\Delta[\vartheta] \times \mathbf{1}_{\Delta[r]})$. In other words, it is precisely the inclusion

$$\mathrm{Tot}(X^\bullet)_r = \mathrm{Hom}_{\mathsf{sSet}^{\mathbf{\Delta}}}(\Delta[\bullet] \times \Delta[r], X^\bullet) \hookrightarrow \prod_{n\geq 0}\mathrm{Hom}_{\mathsf{sSet}}(\Delta[n] \times \Delta[r], X^n).$$

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Proposition 6. *Let $X^\bullet \in \mathsf{sSet}^{\mathbf{\Delta}}$ be a fibrant object for the Reedy model structure. There is an isomorphism in* $\mathbf{Ho}(\mathsf{sSet})$ *:*

$$\mathrm{Tot}(X^\bullet) \simeq \mathrm{holim}_{\mathbf{\Delta}}(X^\bullet).$$

Proof. Let $X^\bullet \overset{u}{\underset{\sim}{\hookrightarrow}} R_{\text{inj}}(X^\bullet) = Z^\bullet$ be a fibrant replacement of $X^\bullet$ for the injective model structure of sSet so that

$$\operatorname{holim}_{\Delta}(X^\bullet) \simeq \lim_{\Delta}(Z^\bullet).$$

Then,

(i) $\Delta[\bullet]$ is cofibrant for the Reedy model structure on sSet : a map $K^\bullet \to L^\bullet$ of sSet^{Δ} is a Reedy cofibration if and only if it is degreewise a cofibration in sSet and the induced map of sSet

$$\operatorname{Eq}\left(K^0 \overset{d^0}{\underset{d^1}{\rightrightarrows}} K^1 \right) \to \operatorname{Eq}\left(L^0 \overset{d^0}{\underset{d^1}{\rightrightarrows}} L^1 \right)$$

is a isomorphism ; so, it suffices to observe that $\varnothing \to X^\bullet$ is degreewise a cofibration (every object being cofibrant in sSet) and that

$$\operatorname{Eq}\left(\Delta[0] \overset{d^0}{\underset{d^1}{\rightrightarrows}} \Delta[1] \right) \simeq \varnothing.$$

(ii) $X^\bullet$ and $Z^\bullet$ are Reedy fibrant : $X^\bullet$ by hypothesis ; and the adjunction

$$\mathbf{1}_{\text{sSet}^{\Delta}} : \left(\text{sSet}^{\Delta}\right)_{\text{Reedy}} \leftrightarrows \left(\text{sSet}^{\Delta}\right)_{\text{inj}} : \mathbf{1}_{\text{sSet}^{\Delta}}$$

being a Quillen equivalence (in particular a Quillen adjunction), the injective fibrant object $Z^\bullet$ also is Reedy fibrant.

Therefore the induced map

$$\operatorname{Map}_{\text{Reedy}}(\Delta[\bullet], X^\bullet) \overset{u^*}{\to} \operatorname{Map}_{\text{Reedy}}(\Delta[\bullet], Z^\bullet)$$

is an isomorphism in $\mathbf{Ho}(\text{sSet})$. Hence the sequence of isomorphism in $\mathbf{Ho}(\text{sSet})$:

$$\begin{aligned} \operatorname{Tot}(X^\bullet) = \underline{\operatorname{Hom}}(\Delta[\bullet], X^\bullet) &\simeq \operatorname{Map}_{\text{Reedy}}(\Delta[\bullet], X^\bullet) \\ &\simeq \operatorname{Map}_{\text{Reedy}}(\Delta[\bullet], Z^\bullet) \\ &\simeq \underline{\operatorname{Hom}}(\Delta[\bullet], Z^\bullet) = \operatorname{Tot}(Z^\bullet). \end{aligned}$$

