Nominal sets for quantum memory

An algebraic account of linear parameters

Pierre Cagne

Laboratoire PPS, Université Paris Diderot

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1

Nominal sets

Definitions

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Denote $\mathfrak{S}_A$ for the permutation group of A.
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$A = \{x_1, x_2, \dots\}$
$X = \{\lambda\text{-terms over } A\}$
Action: $\sigma \cdot t = t[x_i/\sigma(x_i)]$

t has support S iff $S \supseteq \mathrm{FV}(t)$

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λ-terms example:
X is a nominal set because every term has a finite number of (free) variables

$$\text{supp}\,(t) = \mathrm{FV}(t)$$

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Sheaf-rephrasing

Given X a nominal set, define

$$\begin{aligned} NX \colon \mathsf{Inj}_A &\to \mathsf{Set} \\ S &\mapsto \{x \in X : S \supseteq \operatorname{supp}(x)\} \end{aligned}$$

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Theorem

N identifies Nom with those functors $\mathrm{Inj} \to \mathrm{Set}$ which preserve pullbacks.

Remark: It makes the nominal sets exactly the sheaves for a certain Grothendieck topology on Inj°. The topos of nominal sets is *Schanuel's topos*.

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λ-terms example:

View X as

$$\{x_j : j \in J \subset_f \mathbb{N}\} \mapsto \{\lambda\text{-terms with fv among the } x_j\text{'s}\}$$

Then pass to de Bruijn indices.

Theorem

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2

Lawvere theories with arities

Classical algebraic theories

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- a set M,
- a 2-ary multiplication $M^2 \to M$,
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Denote $\aleph_0$ for the category of finite set $\mathbf{n} = \{0, \ldots, n-1\}$ and set-maps between them.

Lawvere theory

An algebraic structure is just a finite-product-preserving functor $A \colon \Theta^\circ \to \mathrm{Set}$ where

- there is a finite-sum-preserving bijective-on-objects functor $\aleph_0 \to \Theta$,
- morphisms $\mathbf{n} \to 1$ in Θ° are the n-ary operation.

The axioms are the commutative diagrams of Θ°.

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À la Lawvere

A category is just a functor $C\colon \Theta^\circ \to \mathsf{Set}$ where

- there is a "nice" functor $\boldsymbol{\Lambda} \to \Theta$, ($\boldsymbol{\Lambda}$ = finite linear graphs)
- morphisms $\Lambda_n \to \Lambda_1$ in Θ° are the compositions,

and such that $\exists G \in \mathsf{Graph},\ C(\Lambda_n) = \mathsf{Graph}\,(\Lambda_n, G)$.

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(Berger, Melliès, Weber)

A generalized algebraic structure of arity $\mathbf{A} \hookrightarrow \mathbf{E}$ is just a functor $F \colon \Theta^\circ \to \mathrm{Set}$ where

- there is a "nice" functor $\mathbf{A} \to \Theta$,
- morphisms $a \to b$ in Θ° are the (a, b)-ary operations,

and such that $\exists E \in \mathbf{E},\ F(a) = \mathbf{E}(a, E)$.

The axioms are the commutative diagrams of Θ°.

Warnings!

Technicalities have been swept under the rug.

- The arity $\mathbf{A} \hookrightarrow \mathbf{E}$ can not be any embedding: there must exists $\mathbf{E} \hookrightarrow \mathrm{Psh}(\mathbf{A})$ such that the composite functor

$$\mathbf{A} \hookrightarrow \mathbf{E} \hookrightarrow \mathrm{Psh}(\mathbf{A})$$

  is the Yoneda embedding.
- The "niceness" of $j\colon \mathbf{A} \to \Theta$ is that the monad

$$j_* j_! \colon \mathrm{Psh}(\mathbf{A}) \to \mathrm{Psh}(\mathbf{A})$$

  should restict to a monad on $\mathbf{E}$.

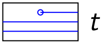
3

Quantum memory

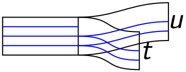
Quantum computation syntax

One construct ternary judgments: $\Delta \mid \Gamma \vdash t$.

