Exercises Lecture 2

Exercise 1: Linear map.

1) $x = (x_1, x_2)$ and $x' = (x_1', x_2')$

$f(x) = (0,1)$ and $f(x') = (0,1)$ so $f(x) + f(x') = (0,2)$

$f(x+x') = f(x_1 + x_1', x_2 + x_2') = (0,1)$

$f(x+x') \neq f(x) + f(x')$: f <u>is not</u> a linear map.

2) $g(x) = (x_1 + 2x_2, -2x_2)$, $g(x') = (x_1' + 2x_2', -2x_2')$

$g(x+x') = ((x_1 + x_1') + 2(x_2 + x_2'), -2(x_2 + x_2'))$

So $g(x+x') = g(x) + g(x')$

Given $\lambda \in \mathbb{R}$, $g(\lambda x) = (\lambda x_1 + 2\lambda x_2, -2\lambda x_2) = \lambda g(x)$

So g <u>is</u> a linear map.

3) $h(x) = (2x_1^2, 3x_2)$ and $h(x') = (2x_1'^2, 3x_2')$

$h(x+x') = (2(x_1 + x_1')^2, 3(x_2 + x_2'))$

$= (2x_1^2 + 4x_1x_1' + 2x_1'^2, 3x_2 + 3x_2') \neq h(x) + h(x')$

h <u>is not</u> a linear map.

4) Given $\lambda \in \mathbb{R}$, $i(\lambda x) = (2\lambda x_1 - \lambda x_2, 4\ln(\lambda x_3), 3\lambda x_2)$

and $\lambda i(x) = (2\lambda x_1 - \lambda x_2, 4\lambda \ln(x_3), 3\lambda x_2)$

so i <u>is not</u> a linear map

5) $j(x) = (0, 4x_3 - x_2, \sqrt{6}x_3)$ ②

$j(x+x') = (0, 4(x_3 + x_3') - (x_2 + x_2'), \sqrt{6}(x_3 + x_3'))$

$j(x+x') = j(x) + j(x')$.

Given $\lambda \in \mathbb{R}$, $j(\lambda x) = (0, 4\lambda x_3 - \lambda x_2, \sqrt{6}\lambda x_3) = \lambda j(x)$

So j is a linear map

6) $k(x) = (0, 4x_1 - x_2)$ $\quad k(x+x') = (0, 4(x_1+x_1') - (x_2+x_2'))$

So $k(x+x') = k(x) + k(x')$

$k(\lambda x) = (0, 4\lambda x_1 - \lambda x_2) = \lambda k(x)$

So k is a linear map.

Exercise 2 :

1) $g : G = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$ $\quad k : K = \begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix}$

$j : J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 4 \\ 0 & 0 & \sqrt{6} \end{pmatrix}$

2) $a : \begin{cases} \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ (x_1, x_2) \longmapsto (3x_1 - x_2, 2x_2) \end{cases}$

$b : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^1 \\ (x_1, x_2, x_3) \longmapsto (3x_1 - x_2 + 2x_3) \end{cases}$

$c : \begin{cases} \mathbb{R}^1 \longrightarrow \mathbb{R}^3 \\ x_1 \longmapsto (3x_1, 2x_1, -1x_1) \end{cases}$

$d : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ (x_1, x_2, x_3) \longmapsto (3x_1 - x_2, 2x_2 + x_3, x_1 + 3x_3) \end{cases}$

Exercise 3 :

1) $g(5,-3) = (5-6, +6) = (-1, 6)$

2) $j(2,-1,3) = (0, 13, 3\sqrt{6})$.

3) $(k+g)$: $M = K + G = \begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix}$

$(k+g)(1,-1)$: $\begin{pmatrix} 1 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \end{pmatrix}$

4) $j \circ j$: $J^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 4\sqrt{6}-8 \\ 0 & 0 & 6 \end{pmatrix}$

$j \circ j(2,-1,0)$: $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 4\sqrt{6}-8 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$

5) $k \circ g$: $N = K\,G = \begin{pmatrix} 0 & 0 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4 & 10 \end{pmatrix}$

$k \circ g(2,-1)$: $\begin{pmatrix} 0 & 0 \\ 4 & 10 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$

Exercise ~~6~~ 4 :

1) $A = \begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix}$ $\det(A) = 7$ so A is invertible.

$A^{-1} = \dfrac{1}{7} \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$

2) $B = \begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix}$ $\det(B) = 0$ so B <u>is not</u> invertible.

3) $C = \begin{pmatrix} 3 & -1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

$\det(C) = 3(6-0) + 1(0-1) + 0 = 17$: C is invertible.

We are looking for $C^{-1} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ such as $CC^{-1} = I_3$.

i.e, $\begin{pmatrix} 3a-d & 3b-e & 3c-f \\ 2d+g & 2e+h & 2f+i \\ a+3g & b+3h & c+3i \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\iff \begin{cases} 3a-d=1 \\ 3b-e=0 \\ 3c-f=0 \\ 2d+g=0 \\ 2e+h=1 \\ 2f+i=0 \\ a+3g=0 \\ b+3h=0 \\ c+3i=1 \end{cases} \iff (\dots) \iff C^{-1} = \frac{1}{17}\begin{pmatrix} 6 & 3 & -1 \\ 1 & 9 & -3 \\ -2 & -1 & 6 \end{pmatrix}$$

Exercise 5: Given E the canonical basis:

1) $\underset{E\to\beta}{P} = \begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}$ and $\underset{\beta\to E}{P} = \underset{E\to\beta}{P^{-1}} = +\frac{1}{3}\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$

2) $A = \underset{E\to\beta}{P} B \underset{\beta\to E}{P}$ with B the matrix of g in β.