Consider $\Delta[\bullet] \to *$ in sSet^{Δ}. This is a weak equivalence : indeed, it is degreewise the map $\Delta[n] \to *$ whose topological realisation is an homotopy equivalence (the standard topological simplexes being contractible). Plus, $*$ and $\Delta[\bullet]$ both are cofibrant in the injective model structure. So, $Z^\bullet$ being fibrant, one has isomorphisms in $\mathbf{Ho}(\text{sSet})$:

$$\underline{\operatorname{Hom}}(*, Z^\bullet) \simeq \operatorname{Map}_{\text{inj}}(*, Z^\bullet) \overset{p^*}{\to} \operatorname{Map}_{\text{inj}}(\Delta[\bullet], Z^\bullet) \simeq \underline{\operatorname{Hom}}(\Delta[\bullet], Z^\bullet) = \operatorname{Tot}(Z^\bullet).$$

Showing an isomorphism in $\mathbf{Ho}(\text{sSet})$ between $\underline{\operatorname{Hom}}(*, Z^\bullet)$ and $\lim_{\Delta}(Z^\bullet)$ will then conclude. Playing with tensors and adjunctions, we even show that there is such an isomorphisms in sSet : for any simplicial set K,

$$\begin{aligned} \operatorname{Hom}_{\text{sSet}}(K, \underline{\operatorname{Hom}}_{\text{sSet}^{\Delta}}(*, Z^\bullet)) &\simeq \operatorname{Hom}_{\text{sSet}^{\Delta}}(K \otimes *, Z^\bullet) \\ &\simeq \operatorname{Hom}_{\text{sSet}^{\Delta}}(\text{const}^{\Delta}(K), Z^\bullet) \\ &\simeq \operatorname{Hom}_{\text{sSet}}(K, \lim_{\Delta}(Z^\bullet)). \end{aligned}$$

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Now recall the Quillen adjunction $\Pi_1 : \mathsf{sSet} \leftrightarrows \mathsf{Grpd} : \mathrm{N}$. Let $\mathcal{G}^\bullet$ be a cosimplicial groupoid and denote $X^\bullet$ the cosimplicial simplicial set

$$[n] \mapsto \mathrm{N}(\mathcal{G}^n).$$

We would like to apply proposition 6 to $X^\bullet$. But for that it must be Reedy fibrant. This is not always the case. However, one can find a cosimplical groupoid $\mathcal{H}^\bullet$ and a map $\mathcal{G}^\bullet \to \mathcal{H}^\bullet$ such that $\mathrm{N}(\mathcal{H}^\bullet)$ is injective fibrant (so Reedy fibrant) and $\mathrm{N}(\mathcal{G}^\bullet) \to \mathrm{N}(\mathcal{H}^\bullet)$ a weak equivalence. We refer to [Jar10] (end of section 2) for the detail. Propositions 6 and 5 then leads to compute $\mathrm{holim}_{\mathbf{\Delta}}(X^\bullet)$ as the equalizer in sSet

$$\mathrm{Eq}\left(\prod_{n\geq 0} \mathrm{N}(\mathcal{G}^n)^{\Delta[n]} \overset{a}{\underset{b}{\rightrightarrows}} \prod_{[m]\to[n]} \mathrm{N}(\mathcal{G}^n)^{\Delta[m]} \right).$$

In the other hand, the nerve functor N being right Quillen, it commutes with $\mathrm{holim}_{\mathbf{\Delta}}(-)$; and so, in $\mathbf{Ho}(\mathsf{sSet})$,

$$\mathrm{holim}_{\mathbf{\Delta}}(X^\bullet) \simeq \mathrm{N}\left(\mathrm{holim}_{\mathbf{\Delta}}(\mathcal{G}^\bullet)\right).$$

To go any further, we need the following lemma.

Lemma 7. *Let $\mathcal{H}$ be a groupoid, and K a simplicial set. There is a isomorphism in* sSet

$$\mathrm{N}(\mathcal{H})^K \simeq \mathrm{N}\left(\mathcal{H}^K\right).$$

Proof. For any simplicial set L, one has bijection natural in L

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{sSet}}(L, \mathrm{N}(\mathcal{H})^K) &\simeq \mathrm{Hom}_{\mathsf{sSet}}(L \times K, \mathrm{N}(\mathcal{H})) \\
&\simeq \mathrm{Hom}_{\mathsf{Grpd}}(\Pi_1(L \times K), \mathcal{H}) \\
&\simeq \mathrm{Hom}_{\mathsf{Grpd}}(\Pi_1(L) \times \Pi_1(K), \mathcal{H}) \\
&\simeq \mathrm{Hom}_{\mathsf{Grpd}}(\Pi_1(L), \mathcal{H}^{\Pi_1(K)}) \\
&\simeq \mathrm{Hom}_{\mathsf{sSet}}(L, \mathrm{N}\left(\mathcal{H}^{\Pi_1(K)}\right)).
\end{aligned}$$