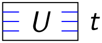
$$\frac{\Delta, a \mid \Gamma \vdash t}{\Delta \mid \Gamma \vdash \mathsf{new}(a.t)}$$



$$\frac{\Delta \mid \Gamma \vdash t \qquad \Delta \mid \Gamma \vdash u}{\Delta, a \mid \Gamma \vdash \mathsf{measure}(a, t, u)}$$



$$\frac{\Delta, \vec{b} \mid \Gamma \vdash t}{\Delta, \vec{a} \mid \Gamma \vdash \mathsf{apply}_U(\vec{a}, \vec{b}.t)} \qquad \forall U \in \mathbb{U}_{2^n}(\mathbb{C})$$



+ contraction/weakening rules on Γ (not Δ!) + permutations on both Δ et Γ.

Theory of quantum computation

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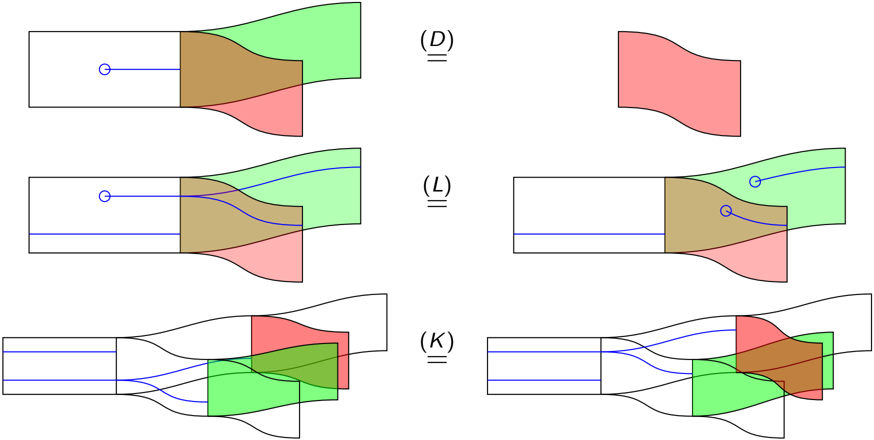

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$\Sigma \mathsf{Bij} \hookrightarrow \mathsf{Psh}(\mathsf{Bij})$: the full subcategory of finite sum of representable presheaves. So the objects of ΣBij are those finite sequence $(n_1, \ldots, n_k)$ of integers.

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Θ_Q: the category freely generated over ΣBij by

$$\begin{aligned} \mathsf{new} &\colon (0) \to (1) \\ \mathsf{measure} &\colon (1) \to (0, 0) \\ \mathsf{apply}_U &\colon (n) \to (n) \end{aligned}$$

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Why is it the theory of quantum computation?

Because $j_Q : \Sigma\mathrm{Bij} \to \Theta_Q$ is a generalized algebraic theory and its structures are exactly the models described in *Algebraic Effects, Linearity, and Quantum Programming Languages, by S.Staton*.

Back to nominal sets

By the same process, one can construct Θ_{coll} the category associated to the theory whose only operator is coll satisfying the axiom:

$$\mathsf{coll}(a, \mathsf{coll}(b, x)) = \mathsf{coll}(b, \mathsf{coll}(a, x)).$$

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Fact

Θ_{coll} is isomorphic to $\Sigma(\mathrm{Inj}^{\circ})$. A generalized structure for $j_{\mathrm{coll}} : \Sigma\mathrm{Bij} \to \Theta_{\mathrm{coll}}$ is then identified with just a functor $\mathrm{Inj} \to \mathrm{Set}$.

