Exercises Lecture 1

Exercise 1 : Derivatives.

1) $\forall x \in \mathbb{R}, \; f'(x) = 3e^{3x} + 2$

2) $\forall x \in \mathbb{R}_+^*, \; g'(x) = \dfrac{3}{3x+4}$

3) $\forall x \in \mathbb{R}, \; h'(x) = 2e^{-x} - 2xe^{-x} = 2e^{-x}(1-x)$.

4) E ? $3 - 2x \geqslant 0 \iff \frac{3}{2} \geqslant x$.

Moreover, $x \longmapsto \sqrt{x}$ is not differentiable in $\{0\}$.

So $E =]-\infty ; \frac{3}{2}[$.

$\forall x \in]-\infty ; \frac{3}{2}[, \; i'(x) = \dfrac{-2}{2\sqrt{3-2x}} = \dfrac{-1}{\sqrt{3-2x}}$.

5) $\forall (x,y) \in \mathbb{R}^2, \; \dfrac{\partial j}{\partial x} = 3x^2 y + y^2 e^{xy^2}$ $\quad \dfrac{\partial j}{\partial y} = x^3 + 2xy\, e^{xy^2}$

Exercise 2 : Matrices

1) $$\begin{array}{cc} & \begin{pmatrix}1 & 2\\ 3 & -1\end{pmatrix} \\ \begin{pmatrix}1 & 2\\ 2 & 2\end{pmatrix} & = \begin{pmatrix}7 & 0\\ 8 & 2\end{pmatrix}\end{array}$$

2) $$\begin{array}{cc} & \begin{pmatrix}1\\ 2\end{pmatrix} \\ \begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix} & = \begin{pmatrix}5\\ 11\end{pmatrix}\end{array}$$

3) $$\begin{array}{cc} & \begin{pmatrix}1 & 2\\ 0 & -1\end{pmatrix} \\ \begin{pmatrix}1 & 2\\ 3 & 4\\ -1 & 2\end{pmatrix} & = \begin{pmatrix}1 & 0\\ 3 & 2\\ -1 & -4\end{pmatrix}\end{array}$$

4) $$\begin{array}{cc} & \begin{pmatrix}1 & 2\\ 4 & 5\end{pmatrix} \\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} & = \begin{pmatrix}1 & 2\\ 4 & 5\end{pmatrix}\end{array}$$

5) $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}^3 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$

6) $$\begin{array}{cc} & \begin{pmatrix}1 & 2\\ 3 & -2\end{pmatrix} \\ \begin{pmatrix}1 & 2\\ 3 & -2\end{pmatrix} & = \begin{pmatrix}7 & -2\\ -3 & 2\end{pmatrix}\end{array}$$

7) $\begin{pmatrix}1 & 6\\ 0 & -8\end{pmatrix}$

Exercise 3 : Determinant

1) $\det(A) = \begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} = 5 - 12 = -7$

2) $\det(B) = \begin{vmatrix} 1 & -1 \\ 0 & 5 \end{vmatrix} = 5$.

3) $\det(C) = \begin{vmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} + 1\begin{vmatrix} 0 & 2 \\ 1 & -1 \end{vmatrix}$

$\det(C) = 3(4+1) + 1(0-1) + 1(0-2) = 12$

Exercise 4 :

1) $\forall x \in \mathbb{R}$, $e^{x} + 1 > 0$ so f is defined on $\mathbb{R}$.

$$\forall x \in \mathbb{R},\ f(-x) = \frac{e^{-x} - 1}{e^{-x} + 1} = \frac{e^{-x}(1 - e^{x})}{e^{-x}(1 + e^{x})} = \frac{1 - e^{x}}{1 + e^{x}}$$

$$f(-x) = -\frac{e^{x} - 1}{e^{x} + 1} = -f(x)$$: f is a odd function.

2) $$\forall x \in \mathbb{R},\ f'(x) = \frac{e^{x}(e^{x}+1) - (e^{x}-1)e^{x}}{(e^{x}+1)^2}$$

$$= \frac{e^{2x} + e^{x} - e^{2x} + e^{x}}{(e^{x}+1)^2} = \frac{2e^{x}}{(e^{x}+1)^2}$$

$\forall x \in \mathbb{R}$, $f'(x) > 0$ so f is a strictly increasing function.

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
f		↗	

Exercise 5: Differential equations.

1) $(E): y' - 3y = 1 \iff \forall x \in \mathbb{R}, \forall \lambda \in \mathbb{R}, y(x) = \lambda e^{3x} - \frac{1}{3}$

Moreover, $y(1) = -2 \iff \lambda e^{3} - \frac{1}{3} = -2$

$$\iff \lambda e^{3} = -\frac{5}{6}$$

$$\iff \lambda = -\frac{5}{6} e^{-3}$$

So, the solution of (E) is $\forall x \in \mathbb{R}, y(x) = -\frac{5}{6} e^{3x-3} - \frac{1}{3}$

2) $(E): 3y' - y = x + 2$

$y = ax + b \iff 3a - ax - b = x + 2$

$$\iff \begin{cases} -a = 1 \\ 3a - b = 2 \end{cases}$$

$$\iff \begin{cases} a = -1 \\ b = -5 \end{cases}$$

$(E) \iff y = -x - 5$.

More general correction: (if we are not in the case $x \to ax + b$)

$3y' - y = x + 2$

$3y' - (y + x) = 2$

$\underbrace{3(y+x)'}_{f'} - 3 - \underbrace{(y+3)}_{f} = 2$

$\forall \lambda \in \mathbb{R}, f(x) = \lambda e^{x/3} - 5$

$y(x) = \lambda e^{x/3} - 5 - x$

For $\lambda = 0$, $y(x) = -5 - x$

Exercise 6 :

1) They are 4 kings among the 32 cards, the probability of this event is $\frac{4}{32} = \frac{1}{8}$

2) $\binom{32}{5} = 201,376$ draws possible.

Number of draws with 4 kings : $\binom{28}{1} = 28$

$$\text{proba (having 4 kings)} = \frac{28}{201,376} = 1.4 . 10^{-4}.$$

3) Number of draws with red cards : $\binom{16}{5} \simeq 4,368$

$$\text{proba (red cards)} = \frac{4,368}{201,376} \simeq 0.02 .$$

4) Number of draws with 2 diamonds and 3 hearts : $\binom{8}{2} \times \binom{8}{3} = 1,568$

$$\text{proba} (2 \diamondsuit\, 3 \heartsuit) = \frac{1,568}{201,376} = 7.8\ 10^{-3} .$$

5) Number of draws without one card of each color : $4 \times \binom{24}{5} = 170,016$

$$\text{proba (every colors)} = \frac{201,376 - 170,016}{201,376} \simeq 0,15$$

6) Number of draws without a king : $\binom{28}{5} = 98,280$.

$$\text{proba (at least one king)} = \frac{201,376 - 98,280}{201,376} \simeq 0.51 .$$

7) 2 types of draws depending on the king of hearts.

- without the king of hearts : $\binom{3}{2} \times \binom{7}{3} = 105$.
- with the king of hearts : $\binom{3}{1} \times \binom{7}{2} \times \binom{21}{1} = 1,323$

$$\text{proba (2 kings, 3 hearts)} = \frac{105 + 1,323}{201,376} = 7.8 . 10^{-3} .$$