The cotensor in the sSet-enrichment of Grpd is by definition $\mathcal{H}^K = \mathcal{H}^{\Pi_1(K)}$, hence the result. ∎

Moreover, the nerve functor N is a right adjoint, so is left exact : it commutes with products and equalizers. Plus, N is fully faithfull. One then has

$$\mathrm{N}\left(\mathrm{holim}_{\mathbf{\Delta}}(\mathcal{G}^\bullet)\right) \simeq \mathrm{holim}_{\mathbf{\Delta}}(X^\bullet) \simeq \mathrm{N}\left(\mathrm{Eq}\left(\prod_{n\geq 0} (\mathcal{G}^n)^{\Delta[n]} \overset{\tilde{a}}{\underset{\tilde{b}}{\rightrightarrows}} \prod_{[m]\to[n]} (\mathcal{G}^n)^{\Delta[m]} \right)\right).$$

The full faithfullness of N can be expressed as $\Pi_1 \circ \mathrm{N} \simeq \mathbf{1}_{\mathsf{Grpd}}$. It leads to

$$\mathrm{holim}_{\mathbf{\Delta}}(\mathcal{G}^\bullet) \simeq \mathrm{Eq}\left(\prod_{n\geq 0} (\mathcal{G}^n)^{\Delta[n]} \overset{\tilde{a}}{\underset{\tilde{b}}{\rightrightarrows}} \prod_{[m]\to[n]} (\mathcal{G}^n)^{\Delta[m]} \right). \qquad (\star)$$

We have successfully express $\mathrm{holim}_{\mathbf{\Delta}}(\mathcal{G}^\bullet)$ as an equalizer in Grpd. It remains to see how to compute this equalizer. First, remember that $(\mathcal{G}^n)^{\Delta[m]}$ is by definition the groupoid

$$(\mathcal{G}^n)^{\Pi_1(\Delta[m])} = \underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^m, \mathcal{G}^n)$$

where $\mathcal{I}^m$ is the groupoid $\Pi_1(\Delta[m])$ (the groupoid with $m+1$ objects and exactly one morphism between any two objects), and where the enrichment of Grpd is the Grpd-one. Then rewrite ($\star$) :

$$\mathrm{holim}_{\mathbf{\Delta}}(\mathcal{G}^\bullet) \simeq \mathrm{Eq}\left(\prod_{n\geq 0} \underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^n, \mathcal{G}^n) \underset{\tilde{b}}{\overset{\tilde{a}}{\rightrightarrows}} \prod_{[m]\to[n]} \underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^m, \mathcal{G}^n) \right).$$

Denote $\mathbf{\Delta}_2$ the full subcategory of $\mathbf{\Delta}$ whose objects are $[0]$, $[1]$ and $[2]$. The inclusion functor $\mathbf{\Delta}_2 \hookrightarrow \mathbf{\Delta}$ induces for any category $\mathcal{C}$ a truncation functor

$$\mathrm{tr}_2^{\mathcal{C}} : \mathcal{C}^{\mathbf{\Delta}} \to \mathcal{C}^{\mathbf{\Delta}_2}$$

which admits a left adjoint $(\mathrm{tr}_2^{\mathcal{C}})^*$, given by

$$\begin{aligned}(\mathrm{tr}_2^{\mathcal{C}})^*(H^\bullet : \mathbf{\Delta}_2 \to \mathcal{C}) : \mathbf{\Delta} &\to \mathcal{C} \\ [n] &\mapsto \mathrm{colim}_{\mathbf{\Delta}^{\leq 2}_{/[n]}}(H^\bullet)\end{aligned}$$

where $\mathbf{\Delta}^{\leq 2}_{/[n]}$ is the full subcategory of $\mathbf{\Delta}_{/[n]}$ with objects $[i] \to [n], i \in \{0,1,2\}$ and where we made the abuse to denote $H^\bullet$ again the functor $\mathbf{\Delta}^{\leq 2}_{/[n]} \to \mathbf{\Delta}_2 \to \mathcal{C}$ induced by $H^\bullet$. We define the 2-skeleton functor as $\mathrm{sk}_2^{\mathcal{C}} = (\mathrm{tr}_2^{\mathcal{C}})^* \mathrm{tr}_2^{\mathcal{C}}$.