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Corollary

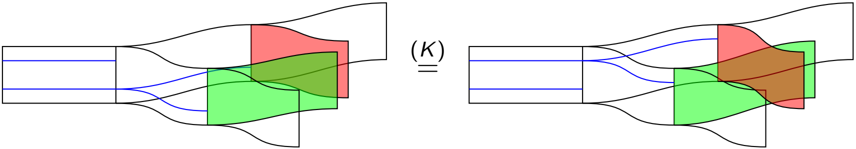
A nominal set is a model of j_{coll} sending the commutative diagrams

$$\begin{array}{ccc} n & \xrightarrow{\iota_i^n} & n+1 \\ {\scriptstyle \iota_{j-1}^n}\downarrow & & \downarrow{\scriptstyle \iota_j^{n+1}} \\ n+1 & \xrightarrow[\iota_i^{n+1}]{} & n+2 \end{array} \qquad (i < j)$$

to pullbacks.

Back to nominal sets

By the same process, one can construct Θ_{meas} the category associated to the theory whose only operator is meas satisfying the axiom:



Definition

A qnominal set is a model of j_{meas} sending the commutative diagrams

$$\begin{array}{ccccc}
(n,n,n,n) & \xleftarrow{(1\ 3\ 2\ 4)} & (n,n,n,n) & \xleftarrow{\mathrm{meas}_i^n \oplus \mathrm{meas}_i^n} & (n+1,n+1) \\
{\scriptstyle \mathrm{meas}_{j-1}^n \oplus \mathrm{meas}_{j-1}^n}\uparrow & & & & \uparrow{\scriptstyle \mathrm{meas}_j^{n+1}} \\
(n+1,n+1) & & \xleftarrow[\mathrm{meas}_i^{n+1}]{} & & n+2
\end{array}$$

to pullbacks.

Quantum support

With quantum nominality comes the notion of quantum support:

Support

For M a qnominal set, the quantum support of a element $x \in M_n$ is defined as

$$\operatorname{supp}(x) = \left\{ i \in \{0, \ldots, n-1\} : \forall y,\, x \neq \operatorname{meas}_i^{n-1}(y) \right\}.$$

Interpretation: the quantum support of a program is then the set of memory location for which the program is not a direct measure on it.

Quantum nominality is relevant

The theory Q contains meas and its axiom, so it makes sense to ask whether a model of Q is qnominal or not.

Proposition

Any model of Q is a qnominal set.

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Preuve.

All is left to show is that if $x \in M_2$ can be written

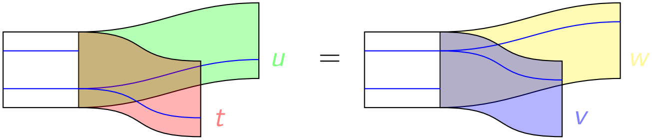
$$\text{measure}(a, t, u) = x = \text{measure}(b, v, w),$$

then $\exists y_1, y_2, y_3, y_4 \in M_0$,

$$t = \text{measure}(b, y_1, y_2), \quad u = \text{measure}(b, y_3, y_4)$$
$$v = \text{measure}(a, y_1, y_3), \quad w = \text{measure}(a, y_2, y_4)$$

□

Draw the proof!



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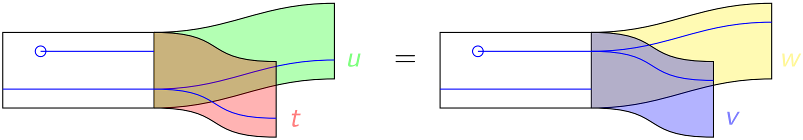

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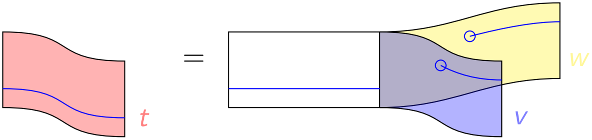

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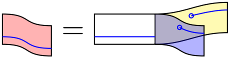

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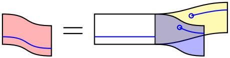

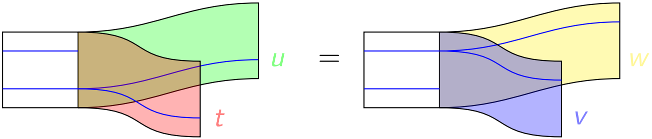

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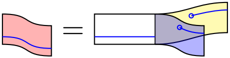

