MASTER IMALIS - ENS PSL

Training in Mathematics and Statistics

Exercises

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Planning

Exercises - Lecture 1

Exercise 1: Calculate the derivatives of the following functions:
1. $\forall x \in \mathbb{R},\ f(x) = e^{3x} + 2x - 6$
2. $\forall x \in \mathbb{R}_+^*,\ g(x) = \ln(3x + 4)$
3. $\forall x \in \mathbb{R},\ h(x) = 2xe^{-x}$
4. $\forall x \in \mathbb{E}$ (to define), $i(x) = \sqrt{3 - 2x}$
5. $\forall (x, y) \in \mathbb{R}^2,\ j(x, y) = x^3y + e^{xy^2}$

Exercise 2: Operations on matrices:

$$1.\ \begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix} \quad 2.\ \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}\begin{pmatrix}1\\ 2\end{pmatrix} \quad 3.\ \begin{pmatrix}1 & 2\\ 3 & 4\\ -1 & 2\end{pmatrix}\begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix}$$

$$4.\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 4 & 5\end{pmatrix} \quad 5.\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}^3 \quad 6.\ \begin{pmatrix}1 & 2\\ 3 & -2\end{pmatrix}^2 \quad 7.\ \begin{pmatrix}1 & 2\\ 0 & -2\end{pmatrix}^3$$

Exercise 3: Calculate the determinant associated with the following matrices:

$$1.\ A = \begin{pmatrix}1 & 3\\ 4 & 5\end{pmatrix} \quad 2.\ B = \begin{pmatrix}1 & -1\\ 0 & 5\end{pmatrix} \quad 3.\ C = \begin{pmatrix}3 & -1 & 1\\ 0 & 2 & 1\\ 1 & -1 & 2\end{pmatrix}$$

Exercise 4: $\forall x \in E, f(x) = \dfrac{e^x - 1}{e^x + 1}$,
1. Determine (E), the domain of definition of f , and demonstrate that f is an odd function.
2. Study the variations of f (increasing, decreasing, ...).

Exercise 5: Solve the following differential equations:
1. $y' - 3y = 1$ and $y(1) = -2$.
2. $3y' - y = x + 2$ with solution(s) verifying $x \longrightarrow ax + b$.

Exercise 6: Discrete probabilities. Let's consider 32 cards (8 spades, 8 hearts, 8 diamonds, and 8 clubs):
1. We randomly distribute one card from a game with 32 cards. What is the probability of having one king?
2. We randomly distribute five cards from a game with 32 cards. What is the probability of having four kings?
3. We randomly distribute five cards from a game with 32 cards. What is the probability of having only red cards?
4. We randomly distribute five cards from a game with 32 cards. What is the probability of having 2 diamonds and 3 hearts?

5. We randomly distribute five cards from a game with 32 cards. What is the probability of having at least one card of each category (at least 1 spade, 1 heart, 1 diamond, and 1 club)?
6. We randomly distribute five cards from a game with 32 cards. What is the probability of having at least one king?
7. We randomly distribute five cards from a game with 32 cards. What is the probability of having two king and 3 hearts?

Exercises - Lecture 2

Exercise 1: Determine if the following functions are linear maps:

1. $f : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x_1, x_2) & \longmapsto & (0, 1) \end{cases}$

2. $g : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x_1, x_2) & \longmapsto & (x_1 + 2x_2, -2x_2) \end{cases}$

3. $h : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x_1, x_2) & \longmapsto & (2x_1^2, 3x_2) \end{cases}$

4. $i : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x_1, x_2, x_3) & \longmapsto & (2x_1 - x_2, 4\ln(x_3), 3x_2) \end{cases}$

5. $j : \begin{cases} \mathbb{R}^3 & \longrightarrow & \mathbb{R}^3 \\ (x_1, x_2, x_3) & \longmapsto & (0, 4x_3 - x_2, \sqrt{6}x_3) \end{cases}$

6. $k : \begin{cases} \mathbb{R}^2 & \longrightarrow & \mathbb{R}^2 \\ (x_1, x_2) & \longmapsto & (0, 4x_1 - x_2) \end{cases}$

Exercise 2: Link between linear maps and matrices:
1. Determine the matrices associated with the linear maps from *Exercise 1.*
2. Determine the linear maps associated with the following matrices in the canonical basis:

$$A = \begin{pmatrix} 3 & -1 \\ 0 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 3 & -1 & 2 \end{pmatrix}, \ C = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix}, \ \text{and} \ D = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Exercise 3: Compute the following (from *Exercise 1*):
1. $g\left((5, -3)\right)$
2. $j\left((2, -1, 3)\right)$
3. $(k + g)\left((1, -1)\right)$
4. $j \circ j\left((2, -1, 0)\right)$
5. $k \circ g\left((2, -1)\right)$.