Lemma 8. *Let $\mathcal{I}^\bullet$ be the cosimplicial groupoid $[n] \mapsto \mathcal{I}^n$. The unit component of the adjunction* $(\mathrm{tr}_2^{\mathsf{Grpd}})^* : \mathsf{Grpd}^{\mathbf{\Delta}} \leftrightarrows \mathsf{Grpd}^{\mathbf{\Delta}_2} : \mathrm{tr}_2^{\mathsf{Grpd}}$

$$\mathrm{sk}_2^{\mathsf{Grpd}}(\mathcal{I}^\bullet) \to \mathcal{I}^\bullet$$

is a isomorphism.

Proof. Given the formula for $(\mathrm{tr}_2^{\mathsf{Grpd}})^*$, one only has to prove that the map

$$\mathrm{colim}_{\mathbf{\Delta}^{\leq 2}_{/[n]}}(\mathcal{I}^\bullet) \to \mathcal{I}^n,$$

induced by the maps $\mathcal{I}^i \to \mathcal{I}^n, i \in \{0,1,2\}$, is an isomorphism. But for any groupoid $\mathcal{H}$,

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Grpd}}(\mathrm{colim}_{\mathbf{\Delta}^{\leq 2}_{/[n]}}(\mathcal{I}^\bullet), \mathcal{H}) &\simeq \lim_{\mathbf{\Delta}^{\leq 2}_{/[n]}}\left(\mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{I}^\bullet, \mathcal{H})\right) \\ &\simeq \lim_{\mathbf{\Delta}^{\leq 2}_{/[n]}}(\mathrm{N}(\mathcal{H})) \\ &\simeq (\mathrm{N}(\mathcal{H}))_n \\ &\simeq \mathrm{Hom}_{\mathsf{Grpd}}(\mathcal{I}^n, \mathcal{G}).\end{aligned}$$

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The sSet-enrichment of $\mathsf{sSet}^{\mathbf{\Delta}}$ and the adjunction $\Pi_1 \dashv \mathrm{N}$ allows us to enrich $\mathsf{Grpd}^{\mathbf{\Delta}}$ over Grpd (with tensor and cotensor) as

$$\underline{\mathrm{Hom}}_{\mathsf{Grpd}^{\mathbf{\Delta}}}(\mathcal{H}^\bullet, \mathcal{K}^\bullet) = \Pi_1\left(\underline{\mathrm{Hom}}_{\mathsf{sSet}^{\mathbf{\Delta}}}(\mathrm{N}(\mathcal{H}^\bullet), \mathrm{N}(\mathcal{K}^\bullet))\right)$$

Then, denoting by $\mathrm{Tot}(\mathcal{G}^\bullet)$ the groupoid $\underline{\mathrm{Hom}}_{\mathsf{Grpd}^\Delta}(\mathcal{I}^\bullet,\mathcal{G}^\bullet)$ and by $\mathrm{Tot}_2(\mathcal{G}^\bullet)$ the groupoid $\underline{\mathrm{Hom}}_{\mathsf{Grpd}^\Delta}(\mathrm{sk}_2(\mathcal{I}^\bullet),\mathcal{G}^\bullet)$, lemma 8 notably states that

$$\mathrm{Tot}(\mathcal{G}^\bullet) \simeq \mathrm{Tot}_2(\mathcal{G}^\bullet).$$