$B = \underset{\beta\to E}{P} A \underset{E\to\beta}{P}$

$\begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$ $\begin{pmatrix} -2 & 1 \\ 3 & 0 \end{pmatrix}$

$B = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix}$ $P = \frac{1}{3}\begin{pmatrix} 0 & -2 \\ 3 & 2 \end{pmatrix}$ $= \frac{1}{3}\begin{pmatrix} -6 & 0 \\ 0 & 3 \end{pmatrix}$

$B = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$

Exercise 6 :

1) $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix}$

$\det(A - \lambda I) = (2-\lambda)^2 - 1 = 4 - 4\lambda + \lambda^2 - 1 = 3 - 4\lambda + \lambda^2$

$\det(A - \lambda I) = 0 \iff 3 - 4\lambda + \lambda^2 = 0$

$\iff \Delta = 16 - 12 =$, ie. $\lambda_1 = 1$, $\lambda_2 = 3$

The eigenvalues of A are 3 and 1 .

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 1\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} 2x + y = x \\ x + 2y = y \end{cases}$$

$$\iff \begin{cases} y = -x \\ y = -x \end{cases}$$

$$A\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} 2x + y = 3x \\ x + 2y = 3y \end{cases}$$

$$\iff \begin{cases} x = y \\ x = y \end{cases}.$$

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are two eigenvectors of A .

$B = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}$ $\quad \det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 3 & -\lambda \end{vmatrix}$

$\det(B - \lambda I) = -(2-\lambda)\lambda - 3 = \lambda^2 - 2\lambda - 3$

$\det(B - \lambda I) = 0 \iff \lambda^2 - 2\lambda - 3 = 0$

$\iff \Delta = 4 + 12 = 16$, $\lambda_1 = 3$ and $\lambda_2 = -1$

The eigenvalues of B are 3 and -1.

$$B\begin{pmatrix}x\\y\end{pmatrix} = 3\begin{pmatrix}x\\y\end{pmatrix} \iff \begin{cases}2x+y=3x\\3x=3y\end{cases} \iff x=y$$

$$B\begin{pmatrix}x\\y\end{pmatrix} = -1\begin{pmatrix}x\\y\end{pmatrix} \iff \begin{cases}2x+y=-x\\3x=-y\end{cases} \iff \left\{ x=-\frac{y}{3}\right.$$

$\begin{pmatrix}1\\1\end{pmatrix}$ and $\begin{pmatrix}-1/3\\1\end{pmatrix}$ are eigenvectors of B.

$$C=\begin{pmatrix}0&1\\1&-1\end{pmatrix} \qquad \det(C-\lambda I)=\begin{vmatrix}-\lambda&1\\1&-1-\lambda\end{vmatrix}=\lambda(1+\lambda)-1$$

$$\det(C-\lambda I)=0 \iff \lambda^2+\lambda-1=0 \iff \Delta=5,\ \lambda=\frac{-1\pm\sqrt5}{2}$$

$$C\begin{pmatrix}x\\y\end{pmatrix}=\frac{-1+\sqrt5}{2}\begin{pmatrix}x\\y\end{pmatrix} \iff \begin{cases}y=\frac{-1+\sqrt5}{2}x\\ x-y=\frac{-1+\sqrt5}{2}y\end{cases}$$

$$C\begin{pmatrix}x\\y\end{pmatrix}=\frac{-1-\sqrt5}{2}\begin{pmatrix}x\\y\end{pmatrix} \iff \begin{cases}y=-\frac{1-\sqrt5}{2}x\\ x-y=\frac{-1-\sqrt5}{2}y\end{cases}$$

$\begin{pmatrix}1\\ \frac{-1+\sqrt5}{2}\end{pmatrix}$ and $\begin{pmatrix}1\\ \frac{-1-\sqrt5}{2}\end{pmatrix}$ are eigenvectors of C.

2) $D_A=\begin{pmatrix}1&0\\0&3\end{pmatrix}$, $P_A=\begin{pmatrix}1&1\\-1&1\end{pmatrix}$, $P_A^{-1}=\frac{1}{2}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$

$D_B=\begin{pmatrix}3&0\\0&-1\end{pmatrix}$, $P_B=\begin{pmatrix}1&-1\\1&3\end{pmatrix}$, $P_B^{-1}=\frac{1}{4}\begin{pmatrix}3&1\\-1&1\end{pmatrix}$

$D_C=\begin{pmatrix}\frac{-1+\sqrt5}{2}&0\\0&\frac{-1-\sqrt5}{2}\end{pmatrix}$, $P_C=\begin{pmatrix}1&1\\ \frac{-1+\sqrt5}{2}&\frac{-1-\sqrt5}{2}\end{pmatrix}$, $P_C^{-1}=\frac{1}{-\sqrt5}\begin{pmatrix}\frac{-1-\sqrt5}{2}&-1\\ \frac{1-\sqrt5}{2}&1\end{pmatrix}$