Exercise 4: Invert (if possible) the matrices :

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}, \ B = \begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix}, \ \text{and} \ C = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$$

Exercise 5: Change of basis. Let's consider $A = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$ the matrix of the linear map g in the canonical basis. Given $\mathcal{B} = (v_1, v_2)$ with $v_1 = (-2, 3)$ and $v_2 = (1, 0)$:

1. Determine the matrix of change of basis P from the canonical basis to $\mathcal{B}$.
2. Write the matrix associated with g in the basis $\mathcal{B}$.

Exercise 6: Diagonalizable matrix.
1. Determine the eigenvalues and eigenvectors of the following matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \; B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}, \; \text{and} \; C = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

2. If diagonalizable, determine the corresponding diagonal matrices of the previous matrices and determine the matrices of change of basis P and P^{-1}.

Exercises - Lecture 3

Exercise 1: Logistic model (one-dimensional non-linear dynamical system): determine the equilibrium and the stability of the following systems:

1. $N' = rN(1 - \frac{N}{K})$ with r and $K > 0$.

2. $N' = rN\ln(\frac{K}{N})$ with r and $K > 0$.

Exercise 2: Two-dimensional linear dynamical systems: determine the fixed points and the behaviors of the systems in the neighborhood of these fixed points for the following dynamical systems:

1. $(E_1) : \begin{cases} x' = 4x - 2y \\ y' = -x + 2y \end{cases}$ 2. $(E_2) : \begin{cases} x' = -2x - y \\ y' = -x + 2y \end{cases}$ 3. $(E_3) : \begin{cases} x' = -2x \\ y' = -x + y/2 \end{cases}$

4. $(E_4) : \begin{cases} x' = -x + y \\ y' = 2x - 2y \end{cases}$ 5. $(E_5) : \begin{cases} x' = 4x + 2y \\ y' = -x + 2y \end{cases}$ 6. $(E_6) : \begin{cases} x' = x - 2y \\ y' = 2x - 2y \end{cases}$

Exercise 3: Lotka-Volterra: let's consider the following prey-predator systems:

1.
$$\begin{cases} x' = 3x - y \\ y' = 6x - 4y \end{cases}$$

Given the initial conditions $x_0 = y_0 = 1$, determine $x(t)$ and $y(t)$.

2.
$$\begin{cases} x' = 3x - xy \\ y' = 6xy - 4y \end{cases}$$

Determine the fixed points and their stability.

Exercise 4: The FitzHugh-Nagumo model describes the temporal behavior of an excitable system such as a neuron. It describes the evolution of two variables: the membrane voltage v and the recovery variable w.

$$\begin{cases} v' = v - v^3 - w \\ \tau w' = v + a - bw \end{cases}$$

Here τ is small and indicates that the evolution of v is faster than the evolution of w.

1. Determine the isoclines of the systems. Plot them on the phase space (v, w) for $a = 1$ and $b = 2$. Indicate on the phase space the fixed point of the system.

2. Draw on the phase space the effect of a pulse (i.e. a quick increase of v) on the system. Draw $v(t)$.

Exercise 5: Bifurcation. Let's consider the system $x' = rx - 2x^2 + x^3$:
1. Demonstrate that x=0 is a fixed point and study its stability according to r.
2. Determine the other equilibrium according to the value of r and study its stability.
3. Represent the diagram of bifurcation and identify the types of bifurcations.

Exercise 6: Leslie matrix: an age-structured population in discrete time. We count the number of individuals in a population every year. Let x_J and x_A respectively indicate the number of juveniles and adults. Every year, the following events take place:
– A fraction Δ of the juveniles dies during the year and the juveniles that stay alive mature with probability M or remain juveniles with probability $1 - M$.
– Each adult produces B offspring on average and then a fraction Δ of the adults die.
1. Construct the graph and the matrix C that corresponds to this life cycle.
2. Modify the model such that instead of maturing with probability M every year, all juveniles mature every year.
3. A new population is formed with $x_J(0) = 100$ individuals. We consider that 50% of the adults reproduce every year, and that the mortality $\Delta = 25\%$. How is the population after 10 years? What if $\Delta = 50\%$? What is the condition on the eigenvalues of the matrix C for the population to persist?

