Exercises - Lecture 4

Exercise 1 :

1) if the sequences are ordered : 4^n sequences of n-nucleotides.

2) proba (AAA) = $\frac{1}{4^3} = \frac{1}{64}$

3) probability of <u>not having</u> a A : $\left(\frac{3}{4}\right)^3$

proba of having at least one A : $1-\left(\frac{3}{4}\right)^3 = 1 - \frac{27}{64} = \frac{37}{64}$

4) number of A $\hookrightarrow \mathcal{B}\left(\frac{1}{4}, n\right)$: binomial distribution.

Here $n = 100$, by applying the Central limit theorem.

number of A $\hookrightarrow \mathcal{N}\left(\frac{n}{4}, \sigma^2\right)$ with $\sigma^2 = \frac{3}{16}n$

(binomial's variance).

$\hookrightarrow \mathcal{N}(25, 18.75)$.

(NB: the CLT can be applied because $np = n/4 \geqslant 5$ and $n(1-p) \geqslant 5$)

Exercise 2 :

1) C : the person is contaminated and T : the test is positive.

$P(C \cap T) = P(C|T)P(T) = P(T|C)P(C) = 0.99 \times 0.02$

$P(C \cap T) = 0.0198$

2) $P(T) = P(T \cap C) + P(T \cap \bar{C})$

with $P(T \cap \bar{C}) = P(T|\bar{C})P(\bar{C}) = \left(1 - P(\bar{T}|\bar{C})\right)\left(1 - P(C)\right)$

$P(T \cap \bar{C}) = (1 - 0.97)(1 - 0.02) = 0.0294$

so $P(T) = 0.0492$.

3) $P(C|T) = \frac{P(C \cap T)}{P(T)} = \frac{0.0198}{0.0492} \simeq 0.40$

4) $P(\bar{C}|\bar{T}) = \frac{P(\bar{C} \cap \bar{T})}{P(\bar{T})} = \frac{P(\bar{T}|\bar{C})P(\bar{C})}{P(\bar{T})} = \frac{0.97 \times 0.98}{1 - 0.0492} = 0.9998$

5) N : number of contaminated persons $\hookrightarrow \mathcal{B}(n = 10, p = 0.02)$

$P(N=2) = \binom{10}{2} 0.02^2 \, 0.98^8 = 0.015$.

Exercise 3 : Expected values

1) X : gain of the game , $X(\Omega) = \{-x ; 35x\}$.

$$E(X) = \sum_{k \in X(\Omega)} k\, P(X=k) = -x \times \frac{36}{37} + 35x \frac{1}{37} = \frac{-x}{37}$$

2) 6-sided dice : $X(\Omega) = \{1, 2, 3, 4, 5, 6\}$

$$E(X) = \sum_{k \in X(\Omega)} k\, P(X=k) = \sum_{k \in X(\Omega)} k \frac{1}{6} = \frac{1}{6} \sum_{k=1}^{6} k = 3.5$$

n-sided dice : $X(\Omega) = \{1 \dots n\}$.

$$E(X) = \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{n} \sum_{k=1}^{n} k = \frac{1}{n} \frac{(n+1)n}{2} = \frac{n+1}{2}$$

Exercise 4 : Uniform distribution

1) B : position of the breakpoint $\in [0 ; L]$: $B \hookrightarrow \mathcal{U}(0, L)$

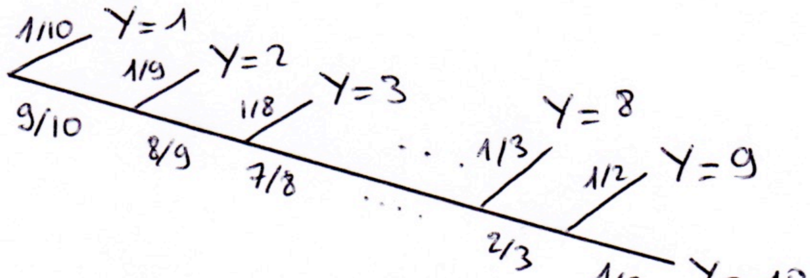
$$E(B) = \frac{L}{2} \quad \text{and} \quad V(B) = \frac{L^2}{12}$$

2) $P\left(B > \frac{L}{3}\right) = 1 - P\left(B \leq \frac{L}{3}\right) = 1 - F_B\left(\frac{L}{3}\right) = 1 - \frac{L/3}{L} = \frac{2}{3}$

Exercise 5 : Geometric distribution.

1) $X \hookrightarrow \mathcal{G}\left(\frac{1}{10}\right)$ and $E(X) = 10$

2) $Y \in [1 : 10]$



$P(Y=1) = 1/10$

$P(Y=2) = 1/10$

$P(Y=3) = 1/10$ (...) $P(Y=10) = 1/10$

$$E(Y) = \sum_{y=1}^{10} y\, P(Y=y) = \frac{1}{10} \sum_{y=1}^{10} y = \frac{11}{2} = 5.5$$

3) D: the jailer is drunk ; Z: he tries 9 keys.

$$P(D|Z) = \frac{P(D\cap Z)}{P(Z)} = \frac{P(D)\,P(Z|D)}{P(Z)}$$

$$= \frac{P(D)\,P(X=9)}{P(D\cap Z) + P(\bar{D}\cap Z)} = \frac{P(D)\,P(X=9)}{P(D)P(Z|D) + P(\bar{D})P(Z|\bar{D})}$$

$$= \frac{P(D)\,P(X=9)}{P(D)P(X=9) + P(\bar{D})P(Y=9)} = 0.18 .$$

with $P(X=9) = \left(\frac{9}{10}\right)^8 \frac{1}{10} = 0.043$

Exercise 6 : Normal distribution.