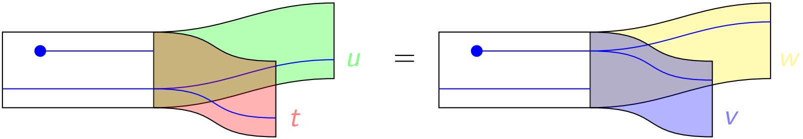

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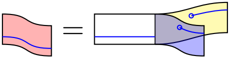

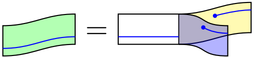

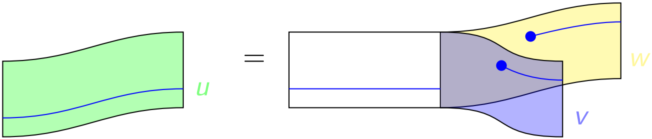

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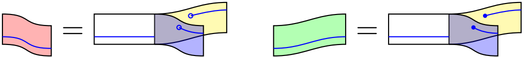

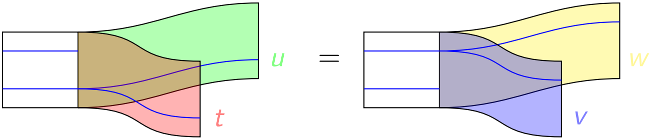

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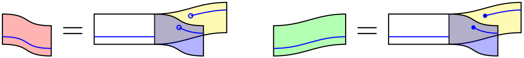

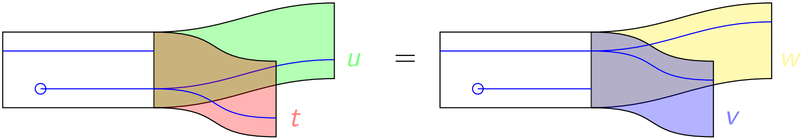

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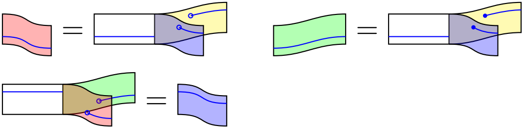

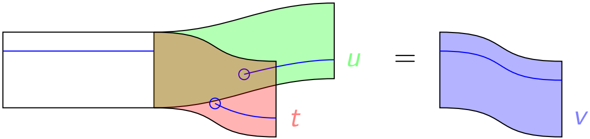

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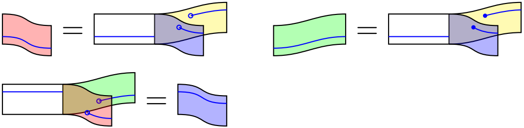

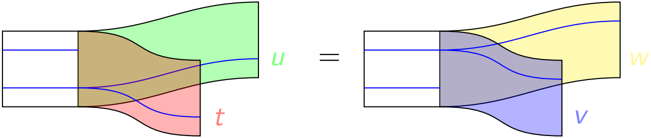

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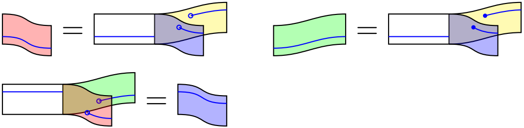

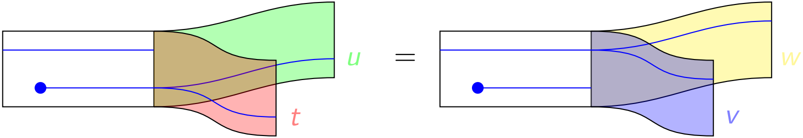

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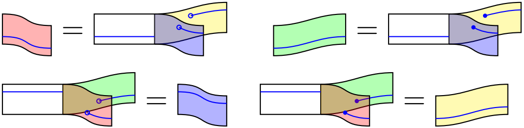
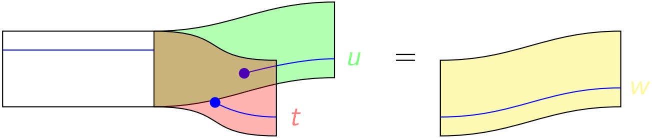
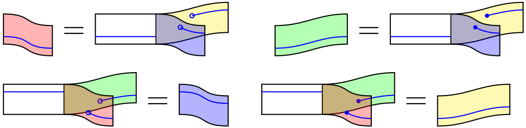




Draw the proof!