By the definition of the enrichment, $\mathrm{Tot}(\mathcal{G}^\bullet) \simeq \Pi_1(\mathrm{Tot}(\mathrm{N}(\mathcal{G}^\bullet)))$. And proposition 5 then gives $\mathrm{holim}_\Delta(\mathcal{G}^\bullet) \simeq \mathrm{Tot}(\mathcal{G}^\bullet)$ (going to sSet^Δ with the nerve, using proposition 5, and then going back to Grpd^Δ with Π_1). So in the end,

$$\mathrm{holim}_\Delta(\mathcal{G}^\bullet) \simeq \mathrm{Tot}_2(\mathcal{G}^\bullet).$$

We are left to compute $\mathrm{Tot}_2(\mathcal{G}^\bullet)$. A proof similar to the one of proposition 5 show that the diagram

$$\underline{\mathrm{Hom}}_{\mathsf{sSet}^\Delta}(\mathrm{sk}_2^{\mathsf{sSet}}(\Delta[\bullet]),X^\bullet) \longrightarrow \prod_{n=0,1,2}(X^n)^{\Delta[n]} \overset{a}{\underset{b}{\rightrightarrows}} \prod_{\substack{[m]\to[n]\\ n,m=0,1,2}}(X^n)^{\Delta[m]}$$

is an equalizer in sSet for any cosimplicial simplicial set $X^\bullet$. Indeed, in degree $r\in\mathbb{N}$, the equalizer of

$$\prod_{n=0,1,2}\left((X^n)^{\Delta[n]}\right)_r \overset{(a_{\theta,r})}{\underset{(b_{\theta,r})}{\rightrightarrows}} \prod_{\substack{[m]\to[n]\\ n,m=0,1,2}}\left((X^n)^{\Delta[m]}\right)_r$$

is the set $\mathrm{Hom}_{\mathsf{sSet}^{\Delta_2}}(\mathrm{tr}_2^{\mathsf{sSet}}(\Delta[\bullet]\times\Delta[r]),\mathrm{tr}_2^{\mathsf{sSet}}(X^\bullet))$ which is by adjunction naturally isomorphic to

$$\mathrm{Hom}_{\mathsf{sSet}^\Delta}(\mathrm{sk}_2^{\mathsf{sSet}}(\Delta[\bullet]\times\Delta[r]),X^\bullet) \simeq \mathrm{Hom}_{\mathsf{sSet}^\Delta}(\mathrm{sk}_2^{\mathsf{sSet}}(\Delta[\bullet])\times\Delta[r],X^\bullet).$$

Proving the equalizer diagram. Apply to the the cosimplicial simplicial set $\mathrm{N}(\mathcal{G}^\bullet)$ and take the image by Π_1 to get an equalizer diagram

$$\begin{array}{c}\Pi_1\left(\underline{\mathrm{Hom}}_{\mathsf{sSet}^\Delta}(\mathrm{sk}_2^\Delta(\mathrm{N}(\mathcal{I}^\bullet)),\mathrm{N}(\mathcal{G}^\bullet))\right)\\ \downarrow \\ \prod_{n=0,1,2}\underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^n,\mathcal{G}^n) \rightrightarrows \prod_{\substack{[m]\to[n]\\ n,m=0,1,2}}\underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^m,\mathcal{G}^n).\end{array}$$

Now observe that $\mathrm{sk}_2^\Delta(\mathrm{N}(\mathcal{I}^\bullet)) = \mathrm{N}\left(\mathrm{sk}_2^{\mathsf{Grpd}}(\mathcal{I}^\bullet)\right)$. Hence the equalizer diagram

$$\mathrm{Tot}_2(\mathcal{G}^\bullet) \longrightarrow \prod_{n=0,1,2}\underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^n,\mathcal{G}^n) \rightrightarrows \prod_{\substack{[m]\to[n]\\ n,m=0,1,2}}\underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^m,\mathcal{G}^n).$$

Recall now that one had an isomorphism in $\mathbf{Ho}(\mathsf{Grpd})$ between $\mathrm{holim}_\Delta(\mathcal{G}^\bullet)$ and $\mathrm{Tot}_2(\mathcal{G}^\bullet)$. So in $\mathbf{Ho}(\mathsf{Grpd})$:

$$\mathrm{holim}_\Delta(\mathcal{G}^\bullet) \simeq \mathrm{Eq}\left(\prod_{n=0,1,2}\underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^n,\mathcal{G}^n) \rightrightarrows \prod_{\substack{[m]\to[n]\\ n,m=0,1,2}}\underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^m,\mathcal{G}^n)\right).$$

To simplify a little the computation of this equalizer, let us show that one can even ditch the arrows $[m] \to [n]$ with $m = 2$ in the product of the codomain.