Exercises - Lecture 4

Exercise 1: Discrete probabilities. Let's consider the nucleotides {A, C, G, T} uniformly distributed on a DNA sequence.
1. How many different n-nucleotide sequences can be done?
2. What is the probability of having 'AAA'?
3. What is the probability of having at least one 'A' in a 3-nucleotide sequence? Or just one 'A'?
4. On a 100-nucleotide sequence, what is the distribution of the number of nucleotide "A"?

Exercise 2: Conditional probability. 2% of the population is contaminated by a virus. We have a medical test for detecting the viral infection. The test has the following properties: the probability of having a positive test for a contaminated person equals 0.99 (test sensitivity), and the probability of having a negative test for a healthy person equals 0.97 (test specificity). We do the medical test on a random individual from the population:
1. What is the probability of being contaminated and having a positive test?
2. What is the probability of having a positive test?
3. What is the probability of being contaminated if the test is positive?
4. What is the probability of being healthy if the test is negative?
5. We randomly sample 10 individuals from the population. What is the probability of having 2 contaminated persons?

Exercise 3: Expected values:
1. 37 numbers have the same probability to be picked. We bet x€ on one particular number: if this number is picked, we win 36 times the bet. What is the mean gain?
2. What is the expected value of rolling a 6-sided dice? an n-sided dice?

Exercise 4: Uniform distribution. The break of chromosome 9 is linked to chronic leukemia. The breakpoint occurs between the exons 12 and 16 of a particular gene. One denotes L the distance between these two exons and we suppose that the breakpoint is uniformly distributed between these two points.
1. What is the distribution of the position of the breakpoint? Its expected value and variance?
2. What is the probability that the breakpoint occurs at a distance greater than L/3 from the exon 12?

Exercise 5: Geometric distribution. A jailer wants to open a locked door and have 10 keys (only one opens the door). If he is drunk, he tries the keys one after one, mixing them after every trial. If he is sober, he tries them one by one, and never tries again the same key.
1. Let X be the number of keys he tried when he is drunk. Determine the distribution of X and its expected value.
2. Let Y be the number of keys he tried when he is sober. Determine the distribution of Y

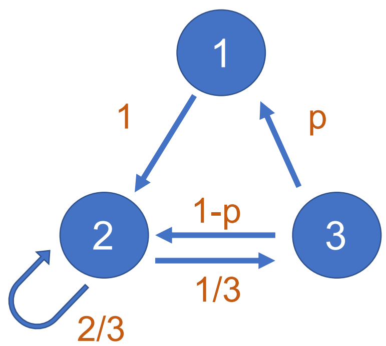
and its expected value.
3. The jailer is drunk on average once every three days. He has tried 9 keys, what is the probability that he is drunk?

Exercise 6: Normal distribution. The length of male northern pikes from the Bleury river is on average equal to 467 mm, with a standard deviation of 47.9 mm. Let's admit that the distribution of these fishes follows a normal distribution. What is the probability of catching a male with a length greater than 460 mm and lower than 480 mm?

Exercise 7: Representation of a Markov chain.
1. Give the state diagram of the Markov chain defined by the following transition matrix:

$$Q = \begin{pmatrix} 0.8 & 0.1 & 0.1 \\ 0.5 & 0 & 0.5 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}$$

2. Give the transition matrix of the Markov chain represented by the following graph. Define the different states according to p.
Then, calculate $P(X_1 = 1|X_0 = 1)$, $P(X_2 = 1|X_0 = 1)$, $P(X_3 = 1|X_0 = 1)$, and $P(X_4 = 1|X_0 = 1)$, $P(X_1 = 2|X_0 = 2)$, $P(X_2 = 2|X_0 = 2)$, and $P(X_3 = 2|X_0 = 2)$.



