Exercises Lecture 3 :

Exercise 1 : Logistic model.

1) equilibrium : $N' = 0 \iff rN\left(1 - \frac{N}{K}\right) = 0$ with r and $K > 0$

$\iff N = 0$ or $N = K$.

stability : $J = \frac{dN'}{dN} = r\left(1 - \frac{2N}{K}\right)$

- $J(0) = r > 0$: instable equilibrium.
- $J(K) = -r < 0$: stable equilibrium.

2) equilibrium : $N' = 0 \iff N = K$

stability : $J = \frac{dN'}{dN} = r \ln\left(\frac{K}{N}\right) - r$

$J(K) = -r < 0$: stable equilibrium.

Exercise 2 : Dynamical systems.

1) $\begin{cases} x' = 0 \\ y' = 0 \end{cases} \iff \begin{cases} 4x - 2y = 0 \\ -x + 2y = 0 \end{cases} \iff \begin{cases} y = 2x \\ x = 2y \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$ linear system

$M = \begin{pmatrix} 4 & -2 \\ -1 & 2 \end{pmatrix}$: $\begin{cases} \det(M) = 6 = \lambda_1 \lambda_2 \\ Tr(M) = 6 = \lambda_1 + \lambda_2 \end{cases} \underset{(\dots)}{\iff} \begin{cases} \lambda_1 = 3 - \sqrt{3} \\ \lambda_2 = 3 + \sqrt{3} \end{cases}$

$\lambda_1 > 0$ and $\lambda_2 > 0$: $(0,0)$ is an unstable fixed point.

2) $(0,0)$ is the fixed point of this linear system.

$M = \begin{pmatrix} -2 & -1 \\ -1 & 2 \end{pmatrix}$: $\begin{cases} \det(M) = -5 \\ Tr(M) = 0 \end{cases} \iff \begin{cases} \lambda_1 = \sqrt{5} \\ \lambda_2 = -\sqrt{5} \end{cases}$

$\lambda_1 > 0$ and $\lambda_2 < 0$: $(0,0)$ is a saddle point.

3) $M = \begin{pmatrix} -2 & 0 \\ -1 & 1/2 \end{pmatrix}$: $\begin{cases} \det(M) = -1 = \lambda_1 \lambda_2 \\ Tr(M) = -\frac{3}{2} = \lambda_1 + \lambda_2 \end{cases} \underset{(\dots)}{\iff} \begin{cases} \lambda_1 = -2 \\ \lambda_2 = \frac{1}{2} \end{cases}$

$\lambda_1 < 0$ $\lambda_2 > 0$: $(0,0)$ is a saddle point

4) $M = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$: $\begin{cases} \det(M) = 0 \\ Tr(M) = -3 \end{cases} \iff \begin{cases} \lambda_1 = 0 \\ \lambda_2 = -3 \end{cases}$

fixed points : $\begin{cases} x' = 0 \\ y' = 0 \end{cases} \iff x = y$

$\lambda_2 = -3 < 0$: so all the fixed points $(x = y)$ are stable.

5) $M = \begin{pmatrix} 4 & 2 \\ -1 & 2 \end{pmatrix}$: $\begin{cases} \det(M) = 10 \\ Tr(M) = 6 \end{cases} \underset{(\dots)}{\iff} \begin{cases} \lambda_1 = 3 + i \\ \lambda_2 = 3 - i \end{cases}$

$Re(\lambda_1) = Re(\lambda_2) = 3 > 0$: $(0,0)$ is a unstable fixed point.

6) $M = \begin{pmatrix} 1 & -2 \\ 2 & -2 \end{pmatrix}$: $\begin{cases} \det(M) = 2 \\ Tr(M) = -1 \end{cases} \underset{(\dots)}{\iff} \begin{cases} \lambda_1 = \dfrac{-1 + i\sqrt{7}}{2} \\ \lambda_2 = \dfrac{-1 - i\sqrt{7}}{2} \end{cases}$

$Re(\lambda_1) = Re(\lambda_2) = -\frac{1}{2} < 0$: $(0,0)$ is a stable fixed point.

Exercise 3: Lotka-Volterra :

fixed point : $\begin{cases} x' = 0 \\ y' = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 0 \end{cases}$.