Lemma 9. *The equalizer of*

$$\prod_{n=0,1,2} \underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^n, \mathcal{G}^n) \rightrightarrows \prod_{\substack{[m]\to[n] \\ n,m=0,1,2}} \underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^m, \mathcal{G}^n)$$

is isomorphic in Grpd *to the equalizer of*

$$\prod_{n=0,1,2} \underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^n, \mathcal{G}^n) \rightrightarrows \prod_{\substack{[m]\to[n] \\ n=0,1,2 \\ m=0,1}} \underline{\mathrm{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^m, \mathcal{G}^n).$$

Proof. Obviously, the first equalizer is a full subgroupoid of the second one. It just then remains to show that every object of the second equalizer is actually a object of the first one.

An object of the second equalizer is a triple $(\varphi^0, \varphi^1, \varphi^2)$ of functors $\varphi^n \colon \mathcal{I}^n \to \mathcal{G}^n$, $n \in \{0,1,2\}$ such that for any map $\vartheta \colon [m] \to [n]$ of $\boldsymbol{\Delta}$ with $n \in \{0,1,2\}$ and $m \in \{0,1\}$,

$$\varphi^n \circ \mathcal{I}^\bullet(\vartheta) = \mathcal{G}^\bullet(\vartheta) \circ \varphi^m. \tag{$\star\star$}$$

We want the equality $(\star\star)$ to hold for maps $\vartheta \colon [m] \to [n]$ with m possibly equal to 2 also. Of course, we can prove it only for the generating maps

$$s_2^0 \colon [2] \to [1] : 0 \mapsto 0, 1 \mapsto 0, 2 \mapsto 1,$$
$$s_2^1 \colon [2] \to [1] : 0 \mapsto 0, 1 \mapsto 1, 2 \mapsto 1.$$

Let us do it for s_1^1 (the other is similar). Denote

$$g_0 \circlearrowleft \mathbf{1}_{g_0} \qquad g_1 \underset{u^{-1}}{\overset{u}{\rightleftarrows}} g_1' \qquad g_2 \underset{v^{-1}}{\overset{v}{\rightleftarrows}} g_2' \underset{w^{-1}}{\overset{w}{\rightleftarrows}} g_2''$$

the diagram of $\mathcal{G}^0$, $\mathcal{G}^1$ and $\mathcal{G}^2$ respectively defined by φ^0, φ^1 and φ^2. Then, one has to show that

$$\mathcal{G}^\bullet(s_2^1)(g_2) = g_1 \tag{1}$$
$$\mathcal{G}^\bullet(s_2^1)(g_2') = g_1' \tag{2}$$
$$\mathcal{G}^\bullet(s_2^1)(g_2'') = g_1' \tag{3}$$
$$\mathcal{G}^\bullet(s_2^1)(v) = u \tag{4}$$
$$\mathcal{G}^\bullet(s_2^1)(w) = \mathbf{1}_{g_1'}. \tag{5}$$

But s_2^1 admits $d_1^2 \colon [1] \to [2] : 0 \mapsto 0, 1 \mapsto 1$ as a right inverse. So one has $\mathcal{G}^\bullet(s_2^1) \circ \mathcal{G}^\bullet(d_1^2) = \mathbf{1}_{\mathcal{G}^1}$, and we already know (by hypothesis) that $\mathcal{G}^\bullet(d_1^2) \circ \varphi^1 = \varphi^2 \circ \mathcal{I}^\bullet(d_1^2)$, giving (1), (2) and (4).