Exercise 8: Markov chain. Let's suppose that a trait is governed by a gene with two alleles, G and g. G is a dominant allele and g is a recessive allele: Gg is the hybrid state, GG is the dominant state, and gg is the recessive state.
1. A breeder adopts the following strategy: each time, he matches the individual of the n^{th}-generation with a hybrid. Model the situation using a Markov chain and classify the states.
2. Another breeder adopts the following strategy: each time, he matches the individual of the n^{th}-generation with a dominant. Model the situation using a Markov chain and classify the states.

Exercises - Lecture 5

Exercise 1: Normal distribution. Using the table of the standard normal distribution in the Appendix:
1. $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 9)$, calculate $P(|X - 3| > 6)$.
2. $X \sim \mathcal{N}(\mu = 5, \sigma^2 = 4)$, calculate $P(2.5 < X < 6.5)$.
3. $X \sim \mathcal{N}(\mu = 3, \sigma^2 = 2.25)$, determinate x such as $P(X < x) = 0.42$.
4. $X \sim \mathcal{N}(\mu = 6, \sigma^2 = 4)$, determinate x such as $P(\mu - x < X < \mu + x) = 0.9$.

Exercise 2: Estimations.
1. A drug indicates that every pill has to contain 2.5 grams of an active substance. 100 pills are randomly sampled and analyzed. They contain on average 2.6 grams of the active substance with an unbiased estimated variance equals to 0.16. Does the drug respect the requirement? Calculate the level of significance of the test.

2. Two samples of apples are collected in order to study their weight. The first sample is constituted by 100 apples with on average a mass of 120 grams and an unbiased estimated variance of 400. The second sample (collected one month after) is constituted by 150 apples with on average a mass of 150 grams and an unbiased estimated variance of 100. Is the difference between the mean weight of the two samples significant (with an alpha risk of 0.05 and 0.01)? Calculate the level of significance of the test.

3. Two radiologists compare their ability to diagnose the same intensity of a disease (range from 0 to 10). They have examined the same 36 radiographs and independently have assigned an intensity. For each radiograph, the difference between the intensities is calculated. The differences are on average equal to 0.5 and their unbiased estimated variance is equal to 1. Are both radiologists diagnose the disease the same way? Calculate the level of significance of the test.

Exercise 3: Confidence interval. We know that vaccination can fail with a probability rate between 10 and 15%. We want to prepare a study to estimate the percentage of immunized persons with $\pm 1\%$ (alpha risk $\alpha = 0.05$). What's the minimal sample size for this experiment?

Exercise 4: Chi-2 test. Two treatments, A and B, are used to cure a particular disease. The results of the treatments are recapitulated in the following table indicating the number of individuals in the different categories (cured, improved, and stationary). Can we say that the two treatments have different effects?

Treatment	Cured	Improved	Stationary	Sum
A	280	210	110	600
B	220	90	90	400
Sum	500	300	200	1000

Exercise 5: ANOVA. Three wheat varieties are studied (A, B, and C) in 5 different localities. Let's consider the quantitative variable X that measures the productivity of a certain wheat variety. Does productivity change according to the wheat variety?

Localities	Variety A	Variety B	Variety C
Locality 1	3	6	3
Locality 2	6	8	3
Locality 3	5	7	2
Locality 4	6	8	2
Locality 5	5	6	5
Sum	25	35	15
Mean	5	7	3

Exercise 6: Predicting the power of a t-test:
We want to test the effect of a drug to decrease body temperature. We assume that the drug will decrease the body temperature by 1.5°C, and that the variance will be equal to 1.

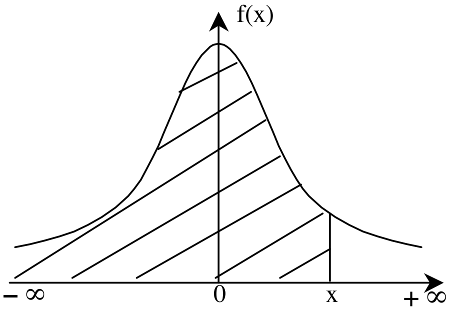
1. What would be the statistical power of the test if you sample 5 individuals?

2. Represent the statistical power of the test as a function of the number of individuals n. Indicate the number of individuals n that needs to be sampled in order to have a statistical power of at least 50% to assess the efficiency of the drug.

3. In March 2020, Pr. D. Raoult declared: "It's counterintuitive, but the smaller the sample size of a clinical test, the more significant its results". Statistically speaking, what do you think about this sentence?