$M = \begin{pmatrix} 3 & -1 \\ 6 & -4 \end{pmatrix}$: $\begin{cases} \det(M) = -6 \\ Tr(M) = -1 \end{cases} \iff \begin{cases} \lambda_1 = -3 \\ \lambda_2 = 2 \end{cases}$

$\lambda_1 < 0 < \lambda_2$: $(0,0)$ is a saddle point.

eigenvectors : $M\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} 3x - y = -3x \\ 6x - 4y = -3y \end{cases} \iff y = 6x$

$M\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} 3x - y = 2x \\ 6x - 4y = 2x \end{cases} \iff y = x$.

so $\begin{pmatrix} 1 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are two eigenvectors of M.

diagonalization : $M = PDP^{-1}$ with $D = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$ and $P = \begin{pmatrix} 1 & 1 \\ 6 & 1 \end{pmatrix}$

and $P^{-1} = \dfrac{1}{-5}\begin{pmatrix} 1 & -1 \\ -6 & 1 \end{pmatrix}$.

solving : $\begin{pmatrix} x' \\ y' \end{pmatrix} = M\begin{pmatrix} x \\ y \end{pmatrix} = PDP^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$

$\iff P^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix} = DP^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$

Let's consider $N = P^{-1}\begin{pmatrix} x \\ y \end{pmatrix}$ and $N' = P^{-1}\begin{pmatrix} x' \\ y' \end{pmatrix}$ ③

Then, $N' = DN \iff N(t) = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{pmatrix} N(0)$

So, $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = P\begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{2t} \end{pmatrix} P^{-1} \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$

$$= \frac{-1}{5}\begin{pmatrix} e^{-3t} & e^{2t} \\ 6e^{-3t} & e^{2t} \end{pmatrix} \begin{pmatrix} x(0) - y(0) \\ -6x(0) + y(0) \end{pmatrix}$$

$$= -\frac{1}{5}\begin{pmatrix} (x(0)-y(0))e^{-3t} + (y(0)-6x(0))e^{2t} \\ (x(0)-y(0))6e^{-3t} + (y(0)-6x(0))e^{2t} \end{pmatrix}$$

with $x(0) = y(0) = 1$:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

2) <u>Fixed point</u>: $\begin{cases} x' = 0 \\ y' = 0 \end{cases} \iff \begin{cases} x' = x(3-y) \\ y' = 2y(3x-2) \end{cases} \iff \begin{cases} (0,0) \\ (\frac{2}{3}, 3) \end{cases}$ are fixed point

$$J_{(\hat{x},\hat{y})} = \begin{pmatrix} 3-y & -x \\ 6y & 6x-4 \end{pmatrix}$$

$J_{(0,0)} = \begin{pmatrix} 3 & 0 \\ 0 & -4 \end{pmatrix}$: $\lambda = \{3, -4\}$: $(0,0)$ is a <u>saddle point</u>.

$J_{(\frac{2}{3},3)} = \begin{pmatrix} 0 & -2/3 \\ 18 & 0 \end{pmatrix}$ $\det(J_{(\frac{2}{3},3)} - \lambda I) = 0 \iff \lambda^2 + 12 = 0$

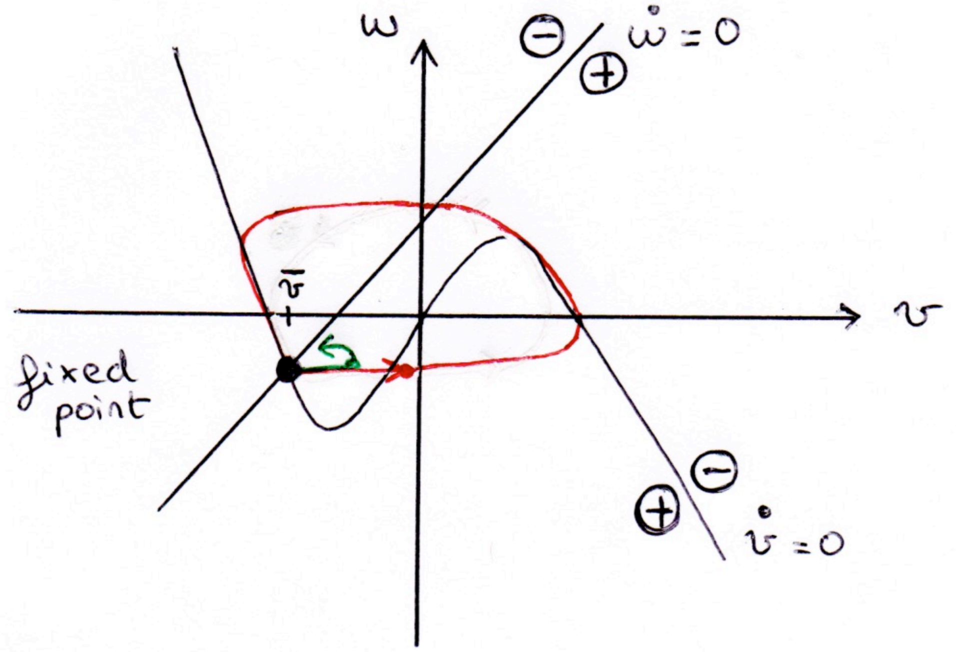
$\iff \lambda = \pm i\sqrt{12}$

So $(\frac{2}{3}, 3)$ isn't stable or unstable: the system oscillates in this neighborhood.