Similarly, s_2^1 admits $d_1^0 \colon [1] \to [2] : 0 \mapsto 0, 1 \mapsto 2$ as a right inverse . So one has $\mathcal{G}^\bullet(s_2^1) \circ \mathcal{G}^\bullet(d_1^0) = \mathbf{1}_{\mathcal{G}^1}$, and we already know (by hypothesis) that $\mathcal{G}^\bullet(d_1^0) \circ \varphi^1 = \varphi^2 \circ \mathcal{I}^\bullet(d_1^0)$, giving (3) and $\mathcal{G}^\bullet(s_2^1)(wv) = u$ which in turn leads to (5) (using (4) that we already have).

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So now, we are able to give a combinatorial description of $\operatorname{holim}_{\Delta}(\mathcal{G}^\bullet)$. This is the second equalizer of lemma 9, that is the full subgroupoid of $\prod_{n=0,1,2} \underline{\operatorname{Hom}}_{\mathsf{Grpd}}(\mathcal{I}^n, \mathcal{G}^n)$ with objects those triples $(\varphi^0, \varphi^1, \varphi^2)$ making

$$
\begin{array}{ccccc}
\mathcal{I}^0 & \rightleftarrows & \mathcal{I}^1 & \begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix} & \mathcal{I}^2 \\
\downarrow{\scriptstyle \varphi^0} & & \downarrow{\scriptstyle \varphi^1} & & \downarrow{\scriptstyle \varphi^2} \\
\mathcal{G}^0 & \rightleftarrows & \mathcal{G}^1 & \begin{smallmatrix}\rightarrow\\ \rightarrow\\ \rightarrow\end{smallmatrix} & \mathcal{G}^2
\end{array}
$$

commutes, where the horizontal arrows are the codegeneracies and cofaces maps with domain in $\{0,1\}$ and codomain in $\{0,1,2\}$. Remark that then φ^2 is redundant (see proof of lemma 9). So $\operatorname{holim}_{\Delta}(\mathcal{G}^\bullet)$ is the groupoid

- with **objects** $(g_0 \in \operatorname{Ob}\mathcal{G}^0, u\colon g_1 \to g_1' \in \operatorname{Mor}\mathcal{G}^1)$ such that

$$
\begin{aligned}
\mathcal{G}^\bullet(d_0^1)(g_0) &= g_1, \\
\mathcal{G}^\bullet(d_0^0)(g_0) &= g_1', \\
\mathcal{G}^\bullet(s_1^0)(u) &= \mathbf{1}_{g_0}, \\
\mathcal{G}^\bullet(d_1^1)(u) &= \mathcal{G}^\bullet(d_1^0)(u) \circ \mathcal{G}^\bullet(d_1^2)(u).
\end{aligned}
$$

- with **arrows** $(g_0, u\colon g_1 \to g_1') \to (h_0, v\colon h_1 \to h_1')$ those $t\colon g_0 \to h_0 \in \operatorname{Mor}\mathcal{G}^0$ making commute

$$
\begin{array}{ccc}
g_1 & \xrightarrow{\mathcal{G}^\bullet(d_0^1)(t)} & h_1 \\
{\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\
g_1' & \xrightarrow[\mathcal{G}^\bullet(d_0^0)(t)]{} & h_1'.
\end{array}
$$

We are now set to prove theorem 3.

Proof of theorem 3. Let $p\colon \mathcal{F} \to \mathcal{C}$ be a category fibered in groupoids and F be its global sections $\Gamma(p)$. We suppose we have a Grothendieck (pre)topology τ on $\mathcal{C}$. Our goal is to prove that p is a stack in groupoids over $(\mathcal{C}, \tau)$ if and only if F has τ-descent.

Take $X \in \operatorname{Ob}\mathcal{C}$ and $\mathcal{U} = \{r_i\colon U_i \to X\}$ a τ-covering family of X. Consider the natural functor

$$\Lambda\colon F(X) \to \operatorname{holim}_{\Delta}(F(\mathcal{U}^\bullet)).$$

We will show that Λ is fully faithful if and only if **ST1** holds, and that Λ is essentially surjective if and only if **ST2** holds. So finally, Λ will be an equivalence of groupoids (that is an isomorphism in $\mathbf{Ho}(\mathsf{Grpd})$) if anf only if the two stack conditions holds for the cover $\mathcal{U}$ of X. Proving the theorem.