Appendix

Standard normal distribution

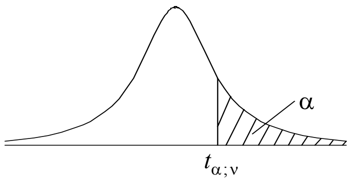


$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

Probability of having a value lower than x

X	0,00	0,01	0,02	0,03	0,04	0,05	0,06	0,07	0,08	0,09
0,0	0,5000	0,5040	0,5080	0,5120	0,5160	0,5199	0,5239	0,5279	0,5319	0,5359
0,1	0,5398	0,5438	0,5478	0,5517	0,5557	0,5596	0,5636	0,5675	0,5714	0,5753
0,2	0,5793	0,5832	0,5871	0,5910	0,5948	0,5987	0,6026	0,6064	0,6103	0,6141
0,3	0,6179	0,6217	0,6255	0,6293	0,6331	0,6368	0,6406	0,6443	0,6480	0,6517
0,4	0,6554	0,6591	0,6628	0,6664	0,6700	0,6736	0,6772	0,6808	0,6844	0,6879
0,5	0,6915	0,6950	0,6985	0,7019	0,7054	0,7088	0,7123	0,7157	0,7190	0,7224
0,6	0,7257	0,7291	0,7324	0,7357	0,7389	0,7422	0,7454	0,7486	0,7517	0,7549
0,7	0,7580	0,7611	0,7642	0,7673	0,7704	0,7734	0,7764	0,7794	0,7823	0,7852
0,8	0,7881	0,7910	0,7939	0,7967	0,7995	0,8023	0,8051	0,8078	0,8106	0,8133
0,9	0,8159	0,8186	0,8212	0,8238	0,8264	0,8289	0,8315	0,8340	0,8365	0,8389
1,0	0,8413	0,8438	0,8461	0,8485	0,8508	0,8531	0,8554	0,8577	0,8599	0,8621
1,1	0,8643	0,8665	0,8686	0,8708	0,8729	0,8749	0,8770	0,8790	0,8810	0,8830
1,2	0,8849	0,8869	0,8888	0,8907	0,8925	0,8944	0,8962	0,8980	0,8997	0,9015
1,3	0,9032	0,9049	0,9066	0,9082	0,9099	0,9115	0,9131	0,9147	0,9162	0,9177
1,4	0,9192	0,9207	0,9222	0,9236	0,9251	0,9265	0,9279	0,9292	0,9306	0,9319
1,5	0,9332	0,9345	0,9357	0,9370	0,9382	0,9394	0,9406	0,9418	0,9429	0,9441
1,6	0,9452	0,9463	0,9474	0,9484	0,9495	0,9505	0,9515	0,9525	0,9535	0,9545
1,7	0,9554	0,9564	0,9573	0,9582	0,9591	0,9599	0,9608	0,9616	0,9625	0,9633
1,8	0,9641	0,9649	0,9656	0,9664	0,9671	0,9678	0,9686	0,9693	0,9699	0,9706
1,9	0,9713	0,9719	0,9726	0,9732	0,9738	0,9744	0,9750	0,9756	0,9761	0,9767
2,0	0,9772	0,9778	0,9783	0,9788	0,9793	0,9798	0,9803	0,9808	0,9812	0,9817
2,1	0,9821	0,9826	0,9830	0,9834	0,9838	0,9842	0,9846	0,9850	0,9854	0,9857
2,2	0,9861	0,9864	0,9868	0,9871	0,9875	0,9878	0,9881	0,9884	0,9887	0,9890
2,3	0,9893	0,9896	0,9898	0,9901	0,9904	0,9906	0,9909	0,9911	0,9913	0,9916
2,4	0,9918	0,9920	0,9922	0,9925	0,9927	0,9929	0,9931	0,9932	0,9934	0,9936
2,5	0,9938	0,9940	0,9941	0,9943	0,9945	0,9946	0,9948	0,9949	0,9951	0,9952
2,6	0,9953	0,9955	0,9956	0,9957	0,9959	0,9960	0,9961	0,9962	0,9963	0,9964
2,7	0,9965	0,9966	0,9967	0,9968	0,9969	0,9970	0,9971	0,9972	0,9973	0,9974
2,8	0,9974	0,9975	0,9976	0,9977	0,9977	0,9978	0,9979	0,9979	0,9980	0,9981
2,9	0,9981	0,9982	0,9982	0,9983	0,9984	0,9984	0,9985	0,9985	0,9986	0,9986
3,0	0,9987	0,9987	0,9987	0,9988	0,9988	0,9989	0,9989	0,9989	0,9990	0,9990
3,1	0,9990	0,9991	0,9991	0,9991	0,9992	0,9992	0,9992	0,9992	0,9993	0,9993
3,2	0,9993	0,9993	0,9994	0,9994	0,9994	0,9994	0,9994	0,9995	0,9995	0,9995
3,3	0,9995	0,9995	0,9995	0,9996	0,9996	0,9996	0,9996	0,9996	0,9996	0,9997
3,4	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9997	0,9998
3,5	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998	0,9998