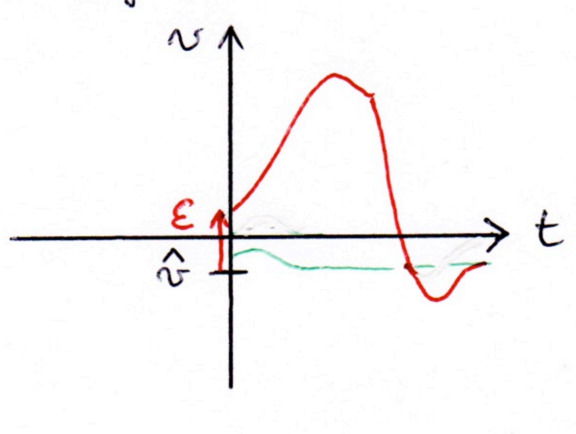
Exercise 4: FitzHugh - Nagumo

1) isoclines:
$$\begin{cases} \dot{v} = 0 \\ \dot{w} = 0 \end{cases} \iff \begin{cases} v - v^3 - w = 0 \\ v + a - bw = 0 \end{cases}$$
$$\iff \begin{cases} w = v - v^3 \\ w = \frac{1}{b}(v+a) \end{cases}$$

fixed point: solution of $v - v^3 = \frac{1}{b}(v+a)$ called $\bar{v}$



2) By applying a pulse $(v + \varepsilon)$ on the system at the equilibrium:
If a and b have values that guarantee the stability of the fixed point, nothing happens if the pulse is low (green arrow).
But if the pulse is strong enought, the system is out of equilibrium and follows a trajectories close to both isoclines (red arrow): it's the action potential of a neuron.



Exercise 5: Bifurcation.

1) $x' = 0 \iff rx - 2x^2 + x^3 = 0 \Rightarrow x = 0$ is a fixed point.

$\frac{dx'}{dx} = r - 4x + 3x^2$ and $\frac{dx'}{dx}(0) = r$

if $r > 0$: $(0,0)$ is unstable

if $r < 0$: $(0,0)$ is stable

2) $x' = 0 \iff x(r - 2x + x^2) = 0 \Rightarrow \Delta = 4 - 4r$

the system has other fixed points if $\Delta \geq 0 \iff 4 - 4r \geq 0 \iff 1 \geq r$

if $r \geqslant 1$, $x_1 = \frac{2 + 2\sqrt{1-r}}{2} = 1 + \sqrt{1-r}$ and $x_2 = 1 - \sqrt{1-r}$

if $r = 1$, $x_3 = 1$.

stability of x_1: $\frac{dx'}{dx}(1+\sqrt{1-r}) = r - 4(1+\sqrt{1-r}) + 3(1+\sqrt{1-r})^2$