So first, let us show that the fully faithfulness of Λ is equivalent to the condition **ST1**. Let $Y, Z \in \mathcal{F}_X$ be two objects of the fiber at X. Recall that there is an isomorphism (natural in X) $\mathcal{F}_X \simeq F(X)$ and denote y, z the image of Y, Z in $F(X)$. Then the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Hom}_{\mathcal{F}_X}(Y,Z) & \longrightarrow & \prod_i \mathrm{Hom}_{\mathcal{F}_{U_i}}(r_i^*Y, r_i^*Z) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{\mathcal{F}_{U_{ij}}}(r_{ij}^*Y, r_{ij}^*Z) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{Hom}_{F(X)}(y,z) & \longrightarrow & \prod_i \mathrm{Hom}_{F(U_i)}(Fr_i(y), Fr_i(z)) & \rightrightarrows & \prod_{i,j} \mathrm{Hom}_{F(U_{ij})}(Fr_{ij}(y), Fr_{ij}(z))
\end{array}$$

says in particular that the above line is an equalizer if and only if the bottom line is an equalizer. By definition, **ST1** is exactly the statement "the above line is an equalizer". So it remains to show that the statement "the bottom line is an equalizer" is equivalent to the fact that Λ is fully faithfull. Since $F(\mathcal{U}^\bullet)(d_0^1)$ is precisely the map $F(\mathcal{U}^0) \to F(\mathcal{U}^1)$ induced by the $F(U_{ij} \to U_i)$, and $F(\mathcal{U}^\bullet)(d_0^0)$ the map $F(\mathcal{U}^0) \to F(\mathcal{U}^1)$ induced by the $F(U_{ij} \to U_j)$, the equalizer of

$$\prod_i \mathrm{Hom}_{F(U_i)}(Fr_i(y), Fr_i(z)) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{F(U_{ij})}(Fr_{ij}(y), Fr_{ij}(z))$$

is isomorphic to $\mathrm{Hom}_{\mathrm{holim}_\Delta(F(\mathcal{U}^\bullet))}(\Lambda(y), \Lambda(z))$ (see the combinatorial description of $\mathrm{holim}_\Delta(F(\mathcal{U}^\bullet))$).

We now prove that **ST2** is equivalent to the essential surjectivity of Λ. For that purpose, let us emphasize a little what is an object of $\mathrm{holim}_\Delta(F(\mathcal{U}^\bullet))$. From the combinatorial description, an object of $\mathrm{holim}_\Delta(F(\mathcal{U}^\bullet))$ is a pair $(x, \alpha\colon d_0^1 x \to d_0^0 x)$ with $x = (x_i)_{i \in I}$ an object of $F(\mathcal{U}^0) = \prod_i F(U_i)$ and

$$\alpha = \Big(\alpha_{ij}\colon F(U_{ij} \to U_i)(x_i) \to F(U_{ij} \to U_j)(x_j)\Big)_{i,j}$$

a map of $F(\mathcal{U}^1) = \prod_{(i,j)} F(U_{ij})$ satisfying $d_1^0(\alpha) \circ d_1^2(\alpha) = d_1^1(\alpha)$, that is

$$\forall i,j,k, \quad F(U_{ijk} \to U_{ik})(\alpha_{ik}) = F(U_{ijk} \to U_{jk})(\alpha_{jk}) \circ F(U_{ijk} \to U_{ij})(\alpha_{ij}),$$

and $s_1^0(\alpha) = \mathbf{1}_x$, that is

$$\forall i, \quad F(U_i \to U_{ii})(\alpha_{ii}) = \mathbf{1}_{x_i}.$$

Also recall that all the $F(V)$ are groupoids, so the α_{ij} are all isomorphisms by definition. So the natural (in V) isomorphism $F(V) \simeq \mathcal{F}_V$ gives a one-to-one correpondance between the objects of $\mathrm{holim}_\Delta(F(\mathcal{U}^\bullet))$ and the families satisfying the hypothesis of **ST2**. The essential surjectivity of Λ is then exactly the statement "every families satisfaying the hypothesis of **ST2** has a glueing in $\mathcal{F}_X$".

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References.

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