Table of the Student's *t*-distribution

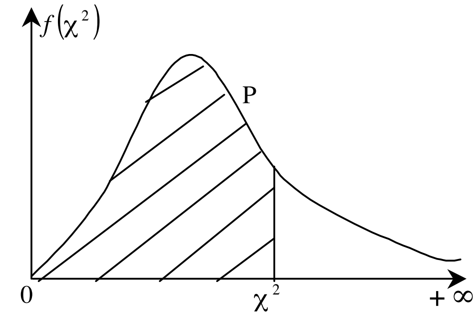


The table gives the values of $t_{\alpha;\nu}$ where
$\Pr(T_\nu > t_{\alpha;\nu}) = \alpha$, with ν degrees of freedom

ν \ α	0.1	0.05	0.025	0.01	0.005	0.001	0.0005
1	3.078	6.314	12.076	31.821	63.657	318.310	636.620
2	1.886	2.920	4.303	6.965	9.925	22.326	31.598
3	1.638	2.353	3.182	4.541	5.841	10.213	12.924
4	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	1.319	1.714	2.069	2.500	2.807	3.485	3.767
24	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	1.310	1.697	2.042	2.457	2.750	3.385	3.646
40	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	1.282	1.645	1.960	2.326	2.576	3.090	3.291

χ^2 distribution

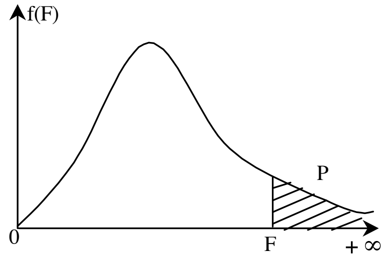
Value of X2 having the corresponding probability of being exceeded