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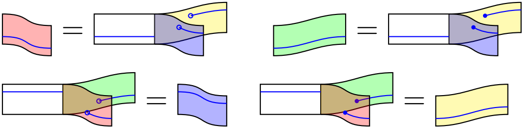

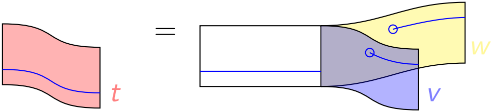

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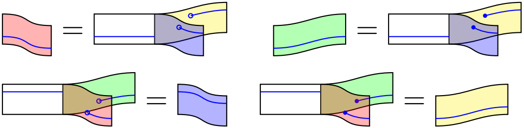

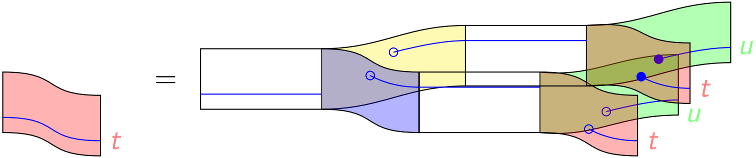

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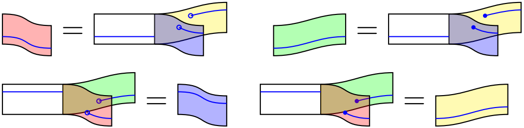

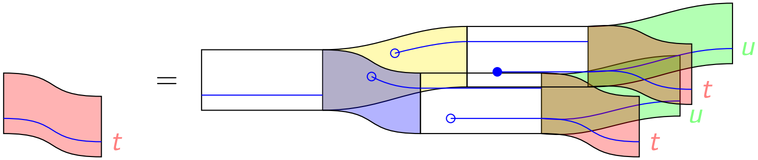

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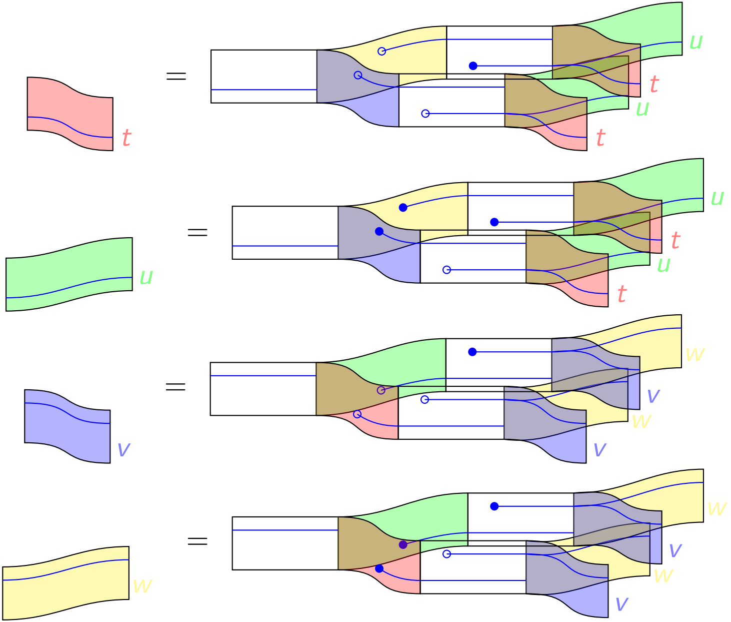

Thanks.

If interested, one can find a complete internship report on my webpage:
`http://www.eleves.ens.fr/home/cagne/`