$N \hookrightarrow \mathcal{N}(\mu = 467, \sigma^2 = 2294)$

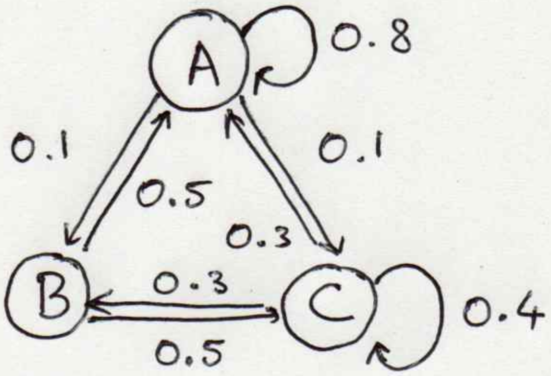
$P(460 \leqslant N \leqslant 480) = P(-7 \leqslant N-\mu \leqslant 13) = P(-0.15 \leqslant \frac{N-\mu}{\sigma} \leqslant 0.27)$

with $\frac{N-\mu}{\sigma} = Z \hookrightarrow \mathcal{N}(0,1)$

$$\begin{aligned} P(460 \leqslant N \leqslant 480) &= P(-0.15 \leqslant Z \leqslant 0.27) = P(Z \leqslant 0.27) - P(Z \leqslant -0.15) \\ &= 0.6064 - (1 - P(Z \leqslant 0.15)) \\ &= 0.6064 - (1 - 0.5596) \\ &= 0.166 \end{aligned}$$

Exercise 7 :

1) State diagram : let's define the states A, B, C



All states are recurrent and aperiodic : the chain is ergodic.

2) Transition matrix : $Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 2/3 & 1/3 \\ p & 1-p & 0 \end{pmatrix}$

if $p = 0$, state 1 is transient, states 2 and 3 are recurrent. state 2 is periodic

if $p = 1$, all states are recurrent, and the state 2 is periodic.

$P(X_1 = 1 \mid X_0 = 1) = 0$

$P(X_2 = 1 \mid X_0 = 1) = 0$

$P(X_3 = 1 \mid X_0 = 1) = \frac{p}{3}$

$P(X_4 = 1 \mid X_0 = 1) = 1 \times \frac{2}{3} \times \frac{1}{3} \times p = \frac{2p}{9}$

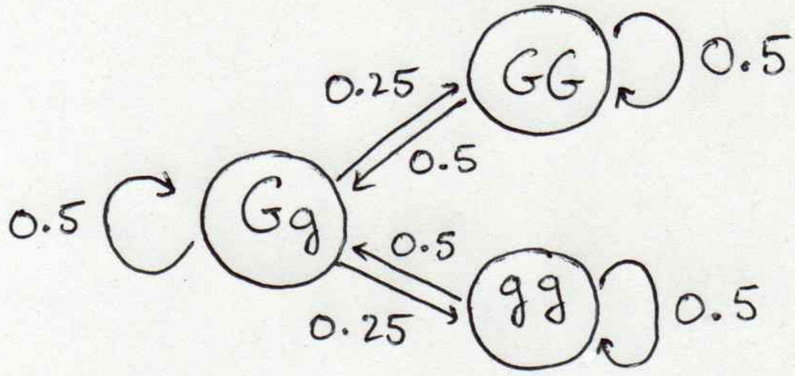
$P(X_1 = 2 \mid X_0 = 2) = \frac{2}{3}$

$P(X_2 = 2 \mid X_0 = 2) = \frac{2}{3} \times \frac{2}{3} + \frac{1}{3}(1-p) = \frac{7}{9} - \frac{p}{3}$

$$\begin{aligned} P(X_3 = 2 \mid X_0 = 2) &= \frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} + \frac{1}{3} \times p \times 1 + 2\left(\frac{1}{3} \times (1-p) \times \frac{2}{3}\right) \\ &= \frac{8}{27} + \frac{p}{3} + \frac{4}{9}(1-p) \\ &= \frac{20}{27} - \frac{p}{9} \end{aligned}$$

Exercise 8 :

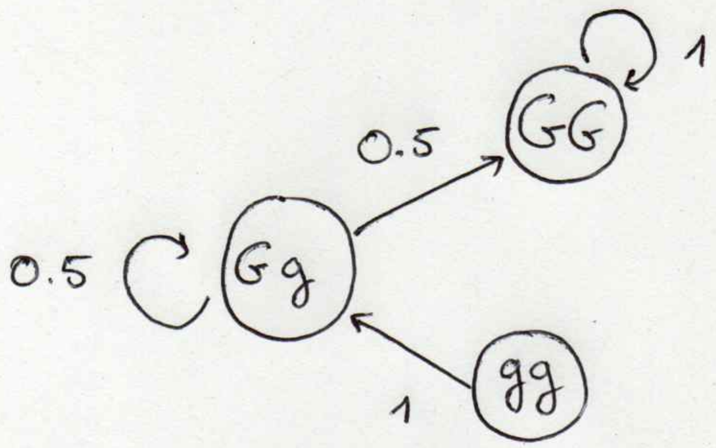
1)



$$Q = \begin{pmatrix} 0.5 & 0.5 & 0 \\ 0.25 & 0.5 & 0.25 \\ 0 & 0.5 & 0.5 \end{pmatrix}$$

All states are recurrent and aperiodic, the chain is ergodic.

2)



$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

GG is an absorbing state, gg and Gg are transient