ddl/P	0,5%	1,0%	2,5%	5,0%	10,0%	50,0%	90,0%	95,0%	97,5%	99,0%	99,5%
1	0,000	0,000	0,001	0,004	0,016	0,455	2,706	3,841	5,024	6,635	7,879
2	0,010	0,020	0,051	0,103	0,211	1,386	4,605	5,991	7,378	9,210	10,597
3	0,072	0,115	0,216	0,352	0,584	2,366	6,251	7,815	9,348	11,345	12,838
4	0,207	0,297	0,484	0,711	1,064	3,357	7,779	9,488	11,143	13,277	14,860
5	0,412	0,554	0,831	1,145	1,610	4,351	9,236	11,070	12,832	15,086	16,750
6	0,676	0,872	1,237	1,635	2,204	5,348	10,645	12,592	14,449	16,812	18,548
7	0,989	1,239	1,690	2,167	2,833	6,346	12,017	14,067	16,013	18,475	20,278
8	1,344	1,647	2,180	2,733	3,490	7,344	13,362	15,507	17,535	20,090	21,955
9	1,735	2,088	2,700	3,325	4,168	8,343	14,684	16,919	19,023	21,666	23,589
10	2,156	2,558	3,247	3,940	4,865	9,342	15,987	18,307	20,483	23,209	25,188
11	2,603	3,053	3,816	4,575	5,578	10,341	17,275	19,675	21,920	24,725	26,757
12	3,074	3,571	4,404	5,226	6,304	11,340	18,549	21,026	23,337	26,217	28,300
13	3,565	4,107	5,009	5,892	7,041	12,340	19,812	22,362	24,736	27,688	29,819
14	4,075	4,660	5,629	6,571	7,790	13,339	21,064	23,685	26,119	29,141	31,319
15	4,601	5,229	6,262	7,261	8,547	14,339	22,307	24,996	27,488	30,578	32,801
16	5,142	5,812	6,908	7,962	9,312	15,338	23,542	26,296	28,845	32,000	34,267
17	5,697	6,408	7,564	8,672	10,085	16,338	24,769	27,587	30,191	33,409	35,718
18	6,265	7,015	8,231	9,390	10,865	17,338	25,989	28,869	31,526	34,805	37,156
19	6,844	7,633	8,907	10,117	11,651	18,338	27,204	30,144	32,852	36,191	38,582
20	7,434	8,260	9,591	10,851	12,443	19,337	28,412	31,410	34,170	37,566	39,997
21	8,034	8,897	10,283	11,591	13,240	20,337	29,615	32,671	35,479	38,932	41,401
22	8,643	9,542	10,982	12,338	14,041	21,337	30,813	33,924	36,781	40,289	42,796
23	9,260	10,196	11,689	13,091	14,848	22,337	32,007	35,172	38,076	41,638	44,181
24	9,886	10,856	12,401	13,848	15,659	23,337	33,196	36,415	39,364	42,980	45,558
25	10,520	11,524	13,120	14,611	16,473	24,337	34,382	37,652	40,646	44,314	46,928
26	11,160	12,198	13,844	15,379	17,292	25,336	35,563	38,885	41,923	45,642	48,290
27	11,808	12,878	14,573	16,151	18,114	26,336	36,741	40,113	43,195	46,963	49,645
28	12,461	13,565	15,308	16,928	18,939	27,336	37,916	41,337	44,461	48,278	50,994
29	13,121	14,256	16,047	17,708	19,768	28,336	39,087	42,557	45,722	49,588	52,335
30	13,787	14,953	16,791	18,493	20,599	29,336	40,256	43,773	46,979	50,892	53,672
31	14,458	15,655	17,539	19,281	21,434	30,336	41,422	44,985	48,232	52,191	55,002
32	15,134	16,362	18,291	20,072	22,271	31,336	42,585	46,194	49,480	53,486	56,328
33	15,815	17,073	19,047	20,867	23,110	32,336	43,745	47,400	50,725	54,775	57,648
34	16,501	17,789	19,806	21,664	23,952	33,336	44,903	48,602	51,966	56,061	58,964
35	17,192	18,509	20,569	22,465	24,797	34,336	46,059	49,802	53,203	57,342	60,275