$= r - 4(1+\sqrt{1-r}) + 3(1 + 2\sqrt{1-r} + 1 - r)$

$= 2 - 2r + 2\sqrt{1-r}$

$= 2(\sqrt{1-r} + 1)\sqrt{1-r} > 0$

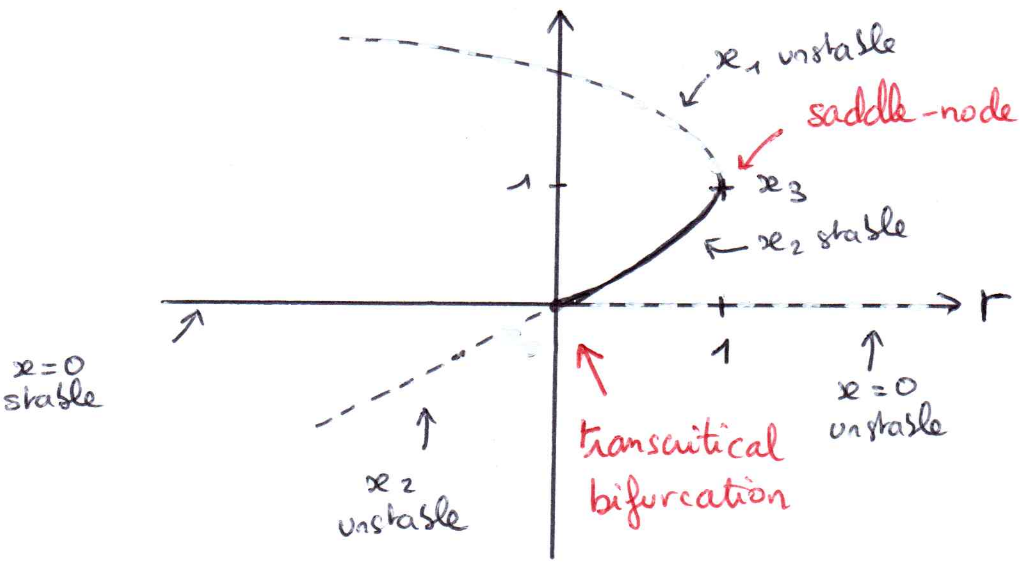
so x_1 is unstable.

stability of x_2: $\frac{dx'}{dx}(1 - \sqrt{1-r}) = 2\sqrt{1-r}(\sqrt{1-r} - 1)$

the stability of x_2 depends on the sign of $(\sqrt{1-r} - 1)$:

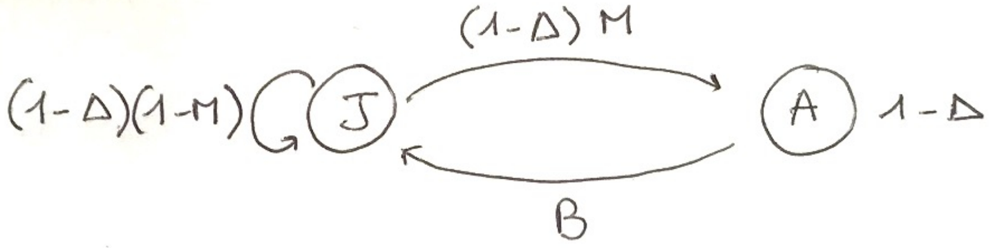
if $0 \leq r \leq 1$, $\sqrt{1-r} - 1 \leq 0$: x_2 is stable

if $r < 0$, $\sqrt{1-r} - 1 > 0$: x_2 is unstable.



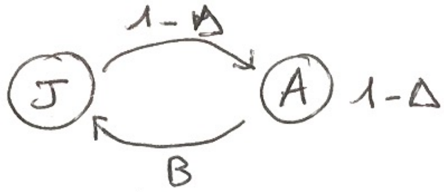
Exercise 6 :

1)



the system can be described by the matrix: $C = \begin{pmatrix} (1-M)(1-\Delta) & B \\ (1-\Delta)M & 1-\Delta \end{pmatrix}$

2) if $M = 1$:



$C = \begin{pmatrix} 0 & B \\ 1-\Delta & 1-\Delta \end{pmatrix}$

3) let's define $X(t) = \begin{pmatrix} x_J(t) \\ x_A(t) \end{pmatrix}$, $C = \begin{pmatrix} 0 & 1/2 \\ 3/4 & 3/4 \end{pmatrix}$, $X(0) = \begin{pmatrix} 100 \\ 0 \end{pmatrix}$

we have $X(t+1) = C\,X(t)$

and we can calculate $X(1) = \begin{pmatrix} 0 \\ 75 \end{pmatrix}$, $X(2) = \begin{pmatrix} 75/2 \\ 75/4 \end{pmatrix}$, ...

also, we have: $X(t) = C^t X(0)$, so the behavior of the system will depend on C^t when $t \to +\infty$.

<u>eigenvalues of C</u>: $\det(C - \lambda I_2) = 0 \iff \lambda^2 - \frac{3}{4}\lambda - \frac{3}{8} = 0$

with $\Delta = \frac{33}{16}$ we have $\lambda_1 = \frac{3}{8} + \frac{\sqrt{33}}{8}$ and $\lambda_2 = \frac{3}{8} - \frac{\sqrt{33}}{8}$

with $\lambda_1 > 0$

You can see with simulation on R (see script on R) that when $\Delta = 0.25$ the system diverge toward $+\infty$, but if $\Delta = 0.5$, it converges toward 0. Actually for $\Delta = 0.5$, the eigenvalues of the system are both < 1. You can demonstrate that the condition for the population to persist is that the dominant eigenvalue λ_1 as to be greater (or equal) than 1.