Fisher distribution

Values of F having a probability of 5% of being exceeded



ν_2 \ ν_1	1	2	3	4	5	6	8	10	12	18	24	30	50	60	120
1	161,446	199,499	215,707	224,583	230,160	233,988	238,884	241,882	243,905	247,324	249,052	250,096	251,774	252,196	253,254
2	18,513	19,000	19,164	19,247	19,296	19,329	19,371	19,396	19,412	19,440	19,454	19,463	19,476	19,479	19,487
3	10,128	9,552	9,277	9,117	9,013	8,941	8,845	8,785	8,745	8,675	8,638	8,617	8,581	8,572	8,549
4	7,709	6,944	6,591	6,388	6,256	6,163	6,041	5,964	5,912	5,821	5,774	5,746	5,699	5,688	5,658
5	6,608	5,786	5,409	5,192	5,050	4,950	4,818	4,735	4,678	4,579	4,527	4,496	4,444	4,431	4,398
6	5,987	5,143	4,757	4,534	4,387	4,284	4,147	4,060	4,000	3,896	3,841	3,808	3,754	3,740	3,705
7	5,591	4,737	4,347	4,120	3,972	3,866	3,726	3,637	3,575	3,467	3,410	3,376	3,319	3,304	3,267
8	5,318	4,459	4,066	3,838	3,688	3,581	3,438	3,347	3,284	3,173	3,115	3,079	3,020	3,005	2,967
9	5,117	4,256	3,863	3,633	3,482	3,374	3,230	3,137	3,073	2,960	2,900	2,864	2,803	2,787	2,748
10	4,965	4,103	3,708	3,478	3,326	3,217	3,072	2,978	2,913	2,798	2,737	2,700	2,637	2,621	2,580
11	4,844	3,982	3,587	3,357	3,204	3,095	2,948	2,854	2,788	2,671	2,609	2,570	2,507	2,490	2,448
12	4,747	3,885	3,490	3,259	3,106	2,996	2,849	2,753	2,687	2,568	2,505	2,466	2,401	2,384	2,341
13	4,667	3,806	3,411	3,179	3,025	2,915	2,767	2,671	2,604	2,484	2,420	2,380	2,314	2,297	2,252
14	4,600	3,739	3,344	3,112	2,958	2,848	2,699	2,602	2,534	2,413	2,349	2,308	2,241	2,223	2,178
15	4,543	3,682	3,287	3,056	2,901	2,790	2,641	2,544	2,475	2,353	2,288	2,247	2,178	2,160	2,114
16	4,494	3,634	3,239	3,007	2,852	2,741	2,591	2,494	2,425	2,302	2,235	2,194	2,124	2,106	2,059
17	4,451	3,592	3,197	2,965	2,810	2,699	2,548	2,450	2,381	2,257	2,190	2,148	2,077	2,058	2,011
18	4,414	3,555	3,160	2,928	2,773	2,661	2,510	2,412	2,342	2,217	2,150	2,107	2,035	2,017	1,968
19	4,381	3,522	3,127	2,895	2,740	2,628	2,477	2,378	2,308	2,182	2,114	2,071	1,999	1,980	1,930
20	4,351	3,493	3,098	2,866	2,711	2,599	2,447	2,348	2,278	2,151	2,082	2,039	1,966	1,946	1,896
21	4,325	3,467	3,072	2,840	2,685	2,573	2,420	2,321	2,250	2,123	2,054	2,010	1,936	1,916	1,866
22	4,301	3,443	3,049	2,817	2,661	2,549	2,397	2,297	2,226	2,098	2,028	1,984	1,909	1,889	1,838
23	4,279	3,422	3,028	2,796	2,640	2,528	2,375	2,275	2,204	2,075	2,005	1,961	1,885	1,865	1,813
24	4,260	3,403	3,009	2,776	2,621	2,508	2,355	2,255	2,183	2,054	1,984	1,939	1,863	1,842	1,790
25	4,242	3,385	2,991	2,759	2,603	2,490	2,337	2,236	2,165	2,035	1,964	1,919	1,842	1,822	1,768
26	4,225	3,369	2,975	2,743	2,587	2,474	2,321	2,220	2,148	2,018	1,946	1,901	1,823	1,803	1,749
27	4,210	3,354	2,960	2,728	2,572	2,459	2,305	2,204	2,132	2,002	1,930	1,884	1,806	1,785	1,731
28	4,196	3,340	2,947	2,714	2,558	2,445	2,291	2,190	2,118	1,987	1,915	1,869	1,790	1,769	1,714
29	4,183	3,328	2,934	2,701	2,545	2,432	2,278	2,177	2,104	1,973	1,901	1,854	1,775	1,754	1,698
30	4,171	3,316	2,922	2,690	2,534	2,421	2,266	2,165	2,092	1,960	1,887	1,841	1,761	1,740	1,683
31	4,160	3,305	2,911	2,679	2,523	2,409	2,255	2,153	2,080	1,948	1,875	1,828	1,748	1,726	1,670
32	4,149	3,295	2,901	2,668	2,512	2,399	2,244	2,142	2,070	1,937	1,864	1,817	1,736	1,714	1,657
33	4,139	3,285	2,892	2,659	2,503	2,389	2,235	2,133	2,060	1,926	1,853	1,806	1,724	1,702	1,645
34	4,130	3,276	2,883	2,650	2,494	2,380	2,225	2,123	2,050	1,917	1,843	1,795	1,713	1,691	1,633
35	4,121	3,267	2,874	2,641	2,485	2,372	2,217	2,114	2,041	1,907	1,833	1,786	1,703	1,681	1,623
40	4,085	3,232	2,839	2,606	2,449	2,336	2,180	2,077	2,003	1,868	1,793	1,744	1,660	1,637	1,577
50	4,034	3,183	2,790	2,557	2,400	2,286	2,130	2,026	1,952	1,814	1,737	1,687	1,599	1,576	1,511
80	3,960	3,111	2,719	2,486	2,329	2,214	2,056	1,951	1,875	1,734	1,654	1,602	1,508	1,482	1,411
100	3,936	3,087	2,696	2,463	2,305	2,191	2,032	1,927	1,850	1,708	1,627	1,573	1,477	1,450	1,376
120	3,920	3,072	2,680	2,447	2,290	2,175	2,016	1,910	1,834	1,690	1,608	1,554	1,457	1,429	1,352